Assignment 1.4 CFD

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Navier-Stokes on a staggered grid

In this section we will be showing some properties of Navier stokes on a staggered grid.Her we start by noting the Navier Stokes equations for an in compressible, constant density and viscosity fluid. The discretised Navier stokes equations can be seen in (1).

$$\begin{gathered} \frac{d\mathbf{v}_h}{dt} + N(\mathbf{v}_h) = -G\mathbf{p}_h + \mathbf{f}, \\ D\mathbf{v}_h = \mathbf{g} \end{gathered} \tag{1}$$

Here we assumed that for for the 2D grid that can be seen in Figure 1 the grid spacing $h_1 = h_2 = h$ is constant over the entire grid. Note here that any contributions of in homogeneous BC are contained in the vectors $\mathbf{f}$ and $\mathbf{g}$ and for the velocity field, we assume homogeneous BC.

1. In Figure 1 the mesh can be seen. For this grid we want to show that $D = -G^T$. For this we will start by deriving the divergence matrix D. For this matrix there are 7 unknowns (in the figure there are a total of 17 velocity components but 10 belong to the homogeneous Dirichlet BC and are thus zero and known). The discretization of the continuity equation gives us the following 6 equations:

$$\begin{aligned}
\Omega_{1,1}: &\quad -v^2_{1,\frac{1}{2}}h + v^1_{\frac{3}{2},1}h + v^2_{1,\frac{3}{2}}h - v^1_{\frac{1}{2},1}h = & v^1_{\frac{3}{2},1}h + v^2_{1,\frac{3}{2}}h = 0 \\
\Omega_{2,1}: &\quad -v^2_{2,\frac{1}{2}}h + v^1_{\frac{5}{2},1}h + v^2_{2,\frac{3}{2}}h - v^1_{\frac{3}{2},1}h = & v^1_{\frac{5}{2},1}h + v^2_{2,\frac{3}{2}}h = v^1_{\frac{3}{2},1}h = 0 \\
\Omega_{3,1}: &\quad -v^2_{3,\frac{1}{2}}h + v^1_{\frac{7}{2},1}h + v^2_{3,\frac{3}{2}}h - v^1_{\frac{5}{2},1}h = & v^2_{3,\frac{3}{2}}h - v^1_{\frac{5}{2},1}h = 0 \\
\Omega_{1,2}: &\quad -v^2_{1,\frac{3}{2}}h + v^1_{\frac{3}{2},2}h + v^2_{1,\frac{5}{2}}h - v^1_{\frac{1}{2},2}h = & -v^2_{1,\frac{3}{2}}h + v^1_{\frac{3}{2},2}h = 0 \\
\Omega_{2,2}: &\quad -v^2_{2,\frac{3}{2}}h + v^1_{\frac{5}{2},2}h + v^2_{2,\frac{5}{2}}h - v^1_{\frac{3}{2},2}h = & -v^2_{2,\frac{3}{2}}h + v^1_{\frac{5}{2},2}h - v^1_{\frac{3}{2},2}h = 0 \\
\Omega_{3,2}: &\quad -v^2_{3,\frac{3}{2}}h + v^1_{\frac{7}{2},2}h + v^2_{3,\frac{5}{2}}h - v^1_{\frac{5}{2},2}h = & -v^2_{3,\frac{3}{2}}h - v^2_{\frac{5}{2},2}h = 0
\end{aligned}$$

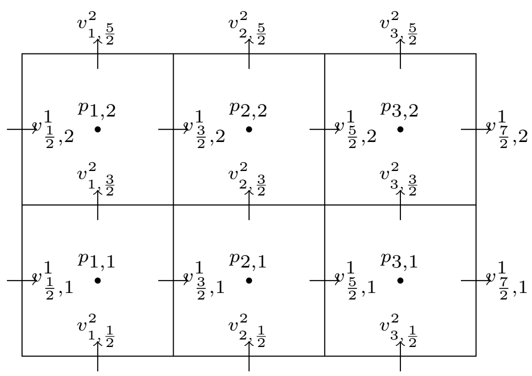

Figure 1: The mesh used to discritise the Navier Stokes equation.

Putting this all in matrix form and combining the fact that $h_1 = h_2 = h$ we obtain the following matrix equation:

$$h\begin{bmatrix} 1 & & & & 1 & & \\ -1 & 1 & & & & 1 & \\ & -1 & & & & & 1 \\ & & 1 & & -1 & & \\ & & -1 & 1 & & -1 & \\ & & & -1 & & & -1 \end{bmatrix}\begin{bmatrix} v^1_{\frac{3}{2},1} \\ v^1_{\frac{5}{2},1} \\ v^1_{\frac{3}{2},2} \\ v^1_{\frac{5}{2},2} \\ v^2_{1,\frac{3}{2}} \\ v^2_{2,\frac{3}{2}} \\ v^2_{3,\frac{3}{2}} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \tag{2}$$

Here the matrix is the desired divergence operator D.

Next we will derive the gradient matrix G for the pressure. From the discretization of the momentum equation we obtain 7 equations with 6 unknowns since there are 7 staggered grids. The 7 equations are:

$$\begin{aligned}
\Omega_{\frac{3}{2},1}:\quad & hp_{2,1} - hp_{1,1} && = 0 \\
\Omega_{\frac{5}{2},1}:\quad & hp_{3,1} - hp_{2,1} && = 0 \\
\Omega_{\frac{3}{2},2}:\quad & hp_{2,2} - hp_{1,2} && = 0 \\
\Omega_{\frac{5}{2},2}:\quad & hp_{3,2} - hp_{2,2} && = 0 \\
\Omega_{1,\frac{3}{2}}:\quad & hp_{1,2} - hp_{1,1} && = 0 \\
\Omega_{2,\frac{3}{2}}:\quad & hp_{2,2} - hp_{2,1} && = 0 \\
\Omega_{3,\frac{3}{2}}:\quad & hp_{3,2} - hp_{3,1} && = 0
\end{aligned}$$

Note that here we set the gradient to zero. E.g. $\Delta p = 0$. The zeros are just there to make sense of the equation and have no significance. As opposed to in the derivation of the divergence before (where the zeros came from the incompressibility constraint). Putting this in to matrix form we obtain:

$$h\begin{bmatrix} -1 & 1 & & & & \\ & -1 & 1 & & & \\ & & & -1 & 1 & \\ & & & & -1 & 1 \\ -1 & & & 1 & & \\ & -1 & & & 1 & \\ & & -1 & & & 1 \end{bmatrix}\begin{bmatrix} p_{1,1} \\ p_{2,1} \\ p_{3,1} \\ p_{1,2} \\ p_{2,2} \\ p_{3,2} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \tag{3}$$

Here the first matrix is the desired gradient operator G. Note that by observation we can see that $D = -G^T$.

2. • For the velocity solution, we wish to find a stream function ϕ which is defined as follows:

$$\begin{bmatrix} v^1 \\ v^2 \end{bmatrix} = \vec{v} = \mathbf{curl}\phi = \begin{bmatrix} \frac{\partial \phi}{\partial y} \\ -\frac{\partial \phi}{\partial x} \end{bmatrix}, \tag{4}$$

   – We would like to see what kind of boundary conditions are automatically put on the constraint function from the Navier Stokes problem stated earlier. Note that on the boundary we have a homogeneous Dirichlet BC. Lets investigate the boundary with normal in the 1 direction ($\hat{x}$). Here we have that $v^1 = 0$ from the BC, so we find have that:

$$\frac{\partial \phi}{\partial y} = v^1 = 0$$

   This means that the stream function ϕ is constant over the boundary. Furthermore the actual value of the stream function has no physical significance. It can be chosen arbitrary as in the definition of the stream function the constant disappears when taking the derivatives. This means that we can set $\phi = 0$ on the boundary. In other words, we impose a homogeneous Dirichlet boundary conditions to the stream-functions. Note that if you choose the constant differently you still obtain a Dirichlet boundary condition.

