Automatica 79 (2017) 144–151

Contents lists available at ScienceDirect

Automatica

journal homepage: www.elsevier.com/locate/automatica

Multi-sinusoidal disturbance rejection for discrete-time uncertain stable systems ☆

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ARTICLE INFO

Article history:
Received 24 November 2015
Received in revised form
12 September 2016
Accepted 14 December 2016
Available online 3 March 2017

Keywords:
Linear discrete-time systems
Disturbance rejection
Uncertain systems
Unknown frequencies

ABSTRACT

Linear single-input single-output discrete-time systems $P(z)$ with unknown parameters, order and relative degree are considered, which are perturbed by biased sinusoidal disturbances. Under the assumption that the disturbance frequencies, the sign of the static gain $P(1)$, the sign of either $\mathrm{Re}[P(e^{j\omega_i})]$ or $\mathrm{Im}[P(e^{j\omega_i})]$, for any disturbance frequency ω_i, are known, linear disturbance compensators are proposed which achieve exponential disturbance suppression. Such results are then locally extended to the case of disturbances with unknown frequencies.

1. Introduction and problem statement

The problem of disturbance rejection or disturbance attenuation on the controlled output of dynamical systems is one of the main problems which arise in control applications. The disturbance signal may be of stochastic type or of deterministic type and in the latter case very often it can be modeled as the output of an exogenous system (exosystem). For example, sinusoidal disturbances and periodic disturbances (with a limited number of harmonics) belong to such a class and are largely found in practical control problems (Ben Amara, Kabamba, & Ulsoy, 1999; Hong, Du, Tee, & Ge, 2010; Landau, Alma, Constantinescu, Martinez, & Noe, 2011). The disturbances may be either of known frequencies or of unknown (or partially known) frequencies so that the uncertainty on the exosystem may refer only to the initial conditions or also to the exosystem parameters.

Even though a large amount of papers is available for continuous-time systems (see Brown & Zhang, 2004; Esbrook, Tan, & Khalil, 2013; Fedele & Ferrise, 2013; Pigg & Bodson, 2010 and references contained therein), only a few are available for the discrete-time counterpart. Since the control algorithms are generally discretized to obtain digital controllers, the study of discrete-time control laws is of great interest. In fact, if we consider the linear scalar system

$$\dot{x} = -x + u + d(t) \tag{1}$$

and the disturbance compensator

$$\begin{aligned} \dot{\eta}_1 &= \eta_2 + 2x \\ \dot{\eta}_2 &= -\omega^2 \eta_1 \\ u &= -\eta_1 \end{aligned} \tag{2}$$

in which ω is the frequency of the sinusoidal disturbance $d(t) = d_0 \sin(\omega t + \phi)$, it is easy to see that the closed loop system is asymptotically stable. However, if an Euler first-order discretized version of (2) is applied to system (1), an easy computation shows that when $d = 0$ the zero-order hold discrete-time system obtained from (1), in closed-loop with the Euler discretized version of (2) is unstable, for some sampling time T and for some frequency ω. For example, for $\omega = 3$, $T = 0.1$ we obtain the following eigenvalues: $(-0.8289, -0.0856 \pm j3.2941)$ in continuous-time, $(0.9205, 0.9921 \pm j0.3282)$ in discrete-time.

The most interesting contributions to the solution of the disturbance rejection problem for discrete-time linear systems are discussed in the sequel. Under the assumption that the plant model is known while the disturbance model is of known order but unknown, two adaptive regulators are proposed in Landau et al. (2011). In Hoagg, Santillo, and Bernstein (2008), the parameters of the process and those of the exosystem are supposed to be unknown and a fully adaptive control law is proposed for minimum-phase multi-input multi-output linear systems

☆ The material in this paper was not presented at any conference. This paper was recommended for publication in revised form by Associate Editor Er-Wei Bai under the direction of Editor Torsten Söderström.

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http://dx.doi.org/10.1016/j.automatica.2017.01.038

with known relative degree. The proposed controller achieves global asymptotic output error vanishing but the convergence is not exponential, so that robustness with respect to unmodeled disturbances is not guaranteed. The approach followed in Guo and Bodson (2009) consists in estimating amplitudes, phases and frequencies of the unknown periodic disturbance so that it can be compensated by the control input. Even in this case, the plant model is required to be known. In Marino and Santosuosso (2011), an exponentially converging global regulator is designed for known systems affected by an unknown number (with known upper bound) of sinusoidal disturbances. Several contributions referring to a benchmark example are given in Airimitoaie, Castellanos Silva, and Landau (2013), Aranovskiy and Freidovich (2013), Castellanos Silva, Landau, and Airimitoaie (2013), Chen and Tomizuka (2013), de Callafon and Fang (2013), Karimi and Emedi (2013) and Wu and Ben Amara (2013). All of them assume that the benchmark is accurately identified and robustness to model uncertainties are analyzed only for the case of known frequencies in Aranovskiy and Freidovich (2013) and de Callafon and Fang (2013), where it is shown that stability is preserved for a little amount of parameter uncertainties. In Jafari, Ioannou, Fitzpatrick, and Wang (2015), small model uncertainties are allowed when the disturbance frequencies are known, while ultimately bounded error is guaranteed in the case of unknown frequencies.

The assumption of known plant model is very restrictive since the plant parameters may be different from the nominal ones for several reasons: different operating conditions, abruptly changes as a consequence of faults, tolerance range allowed by manufacturing industries. For instance, the system

$$y(z) = \frac{(1+z)[u(z)+d(z)]}{(z-0.1)(z-0.5)} \tag{3}$$

may reduce to

$$y(z) = \frac{u(z)+(1+z)d(z)}{(z-0.1)(z-0.5)} \tag{4}$$

as a consequence of sensor/actuator fault, so that even its relative degree may be different (see further details in Section 5). In this paper, motivated by the recent results obtained for linear continuous-time systems (Marino & Tomei, 2015, 2016), we consider single-input single-output linear stable discrete-time systems whose order, relative degree and parameters are completely unknown, perturbed by additive biased multi-sinusoidal disturbances. However, the sign of the static gain $P(1)$ of the transfer function $P(z)$ along with either the sign of $\mathrm{Re}[P(e^{j\omega_i})]$ or the sign of $\mathrm{Im}[P(e^{j\omega_i})]$, for any disturbance frequency ω_i, must be known. The problem we are considering is precisely stated in the following definition.

