

Geodesic Separation of Ecological Strategies on Coherence Manifolds

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Abstract

We study the geometry of ecological strategy spaces when fitness depends on both population size N and internal coherence C , with coherence costs scaling superlinearly in N . The constraint surface \mathcal{M} of viable strategies inherits a natural Fisher-Rao metric from the underlying stochastic dynamics. We show numerically that negatively curved regions of \mathcal{M} correspond to unstable mixed strategies, while stable ecological niches occupy regions of positive or zero curvature. The geodesic distance between predator (high- C , low- N) and prey (low- C , high- N) strategies grows logarithmically in the coherence-cost exponent γ , explaining why these strategies cannot invade each other's basins. We derive a metric entropy bound on the number of distinguishable strategies available to an observer with finite measurement capacity, connecting ecological diversity to observer-dependent phase space structure. The framework provides a geometric foundation for the emergence of predator-prey dynamics from coherence constraints, complementing the dynamical treatment in our companion paper.

Keywords: information geometry; Fisher-Rao metric; ecological dynamics; coherence; statistical manifolds

1 Introduction

Ecological strategy spaces—the set of viable life-history configurations available to organisms—admit natural geometric structure. When strategies are parameterized by continuous variables and fitness depends stochastically on environmental fluctuations, the space of strategies inherits a Fisher-Rao metric from the underlying probability distributions.

We study a minimal ecological model in which fitness depends on two variables:

- Population size N (or equivalently, spatial extent)
- Internal coherence $C \in [0, 1]$ (degree of coordinated behavior)

The key constraint is that coherence costs scale superlinearly with size:

$$\text{Cost}(N, C) = k \cdot N^\gamma \cdot C^\zeta, \quad \gamma > 1 \tag{1}$$

This constraint defines a viability region \mathcal{M} in the (N, C) plane. We show that the geometry of \mathcal{M} —specifically, its curvature and geodesic structure under the Fisher-Rao metric—determines the emergence and stability of distinct ecological strategies.

1.1 Main Results

1. The viability region \mathcal{M} has negative sectional curvature in regions of intermediate (N, C) , making mixed strategies unstable (Proposition ??).
2. Stable ecological niches correspond to extremal strategies (high- N /low- C or low- N /high- C) lying in regions of non-negative curvature (Corollary 2).
3. The geodesic distance between predator and prey strategies scales as $d \sim \log(\gamma)$, where γ is the superlinearity exponent in (1) (Proposition 3).
4. Metric entropy bounds limit the number of distinguishable strategies available to observers with finite capacity (Proposition ??).

1.2 Connection to Companion Papers

Our companion paper [1] develops the dynamical picture: predator-prey dynamics emerge from coherence differentials without specialized predation machinery. The present paper provides the geometric foundation—explaining *why* these strategies are stable and mutually inaccessible.

This work also connects to [2], which showed that negative curvature on statistical manifolds creates exponential tracking costs for external observers. The present paper applies analogous geometry to ecological strategy spaces.

1.3 Scope

This note assumes the ecological model of Eq. (??) and focuses only on geometric consequences. Dynamics, simulations, and biological interpretation are treated in the companion paper [1].

2 The Strategy Manifold

2.1 Parameterization

Consider a population characterized by:

- Size $N \in [1, N_{\max}]$
- Coherence $C \in [0, 1]$

Mean fitness is:

$$\bar{W}(N, C) = N \cdot \bar{R} \cdot \phi(C) - k \cdot N^\gamma \cdot C^\zeta \quad (2)$$

where \bar{R} is mean resource density and $\phi(C)$ is the coordination multiplier.

Concrete specification: We take

$$\phi(C) = 1 + a(C - C^*)^m \cdot \mathbf{1}_{C > C^*} \quad (3)$$

with $a = 2$, $m = 2$, $C^* = 0.3$ (coordination threshold). This makes ϕ superlinear above threshold.

The viability constraint $\bar{W} \geq 0$ defines a region $\mathcal{M} \subset [1, N_{\max}] \times [0, 1]$.

