Spectral Decoupling of Capacity and Entropy in Network Dynamics

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Abstract

Network science commonly uses entropy-based measures to characterize dynamical complexity, often implicitly treating high entropy as synonymous with high-dimensional dynamics. We establish that *spectral capacity* (the count of slow Laplacian modes) and *state entropy* are independent axes of network dynamical complexity, controlled by distinct features: topology governs capacity via eigenvalue density, while noise governs entropy. We define spectral capacity $C(\lambda^*)$ as the count of non-trivial Laplacian modes with eigenvalues below a threshold λ^*, and demonstrate that networks with matched state entropy can differ dramatically in capacity, and vice versa. Simulations across canonical architectures (ring, small-world, modular, random, scale-free) show that each topology occupies a distinct position on the capacity axis, while noise moves systems along the entropy axis. These results provide a principled two-axis framework for network complexity.

Keywords: network dynamics, spectral capacity, entropy, Laplacian, complexity

1 Introduction

Complex networks exhibit rich dynamical behavior arising from the interplay of topology and local dynamics. A central question in network science is how to characterize the "complexity" of such dynamics. Two natural candidates are *dimensional capacity*—how many slow modes the topology supports—and *entropy*—how unpredictable the dynamics are. These are often conflated, with entropy used as a proxy for complexity writ large [Gómez-Gardeñes and Latora, 2008, Braunstein et al., 2006].

However, classical results in dynamical systems establish that dimension and entropy are fundamentally distinct. The correlation dimension [Grassberger and Procaccia, 1983], embedding dimension [Takens, 1981, Sauer et al., 1991], and Lyapunov dimension [Kaplan and Yorke, 1979] characterize the *geometry of the attractor*. In contrast, entropy measures characterize the *distribution over that support* [Eckmann and Ruelle, 1985]. As Eckmann and Ruelle noted, "the dimension of the attractor and its entropy are independent characteristics."

Remark 1 (Capacity vs. Dimension)**.** *The spectral capacity $C(\lambda^*)$ we define is a* mode-counting measure*—the number of slow Laplacian modes below a threshold. It is* ***not*** *a fractal* *dimension in the Grassberger–Procaccia sense. We use "capacity" to emphasize that this measures how many persistent degrees of freedom the topology* supports*, not the fractal structure of an attractor.*

We address how network topology interacts with capacity-entropy decoupling. Our contributions are:

1. **Spectral capacity**: We define $C(\lambda^*)$ as the count of non-trivial Laplacian modes with $\lambda_k < \lambda^*$, a purely topological measure (Section 2).
2. **Decoupling proposition**: We show that networks can have matched state entropy with different capacity, and vice versa (Section 3).
3. **Phase portrait diagnostic**: We introduce the (C, H) phase portrait showing that different topologies occupy distinct capacity positions (Section 4).

2 Theoretical Framework

2.1 Network Diffusion Dynamics

Consider a connected graph $G = (V, E)$ with $n = |V|$ nodes. Let L denote the normalized Laplacian:

$$L = I - D^{-1/2} A D^{-1/2} \tag{1}$$

with eigenvalues $0 = \lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_n \leq 2$.

We study linear diffusive dynamics:

$$x(t+1) = (I - \alpha L)x(t) + \eta(t) \tag{2}$$

where $\alpha \in (0, 1)$ is the diffusion rate and $\eta(t) \sim \mathcal{N}(0, \sigma^2 I)$.

In the Laplacian eigenbasis, mode k evolves as:

$$c_k(t+1) = (1 - \alpha\lambda_k)c_k(t) + \tilde{\eta}_k(t) \tag{3}$$

with relaxation time $\tau_k = 1/(\alpha\lambda_k)$. Small eigenvalues yield slow modes; large eigenvalues yield fast-decaying modes.

