$$\vec{x} = x_1, x_2, ..., x_N$$

$$x_i \sim power(\alpha, x_{\min})$$

$$p_x(x) = \frac{x^{-\alpha}}{\zeta(\alpha, x_{\min})} \qquad x_{\min} \leq x \leq x_{\max} \ (discrete)$$

$$\zeta(s, q) = \sum_{i=0}^{\infty} \frac{1}{(i+q)^s}$$

$$P(x) = P(X \leq x) = \frac{\zeta(\alpha, x)}{\zeta(\alpha, x_{\min})} = \frac{\sum_{i=0}^{\infty} \frac{1}{(i+x)^{\alpha}}}{\sum_{i=0}^{\infty} \frac{1}{(i+x_{\min})^{\alpha}}}$$

$$L(\alpha) = -\ \mathrm{n}\ \ln\varsigma(\alpha, x_{\min})\ \text{-}\alpha \sum_{i \in \mathrm{xmins} \leq x \leq \mathrm{xmax}}^{\mathrm{xmax}} \ln(x_i)$$

$$L(\alpha_k, x\min s_{xm}) = -\ \mathrm{n}\ \ln(\mathrm{z}(\alpha_k) - \sum_{i=1}^{\mathrm{xmins}_{xm}} \mathrm{i}^{-\alpha_k})\ \text{-}\alpha_k \sum_{i \in \mathrm{xmins}_{xm} \leq x \leq \mathrm{xmax}} \ln(x_i)$$

$$\vec{\alpha} = \begin{bmatrix} \alpha_0 \\ \alpha_1 \\ \vdots \\ \alpha_w \end{bmatrix}, \qquad \overrightarrow{zvec} = \begin{bmatrix} zvec_0 = \zeta(\alpha_0, 0) \\ zvec_1 = \zeta(\alpha_1, 0) \\ \vdots \\ zvec_w = \zeta(\alpha_w, 0) \end{bmatrix}$$

$$\overrightarrow{\mathrm{xmins}} = \begin{bmatrix} \mathrm{xmins}_0 \\ \mathrm{xmins}_1 \\ \vdots \\ \mathrm{xmins}_v \end{bmatrix} \text{ where v = length(unique(}\vec{x}\text{)),} \qquad Z = \begin{bmatrix} \vec{z}_0 = \{i \in \mathbb{Z} \mid x \geq x\min s_0\} \\ \vec{z}_1 = \{i \in \mathbb{Z} \mid x \geq x\min s_1\} \\ \vdots \\ \vec{z}_v = \{i \in \mathbb{Z} \mid x \geq x\min s_v\} \end{bmatrix}, \qquad \mathrm{XMINVEC} = \begin{bmatrix} 1 & 2 & \cdots & \mathrm{xmins}_0 \\ 1 & 2 & \cdots & \mathrm{xmins}_1 \\ \vdots & \vdots & \ddots & \vdots \\ 1 & 2 & \cdots & \mathrm{xmins}_v \end{bmatrix}, \quad \vec{\mathrm{n}} = \begin{bmatrix} \mathrm{n}_0 = \mathrm{len}(\vec{z}_0) \\ \mathrm{n}_1 = \mathrm{len}(\vec{z}_1) \\ \vdots \\ \mathrm{n}_v = \mathrm{len}(\vec{z}_v) \end{bmatrix}$$