Definition 1.1 (*Disturbance Rejection Problem*). Consider the linear system

$$\begin{aligned} x(k+1) &= Ax(k) + B[u(k)+d(k)],\ x(0)=x_0 \\ y(k) &= Cx(k) + D[u(k)+d(k)] \end{aligned} \tag{5}$$

in which $x \in \mathbb{R}^n$, $y \in \mathbb{R}$, $u \in \mathbb{R}$, $d(k) = d_0 + \sum_{i=1}^{q} d_i \cos(\Omega_i kT + \phi_i)$ is a matching biased multi-sinusoidal disturbance, with $d_i \geq 0$, $0 \leq i \leq q$, $\Omega_i > 0$, $1 \leq i \leq q$, $0 \leq \phi_i < 2\pi$, $1 \leq i \leq q$ and T being the sampling time. Assume that A is a Schur matrix (i.e. all its eigenvalues are strictly inside the unit disk) and denote by

$$P(z) = C(zI-A)^{-1}B + D \tag{6}$$

the transfer function between $u(z)$ and $y(z)$. The disturbance rejection problem is solvable for the linear system (5), if there exists a linear output feedback compensator (see Fig. 1)

$$\xi(k+1) = A_u\xi(k) + B_u y(k), \qquad \xi(0) = \xi_0$$

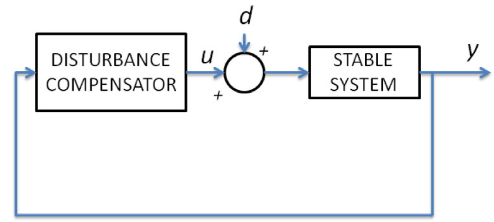

Fig. 1. Block diagram of the disturbance compensator.

$$u(k) = C_u\xi(k) + D_u y(k) \tag{7}$$

such that for the closed-loop system (5), (7), the state vector $x(k)$ and the disturbance compensator error $u(k)+d(k)$ converge exponentially to zero as k tends to infinity, for any initial condition (x_0, ξ_0). □

Remark 1.1. Note that also systems with not-matching disturbances can be reduced to the form (5). In fact, consider the system

$$\begin{aligned} x(k+1) &= Ax(k) + Bu(k) + M\bar{d}(k) \\ y(k) &= Cx(k) + Du(k) + N\bar{d}(k) \end{aligned} \tag{8}$$

in which $\bar{d}(k)$ is a biased multi-sinusoidal disturbance. Assume that an input $u_r(k)$ exists such that (for suitable initial conditions) the disturbance has no effect on the output, i.e.

$$\begin{aligned} x_r(k+1) &= Ax_r(k) + Bu_r(k) + M\bar{d}(k) \\ 0 &= Cx_r(k) + Du_r(k) + N\bar{d}(k). \end{aligned} \tag{9}$$

Defining $\tilde{x} = x - x_r$, from (8) and (9), we obtain

$$\begin{aligned} \tilde{x}(k+1) &= A\tilde{x}(k) + B[u(k) - u_r(k)] \\ y(k) &= C\tilde{x}(k) + D[u(k) - u_r(k)]. \end{aligned} \tag{10}$$

so that by setting $d(k) = -u_r(k)$, we re-obtain (5). □

We show that a copy of the disturbance exosystem (internal model, Francis & Wonham, 1976) driven by the output error fed back by a sufficiently small gain is sufficient to achieve global exponential disturbance rejection. We begin in Section 2 by considering constant and pure sinusoidal disturbances while the general case of biased multi-sinusoidal disturbances is addressed in Section 3. In Section 4, we consider disturbances with unknown frequencies. A local solution for the disturbance rejection problem is proposed in which the unknown frequencies are estimated by a first order updating difference equation for each frequency to be estimated. By means of the averaging method, local converging properties are demonstrated. Finally, two numerical examples are simulated in Section 5 to illustrate the performance of the proposed compensator: the first one is referred to a system with variable relative degree as a consequence of fault; the second one deals with a learning problem in the case of unknown period.

2. Constant and single-frequency disturbances

In this section, some preliminary results are stated and demonstrated. In particular, the problem of disturbance rejection is examined when disturbances are either constant or purely sinusoidal. The constructive proofs of the theorems proposed in this section will be useful to gain a better understanding of the more complex results given in the next section.

Theorem 2.1 (*Constant Disturbance*)**.** *Consider the linear system (5) with $d(k)$ a constant disturbance. Assume that $P(1) \neq 0$ with known sign. Then, there exists a $g^* > 0$ such that for any $0 < g \leq g^*$, the dynamic output feedback compensator*

$$\hat{\eta}(k+1) = \hat{\eta}(k) + g\,\mathrm{sign}[P(1)]y(k)$$

$$u(k) = -\hat{\eta}(k) \tag{11}$$

solves the disturbance rejection problem, according to Definition 1.1.

Proof. Let the constant disturbance be modeled as

$$\begin{aligned} \eta(k+1) &= \eta(k),\ \eta(0) = d_0 \\ d(k) &= \eta(k). \end{aligned} \tag{12}$$

Defining $\tilde{\eta} = \eta - \hat{\eta}$, we can write for the closed-loop system (5), (11), (12)

$$\begin{aligned} y(z) &= P(z)\tilde{\eta}(z) \\ \tilde{\eta}(z) &= -\frac{1}{z-1} g \operatorname{sign}[P(1)] y(z) \end{aligned} \tag{13}$$

from which we obtain

$$y(z)\left\{1 + \frac{1}{z-1} g \operatorname{sign}[P(1)]P(z)\right\} = 0.$$

The closed-loop poles are the solutions of the equation

$$z - 1 + g \operatorname{sign}[P(1)]P(z) = 0. \tag{14}$$

For $g = 0$, the solutions coincide with $z = 1$ and the poles of $P(z)$. Since the poles of $P(z)$ are strictly inside the unit disk, for sufficiently small $g > 0$, the branches of the root locus starting from the poles of $P(z)$ remain inside the unit disk, while the branch starting from $z = 1$ should satisfy the phase condition

$$\arg\{\operatorname{sign}[P(1)]P(z)\} - \arg\{z-1\} = -\pi. \tag{15}$$

Since in a neighborhood of $z = 1$

$$-\pi/2 < \arg\{\operatorname{sign}[P(1)]P(z)\} < \pi/2$$

it follows that in the same neighborhood $\pi/2 < \arg\{z-1\} < 3\pi/2$, which implies that the branch starting from $z = 1$, for sufficiently small $g > 0$, is strictly inside the unit disk. Therefore, for any initial condition, $y(k)$ and $\tilde{\eta}(k)$ tend exponentially to zero. Since by hypothesis A is a Schur matrix, it follows that $x(k)$ tend exponentially to zero. □

In the next theorem, we refer to a sampled sinusoidal disturbance $d(k) = A_1 \sin(k\Omega T + \phi_1)$ in which $A_1 \geq 0$, $\Omega > 0$ and $0 \leq \phi_1 < 2\pi$ are amplitude, frequency and phase, respectively, of the corresponding continuous-time sinusoidal signal. From now on, we define $\omega = \Omega T$ and assume that $\omega \in [0, 2\pi]$, without loss of generality.

