Introduction to Orthogonal Polynomials of Non-Polynomial Type and Their Applications

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Special Functions

1. Introduction

Orthogonal polynomials of non-polynomial type are frequently used in fields of physics with specific applications in geodesy and seismology, but can also be useful in areas of mathematics such as spectral analysis. Orthogonal polynomials of any kind are defined by the equality:

$$\int_a^b w(x)p_m(x)p_n(x)\,dx = \delta_{mn}c_n,$$

with $\{p_n(x)\}$ being a sequence of polynomials defined over an interval $[a, b]$, $w(x)$ a weight function, and δ_{mn} the Kronecker delta. A special case arises whenever $c_n = 1$. We call orthogonal polynomials with this special case orthonormal polynomials.

Orthogonal polynomials of the non-polynomial type originated from orthogonal polynomials themselves. Starting in 1784, Adrien-Marie Legendre published a paper on equilibrium shapes of celestial objects titled *Recherches sur la figure des planètes*. He took the Newtonian gravitational potential of a point mass and expanded the term into polar coordinates to derive what were to be the first orthogonal polynomials: the Legendre polynomials. In the coming years, a more general theory of orthogonal polynomials began to emerge. We began to see orthogonal polynomials of the non-polynomial type be derived from the Legendre polynomials, as well as others. In recent times, orthogonal polynomials of the non-polynomial type have been heavily researched in an area of mathematics and physics known as spherical harmonics.

2. Narrative

We will describe in detail both the associated Legendre polynomials and Legendre polynomials of the second kind. After reading this paper, the reader should understand how to properly use both orthogonal polynomials of non-polynomial type.

As mentioned in the introduction, The associated Legendre polynomials are naturally found in mathematics and physics. In the case of spherical harmonics, associated Legendre polynomials arise in the derivation of spherical coordinates after using the method of separation of variables on Laplace's equation in spherical coordinates. From this information, we begin to see the importance of associated Legendre polynomials in mathematics and physics.

Like all orthogonal polynomials, associated Legendre polynomials have an orthogonality relation, that is:

$$\int_{-1}^{1} P_k^m(x)P_l^m(x)\, dx = \frac{2(l+m)!}{(2l+1)(l-m)!}\delta_{kl},$$

with $\{P_l^m(x)\}$ being a sequence of associated Legendre polynomials. Referring back to the definition of orthogonal polynomials in the introduction, we see that $a = -1$, $b = 1$, $w(x) = 1$, and $c_n = \frac{2(l+m)!}{(2l+1)(l-m)!}$. It is interesting to note that the weight function is equal to 1 because nearly all weight functions for orthogonal polynomials are not constant functions.

The actual origin of the associated Legendre polynomials is the general Legendre differential equation. They are the general solution to the differential equation. Written out, the general Legendre differential equation is:

$$\frac{d}{dx}[(1 - x^2)\frac{d}{dx}P_l^m(x)] + [l(l + 1) - \frac{m^2}{1-x^2}]P_l^m(x) = 0.$$

Now, the most important fact of all for the associated Legendre polynomials in this paper is that associated Legendre polynomials are of non-polynomial type only when m is odd.

To define the associated Legendre polynomials recursively, we need a generating function. From [3], we have:

$$(2m - 1)!!\frac{(1-x^2)^{m/2}t^m}{(1-2xt+t^2)^{m+1/2}} = \sum_{l=0}^{\infty} P_l^m(x)t^l.$$

(1)

Now, by differentiating (1) with respect to t and then substituting part of the left hand side with (1), we see that comparing the coefficients of the powers of t term-by-term gives us a simple recurrence relation:

$$(2l + 1)xP_l^m(x) = (l + m)P_{l-1}^m(x) + (l - m + 1)P_{l+1}^m(x).$$

This recurrence relation is helpful when trying to find a specific associated Legendre function.

