



IMPUTATION OF MISSING DATA WITH BAYESIAN ADDITIVE REGRESSION TREES

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INTRODUCTION TO THE PROBLEM OF MISSING DATA

- ▶ Historically, **missingness** was commonly ignored.¹
- ▶ **Donald Rubin** formalised the analysis of missing data.
- ▶ **Multiple Imputation** proposed to capture the uncertainty caused by the missing data.²

¹Rubin, 1976

²Rubin, 1978

TREE BASED REGRESSION

CLASSIFICATION AND REGRESSION TREES³

- ▶ Statistical **classification** or **regression** that partitions the predictor space into subgroups.
- ▶ **Non-parametric** and can model high-level interactions and **non-linearities**.

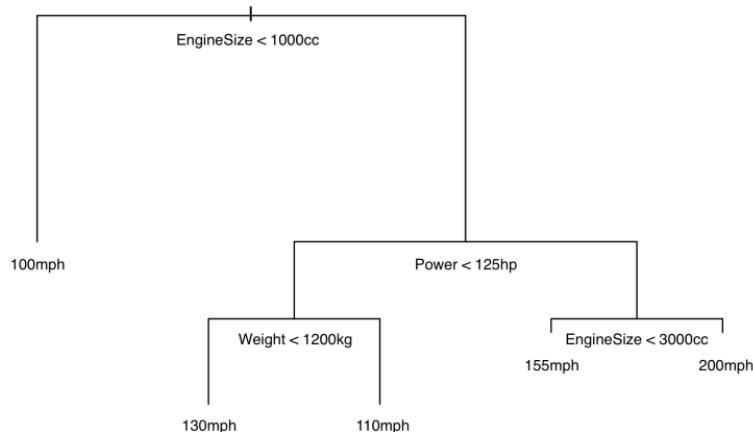


Figure. A simple regression tree to predict the top speed of a vehicle, given the engine size, power and weight.

³Formalised by Breiman et al., 1984

BAYESIAN ADDITIVE REGRESSION TREES

AN INTRODUCTION TO BART⁴

- ▶ BART utilises a **sum-of-trees** structure, $Y = \sum_{j=1}^m g(\mathbf{X}; T_j, M_j) + \epsilon$, where for tree j
 - ▶ T_j is the **tree structure**.
 - ▶ $M_j = \{\mu_{1j}, \mu_{2j}, \dots, \mu_{bj}\}$ is the set of b_j **terminal parameters**.
- ▶ Trees built via a **Bayesian backfitting** algorithm.
- ▶ Can form the basis of an **imputation** model.

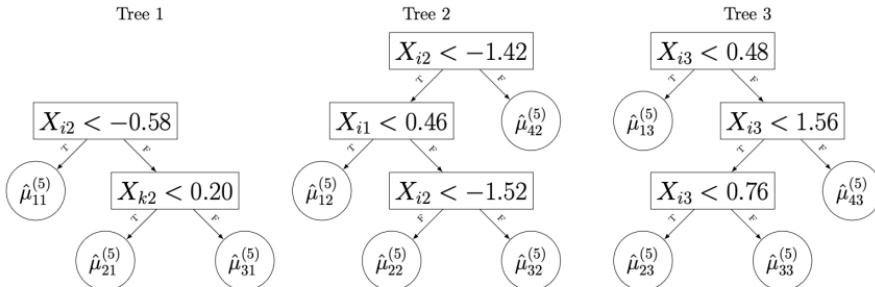


Figure. A sum-of-trees structure. Extracted from Tan and Roy, 2019.

⁴Chipman et al., 2010

THE BART PROCESS

THE PRIOR

- ▶ A strongly influential **regularising** prior guides the tree building process, tailoring the fit.

- ▶ **The Tree Prior** – $p(T_j)$.
- ▶ **The Terminal Parameter Prior** – $p(\mu_{ij} \mid T_j)$.
- ▶ **The Error Variance Prior** – $p(\sigma)$.

