Coupling-Induced Rank Transitions in Statistical Manifolds

Ian Todd

Abstract Many information-theoretic bounds assume a fixed statistical model class. We identify a regime where this assumption fails: when high-dimensional systems couple, the identifiable parameter set changes because the image rank of the dynamics-to-distribution map changes. We formalize this as *manifold expansion*: coupling-induced increase in Fisher rank. Operationally, this means new score directions become nonzero—statistics that were previously insensitive to parameter variation become informative under coupling. Our main theorem provides checkable criteria for rank transitions: the *transversality criterion* (coupling moves the accessible family off a constraint submanifold) and the *symmetry-breaking criterion* (coupling breaks a group invariance). Since Fisher rank equals Jacobian rank (via the pullback identity), transitions are coordinate-invariant and detectable via eigenvalue emergence in the Fisher information matrix. Consequently, Cramér–Rao bounds shift: parameters that were formally unidentifiable at zero coupling (infinite Cramér–Rao variance) become estimable under interaction. Two examples demonstrate superadditive complexity: coupled Ornstein–Uhlenbeck processes (transversality, $\kappa_c = 0$) and Kuramoto oscillators (symmetry breaking, $\kappa_c > 0$).

Keywords Fisher information metric · Parameter identifiability · Superadditivity · Coupled dynamical systems · Statistical manifolds

1 Introduction

Consider a family of dynamical systems parameterized by coupling strength $\kappa \geq 0$ and physical parameters $\beta \in \mathcal{B}$. Each configuration (κ, β) induces a stationary distribution $Q_{(\kappa,\beta)}$ on observations. The *dynamics-to-distribution*

I. Todd
Sydney Medical School, University of Sydney
Sydney, NSW, Australia
E-mail: itod2305@uni.sydney.edu.au

map $F_\kappa : \mathcal{B} \to \mathcal{P}$, $\beta \mapsto Q_{(\kappa,\beta)}$, determines which distributions are accessible at coupling κ.

Question: When does coupling increase the rank of dF_κ?

An increase in rank dF_κ means new parameter directions become statistically identifiable—the Cramér–Rao bound transitions from infinite to finite for those directions. We call this *manifold expansion*: the accessible statistical family gains dimension under coupling.

Main result (Theorem 1). We provide two checkable criteria for rank increase:

- *Constraint-release criterion* (Part I): Coupling moves the accessible family off a constraint submanifold $\mathcal{M}_0 \subset \mathcal{P}$, activating coordinates normal to $\mathcal{M}_0$.
- *Symmetry-breaking criterion* (Part II): Coupling breaks a group symmetry G, making G-invariant observables parameter-sensitive.

Since Fisher rank equals Jacobian rank (via the pullback identity $I_\mathcal{B} = F_\kappa^* g$), rank transitions are coordinate-invariant and detectable as eigenvalue emergence in the Fisher information matrix.

Examples. Two canonical systems demonstrate the phenomena:

- *Coupled Ornstein–Uhlenbeck processes*: At $\kappa = 0$, the cross-covariance $\Sigma_{12} = 0$ regardless of parameters; rank is 2. At any $\kappa > 0$, Σ_{12} becomes parameter-sensitive; rank increases to 3. This is constraint-release with $\kappa_c = 0$.
- *Kuramoto oscillators*: Below K_c, the order parameter $r = 0$ regardless of noise/frequency parameters; rank is 0. Above K_c, $r > 0$ and parameter-sensitive; rank increases to 1. This is symmetry-breaking with $\kappa_c > 0$.

Relation to existing work. The pullback characterization of Fisher information is standard (Amari 2016). Our contribution is identifying *when* coupling increases Jacobian rank—the constraint-release and symmetry-breaking criteria (Theorem 1, Parts I–II). This connects to identifiability theory (Rothenberg 1971), observability in control (Hermann and Krener 1977), and sloppy parameter analysis (Transtrum et al. 2015), but focuses specifically on coupling-induced transitions.

What we are not claiming. We do not argue that data-processing inequalities are violated or that channel capacity bounds fail. These hold within their stated assumptions. We study how coupling changes the *reachable subset* of distributions—the accessible family $\mathcal{M}_{acc}(\kappa) = \mathrm{Image}(F_\kappa)$. Information-theoretic bounds apply to each fixed κ; what changes across κ is the dimension of the family to which they apply.

Organization. Section 2 reviews information geometry. Section 3 presents the formal setup and main theorem. Section 4 discusses consequences and eigenvalue detection. Section 5 addresses scope and limitations.

2 Background: Information geometry

We briefly review the relevant information-geometric concepts; see Amari (2016) for a comprehensive treatment.

A *statistical manifold* $\mathcal{M}$ is a smooth family of probability distributions parameterized by $\theta \in \Theta$. The *Fisher information metric* is:

$$g_{ij}(\theta) = \mathbb{E}\left[\frac{\partial \log p(x|\theta)}{\partial \theta_i} \frac{\partial \log p(x|\theta)}{\partial \theta_j}\right] \tag{1}$$

This Riemannian metric is intrinsic: it transforms correctly under reparameterization and locally approximates KL divergence to second order.

Hierarchical decomposition. For joint distributions $p(x_1, x_2)$, Amari (2001)'s log-linear decomposition separates marginal and interaction terms:

$$\log p(x_1, x_2) = \theta_1(x_1) + \theta_2(x_2) + \eta(x_1, x_2) + \psi \tag{2}$$

The *independence submanifold* $\mathcal{M}_{\mathrm{ind}} = \{\eta = 0\}$ is e-flat. Interaction coordinates η are structurally present in any joint model—they parameterize directions transverse to independence.

What we add. Standard information geometry studies the geometry of a fixed model class. Interaction coordinates either belong to the model or don't—a modeling choice. We study a different question: given that dynamics determine which distributions are reachable (the *accessible family*), when does coupling expand this family? The interaction coordinates may be present in the ambient model yet dynamically inaccessible at zero coupling. Manifold expansion occurs when coupling makes them accessible.

This connects to singular learning theory (Watanabe 2009), where Fisher degeneracy is a fixed property of the model. Here, degeneracy can be *lifted* by coupling: directions in $\ker(dF_0)$ at zero coupling can leave the kernel at positive coupling. This "identifiability activation" is dynamic, not static.

3 Manifold expansion under coupling

3.1 Setup: three distinct objects

Consider two dynamical systems with state spaces $X_1 \subset \mathbb{R}^{n_1}$ and $X_2 \subset \mathbb{R}^{n_2}$. We distinguish three objects that are often conflated:

1. **Ambient manifold** $\mathcal{P}(X_1 \times X_2)$: The space of all probability distributions over the joint state space. This is fixed.
2. **Chosen model class** $\mathcal{M} \subset \mathcal{P}$: A parametric family the analyst uses for inference (e.g., "all bivariate Gaussians"). Standard bounds assume this is fixed.
3. **Accessible family** $\mathcal{M}_{\mathrm{acc}}(\kappa)$: The distributions that the *dynamics* can actually produce at coupling strength κ. This may change with κ.

The independence submanifold: A key example of a submodel is $\mathcal{M}_{\text{ind}} = \{p_1 \otimes p_2 : p_i \in \mathcal{M}_i\}$—the product distributions with zero interaction. In information geometry terminology, this is an *e-flat* submanifold corresponding to zero interaction coordinates in the hierarchical log-linear decomposition (Amari 2016).

The fixed-model assumption: Many learning and communication bounds assume the accessible family is contained in a fixed model class. We do *not* claim this assumption is wrong in general—only that it need not hold in the regime we study.