Theorem 2.2 (*Pure Sinusoidal Disturbance*)**.** *Consider the linear system (5) with $d(k)$ a pure sinusoidal disturbance given by $d(k) = A_1 \sin(k\omega + \phi_1)$. Let $\bar{z} = \cos\omega + j\sin\omega = e^{j\omega}$, with $\sin\omega \geq 0$, without loss of generality, and assume that either* (A) $\operatorname{Im}[P(\bar{z})] \neq 0$ *with known sign or* (B) $\operatorname{Re}[P(\bar{z})] \neq 0$ *with known sign. Then, there exists a $g^* > 0$ such that for any $0 < g \leq g^*$ the dynamic output feedback compensator*

$$\begin{aligned} \hat{\eta}_1(k+1) &= \hat{\eta}_2(k) + gI_\omega y(k) \\ \hat{\eta}_2(k+1) &= -\hat{\eta}_1(k) + 2(\cos\omega)\hat{\eta}_2(k) + 2(\cos\omega) gI_\omega y(k) \\ u(k) &= -\hat{\eta}_1(k) \end{aligned} \tag{16}$$

in case (A) *or*

$$\begin{aligned} \hat{\eta}_1(k+1) &= \hat{\eta}_2(k) + gR_\omega(\cos\omega) y(k) \\ \hat{\eta}_2(k+1) &= -\hat{\eta}_1(k) + 2(\cos\omega)\hat{\eta}_2(k) - gR_\omega y(k) \\ u(k) &= -\hat{\eta}_1(k) - gR_\omega y(k) \end{aligned} \tag{17}$$

in case (B), *with $I_\omega = \operatorname{sign}\{\operatorname{Im}[P(\bar{z})]\}$ and $R_\omega = \operatorname{sign}\{\operatorname{Re}[P(\bar{z})]\}$, solves the disturbance rejection problem, according to Definition 1.1.*

Proof. Let the disturbance $d(k)$ be modeled as

$$\begin{aligned} \eta_1(k+1) &= \eta_2(k) \\ \eta_2(k+1) &= -\eta_1(k) + 2(\cos\omega)\eta_2(k) \\ d(k) &= \eta_1(k). \end{aligned} \tag{18}$$

Defining $\tilde{\eta}_i = \eta_i - \hat{\eta}_i$, $i = 1, 2$, we can write for the closed-loop system in case (A)

$$\begin{aligned} y(z) &= P(z)\tilde{\eta}_1(z) \\ \tilde{\eta}_1(z) &= -g\frac{I_\omega z}{z^2 - 2(\cos\omega)z + 1} y(z) \end{aligned} \tag{19}$$

from which we have

$$y(z)\left\{1 + g\frac{I_\omega z}{z^2 - 2(\cos\omega)z + 1}P(z)\right\} = 0. \tag{20}$$

The closed-loop poles are the solutions of the equation

$$z^2 - 2(\cos\omega)z + 1 + gI_\omega P(z)z = 0. \tag{21}$$

The solutions for $g = 0$ coincide with the poles of $P(z)$ and with $\cos\omega \pm j\sin\omega$. For sufficiently small $g > 0$, the branches of the root locus starting from the poles of $P(z)$ (which, by hypothesis, are strictly inside the unit circle), remain strictly inside the unit circle, while the branches starting from $\cos\omega \pm j\sin\omega$ (which are on the unit circumference) should satisfy the phase condition

$$\begin{aligned} &\arg\left\{\frac{I_\omega P(z)z}{[z - \cos\omega + j\sin\omega][z - \cos\omega - j\sin\omega]}\right\} \\ &\quad = -\pi \end{aligned} \tag{22}$$

which in the neighborhood of $\bar{z} = \cos\omega + j\sin\omega$ becomes

$$\arg(z) + \arg\{I_\omega P(z)\} - \pi/2 - \arg(z - \bar{z}) = -\pi$$

which implies that

$$\arg(z) - \arg(z - \bar{z}) = -\pi/2 - \arg\{I_\omega P(z)\}. \tag{23}$$

Since, in a neighborhood of $\bar{z}$, $0 < \arg\{I_\omega P(z)\} < \pi$, from (23), it follows that

$$\pi/2 < \arg(z) - \arg(z - \bar{z}) < 3\pi/2 \tag{24}$$

which implies that, for sufficiently small $g > 0$, the branch starting from $\bar{z}$ is strictly inside the unit circle. By the symmetry of the root locus with respect to the real axis, we obtain that the branch starting from $\bar{z}^*$ is strictly inside the unit circle. For the case (B), the closed-loop system is

$$\begin{aligned} y(z) &= P(z)[\tilde{\eta}_1(z) - gR_\omega y(z)] \\ \tilde{\eta}_1(z) &= -g\frac{R_\omega[z(\cos\omega) - 1]}{z^2 - 2(\cos\omega)z + 1} y(z) \end{aligned} \tag{25}$$

from which we obtain

$$y(z)\left\{1 + g\frac{R_\omega z[z - (\cos\omega)]P(z)}{z^2 - 2(\cos\omega)z + 1}\right\} = 0 \tag{26}$$

so that the phase condition (23) in the neighborhood of $\bar{z}$ is substituted by

$$\arg(z) - \arg(z - \bar{z}) = -\pi - \arg\{R_\omega P(z)\}. \tag{27}$$

Since, in a neighborhood of $\bar{z}$,

$$-\pi/2 < \arg\{R_\omega P(z)\} < \pi/2 \tag{28}$$

from (27) and (28), we re-obtain (24). □

Remark 2.1. The sign of the static gain $P(1)$ determines the inversion property of $P(z)$ with respect to constant inputs, while the sign of $\mathrm{Re}[P(\bar{z})] = \mathrm{Re}[P(e^{j\omega})]$ is equal to the sign of the average power (Chua, Desoer, & Kuh, 1987)

$$\lim_{N\to\infty} \frac{1}{2N+1} \sum_{k=-N}^{N} y(k)u(k) \tag{29}$$

which can be either dissipated (positive sign) or generated (negative sign) by the process $P(z)$, in response to a sampled sinusoidal input $u(k) =$ of frequency ω. If $P(z)$ is available for off-line measurements, (29) can be used to obtain the sign of $\mathrm{Re}[P(\bar{z})]$. The sign of $\mathrm{Im}[P(j\omega)]$ has the same interpretation with reference to $\frac{1}{z-(\cos\omega)}P(z)$.