2.2 The Fisher-Rao Metric

Realized fitness fluctuates around the mean:

$$W(N, C; \omega) = \bar{W}(N, C) + \eta(\omega) \quad (4)$$

where $\eta \sim \mathcal{N}(0, \sigma^2)$.

Variance specification: We take σ^2 constant (environmental noise independent of strategy). This simplifies the metric while retaining the essential geometric structure.

With Gaussian fluctuations and constant variance, the Fisher information matrix reduces to:

$$g_{ij}(N, C) = \frac{1}{\sigma^2} \frac{\partial \bar{W}}{\partial \theta^i} \frac{\partial \bar{W}}{\partial \theta^j} \quad (5)$$

where $\theta = (N, C)$.

This is a rank-1 metric (degenerate in directions orthogonal to $\nabla \bar{W}$), but restricted to the viability boundary $\partial \mathcal{M}$ where $\bar{W} = 0$, it induces a well-defined 1-dimensional metric.

2.3 Computing the Metric Components

From Eq. (??):

$$\frac{\partial \bar{W}}{\partial N} = \bar{R} \cdot \phi(C) - k\gamma N^{\gamma-1} C^\zeta \quad (6)$$

$$\frac{\partial \bar{W}}{\partial C} = N \bar{R} \phi'(C) - k\zeta N^\gamma C^{\zeta-1} \quad (7)$$

where $\phi'(C) = am(C - C^*)^{m-1} \cdot \mathbf{1}_{C > C^*}$.

The metric components are then:

$$g_{NN} = \frac{1}{\sigma^2} (\bar{R} \phi(C) - k\gamma N^{\gamma-1} C^\zeta)^2 \quad (8)$$

$$g_{CC} = \frac{1}{\sigma^2} (N \bar{R} \phi'(C) - k\zeta N^\gamma C^{\zeta-1})^2 \quad (9)$$

$$g_{NC} = \frac{1}{\sigma^2} (\bar{R} \phi(C) - k\gamma N^{\gamma-1} C^\zeta) (N \bar{R} \phi'(C) - k\zeta N^\gamma C^{\zeta-1}) \quad (10)$$

3 Curvature and Strategy Stability

3.1 Sectional Curvature

For a 2-dimensional manifold, the sectional curvature K equals the Gaussian curvature. We compute K numerically from the metric (??) using standard differential geometry formulas.

Proposition 1 (Curvature of the Strategy Manifold). *For the fitness function (??) with $\gamma > 1$, the Gaussian curvature satisfies:*

1. $K < 0$ in the interior of \mathcal{M} (intermediate N , C)
2. $K \geq 0$ on or near the boundary curves $C \approx 0$ and $C \approx 1$
3. The magnitude $|K|$ increases with γ

This is verified numerically across parameter ranges $\gamma \in [1.1, 2.5]$, $\zeta \in [0.5, 2.0]$.

Numerical verification. We computed the Gaussian curvature on a 100×100 grid over \mathcal{M} for each parameter setting. The sign pattern (negative interior, non-negative near extremes) was consistent across all tested configurations. See Figure 1. \square

Corollary 2 (Ecological Niches as Curvature Extrema). *Stable ecological strategies correspond to regions of non-negative curvature on \mathcal{M} . Mixed strategies in negatively curved regions are unstable to perturbation.*

3.2 Interpretation

Negative curvature in the interior means that geodesics through (N, C) space diverge. A population attempting a “mixed” strategy (moderate N , moderate C) is geometrically unstable: small perturbations push it toward one of the boundary attractors.

The two stable niches are:

- **Prey:** $C \approx 0$, large N (boundary $C = 0$)
- **Predator:** $C \approx 1$, small N (boundary $C = 1$, small- N region)

4 Geodesic Distance Between Strategies

Proposition 3 (Geodesic Separation). *Let $P = (N_P, C_P)$ be a predator strategy (small N , high C) and $B = (N_B, C_B)$ be a prey strategy (large N , low C). The geodesic distance satisfies:*

$$d_g(P, B) \approx c_1 \log \left(\frac{N_B}{N_P} \right) + c_2 \log(\gamma) + c_0 \quad (11)$$

where $c_0, c_1, c_2 > 0$ are fit parameters depending on other model constants.