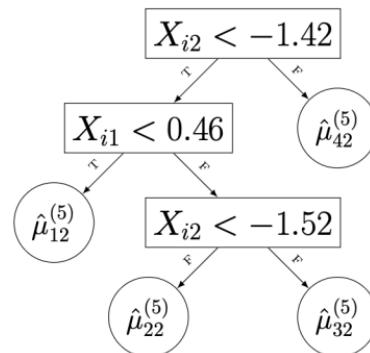


Figure. A single tree. Extracted from **Tan and Roy, 2019**.

THE BART PROCESS

TREE BUILDING

- ▶ The **backfitting** algorithm draws each tree (T_j, M_j) conditional on $[(T_{(j)}, M_{(j)}), \sigma]$.
- ▶ The conditional distribution of the j^{th} tree depends only on the other trees and the training data via,

$$R_j \equiv y - \sum_{k \neq j} g(\mathbf{x}; T_k, M_k).$$

- ▶ Tree design proposed iteratively by a **stochastic search** procedure that repeatedly **perturbs** the trees in one of four ways.

▶ **Grow**

▶ **Prune**

▶ **Change**

▶ **Swap**

MISSBART IMPUTATION METHOD

- ▶ Based on the popular **missForest** method of **Stekhoven and Bühlmann, 2012**.

missBART Algorithm

Require: \mathbf{X} an $n \times p$ matrix, stopping criterion γ

- 1: Initialise the missing values with a simple mean imputation
- 2: $\mathbf{k} \leftarrow$ vector of the indices of the columns in \mathbf{X} in order of increasing missingness
- 3: **while** not γ **do**
- 4: $\mathbf{X}_{\text{old}}^{\text{imp}} \leftarrow$ current imputed matrix
- 5: **for** each column, c in \mathbf{k} **do**
- 6: Fit a **BART** model: $\mathbf{y}_{\text{obs}}^{(c)} \sim \mathbf{x}_{\text{obs}}^{(c)}$
- 7: Predict $\mathbf{y}_{\text{mis}}^{(c)}$ using $\mathbf{x}_{\text{mis}}^{(c)}$
- 8: $\mathbf{X}_{\text{new}}^{\text{imp}} \leftarrow$ new imputed matrix,
 with updated $\mathbf{y}_{\text{mis}}^{(c)}$
- 9: **end for**
- 10: Update stopping criterion γ
- 11: **end while**
- 12: **return** the imputed matrix \mathbf{X}^{imp}

Table. Data based on the Theoph dataset [R Core Team, 2024].

Y	X_1	X_2	X_3	X_4
70.5	4.53	7.07	–	–
86.4	3.10	–	4.24	–
–	–	6.98	5.25	74
70.5	4.53	7.15	4.73	85
–	4.40	12.00	3.01	–
65.0	4.92	–	5.87	116
70.5	4.53	0.98	7.31	81
79.6	4.02	3.82	–	–
–	4.40	9.02	5.33	42
64.6	4.95	0.25	0.85	68

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86.4	3.10	6.98	4.24	88.4
65.2	4.92	6.98	5.25	74
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DISCUSSION AND RESEARCH

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- ▶ Yes – not multiple imputation! ⁵

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IMPUTATION MODELS

- ▶ **missForest/BART** produces a single imputed dataset.
 - ▶ Is this a disadvantage?
 - ▶ Yes – not multiple imputation! ⁶
-
- ▶ Bootstrap the data and produce multiple imputed datasets.
 - ▶ Compare missForest vs missBART vs both bootstrapped.
 - ▶ Utilise BART's intrinsic **Bayesian probability model** to generate multiple imputations from **posterior predictive** draws and combine via **Rubin's rules**.

⁶van Buuren, 2018

DISCUSSION AND RESEARCH

IMPUTATION METRICS

- ▶ Evaluating accuracy alone **fails** to detect **variability** issues.
- ▶ Calculate the **empirical bias**.
- ▶ Calculate the **coverage**.