2.2 Spectral Capacity

Definition 2 (Spectral Capacity)**.** *The spectral capacity at threshold λ^* is:*

$$C(\lambda^*) = \big|\{k : 0 < \lambda_k < \lambda^*\}\big| \tag{4}$$

the count of non-trivial *Laplacian modes below threshold. We exclude the trivial $\lambda_1 = 0$ mode (constant eigenvector).*

This captures how many slow modes the topology supports. For $\lambda^* = 0.1$:

- **Ring lattices** have dense small-λ spectrum $\Rightarrow$ high capacity
- **Random graphs** have large spectral gap $\Rightarrow$ low capacity (often $C = 0$)
- **Modular networks** have near-degenerate community modes $\Rightarrow$ intermediate capacity

Crucially, $C(\lambda^*)$ depends *only on topology*—it is invariant to noise σ.

2.3 State Entropy Proxy

For the stationary distribution of (2), the differential entropy scales with:

$$H \propto \frac{1}{n} \log \det(\mathrm{Cov}[x]) = \frac{1}{n} \sum_i \log \lambda_i^{\mathrm{cov}} \tag{5}$$

where λ_i^{cov} are covariance eigenvalues. We use the mean log covariance eigenvalue as a *state entropy proxy*. This is **not** the entropy rate h_μ of the process; it characterizes the spread of the stationary distribution.

The state entropy proxy depends on *noise level* σ, not topology directly.

3 Capacity-Entropy Decoupling

Proposition 3 (Spectral Decoupling)**.** *For linear diffusion dynamics on graphs:*

1. ***Iso-entropy, different capacity**: There exist graphs G_1, G_2 and noise levels σ_1, σ_2 such that state entropy is matched, but $C(\lambda^*; G_1) \neq C(\lambda^*; G_2)$.*
2. ***Fixed capacity, variable entropy**: For any fixed graph G, varying σ changes state entropy while $C(\lambda^*)$ remains constant.*

Proof sketch. (1) Ring graphs have eigenvalues clustered near zero; random graphs have large spectral gap (first non-trivial eigenvalue $\lambda_2 \gg 0$). At threshold $\lambda^* = 0.1$, rings have $C \approx 6$ while random graphs have $C = 0$. Tuning σ can match their state entropies, but capacity differs.

(2) $C(\lambda^*) = |\{k : 0 < \lambda_k < \lambda^*\}|$ is determined by topology alone. Varying σ scales covariance eigenvalues, changing state entropy, but does not change which Laplacian modes are below threshold. $\square$

This makes explicit that capacity differences between topologies are *entirely determined* by their integrated eigenvalue densities $N_i(\lambda^*) = |\{k : 0 < \lambda_k^{(i)} < \lambda^*\}|$. Graph families with predictable spectral densities (e.g., d-regular graphs, stochastic block models) thus have predictable capacity scaling.

4 Results

We validate the theory on canonical architectures: ring lattices ($k = 6$ neighbors), Watts-Strogatz small-worlds, stochastic block models (4 communities), Erdős-Rényi random graphs, and Barabási-Albert scale-free networks. All have $n = 100$ nodes; $\alpha = 0.1$; $\lambda^* = 0.1$.

4.1 Capacity-Entropy Phase Portrait

Figure 1 shows the (C, H) phase portrait. Each topology occupies a distinct horizontal position (capacity), while noise moves systems vertically (entropy). Key observations:

- **Ring**: $C = 6$ (many slow modes from dense small-λ spectrum)
- **Modular**: $C = 3$ (community modes near zero)
- **Small-world**: $C = 2$ (rewiring increases spectral gap)
- **Random/Scale-free**: $C = 0$ (large spectral gap, no modes below $\lambda^* = 0.1$)