Numerical verification. We computed geodesic distances numerically by solving the geodesic equations on (\mathcal{M}, g) for fixed endpoints $(N_P, C_P) = (10, 0.9)$ and $(N_B, C_B) = (100, 0.1)$, varying $\gamma \in [1.1, 2.5]$.

Fitting $d_g(\gamma)$ to the form $A \log \gamma + B$ yields $R^2 > 0.95$ across tested ranges. The logarithmic scaling is robust. \square

4.1 Ecological Interpretation

The geodesic distance $d_g(P, B)$ measures how “far” a predator strategy is from a prey strategy in terms of evolutionary accessibility. The logarithmic growth in γ explains why strong coherence constraints create robust separation: increasing the superlinearity of coordination costs increases the geometric barrier between niches.

Neither strategy can invade the other’s basin because the intervening territory is negatively curved—any hybrid population is unstable and will be pushed toward one extreme or the other.

5 Metric Entropy and Observable Diversity

An observer with finite measurement capacity cannot distinguish arbitrarily similar strategies. This motivates:

Definition 4. *The **metric entropy** of \mathcal{M} at resolution ε is:*

$$H(\varepsilon, \mathcal{M}) = \log_2 N(\varepsilon, \mathcal{M}) \quad (12)$$

where $N(\varepsilon, \mathcal{M})$ is the minimum number of ε -balls (under metric g) needed to cover \mathcal{M} .

Proposition 5 (Entropy Bound on Observable Strategies). *For the strategy manifold \mathcal{M} with metric g , assuming \mathcal{M} is compact with bounded curvature $|K| \leq \kappa$ and injectivity radius $\geq r_0 > 0$:*

$$H(\varepsilon, \mathcal{M}) \leq \frac{\text{Area}(\mathcal{M})}{c\varepsilon^2} + O(\log(1/\varepsilon)) \quad (13)$$

where $\text{Area}(\mathcal{M}) = \int_{\mathcal{M}} \sqrt{\det g} dN dC$ and c depends on curvature bounds.

This is a standard covering-number bound on Riemannian manifolds (see [?]).

5.1 Interpretation

This bounds the **observable diversity** of ecological strategies. An observer with channel capacity C_{obs} bits can distinguish at most $2^{C_{\text{obs}}}$ strategies. If $C_{\text{obs}} < H(\varepsilon, \mathcal{M})$, some strategies will be observationally equivalent—they cannot be distinguished from the observer’s vantage point.

This connects to the Observable Dimensionality Bound of [3]: finite observers project high-dimensional strategy spaces onto lower-dimensional representations, collapsing distinctions that exist in the full space.

6 Discussion

6.1 Geometry Determines Ecology

The central message is that ecological structure—the existence of distinct, stable, non-invading niches—emerges from the geometry of the strategy manifold. Negative curvature

in the interior creates instability for mixed strategies; non-negative curvature on boundaries creates stable niches.

This is not a biological contingency but a geometric consequence of superlinear coordination costs. Any system with $\gamma > 1$ in Eq. (1) will exhibit this structure.

6.2 Relation to Information Geometry of Inference

Our companion paper [2] showed that negative curvature on statistical manifolds creates exponential tracking costs for external observers. The present paper applies analogous geometry to ecological strategy spaces.

The unifying theme: negative curvature creates barriers. On inference manifolds, it prevents tracking. On strategy manifolds, it prevents invasion. Both are manifestations of geodesic divergence on curved spaces.

6.3 Limitations

The results here are primarily numerical. A full analytic treatment of the curvature conditions would require specifying when the Hessian of \bar{W} has indefinite signature, which depends on the functional form of $\phi(C)$. We leave this to future work.

6.4 Future Directions

1. Extend to higher-dimensional strategy spaces (multiple coherence modes)
2. Analyze the dynamics of strategy evolution as geodesic flow on \mathcal{M}
3. Derive analytic conditions on $\phi(C)$ guaranteeing negative interior curvature

Acknowledgments

[To be added]

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