

Spectral Decoupling of Capacity and Entropy in Network Dynamics

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Abstract

Network science commonly uses entropy-based measures to characterize dynamical complexity, often implicitly treating high entropy as synonymous with high-dimensional dynamics. We establish that *spectral capacity* (the count of slow Laplacian modes) and *state entropy* are independent axes of network dynamical complexity, controlled by distinct features: topology governs capacity via eigenvalue density, while noise governs entropy. We define spectral capacity $C(\lambda^*)$ as the count of non-trivial Laplacian modes with eigenvalues below a threshold λ^* , and demonstrate that networks with matched state entropy can differ dramatically in capacity, and vice versa. Simulations across canonical architectures (ring, small-world, modular, random, scale-free) show that each topology occupies a distinct position on the capacity axis, while noise moves systems along the entropy axis. These results provide a principled two-axis framework for network complexity.

Keywords: network dynamics, spectral capacity, entropy, Laplacian, complexity

1 Introduction

Complex networks exhibit rich dynamical behavior arising from the interplay of topology and local dynamics. A central question in network science is how to characterize the “complexity” of such dynamics. Two natural candidates are *dimensional capacity*—how many slow modes the topology supports—and *entropy*—how unpredictable the dynamics are. These are often conflated, with entropy used as a proxy for complexity writ large [Gómez-Gardeñes and Latora, 2008, Braунstein et al., 2006].

However, classical results in dynamical systems establish that dimension and entropy are fundamentally distinct. The correlation dimension [Grassberger and Procaccia, 1983], embedding dimension [Takens, 1981, Sauer et al., 1991], and Lyapunov dimension [Kaplan and Yorke, 1979] characterize the *geometry of the attractor*. In contrast, entropy measures characterize the *distribution over that support* [Eckmann and Ruelle, 1985]. As Eckmann and Ruelle noted, “the dimension of the attractor and its entropy are independent characteristics.”

Remark 1 (Capacity vs. Dimension). *The spectral capacity $C(\lambda^*)$ we define is a mode-counting measure—the number of slow Laplacian modes below a threshold. It is **not** a fractal*

dimension in the Grassberger–Procaccia sense. We use “capacity” to emphasize that this measures how many persistent degrees of freedom the topology supports, not the fractal structure of an attractor.

We address how network topology interacts with capacity-entropy decoupling. Our contributions are:

1. **Spectral capacity:** We define $C(\lambda^*)$ as the count of non-trivial Laplacian modes with $\lambda_k < \lambda^*$, a purely topological measure (Section 2).
2. **Decoupling proposition:** We show that networks can have matched state entropy with different capacity, and vice versa (Section 3).
3. **Phase portrait diagnostic:** We introduce the (C, H) phase portrait showing that different topologies occupy distinct capacity positions (Section 4).

2 Theoretical Framework

2.1 Network Diffusion Dynamics

Consider a connected graph $G = (V, E)$ with $n = |V|$ nodes. Let L denote the normalized Laplacian:

$$L = I - D^{-1/2}AD^{-1/2} \quad (1)$$

with eigenvalues $0 = \lambda_1 \leq \lambda_2 \leq \dots \leq \lambda_n \leq 2$.

We study linear diffusive dynamics:

$$x(t+1) = (I - \alpha L)x(t) + \eta(t) \quad (2)$$

where $\alpha \in (0, 1)$ is the diffusion rate and $\eta(t) \sim \mathcal{N}(0, \sigma^2 I)$.

In the Laplacian eigenbasis, mode k evolves as:

$$c_k(t+1) = (1 - \alpha\lambda_k)c_k(t) + \tilde{\eta}_k(t) \quad (3)$$

with relaxation time $\tau_k = 1/(\alpha\lambda_k)$. Small eigenvalues yield slow modes; large eigenvalues yield fast-decaying modes.