Remark 2.2. From (5) and (16), we have in case (A)

$$\frac{y(z)}{d(z)} = \frac{P(z)(z^2 - 2z\cos\omega + 1)}{z^2 - 2z\cos\omega + 1 + P(z)zgI_\omega}$$

and from (5) and (17) in case (B)

$$\frac{y(z)}{d(z)} = \frac{P(z)(z^2 - 2z\cos\omega + 1)}{z^2 - 2z\cos\omega + 1 + P(z)z(z-\cos\omega)gR_\omega}$$

which in the neighborhood of $\bar{z}$ become in case (A)

$$\frac{y(z)}{d(z)} = \frac{(z-\bar{z})(z-\bar{z}^*)}{g\bar{z}I_\omega} \triangleq G_I(z)$$

and in case (B)

$$\frac{y(z)}{d(z)} = \frac{(z-\bar{z})(z-\bar{z}^*)}{g\bar{z}(\bar{z}-\cos\omega)R_\omega} \triangleq G_R(z)$$

so that the amplitudes of the output for disturbances whose frequency ω_a is not exactly equal to ω in the two cases (A) and (B) are, respectively, multiplied by

$$\begin{aligned} |G_I(e^{j\omega_a})| &\simeq \frac{2|\sin\omega|}{g}[(\cos\omega_a - \cos\omega)^2 \\ &\quad + (\sin\omega_a - \sin\omega)^2]^{\frac{1}{2}} \simeq k_I|\omega_a - \omega| \\ |G_R(e^{j\omega_a})| &\simeq \frac{2}{g}[(\cos\omega_a - \cos\omega)^2 \\ &\quad + (\sin\omega_a - \sin\omega)^2]^{\frac{1}{2}} \simeq k_R|\omega_a - \omega|. \end{aligned} \tag{30}$$

From (30), it follows that the sinusoidal disturbances which are in the neighborhood of ω are still attenuated but not rejected. However, according to the Bode sensitivity integral (Bode, 1945; Mohtadi, 1990), there could be amplification instead of attenuation for frequencies not sufficiently close to the nominal one (waterbed effect).

3. Biased multi-frequency disturbances

This section is devoted to the design of control algorithms for the rejection of biased multi-sinusoidal disturbances of known frequencies and unknown amplitudes and phases.

Theorem 3.1 (*Biased Multi-Sinusoidal Disturbance*). *Consider the linear system (5) with $d(k)$ a biased multi-sinusoidal disturbance given by $d(k) = A_0 + \sum_{i=1}^{q} A_i \sin(\omega_i + \phi_i)$, in which $A_i \geq 0$, $0 \leq i \leq q$, $\omega_i \geq 0$, $1 \leq i \leq q$ and $0 \leq \phi_i < 2\pi$, $1 \leq i \leq q$. Let $\bar{z}_i = \cos\omega_i + j\sin\omega_i$ with $\sin\omega_i \geq 0$, without loss of generality, and assume that either* (A) *$\mathrm{Im}P(\bar{z}_i) \neq 0$ with known sign, $1 \leq i \leq q$, or* (B) *$\mathrm{Re}[P(\bar{z}_i)] \neq 0$ with known sign, $1 \leq i \leq q$. Assume also that $P(1) \neq 0$ with known sign. Then, there exists a $g^* > 0$ such that for any $0 < g \leq g^*$ the dynamic output feedback compensator*

$$\begin{aligned} \hat{\eta}_1(k+1) &= \hat{\eta}_1(k) + gR_0 y(k) \\ \hat{\eta}_{2i}(k+1) &= \hat{\eta}_{2i+1}(k) + gI_{\omega_i} y(k) \\ \hat{\eta}_{2i+1}(k+1) &= -\hat{\eta}_{2i}(k) + 2(\cos\omega_i)\hat{\eta}_{2i+1}(k) \\ &\quad + 2(\cos\omega_i)gI_{\omega_i}y(k), \quad 1 \leq i \leq q \\ u &= -\hat{\eta}_1(k) - \sum_{i=1}^{q} \hat{\eta}_{2i}(k) \end{aligned} \tag{31}$$

in case (A) *or*

$$\begin{aligned} \hat{\eta}_1(k+1) &= \hat{\eta}_1(k) + gR_0 y(k) \\ \hat{\eta}_{2i}(k+1) &= \hat{\eta}_{2i+1}(k) + gR_{\omega_i}(\cos\omega_i) y(k) \\ \hat{\eta}_{2i+1}(k+1) &= -\hat{\eta}_{2i}(k) + 2(\cos\omega_i)\hat{\eta}_{2i+1}(k) \\ &\quad - gR_{\omega_i}y(k), \quad 1 \leq i \leq q \\ u &= -\hat{\eta}_1(k) - \sum_{i=1}^{q} [\hat{\eta}_{2i}(k) + gR_{\omega_i}y(k)] \end{aligned} \tag{32}$$

in case (B), *with $R_0 = \mathrm{sign}[P(1)]$, $R_{\omega_i} = \mathrm{sign}\{\mathrm{Re}[P(\bar{z}_i)]\}$ and $I_{\omega_i} = \mathrm{sign}\{\mathrm{Im}[P(\bar{z}_i)]\}$, solves the disturbance rejection problem, according to Definition 1.1.*

Proof. Let the disturbance $d(k)$ be modeled as

$$\begin{aligned} \eta_1(k+1) &= \eta_1(k) \\ \eta_{2i}(k+1) &= \eta_{2i+1}(k), \quad 1 \leq i \leq q \\ \eta_{2i+1}(k+1) &= -\eta_{2i}(k) + 2(\cos\omega_i)\eta_{2i+1}(k) \\ d(k) &= \eta_1(k) + \sum_{i=1}^{q} \eta_{2i}(k). \end{aligned} \tag{33}$$

Defining $\tilde{\eta}_i = \eta_i - \hat{\eta}_i$, $1 \leq i \leq 2q+1$, we can write for the closed-loop system in case (A)

$$y(z) = P(z)\left[\tilde{\eta}_1(z) + \sum_{i=1}^{q} \tilde{\eta}_{2i}(z)\right]$$

$$\tilde{\eta}_1(z) + \sum_{i=1}^{q} \tilde{\eta}_{2i}(z) = -gy(z)\left\{\frac{R_0}{z-1} + \sum_{i=1}^{q} \frac{I_{\omega_i} z}{z^2 - 2(\cos\omega_i)z + 1}\right\}$$

from which we obtain

$$y(z)\left\{1 + gP(z)\left\{\frac{R_0}{z-1} + \sum_{i=1}^{q} \frac{I_{\omega_i} z}{z^2 - 2(\cos\omega_i)z + 1}\right\}\right\} = 0 \tag{34}$$

or, equivalently,

$$y(z)\left\{1 + gP(z)\frac{N(z)}{D(z)}\right\} = 0 \tag{35}$$

with

$$N(z) = R_0 \prod_{i=1}^{q}(z^2 - 2(\cos\omega_i) + 1) + z(z-1)\sum_{i=1}^{q} \prod_{\substack{j=1 \\ j\neq i}}^{q} (z^2 - 2(\cos\omega_j) + 1) I_{\omega_i}$$

$$D(z) = (z-1)\prod_{i=1}^{q}(z^2 - 2z(\cos\omega_i) + 1).$$

The closed-loop poles of (34) can be found by using the positive root locus with $g > 0$ treated as the gain. Since the poles of $P(z)$ are strictly inside the unit circle, for sufficiently small $g > 0$ the branches of the root locus which start from the poles of $P(z)$ will stay strictly inside the unit circle. The branches starting from $\cos\omega_i \pm j\sin\omega_i$ on the unit circumference should satisfy the phase condition which in the neighborhood of $z = 1$ becomes

$$\arg\{R_0 P(z)\} - \arg(z-1) = -\pi \tag{36}$$

and in the neighborhood of $z = \bar{z}_i = \cos\omega_i + j\sin\omega_i$ becomes $(1 \le i \le q)$

$$\arg(z) - \arg(z - \bar{z}_i) = -\pi/2 - \arg\{I_{\omega_i} P(z)\}. \tag{37}$$

Recalling the proofs of Theorems 2.1 and 2.2, (36) and (37) imply that the branches of the root locus starting from the poles $z = 1$, $z = \bar{z}_i$, $1 \le i \le q$, are strictly inside the unit circle for sufficiently small $g > 0$. By the symmetry of the root locus with respect to the real axis, this is true also for the branches starting from $\bar{z}_i^*$, $1 \le i \le q$, so that part (A) of this theorem is proved. Part (B) can be proved in a very similar way. □

