

# Quantum Optics

## lecture course

Petrov M. I.

# 1 Secondary quantization

## 1.1 Introduction

Second quantization starts with an expansion of a scalar or vector field (or wave functions) in a basis consisting of a complete set of functions. These expansion functions depend on the coordinates of a single particle. The expansion coefficients have been promoted from ordinary numbers to operators, creation and annihilation operators. A creation operator creates a particle in the corresponding basis function and an annihilation operator annihilates a particle in this function.

## 1.2 System for $\mathbf{A}$

Let us write Maxwell equations in vacuum without any charge in the system (in CGS units):

$$\begin{cases} \text{rot } \mathbf{E} = -\frac{1}{c} \frac{\partial \mathbf{H}}{\partial t}, & (1a) \\ \text{rot } \mathbf{H} = \frac{1}{c} \frac{\partial \mathbf{E}}{\partial t}, & (1b) \\ \text{div } \mathbf{E} = 0, & (1c) \\ \text{div } \mathbf{H} = 0. & (1d) \end{cases}$$

It's more convenient to work with potentials but not the fields itself. If we know  $\mathbf{A}$  и  $\varphi$ , we know the field

$$\mathbf{E} = -\frac{1}{c} \frac{\partial \mathbf{A}}{\partial t} - \nabla \varphi, \quad (2)$$

$$\mathbf{H} = \text{rot } \mathbf{A}. \quad (3)$$

It's easy to construct two (generally speaking infinitely many) different potential which can lead to the same EM fields. So we can put one arbitrary condition for  $\mathbf{A}$  and  $\varphi$ . Let us use Lorentz gauge:

$$\text{div } \mathbf{A} = 0. \quad (4)$$

Let us obtain an equation for  $\mathbf{A}$ . Substitution of field to the (1b) gives us

$$\text{rot rot } \mathbf{A} = -\frac{1}{c^2} \frac{\partial^2 \mathbf{A}}{\partial t^2} - \frac{1}{c} \nabla \frac{\partial \varphi}{\partial t} \quad (5)$$

and since

$$\text{rot rot } \mathbf{A} = \underbrace{\text{grad div } \mathbf{A} - \text{div grad } \mathbf{A}}_{\hookrightarrow=0} = -\Delta \mathbf{A} \quad (6)$$

then

$$\Delta \mathbf{A} - \frac{1}{c^2} \frac{\partial^2 \mathbf{A}}{\partial t^2} = \frac{1}{c} \nabla \frac{\partial \varphi}{\partial t}. \quad (7)$$

If we do  $\nabla \cdot (2)$  then we get

$$\underbrace{\text{div } \mathbf{E}}_{\hookrightarrow=0} = -\underbrace{\frac{1}{c} \frac{\partial}{\partial t} \text{div } \mathbf{A}}_{\hookrightarrow=0} - \Delta \varphi \quad \rightarrow \quad \Delta \varphi = 0 \quad \rightarrow \quad \nabla \varphi = 0. \quad (8)$$

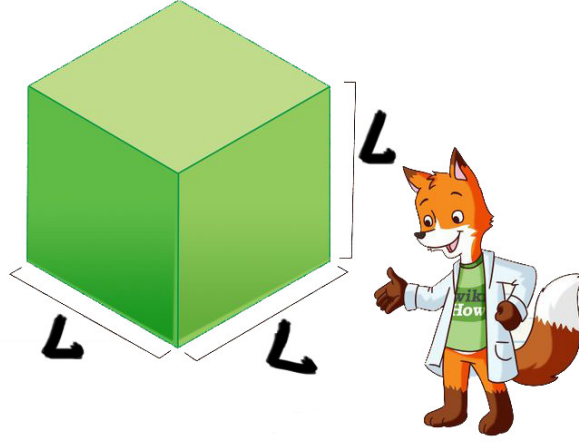


Рис. 1: Formulation of the problem

So we have system for  $\mathbf{A}$ :

$$\begin{cases} \Delta \mathbf{A} - \frac{1}{c^2} \frac{\partial^2 \mathbf{A}}{\partial t^2} = 0, \\ \operatorname{div} \mathbf{A} = 0. \end{cases} \quad (9)$$

*Remark:* in deriving this system we put  $\rho = 0$  and  $\mathbf{j} = 0$ .

### 1.3 Formulation of the problem

Let us consider a cube with length of the edge  $L$  (fig 1). Boundary conditions **should be zero or periodic** with the period  $L$ . System is considered to be conservative. The main idea is to solve system (9) and then just find the limit  $L \rightarrow \infty$ .

