Multiple Equilibria in a Simple Asset Pricing Model[*]

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Abstract

Multiple stationary equilibria are often encountered in standard asset pricing models when one assumes negative-exponential (CARA) utility with Gaussian uncertainty. This paper demonstrates that in a homogenous information time series model there are exactly two stationary equilibria. It is shown that these equilibria are due solely to the presence of nonlinearities in the model. The results established here extend to both common knowledge and heterogenous information models.

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1   Introduction

Multiple stationary equilibria are often encountered in time-series models when one assumes negative-exponential (CARA) utility coupled with Gaussian random variables [*e.g.,* Bacchetta and van Wincoop (2006), Grundy and McNichols (1989), Singleton (1987)]. This note shows that multiple equilibria arise even in simple informational environments and thus are not an artifact of asymmetric information *per se.* It is the presence of the conditional variance parameter that introduces a nonlinearity into the model and gives rise to issues of existence and uniqueness [McCafferty and Driskill (1980)]. The results and methodology established here extend to both common knowledge and heterogenous information models.[1]

2   Model and Analysis

A myopic investor may choose to hold wealth in either a riskless asset which earns the return r or a risky asset.[2] The wealth of the agent evolves according to

$$w_{t+1} = z_t(p_{t+1} + d_{t+1}) + (w_t - z_t p_t)(1 + r)$$

where p_t is the (ex-dividend) price of the risky asset at time t, d_t is the dividend payment received at the beginning of period t, and z_t is the number of units of the risky asset held at time t. The dividend process $\{d_t\}$ is assumed to be normally distributed and given by

$$d_t = \bar{d} + D(L)\varepsilon_t^d, \qquad \qquad \varepsilon_t^d \sim N(0, \sigma_d^2) \qquad (2.1)$$

where $D(L)$ is (possibly) an infinite-order square summable polynomial in the lag operator L (i.e., $\sum_{j=0}^{\infty} D_j^2 < \infty$). It is important to note that (2.1) places no restrictions on the serial correlation properties of $\{d_t\}$; thus the results established below pertain to *any* covariance stationary process. The Wold Decomposition Theorem allows for such a general structure [see, Sargent (1987)].

The agent seeks to maximize, by choice of z_t, the expected value of a constant absolute risk aversion (CARA) utility function

$$-E_t \exp(-\gamma w_{t+1}), \qquad (2.2)$$

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[1]It is important to note that these results pertain to stationary environments only and do not apply to models with assets that have a terminal value [*e.g.,* He and Wang (1995)].

[2]The model has an alternative interpretation as an overlapping generations model in which agents are endowed with w_{it} in the first period of life, and value consumption only in the second period of life.

where γ is the risk aversion parameter, and E_t denotes the time t conditional expectation. All random variables in the model are assumed to be distributed normally, so that (2.2) can be calculated from the (conditional) moment generating function for the normal random variable $-\gamma w_{t+1}$. That is,

$$-E_t \exp(-\gamma w_{t+1}) = -\exp\{-\gamma E_t(w_{t+1}) + (1/2)\gamma^2 v_t(w_{t+1})\}$$

where v_t denotes conditional variance. Note that $v_t(w_{t+1}) = z_t^2 v_t(p_{t+1} + d_{t+1})$. Stationarity implies the conditional variance term will be a constant; thus write $v_t(w_{t+1}) \equiv z_t^2 \delta$. The agent's demand function for the risky asset follows from the first-order necessary conditions for maximization and is given by

$$z_t = \frac{1}{\gamma\delta}[E_t p_{t+1} - \alpha p_t + E_t d_{t+1}] \tag{2.3}$$

where $\alpha \equiv 1 + r > 1$.

The supply of the risky asset, denoted s_t, is assumed to be a random variable. This assumption, common to the literature, is equivalent to the presence of noise traders. That is, supply represents the behavior of liquidity traders who participate in the market primarily for nonspeculative reasons. Changes in liquidity traders' demand constitutes noise trading and will change the number of shares supplied to the market. Thus let supply be given by

$$s_t = \bar{s} + A(L)\varepsilon_t^s \qquad\qquad \varepsilon_t^s \sim N(0, \sigma_s^2)$$

where $A(L)$ is (possibly) an infinite-order square summable polynomial in the lag operator L, and the shocks $\{\varepsilon^s, \varepsilon^d\}$ are assumed orthogonal at all leads and lags.