Remark 3.1. The control algorithms (31), (32) can be also used to perform learning of periodic references with known period $T_p = 2\pi/\omega$ for stable systems. Indeed, let $y_r(k) = A_0 + \sum_{n=1}^{N} A_n \cos(kn\omega + \phi_n)$ be the periodic sequence to be followed by the output $y(k)$ and let $u_r(k) = B_0 + \sum_{n=1}^{N} B_n \cos(kn\omega + \psi_n)$ be the corresponding input periodic sequence which, with proper initial conditions, ensures that $y(k) = y_r(k)$, for any $k \ge 0$. Assume that either $\mathrm{Im}[P(\bar{z}_i)] \ne 0$ with known sign, $1 \le i \le q$, or $\mathrm{Re}[P(\bar{z}_i)] \ne 0$ with known sign, $1 \le i \le q$, with $\bar{z}_i = \cos(i\omega) + j\sin(i\omega)$. Then, the control laws (31), (32) with $y(k) - y_r(k)$ in place of $y(k)$ and $\omega_i = i\omega$ guarantee exponential tracking of the given periodic reference if $q \ge N$. If $q < N$, a residual error $y(k) - y_r(k)$ will be present due to the higher order harmonics of $u_r(k)$ which are not modeled in (31), (32).

Remark 3.2. The control algorithms given in Theorem 3.1 can be also used to reject disturbances acting on nonlinear systems. In fact, it is easy to show by following the proof of Theorem 3.1 that a local convergence result can be obtained for single-input single-output nonlinear systems

$$\begin{aligned} x(k+1) &= f(x(k), u(k) + d(k)) \\ y(k) &= h(x(k), u(k) + d(k)) \end{aligned} \tag{38}$$

provided that the hypotheses in Theorem 3.1 are satisfied for the transfer function $P(z)$ of the linear approximation about the origin $(x, u + d) = 0$ of (38).

4. Unknown frequencies

In this section, the case of disturbances with unknown frequencies is addressed. Two theorems are stated and proved in which are treated the cases of a single unknown frequency disturbance and of a biased multi-sinusoidal disturbance with unknown frequencies, respectively. We show that the disturbance compensators given in the previous sections are still working provided that suitable frequency estimators are added to the dynamic equations of the control algorithms. However, only local convergence results are obtained while in the previous Theorems 2.1, 2.2 and 3.1, global results are stated.

Theorem 4.1 (*Sinusoidal Disturbance with Unknown Frequency*). *Consider the linear system (5) with $d(k) = A_1 \sin(k\omega + \phi_1)$ generated by (18), and $A_1 > 0$, $\omega > 0$, $0 \le \phi_1 < 2\pi$ unknown amplitude, frequency and phase. Let $\bar{z} = \cos\omega + j\sin\omega$ with $\sin\omega > 0$, without loss of generality, and assume that either (A) $\mathrm{Im}[P(\bar{z})] \ne 0$ with known sign or (B) $\mathrm{Re}[P(\bar{z})] \ne 0$ with known sign. Then, there exist $g^* > 0$ and $\epsilon^*(g) > 0$ such that for any $0 < g \le g^*$, $0 < \epsilon \le \epsilon^*(g)$ the dynamic output feedback compensator*

$$\begin{aligned} \hat{\eta}_1(k+1) &= \hat{\eta}_2(k) + g I_\omega y(k) \\ \hat{\eta}_2(k+1) &= -\hat{\eta}_1(k) + 2\hat{\theta}\hat{\eta}_2(k) + 2\hat{\theta} g I_\omega y(k) \\ u &= -\hat{\eta}_1(k) \\ \hat{\theta}(k+1) &= \hat{\theta}(k) + \epsilon I_\omega \hat{\eta}_1(k) y(k) \end{aligned} \tag{39}$$

in case (A) or

$$\begin{aligned} \hat{\eta}_1(k+1) &= \hat{\eta}_2(k) + g R_\omega \hat{\theta} y(k) \\ \hat{\eta}_2(k+1) &= -\hat{\eta}_1(k) + 2\hat{\theta}\hat{\eta}_2(k) - g R_\omega y(k) \\ u &= -\hat{\eta}_1(k) - g R_\omega y(k) \\ \hat{\theta}(k+1) &= \hat{\theta}(k) + \epsilon R_\omega \hat{\eta}_1(k) y(k) \end{aligned} \tag{40}$$

in case (B), with $I_\omega = \mathrm{sign}\{\mathrm{Im}[P(\bar{z})]\}$ and $R_\omega = \mathrm{sign}\{\mathrm{Re}[P(\bar{z})]\}$, guarantees that the equilibrium point $(x, \eta_1 - \hat{\eta}_1, \eta_2 - \hat{\eta}_2, \cos\omega - \hat{\theta}) = 0$ of (5), (18), (39) in case (A) or (5), (18), (40) in case (B) is locally exponentially stable.

Proof. Define, $\tilde{\eta}_i = \eta_i - \hat{\eta}_i$, $\tilde{\theta} = (\cos\omega) - \hat{\theta}$ so that we can write in case (A)

$$\begin{aligned} x(k+1) &= Ax(k) + B\tilde{\eta}_1(k) \\ y(k) &= Cx(k) + D\tilde{\eta}_1(k) \\ \tilde{\eta}_1(k+1) &= \tilde{\eta}_2(k) - g I_\omega y(k) \\ \tilde{\eta}_2(k+1) &= -\tilde{\eta}_1(k) + 2\tilde{\theta}(k)\eta_2(k) + 2\hat{\theta}(k)\tilde{\eta}_2(k) \\ &\quad + 2\hat{\theta}(k) g I_\omega y(k) \\ \tilde{\theta}(k+1) &= \tilde{\theta}(k) + \epsilon I_\omega[\tilde{\eta}_1(k) - \eta_1(k)] y(k). \end{aligned} \tag{41}$$