$$L(\vec{\alpha}, \overrightarrow{\mathrm{xmins}}) = \begin{bmatrix}
\left[-\mathrm{n}_0\ \log(\mathrm{zvec}(\alpha_0) - \sum_{i=1}^{\mathrm{xmins}_0} \mathrm{i}^{-\alpha_0})\ \text{-}\alpha_0 \sum_{p \in x \geq \mathrm{xmins}_0}^{\mathrm{xmax}} \log(x_p)\right] & \left[-\mathrm{n}_0\ \log(\mathrm{zvec}(\alpha_1) - \sum_{i=1}^{\mathrm{xmins}_0} \mathrm{i}^{-\alpha_1})\ \text{-}\alpha_1 \sum_{p \in x \geq \mathrm{xmins}_0}^{\mathrm{xmax}} \log(x_p)\right] & \cdots & \left[-\mathrm{n}_0\ \log(\mathrm{zvec}(\alpha_w) - \sum_{i=1}^{\mathrm{xmins}_0} \mathrm{i}^{-\alpha_w})\ \text{-}\alpha_w \sum_{p \in x \geq \mathrm{xmins}_0}^{\mathrm{xmax}} \log(x_p)\right] \\
\left[-\mathrm{n}_1\ \log(\mathrm{zvec}(\alpha_0) - \sum_{i=1}^{\mathrm{xmins}_1} \mathrm{i}^{-\alpha_0})\ \text{-}\alpha_0 \sum_{p \in x \geq \mathrm{xmins}_1}^{\mathrm{xmax}} \log(x_p)\right] & \left[-\mathrm{n}_1\ \log(\mathrm{zvec}(\alpha_1) - \sum_{i=1}^{\mathrm{xmins}_1} \mathrm{i}^{-\alpha_1})\ \text{-}\alpha_1 \sum_{p \in x \geq \mathrm{xmins}_1}^{\mathrm{xmax}} \log(x_p)\right] & \cdots & \left[-\mathrm{n}_1\ \log(\mathrm{zvec}(\alpha_w) - \sum_{i=1}^{\mathrm{xmins}_1} \mathrm{i}^{-\alpha_w})\ \text{-}\alpha_w \sum_{p \in x \geq \mathrm{xmins}_1}^{\mathrm{xmax}} \log(x_p)\right] \\
\vdots & \vdots & \ddots & \\
\left[-\mathrm{n}_v\ \log(\mathrm{zvec}(\alpha_0) - \sum_{i=1}^{\mathrm{xmins}_v} \mathrm{i}^{-\alpha_0})\ \text{-}\alpha_0 \sum_{p \in x \geq \mathrm{xmins}_v}^{\mathrm{xmax}} \log(x_p)\right] & \left[-\mathrm{n}_v\ \log(\mathrm{zvec}(\alpha_1) - \sum_{i=1}^{\mathrm{xmins}_v} \mathrm{i}^{-\alpha_1})\ \text{-}\alpha_1 \sum_{p \in x \geq \mathrm{xmins}_v}^{\mathrm{xmax}} \log(x_p)\right] & \cdots & \left[-\mathrm{n}_v\ \log(\mathrm{zvec}(\alpha_w) - \sum_{i=1}^{\mathrm{xmins}_v} \mathrm{i}^{-\alpha_w})\ \text{-}\alpha_w \sum_{p \in x \geq \mathrm{xmins}_v}^{\mathrm{xmax}} \log(x_p)\right]
\end{bmatrix}$$

$$\widehat{\vec{\alpha}} = \begin{bmatrix} \widehat{\alpha_0} = \max_{\alpha_j, j \in [0,w]}(L_{0,j}) \\ \widehat{\alpha_1} = \max_{\alpha_j, j \in [0,w]}(L_{1,j}) \\ \vdots \\ \widehat{\alpha_v} = \max_{\alpha_j, j \in [0,w]}(L_{v,j}) \end{bmatrix}, \qquad \vec{I} = \begin{bmatrix} I_0 = \max_{j \in [0,w]}(L_{0,j}) \\ I_1 = \max_{j \in [0,w]}(L_{1,j}) \\ \vdots \\ I_v = \max_{j \in [0,w]}(L_{v,j}) \end{bmatrix}$$