  – Note that before we have seen that there are 7 unknown quantities of velocity. The vector space of velocity unknowns thus has 7 dimensions.
  – Note that for the stream function ϕ, we will only post process them to exist on the points $(i+\frac{1}{2}, j+\frac{1}{2})$. In other words, only on the grid line intersections. Since we have found that the boundary is known (homogeneous BC), the only unknown points are $\phi_{\frac{3}{2},\frac{3}{2}}$ and $\phi_{\frac{5}{2},\frac{3}{2}}$. The dimension of the unknown stream function values is thus 2.

- We now wish to find a matrix K and vector $\mathbf{r}$ such that the following equation is satisfied:

$$K\Psi_h = \mathbf{v}_h + \mathbf{r},$$

In order to find these, we will start by writing out 7 equations for the unknown velocity components of $\mathbf{v}_h$ in terms of ϕ, these are:

$$
\begin{aligned}
v^1_{\frac{3}{2},1} &= \frac{\partial \phi}{\partial y} = \frac{1}{h}\left(\phi_{\frac{3}{2},\frac{3}{2}} - \phi_{\frac{3}{2},\frac{1}{2}}\right) \to & \frac{1}{h}\phi_{\frac{3}{2},\frac{3}{2}} &= v^1_{\frac{3}{2},1} + \frac{C}{h} \\
v^1_{\frac{5}{2},1} &= \frac{\partial \phi}{\partial y} = \frac{1}{h}\left(\phi_{\frac{5}{2},\frac{3}{2}} - \phi_{\frac{5}{2},\frac{1}{2}}\right) \to & \frac{1}{h}\phi_{\frac{5}{2},\frac{3}{2}} &= v^1_{\frac{5}{2},1} + \frac{C}{h} \\
v^1_{\frac{3}{2},2} &= \frac{\partial \phi}{\partial y} = \frac{1}{h}\left(\phi_{\frac{3}{2},\frac{5}{2}} - \phi_{\frac{3}{2},\frac{3}{2}}\right) \to & -\frac{1}{h}\phi_{\frac{3}{2},\frac{3}{2}} &= v^1_{\frac{3}{2},2} - \frac{C}{h} \\
v^1_{\frac{5}{2},2} &= \frac{\partial \phi}{\partial y} = \frac{1}{h}\left(\phi_{\frac{5}{2},\frac{5}{2}} - \phi_{\frac{5}{2},\frac{3}{2}}\right) \to & -\frac{1}{h}\phi_{\frac{5}{2},\frac{3}{2}} &= v^1_{\frac{5}{2},2} - \frac{C}{h} \\
v^2_{1,\frac{3}{2}} &= -\frac{\partial \phi}{\partial x} = \frac{1}{h}\left(\phi_{\frac{3}{2},\frac{3}{2}} - \phi_{\frac{1}{2},\frac{3}{2}}\right) \to & -\frac{1}{h}\phi_{\frac{3}{2},\frac{3}{2}} &= v^2_{1,\frac{3}{2}} - \frac{C}{h} \\
v^2_{2,\frac{3}{2}} &= -\frac{\partial \phi}{\partial x} = \frac{1}{h}\left(\phi_{\frac{5}{2},\frac{3}{2}} - \phi_{\frac{3}{2},\frac{3}{2}}\right) \to & \frac{1}{h}\left(\phi_{\frac{3}{2},\frac{3}{2}} - \phi_{\frac{5}{2},\frac{3}{2}}\right) &= v^2_{2,\frac{3}{2}} \\
v^2_{3,\frac{3}{2}} &= -\frac{\partial \phi}{\partial x} = \frac{1}{h}\left(\phi_{\frac{7}{2},\frac{3}{2}} - \phi_{\frac{5}{2},\frac{3}{2}}\right) \to & \frac{1}{h}\phi_{\frac{5}{2},\frac{3}{2}} &= v^2_{3,\frac{3}{2}} + \frac{C}{h}
\end{aligned}
$$

Here we used that on the boundary ϕ is constant, namely C. We can rewrite this in matrix form to find:

$$
\frac{1}{h}\begin{bmatrix} 1 & \\ & 1 \\ -1 & \\ & -1 \\ -1 & \\ 1 & -1 \\ & 1 \end{bmatrix}
\begin{bmatrix} \phi_{\frac{3}{2},\frac{3}{2}} \\ \phi_{\frac{5}{2},\frac{3}{2}} \end{bmatrix}
=
\begin{bmatrix} v^1_{\frac{3}{2},1} \\ v^1_{\frac{5}{2},1} \\ v^1_{\frac{3}{2},2} \\ v^1_{\frac{5}{2},2} \\ v^2_{1,\frac{3}{2}} \\ v^2_{2,\frac{3}{2}} \\ v^2_{3,\frac{3}{2}} \end{bmatrix}
+
\begin{bmatrix} \frac{C}{h} \\ \frac{C}{h} \\ -\frac{C}{h} \\ -\frac{C}{h} \\ -\frac{C}{h} \\ 0 \\ \frac{C}{h} \end{bmatrix}
\tag{5}
$$

Here the first matrix is the desired matrix K, the vector containing the unknowns for ϕ is abriviated as Ψ_h. The second vector containing the unknown velocities is given by $\mathbf{v}_h$ and the last vector is $\mathbf{r}$. Note that $\mathbf{r} = \mathbf{0}$ when C is chosen to be 0. Next we will calculate DK:

$$
\frac{h}{h}\begin{bmatrix}
1 & & & & 1 & & \\
-1 & 1 & & & & 1 & \\
 & -1 & & & & & 1 \\
 & & 1 & & -1 & & \\
 & & -1 & 1 & & -1 & \\
 & & & -1 & & & -1
\end{bmatrix}
\begin{bmatrix} 1 & \\ & 1 \\ -1 & \\ & -1 \\ -1 & \\ 1 & -1 \\ & 1 \end{bmatrix}
=
\begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}
$$

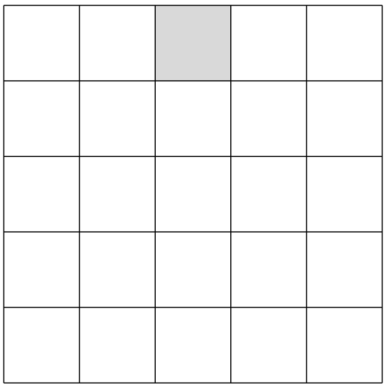

Figure 2: A 5×5 grid.

This means that the discretized ϕ is solenoidal. Since this is all derived for this specific grid we cant claim that this means that the continuous version of ϕ is soleniodal. However:

$$\nabla(\frac{d\phi}{dy}, -\frac{d\phi}{dx}) = \frac{d^2\phi}{dydx} - \frac{d^2\phi}{dxdy}$$

is only zero when $\frac{d^2\phi}{dydx} = \frac{d^2\phi}{dxdy}$ which only occurs when ϕ is twice continuously differentiable.