Eqs. (41) may be written as

$$\begin{aligned} \chi'(k+1) &= F(\tilde{\theta}(k))\chi'(k) + h(k, \tilde{\theta}(k)) \\ \tilde{\theta}(k+1) &= \tilde{\theta}(k) + \epsilon f'(k, \chi'(k)) \end{aligned} \tag{42}$$

in which $\chi' = [x^T, \tilde{\eta}_1, \tilde{\eta}_2]$ and

$$\begin{aligned} F(\tilde{\theta}(k)) &= \begin{bmatrix} A & B & 0 \\ -g I_\omega C & -g I_\omega D & 1 \\ 2\hat{\theta}(k) g I_\omega C & 2\hat{\theta}(k) I_\omega D - 1 & 2\hat{\theta}(k) \end{bmatrix} \\ h^T(k, \tilde{\theta}(k)) &= \begin{bmatrix} 0 & 0 & 2\eta_2(k) \end{bmatrix} \tilde{\theta}(k) \\ f'(k, \chi'(k)) &= I_\omega \tilde{\eta}_1(k) y(k) - I_\omega \eta_1(k) y(k) \\ &= -I_\omega \hat{\eta}_1(k) y(k). \end{aligned} \tag{43}$$

Note that, by virtue of Theorem 2.2, $F(0)$ is a Schur matrix, so that by continuity also $F(\tilde{\theta})$ is a Schur matrix for sufficiently small $\tilde{\theta}$. According to Bai, Fu, and Sastry (1988), we apply the transformation

$$\chi(k) = \chi'(k) - \nu(k, \tilde{\theta}(k)) \tag{44}$$

with

$$\nu(k, \tilde{\theta}) = \sum_{i=0}^{k-1} F(\tilde{\theta})^{k-i-1} h(i, \tilde{\theta}) \tag{45}$$

to transform the mixed time scale system (42) into the two time scale system

$$\begin{aligned} \chi(k+1) &= F(\tilde{\theta}(k))\chi(k) + \epsilon g(k, \tilde{\theta}(k), \chi(k)) \\ \tilde{\theta}(k+1) &= \tilde{\theta}(k) + \epsilon f(k, \tilde{\theta}(k), \chi(k)) \end{aligned} \tag{46}$$

in which

$$f(k,\tilde{\theta},\chi)=f'(k,\chi+\nu(k,\tilde{\theta}))$$
$$g(k,\tilde{\theta},\chi)=-\int_0^1\frac{\partial\nu}{\partial\tilde{\theta}}(k+1,s\tilde{\theta}(k+1)+(1-s)\tilde{\theta}(k))ds \cdot[f'(k,\chi+\nu(k,\tilde{\theta}))]. \quad (47)$$

The averaged system for (46) is defined as

$$\tilde{\theta}(k+1)=\tilde{\theta}(k)+\epsilon f_{av}(\tilde{\theta}(k)) \quad (48)$$

with

$$f_{av}(\tilde{\theta})=\lim_{T\to\infty}\frac{1}{T}\sum_{k=s+1}^{s+T}f(k,\tilde{\theta},0) \quad (49)$$

in which the limit must exist uniformly in s and for any $\tilde{\theta}$ belonging to a neighborhood of the origin. To compute such a limit, we consider $\tilde{\theta}$ (and, consequently, $\hat{\theta}$) as a constant, so that from (41) we can write

$$y(z)=P(z)[u(z)+\eta_1(z)]$$
$$u(z)=-\frac{gI_\omega z}{z^2-2\hat{\theta}z+1}y(z)$$

which imply

$$y(z)=\left[1+\frac{gI_\omega zP(z)}{z^2-2\hat{\theta}z+1}\right]^{-1}P(z)\eta_1(z) = \frac{z^2-2\hat{\theta}z+1}{z^2-2\hat{\theta}z+1+gI_\omega zP(z)}P(z)\eta_1(z) \triangleq G(z)(z^2-2\hat{\theta}z+1)\eta_1(z). \quad (50)$$

Moreover, from (39),

$$z^2\hat{\eta}_1(z)-2\hat{\theta}z\hat{\eta}_1(z)+\hat{\eta}_1(z)=gI_\omega zy(z)$$

which implies that

$$\hat{\eta}_1(z)=\frac{gI_\omega z}{z^2-2\hat{\theta}z+1}y(z)=G(z)gI_\omega z\eta_1(z). \quad (51)$$

Note that, from (43),

$$f_{av}(\tilde{\theta})=-I_\omega\lim_{T\to\infty}\frac{1}{T}\sum_{k=s+1}^{s+T}y_{ss}(k)\hat{\eta}_{1ss}(k) \quad (52)$$

in which $y_{ss}(k)$ and $\hat{\eta}_{1ss}(k)$ are the steady-state responses to the sinusoidal sequence $\eta_2(k)$ of the following transfer functions:

$$G_y(z)=G(z)\frac{z^2-2\hat{\theta}z+1}{z}$$
$$G_{\hat{\eta}_2}(z)=G(z)gI_\omega. \quad (53)$$

Since $\bar{z}^2-2\hat{\theta}\bar{z}+1=2\tilde{\theta}\bar{z}$, (52) and (53) imply that

$$f_{av}(\tilde{\theta})=-g|G(\bar{z})|^2A_1^2\tilde{\theta}\triangleq-k_1(A_1,\omega,\hat{\theta})\tilde{\theta} \quad (54)$$

with $k_1(A_1,\omega,\hat{\theta})>0$. From (48) and (54), it follows that

$$\tilde{\theta}_{av}(k+1)=(1-\epsilon k_1)\tilde{\theta}_{av}(k) \quad (55)$$

so that for $0<\epsilon<1/k_1$ the averaged system (48) is exponentially stable. Moreover, it is easy to check that conditions (B1)–(B6) in Bai et al. (1988) are satisfied, so that by applying Theorem 2.2.4 in Bai et al. (1988) the proof for case (A) is completed. The proof for case (B) is very similar and is, therefore, omitted. □

Now, we consider the case of a general biased multi-sinusoidal disturbance with q unknown frequencies.

Theorem 4.2 (*Biased Multi-Sinusoidal Disturbance with Unknown Frequencies*). *Consider the linear system (5) with $d(k)=A_0+\sum_{i=1}^{q}A_i\sin(\omega_i+\phi_i)$ generated by (33), and $A_i>0$, $\omega_i>0$, $0\le\phi_i<2\pi$, $1\le i\le q$, unknown amplitudes, frequencies and phases. Let $\bar{z}_i=\cos\omega_i+j\sin\omega_i$ with $\sin\omega_i>0$, without loss of generality, and assume that either (A) $\mathrm{Im}[P(\bar{z}_i)]\neq0$ with known sign, $1\le i\le q$, or (B) $\mathrm{Re}[P(\bar{z}_i)]\neq0$ with known sign, $1\le i\le q$. Assume also that $P(1)\neq0$ with known sign. Then, there exist $g^*>0$ and $\epsilon^*(g)>0$ such that for any $0<g\le g^*$, $0<\epsilon\le\epsilon^*(g)$ the dynamic output feedback compensator*