Formal setup: Let $\varphi_\kappa : X_1 \times X_2 \to X_1 \times X_2$ denote coupled dynamics with coupling strength $\kappa \geq 0$, and let $h : X_1 \times X_2 \to \mathbb{R}^m$ be an observation map. Let $\mathcal{B}$ be a smooth d-dimensional manifold of *physical parameters* β (e.g., damping rates, noise intensities, frequencies).

Key convention. The coupling strength κ is an *index* selecting a dynamical system, not a coordinate in $\mathcal{B}$. This prevents trivial rank increases from simply adding κ as a new parameter. Rank changes must arise from how coupling restructures the parameter→distribution map, not from expanding the parameter space.

For each fixed κ, we have a map:

$$F_\kappa : \mathcal{B} \to \mathcal{P}(\mathbb{R}^m), \quad \beta \mapsto Q_{(\kappa,\beta)} = (h)_\# P_{(\kappa,\beta)} \tag{3}$$

where $P_{(\kappa,\beta)}$ is the stationary distribution of φ_κ at parameters β, and $(h)_\#$ denotes the pushforward through h. The *accessible family* at coupling κ is the image:

$$\mathcal{M}_{\text{acc}}(\kappa) = \text{Image}(F_\kappa) \subset \mathcal{P}(\mathbb{R}^m)$$

Role of the observation map. The observation map $h : X_1 \times X_2 \to \mathbb{R}^m$ determines which features of the joint state are statistically accessible. When h is the identity (full state observation), identifiability depends only on dynamics. When h is a low-dimensional projection (partial observation), some coordinates may remain unidentifiable even if the dynamics distinguish them. Our rank-change theorems concern the map $F_\kappa = (h)_\# \circ P_{(\kappa,\cdot)}$; both dynamics and observation contribute to the accessible family. In the examples, we specify h explicitly and note when identifiability depends on observation richness.

Snapshot vs. path distributions. Throughout this paper, $Q_{(\kappa,\beta)}$ is the distribution of a *single observation* (snapshot) from the stationary measure—not the distribution over time-series paths. We analyze identifiability from i.i.d. samples after mixing; temporal autocorrelations are ignored unless stated otherwise. For dynamical systems where time correlations make additional parameters identifiable (e.g., drift rates from trajectory data), a richer observation map h that extracts lagged covariances or path statistics would be appropriate; our framework accommodates this by enlarging the codomain of h.

Standing assumptions. Throughout, we assume:

(A1) For each (κ, β), the dynamics φ_κ admit a (locally) unique stationary distribution $P_{(\kappa, \beta)}$.
(A2) The map $\beta \mapsto P_{(\kappa, \beta)}$ (and hence F_κ) is smooth.
(A3) The observation map h is smooth with sufficient regularity that $Q = (h)_\# P$ has a smooth density on its support.
(A4) **Finite-dimensional target.** The accessible family $F_\kappa(\mathcal{B})$ is (or is embedded in) a finite-dimensional *regular* statistical manifold $\mathcal{P} \subset \mathcal{P}(\mathbb{R}^m)$, where the Fisher metric is positive-definite on each tangent space.

These assumptions exclude bifurcation points where stationary distributions split or vanish. Our results apply to parameter regions where (A1)–(A4) hold. At the critical threshold κ_c itself, the accessible family may form a singular variety rather than a smooth manifold, and the Fisher metric can degenerate—a setting studied in singular learning theory (Watanabe 2009). The novelty here is that coupling can *resolve* such singularities: directions that are rank-degenerate at $\kappa < \kappa_c$ become regular at $\kappa > \kappa_c$.

Our claim: For high-dimensional coherent systems, the rank of dF_κ can increase as κ crosses a threshold—the image manifold gains dimension.

Scope: finite-dimensional models. The ambient space $\mathcal{P}(\mathbb{R}^m)$ of all distributions is infinite-dimensional. **All results in this paper are stated for finite-dimensional statistical manifolds** (Assumption A4), which is the setting of classical information geometry (Amari 2016). This is achieved either by working with parametric families (OU: Gaussians) or by reducing via the observation map h to finite-dimensional summaries (Kuramoto: order parameter). The core argument—Jacobian rank of F_κ determines identifiable dimension—could in principle be extended to Banach/Hilbert manifold settings with appropriate infinite-dimensional Fisher geometry, but this is beyond our scope.

Definition 1 (Manifold expansion) Define the *maximal rank* of F_κ as:

$$R(\kappa) := \sup_{\beta \in \mathcal{B}} \operatorname{rank} dF_\kappa(\beta) \tag{4}$$

By lower semicontinuity of rank (the sets $\{\beta : \operatorname{rank} dF_\kappa(\beta) \geq r\}$ are open, since rank $\geq r$ iff some $r \times r$ minor is nonzero), the set where rank equals the supremum is open. For analytic maps or generic smooth maps this set is also dense; in general applications we restrict to parameter regions where the supremum is attained. When $\mathcal{B}$ is compact, the supremum is a maximum.

Manifold expansion occurs at κ_c if the maximal rank increases as κ crosses κ_c:

$$R(\kappa_c^+) > R(\kappa_c^-) \tag{5}$$

Equivalently, the dimension of $\mathcal{M}_{\mathrm{acc}}(\kappa)$ increases: a parameter direction in $\mathcal{B}$ that was generically in the kernel of dF_κ becomes generically non-degenerate under coupling.

In Fisher-metric terms: manifold expansion corresponds to an increase in Fisher rank—the number of generically linearly independent score directions.

Remark (Generic vs. maximal rank). Theorem 1 is stated for maximal rank $R(\kappa) = \sup_\beta \operatorname{rank} dF_\kappa(\beta)$. The *generic rank* (the rank achieved on a dense open set) satisfies generic rank $\leq$ maximal rank, with equality for analytic maps. For the OU process, the rank is constant across all (γ, σ) with $\gamma, \sigma > 0$, so generic and maximal coincide. For Kuramoto, rank may vary with parameters, but the maximal rank still exhibits the phase transition. Our results guarantee that the maximal rank increases; whether all of $\mathcal{B}$ achieves the new rank depends on the specific model.

This definition is coordinate-invariant: rank is preserved under reparameterization. It captures "new identifiable directions appear" in an intrinsic way.

The novelty is not that interaction parameters exist in the full parametric family—Amari (2001)'s hierarchical decomposition already accounts for this. The novelty is that **dynamics and observation determine which interaction coordinates are identifiable**. Coupling can *activate* coordinates that were structurally present but dynamically inaccessible—a phenomenon we term **coupling-induced identifiability activation**.
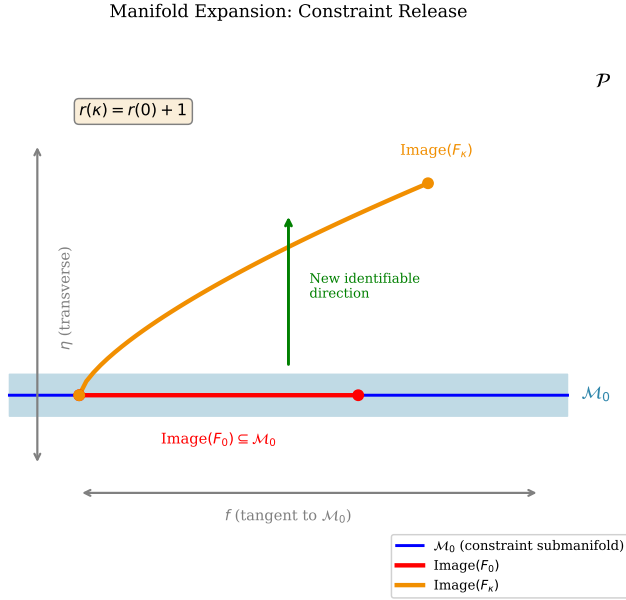

Fig. 1 Geometric schematic of manifold expansion. At $\kappa = 0$, the image of F_0 lies on the constraint submanifold $\mathcal{M}_0$. Under coupling ($\kappa > 0$), the image $F_\kappa(\mathcal{B})$ escapes into the transverse direction η, increasing Fisher rank by at least one.