- In the previous example we created an equation that puts $\mathbf{v}_h$ in term of Ψ. We would however like to put Ψ_h in terms of $\mathbf{v}_h$. For this we start by noting that K^TK is invertible and has inverse:

$$K^TK = \frac{1}{h^2}\begin{bmatrix} 4 & -1 \\ -1 & 4 \end{bmatrix} \qquad (K^TK)^{-1} = \frac{h^2}{15}\begin{bmatrix} 4 & 1 \\ 1 & 4 \end{bmatrix}$$

From this we find that by multiplying (5) by $(K^TK)^{-1}K^T$ we find:

$$\Psi_h = (K^TK)^{-1}K^TK\Psi_h = (K^TK)^{-1}K^T\mathbf{v}_h + (K^TK)^{-1}K^T\mathbf{r}$$

By rewriting this equation we find the following matrix equation:

$$\frac{h}{15}\begin{bmatrix} 4 & 1 & -4 & -1 & -4 & 3 & 1 \\ 1 & 4 & -1 & -4 & -1 & -3 & 4 \end{bmatrix}\vec{v}_h = \Psi - \begin{bmatrix} C \\ C \end{bmatrix} \tag{6}$$

From this we find that the dimension of the stream function Ψ_h is 2.

3. Next we will constrain our efforts to the larger grid that can be seen in Figure 2.
In this section we will solve (1) by using staggered grid discretization for the spatial dimensions and Euler-Forward for the temporal discretization. This discretization gives us the following system of equations:

$$\begin{aligned} \mathbf{v}_h^{(n)} + \Delta t G\mathbf{p}_h^{(n)} &= \mathbf{r}_h^{(n-1)} \\ D\mathbf{v}_h^{(n)} &= \mathbf{g}_h^{(n)} \end{aligned} \tag{7}$$

Here Δt is the difference in time between step n and $n-1$. G is the gradient operator and D is the divergence operator. $\mathbf{r}^{(n-1)}$ contains the forcing terms and all non linear terms depending on $\mathbf{v}_h^{(n-1)}$ and $\mathbf{g}_h^{(n)}$ contains the contributions from the Dirichlet boundary condition.
This system is solved by using pressure-correction. Firstly, the following modified momentum equation is solved:

$$\mathbf{v}_h^* = \mathbf{r}_h^{(n-1)} - \Delta t G\mathbf{p}_h^{(n-1)} \tag{8}$$

Next we want to make sure that mass conservation is still ensured. For this we will solve the following equation:

$$DG\mathbf{p}_h^* = \frac{1}{\Delta t}\left(D\mathbf{v}_h^* - \mathbf{g}_h^{(n)}\right) \tag{9}$$

Now we update both $\mathbf{v}_h^{(n)}$ and $\mathbf{p}_h^{(n)}$ as follows:

$$\begin{aligned}\mathbf{v}_h^{(n)} &= \mathbf{v}_h^* - \Delta t G \mathbf{p}_h^* \\ \mathbf{p}_h^{(n)} &= \mathbf{p}_h^{(n-1)} + \mathbf{p}_h^*\end{aligned} \tag{10}$$

To see why this method ensures mass conservation we will calculate the divergence of the updated velocity. If it conserves mass it should give the following which can be found in (7):

$$D\mathbf{v}_h^{(n)} = \mathbf{g}_h^{(n)}$$

Applying the divergence to the first equation of (10) we find:

$$\begin{aligned}D\mathbf{v}_h^{(n)} =& D\mathbf{v}_h^* - \Delta T DG \mathbf{p}_h^* \\ =& D\mathbf{v}_h^* - D\mathbf{v}_h^* + \mathbf{g}_h^{(n)} \\ =& \mathbf{g}_h^{(n)}\end{aligned}$$

So mass is conserved.

Next we would like to write (9) in to stencil notation for the grey coloured grid of Figure 2:

$$\sum_{i,j} s_{i,j} p_{i,j}^* = k \tag{11}$$

For this we want to derive the DG operator. For this we will first derive terms for the gradient operator. Since we are working on a staggered grid, the gradient values are only calculated for grid borders that are not boundary borders (this is an effect from the staggered grid method). So for the grey grid volume, we only have to consider the $\Omega_{2+\frac{1}{2},5}, \Omega_{3,4+\frac{1}{2}}$ and $\Omega_{3+\frac{1}{2},5}$ staggered volumes (the $\Omega_{3,5+\frac{1}{2}}$ is not an staggered grid volume). So we find the following equations:

$$\begin{aligned}(G\mathbf{p}_h^*)_{2+\frac{1}{2},5} &= h\left(p_{3,5}^* - p_{2,5}^*\right) \\ (G\mathbf{p}_h^*)_{3+\frac{1}{2},5} &= h\left(p_{4,5}^* - p_{3,5}^*\right) \\ (G\mathbf{p}_h^*)_{3,4+\frac{1}{2}} &= h\left(p_{3,5}^* - p_{3,4}^*\right)\end{aligned}$$

The Divergence operator acts only on the non staggered grids. Note however that the boundary is not included in the operator D but is instead taken into account in the $g_h^{(n)}$. So the boundary can be neglected. This means that we only have to consider 3 borders for the gradient operator. Applying the gradient operator directly to Gp^* we find:

$$(DG\mathbf{p}_h^*)_{3,5} = h\left((Gp^*)_{3+\frac{1}{2},5} - (Gp^*)_{2+\frac{1}{2},5} - (Gp^*)_{3,4+\frac{1}{2}}\right) \tag{12}$$

This gives:

$$(DG\mathbf{p}_h^*)_{3,5} = h^2\left(-3p_{3,5}^* + p_{2,5}^* + p_{4,5}^* + p_{3,4}^*\right) \tag{13}$$

This means that we have found the left hand side of (11). We only have to find an expression for the right hand side. For this we can apply the divergence operator to (8), this gives us:

$$D\mathbf{v}_h^* = D\mathbf{r}_h^{(n-1)} - \Delta t DG \mathbf{p}_h^{(n-1)},$$

Substituting this into the right hand side of (9) we obtain with some rewriting:

$$DG\mathbf{p}_h^* = \frac{1}{\Delta t}\left(D\mathbf{r}_h^{(n-1)} - \mathbf{g}_h^{(n)}\right) - DG\mathbf{p}_h^{(n-1)}$$

We can now restrict our self to $(i,j) = (3,5)$ to find:

$$k = \frac{1}{\Delta t}\left(\left(D\mathbf{r}_h^{(n-1)}\right)_{3,5} - g_{3,5}^{(n)}\right) - \left(DG\mathbf{p}_h^{(n-1)}\right)_{3,5}$$

Here $DGp_{3,5}^{(n-1)}$ can be found by replacing $\mathbf{p}_h^*$ by $\mathbf{p}_h^{(n-1)}$ in (13). Furthermore $\left(D\mathbf{r}_h^{(n-1)}\right)_{3,5}$ can be found by replacing $G\mathbf{p}_h^*$ by $\mathbf{r}_h^{(n-1)}$ in (12). This means that the expression for k is given by:

$$k = \frac{1}{\Delta t}\left(hr_{3+\frac{1}{2},5}^{(n-1)} - hr_{2+\frac{1}{2},5}^{(n-1)} - hr_{3,4+\frac{1}{2}}^{(n-1)} - g_{3,5}^{(n-1)}\right) + h^2\left(-3p_{3,5}^{(n-1)} + p_{2,5}^{(n-1)} + p_{4,5}^{(n-1)} + p_{3,4}^{(n-1)}\right)$$