(1) Compute Likelihood for single

$$L_k = -\alpha_k \sum_{x \ge x\min} \log(x) - \text{ n } \log(\text{zvec}_k - \sum_{i=1}^{x\min-1} x\min vec_i^{-\alpha_k}) \qquad \textit{code}$$

$$= - \text{ n } \log(\text{zvec}_k - \sum_{i=1}^{x\min-1} x\min vec_i^{-\alpha_k}) - \alpha_k \sum_{x \ge x\min} \log(x) \qquad (\textit{eq } 3.5 \text{ powerlaw in empirical data})$$

$$\text{zvec}_k - \sum_{i=1}^{x\min-1} x\min vec_i^{-\alpha_k} = \sum_{i=1}^{\infty} \frac{1}{i^{\alpha_k}} - \sum_{i=1}^{x\min-1} \frac{1}{i^{\alpha_k}} = \frac{1}{1^{\alpha_k}} + \frac{1}{2^{\alpha_k}} + \cdots + \frac{1}{\infty^{\alpha_k}} - \left(\frac{1}{1^{\alpha_k}} + \frac{1}{2^{\alpha_k}} + \cdots + \frac{1}{(x\min_{xm} - 1)^{\alpha_k}} \right)$$

$$= \sum_{i=0}^{\infty} \frac{1}{(i + x\min_{xm})^{\alpha_k}} = \varsigma(\alpha_k, x_{\min})$$

(2) fit is an array of arrays where each inner array is a cdf of power law using the MLE of α found for a particular value of $x_{\min}$.

Cummulative distribution function of power law :=$P(x_r|\alpha, x_{\min}) = \begin{cases} 0 & r < x_{\min} \\ \dfrac{\widehat{\varsigma(\alpha, x_r)}}{\varsigma(\alpha, x_{\min})} & x_{\min} \le r < x_{\max} \\ 1 & r \ge x_{\max} \end{cases}$

$$P(x_r|\widehat{\alpha_j}, \text{xmins}_j) = \frac{\varsigma(\widehat{\alpha_j}, x_r)}{\varsigma(\widehat{\alpha_j}, x\min s_j)} = \frac{zvec_{I_j} - \sum_{i=1}^{x_r-1}\frac{1}{x_r^{\widehat{\alpha_j}}}}{zvec_{I_j} - \sum_{i=1}^{\text{xmins}_j-1}\frac{1}{\text{i}^{\alpha_k}}}$$

Proof :

$$\frac{zvec_{I_j} - \sum_{i=1}^{x_r-1}\frac{1}{x_r^{\widehat{\alpha_j}}}}{zvec_{I_j} - \sum_{i=1}^{\text{xmins}_j-1}\frac{1}{\text{i}^{\hat{\alpha}_{I_j}}}} = \frac{\sum_{i=1}^{\infty}\frac{1}{i^{\widehat{\alpha_j}}} - \sum_{i=1}^{x_r-1}\frac{1}{x_r^{\widehat{\alpha_j}}}}{\sum_{i=1}^{\infty}\frac{1}{i^{\hat{\alpha}_{I_j}}} - \sum_{i=1}^{x\min s_j-1}\frac{1}{i^{\hat{\alpha}_{I_j}}}} = \frac{\frac{1}{1^{\widehat{\alpha_j}}}+\frac{1}{2^{\widehat{\alpha_j}}}+\cdots+\frac{1}{\infty^{\widehat{\alpha_j}}} - \left(\frac{1}{1^{\widehat{\alpha_j}}}+\frac{1}{2^{\widehat{\alpha_j}}}+\cdots+\frac{1}{(x_r-1)^{\widehat{\alpha_j}}}\right)}{\frac{1}{1^{\hat{\alpha}_{I_j}}}+\frac{1}{2^{\hat{\alpha}_{I_j}}}+\cdots+\frac{1}{\infty^{\hat{\alpha}_{I_j}}} - \left(\frac{1}{1^{\hat{\alpha}_{I_j}}}+\frac{1}{2^{\hat{\alpha}_{I_j}}}+\cdots+\frac{1}{(\text{xmins}_j-1)^{\hat{\alpha}_{I_j}}}\right)} = \frac{\sum_{i=0}^{\infty}\frac{1}{(i+x_r)^{\widehat{\alpha_j}}}}{\sum_{i=0}^{\infty}\frac{1}{(i+\text{xmins}_j)^{\widehat{\alpha_j}}}} = \frac{\varsigma(\widehat{\alpha_j}, x_r)}{\varsigma(\widehat{\alpha_j}, \text{xmins}_j)} \square$$