3.2 Mechanism: collective coordinates

Let $\varphi : X_1 \times X_2 \to X_1 \times X_2$ be the coupled dynamics. Under coupling, new *collective coordinates* can become accessible:

Definition 2 (Collective coordinate) A collective coordinate is a function $\psi : X_1 \times X_2 \to \mathbb{R}$ that is:

1. Not reducible to functions of X_1 or X_2 alone
2. *Dynamically stable*: slow manifold, metastable basin, or conserved quantity
3. *Statistically identifiable*: non-degenerate Fisher information along ψ under observation h

   Examples:

– Phase difference between coupled oscillators (Strogatz 2000; Acebrón et al. 2005)
– Synchronization manifold coordinates (Pikovsky et al. 2001)
– Order parameters of collective states (e.g., chimera states) (Panaggio and Abrams 2015)
– Interface modes at boundaries between systems

   These coordinates represent genuinely new degrees of freedom: they were not accessible to either system in isolation.

3.3 Main theorem: coupling-induced rank transitions

We now state our main result. First, a standard lemma connecting Fisher rank to Jacobian rank.

Lemma 1 (Pullback identity) *Let $\mathcal{P}$ be a regular statistical manifold with Fisher metric g, and let $F : \mathcal{B} \to \mathcal{P}$ be a smooth map. The Fisher information matrix on $\mathcal{B}$ induced by the family* Image(F) *is the pullback:*

$$I_{\mathcal{B}} = F^* g, \qquad \operatorname{rank} I_{\mathcal{B}} = \operatorname{rank} dF. \tag{6}$$

Proof The pullback $(F^* g)_{ij} = g(dF(\partial_i), dF(\partial_j))$ equals $(I_{\mathcal{B}})_{ij}$ by the chain rule for score functions. The kernel of $F^* g$ equals the kernel of dF (since g is positive-definite), giving rank equality.

Thus Fisher rank = Jacobian rank of the dynamics-to-distribution map. The main theorem provides criteria for when coupling increases this rank.

Why this is nontrivial. One might object: "of course coupling adds identifiable directions—κ is itself a parameter." But our theorem concerns rank on the *fixed* parameter space $\mathcal{B}$, not the extended space $\mathcal{B} \times \mathbb{R}_\kappa$. Many couplings preserve rank; rank increase requires that coupling *activates* a previously null direction in parameter space.

Theorem 1 (Coupling-induced rank transitions) *Let $\mathcal{B}$ be a smooth d-dimensional parameter manifold, let $\mathcal{P}$ be a finite-dimensional regular statistical manifold with Fisher metric g, and let $F_\kappa : \mathcal{B} \to \mathcal{P}$ be a smooth family of maps indexed by coupling strength $\kappa \geq 0$.*

Part I (Constraint-release criterion). *Let $\mathcal{M}_0 \subset \mathcal{P}$ be a smooth submanifold of codimension c (e.g., the independence submanifold). Choose local*

coordinates on $\mathcal{P}$ near $\mathcal{M}_0$ so that $\mathcal{M}_0 = \{\eta = 0\}$ locally, and write $F_\kappa(\beta) = (f_\kappa(\beta), \eta_\kappa(\beta))$ where f are coordinates along $\mathcal{M}_0$ and η are transverse (normal) coordinates. Let $\beta_0 \in \mathcal{B}$ achieve the maximal rank $R(\kappa_c) = \mathrm{rank}\, dF_{\kappa_c}(\beta_0)$. Suppose:

(C1) $F_{\kappa_c}(\mathcal{B}) \subseteq \mathcal{M}_0$, i.e., $\eta_{\kappa_c}(\beta) \equiv 0$ (dynamics constrain to submanifold at $\kappa = \kappa_c$)

(C2) **Kernel activation at β_0:** There exists $v \in T_{\beta_0}\mathcal{B}$ with $df_{\kappa_c}(v) = 0$ (hence $v \in \ker(dF_{\kappa_c}(\beta_0))$) such that $d\eta_\kappa|_{\beta_0}(v) \neq 0$ for κ just above κ_c

Geometrically, (C2) says F_κ escapes the constraint $\mathcal{M}_0$ at $F_\kappa(\beta_0)$: the image acquires a component in the normal direction.

Then **for κ in a right neighborhood of κ_c:**

$$R(\kappa) \geq R(\kappa_c) + 1 \tag{7}$$

If m independent kernel directions are activated (i.e., $\dim \mathrm{span}\{d\eta_\kappa(v_i)\} = m$ for $v_i \in \ker(dF_{\kappa_c})$), then $R(\kappa) \geq R(\kappa_c) + m$.

Perturbative test (when $\kappa_c = 0$). When the critical threshold is zero, condition (C2) can be checked via a mixed derivative: there exists $v \in \ker(dF_0(\beta_0))$ such that

$$\left.\frac{\partial}{\partial \kappa}\right|_{\kappa=0} (d_\beta \eta_\kappa(\beta_0)[v]) \neq 0. \tag{8}$$

In coordinates, this is a nonzero entry of $\partial^2 \eta/(\partial\kappa\,\partial\beta)$ along v. Note that merely checking $\partial_\kappa \eta_\kappa(\beta_0)|_{\kappa=0} \neq 0$ is insufficient: that condition says the image point moves off $\mathcal{M}_0$, but rank increase requires that β-sensitivity (not just position) changes.

Part II (Symmetry-breaking criterion). Let G be a compact Lie group acting smoothly and properly on $\mathcal{P}$, and let $\mathrm{Fix}(G) = \{p \in \mathcal{P} : a \cdot p = p \text{ for all } a \in G\}$ be the fixed-point set (an embedded submanifold by the slice theorem). Let $\beta_0 \in \mathcal{B}$ achieve the maximal rank $R(\kappa_c) = \mathrm{rank}\, dF_{\kappa_c}(\beta_0)$.

Key point. We work on the quotient $\mathcal{P}/G$—equivalently, we assume F_κ is gauge-fixed so its image lies in a slice transversal to G-orbits. Orbit directions are not counted toward rank. The rank increase comes from G-**invariant observables** (e.g., order parameter magnitude r) becoming parameter-sensitive when symmetry breaks. Formally, define $\bar{F}_\kappa = \pi \circ F_\kappa$ where $\pi : \mathcal{P} \to \mathcal{P}/G$ is the quotient map; the rank claim is for $\bar{F}_\kappa$.

Suppose:

(S1) $F_{\kappa_c}(\mathcal{B}) \subseteq \mathrm{Fix}(G)$ (distributions are G-invariant at $\kappa = \kappa_c$)

(S2) For κ just above κ_c, there exists β such that $F_\kappa(\beta) \notin \mathrm{Fix}(G)$ (symmetry broken)

(S3) **Invariant-observable activation:** There exists $v \in \ker(dF_{\kappa_c}(\beta_0))$ and a G-invariant observable $\phi : \mathcal{P} \to \mathbb{R}$ such that $d(\phi \circ F_\kappa)|_{\beta_0}(v) \neq 0$ for κ just above κ_c

*Then **for κ in a right neighborhood of** κ_c:*

$$R(\kappa) \geq R(\kappa_c) + 1 \tag{9}$$

If m independent G-invariant observables are activated by coupling, the rank increase is at least m.