Kinetic energy conservation

- Given the following Navier-Stokes discretization of the general Navier-Stokes equation over a domain Ω.

$$\begin{aligned}\frac{d\mathbf{v}_h}{dt} + C(\mathbf{v}_h)\mathbf{v}_h + B\mathbf{v}_h &= -G\mathbf{p}_h + \mathbf{f},\\ D\mathbf{v}_h &= \mathbf{g},\end{aligned} \tag{14}$$

Since the calculations we want to perform in the section are easier in tensor notation, we will rewrite (14) in to:

$$\begin{aligned}v_{\beta,t} + C(\mathbf{v}_h)_{\beta\alpha}v_\alpha + B_{\beta\alpha}v_\alpha &= -G_{\beta\gamma}p_\gamma + f_\beta,\\ D_{\beta\alpha}v_\alpha &= g_\beta,\end{aligned} \tag{15}$$

Here we have been quite sloppy with covariant and contravariant notation. We will however not perform any transformations so this wont be a problem. Furthermore note that we use Einstein notation convention meaning that summation signs have been omitted. These are redundant since when a product has the same indices, e.g. $B_{\beta,\alpha}\mathbf{v}_\alpha$, we imply that a summation over those indices needs to be taken. e.g. $\sum_\alpha B_{\beta,\alpha}\mathbf{v}_\alpha$. This however means we have to be careful over what indices α, β and γ run. β and α run over all the velocity components. Thus all dimensions and all cells. Thus if these equations have been derived for a $2D$ grid of N cell, α and β have $2N$ components. The pressure is a scalar value, so only has N components. For this reason the pressure has γ as indices as to make the difference more clear.

In this section we want to see how the kinetic energy changes over time. For this the kinetic energy K is defined as follows:

$$K = \frac{1}{2}\mathbf{v}_h^T\mathbf{v}_h = \frac{1}{2}v_\beta v_\beta,$$

To see how the kinetic energy changes over time, we will take the total time derivative of the kinetic energy. From this we find:

$$\frac{DK}{Dt} = \frac{\partial K}{\partial t} + \frac{\partial K}{\partial x_\mu}\frac{\partial x_\mu}{\partial t} = K_{,t} + K_{,\mu}v_\mu = v_\beta v_{\beta,t} + v_\beta v_{\beta,\mu}v_\mu$$

Here μ runs over the dimension indices and β over the grid dimension times grid number indices (just as β was defined before). We now wish to find an expression for $v_{\beta,\mu}v_\mu$. To find this we draw the similarities to equation 5.2 from the lecture notes. Here the Navier-Stokes equation for a domain Ω_β is given by

$$v_{\beta,t} + v_\mu v_{\beta,\mu} + p_{,\beta} - Re^{-1}v_{\beta,\mu\mu} - f_\beta = 0,$$

Here μ again runs over the dimensions, β also runs over the dimensions (not like how it was previously defined), but we can easily extend it to how we defined β previously as this equation must also hold for all grids Ω_β. Only the p_β term needs extra clarification in this extension. Since β indicates a grid and a dimension, we mean with $p_{,\beta}$ the pressure of the β grid differentiated to the β dimension. This equation has been discretized which resulted in the following matrix equation over the entire domain:

$$\frac{\mathbf{v}_h}{dt} + C(\mathbf{v}_h)\mathbf{v}_h + B\mathbf{v}_h + G\mathbf{p}_h - \mathbf{f} = \mathbf{0}$$

Thus the discretization of $v_{\beta,\mu}v_\mu$ is given by $C(\mathbf{v}_h)\mathbf{v}_h = C(\mathbf{v}_h)\mathbf{v}_h = C(\mathbf{v}_h)_{\beta\alpha}v_\alpha$. Here we can use the α indices instead of μ. Combining these previous expressions with (15) we find the following formula for the total time derivative of the kinetic energy:

$$\begin{aligned}\frac{DK}{Dt} &= v_\beta\left(-C(\mathbf{v}_h)_{\beta\alpha}v_\alpha - B_{\beta\alpha}v_\alpha - G_{\beta\gamma}p_\gamma + f_\beta\right) + v_\beta C(\mathbf{v}_h)_{\beta\alpha}v_\alpha \\ &= v_\beta f_\beta - v_\beta B_{\beta\alpha}v_\alpha - v_\beta G_{\beta\gamma}p_\gamma \\ &= \mathbf{v}_h^T\mathbf{f}_h - \mathbf{v}_h^T B\mathbf{v}_h - \mathbf{v}_h^T G\mathbf{p}_h\end{aligned} \tag{16}$$

This is the equation that describes how the kinetic energy evolves over time.

- For an in-viscid liquid, the Reynolds number is very large, in fact $Re^{-1} = 0$. This means that inertia dominates which would imply (comparing (14) to (5.11) from the book) that $B = 0$. Furthermore since we have a staggered grid, we have that $D = -G^T$. So $g_\gamma = D_{\gamma\alpha}v_\alpha = -G_{\alpha\gamma}v_\alpha$. Using this we can rewrite the gradient pressure term in (16) into:

$$-\mathbf{v}_h^T G\mathbf{p}_h = -v_\alpha G_{\alpha\gamma}p_\gamma = g_\gamma p_\gamma = 0$$

Since the boundary conditions on v are homogeneous, we have that $g = 0$. This means that (16) can be updated into:

$$\frac{DK}{Dt} = \mathbf{v}_h^T\mathbf{f}_h = \sum_\alpha v_\alpha f_\alpha = \sum_\alpha P_\alpha$$

Here P_α is the power supplied on the Ω_α domain in that specific direction. By taking the sum over the entire domain and all directions we find the total power acting on the entire domain Ω.

Finite volume implementation of Stokes flow

In this section we concern ourselves with Stokes flow on the square domain $\Omega := [0,1]^2$. The general boundary value problem (BVP), with Dirichlet boundary conditions, is given by

$$\begin{aligned}-\nabla p + \nabla^2\mathbf{v} &= \mathbf{f}, \text{ on } \Omega \\ \nabla\cdot\mathbf{v} &= 0, \text{ on } \Omega \\ \mathbf{v} &= \mathbf{v}_0, \text{ on } \partial\Omega,\end{aligned} \tag{17}$$

where $\mathbf{v}$ is the fluid velocity field, p the pressure, $\mathbf{v}_0$ the known velocity on the boundary of Ω, and $\mathbf{f}$ the known source function. To solve this boundary value problem numerically we will make use of the finite volume method along with a staggered grid, such as depicted in figure 1. In this case we divide the domain into $J_1 \times J_2$ rectangular unit cells (for the x^1 and x^2 direction respectively). We divide Ω uniformly such that each cell is of size $h^1 \times h^2$ with $h^1 = \frac{1}{J_1}$ and $h^2 = \frac{1}{J_2}$, where h^1 is the cell length in the horizontal, or x^1, direction and h^2 the cell length in the vertical, or x^2, direction. Again, we index each unit cells Ω_{j_1,j_2} according to their horizontal index j_1 $(1 \le j_1 \le J_1)$ and vertical index j_2 $(1 \le j_2 \le J_2)$. We now have approximations $p(\mathbf{x}_{j_1,j_2}) \approx p_{j_1,j_2}$, $v^1(\mathbf{x}_{j_1,j_2}) \approx v^1_{j_1,j_2}$, and $v^2(\mathbf{x}_{j_1,j_2}) \approx v^2_{j_1,j_2}$ with $\mathbf{x}_{j_1,j_2} = (\frac{1}{2}h^1 + (j_1-1)h^1, \frac{1}{2}h^2 + (j_2-1)h^2)$.