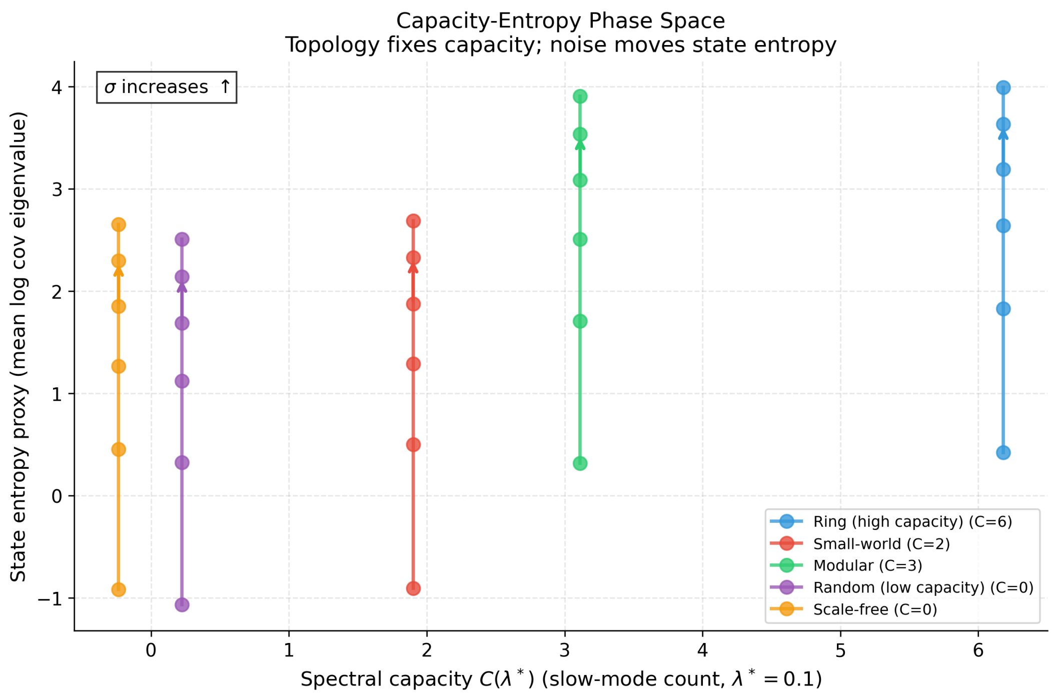

Figure 1: Capacity-entropy phase portrait. Each topology has fixed capacity (horizontal position); noise moves systems vertically. Topology and noise control orthogonal axes.

4.2 Iso-Entropy Comparison

Figure 2 demonstrates Proposition 3(1). We tune σ for each topology to match state entropy at $H \approx 1.0$. Despite matched entropy: Ring has $C = 6$, Modular has $C = 3$, Random has $C = 0$.

4.3 Fixed Capacity, Variable Entropy

Figure 3 demonstrates Proposition 3(2). For a fixed modular network ($C = 3$), varying σ changes state entropy continuously while capacity remains constant.