Proof **Part I.** Work in a local chart $(f, \eta) \in \mathbb{R}^k \times \mathbb{R}^s$ on $\mathcal{P}$ where $\mathcal{M}_0 = \{\eta = 0\}$. Represent the differential as $dF_\kappa = (df_\kappa, d\eta_\kappa) : T_{\beta_0}\mathcal{B} \to \mathbb{R}^k \times \mathbb{R}^s$. Condition (C1) gives $\eta_{\kappa_c} \equiv 0$, hence $d\eta_{\kappa_c} = 0$ identically. Let $u_1, \ldots, u_{R(\kappa_c)} \in T_{\beta_0}\mathcal{B}$ be vectors whose images $dF_{\kappa_c}(u_i) = (df_{\kappa_c}(u_i), 0)$ are linearly independent, spanning $\text{Image}(dF_{\kappa_c}(\beta_0))$.

Let $v \in \ker(df_{\kappa_c}(\beta_0))$ be as in (C2). Since $d\eta_{\kappa_c} = 0$, we have $dF_{\kappa_c}(v) = (0, 0)$, so $v \in \ker(dF_{\kappa_c}(\beta_0))$.

Claim: For κ in a right neighborhood of κ_c, the vectors $dF_\kappa(u_1), \ldots, dF_\kappa(u_{R(\kappa_c)}), dF_\kappa(v)$ are linearly independent.

Proof of claim: Form the $(R(\kappa_c) + 1) \times (k + s)$ matrix M_κ whose rows are these image vectors. Choose $R(\kappa_c)$ columns from the f-block where $df_{\kappa_c}(u_1), \ldots, df_{\kappa_c}(u_{R(\kappa_c)})$ are linearly independent (such columns exist since $\text{rank}\, df_{\kappa_c}|_{\text{span}\{u_i\}} = R(\kappa_c)$), and one column j from the η-block where $(d\eta_\kappa(v))^j \neq 0$ for κ just above κ_c (such a column exists by (C2)). Consider the $(R(\kappa_c) + 1) \times (R(\kappa_c) + 1)$ minor N_κ formed by these columns.

At $\kappa = \kappa_c$: The upper-left $R(\kappa_c) \times R(\kappa_c)$ block (from the u_i rows and f-columns) equals $A_{\kappa_c} := (df_{\kappa_c}(u_i))^{\text{selected columns}}$, which is invertible. The last row is $(df_{\kappa_c}(v), 0) = (0, 0)$ since $v \in \ker(df_{\kappa_c})$ and $d\eta_{\kappa_c} = 0$. The right column is zero since $d\eta_{\kappa_c} = 0$. Thus $\det N_{\kappa_c} = 0$.

At κ just above κ_c: Write N_κ in block form:

$$N_\kappa = \begin{pmatrix} A_\kappa & b_\kappa \\ c_\kappa^T & d_\kappa \end{pmatrix}$$

where A_κ is $R(\kappa_c) \times R(\kappa_c)$ (from u_i and f-columns), b_κ is $R(\kappa_c) \times 1$ (from u_i and the η-column), c_κ^T is $1 \times R(\kappa_c)$ (from v and f-columns), and $d_\kappa = (d\eta_\kappa(v))^j$.

By continuity, A_κ remains invertible for κ in a neighborhood of κ_c (since $\det A_{\kappa_c} \neq 0$). By condition (C2), $d_\kappa \neq 0$ for κ just above κ_c. The Schur complement formula gives:

$$\det N_\kappa = \det(A_\kappa) \cdot (d_\kappa - c_\kappa^T A_\kappa^{-1} b_\kappa).$$

By continuity, $b_\kappa, c_\kappa \to 0$ as $\kappa \to \kappa_c^+$ (since $b_{\kappa_c} = c_{\kappa_c} = 0$). For κ just above κ_c, the correction term $c_\kappa^T A_\kappa^{-1} b_\kappa$ is second-order small while $d_\kappa \neq 0$ by (C2), so $\det N_\kappa \neq 0$. Thus M_κ has rank at least $R(\kappa_c) + 1$, establishing $R(\kappa) \geq R(\kappa_c) + 1$ for κ in a right neighborhood of κ_c.

Part II. We reduce this to Part I by taking $\mathcal{M}_0 = \text{Fix}(G)$ as the constraint submanifold. Since G is compact and acts properly, the fixed-point set $\text{Fix}(G)$ is an embedded submanifold by the slice theorem (Palais 1961). There exists a G-invariant tubular neighborhood with slice coordinates (f, η) where $\text{Fix}(G) = \{\eta = 0\}$ and η transforms nontrivially under G.

Condition (S1) states $F_{\kappa_c}(\mathcal{B}) \subseteq \mathrm{Fix}(G)$, which gives $\eta_{\kappa_c} \equiv 0$—matching (C1). For (S3), let ϕ be G-invariant with $d(\phi \circ F_\kappa)(v) \neq 0$ for κ just above κ_c. Since ϕ is G-invariant, it depends only on the G-invariant coordinates, which include f (restricted to $\mathrm{Fix}(G)$) and functions of $|\eta|$ (or other G-invariant combinations of η). Near $\mathrm{Fix}(G)$, any smooth G-invariant ϕ can be written as $\phi(f, \eta) = \psi(f, \|\eta\|^2)$ for some smooth ψ.

Now, $d(\phi \circ F_\kappa)(v) = d\psi \cdot (df_\kappa(v), 2\eta_\kappa \cdot d\eta_\kappa(v))$. At $\kappa = \kappa_c$: $\eta_{\kappa_c} = 0$, so the second term vanishes and we get $d(\phi \circ F_{\kappa_c})(v) = \partial_f \psi \cdot df_{\kappa_c}(v) = 0$ since $v \in \ker(df_{\kappa_c})$.

At κ just above κ_c: Since $d(\phi \circ F_\kappa)(v) \neq 0$ and $v \in \ker(df_{\kappa_c})$ (hence $df_\kappa(v)$ is small by continuity), the nonzero contribution must come from the η-dependent term. This requires $\eta_\kappa(\beta_0) \neq 0$ (the image has left $\mathrm{Fix}(G)$) and $d\eta_\kappa(v) \neq 0$—precisely condition (C2). The rank increase then follows from Part I.

Corollary 1 (Operational detection of rank transitions) *Given time-series data at coupling strengths $\kappa_1 < \kappa_2 < \cdots < \kappa_m$, rank transitions can be detected as follows:*

1. ***Estimate Fisher information***: *For each κ_j, estimate the Fisher information matrix $\hat{I}(\kappa_j)$ from data—either directly via score function estimation, or via the Hessian of the log-likelihood at the MLE.*
2. ***Track eigenvalues***: *Compute eigenvalues $\lambda_1(\kappa_j) \geq \lambda_2(\kappa_j) \geq \cdots$ of $\hat{I}(\kappa_j)$.*
3. ***Detect emergence***: *A rank transition at κ_c manifests as one or more eigenvalues crossing from $\lambda_k \approx 0$ (within estimation noise) to $\lambda_k > 0$ significantly.*