The representative agent observes current (time t) and past prices and dividends and believes the price process to be a linear function of the underlying shocks,

$$p_t = F(L)\varepsilon_t^d + G(L)\varepsilon_t^s$$

where $F(L)$ and $G(L)$ are (possibly) infinite-order square summable polynomials in the lag operator L with coefficients to be determined in equilibrium. The expectations in (2.3) are given by the Weiner-Komolgorov optimal prediction formulas.

$$E_t(p_{t+1}) = L^{-1}[F(L) - F(0)]\varepsilon_t^d + L^{-1}[G(L) - G(0)]\varepsilon_t^s$$
$$E_t(d_{t+1}) = L^{-1}[D(L) - D(0)]\varepsilon_t^d$$

$$\text{var}_t(p_{t+1}) = E_t\{[p_{t+1} - E_t(p_{t+1})]^2\}$$
$$= E_t\{[F(L)\varepsilon_{t+1}^d + G(L)\varepsilon_{t+1}^s - L^{-1}[F(L) - F(0)]\varepsilon_t^d - L^{-1}[G(L) - G(0)]\varepsilon_t^s]^2\}$$
$$= E_t\{[F(0)\varepsilon_{t+1}^d + G(0)\varepsilon_{t+1}^s]^2\}$$
$$= F(0)^2\sigma_d^2 + G(0)^2\sigma_s^2.$$

A similar calculation for the variance of the dividend process and the covariance between price and dividends yields

$$\delta = \text{var}_t(p_{t+1}) + \text{var}_t(d_{t+1}) + 2\,\text{cov}_t(p_{t+1}, d_{t+1})$$
$$= G(0)^2\sigma_s^2 + (F(0) + D(0))^2\sigma_d^2.$$

Equilibrium in the market for the risky asset equates supply and demand,

$$s_t = (\gamma\delta)^{-1}[E_t p_{t+1} - \alpha p_t + E_t d_{t+1}]. \tag{2.4}$$

Expanding this equation gives[3]

$$\gamma[G(0)^2\sigma_s^2 + (F(0) + D(0))^2\sigma_d^2]A(L)\varepsilon_t^s$$
$$= L^{-1}[F(L) - F(0)]\varepsilon_t^d + L^{-1}[G(L) - G(0)]\varepsilon_t^s + L^{-1}[D(L) - D(0)]\varepsilon_t^d - \alpha F(L)\varepsilon_t^d - \alpha G(L)\varepsilon_t^s.$$

Assuming that this expression holds for all realizations of ε_t^s and ε_t^d, the coefficients on ε_s^d and ε_s^s must match for every s. In lieu of solving this infinite sequential problem, one can employ the powerful Riesz-Fischer Theorem and solve an equivalent functional fixed-point problem by examining the corresponding power series equalities [see, Whiteman (1983)].[4] Focusing on ε_t^d gives

$$F(z)[1 - \alpha z] = D(0) + F(0) - D(z). \tag{2.5}$$

Thus, we seek a *function* $F(z)$ that satisfies (2.5). A stable solution (*i.e.*, ruling out bubble equilibria) requires that the coefficients F_j are square summable. The requirement of square-summability in the time domain corresponds to the requirement that $F(z)$ be analytic on the open unit disk $|z| < 1$ in the frequency domain. Given $\alpha > 1$, this function will not be analytic unless the free parameter $F(0)$ removes the singularity at $z = \alpha^{-1}$. This is achieved

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[3]Note that this equation abstracts from the constants $\bar{d}$ and $\bar{s}$. Including the constants will not alter the analysis.

[4]Recall the Riesz-Fischer Theorem states there is an equivalence (i.e., an isometric isomorphism) between the space of square-summable sequences denoted by $\ell_2(-\infty, \infty)$ and the space of square integrable functions, $L^2[\pi, -\pi]$; the former is referred to as the time domain and the latter the frequency domain.

by setting the residue equal to zero and solving for $F(0)$, which yields

$$\lim_{z \to \alpha^{-1}} F(z)[1 - \alpha z] = D(0) + F(0) - D(z) = 0$$

$$F(0) = -D(0) + D(\alpha^{-1}).$$

This implies $F(z)$ is *unique* and given by

$$F^*(z) = \frac{D(\alpha^{-1}) - D(z)}{(1 - \alpha z)}, \tag{2.6}$$

which is simply an instance of the Hansen and Sargent (1980) prediction formula and clearly demonstrates the cross-equation restrictions typical of rational expectations models.

Examining the power series equality for $G(z)$ yields the following proposition.

Proposition 2.1. *There are exactly two stationary equilibria.*

Proof. The power series is given by

$$G(z)[1 - \alpha z] = G(0) + \gamma \delta z A(z)$$

$$G(z)[1 - \alpha z] = G(0) + \gamma[G(0)^2 \sigma_s^2 + D(\alpha^{-1})^2 \sigma_d^2] z A(z)$$

Again, $G(0)$ must be set to remove the singularity at $z = \alpha^{-1}$, thus

$$(\gamma \alpha^{-1} A(\alpha^{-1}) \sigma_s^2) G(0)^2 + G(0) + \gamma D(\alpha^{-1})^2 \sigma_d^2 \alpha^{-1} A(\alpha^{-1}) = 0 \tag{2.7}$$

Given that there are two roots that satisfy (2.7), there are *exactly* two equilibria. $\qquad \square$

The nonlinearity in (2.7) (and thus the existence of multiple equilibria) is due entirely to the presence of the conditional variance term δ and is independent of any informational assumptions and parameter values. The quadratic formula gives the roots as

$$G_R(0) = \frac{-1 \pm \sqrt{1 - 4\gamma^2 \alpha^{-2} A(\alpha^{-1})^2 D(\alpha^{-1})^2 \sigma_s^2 \sigma_d^2}}{2\gamma \alpha^{-1} A(\alpha^{-1}) \sigma_s^2}$$

Thus,

$$G^*(z) = \frac{G_R(0) + \gamma[G_R(0)^2 \sigma_s^2 + D(\alpha^{-1})^2 \sigma_d^2] z A(z)}{1 - \alpha z} \tag{2.8}$$

Combining Equations (2.6) and (2.8) gives the equilibrium price as $p_t = F^*(L)\varepsilon_t^d + G^*(L)\varepsilon_t^s$.