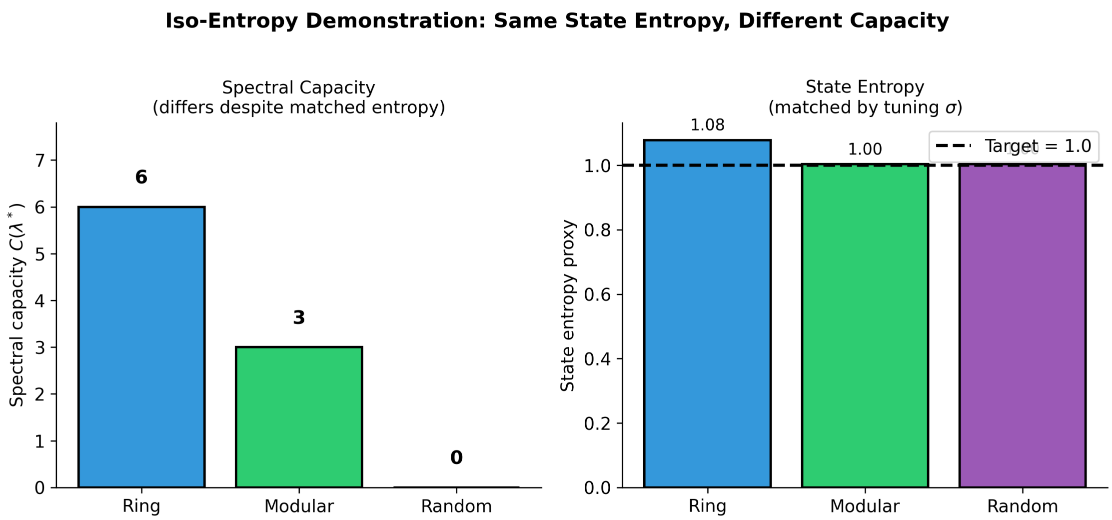

Figure 2: Iso-entropy, different capacity. Left: capacity differs substantially. Right: state entropy matched by tuning σ.

4.4 Spectral Mechanism

Figure 4 shows Laplacian eigenvalue distributions. The threshold $\lambda^* = 0.1$ (dashed line) separates slow from fast modes. Ring lattices have eigenvalues clustered near zero; random graphs have a spectral gap with no eigenvalues below 0.1.

5 Discussion

5.1 Implications

Network complexity is two-dimensional: capacity (topological) and entropy (dynamical). Entropy-only characterizations miss the capacity axis. A network can have:

- High entropy, low capacity: unpredictable dynamics in few modes (e.g., noisy random graph)
- Low entropy, high capacity: structured dynamics across many modes (e.g., coherent ring)

For **neural networks**, this distinguishes high-dimensional activity [Stringer et al., 2019] from high signal diversity [Schartner et al., 2017]. For **network design**, topology selection matters for capacity; noise tuning matters for entropy.

5.2 Relation to Prior Work

The dimension-entropy distinction is classical [Eckmann and Ruelle, 1985, Grassberger and Procaccia, 1983]. Our contribution is making it network-native via Laplacian spectral analysis. Related work includes entropy rate on networks [Gómez-Gardeñes and Latora, 2008],

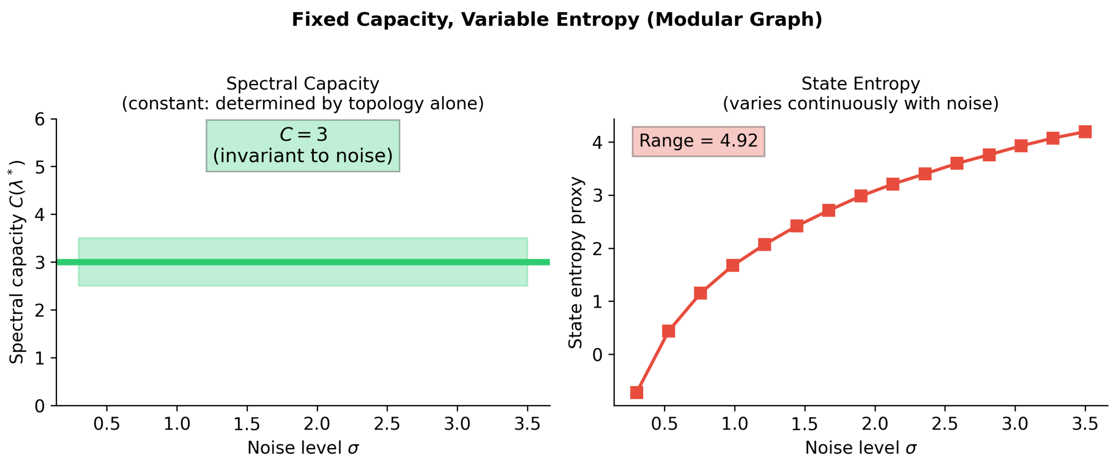

Figure 3: Fixed capacity, variable entropy. Capacity is constant (determined by topology); state entropy varies with noise.

correlation dimension [Lacasa and Gómez-Gardeñes, 2013], and entrograms [Faccin et al., 2018].

5.3 Limitations

We analyzed linear diffusion; nonlinear dynamics may couple capacity and entropy in certain regimes. The threshold λ^* must be chosen appropriately—we recommend either fixing a timescale τ and setting $\lambda^* = 1/(\alpha\tau)$, or choosing λ^* as a fixed percentile of the spectrum to enable cross-network comparisons. Empirical estimation from finite data requires care.

