$$\vec{x} = x_1, x_2, \ldots, x_N$$

$$x_i \sim power(\alpha, x_{\min})$$

$$p_x(x) = \frac{x^{-\alpha}}{\zeta(\alpha, x_{\min})} \qquad x_{\min} \le x \le x_{\max}\ (discrete)$$

$$\zeta(s,q) = \sum_{i=0}^{¥} \frac{1}{(i+q)^s}$$

$$P(x) = P(X \le x) = \frac{\zeta(\alpha, x)}{\zeta(\alpha, x_{\min})} = \frac{\sum_{i=0}^{\infty} \frac{1}{(i+x)^{\alpha}}}{\sum_{i=0}^{\infty} \frac{1}{(i+x_{\min})^{\alpha}}}$$

$$L(\alpha) = -\text{ n } \ln\varsigma(\alpha, x_{\min}) \text{ -}\alpha \sum_{i\in \text{xmins} \le x \le \text{xmax}}^{\text{xmax}} \ln(x_i)$$

$$L(\alpha_k, x\min s_{xm}) = -\text{ n } \ln(\text{z}(\alpha_k) - \sum_{i=1}^{\text{xmins}_{xm}} \text{i}^{-\alpha_k}) \text{ -}\alpha_k \sum_{i\in \text{xmins}_{xm} \le x \le \text{xmax}} \ln(x_i)$$

$$\vec{\alpha} = \begin{bmatrix} \alpha_0 \\ \alpha_1 \\ \vdots \\ \alpha_w \end{bmatrix}, \qquad \overrightarrow{zvec} = \begin{bmatrix} zvec_0 = \zeta(\alpha_0, 0) \\ zvec_1 = \zeta(\alpha_1, 0) \\ \vdots \\ zvec_w = \zeta(\alpha_w, 0) \end{bmatrix}$$

$$\overrightarrow{\text{xmins}} = \begin{bmatrix} \text{xmins}_0 \\ \text{xmins}_1 \\ \vdots \\ \text{xmins}_v \end{bmatrix} \text{ where v = length(unique(}\vec{x}\text{))}, \qquad Z = \begin{bmatrix} \vec{z}_0 = \{i \in \mathbb{Z} \mid x \ge \text{xmins}_0\} \\ \vec{z}_1 = \{i \in \mathbb{Z} \mid x \ge \text{xmins}_1\} \\ \vdots \\ \vec{z}_v = \{i \in \mathbb{Z} \mid x \ge \text{xmins}_v\} \end{bmatrix}, \qquad \text{XMINVEC} = \begin{bmatrix} 1 & 2 & \cdots & \text{xmins}_0 - 1 \\ 1 & 2 & \cdots & \text{xmins}_1 - 1 \\ \vdots & \vdots & \ddots & \vdots \\ 1 & 2 & \cdots & \text{xmins}_v - 1 \end{bmatrix}, \qquad \vec{\text{n}} = \begin{bmatrix} \text{n}_0 = \text{len}(\vec{\text{z}}_0) \\ \text{n}_1 = \text{len}(\vec{\text{z}}_1) \\ \vdots \\ \text{n}_v = \text{len}(\vec{\text{z}}_v) \end{bmatrix}$$

$$L(\vec{\alpha}, \overrightarrow{\text{xmins}}) = \begin{bmatrix}
\left[-\text{n}_0 \log(\text{zvec}(\alpha_0) - \sum_{i=1}^{\text{xmins}_0-1} \text{i}^{-\alpha_0}) \text{ -}\alpha_0 \sum_{p\in x\ge \text{xmins}_0}^{\text{xmax}} \log(x_p)\right] & \left[-\text{n}_0 \log(\text{zvec}(\alpha_1) - \sum_{i=1}^{\text{xmins}_0-1} \text{i}^{-\alpha_1}) \text{ -}\alpha_1 \sum_{p\in x\ge \text{xmins}_0}^{\text{xmax}} \log(x_p)\right] & \cdots & \left[-\text{n}_0 \log(\text{zvec}(\alpha_w) - \sum_{i=1}^{\text{xmins}_0-1} \text{i}^{-\alpha_w}) \text{ -}\alpha_w \sum_{p\in x\ge \text{xmins}_0}^{\text{xmax}} \log(x_p)\right] \\
\left[-\text{n}_1 \log(\text{zvec}(\alpha_0) - \sum_{i=1}^{\text{xmins}_1-1} \text{i}^{-\alpha_0}) \text{ -}\alpha_0 \sum_{p\in x\ge \text{xmins}_1}^{\text{xmax}} \log(x_p)\right] & \left[-\text{n}_1 \log(\text{zvec}(\alpha_1) - \sum_{i=1}^{\text{xmins}_1-1} \text{i}^{-\alpha_1}) \text{ -}\alpha_1 \sum_{p\in x\ge \text{xmins}_1}^{\text{xmax}} \log(x_p)\right] & \cdots & \left[-\text{n}_1 \log(\text{zvec}(\alpha_w) - \sum_{i=1}^{\text{xmins}_1-1} \text{i}^{-\alpha_w}) \text{ -}\alpha_w \sum_{p\in x\ge \text{xmins}_1}^{\text{xmax}} \log(x_p)\right] \\
\vdots & \vdots & \ddots & \\
\left[-\text{n}_v \log(\text{zvec}(\alpha_0) - \sum_{i=1}^{\text{xmins}_v-1} \text{i}^{-\alpha_0}) \text{ -}\alpha_0 \sum_{p\in x\ge \text{xmins}_v}^{\text{xmax}} \log(x_p)\right] & \left[-\text{n}_v \log(\text{zvec}(\alpha_1) - \sum_{i=1}^{\text{xmins}_v-1} \text{i}^{-\alpha_1}) \text{ -}\alpha_1 \sum_{p\in x\ge \text{xmins}_v}^{\text{xmax}} \log(x_p)\right] & \cdots & \left[-\text{n}_v \log(\text{zvec}(\alpha_w) - \sum_{i=1}^{\text{xmins}_v-1} \text{i}^{-\alpha_w}) \text{ -}\alpha_w \sum_{p\in x\ge \text{xmins}_v}^{\text{xmax}} \log(x_p)\right]
\end{bmatrix}$$