$$\begin{aligned}\hat{\eta}_1(k+1)&=\hat{\eta}_1(k)+gR_0y(k)\\ \hat{\eta}_{2i}(k+1)&=\hat{\eta}_{2i+1}(k)+gI_{\omega_i}y(k)\\ \hat{\eta}_{2i+1}(k+1)&=-\hat{\eta}_{2i}(k)+2\hat{\theta}_i\hat{\eta}_{2i+1}(k)+2\hat{\theta}_igI_{\omega_i}y(k)\\ \hat{\theta}_i(k+1)&=\hat{\theta}_i(k)+\epsilon I_{\omega_i}\hat{\eta}_{2i}(k)y(k),\quad 1\le i\le q\\ u&=-\hat{\eta}_1(k)-\sum_{i=1}^{q}\hat{\eta}_{2i}(k)\end{aligned} \quad (56)$$

in case (A) *or*

$$\begin{aligned}\hat{\eta}_1(k+1)&=\hat{\eta}_1(k)+gR_0y(k)\\ \hat{\eta}_{2i}(k+1)&=\hat{\eta}_{2i+1}(k)+gR_{\omega_i}\hat{\theta}_iy(k)\\ \hat{\eta}_{2i+1}(k+1)&=-\hat{\eta}_{2i}(k)+2\hat{\theta}_i\hat{\eta}_{2i+1}(k)-gR_{\omega_i}y(k)\\ \hat{\theta}_i(k+1)&=\hat{\theta}_i(k)+\epsilon R_{\omega_i}\hat{\eta}_{2i}(k)y(k),\quad 1\le i\le q\\ u&=-\hat{\eta}_1(k)-\sum_{i=1}^{q}[\hat{\eta}_{2i}(k)+gR_{\omega_i}y(k)]\end{aligned} \quad (57)$$

in case (B), *with $R_0=\mathrm{sign}[P(1)]$, $I_{\omega_i}=\mathrm{sign}\{\mathrm{Im}[P(\bar{z}_i)]\}$ and $R_{\omega_i}=\mathrm{sign}\{\mathrm{Re}[P(\bar{z}_i)]\}$, guarantees that the equilibrium point $(x,\eta_1-\hat{\eta}_1,\ldots,\eta_{2q+1}-\hat{\eta}_{2q+1},(\cos\omega_1)-\hat{\theta}_1,\ldots,(\cos\omega_q)-\hat{\theta}_q)=0$ of (5), (33), (56) in case (A) or (5), (33), (57) in case (B) is locally exponentially stable.*

Proof. This proof follows the same steps of the proof of Theorem 4.1. The main point is to show the exponential stability of the averaged system which in case (A) becomes ($\tilde{\theta}=[\tilde{\theta}_1,\ldots,\tilde{\theta}_q]^T$)

$$\tilde{\theta}(k+1)=\tilde{\theta}(k)+\epsilon f_{av}(\tilde{\theta}(k))$$

with

$$f_{av}(\tilde{\theta})=-\lim_{T\to\infty}\frac{1}{T}\begin{bmatrix}I_{\omega_1}\sum_{k=s+1}^{s+T}\hat{\eta}_{2ss}(k)y_{ss}(k)\\ \vdots\\ I_{\omega_q}\sum_{k=s+1}^{s+T}\hat{\eta}_{2qss}(k)y_{ss}(k)\end{bmatrix} \quad (58)$$

where $\hat{\eta}_{2iss}(k)$ and $y_{ss}(k)$ are the steady-state responses to the multi-sinusoidal unbiased sequence $\sum_{i=1}^{q}\eta_{2i+1}(k)$ of the transfer functions

$$G_y(z)=G(z)\frac{z-1}{z}\prod_{i=1}^{q}(z^2-2\hat{\theta}_iz+1)$$
$$G_{\hat{\eta}_{2i}}=G(z)gI_{\omega_i}(z-1)\prod_{\substack{j=1\\ j\neq i}}^{q}(z^2-2\hat{\theta}_jz+1). \quad (59)$$

In fact, from (5) and (56), we have

$$y(z) = P(z)\left[u(z) + \eta_1(z) + \sum_{i=1}^{q} \eta_{2i}(z)\right]$$

$$u(z) + \eta_1(z) = -g\left[\frac{R_0}{z-1} + \sum_{i=1}^{q} I_{\omega_i} \frac{z}{z^2 - 2\hat{\theta}_i z + 1}\right] y(z)$$

so that

$$y(z) = \left[1 + g\left(\frac{R_0}{z-1} + \sum_{i=1}^{q} I_{\omega_i} \frac{z}{z^2 - 2\hat{\theta}_i z + 1}\right)\right]^{-1} \cdot P(z) \sum_{i=1}^{q} \eta_{2i}(z)$$

$$\triangleq G(z)(z-1) \prod_{i=1}^{q} (z^2 - 2\hat{\theta}_i z + 1) \sum_{i=1}^{q} \eta_{2i}(z). \tag{60}$$

Moreover, we can write from (56)

$$z^2 \hat{\eta}_{2i}(z) - 2\hat{\theta}_i z \hat{\eta}_{2i}(z) + \hat{\eta}_{2i}(z) = g I_{\omega_i} z y(z)$$

which implies that

$$\hat{\eta}_{2i}(z) = g \frac{I_{\omega_i} z}{z^2 - 2\hat{\theta}_i z + 1} y(z) = G(z) g I_{\omega_i} z (z-1) \cdot \prod_{\substack{j=1 \\ j \neq i}}^{q} (z^2 - 2\hat{\theta}_j z + 1) \sum_{i=1}^{q} \eta_{2i}(z). \tag{61}$$

From (58)–(61), we obtain

$$f_{av}(\tilde{\theta}) = -\begin{bmatrix} k_1(A_1, \ldots, A_q, \omega_1, \ldots, \omega_q, \hat{\theta}_1, \ldots, \hat{\theta}_q)\tilde{\theta}_1 \\ \vdots \\ k_q(A_1, \ldots, A_q, \omega_1, \ldots, \omega_q, \hat{\theta}_1, \ldots, \hat{\theta}_q)\tilde{\theta}_q \end{bmatrix}$$

so that for $0 < \epsilon < \min\{1/k_1, \ldots, 1/k_q\}$ the averaged system (58) is exponentially stable. The proof for case (B) follows exactly the same steps of case (A). □

Remark 4.1. If we set $P(z) = 1$, from (57), we obtain the local frequency estimator

$$\hat{\eta}_1(k+1) = \hat{\eta}_1(k) + g y(k)$$
$$\hat{\eta}_{2i}(k+1) = \hat{\eta}_{2i+1}(k) + g\hat{\theta}_i y(k)$$
$$\hat{\eta}_{2i+1}(k+1) = -\hat{\eta}_{2i}(k) + 2\hat{\theta}_i \hat{\eta}_{2i+1}(k) - g y(k)$$
$$\hat{\theta}_i(k+1) = \hat{\theta}_i(k) + \epsilon \hat{\eta}_{2i}(k) y(k), \quad 1 \le i \le q$$
$$u = -\hat{\eta}_1(k) - \sum_{i=1}^{q} \hat{\eta}_{2i}(k) - gq y(k)$$
$$y(k) = u(k) + d(k)$$
$$\hat{\omega}_i(k) = \begin{cases} \hat{\omega}_i(k-1), & \text{if } \hat{\theta}_i > 1 \text{ or } \hat{\theta}_i < -1 \\ \arccos \hat{\theta}_i(k), & \text{otherwise.} \end{cases}$$