For the OU example, the minimal sufficient statistics are the sample covariance entries $(\hat{\Sigma}_{11}, \hat{\Sigma}_{22}, \hat{\Sigma}_{12})$; rank increase corresponds to $\hat{\Sigma}_{12}$ becoming β-sensitive. For Kuramoto, the order parameter magnitude r suffices (after gauge-fixing $\Psi = 0$); rank increase corresponds to r becoming parameter-sensitive above K_c.
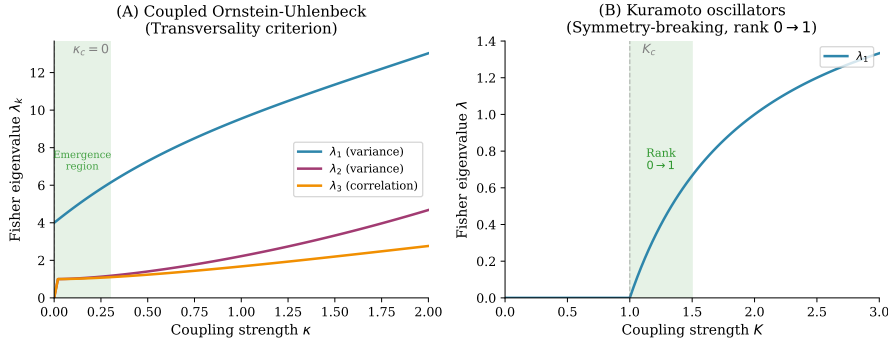
Fig. 2 Eigenvalue emergence under coupling (schematic). Illustrative eigenvalue trajectories showing rank transitions as coupling increases. (A) Coupled OU: the correlation coordinate ρ becomes accessible at any $\kappa > 0$, activating a third identifiable direction (rank $2 \to 3$). (B) Kuramoto (mean-field theory, observing only order parameter r): below K_c, $r = 0$ for all parameter values, so Fisher rank is zero; at K_c, r emerges and becomes parameter-sensitive (rank $0 \to 1$). Since r is a 1D statistic, at most one eigenvalue can be nonzero. These schematics illustrate the qualitative signature of Theorem 1; operational detection from data would use the pullback Fisher information $F_\kappa^* g$ on the physical parameter space $\mathcal{B}$ (see Section 5.4).

3.4 Superadditive accessible complexity

The preceding theorem establishes conditions for rank increase under coupling. We now formalize "the whole exceeds the sum of its parts" as a precise inequality.

Definition 3 (Component accessible families) For each subsystem $i = 1, \ldots, n$, let $\mathcal{B}_i$ be its parameter manifold and let $F_i : \mathcal{B}_i \to \mathcal{P}_i$ be its dynamics-to-distribution map. Define the component accessible family $\mathcal{M}_i := \text{Image}(F_i)$ and its maximal rank $R_i := \max_{\beta_i \in \mathcal{B}_i} \text{rank}\, dF_i(\beta_i)$.

Definition 4 (Superadditive accessible complexity) Let $F_\kappa : \mathcal{B} \to \mathcal{P}$ be the coupled dynamics-to-distribution map with maximal rank $R(\kappa) := \max_{\beta \in \mathcal{B}} \text{rank}\, dF_\kappa(\beta)$. The coupled system exhibits *superadditive accessible complexity* at coupling κ if

$$R(\kappa) > R_\oplus = \sum_i R_i.$$

Lemma 2 (Additivity under independence) *For the independent composite map $F_\oplus$, $R_\oplus = \sum_{i=1}^n R_i$.*

Proof The differential $dF_\oplus$ is block-diagonal with blocks dF_i on $T_{\beta_i} \mathcal{B}_i$. Hence $\text{rank}\, dF_\oplus(\beta) = \sum_i \text{rank}\, dF_i(\beta_i)$. Maximizing over β yields $R_\oplus = \sum_i R_i$.

Proposition 1 (Sufficient condition via non-factorizable coordinates) *Suppose there exists a smooth map $\pi : \mathcal{P} \to \mathbb{R}^k$ such that:*

(A1) $\pi \circ F_\oplus$ is locally constant on some open $U \subset \mathcal{B}$;
(A2) $\pi \circ F_\kappa$ has Jacobian rank k on U.

Then $R(\kappa) \geq R_\oplus + k$, so the coupled system is superadditive whenever $k \geq 1$.

3.5 Example 1: coupled Ornstein–Uhlenbeck (constraint-release)

This example demonstrates Theorem 1 Part I (constraint-release criterion) with $\kappa_c = 0$.

Dynamics. Consider two coupled Ornstein–Uhlenbeck processes:

$$dX_1 = -\gamma_1 X_1\, dt + \kappa(X_2 - X_1)\, dt + \sigma_1\, dW_1 \tag{10}$$

$$dX_2 = -\gamma_2 X_2\, dt + \kappa(X_1 - X_2)\, dt + \sigma_2\, dW_2 \tag{11}$$

where $\gamma_i > 0$ are mean-reversion rates, $\sigma_i > 0$ are noise intensities, and $\kappa \geq 0$ is the coupling strength.

Parameter manifold and observation. The physical parameter manifold is $\mathcal{B} = \{(\gamma_1, \sigma_1, \gamma_2, \sigma_2) : \gamma_i > 0, \sigma_i > 0\} \cong (\mathbb{R}^+)^4$, which is 4-dimensional. The observation map is full state observation: $h(X_1, X_2) = (X_1, X_2)$, so the observable is the bivariate Gaussian with covariance Σ. The map $F_\kappa : \mathcal{B} \to \mathcal{P}$ sends $\beta = (\gamma_1, \sigma_1, \gamma_2, \sigma_2)$ to the centered bivariate Gaussian with covariance $\Sigma(\kappa, \beta)$. We compute the rank of the Jacobian $\partial(\Sigma_{11}, \Sigma_{22}, \Sigma_{12})/\partial\beta$.

Generic vs symmetric. For generic $(\gamma_1, \sigma_1, \gamma_2, \sigma_2)$, the map $\beta \mapsto (\Sigma_{11}, \Sigma_{22}, \Sigma_{12})$ has rank 3 for $\kappa > 0$ and rank 2 for $\kappa = 0$. We present the symmetric case $\gamma_1 = \gamma_2 = \gamma$, $\sigma_1 = \sigma_2 = \sigma$ for intuition. Under this constraint, the parameter manifold is 2-dimensional (coordinates γ, σ), reducing the rank transition to $1 \to 2$; the *mechanism*—activation of the correlation coordinate—remains the same. We use stationary distributions throughout; temporal autocorrelation (which would make drift rates identifiable from dynamics) is not considered.

Generic case Jacobian (verification). For completeness, we verify the generic 4D→3D rank claim. From the Lyapunov equation with drift matrix $A = -\mathrm{diag}(\gamma_1 + \kappa, \gamma_2 + \kappa) + \kappa J$ (where J is the exchange matrix) and diffusion $D = \mathrm{diag}(\sigma_1^2, \sigma_2^2)$, the stationary covariance entries satisfy:

$$\Sigma_{11} = \frac{\sigma_1^2(\gamma_2 + \kappa) + \kappa^2 \sigma_2^2}{2(\gamma_1 + \kappa)(\gamma_2 + \kappa) - 2\kappa^2}, \quad \Sigma_{22} = \frac{\sigma_2^2(\gamma_1 + \kappa) + \kappa^2 \sigma_1^2}{2(\gamma_1 + \kappa)(\gamma_2 + \kappa) - 2\kappa^2} \tag{12}$$

$$\Sigma_{12} = \frac{\kappa(\sigma_1^2 + \sigma_2^2)}{2(\gamma_1 + \kappa)(\gamma_2 + \kappa) - 2\kappa^2} \tag{13}$$

At $\kappa = 0$: $\Sigma_{12} = 0$ identically, and $\Sigma_{ii} = \sigma_i^2/(2\gamma_i)$. The Jacobian $\partial(\Sigma_{11}, \Sigma_{22}, \Sigma_{12})/\partial(\gamma_1, \sigma_1, \gamma_2, \sigma_2)$ is 3×4; at $\kappa = 0$ its rank is 2 (the Σ_{12} row vanishes).

At $\kappa > 0$: Computing the 3×3 minor formed by $(\gamma_1, \sigma_1, \gamma_2)$ at the parameter point $(\gamma_1, \sigma_1, \gamma_2, \sigma_2) = (1, 1, 1, 1)$ with $\kappa = 0.5$ gives a determinant $\approx -0.0074 \neq 0$. Hence rank $= 3$ for $\kappa > 0$: the rank transition $2 \to 3$ is verified.