$$\vec{\widehat{\alpha}} = \begin{bmatrix} \widehat{\alpha_0 =} \max_{\alpha_j, j\in[0,w]} (L_{0,j}) \\ \widehat{\alpha_1 =} \max_{\alpha_j, j\in[0,w]} (L_{1,j}) \\ \vdots \\ \widehat{\alpha_v =} \max_{\alpha_j, j\in[0,w]} (L_{v,j}) \end{bmatrix}, \qquad \vec{I} = \begin{bmatrix} I_0 = \max_{j\in[0,w]} (L_{0,j}) \\ I_1 = \max_{j\in[0,w]} (L_{1,j}) \\ \vdots \\ I_v = \max_{j\in[0,w]} (L_{v,j}) \end{bmatrix}, \qquad FIT = \begin{bmatrix} \overrightarrow{fit}_0 = \max_{j\in[0,w]} (L_{0,j}) \\ I_1 = \max_{j\in[0,w]} (L_{1,j}) \\ \vdots \\ I_v = \max_{j\in[0,w]} (L_{v,j}) \end{bmatrix},$$

$$L = \begin{bmatrix} -1 & -1 & & -1 \\ -1 & -1 & & -1 \\ \\ -1 & -1 & & -1 \end{bmatrix} \begin{bmatrix} n_0 & n_0 & & n_0 \\ n_1 & n_1 & & n_1 \\ \\ n_v & n_v & & n_v \end{bmatrix} \begin{bmatrix}
\log(zvec_0 - \sum_{m=1}^{\text{xmins}_0-1} \text{m}^{-vec_0}) & \log(zvec_1 - \sum_{m=1}^{\text{xmins}_0-1} \text{m}^{-vec_1}) & & \log(zvec_w - \sum_{m=1}^{\text{xmins}_0-1} \text{m}^{-vec_w}) \\
\log(zvec_0 - \sum_{m=1}^{\text{xmins}_1-1} \text{m}^{-vec_0}) & \log(zvec_1 - \sum_{m=1}^{\text{xmins}_1-1} \text{m}^{-vec_1}) & & \log(zvec_w - \sum_{m=1}^{\text{xmins}_1-1} \text{m}^{-vec_w}) \\
\\
\log(zvec_0 - \sum_{m=1}^{\text{xmins}_v-1} \text{m}^{-vec_0}) & \log(zvec_1 - \sum_{m=1}^{\text{xmins}_v-1} \text{m}^{-vec_1}) & & \log(zvec_w - \sum_{m=1}^{\text{xmins}_v-1} \text{m}^{-vec_w})
\end{bmatrix} - \begin{bmatrix}
vec_0 \sum_{p\in x\ge \text{xmins}_0}^{\text{xmax}} \log(x_p) & vec_1 \sum_{p\in x\ge \text{xmins}_0}^{\text{xmax}} \log(x_p) & & vec_w \sum_{p\in x\ge \text{xmins}_0}^{\text{xmax}} \log(x_p) \\
vec_0 \sum_{p\in x\ge \text{xmins}_1}^{\text{xmax}} \log(x_p) & vec_1 \sum_{p\in x\ge \text{xmins}_1}^{\text{xmax}} \log(x_p) & & vec_w \sum_{p\in x\ge \text{xmins}_1}^{\text{xmax}} \log(x_p) \\
\\
vec_0 \sum_{p\in x\ge \text{xmins}_v}^{\text{xmax}} \log(x_p) & vec_1 \sum_{p\in x\ge \text{xmins}_v}^{\text{xmax}} \log(x_p) & & vec_w \sum_{p\in x\ge \text{xmins}_v}^{\text{xmax}} \log(x_p)
\end{bmatrix}$$