1. Let us now start with the divergence condition in BVP (17) on $\mathbf{v}$. We first write

$$\begin{aligned}&\int_\Omega \nabla\cdot\mathbf{v}ds = 0 \xrightarrow{\text{Divergence theorem}} \\ &\int_{\partial\Omega} \mathbf{v}\cdot\hat{\mathbf{n}}dl = 0.\end{aligned} \tag{18}$$

For a general cell Ω_{j_1,j_2}, not touching the boundary, we now get

$$\int_{\partial\Omega_{j_1,j_2}} \mathbf{v}\cdot\hat{\mathbf{n}}dl \approx -h^1 v^2_{j_1,j_2-\frac{1}{2}} + h^1 v^2_{j_1,j_2+\frac{1}{2}} - h^2 v^1_{j_1-\frac{1}{2},j_2} + h^2 v^1_{j_1+\frac{1}{2},j_2}, \tag{19}$$

where we applied the central-point approximation for each of the boundary integrals. Next, we consider the

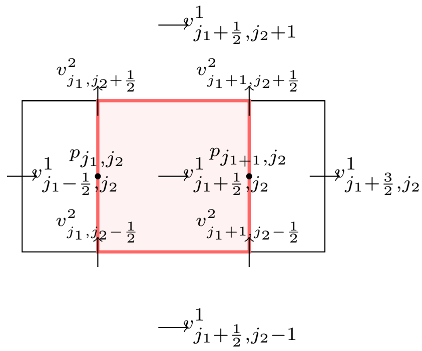

Figure 3: Unit cell $\Omega_{j_1+\frac{1}{2},j_2}$ (red).

first equation of BVP (17) for which we do

$$\begin{gathered}\int_\Omega -\nabla p+\nabla^2\mathbf{v}ds=\int_\Omega \mathbf{f}ds \xrightarrow{\text{Separate equations}}\\ \int_\Omega -\frac{dp}{dx^1}+\frac{dv^1}{dx^1dx^1}+\frac{dv^1}{dx^2dx^2}ds=\int_\Omega f^1ds\\ \int_\Omega -\frac{dp}{dx^2}+\frac{dv^2}{dx^1dx^1}+\frac{dv^2}{dx^2dx^2}ds=\int_\Omega f^2ds \xrightarrow{\text{Divergence theorem}}\\ \int_{\partial\Omega} -p+\frac{dv^1}{dx^1}+\frac{dv^1}{dx^2}ds=\int_\Omega f^1ds\\ \int_{\partial\Omega} -p+\frac{dv^2}{dx^1}+\frac{dv^2}{dx^2}ds=\int_\Omega f^2ds.\end{gathered} \tag{20}$$

Let us now consider a special type of unit cell $\Omega_{j_1+\frac{1}{2},j_2}$ such as depicted in figure 3. Employing this unit cell, away from the boundary, we can write

$$\begin{aligned}&\int_{\partial\Omega_{j_1+\frac{1}{2},j_2}} -p+\frac{dv^1}{dx^1}+\frac{dv^1}{dx^2}ds=\int_{\partial\Omega_{j_1+\frac{1}{2},j_2}} f^1ds \xrightarrow{\text{Central-point approx.}}\\ &h^1\left(\frac{v^1_{j_1+\frac{1}{2},j_2+1}-v^1_{j_1+\frac{1}{2},j_2}}{h^2}-\frac{v^1_{j_1+\frac{1}{2},j_2}-v^1_{j_1+\frac{1}{2},j_2-1}}{h^2}\right)\\ &+h^2\left(p_{j_1,j_2}-p_{j_1+1,j_2}+\frac{v^1_{j_1+\frac{3}{2},j_2}-v^1_{j_1+\frac{1}{2},j_2}}{h^1}-\frac{v^1_{j_1+\frac{1}{2},j_2}-v^1_{j_1-\frac{1}{2},j_2}}{h^1}\right)=h^1h^2f^1(\mathbf{x}_{j_1+\frac{1}{2},j_2})\end{aligned} \tag{21}$$

Similarly, by employing $\Omega_{j_1,j_2+\frac{1}{2}}$ (figure 4), we can write

$$\begin{aligned}&\int_{\partial\Omega_{j_1,j_2+\frac{1}{2}}} -p+\frac{dv^2}{dx^1}+\frac{dv^2}{dx^2}ds=\int_{\partial\Omega_{j_1,j_2+\frac{1}{2}}} f^2ds \xrightarrow{\text{Central-point approx.}}\\ &h^2\left(\frac{v^2_{j_1+1,j_2+\frac{1}{2}}-v^2_{j_1,j_2+\frac{1}{2}}}{h^1}-\frac{v^2_{j_1,j_2+\frac{1}{2}}-v^2_{j_1-1,j_2+\frac{1}{2}}}{h^1}\right)\\ &+h^1\left(p_{j_1,j_2}-p_{j_1,j_2+1}+\frac{v^2_{j_1,j_2+\frac{3}{2}}-v^2_{j_1,j_2+\frac{1}{2}}}{h^2}-\frac{v^2_{j_1,j_2+\frac{1}{2}}-v^2_{j_1,j_2-\frac{1}{2}}}{h^2}\right)=h^1h^2f^2(\mathbf{x}_{j_1,j_2+\frac{1}{2}}).\end{aligned} \tag{22}$$

2. Next, we treat the unit cells at the boundaries. With regards to the discretization presented in equation (19)

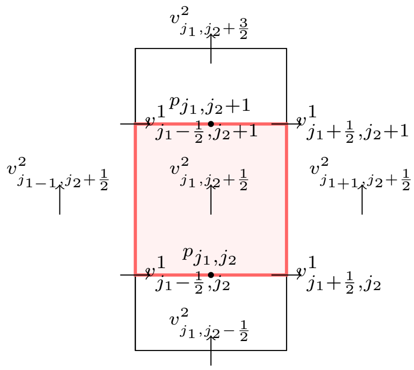

Figure 4: Unit cell $\Omega_{j_1,j_2+\frac{1}{2}}$ (red).

on Ω_{j_1,j_2} we make the following changes on the boundaries:

$$
\begin{aligned}
\Omega_{j_1,1}:\ &-h^1 v^2_{j_1,j_2-\frac{1}{2}} \xrightarrow{\text{Replace by}} -h^1 v^2_0(\mathbf{x}_{j_1,\frac{1}{2}}) \\
\Omega_{J_1,j_2}:\ &h^2 v^1_{j_1+\frac{1}{2},j_2} \xrightarrow{\text{Replace by}} h^2 v^1_0(\mathbf{x}_{J_1+\frac{1}{2},j_2}) \\
\Omega_{j_1,J_2}:\ &h^1 v^2_{j_1,j_2+\frac{1}{2}} \xrightarrow{\text{Replace by}} h^1 v^2_0(\mathbf{x}_{j_1,J_2+\frac{1}{2}}) \\
\Omega_{1,j_2}:\ &-h^2 v^1_{j_1-\frac{1}{2},j_2} \xrightarrow{\text{Replace by}} -h^2 v^1_0(\mathbf{x}_{\frac{1}{2},j_2}).
\end{aligned}
\tag{23}
$$