$xm = \{r \in \mathbb{Z} \mid 0 \le r \le len(unique(X))\}$
$J = \{j \in \mathbb{Z} \mid x\min \le j \le x\max\}$

$$FIT = \begin{bmatrix} 0 & \left\{\frac{\frac{1}{(0+\text{xmins}_0)^{\widehat{\alpha_0}}}}{\sum_{i=0}^{\infty}\frac{1}{(i+\text{xmins}_0)^{\widehat{\alpha_0}}}}\right\} & \left\{\frac{\frac{1}{(0+\text{xmins}_0)^{\widehat{\alpha_0}}}+\frac{1}{(1+\text{xmins}_0)^{\widehat{\alpha_0}}}}{\sum_{i=0}^{\infty}\frac{1}{(i+\text{xmins}_0)^{\widehat{\alpha_0}}}}\right\} & \cdots & \left\{\frac{\frac{1}{(0+\text{xmins}_0)^{\widehat{\alpha_0}}}+\frac{1}{(1+\text{xmins}_0)^{\widehat{\alpha_0}}}+\cdots+\frac{1}{(\text{xmax-xmins}_0\text{+xmin}_0)^{\widehat{\alpha_0}}}}{\sum_{i=0}^{\infty}\frac{1}{(i+\text{xmins}_0)^{\widehat{\alpha_0}}}}\right\} \\ 0 & \left\{\frac{\frac{1}{(0+\text{xmins}_1)^{\widehat{\alpha_1}}}}{\sum_{i=0}^{\infty}\frac{1}{(i+\text{xmins}_1)^{\widehat{\alpha_1}}}}\right\} & \left\{\frac{\frac{1}{(0+\text{xmins}_1)^{\widehat{\alpha_1}}}+\frac{1}{(1+\text{xmins}_1)^{\widehat{\alpha_1}}}}{\sum_{i=0}^{\infty}\frac{1}{(i+\text{xmins}_1)^{\widehat{\alpha_1}}}}\right\} & \cdots & \left\{\frac{\frac{1}{(0+\text{xmin}_{xm})^{\widehat{\alpha_1}}}+\frac{1}{(1+\text{xmins}_1)^{\widehat{\alpha_1}}}+\cdots+\frac{1}{(\text{xmax-xmins}_{\cdot}\text{+xmins}_1)^{\widehat{\alpha_1}}}}{\sum_{i=0}^{\infty}\frac{1}{(i+\text{xmins}_1)^{\widehat{\alpha_1}}}}\right\} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \vdots & \vdots & \vdots & & \\ 0 & \left\{\frac{\frac{1}{(0+\text{xmins}_v)^{\widehat{\alpha_v}}}}{\sum_{i=0}^{\infty}\frac{1}{(i+\text{xmins}_v)^{\widehat{\alpha_v}}}}\right\} & \left\{\frac{\frac{1}{(0+\text{xmins}_v)^{\widehat{\alpha_v}}}+\frac{1}{(1+\text{xmins}_v)^{\widehat{\alpha_v}}}}{\sum_{i=0}^{\infty}\frac{1}{(i+\text{xmins}_v)^{\widehat{\alpha_v}}}}\right\} & \cdots & \left\{\frac{\frac{1}{(0+\text{xmins}_v)^{\widehat{\alpha_v}}}+\frac{1}{(1+\text{xmins}_v)^{\widehat{\alpha_v}}}+\cdots+\frac{1}{(\text{xmax-xmins}_v\text{+xmins}_v)^{\widehat{\alpha_v}}}}{\sum_{i=0}^{\infty}\frac{1}{(i+\text{xmins}_v)^{\widehat{\alpha_v}}}}\right\} \end{bmatrix}$$