Solution of (9) may be written as the sum of all eigen solutions (each of it is a plane wave) in respect that boundary conditions are periodic:

$$\mathbf{A}(\mathbf{r}, t) = \sum_{\mathbf{k}} \mathbf{A}_{\mathbf{k}}(t) e^{i\mathbf{k}\mathbf{r}}, \quad k_{\alpha} = \frac{2\pi n_{\alpha}}{L}, \quad \alpha = x, y, z. \quad (10)$$

Where  $n_{\alpha}$  — is an integer number. As waves are plane then we should write

$$\mathbf{A}_{\mathbf{k}}(t) = \mathbf{c}_{\mathbf{k}} e^{-i\omega_{\mathbf{k}} t} + \mathbf{c}_{-\mathbf{k}}^* e^{i\omega_{\mathbf{k}} t}, \quad (11)$$

where  $\omega_{\mathbf{k}} = ck = c\sqrt{k_x^2 + k_y^2 + k_z^2}$ . Let us make a remark:  $\mathbf{A}_{\mathbf{k}} \in \mathbb{R}$ , so

$$\mathbf{A}_{\mathbf{k}}(t) = \mathbf{A}_{\mathbf{k}}^*(t) = \mathbf{c}_{-\mathbf{k}} e^{-i\omega_{\mathbf{k}} t} + \mathbf{c}_{\mathbf{k}}^* e^{i\omega_{\mathbf{k}} t} = \mathbf{A}_{-\mathbf{k}}(t), \quad (12)$$

. The Lorentz gauge leads to the fact that *waves are transverse*:

$$\operatorname{div} \mathbf{A} = 0 \quad \rightarrow \quad \sum_{\mathbf{k}} \mathbf{k} \mathbf{A}_{\mathbf{k}} e^{i\mathbf{k}\mathbf{r}} = 0 \quad \Longleftrightarrow \quad \boxed{\mathbf{A}_{\mathbf{k}}(t) \cdot \mathbf{k} = 0.} \quad (13)$$

Consider a wave with wave vector  $\mathbf{k}$ . According to Maxwell equations, there are two independent polarizations, so let us introduce two transverse polarization vectors  $\mathbf{e}_{\mathbf{k}1}; \mathbf{e}_{\mathbf{k}2}$ . After that let us make right-handed orthonormal basis from three vectors  $(\mathbf{e}_{\mathbf{k}1}; \mathbf{e}_{\mathbf{k}2}; \mathbf{k}/k)$ . It means that

$$\begin{aligned} \mathbf{k} \cdot \mathbf{e}_{\mathbf{k}s} &= 0, & [\mathbf{e}_{\mathbf{k}1} \times \mathbf{e}_{\mathbf{k}2}] &= \mathbf{k}/k, \\ \mathbf{e}_{\mathbf{k}s} \cdot \mathbf{e}_{\mathbf{k}s'} &= \delta_{ss'}, & \mathbf{c}_{\mathbf{k}} &= \sum_s c_{\mathbf{k}s} \mathbf{e}_{\mathbf{k}s}. \end{aligned} \quad (14)$$

After that we can rewrite decomposition of  $\mathbf{A}$  as

$$\begin{aligned}\mathbf{A} &= \sum_{\mathbf{k},s} \tilde{A}_{\mathbf{k}} \left( c_{\mathbf{k}s} \mathbf{e}_{\mathbf{k}s} e^{-i\omega_{\mathbf{k}}t} + c_{-\mathbf{k}s}^* \mathbf{e}_{-\mathbf{k}s}^* e^{i\omega_{\mathbf{k}}t} \right) \cdot e^{i\mathbf{k}\mathbf{r}} = \\ &= / \text{inverse 2nd sum using (12): } (-k) \rightarrow k / = \sum_{\mathbf{k},s} \tilde{A}_{\mathbf{k}} \left( u_{\mathbf{k}s}(t) \mathbf{e}_{\mathbf{k}s} e^{i\mathbf{k}\mathbf{r}} + u_{\mathbf{k}s}^*(t) \mathbf{e}_{\mathbf{k}s}^* e^{-i\mathbf{k}\mathbf{r}} \right),\end{aligned}\tag{15}$$

where  $u_{\mathbf{k}s}(t) = c_{\mathbf{k}s} e^{-i\omega_{\mathbf{k}}t}$ . Now we can write fields