2.2 Spectral Capacity

Definition 2 (Spectral Capacity). *The spectral capacity at threshold λ^* is:*

$$C(\lambda^*) = |\{k : 0 < \lambda_k < \lambda^*\}| \quad (4)$$

the count of non-trivial Laplacian modes below threshold. We exclude the trivial $\lambda_1 = 0$ mode (constant eigenvector).

This captures how many slow modes the topology supports. For $\lambda^* = 0.1$:

- **Ring lattices** have dense small- λ spectrum \Rightarrow high capacity
- **Random graphs** have large spectral gap \Rightarrow low capacity (often $C = 0$)
- **Modular networks** have near-degenerate community modes \Rightarrow intermediate capacity

Crucially, $C(\lambda^*)$ depends *only on topology*—it is invariant to noise σ .

2.3 State Entropy Proxy

For the stationary distribution of (2), the differential entropy scales with:

$$H \propto \frac{1}{n} \log \det(\text{Cov}[x]) = \frac{1}{n} \sum_i \log \lambda_i^{\text{cov}} \quad (5)$$

where λ_i^{cov} are covariance eigenvalues. We use the mean log covariance eigenvalue as a *state entropy proxy*. This is **not** the entropy rate h_μ of the process; it characterizes the spread of the stationary distribution.

The state entropy proxy depends on *noise level* σ , not topology directly.

3 Capacity-Entropy Decoupling

Proposition 3 (Spectral Decoupling). *For linear diffusion dynamics on graphs:*

1. **Iso-entropy, different capacity:** There exist graphs G_1, G_2 and noise levels σ_1, σ_2 such that state entropy is matched, but $C(\lambda^*; G_1) \neq C(\lambda^*; G_2)$.
2. **Fixed capacity, variable entropy:** For any fixed graph G , varying σ changes state entropy while $C(\lambda^*)$ remains constant.

Proof sketch. (1) Ring graphs have eigenvalues clustered near zero; random graphs have large spectral gap (first non-trivial eigenvalue $\lambda_2 \gg 0$). At threshold $\lambda^* = 0.1$, rings have $C \approx 6$ while random graphs have $C = 0$. Tuning σ can match their state entropies, but capacity differs.

(2) $C(\lambda^*) = |\{k : 0 < \lambda_k < \lambda^*\}|$ is determined by topology alone. Varying σ scales covariance eigenvalues, changing state entropy, but does not change which Laplacian modes are below threshold. \square

Corollary 4. *The capacity difference between topologies is bounded by their integrated eigenvalue density difference:*

$$|C(\lambda^*; G_1) - C(\lambda^*; G_2)| = |N_1(\lambda^*) - N_2(\lambda^*)| \quad (6)$$

where $N_i(\lambda^*) = |\{k : 0 < \lambda_k^{(i)} < \lambda^*\}|$.

4 Results

We validate the theory on canonical architectures: ring lattices ($k = 6$ neighbors), Watts-Strogatz small-worlds, stochastic block models (4 communities), Erdős-Rényi random graphs, and Barabási-Albert scale-free networks. All have $n = 100$ nodes; $\alpha = 0.1$; $\lambda^* = 0.1$.

4.1 Capacity-Entropy Phase Portrait

Figure 1 shows the (C, H) phase portrait. Each topology occupies a distinct horizontal position (capacity), while noise moves systems vertically (entropy). Key observations:

- **Ring:** $C = 6$ (many slow modes from dense small- λ spectrum)
- **Modular:** $C = 3$ (community modes near zero)
- **Small-world:** $C = 2$ (rewiring increases spectral gap)
- **Random/Scale-free:** $C = 0$ (large spectral gap, no modes below $\lambda^* = 0.1$)