$$\text{fit}_{xm} = \underset{\text{xmin}_{xm}\le j\le \text{xmax}}{\text{cumsum}}\left\{\frac{\frac{1}{\text{x}_j^{\alpha_1}}}{\sum_{i=1}^{\infty}\frac{1}{\text{i}^{\hat{\alpha}_{I_j}}} - \sum_{i=1}^{\text{xmins}_{xm}\text{-1}}\frac{1}{\text{i}^{\text{-}\alpha_1}}}\right\} = \left\{\frac{\sum_{j=\text{xmin}_{\text{sxm}}}^{\text{xmin}_{\text{sxm}}}\frac{1}{\text{x}_j^{\alpha_1}}}{\sum_{i=1}^{\infty}\frac{1}{\text{i}^{\hat{\alpha}_{I_j}}} - \sum_{i=1}^{\text{xmin}_{xm}\text{-1}}\frac{1}{\text{i}^{\text{-}\alpha_1}}}, \frac{\sum_{j=\text{xmin}_{\text{sxm}}}^{\text{xmin}_{\text{sxm}}+1}\frac{1}{\text{x}_j^{\alpha_1}}}{\sum_{i=1}^{\infty}\frac{1}{\text{i}^{\hat{\alpha}_{I_j}}} - \sum_{i=1}^{\text{xmins}_{xm}\text{-1}}\frac{1}{\text{i}^{\text{-}\alpha_1}}}, \dots, \frac{\sum_{j=\text{xmin}_{\text{sxm}}}^{\text{xmax}}\frac{1}{\text{x}_j^{\alpha_1}}}{\sum_{i=1}^{\infty}\frac{1}{\text{i}^{\hat{\alpha}_{I_j}}} - \sum_{i=1}^{\text{xmins}_{xm}\text{-1}}\frac{1}{\text{i}^{\text{-}\alpha_1}}}\right\}$$

$$= \left\{\frac{\frac{1}{(0+\text{xmin}_{xm})^{\alpha_1}}}{\sum_{i=0}^{\infty}\frac{1}{(i+\text{xmins}_j)^{\widehat{\alpha_j}}}}, \frac{\frac{1}{(0+\text{xmin}_{xm})^{\alpha_1}}+\frac{1}{(1+\text{xmin}_{xm})^{\alpha_1}}}{\sum_{i=0}^{\infty}\frac{1}{(i+\text{xmins}_j)^{\widehat{\alpha_j}}}}, \dots, \frac{\frac{1}{(0+\text{xmin}_{xm})^{\alpha_1}}+\frac{1}{(1+\text{xmin}_{xm})^{\alpha_1}}+\cdots+\frac{1}{(\text{xmax-xmin}_{xm}\text{+xmin}_{xm})^{\alpha_1}}}{\sum_{i=0}^{\infty}\frac{1}{(i+\text{xmins}_j)^{\widehat{\alpha_j}}}}\right\}$$

$$= \{p_x(x = \text{xmin}_{xm}), p_x(x = \text{xmin}_{xm}) + p_x(x = \text{xmin}_{xm}+1), \cdots, p_x(x = \text{xmin}_{xm}) + p_x(x = \text{xmin}_{xm}+1) + \cdots + p_x(x = \text{xmax}-1) + p_x(x = \text{xmax})\}$$

$$= P(x)$$

Plfit

Create vec which is a list of alphas [1.5,3.5]

Compute zeta(vec) for all elements of vec

xmins <- unique(X)

```
Sort Xmins
xmax <- maximum of X
Z <- X
Sort Z
For xm in 0 to xmins.shape[0]
    xmin=xmins[xm]
    Z = Z >= xmin
    n <- Z.shape[0]
    slogz sum log(z) for all z in Z
    for k in 0 to vec.shape[0]
        compute Likelihood (1) and append it to L
    I = argmax(L)
    (2) fit[xm] <-
```