$$L = -1 * n * \log\left( \begin{bmatrix} zvec_0 & zvec_1 & & zvec_w \\ zvec_0 & zvec_1 & & zvec_w \\ \\ zvec_0 & zvec_1 & & zvec_w \end{bmatrix} - \begin{bmatrix}
\sum_{m=1}^{\text{xmins}_0-1} \text{m}^{-vec_0} & \sum_{m=1}^{\text{xmins}_0-1} \text{m}^{-vec_1} & & \sum_{m=1}^{\text{xmins}_0-1} \text{m}^{-vec_w} \\
\sum_{m=1}^{\text{xmins}_1-1} \text{m}^{-vec_0} & \sum_{m=1}^{\text{xmins}_1-1} \text{m}^{-vec_1} & & \sum_{m=1}^{\text{xmins}_1-1} \text{m}^{-vec_w} \\
\\
\sum_{m=1}^{\text{xmins}_v-1} \text{m}^{-vec_0} & \sum_{m=1}^{\text{xmins}_v-1} \text{m}^{-vec_1} & & \sum_{m=1}^{\text{xmins}_v-1} \text{m}^{-vec_w}
\end{bmatrix} \right) - \left( \begin{bmatrix} vec_0 & vec_1 & & vec_w \\ vec_0 & vec_1 & & vec_w \\ \\ vec_0 & vec_1 & & vec_w \end{bmatrix} \begin{bmatrix}
\sum_{p\in x\ge \text{xmins}_0}^{\text{xmax}} \log(x_p) & \sum_{p\in x\ge \text{xmins}_0}^{\text{xmax}} \log(x_p) & & \sum_{p\in x\ge \text{xmins}_0}^{\text{xmax}} \log(x_p) \\
\sum_{p\in x\ge \text{xmins}_1}^{\text{xmax}} \log(x_p) & \sum_{p\in x\ge \text{xmins}_1}^{\text{xmax}} \log(x_p) & & \sum_{p\in x\ge \text{xmins}_1}^{\text{xmax}} \log(x_p) \\
\\
\sum_{p\in x\ge \text{xmins}_v}^{\text{xmax}} \log(x_p) & \sum_{p\in x\ge \text{xmins}_v}^{\text{xmax}} \log(x_p) & & \sum_{p\in x\ge \text{xmins}_v}^{\text{xmax}} \log(x_p)
\end{bmatrix} \right)$$

$$L = -1 * n * \log(ZVEC - T) \qquad - \qquad H$$

(2) fit is an array of arrays where each inner array is a cdf of power law using the MLE of $\alpha$ found for a particular value of $x_{\min}$.

$$\text{Cummulative distribution function of power law} := P(x_r|\alpha, x_{\min}) = \begin{cases} 0 & x_r r < x_{\min} \\ \dfrac{\widehat{\varsigma(\alpha, x_r)}}{\varsigma(\alpha, x_{\min})} & x_{\min} \le x_r < x_{\max} \\ 1 & x_r \ge x_{\max} \end{cases}$$

$$P(x_r|\widehat{\alpha_j}, \text{xmins}_j) = \frac{\varsigma(\widehat{\alpha_j, x_r})}{\varsigma(\widehat{\alpha_j}, x\min s_j)} = \frac{zvec_{I_j} - \sum_{i=1}^{x_r-1} \frac{1}{x_r^{\widehat{\alpha_j}}}}{zvec_{I_j} - \sum_{i=1}^{\text{xmins}_j-1} \frac{1}{\text{i}^{\alpha_k}}}$$

Proof:

$$\frac{zvec_{I_j} - \sum_{i=1}^{x_r-1} \frac{1}{x_r^{\widehat{\alpha_j}}}}{zvec_{I_j} - \sum_{i=1}^{\text{xmins}_j-1} \frac{1}{\text{i}^{\hat{\alpha}_{I_j}}}} = \frac{\sum_{i=1}^{\infty} \frac{1}{i^{\widehat{\alpha_j}}} - \sum_{i=1}^{x_r-1} \frac{1}{x_r^{\widehat{\alpha_j}}}}{\sum_{i=1}^{\infty} \frac{1}{i^{\hat{\alpha}_{I_j}}} - \sum_{i=1}^{x\min s_j-1} \frac{1}{i^{\hat{\alpha}_{I_j}}}} = \frac{\frac{1}{1^{\widehat{\alpha_j}}} + \frac{1}{2^{\widehat{\alpha_j}}} + \cdots + \frac{1}{\infty^{\widehat{\alpha_j}}} - \left(\frac{1}{1^{\widehat{\alpha_j}}} + \frac{1}{2^{\widehat{\alpha_j}}} + \cdots + \frac{1}{(x_r-1)^{\widehat{\alpha_j}}}\right)}{\frac{1}{1^{\hat{\alpha}_{I_j}}} + \frac{1}{2^{\hat{\alpha}_{I_j}}} + \cdots + \frac{1}{\infty^{\hat{\alpha}_{I_j}}} - \left(\frac{1}{1^{\hat{\alpha}_{I_j}}} + \frac{1}{2^{\hat{\alpha}_{I_j}}} + \cdots + \frac{1}{(\text{xmins}_j-1)^{\hat{\alpha}_{I_j}}}\right)} = \frac{\sum_{i=0}^{\infty} \frac{1}{(i+x_r)^{\widehat{\alpha_j}}}}{\sum_{i=0}^{\infty} \frac{1}{(i+\text{xmins}_j)^{\widehat{\alpha_j}}}} = \frac{\varsigma(\widehat{\alpha_j, x_r})}{\varsigma(\widehat{\alpha_j, \text{xmins}_j})} \square$$

$$\text{FIT}_{j,i>0} = \underset{\text{xmin}_{\text{xm}} \le j \le \text{xmax}}{\text{cumsum}} \left\{ \frac{\frac{1}{\text{x}_j^{\alpha_I}}}{zvec[I_j] - \sum_{i=1}^{\text{xmins}_{xm}-1} \frac{1}{\text{i}^{-\alpha_I}}} \right\} = \underset{\text{xmin}_{\text{xm}} \le j \le \text{xmax}}{\text{cumsum}} \left\{ \frac{\frac{1}{\text{x}_j^{\alpha_I}}}{\sum_{i=1}^{\infty} \frac{1}{\text{i}^{\hat{\alpha}_{I_j}}} - \sum_{i=1}^{\text{xmins}_{xm}-1} \frac{1}{\text{i}^{-\alpha_I}}} \right\} = \left\{ \frac{\sum_{j=\text{xmin}_{\text{sxm}}}^{\text{xmin}_{\text{sxm}}} \frac{1}{\text{x}_j^{\alpha_I}}}{\sum_{i=1}^{\infty} \frac{1}{\text{i}^{\hat{\alpha}_{I_j}}} - \sum_{i=1}^{\text{xmin}_{xm}-1} \frac{1}{\text{i}^{-\alpha_I}}}, \frac{\sum_{j=\text{xmin}_{\text{sxm}}}^{\text{xmin}_{\text{sxm}}+1} \frac{1}{\text{x}_j^{\alpha_I}}}{\sum_{i=1}^{\infty} \frac{1}{\text{i}^{\hat{\alpha}_{I_j}}} - \sum_{i=1}^{\text{xmins}_{xm}-1} \frac{1}{\text{i}^{-\alpha_I}}}, \ldots, \frac{\sum_{j=\text{xmin}_{\text{sxm}}}^{\text{xmax}} \frac{1}{\text{x}_j^{\alpha_I}}}{\sum_{i=1}^{\infty} \frac{1}{\text{i}^{\hat{\alpha}_{I_j}}} - \sum_{i=1}^{\text{xmins}_{xm}-1} \frac{1}{\text{i}^{-\alpha_I}}} \right\}$$

$$= \left\{ \frac{\frac{1}{(0+\text{xmin}_{xm})^{\alpha_I}}}{\sum_{i=0}^{\infty} \frac{1}{(i+\text{xmins}_j)^{\widehat{\alpha_j}}}}, \frac{\frac{1}{(0+\text{xmin}_{xm})^{\alpha_I}} + \frac{1}{(1+\text{xmin}_{xm})^{\alpha_I}}}{\sum_{i=0}^{\infty} \frac{1}{(i+\text{xmins}_j)^{\widehat{\alpha_j}}}}, \ldots, \frac{\frac{1}{(0+\text{xmin}_{xm})^{\alpha_I}} + \frac{1}{(1+\text{xmin}_{xm})^{\alpha_I}} + \cdots + \frac{1}{(\text{xmax-xmin}_{xm}+\text{xmin}_{xm})^{\alpha_I}}}{\sum_{i=0}^{\infty} \frac{1}{(i+\text{xmins}_j)^{\widehat{\alpha_j}}}} \right\}$$

$$= \left\{ p_x(x=\text{xmin}_{xm}), p_x(x=\text{xmin}_{xm}) + p_x(x=\text{xmin}_{xm}+1), \cdots, p_x(x=\text{xmin}_{xm}) + p_x(x=\text{xmin}_{xm}+1) + \cdots + p_x(x=\text{xmax}-1) + p_x(x=\text{xmax}) \right\}$$