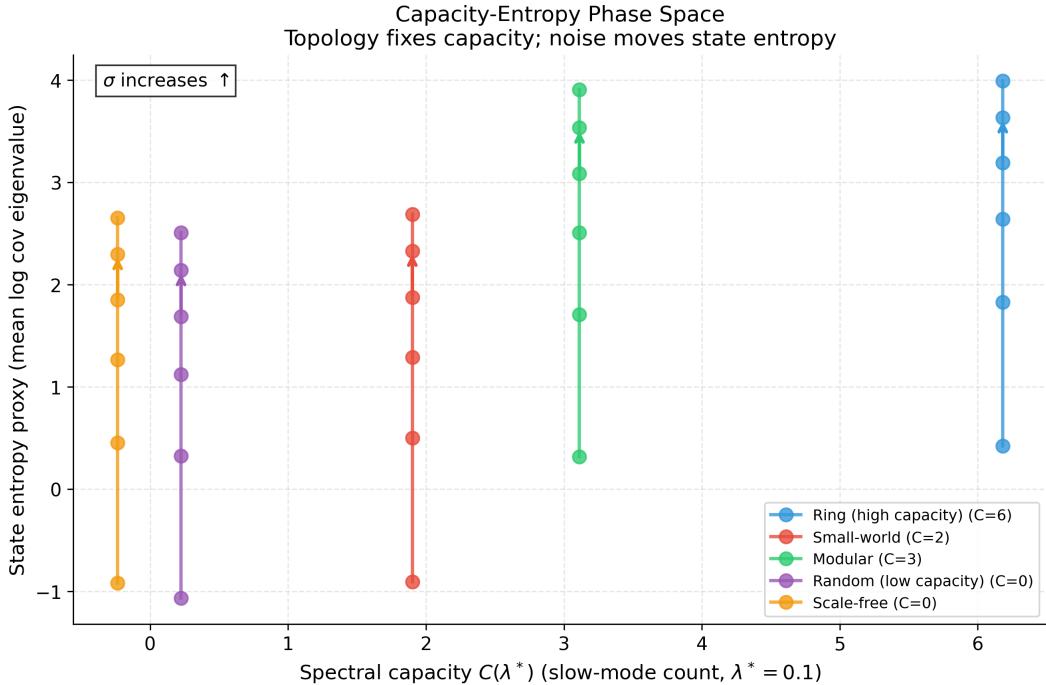


Figure 1: Capacity-entropy phase portrait. Each topology has fixed capacity (horizontal position); noise moves systems vertically. Topology and noise control orthogonal axes.

4.2 Iso-Entropy Comparison

Figure 2 demonstrates Proposition 3(1). We tune σ for each topology to match state entropy at $H \approx 1.0$. Despite matched entropy: Ring has $C = 6$, Modular has $C = 3$, Random has $C = 0$.