For the discretization presented in equation (21) on $\Omega_{j_1+\frac{1}{2},j_2}$ we make the following changes on the boundaries:

$$
\begin{aligned}
\Omega_{j_1+\frac{1}{2},1}:\ &h^1\Big(-\frac{v^1_{j_1+\frac{1}{2},j_2}-v^1_{j_1+\frac{1}{2},j_2-1}}{h^2}\Big) \xrightarrow{\text{Replace by}} h^1\Big(-\frac{v^1_{j_1+\frac{1}{2},1}-v^1_0(\mathbf{x}_{j_1+\frac{1}{2},\frac{1}{2}})}{\frac{1}{2}h^2}\Big) \\
\Omega_{J_1-\frac{1}{2},j_2}:\ &h^2\Big(\frac{v^1_{j_1+\frac{3}{2},j_2}-v^1_{j_1+\frac{1}{2},j_2}}{h^1}\Big) \xrightarrow{\text{Replace by}} h^2\Big(\frac{v^1_0(\mathbf{x}_{J_1+\frac{1}{2},j_2})-v^1_{J_1-\frac{1}{2},j_2}}{h^1}\Big) \\
\Omega_{j_1+\frac{1}{2},J_2}:\ &h^1\Big(\frac{v^1_{j_1+\frac{1}{2},j_2+1}-v^1_{j_1+\frac{1}{2},j_2}}{h^2}\Big) \xrightarrow{\text{Replace by}} h^1\Big(\frac{v^1_0(\mathbf{x}_{j_1+\frac{1}{2},J_2+\frac{1}{2}})-v^1_{j_1+\frac{1}{2},J_2}}{\frac{1}{2}h^2}\Big) \\
\Omega_{\frac{3}{2},j_2}:\ &h^2\Big(-\frac{v^1_{j_1+\frac{1}{2},j_2}-v^1_{j_1-\frac{1}{2},j_2}}{h^1}\Big) \xrightarrow{\text{Replace by}} h^2\Big(-\frac{v^1_{\frac{3}{2},j_2}-v^1_0(\mathbf{x}_{\frac{1}{2},j_2})}{h^1}\Big).
\end{aligned}
\tag{24}
$$

For the discretization presented in equation (22) on $\Omega_{j_1,j_2+\frac{1}{2}}$ we make the following changes on the boundaries:

$$
\begin{aligned}
\Omega_{j_1,\frac{3}{2}}:\ &h^1\Big(-\frac{v^2_{j_1,j_2+\frac{1}{2}}-v^2_{j_1,j_2-\frac{1}{2}}}{h^2}\Big) \xrightarrow{\text{Replace by}} h^1\Big(-\frac{v^2_{j_1,\frac{3}{2}}-v^2_0(\mathbf{x}_{j_1,\frac{1}{2}})}{h^2}\Big) \\
\Omega_{J_1,j_2+\frac{1}{2}}:\ &h^2\Big(\frac{v^2_{j_1+1,j_2+\frac{1}{2}}-v^2_{j_1,j_2+\frac{1}{2}}}{h^1}\Big) \xrightarrow{\text{Replace by}} h^2\Big(\frac{v^2_0(\mathbf{x}_{J_1+\frac{1}{2},j_2+\frac{1}{2}})-v^2_{J_1,j_2+\frac{1}{2}}}{\frac{1}{2}h^1}\Big) \\
\Omega_{j_1,J_2-\frac{1}{2}}:\ &h^1\Big(\frac{v^2_{j_1,j_2+\frac{3}{2}}-v^2_{j_1,j_2+\frac{1}{2}}}{h^2}\Big) \xrightarrow{\text{Replace by}} h^1\Big(\frac{v^2_0(\mathbf{x}_{j_1,J_2+\frac{1}{2}})-v^2_{j_1,J_2-\frac{1}{2}}}{h^2}\Big) \\
\Omega_{1,j_2+\frac{1}{2}}:\ &h^2\Big(-\frac{v^2_{j_1,j_2+\frac{1}{2}}-v^2_{j_1-1,j_2+\frac{1}{2}}}{h^1}\Big) \xrightarrow{\text{Replace by}} h^2\Big(-\frac{v^2_{1,j_2+\frac{1}{2}}-v^2_0(\mathbf{x}_{\frac{1}{2},j_2+\frac{1}{2}})}{\frac{1}{2}h^1}\Big).
\end{aligned}
\tag{25}
$$

When a unit cell meets two conditions both replacements are made.

3. The derived finite volume scheme can be expressed as the following linear system

$$
\begin{aligned}
A\mathbf{v}_h + G\mathbf{p}_h &= \mathbf{f}_h \\
D\mathbf{v}_h &= \mathbf{g}_h,
\end{aligned}
\tag{26}
$$

where the first equation and second equation are the discretized version of the first and second equation in BVP (17), respectively. The approximated velocity and pressure on Ω are now obtained by solving for $\mathbf{v}_h$ and $\mathbf{p}_h$. This can be done by first writing the problem as

$$\begin{bmatrix} A & G \\ D & \end{bmatrix} \begin{bmatrix} \mathbf{v}_h \\ \mathbf{p}_h \end{bmatrix} = \begin{bmatrix} \mathbf{f}_h \\ \mathbf{g}_h \end{bmatrix} \tag{27}$$

using the previously described discretizations. Before solving this system we add the following constraint:

$$\int_\Omega p ds = 0. \tag{28}$$

Practically, this boils down to

$$\begin{bmatrix} A & G \\ D & \\ \mathbf{0}^T & \mathbf{1}^T \end{bmatrix} \begin{bmatrix} \mathbf{v}_h \\ \mathbf{p}_h \end{bmatrix} = \begin{bmatrix} \mathbf{f}_h \\ \mathbf{g}_h \\ 0 \end{bmatrix}. \tag{29}$$

The least-squares solution of $\mathbf{v}_h$ and $\mathbf{p}_h$ is now obtained by solving

$$\begin{bmatrix} A & G \\ D & \\ \mathbf{0}^T & \mathbf{1}^T \end{bmatrix}^T \begin{bmatrix} A & G \\ D & \\ \mathbf{0}^T & \mathbf{1}^T \end{bmatrix} \begin{bmatrix} \mathbf{v}_h \\ \mathbf{p}_h \end{bmatrix} = \begin{bmatrix} A & G \\ D & \\ \mathbf{0}^T & \mathbf{1}^T \end{bmatrix}^T \begin{bmatrix} \mathbf{f}_h \\ \mathbf{g}_h \\ 0 \end{bmatrix}. \tag{30}$$

Our implementation of this procedure is included in our submitted **Jupyter notebook**.

4. In order to asses the accuracy of our numerical solver we employ the method of manufactured solutions. For this we take $\mathbf{v} = (\sin(x)\sin(y), \cos(x)\cos(y)$ and $p = \sin(x)\sin(y)$. This choice of $\mathbf{v}$ ensures that

$$\nabla \cdot \mathbf{v} = \cos(x)\sin(y) - \cos(x)\sin(y) = 0. \tag{31}$$

For $\mathbf{f}$ we now have

$$-\nabla p + \nabla^2 \mathbf{v} = (-\cos(x)\sin(y) - 2\sin(y)\sin(x), -\sin(x)\cos(y) - 2\cos(x)\cos(y)). \tag{32}$$

Lastly we take $\mathbf{v}_0 = \mathbf{v}$.