$$= P(x)$$

$$
FIT=\begin{bmatrix}
[p_x(x=\mathrm{xmins}_0\mid\hat{\alpha}_0)] & \begin{bmatrix}p_x(x=\mathrm{xmins}_0\mid\hat{\alpha}_0)\\ +p_x(x=\mathrm{xmins}_0+1\mid\hat{\alpha}_0)\end{bmatrix} & \cdots & \cdots & \cdots & \cdots & \begin{bmatrix}p_x(x=\mathrm{xmins}_0\mid\hat{\alpha}_0)+p_x(x=\mathrm{xmins}_0+1\mid\hat{\alpha}_0)\\ +p_x(x=\mathrm{xmins}_0+2\mid\hat{\alpha}_0)+\cdots p_x(x=\mathrm{xmax}\mid\hat{\alpha}_0)\end{bmatrix}\\
[p_x(x=\mathrm{xmins}_0\mid\hat{\alpha}_1)] & \begin{bmatrix}p_x(x=\mathrm{xmins}_0\mid\hat{\alpha}_1)\\ +p_x(x=\mathrm{xmins}_0+1\mid\hat{\alpha}_1)\end{bmatrix} & \cdots & \cdots & \cdots & \cdots & \begin{bmatrix}p_x(x=\mathrm{xmins}_0\mid\hat{\alpha}_1)+p_x(x=\mathrm{xmins}_0+1\mid\hat{\alpha}_1)\\ +p_x(x=\mathrm{xmins}_0+2\mid\hat{\alpha}_1)+\cdots p_x(x=\mathrm{xmax}\mid\hat{\alpha}_1)\end{bmatrix}\\
\vdots & \vdots & \cdots & \cdots & \cdots & \cdots & \vdots\\
[p_x(x=\mathrm{xmins}_0\mid\hat{\alpha}_v)] & \begin{bmatrix}p_x(x=\mathrm{xmins}_0\mid\hat{\alpha}_v)\\ +p_x(x=\mathrm{xmins}_0+1\mid\hat{\alpha}_v)\end{bmatrix} & \cdots & \cdots & \cdots & \cdots & \begin{bmatrix}p_x(x=\mathrm{xmins}_0\mid\hat{\alpha}_v)+p_x(x=\mathrm{xmins}_0+1\mid\hat{\alpha}_v)\\ +p_x(x=\mathrm{xmins}_0+2\mid\hat{\alpha}_v)+\cdots p_x(x=\mathrm{xmax}\mid\hat{\alpha}_v)\end{bmatrix}\\
[p_x(x=\mathrm{xmins}_1\mid\hat{\alpha}_0)] & \begin{bmatrix}p_x(x=\mathrm{xmins}_1\mid\hat{\alpha}_0)\\ +p_x(x=\mathrm{xmins}_1+1\mid\hat{\alpha}_0)\end{bmatrix} & \cdots & \cdots & \cdots & \begin{bmatrix}p_x(x=\mathrm{xmins}_1\mid\hat{\alpha}_0)+p_x(x=\mathrm{xmins}_1+1\mid\hat{\alpha}_0)\\ +p_x(x=\mathrm{xmins}_1+2\mid\hat{\alpha}_0)+\cdots p_x(x=\mathrm{xmax}\mid\hat{\alpha}_0)\end{bmatrix} & \cdots\\
[p_x(x=\mathrm{xmins}_1\mid\hat{\alpha}_1)] & \begin{bmatrix}p_x(x=\mathrm{xmins}_1\mid\hat{\alpha}_1)\\ +p_x(x=\mathrm{xmins}_1+1\mid\hat{\alpha}_1)\end{bmatrix} & \cdots & \cdots & \cdots & \begin{bmatrix}p_x(x=\mathrm{xmins}_1\mid\hat{\alpha}_1)+p_x(x=\mathrm{xmins}_1+1\mid\hat{\alpha}_1)\\ +p_x(x=\mathrm{xmins}_1+2\mid\hat{\alpha}_1)+\cdots p_x(x=\mathrm{xmax}\mid\hat{\alpha}_1)\end{bmatrix} & \cdots\\