$$\mathbf{E} = -\frac{1}{c} \frac{\partial \mathbf{A}}{\partial t} = \frac{i}{c} \sum_{\mathbf{k},s} \tilde{A}_{\mathbf{k}} \omega_{\mathbf{k}} \left( u_{\mathbf{k}s}(t) \mathbf{e}_{\mathbf{k}s} e^{i\mathbf{k}\mathbf{r}} - u_{\mathbf{k}s}^*(t) \mathbf{e}_{\mathbf{k}s}^* e^{-i\mathbf{k}\mathbf{r}} \right),\tag{16}$$

$$\mathbf{H} = \text{rot } \mathbf{A} = i \sum_{\mathbf{k},s} \tilde{A}_{\mathbf{k}} \left( u_{\mathbf{k}s} [\mathbf{k} \times \mathbf{e}_{\mathbf{k}s}] e^{i\mathbf{k}\mathbf{r}} - u_{\mathbf{k}s}^* [\mathbf{k} \times \mathbf{e}_{\mathbf{k}s}^*] e^{-i\mathbf{k}\mathbf{r}} \right).\tag{17}$$

The energy of EM field

$$\mathcal{H} = \frac{1}{8\pi} \int (\mathbf{H}^2 + \mathbf{E}^2) dV.\tag{18}$$

*Remark:* there is no averaging over time!

To make next calculations easier, let us notice that

$$\int_{L^3} e^{i(\mathbf{k}-\mathbf{k}')\mathbf{r}} dV = L^3 \delta_{\mathbf{k}\mathbf{k}'}.\tag{19}$$

This feature vanish the  $\sum_{\mathbf{k}}$ . Besides, it's convenient to notice that

$$\mathbf{e}_{\mathbf{k}s}^* \cdot \mathbf{e}_{\mathbf{k}s'} = \delta_{ss'} \quad \rightarrow \quad [\mathbf{k} \times \mathbf{e}_{\mathbf{k}s}^*] \cdot [\mathbf{k} \times \mathbf{e}_{\mathbf{k}s'}] = k^2 \delta_{ss'}.\tag{20}$$

Then we get

$$\mathcal{H} = \frac{L^3}{8\pi} \sum_{\mathbf{k},s} \tilde{A}_{\mathbf{k}}^2 \left( \underbrace{\frac{\omega_{\mathbf{k}}^2}{c^2} |u_{\mathbf{k}s}|^2}_{\hookrightarrow E^2} + \underbrace{k^2 |u_{\mathbf{k}s}|^2}_{\hookrightarrow H^2} \right), \quad k^2 = \frac{\omega_{\mathbf{k}}^2}{c^2} \quad (\text{for each mode!})\tag{21}$$

$$\mathcal{H} = \frac{L^3}{2\pi} \sum_{\mathbf{k},s} \tilde{A}_{\mathbf{k}}^2 k^2 |u_{\mathbf{k}s}|^2.\tag{22}$$

Segregation if real and imaginary part of mode can be done by introducing new variables

$$q_{\mathbf{k}s}(t) = u_{\mathbf{k}s}(t) + u_{\mathbf{k}s}^*(t),\tag{23}$$

$$p_{\mathbf{k}s}(t) = -i\omega_{\mathbf{k}} (u_{\mathbf{k}s}(t) - u_{\mathbf{k}s}^*(t)).\tag{24}$$

It's obvious that

$$u_{\mathbf{k}s}(t) = \frac{1}{2} q_{\mathbf{k}s}(t) - \frac{1}{2i\omega_{\mathbf{k}}} p_{\mathbf{k}s}(t) \quad \rightarrow \quad |u_{\mathbf{k}s}|^2 = \frac{1}{4\omega_{\mathbf{k}}^2} (p_{\mathbf{k}s}^2 + \omega_{\mathbf{k}}^2 q_{\mathbf{k}s}^2).\tag{25}$$

The Hamiltonian function will be as follows

$$\mathcal{H} = \frac{L^3}{4\pi c} \sum_{\mathbf{k},s} \frac{\tilde{A}_{\mathbf{k}}^2}{2} (p_{\mathbf{k}s}^2 + \omega_{\mathbf{k}}^2 q_{\mathbf{k}s}^2).\tag{26}$$

Let us boldly put  $\tilde{A}_{\mathbf{k}} = \sqrt{4\pi c^2/L^3}$ , then finally

$$\mathcal{H} = \sum_{\mathbf{k},s} \left( \frac{p_{\mathbf{k}s}^2}{2} + \frac{\omega_{\mathbf{k}}^2 q_{\mathbf{k}s}^2}{2} \right).\tag{27}$$