Other orthogonal polynomials of non-polynomial type we will look at are Legendre polynomials of the second kind. Because the polynomials are of non-polynomial type, I will refer to them as functions from here on to reduce confusion. It is also not uncommon to hear them be called Legendre functions of the second kind. The functions are used in spherical harmonics like associated Legendre polynomials, but are also applied in mathematical chemistry and in potential theory. Legendre's Differential equation, that Legendre polynomials of the second kind are a solution of, is:

$$\frac{d}{dx}[(1 - x^2)\frac{d}{dx}f(x)] + n(n + 1)f(x) = 0.$$

By using the first two Legendre polynomials of the second kind (that satisfy the above equation):

$$Q_0(x) = \frac{1}{2}ln(\frac{1+x}{1-x})$$

and

$$Q_1(x) = \frac{x}{2}ln(\frac{1+x}{1-x}) - 1,$$

We will be able to derive a recurrence relation. According to source [2], all the recurrence relations for Legendre polynomials coincide with the recurrence relations for Legendre polynomials of the second kind. In that case, all we need to define the functions recursively is a generating function for $P_n(x)$, that is:

$$\frac{1}{\sqrt{1-2xt+t^2}} = \sum_{n=0}^{\infty} P_n(x)t^n.$$

(2)

We will now differentiate (2) and and substitute a part of the left-hand side with (2):

$$\frac{\partial}{\partial t}\frac{1}{\sqrt{1-2xt+t^2}} = \frac{\partial}{\partial t}\sum_{n=0}^{\infty} P_n(x)t^n \quad \Rightarrow \quad \frac{x-t}{(1-2xt+t^2)^{\frac{3}{2}}} = \sum_{n=0}^{\infty} nP_n(x)t^{n-1}$$

$$\Rightarrow \frac{x-t}{1-2xt+t^2} \sum_{n=0}^{\infty} P_n(x)t^n = \sum_{n=0}^{\infty} nP_n(x)t^{n-1}.$$

Now, we will write out each term as a summation and then proceed with comparing the coefficients of the powers of t term-by-term:

$$\Rightarrow (x - t) \sum_{n=0}^{\infty} P_n(x)t^n = (1 - 2xt + t^2) \sum_{n=0}^{\infty} nP_n(x)t^{n-1}$$

$$\Rightarrow \sum_{n=1}^{\infty} xP_{n-1}(x)t^{n-1} - \sum_{n=2}^{\infty} P_{n-2}(x)t^{n-1} = \sum_{n=0}^{\infty} nP_n(x)t^{n-1}$$

$$- \sum_{n=1}^{\infty} 2(n - 1)xP_{n-1}(x)t^{n-1} + \sum_{n=2}^{\infty} (n - 2)P_{n-2}(x)t^{n-1}$$

$$\Rightarrow xP_n(x) - P_{n-1}(x) = (n + 1)P_{n+1}(x) - 2nxP_n(x) + (n - 1)P_{n-1}(x).$$

Simplifying more, we derive a recurrence relation:

$$(n + 1)P_{n+1}(x) = (2n + 1)xP_n(x) - nP_{n-1}(x) \ \mid \ P_0(x) = 1,\ P_1(x) = x.$$

With this recursive definition, we can now define the recurrence equation for Legendre's polynomial of the second kind, that is:

$$(n + 1)Q_{n+1}(x) = (2n + 1)xQ_n(x) - nQ_{n-1}(x).$$

Where

$$Q_0(x) = \frac{1}{2} ln(\frac{1+x}{1-x})$$

and

$$Q_1(x) = \frac{x}{2} ln(\frac{1+x}{1-x}) - 1.$$

With this recurrence relation, we can now find the nth Legendre polynomial of the second kind that solves the Legendre differential equation.

3. Conclusion

As we have now seen, orthogonal polynomials of non-polynomial type have many properties similar to regular orthogonal polynomials. They both have generating functions, recurrence relations, orthogonality relations, and their differential equations. The applications of these properties range widely from various areas of mathematics to diverse fields in physics. With an ever growing popularity of orthogonal polynomials, we hope to find more rich connections between orthogonal polynomials and pure mathematics.

Bibliography

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