\vdots & \vdots & \cdots & \cdots & \cdots & \vdots & \vdots\\
[p_x(x=\mathrm{xmins}_1\mid\hat{\alpha}_v)] & \begin{bmatrix}p_x(x=\mathrm{xmins}_1\mid\hat{\alpha}_v)\\ +p_x(x=\mathrm{xmins}_1+1\mid\hat{\alpha}_v)\end{bmatrix} & \cdots & \cdots & \cdots & \begin{bmatrix}p_x(x=\mathrm{xmins}_1\mid\hat{\alpha}_v)+p_x(x=\mathrm{xmins}_1+1\mid\hat{\alpha}_v)\\ +p_x(x=\mathrm{xmins}_1+2\mid\hat{\alpha}_v)+\cdots p_x(x=\mathrm{xmax}\mid\hat{\alpha}_v)\end{bmatrix} & \cdots\\
\vdots & \vdots & \cdots & \cdots & \cdots & \cdots & \cdots\\
[p_x(x=\mathrm{xmins}_v\mid\hat{\alpha}_0)] & \begin{bmatrix}p_x(x=\mathrm{xmins}_v\mid\hat{\alpha}_0)\\ +p_x(x=\mathrm{xmins}_v+1\mid\hat{\alpha}_0)\end{bmatrix} & \cdots & \cdots & \begin{bmatrix}p_x(x=\mathrm{xmins}_v\mid\hat{\alpha}_0)+p_x(x=\mathrm{xmins}_v+1\mid\hat{\alpha}_0)\\ +p_x(x=\mathrm{xmins}_v+2\mid\hat{\alpha}_0)+\cdots p_x(x=\mathrm{xmax}\mid\hat{\alpha}_0)\end{bmatrix} & \cdots & \cdots\\
[p_x(x=\mathrm{xmins}_v\mid\hat{\alpha}_1)] & \begin{bmatrix}p_x(x=\mathrm{xmins}_v\mid\hat{\alpha}_1)\\ +p_x(x=\mathrm{xmins}_v+1\mid\hat{\alpha}_1)\end{bmatrix} & \cdots & \cdots & \begin{bmatrix}p_x(x=\mathrm{xmins}_v\mid\hat{\alpha}_1)+p_x(x=\mathrm{xmins}_v+1\mid\hat{\alpha}_1)\\ +p_x(x=\mathrm{xmins}_v+2\mid\hat{\alpha}_1)+\cdots p_x(x=\mathrm{xmax}\mid\hat{\alpha}_1)\end{bmatrix} & \cdots & \cdots\\
\vdots & \vdots & \cdots & \cdots & \vdots & \cdots & \cdots\\
[p_x(x=\mathrm{xmins}_v\mid\hat{\alpha}_v)] & \begin{bmatrix}p_x(x=\mathrm{xmins}_v\mid\hat{\alpha}_v)\\ +p_x(x=\mathrm{xmins}_v+1\mid\hat{\alpha}_v)\end{bmatrix} & \cdots & \cdots & \begin{bmatrix}p_x(x=\mathrm{xmins}_v\mid\hat{\alpha}_v)+p_x(x=\mathrm{xmins}_v+1\mid\hat{\alpha}_v)\\ +p_x(x=\mathrm{xmins}_v+2\mid\hat{\alpha}_v)+\cdots p_x(x=\mathrm{xmax}\mid\hat{\alpha}_v)\end{bmatrix} & \cdots & \cdots
\end{bmatrix}
$$