Let us now first look at the numerical solution for $J_1 = 35$ and $J_2 = 30$ presented in figure 5. Regarding the visualized velocities we take

$$\begin{aligned} v^1_{j_1,j_2} &= \frac{v^1_{j_1+\frac{1}{2},j_2} + v^1_{j_1-\frac{1}{2},j_2}}{2} \\ v^2_{j_1,j_2} &= \frac{v^2_{j_1,j_2+\frac{1}{2}} + v^2_{j_1,j_2-\frac{1}{2}}}{2}. \end{aligned} \tag{33}$$

From this figure we can see that the approximate solution matches the behavior of the intended manufactured solution. This serves as confirmation of the correctness of our implementation. We can now evaluate the convergence behavior of the solution as a function of $h = \max(h^1, h^2)$. The error measures are:

$$\begin{aligned} e_p &= \max_{j_1,j_2} |p(\mathbf{x}_{j_1,j_2}) - p_{j_1,j_2}| \\ e_\mathbf{v} &= \max[\max_{j_1,j_2} |v^1(\mathbf{x}_{j_1+\frac{1}{2}}) - v^1_{j_1+\frac{1}{2},j_2}|, \max_{j_1,j_2} |v^2(\mathbf{x}_{j_1,j_2+\frac{1}{2}}) - v^2_{j_1,j_2+\frac{1}{2}}|]. \end{aligned} \tag{34}$$

Note here that p is only defined up to a constant, as BVP (17) only includes its gradient. This is why, in order to attempt to make a fair comparison between $p(\mathbf{x}_{j_1,j_2})$ and p_{j_1,j_2}, we make sure both are centred around the average pressure on Ω before evaluating e_p:

$$\begin{aligned} p(\mathbf{x}_{j_1,j_2}) &\to p(\mathbf{x}_{j_1,j_2}) - \sum_{j_1,j_2=1}^{J_1,J_2} \frac{p(\mathbf{x}_{j_1,j_2})}{J_1 J_2} \\ pj_1, j_2 &\to p_{j_1,j_2} - \sum_{j_1,j_2=1}^{J_1,J_2} \frac{p_{j_1,j_2}}{J_1 J_2}. \end{aligned} \tag{35}$$

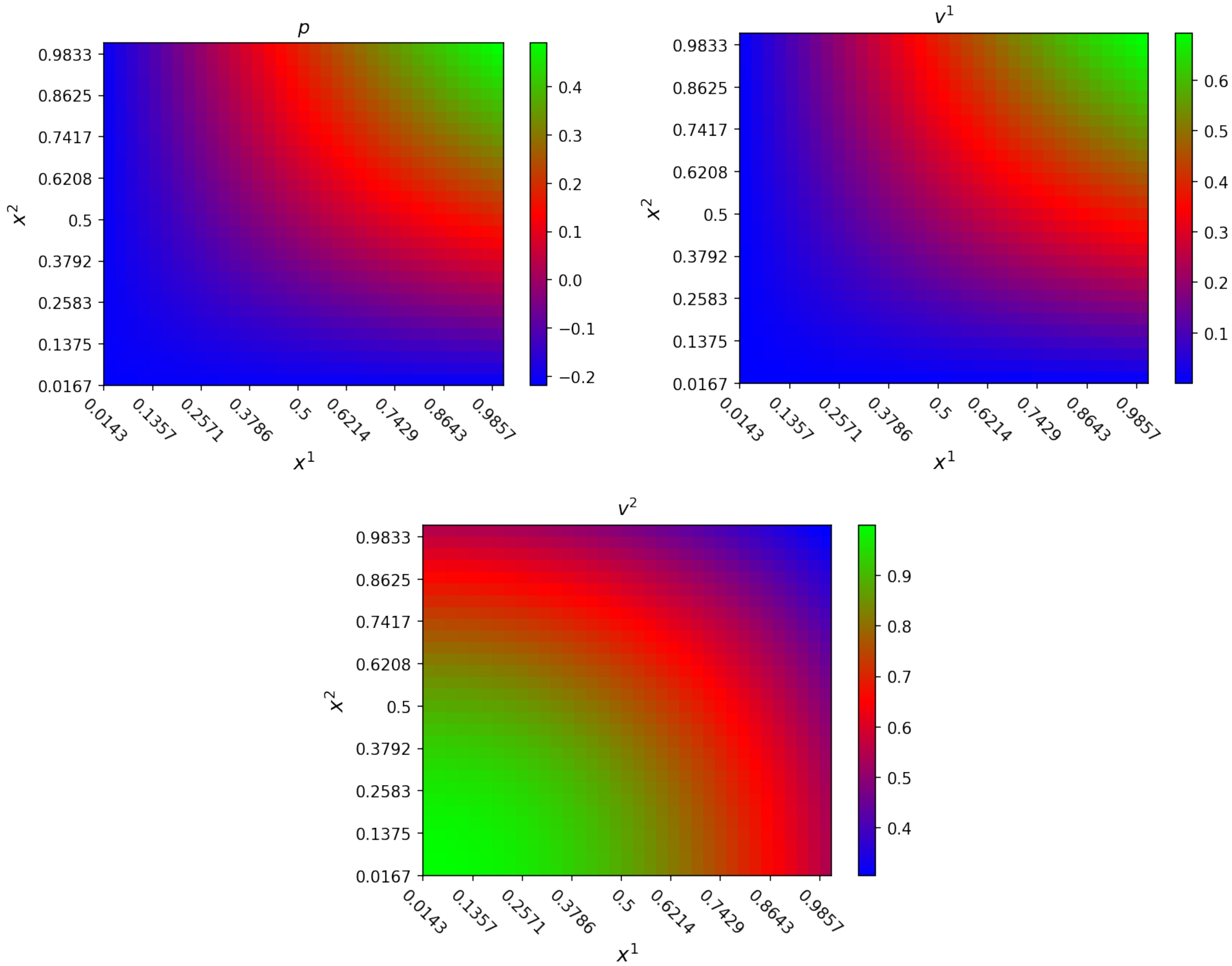

Figure 5: Finite volume solution of BVP (17) with $\mathbf{f} = (-\cos(x)\sin(y) - 2\sin(y)\sin(x), -\sin(x)\cos(y) - 2\cos(x)\cos(y))$ and $\mathbf{v}_0 = (\sin(x)\sin(y), \cos x \cos y)$ for $J_1 = 35$ and $J_2 = 30$.