Within this framework, it is easy to see how removing uncertainty in the dividend or supply process yields a unique equilibrium. For example, removing uncertainty in the dividend process ($\sigma_d^2 = 0$) implies (2.7) becomes

$$G(0)\{1 + G(0)[\gamma\alpha^{-1}A(\alpha^{-1})\sigma_s^2]\} = 0.$$

If $G(0) = 0$ then $G(z) = 0$, and the price is not speculative. This corresponds to the case where dividends are known to be constant over time, and the asset behaves as a risk-free security. However, the unique equilibrium that reinstates market dynamics is achieved by setting $G(0) = -[\gamma\alpha^{-1}A(\alpha^{-1})\sigma_s^2]^{-1}$. That is, if investors *believe* the stock price will be volatile due to supply shocks, then the self-fulfilling nature of rational expectations ensures this outcome. Similarly, setting $\sigma_s^2 = 0$ removes the uncertainty in the supply process and implies that $G^*(z)$ is unique with $G(0) = -\gamma D(\alpha^{-1})^2\sigma_d^2\alpha^{-1}A(\alpha^{-1})$. Of course, price volatility in this scenario is driven entirely by dividend dynamics.

When two equilibria are encountered, it is sometimes convenient to effectively rule out an equilibrium. One method for doing so is to examine the stability of the equilibria [see, *e.g.*, Bacchetta and van Wincoop (2006)]. Unlike the existence of multiple equilibria, stability is contingent upon the assumed parameter values of the model. As an illustrative example, suppose the cum-dividend equilibrium price of the asset is given by

$$p_t = \alpha^{-1}[E_t(p_{t+1}) + d_t - \gamma\delta_t s_t] \tag{2.9}$$

where $\delta_t = \text{var}_t(w_{t+1}) = \text{var}_t(p_{t+1})$.[5] Suppose further that the dividend and supply processes are given by

$$d_t = \bar{d} + \left[\frac{1}{1 - \rho_d L}\right]\varepsilon_t^d, \quad s_t = \bar{s} + \left[\frac{1}{1 - \rho_s L}\right]\varepsilon_t^s.$$

Solving (2.9) forward and taking expectations yields

$$p_t = \left[\frac{\alpha^{-1}}{1 - \alpha^{-1}\rho_d}\right]d_t - \delta_t\left[\frac{\alpha^{-1}\gamma}{1 - \alpha^{-1}\rho_s}\right]s_t. \tag{2.10}$$

We must now "pin down" the endogenous variance δ_t. Forwarding (2.10) ahead one period and applying the variance operator gives

$$\delta_t = c - a\delta_{t+1}^2 \tag{2.11}$$

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[5]The slight change in the timing of dividend payment will not change the analysis above. Proposition 2.1 continues to hold here.

where $a = \left[\frac{\alpha^{-1}\gamma}{1-\alpha^{-1}\rho_s}\right]^2 \frac{\sigma_s^2}{1-\rho_s^2}$ and $c = \left[\frac{\alpha^{-1}}{1-\alpha^{-1}\rho_d}\right]^2 \frac{\sigma_d^2}{1-\rho_d^2}$. Stationarity implies $\delta_t = \delta_{t+1}$ and the multiple equilibria are now given by the roots of (2.11),

$$\delta = \frac{-1 \pm \sqrt{1+4ac}}{2a}.$$

The advantage of solving the model in the time domain, however, is the ability to assess the stability of the equilibria. That is,

$$\frac{d\delta_t}{d\delta_{t+1}} = -2a\delta = 1 \pm \sqrt{1+4ac} \tag{2.12}$$

An equilibrium is stable if $\left|d\delta_t/d\delta_{t+1}\right| < 1$, (see, Blanchard and Fischer (1989), ch. 5). It is clear that the high-variance equilibrium is not stable. This implies that the high-variance equilibrium is a knife-edge and can only be sustained if the agent believes the price will remain in the high-volatility regime indefinitely.

3 Conclusion

The existence of multiple equilibria in the CARA-Gaussian framework extends to models with asymmetric information [Walker (2007)]. Although most dynamic models that incorporate asymmetric information are often too complex to solve analytically, numerical solution methods have borne out the results established here [Bacchetta and van Wincoop (2006), Singleton (1987)]. It is the nonlinearity introduced by the conditional variance term that leads to multiple equilibria.[6] The existence of multiple equilibria is independent of any informational or parameter assumptions. McCafferty and Driskill (1980) discovered this working within the context of Muth (1960). This note extends McCafferty and Driskill (1980) to the popular CARA-Gaussian model.

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[6]The results and methodology established extend to other forms of nonlinearities (e.g., second-order log-linearized CRRA model).

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