Summary: rank transitions in coupled OU.
Generic case ($\mathcal{B} \cong \mathbb{R}_+^4$): outputs $(\Sigma_{11}, \Sigma_{22}, \Sigma_{12})$; rank $2 \to 3$.
Symmetric slice ($\mathcal{B} \cong \mathbb{R}_+^2$): same mechanism; rank $1 \to 2$.
In both cases, the correlation coordinate $\rho = \Sigma_{12}/\sqrt{\Sigma_{11}\Sigma_{22}}$ is activated.

Stationary distribution. The stationary distribution is bivariate Gaussian with zero mean and covariance matrix Σ satisfying the Lyapunov equation $A\Sigma + \Sigma A^T + D = 0$. For the symmetric case $\gamma_1 = \gamma_2 = \gamma$ and $\sigma_1 = \sigma_2 = \sigma$:

$$\Sigma_{11} = \Sigma_{22} = \frac{\sigma^2(\gamma + \kappa)}{2\gamma(\gamma + 2\kappa)}, \qquad \Sigma_{12} = \frac{\kappa\sigma^2}{2\gamma(\gamma + 2\kappa)} \tag{14}$$

giving correlation $\rho = \Sigma_{12}/\Sigma_{11} = \kappa/(\gamma + \kappa)$.

This confirms $\rho(\kappa = 0) = 0$ and $\rho \to 1$ as $\kappa \to \infty$ (strong coupling induces perfect correlation, as expected physically). Computing $\partial\rho/\partial\gamma$:

$$\frac{\partial\rho}{\partial\gamma} = -\frac{\kappa}{(\gamma + \kappa)^2} \neq 0 \quad \text{for } \kappa > 0 \tag{15}$$

This nonzero derivative demonstrates kernel activation: a direction that left ρ unchanged at $\kappa = 0$ now varies ρ at $\kappa > 0$.

Verification of Theorem 1 Part I. The ambient manifold $\mathcal{P}$ of centered bivariate Gaussians is 3-dimensional, parameterized by (v_1, v_2, ρ). The independence submanifold is $\mathcal{M}_0 = \{(v_1, v_2, 0)\}$, which has codimension $c = 1$.

At $\kappa = 0$: $\rho = 0$ for all β, so $F_0(\mathcal{B}) \subseteq \mathcal{M}_0$ and $R(0) = 2$.

At $\kappa > 0$: $\rho > 0$, so $F_\kappa(\mathcal{B})$ escapes $\mathcal{M}_0$ and $R(\kappa) = 3$.

Role of observation. The rank increase from 2 to 3 assumes the observation map h is rich enough to identify the cross-covariance Σ_{12}. If h only observes X_1 (marginal observation), or only observes variances $(\Sigma_{11}, \Sigma_{22})$ without cross-terms, the correlation ρ remains unidentifiable and the rank increase is not observable. This illustrates that both dynamics and observation contribute to the accessible family.

Superadditivity. Each OU process in isolation has a 1-dimensional accessible family (variance only), so $R_1 = R_2 = 1$ and $R_\oplus = 2$. The coupled system achieves $R(\kappa) = 3 > R_\oplus = 2$: *strict superadditivity.*

MI vs. rank: a quantitative contrast. For a bivariate Gaussian with correlation ρ, the mutual information is $I(X_1; X_2) = -\frac{1}{2}\log(1 - \rho^2) \approx \rho^2/2$ for small ρ. At weak coupling, $\rho = \kappa/(\gamma + \kappa) \approx \kappa/\gamma$ is first-order small, so $I(X_1; X_2) \approx \kappa^2/(2\gamma^2)$ is *second-order* small. Yet the rank transition—from 2 to 3—occurs immediately at any $\kappa > 0$. This illustrates the paper's central point: manifold expansion (new identifiable coordinates) can occur even when mutual information through the coupling is negligible.
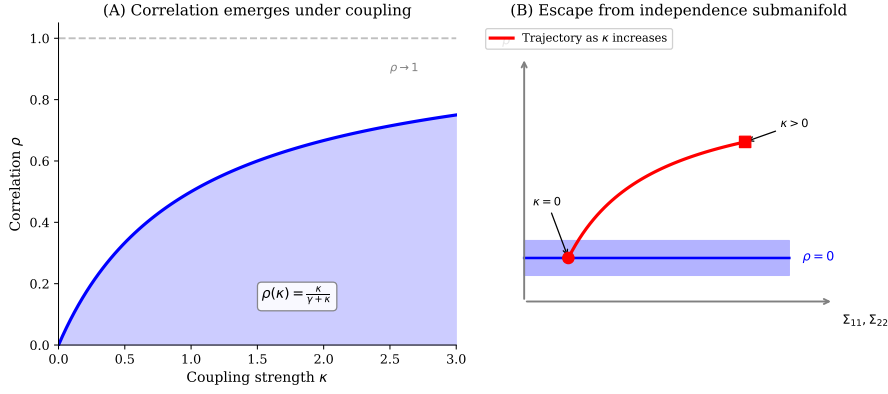
Fig. 3 Coupled OU: correlation emerges under coupling. (A) The correlation $\rho = \kappa/(\gamma + \kappa)$ increases monotonically from zero, asymptoting to 1 as $\kappa \to \infty$. (B) Geometric interpretation: at $\kappa = 0$, the accessible family lies on the independence submanifold ($\rho = 0$); coupling moves the trajectory into the transverse direction.
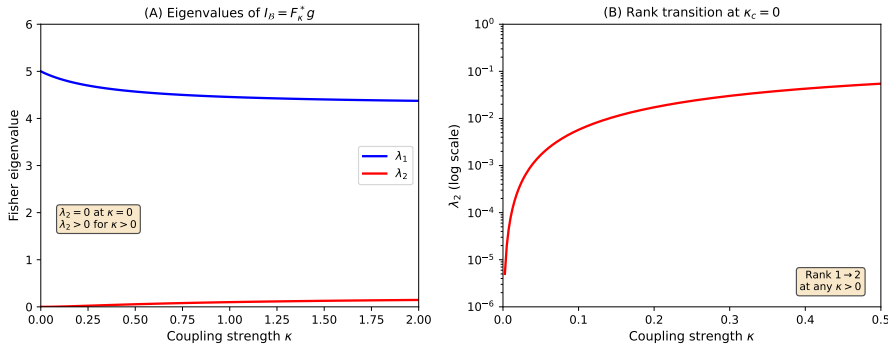

Fig. 4 Operational validation: pullback Fisher eigenvalues on $\mathcal{B}$. Unlike the schematic in Fig. 2, this figure shows the *actual* eigenvalues of the pullback Fisher information $I_{\mathcal{B}} = F_\kappa^* g$ computed on the physical parameter space $\mathcal{B} = \{(\gamma, \sigma)\}$ for symmetric coupled OU. (A) Both eigenvalues as coupling increases. (B) Log-scale view of λ_2 near $\kappa = 0$: the second eigenvalue is exactly zero at $\kappa = 0$ and becomes positive immediately for any $\kappa > 0$, confirming the rank transition $1 \to 2$ predicted by Theorem 1 Part I with $\kappa_c = 0$.

Detectability scaling. For small κ, $\lambda_2(\kappa) = O(\kappa^2)$: the second eigenvalue emerges quadratically. This has a practical implication: detecting the rank transition in finite samples requires signal-to-noise that scales like $1/\kappa^2$. The *dimension* changes immediately at any $\kappa > 0$, but finite-sample detectability has a coupling-dependent threshold. This parallels the MI comparison: both $I(X_1; X_2) \approx \kappa^2/(2\gamma^2)$ and the new Fisher eigenvalue are $O(\kappa^2)$ for small κ.