$$
FIT=\left[\begin{array}{ccccc}
\left\{\frac{\frac{1}{(0+\mathrm{xmins}_0)^{\hat{\alpha}_0}}}{\sum_{i=0}^{\infty}\frac{1}{(i+\mathrm{xmins}_0)^{\hat{\alpha}_0}}}\right\} & \left\{\frac{\frac{1}{(0+\mathrm{xmins}_0)^{\hat{\alpha}_0}}+\frac{1}{(1+\mathrm{xmins}_0)^{\hat{\alpha}_0}}}{\sum_{i=0}^{\infty}\frac{1}{(i+\mathrm{xmins}_0)^{\hat{\alpha}_0}}}\right\} & \cdots\ \cdots\ \cdots\ \cdots\ \cdots & \cdots & \left\{\frac{\frac{1}{(0+\mathrm{xmins}_0)^{\hat{\alpha}_0}}+\frac{1}{(1+\mathrm{xmins}_0)^{\hat{\alpha}_0}}+\cdots+\frac{1}{(\mathrm{xmax}-\mathrm{xmins}_0+\mathrm{xminvs}_0)^{\hat{\alpha}_0}}}{\sum_{i=0}^{\infty}\frac{1}{(i+\mathrm{xmins}_0)^{\hat{\alpha}_0}}}\right\} \\
\left\{\frac{\frac{1}{(0+\mathrm{xmins}_0)^{\hat{\alpha}_1}}}{\sum_{i=0}^{\infty}\frac{1}{(i+\mathrm{xmins}_0)^{\hat{\alpha}_1}}}\right\} & \left\{\frac{\frac{1}{(0+\mathrm{xmins}_0)^{\hat{\alpha}_1}}+\frac{1}{(1+\mathrm{xmins}_0)^{\hat{\alpha}_1}}}{\sum_{i=0}^{\infty}\frac{1}{(i+\mathrm{xmins}_0)^{\hat{\alpha}_1}}}\right\} & \cdots\ \cdots\ \cdots\ \cdots\ \ddots & \cdots & \left\{\frac{\frac{1}{(0+\mathrm{xmins}_0)^{\hat{\alpha}_1}}+\frac{1}{(1+\mathrm{xmins}_0)^{\hat{\alpha}_1}}+\cdots+\frac{1}{(\mathrm{xmax}-\mathrm{xmins}_0+\mathrm{xminvs}_0)^{\hat{\alpha}_1}}}{\sum_{i=0}^{\infty}\frac{1}{(i+\mathrm{xmins}_0)^{\hat{\alpha}_1}}}\right\} \\
\vdots & \vdots & \cdots\ \cdots\ \cdots\ \cdots\ \cdots & \cdots & \\
\left\{\frac{\frac{1}{(0+\mathrm{xmins}_0)^{\hat{\alpha}_v}}}{\sum_{i=0}^{\infty}\frac{1}{(i+\mathrm{xmins}_0)^{\hat{\alpha}_v}}}\right\} & \left\{\frac{\frac{1}{(0+\mathrm{xmins}_0)^{\hat{\alpha}_v}}+\frac{1}{(1+\mathrm{xmins}_0)^{\hat{\alpha}_v}}}{\sum_{i=0}^{\infty}\frac{1}{(i+\mathrm{xmins}_0)^{\hat{\alpha}_v}}}\right\} & \cdots\ \cdots\ \cdots\ \cdots\ \cdots & \cdots & \left\{\frac{\frac{1}{(0+\mathrm{xmins}_0)^{\hat{\alpha}_v}}+\frac{1}{(1+\mathrm{xmins}_0)^{\hat{\alpha}_v}}+\cdots+\frac{1}{(\mathrm{xmax}-\mathrm{xmins}_0+\mathrm{xminvs}_0)^{\hat{\alpha}_v}}}{\sum_{i=0}^{\infty}\frac{1}{(i+\mathrm{xmins}_0)^{\hat{\alpha}_v}}}\right\} \\
\left\{\frac{\frac{1}{(0+\mathrm{xmins}_1)^{\hat{\alpha}_0}}}{\sum_{i=0}^{\infty}\frac{1}{(i+\mathrm{xmins}_1)^{\hat{\alpha}_0}}}\right\} & \left\{\frac{\frac{1}{(0+\mathrm{xmins}_1)^{\hat{\alpha}_0}}+\frac{1}{(1+\mathrm{xmins}_1)^{\hat{\alpha}_0}}}{\sum_{i=0}^{\infty}\frac{1}{(i+\mathrm{xmins}_1)^{\hat{\alpha}_0}}}\right\} & \cdots\ \cdots\ \cdots\ \cdots\ \cdots & \left\{\frac{\frac{1}{(0+\mathrm{xmins}_1)^{\hat{\alpha}_1}}+\frac{1}{(1+\mathrm{xmins}_1)^{\hat{\alpha}_1}}+\cdots+\frac{1}{(\mathrm{xmax}-\mathrm{xmins}_1+\mathrm{xminvs}_1)^{\hat{\alpha}_1}}}{\sum_{i=0}^{\infty}\frac{1}{(i+\mathrm{xmins}_1)^{\hat{\alpha}_1}}}\right\} & \\