- ▶ **Imputation is not prediction.**⁷

⁷ van Buuren, 2018

REFERENCES AND QUESTIONS

THANK YOU FOR LISTENING, ANY QUESTIONS? ⁸

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⁸<https://github.com/toddburrows>

APPENDIX

APPENDIX A: THE BART MCMC CYCLE

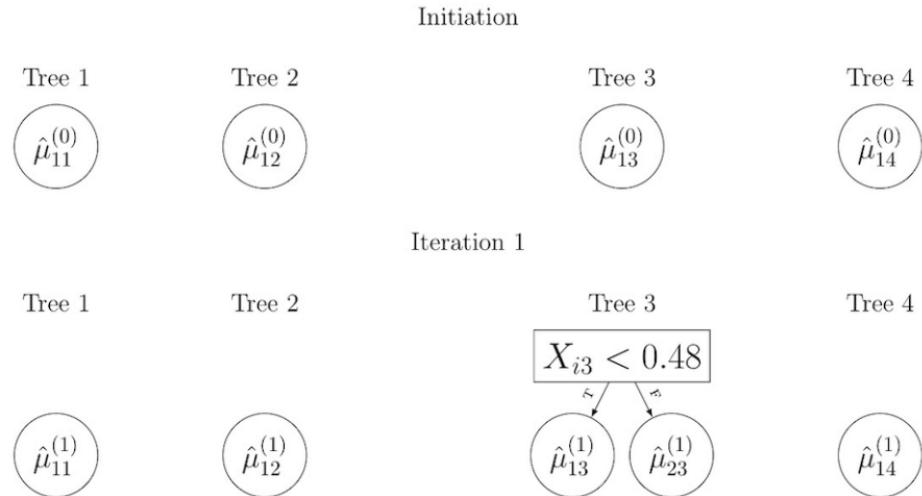
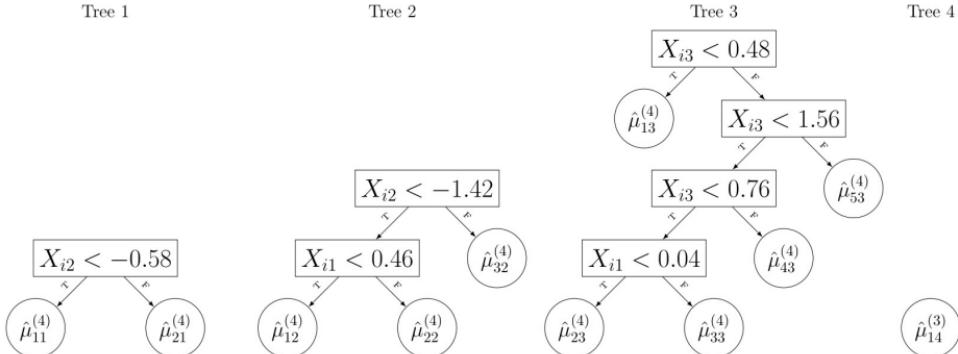


Figure. Initialisation and Iteration 1 of a BART model. Extracted from [Tan and Roy, 2019](#).

APPENDIX

APPENDIX A: THE BART MCMC CYCLE

Iteration 4



Iteration 5

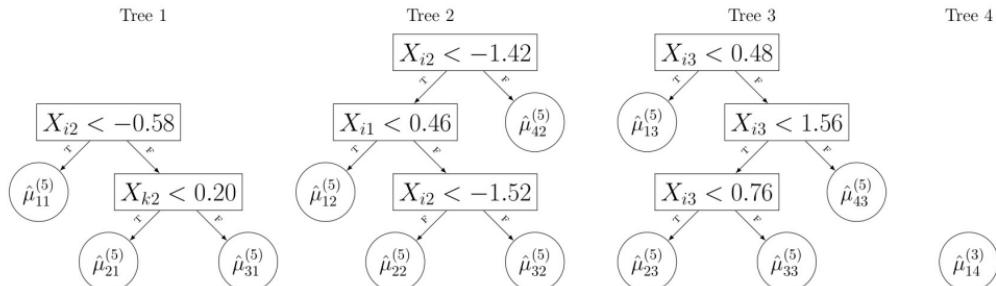


Figure. Iteration 4 and 5 of a BART model. Extracted from Tan and Roy, 2019.