The distinction is that rank is binary (0 vs. nonzero), while MI and eigenvalue magnitude measure degree of dependence.

Cramér–Rao bound transition. Manifold expansion has concrete inferential consequences. Consider estimating the correlation ρ from n i.i.d. samples of (X_1, X_2). At $\kappa = 0$, the correlation is identically zero—no estimator can learn ρ because it doesn't vary with parameters (infinite CRB in the ρ-direction). At $\kappa > 0$, $\rho = \kappa/(\gamma + \kappa)$ becomes a function of parameters. The Fisher information for ρ (as a reparameterization) is $I_\rho = 2n(1 - \rho^2)^{-2}$ for n samples from a bivariate Gaussian, giving CRB of $\mathrm{Var}(\hat{\rho}) \geq (1 - \rho^2)^2/(2n)$. At weak coupling ($\rho \approx 0$), this is approximately $1/(2n)$—finite and achievable by the sample correlation coefficient. The transition from infinite to finite CRB is the statistical signature of rank increase.

3.6 Example 2: Kuramoto synchronization (symmetry breaking)

This example demonstrates Theorem 1 Part II (symmetry-breaking criterion) with $\kappa_c > 0$. We work in the **mean-field limit** ($N \to \infty$) with added noise, where the stationary density is unique up to symmetry (Strogatz 2000; Acebrón et al. 2005).

Notation. Following the Kuramoto literature, we denote coupling strength by K (rather than κ) in this section. The role is identical: K indexes the dynamical system, not a coordinate in the parameter space $\mathcal{B}$.

Parameter space. The physical parameter manifold is $\mathcal{B} = \{(D, \nu) : D > 0, \nu \in \mathcal{P}(\mathbb{R})\}$ where D is noise intensity and ν is the frequency distribution. For concreteness, take $\nu = \mathcal{N}(\omega_0, \sigma_\omega^2)$ so that $\mathcal{B} \cong \mathbb{R}^+ \times \mathbb{R} \times \mathbb{R}^+$ is 3-dimensional. The mean phase Ψ is *not* a coordinate in $\mathcal{B}$—it parameterizes the S^1-orbit of solutions.

Observation map and regularity. In the mean-field limit, the order parameter r is a deterministic function of (K, β). To satisfy the regularity requirements (A3)–(A4), we model observation with additive noise: the observed order parameter is $\tilde{r} = r(K, \beta) + \epsilon$ where $\epsilon \sim \mathcal{N}(0, \tau^2)$ represents measurement uncertainty. The observation distribution is then $Q_{(K,\beta)} = \mathcal{N}(r(K, \beta), \tau^2)$, which has a smooth density on $\mathbb{R}$. After gauge-fixing $\Psi = 0$, the accessible family $\mathcal{M}_{\mathrm{acc}}(K) \subset \mathcal{P}(\mathbb{R})$ is 1-dimensional, parameterized by the mean r. (Alternatively, one could work at finite N where r fluctuates; the mean-field + noise model is analytically simpler while preserving the rank transition.)

Observation-dependence caveat. The rank-zero claim below K_c is *specific to this reduced observation*. With richer observation—e.g., phase histograms, finite-N fluctuations of r, or the full stationary density $\rho(\theta, \omega)$—the noise intensity D or frequency spread σ_ω may affect the observable distribution even in the incoherent phase. Our rank-transition claim concerns the order-parameter observation; more comprehensive observation maps would require separate analysis.

Mean-field dynamics. Let $\rho(\theta, \omega, t)$ be the density of oscillators with natural frequency ω at phase θ. With phase diffusion of strength $D > 0$, the

density evolves according to the nonlinear Fokker–Planck equation:

$$\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial \theta}\left[\rho\left(\omega + Kr\sin(\Psi - \theta)\right)\right] = D\frac{\partial^2 \rho}{\partial \theta^2} \tag{16}$$

where the order parameter $re^{i\Psi} = \int e^{i\theta}\rho(\theta, \omega, t)\,\nu(\omega)\,d\omega\,d\theta$ is determined self-consistently. The system has S^1 symmetry: if $\rho(\theta)$ is a solution, so is $\rho(\theta + \phi)$ for any ϕ.

Stationary solutions and gauge fixing. For $K < K_c$, the unique stable stationary solution is the incoherent S^1-invariant density with $r = 0$. For $K > K_c$, there exists an S^1-*orbit* of synchronized stationary solutions with $r > 0$. To satisfy assumption (A1), we define F_K to return the **gauge-fixed representative** with $\Psi = 0$. Equivalently, one can define F_K to return the equivalence class $[\rho]$ under S^1; the rank analysis is unchanged since we quotient by the symmetry.

Verification of Theorem 1 Part II.

– (S1): For $K < K_c$, $F_K(\mathcal{B})$ is contained in the S^1-fixed point set (the incoherent state with $r = 0$).
– (S2): For $K > K_c$, the synchronized stationary solutions have $r > 0$, breaking S^1 symmetry.
– (S3) **Kernel activation via order parameter**: Consider varying the noise intensity D or the frequency spread σ_ω. For $K < K_c$, these parameters do not affect the (unique) incoherent distribution, so they lie in $\ker(dF_K)$. For $K > K_c$, these parameters affect the order parameter magnitude: $\partial r/\partial D \neq 0$ and $\partial r/\partial \sigma_\omega \neq 0$. The S^1-invariant observable $\phi = r$ becomes parameter-sensitive.

The theorem predicts: $R(K) \geq R(0) + 1$ for $K > K_c$. Since $R(0) = 0$ (observing only $\tilde{r}$, whose mean is zero regardless of parameters below K_c), this gives a rank $0 \to 1$ transition. The "$+1$" comes from the order parameter magnitude becoming identifiable—a G-invariant observable that was constant on the incoherent manifold but varies with parameters on the synchronized manifold.

Fisher information. For the Gaussian observation model $Q_{(K,\beta)} = \mathcal{N}(r(K,\beta), \tau^2)$, the Fisher information matrix on $\mathcal{B}$ is $(I_{\mathcal{B}})_{ij} = \tau^{-2}(\partial r/\partial \beta^i)(\partial r/\partial \beta^j)$. This has rank 0 when $\nabla_\beta r = 0$ (below K_c) and rank 1 when $\nabla_\beta r \neq 0$ (above K_c). Since r determines a 1-dimensional family, rank cannot exceed 1 under this observation map.

Key difference from OU. In the OU example (Part II), any $\kappa > 0$ immediately releases the constraint: $\kappa_c = 0$. In Kuramoto (Part II), the symmetry persists for $0 < K < K_c$; manifold expansion occurs at a *genuine* critical threshold $K_c > 0$.

The strong-coupling limit. Manifold expansion is not monotonic in coupling strength. In the Kuramoto model, as $K \to \infty$, all phases lock to a common value and the system becomes effectively rigid—the N individual phase parameters collapse to a single collective phase. This is *manifold collapse*: excessive coupling destroys the degrees of freedom it initially created.

The "sweet spot" for superadditive complexity lies at intermediate coupling, where interaction coordinates are activated but individual variation persists.

4 Consequences for complexity growth

4.1 Faster than fixed-class bounds predict

Standard bounds on complexity growth assume fixed parameterization, fixed sufficient statistics, and fixed dimensionality.

When coupling creates new coordinates:

- The Fisher geometry changes because the model class changes—new significant Fisher directions appear
- KL-based learning rates underpredict adaptation
- Channel capacity arguments apply at each fixed κ, but across κ the "channel" itself changes; fixed-channel bounds cannot predict cross-κ complexity without tracking how the accessible family evolves

Definition 5 (Model complexity) We define *complexity C* of a statistical family $\mathcal{M}$ as the Fisher rank: the dimension of the identifiable tangent space. This is coordinate-invariant and intrinsic to the manifold geometry.