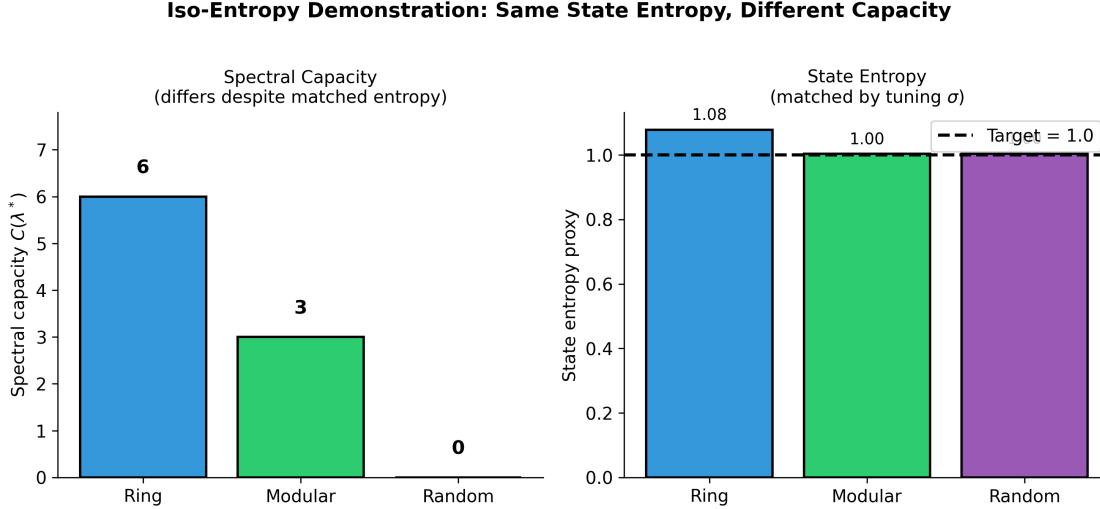


Figure 2: Iso-entropy, different capacity. Left: capacity differs substantially. Right: state entropy matched by tuning σ .

4.3 Fixed Capacity, Variable Entropy

Figure 3 demonstrates Proposition 3(2). For a fixed modular network ($C = 3$), varying σ changes state entropy continuously while capacity remains constant.

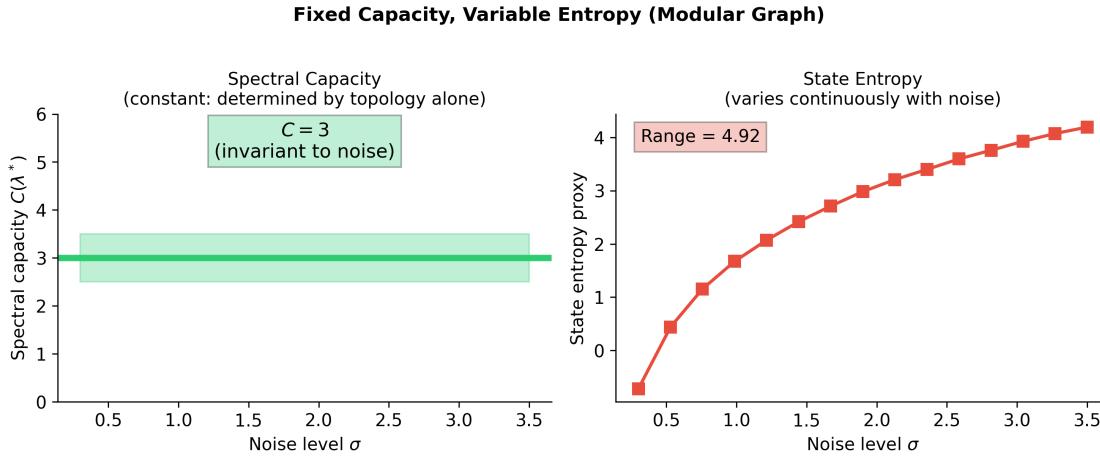


Figure 3: Fixed capacity, variable entropy. Capacity is constant (determined by topology); state entropy varies with noise.

4.4 Spectral Mechanism

Figure 4 shows Laplacian eigenvalue distributions. The threshold $\lambda^* = 0.1$ (dashed line) separates slow from fast modes. Ring lattices have eigenvalues clustered near zero; random graphs have a spectral gap with no eigenvalues below 0.1.

Spectral Capacity: $C(\lambda^*) = |\{k : 0 < \lambda_k < \lambda^*\}|$
Modes below threshold (shaded) are slow modes