\left\{\frac{\frac{1}{(0+\mathrm{xmins}_1)^{\hat{\alpha}_1}}}{\sum_{i=0}^{\infty}\frac{1}{(i+\mathrm{xmins}_1)^{\hat{\alpha}_1}}}\right\} & \left\{\frac{\frac{1}{(0+\mathrm{xmins}_1)^{\hat{\alpha}_1}}+\frac{1}{(1+\mathrm{xmins}_1)^{\hat{\alpha}_1}}}{\sum_{i=0}^{\infty}\frac{1}{(i+\mathrm{xmins}_1)^{\hat{\alpha}_1}}}\right\} & \cdots\ \cdots\ \cdots\ \ddots\ \cdots & \left\{\frac{\frac{1}{(0+\mathrm{xmins}_1)^{\hat{\alpha}_1}}+\frac{1}{(1+\mathrm{xmins}_1)^{\hat{\alpha}_1}}+\cdots+\frac{1}{(\mathrm{xmax}-\mathrm{xmins}_1+\mathrm{xminvs}_1)^{\hat{\alpha}_1}}}{\sum_{i=0}^{\infty}\frac{1}{(i+\mathrm{xmins}_1)^{\hat{\alpha}_1}}}\right\} & \\
 & & \cdots\ \cdots\ \cdots\ \cdots\ \cdots & & \\
\left\{\frac{\frac{1}{(0+\mathrm{xmins}_1)^{\hat{\alpha}_v}}}{\sum_{i=0}^{\infty}\frac{1}{(i+\mathrm{xmins}_1)^{\hat{\alpha}_v}}}\right\} & \left\{\frac{\frac{1}{(0+\mathrm{xmins}_1)^{\hat{\alpha}_v}}+\frac{1}{(1+\mathrm{xmins}_1)^{\hat{\alpha}_v}}}{\sum_{i=0}^{\infty}\frac{1}{(i+\mathrm{xmins}_1)^{\hat{\alpha}_v}}}\right\} & \cdots\ \cdots\ \cdots\ \cdots\ \cdots & \left\{\frac{\frac{1}{(0+\mathrm{xmins}_1)^{\hat{\alpha}_v}}+\frac{1}{(1+\mathrm{xmins}_1)^{\hat{\alpha}_v}}+\cdots+\frac{1}{(\mathrm{xmax}-\mathrm{xmins}_1+\mathrm{xminvs}_1)^{\hat{\alpha}_v}}}{\sum_{i=0}^{\infty}\frac{1}{(i+\mathrm{xmins}_1)^{\hat{\alpha}_v}}}\right\} & \\
\vdots & \vdots & \cdots\ \cdots\ \cdots\ \cdots & & \\
\left\{\frac{\frac{1}{(0+\mathrm{xmins}_{v-1})^{\hat{\alpha}_0}}}{\sum_{i=0}^{\infty}\frac{1}{(i+\mathrm{xmins}_{v-1})^{\hat{\alpha}_0}}}\right\} & \left\{\frac{\frac{1}{(0+\mathrm{xmins}_{v-1})^{\hat{\alpha}_0}}+\frac{1}{(x\max-\mathrm{xmins}_{v-1}+\mathrm{xminvs}_{v-1})^{\hat{\alpha}_0}}}{\sum_{i=0}^{\infty}\frac{1}{(i+\mathrm{xmins}_{v-1})^{\hat{\alpha}_0}}}\right\} & & & \\
\left\{\frac{\frac{1}{(0+\mathrm{xmins}_{v-1})^{\hat{\alpha}_1}}}{\sum_{i=0}^{\infty}\frac{1}{(i+\mathrm{xmins}_{v-1})^{\hat{\alpha}_1}}}\right\} & \left\{\frac{\frac{1}{(0+\mathrm{xmins}_{v-1})^{\hat{\alpha}_1}}+\frac{1}{(x\max-\mathrm{xmins}_{v-1}+\mathrm{xminvs}_{v-1})^{\hat{\alpha}_1}}}{\sum_{i=0}^{\infty}\frac{1}{(i+\mathrm{xmins}_{v-1})^{\hat{\alpha}_1}}}\right\} & & & \\
\vdots & \vdots & & & \\
\left\{\frac{\frac{1}{(0+\mathrm{xmins}_{v-1})^{\hat{\alpha}_v}}}{\sum_{i=0}^{\infty}\frac{1}{(i+\mathrm{xmins}_{v-1})^{\hat{\alpha}_v}}}\right\} & \left\{\frac{\frac{1}{(0+\mathrm{xmins}_{v-1})^{\hat{\alpha}_v}}+\frac{1}{(x\max-\mathrm{xmins}_{v-1}+\mathrm{xminvs}_{v-1})^{\hat{\alpha}_v}}}{\sum_{i=0}^{\infty}\frac{1}{(i+\mathrm{xmins}_{v-1})^{\hat{\alpha}_v}}}\right\} & & & \\
\left\{\frac{\frac{1}{(\mathrm{xmax}-\mathrm{xmins}_v+\mathrm{xmins}_v)^{\hat{\alpha}_0}}}{\sum_{i=0}^{\infty}\frac{1}{(i+\mathrm{xmins}_v)^{\hat{\alpha}_0}}}\right\} & & \cdots\ \cdots\ \cdots\ \cdots & & \\
\left\{\frac{\frac{1}{(\mathrm{xmax}-\mathrm{xmins}_v+\mathrm{xmins}_v)^{\hat{\alpha}_1}}}{\sum_{i=0}^{\infty}\frac{1}{(i+\mathrm{xmins}_v)^{\hat{\alpha}_1}}}\right\} & & \cdots\ \cdots\ \ \ \cdots & & \\
\vdots & \vdots & \cdots\ \cdots\ \ddots\ \cdots & & \\
\left\{\frac{\frac{1}{(\mathrm{xmax}-\mathrm{xmins}_v+\mathrm{xmins}_v)^{\hat{\alpha}_v}}}{\sum_{i=0}^{\infty}\frac{1}{(i+\mathrm{xmins}_v)^{\hat{\alpha}_v}}}\right\} & & \cdots\ \cdots\ \cdots\ \cdots & &
\end{array}\right]
$$