Clarification. Fisher rank measures *identifiability dimension*—the number of statistically distinguishable directions—not computational or algorithmic complexity. We use "complexity" in the sense of "accessible degrees of freedom under observation," following the information-geometric tradition where model dimension is a natural complexity measure (Amari 2016).

Rank vs. effective rank. Fisher rank is a binary notion: a direction is either in the kernel (rank 0 contribution) or not (rank 1 contribution). It does not distinguish "sloppy" directions (small but nonzero eigenvalues) from "stiff" directions (large eigenvalues). In finite-sample settings, practical identifiability depends on whether eigenvalues exceed a noise floor. Our theoretical results concern the transition from $\lambda = 0$ to $\lambda > 0$; Corollary 2 makes this explicit. For applications, one would use an *effective rank* (e.g., number of eigenvalues above noise level) rather than exact rank.

Corollary 2 (Eigenvalue emergence under coupling) *In regimes where the accessible family $\mathcal{M}_{acc}(\kappa)$ can be embedded in a fixed parameterization across κ, rank increase (Theorem 1) manifests as eigenvalue emergence. As coupling strength κ crosses a critical threshold κ_c, one or more Fisher eigenvalues emerge from zero:*

$$\lambda_k(\kappa_c^-) = 0 \quad and \quad \lambda_k(\kappa_c^+) > 0 \tag{17}$$

In finite-sample settings, emergence is detected when eigenvalues cross a noise threshold $\epsilon > 0$.

4.2 Heuristic scaling: colony dynamics

Consider N high-D systems in a "colony" (weakly coupled network). The *ambient space* of possible interaction parameters grows rapidly: with N subsystems of internal dimension n, pairwise cross-covariances alone contribute $O(N^2 \cdot n^2)$ potential coordinates. However, the *accessible* submanifold dimension is constrained by coupling parameters, symmetry constraints, attractor structure, and observation map.

A rough upper bound, if each pairwise coupling can create k new identifiable coordinates:

$$\operatorname{rank} I_{\text{colony}} \leq N \cdot \operatorname{rank} I_{\text{single}} + |E| \cdot k \tag{18}$$

where $|E|$ is the number of coupling edges. For dense networks $|E| \sim N^2$; for sparse networks $|E| \sim N$.

5 Discussion

5.1 Summary of contributions

This paper provides:

1. A formal definition of **manifold expansion** as Fisher-rank increase under coupling (Definition 1)
2. **Lemma 1**: Pullback identity (Fisher rank = Jacobian rank)
3. **Theorem 1**: Coupling-induced rank transitions, with two parts:
   – Part I: Constraint-release criterion for rank increase
   – Part II: Symmetry-breaking criterion for rank increase
4. Worked examples verifying the theorem: coupled OU (Part I, $\kappa_c = 0$) and Kuramoto oscillators (Part II, $\kappa_c > 0$)

The core technical contribution is identifying **coupling-induced identifiability activation**: interaction coordinates that are structurally present but dynamically inaccessible can become identifiable under coupling, increasing the Fisher rank of the observable family.

5.2 Relation to emergence

Manifold expansion provides an information-geometric characterization of emergence: new identifiable coordinates that are properties of the coupled system but not reducible to properties of components in isolation. The superadditivity condition $R(\kappa) > \sum_i R_i$ is a formal criterion for "the whole exceeds the sum of its parts"—not merely more degrees of freedom, but genuinely new coordinates that exist only relationally.

This framing sidesteps metaphysical debates about "strong" versus "weak" emergence by grounding the concept in statistical identifiability: an *emergent*

coordinate is one whose Fisher information is zero for all subsystems in isolation but nonzero for the coupled system. The correlation ρ in coupled OU processes and the order parameter r in Kuramoto oscillators are emergent in precisely this sense—they become identifiable only through interaction.

5.3 Testable predictions

The framework makes specific predictions:

1. **Fisher rank increase**: For coupled dynamical systems, estimate Fisher information from time-series data at varying coupling strengths κ. Predict: rank $I(\kappa)$ increases as κ crosses critical thresholds.
2. **Eigenvalue emergence**: New Fisher eigenvalues should emerge continuously from zero as coupling increases.
3. **Superlinear scaling**: For colonies of N coupled subsystems, the identifiable parameter count should grow faster than N in the manifold-expansion regime.

5.4 Practical estimation protocol

We outline a concrete protocol for detecting manifold expansion from data.

Setup. Given time-series observations $\{y_t^{(\kappa)}\}_{t=1}^T$ at coupling strengths $\kappa \in \{\kappa_1, \ldots, \kappa_m\}$, the goal is to detect whether Fisher rank increases across the κ sweep.

Step 1: Choose sufficient statistics. For each example:

- *Coupled OU*: Use sample covariance $\hat{\Sigma} = (\hat{\Sigma}_{11}, \hat{\Sigma}_{22}, \hat{\Sigma}_{12})$
- *Kuramoto*: Use order parameter (r, Ψ) estimated from phase time-series
- *General*: Use moments or cumulants up to order matching the expected model dimension

Step 2: Estimate Fisher information. Two approaches:

(a) *Score-based*: Estimate partial derivatives $\partial_\beta \log p(y|\beta)$ at the MLE $\hat{\beta}$; compute $\hat{I}_{ij} = \frac{1}{T} \sum_t s_i(y_t) s_j(y_t)$ where $s_i = \partial_{\beta_i} \log p$.
(b) *Hessian-based*: Compute $\hat{I} = -\nabla_\beta^2 \ell(\hat{\beta})$ where ℓ is the log-likelihood.

Step 3: Track eigenvalue spectrum. For each κ_j, compute eigenvalues $\lambda_1 \geq \lambda_2 \geq \cdots$ of $\hat{I}(\kappa_j)$. Plot $\lambda_k(\kappa)$ versus κ.

Step 4: Detect rank transition. A rank increase at κ_c appears as:

- One or more eigenvalues crossing from $\lambda_k < \epsilon$ to $\lambda_k \gg \epsilon$
- The threshold ϵ should account for finite-sample noise (bootstrap confidence intervals recommended)

Expected signatures by example:

- *OU (transversality, $\kappa_c = 0$)*: The third eigenvalue (corresponding to Σ_{12}) emerges immediately at any $\kappa > 0$

– *Kuramoto (symmetry breaking, $\kappa_c > 0$)*: An eigenvalue corresponding to phase sensitivity emerges at $K = K_c$, with the transition sharpening as system size increases

5.5 Scope and limitations

Our analysis assumes that the accessible family forms a smooth parametric model. In practice, the set of stationary distributions induced by dynamics may have irregular structure; our results apply where smooth-family regularity holds at least locally.

The mechanism is geometric, not substrate-specific. It applies wherever high-dimensional coherent systems couple and satisfy the regularity conditions of Theorem 1.

6 Conclusion

Information geometry has been remarkably successful because the fixed-manifold assumption usually holds. We have identified a regime where it does not: when high-dimensional coherent systems couple, the identifiable statistical family can expand.

This is not a rejection of information-theoretic bounds. It is recognition that such bounds apply to fixed model classes, and coupling can change which parameters are identifiable. The key contribution is formalizing this as **manifold expansion**—defined via Fisher-rank increase—and establishing conditions under which superadditive dimensionality occurs.

In the regime described here, a useful description is constraint exchange rather than token exchange: high-dimensional systems reshape each other's accessible state spaces rather than transmitting discrete symbols through a fixed channel.

Statements and declarations

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