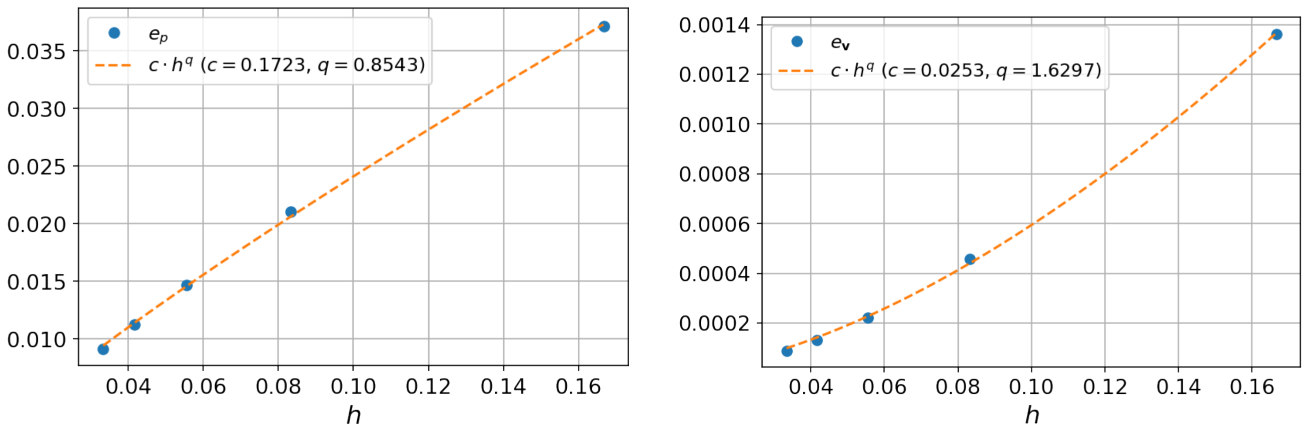

Figure 6: e_p and $e_{\mathbf{v}}$ evaluated for BVP (17) with $\mathbf{f} = (-\cos(x)\sin(y) - 2\sin(y)\sin(x), -\sin(x)\cos(y) - 2\cos(x)\cos(y))$ and $\mathbf{v}_0 = (\sin(x)\sin(y), \cos x \cos y)$. Here, $h = \max(h^1, h^2) = h^2$ for $h^1 = \frac{6}{7}h^2$.

We can finally fit the observed errors to the relation

$$e = c \cdot h^q, \tag{36}$$

where e is a general error measure and c and q the fitted parameters. For the fitting procedure we use the non-linear least-squares procedure provided in the **SciPy** package [1]. The results are shown in figure 6 for $h^1 = \frac{6}{7}h^2$ such that $J_1 = 7k$ and $J_2 = 6k$, for $k = 1, 2, \ldots, 5$. We find $q = 0.8543$ for e_p and faster convergence for $e_{\mathbf{v}}$ with $q = 1.6296$. e_p is also much larger than $e_{\mathbf{v}}$ for all tested values of h. When interpreting these results one has to take into account the correction introduced in (35).

5. After extensively testing our methodology and the convergence of the results, we finally end up at simulating the lid driven cavity flow on Ω. For this we take

$$\mathbf{v}_0 = \begin{cases} (1,0), \ \mathbf{x} = (x^1, 1) \\ (0,0), \ \text{otherwise} \end{cases} . \tag{37}$$

along with $\mathbf{f} = (0, 0)$ for BVP (17). The results for $J_1 = J_2 = 32$ are shown in figure 7.

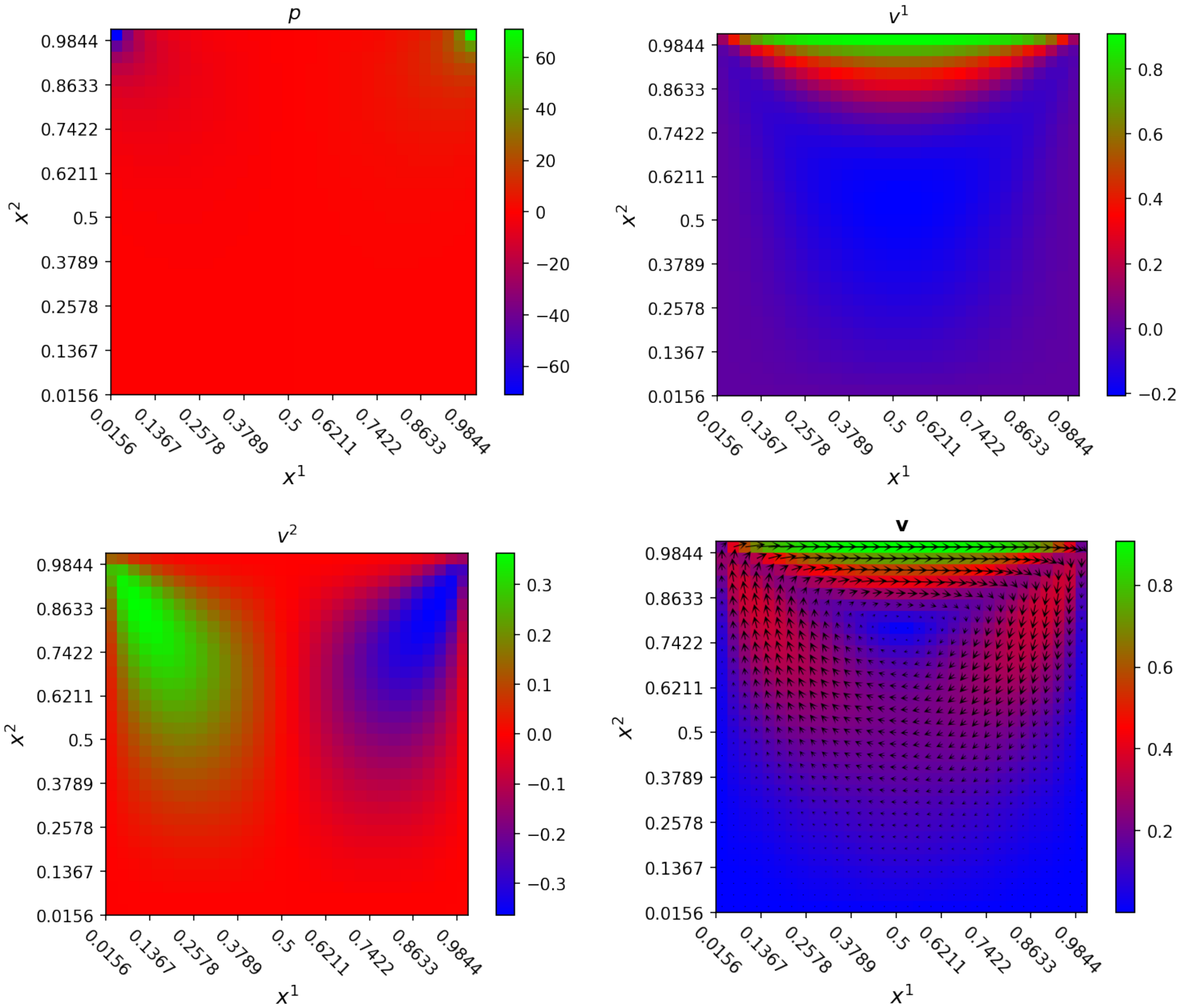

Figure 7: Finite volume solution of BVP (17) with $\mathbf{f} = (0, 0)$ and $\mathbf{v}_0$ given by (37) for $J_1 = J_2 = 32$.

References

[1] Pauli Virtanen, Ralf Gommers, Travis E. Oliphant, Matt Haberland, Tyler Reddy, David Cournapeau, Evgeni Burovski, Pearu Peterson, Warren Weckesser, Jonathan Bright, Stéfan J. van der Walt, Matthew Brett, Joshua Wilson, K. Jarrod Millman, Nikolay Mayorov, Andrew R. J. Nelson, Eric Jones, Robert Kern, Eric Larson, C J Carey, İlhan Polat, Yu Feng, Eric W. Moore, Jake VanderPlas, Denis Laxalde, Josef Perktold, Robert Cimrman, Ian Henriksen, E. A. Quintero, Charles R. Harris, Anne M. Archibald, Antônio H. Ribeiro, Fabian Pedregosa, Paul van Mulbregt, and SciPy 1.0 Contributors. SciPy 1.0: Fundamental Algorithms for Scientific Computing in Python. *Nature Methods*, 17:261–272, 2020.