6 Conclusion

Spectral capacity and state entropy are orthogonal axes of network complexity. Topology sets capacity through eigenvalue density; noise sets entropy. The (C, H) phase portrait provides a diagnostic for characterizing network dynamical regimes.

Data Availability

Code available at `https://github.com/todd866/spectral-decoupling`.

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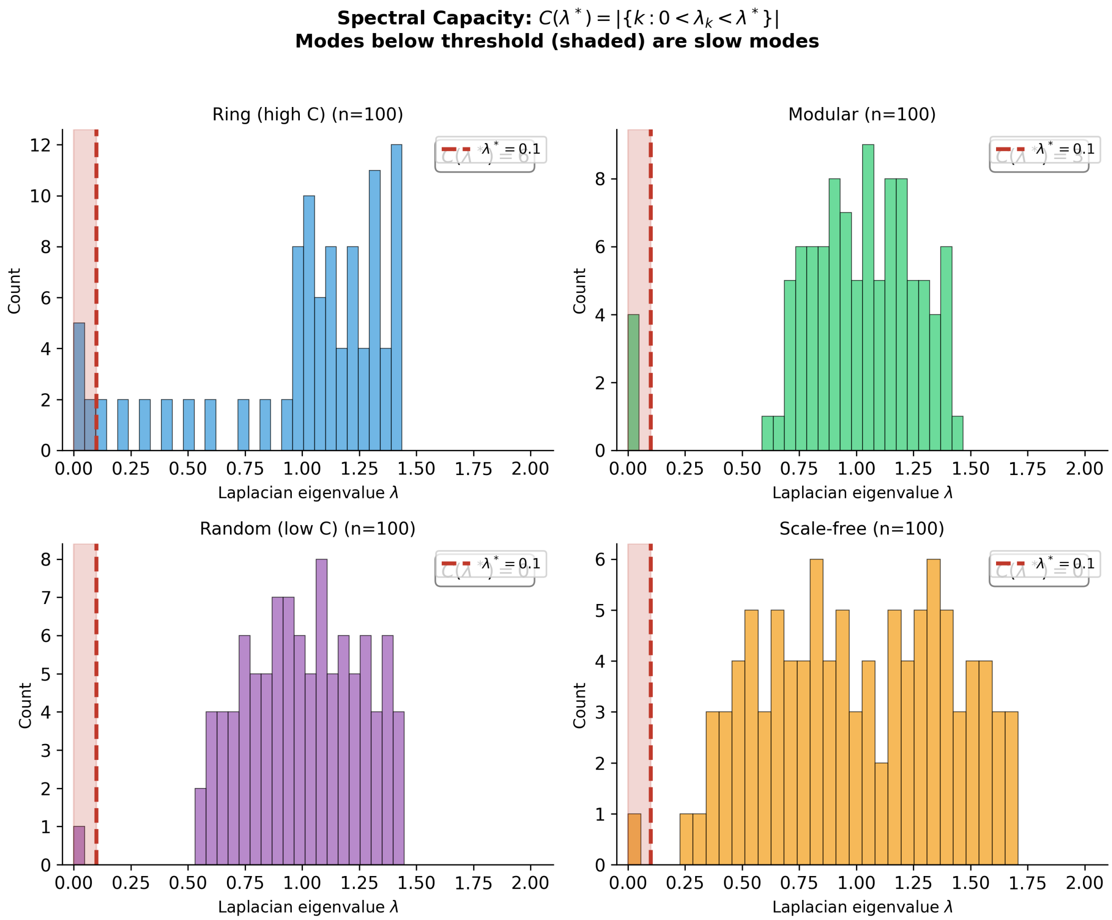

Figure 4: Laplacian spectra. Shaded region shows modes contributing to capacity. Ring has many slow modes; random has none below threshold.

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