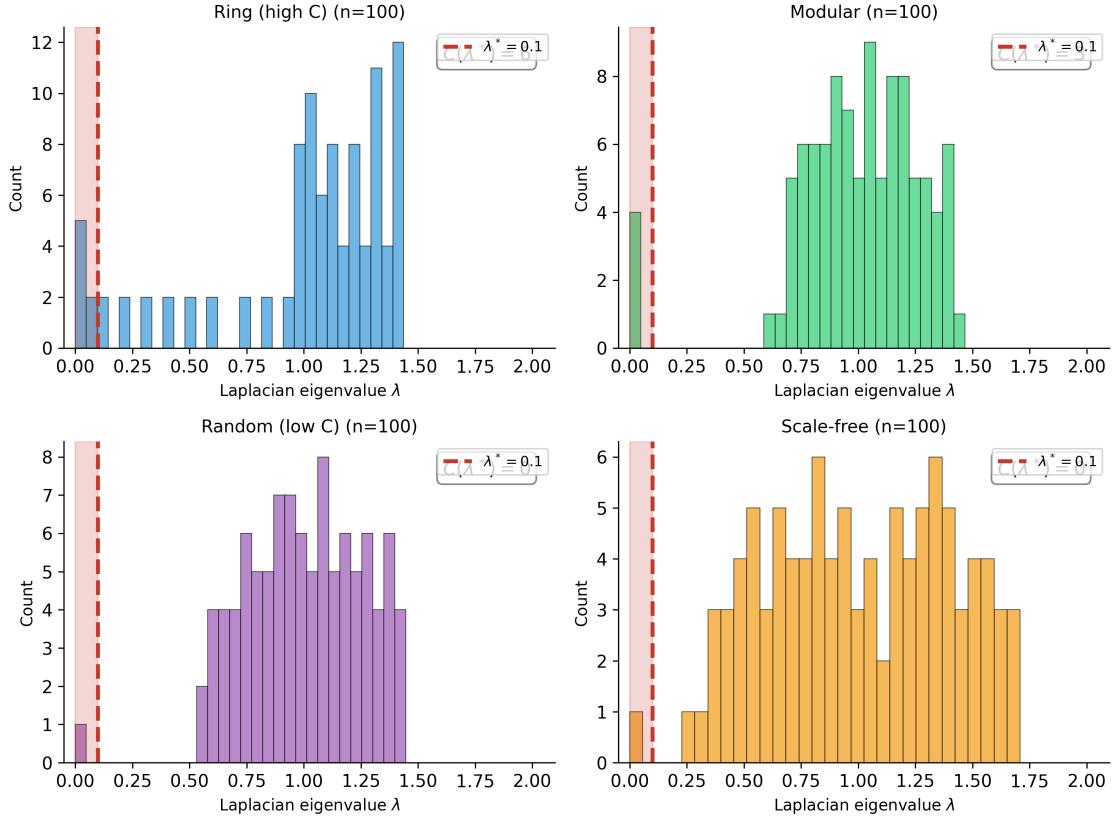


Figure 4: Laplacian spectra. Shaded region shows modes contributing to capacity. Ring has many slow modes; random has none below threshold.

5 Discussion

5.1 Implications

Network complexity is two-dimensional: capacity (topological) and entropy (dynamical). Entropy-only characterizations miss the capacity axis. A network can have:

- High entropy, low capacity: unpredictable dynamics in few modes (e.g., noisy random graph)
- Low entropy, high capacity: structured dynamics across many modes (e.g., coherent ring)

For **neural networks**, this distinguishes high-dimensional activity [Stringer et al., 2019] from high signal diversity [Schartner et al., 2017]. For **network design**, topology selection matters for capacity; noise tuning matters for entropy.

5.2 Relation to Prior Work

The dimension-entropy distinction is classical [Eckmann and Ruelle, 1985, Grassberger and Procaccia, 1983]. Our contribution is making it network-native via Laplacian spectral analysis. Related work includes entropy rate on networks [Gómez-Gardeñes and Latora, 2008], correlation dimension [Lacasa and Gómez-Gardeñes, 2013], and entrograms [Faccin et al., 2018].

5.3 Limitations

We analyzed linear diffusion; nonlinear dynamics may couple capacity and entropy in certain regimes. The threshold λ^* must be chosen appropriately for the network scale. Empirical estimation from finite data requires care.

6 Conclusion

Spectral capacity and state entropy are orthogonal axes of network complexity. Topology sets capacity through eigenvalue density; noise sets entropy. The (C, H) phase portrait provides a diagnostic for characterizing network dynamical regimes.

Data Availability

Code available at <https://github.com/todd866/spectral-decoupling>.

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