5. Examples

As a first example, we consider system (3) in Section 1 perturbed by a sinusoidal disturbance with known frequency $\omega = 0.1$. A state space realization can be given as

$$x_1(k+1) = x_2(k) + 0.6x_1(k) + \alpha u(k) + d(k)$$
$$x_2(k+1) = -0{,}05x_1(k) + u(k) + d(k)$$
$$y(k) = x_1(k) \tag{62}$$

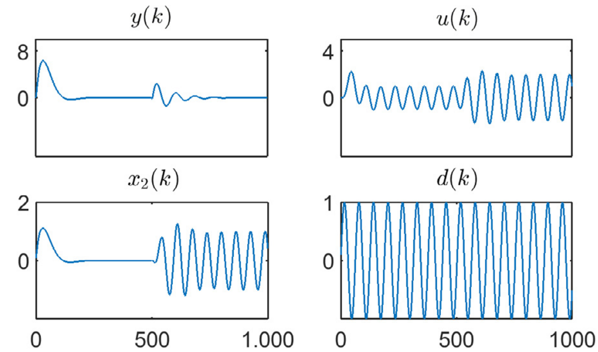

Fig. 2. Simulation results for example (62).

in which $\alpha = 1$ is the nominal value, while $\alpha = 0$ represents a sensor/actuator fault. Following Remark 1.1, we can recover disturbance matching even in the case $\alpha = 0$ by defining $z_1 = x_1$, $z_2 = x_2 + d$. We obtain

$$z_1(k+1) = z_2(k) + 0.6z_1(k)$$
$$z_2(k+1) = -0{,}05z_1(k) + u(k) + [d(k) + d(k+1)]$$
$$y(k) = z_1(k). \tag{63}$$

The transfer functions for (62) and (63) are, respectively, given by $P_1(z) = \frac{1+z}{(z-0.1)(z-0.5)}$ and $P_2(z) = \frac{1}{(z-0.1)(z-0.5)}$, while the matching disturbances are different in the two cases but still have the same frequency $\omega = 0.1$. We assume to know the sign of $\mathrm{Re}[P_i(e^{j0.1})]$, which is positive for $i = 1, 2$. According to Theorem 2.2, we design the dynamic compensator (17) and choose $g = 0.001$. A numerical simulation has been carried out whose results are illustrated in Fig. 2, in which all initial conditions are set to zero and the disturbance is chosen as $d(k) = \sin(0.1k)$. A sensor fault happens at $k = 500$ so that we have an abrupt change from system (62) to system (63). Nevertheless, as one can see, the fault is well tolerated by the control system.

As a second example, we consider the learning problem for the second order system

$$x_1(k+1) = \lambda x_1(k) + x_2(k)$$
$$x_2(k+1) = \lambda x_2(k) + u(k)$$
$$y(k) = x_1(k). \tag{64}$$

The periodic reference y_r for the output y is assumed to be $y_r(k) = \frac{1}{2}[1 + \frac{1}{2}\cos(\omega k + 0.3) - \cos(2\omega k) - \frac{1}{2}\cos(3\omega k + 0.3)]$. By defining $e_1(k) = y(k) - y_r(k)$ and $e_2(k) = x_2(k) - y_r(k+1) + \lambda y_r(k)$, from (64) we have $e_1(k+1) = \lambda e_1(k) + e_2(k)$, $e_2(k+1) = \lambda e_2(k) + d(k) + u(k)$ in which $d(k) = -y_r(k+2) + 2\lambda y_r(k+1) - \lambda^2 y_r(k)$ is viewed as an external disturbance. The nominal frequency is $\omega = 0.1$ and is affected by an uncertainty of 10%. The only a priori informations are the sign of the imaginary part of the transfer function $P(z)$ of the process (64) for $z = e^{j\omega k}$, $k = 1, 2, 3$ in the range $\omega \in [0.09, 0.11]$ (which are negative) and the sign of $P(1)$ (which is positive). For simulation purposes, we select $\lambda = 0.5$, $\omega = 0.1$. We adopt the adaptive disturbance compensator (56) given in Theorem 4.2, with three frequencies in order to compensate the disturbance $d(k)$. We choose $g = 0.001$ and $\epsilon = 0.0001$ in the adaptation equations for $\hat{\theta}_i, i = 1, 2, 3$. The initial conditions were set to $\hat{\theta}_i(0) = \cos(0.11\, i)$, $i = 1, 2, 3$, while all other initial conditions were set to zero. In Fig. 3 are reported the time histories of the tracking error $y - y_r$, the input u and the first two parameter estimate errors $\cos(0.1) - \hat{\theta}_1$, $\cos(0.2) - \hat{\theta}_2$. As one can see, good disturbance rejection is achieved even though the settling time is larger than that achieved in the known frequency case.

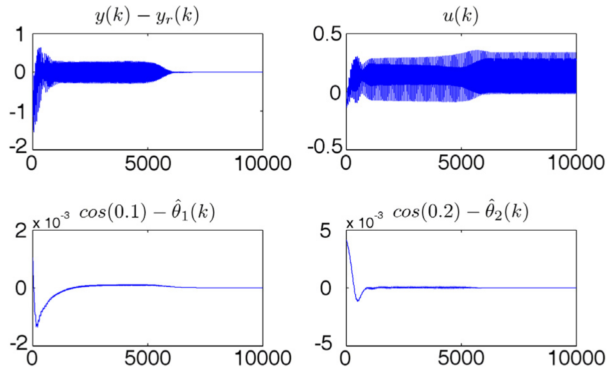

Fig. 3. Simulation results for example (64).

6. Conclusions

Discrete-time linear SISO stable systems have been considered, which are perturbed by multi-sinusoidal disturbances. The parameters, the order and the relative degree of the system are unknown while its steady-state passivity properties with respect to constant or sinusoidal inputs are known. More precisely, the sign of the static gain $P(1)$ and the sign of either $\mathrm{Re}[P(e^{j\omega_i})]$ or $\mathrm{Im}[P(e^{j\omega_i})]$, for each disturbance frequency ω_i are known. It is shown that minimal order linear disturbance compensators can be designed if the disturbance frequencies are known. If such frequencies are not known, it is shown that by adding first order adaptation dynamics for each unknown frequency, disturbance compensators can still be achieved with local convergence properties. Future work will be devoted to study the extension of the local results obtained in the case of disturbances with unknown frequencies to global ones.

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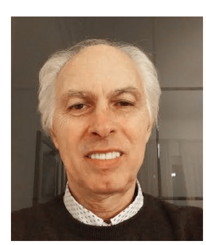

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