Quantum Optics
lecture course

Petrov M. I.

1 Secondary quantization

1.1 Introduction

Second quantization starts with an expansion of a scalar or vector field (or wave functions) in a basis consisting of a complete set of functions. These expansion functions depend on the coordinates of a single particle. The expansion coefficients have been promoted from ordinary numbers to operators, creation and annihilation operators. A creation operator creates a particle in the corresponding basis function and an annihilation operator annihilates a particle in this function.

1.2 System for A

Let us write Maxwell equations in vacuum without any charge in the system (in CGS units):

$$\begin{cases} \operatorname{rot}\mathbf{E} = -\dfrac{1}{c}\dfrac{\partial \mathbf{H}}{\partial t}, & (1a)\\ \operatorname{rot}\mathbf{H} = \dfrac{1}{c}\dfrac{\partial \mathbf{E}}{\partial t}, & (1b)\\ \operatorname{div}\mathbf{E} = 0, & (1c)\\ \operatorname{div}\mathbf{H} = 0. & (1d)\end{cases}$$

It's more convenient to work with potentials but not the fields itself. If we know $\mathbf{A}$ и φ, we know the field

$$\mathbf{E} = -\frac{1}{c}\frac{\partial \mathbf{A}}{\partial t} - \nabla\varphi, \tag{2}$$

$$\mathbf{H} = \operatorname{rot}\mathbf{A}. \tag{3}$$

It's easy to construct two (generally speaking infinitely many) different potential which can lead to the same EM fields. So we can put one arbitrary condition for $\mathbf{A}$ and φ. Let us use Lorentz gauge:

$$\operatorname{div}\mathbf{A} = 0. \tag{4}$$

Let us obtain an equation for $\mathbf{A}$. Substitution of field to the (1b) gives us

$$\operatorname{rot}\operatorname{rot}\mathbf{A} = -\frac{1}{c^2}\frac{\partial^2 \mathbf{A}}{\partial t^2} - \frac{1}{c}\nabla\frac{\partial\varphi}{\partial t} \tag{5}$$

and since

$$\operatorname{rot}\operatorname{rot}\mathbf{A} = \underbrace{\operatorname{grad}\operatorname{div}\mathbf{A}}_{\hookrightarrow=0} - \operatorname{div}\operatorname{grad}\mathbf{A} = -\Delta\mathbf{A} \tag{6}$$

then

$$\Delta\mathbf{A} - \frac{1}{c^2}\frac{\partial^2 \mathbf{A}}{\partial t^2} = \frac{1}{c}\nabla\frac{\partial\varphi}{\partial t}. \tag{7}$$

If we do $\nabla\cdot$(2) then we get

$$\underbrace{\operatorname{div}\mathbf{E}}_{\hookrightarrow=0} = -\underbrace{\frac{1}{c}\frac{\partial}{\partial t}\operatorname{div}\mathbf{A}}_{\hookrightarrow=0} - \Delta\varphi \quad\rightarrow\quad \Delta\varphi = 0 \quad\rightarrow\quad \nabla\varphi = 0. \tag{8}$$

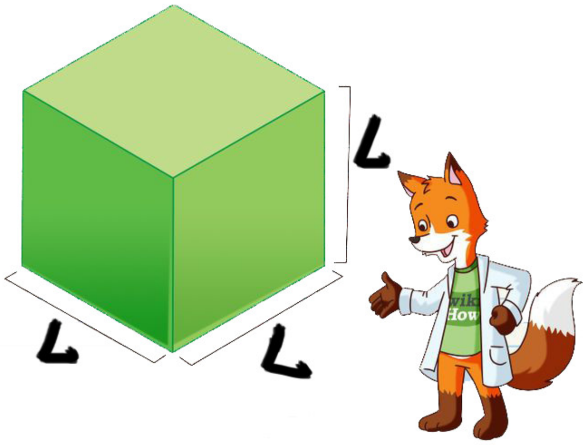

Рис. 1: Formulation of the problem

So we have system for **A**:

$$\begin{cases} \Delta \mathbf{A} - \dfrac{1}{c^2}\dfrac{\partial^2 \mathbf{A}}{\partial t^2} = 0, \\ \\ \operatorname{div} \mathbf{A} = 0. \end{cases} \tag{9}$$

Remark: in deriving this system we put $\rho = 0$ and $\mathbf{j} = 0$.

1.3 Formulation of the problem

Let us consider a cube with length of the edge L (fig 1). Boundary conditions should be zero or periodic with the period L. System is considered to be conservative. The main idea is to solve system (9) and then just find the limit $L \to \infty$.

Solution of (9) may be written as the sum of all eigen solutions (each of it is a plane wave) in respect that boundary conditions are periodic:

$$\mathbf{A}(\mathbf{r}, t) = \sum_{\mathbf{k}} \mathbf{A}_{\mathbf{k}}(t) e^{i\mathbf{k}\mathbf{r}}, \qquad k_\alpha = \frac{2\pi n_\alpha}{L}, \quad \alpha = x, y, z. \tag{10}$$

Where n_α — is an integer number. As waves are plane then we should write

$$\mathbf{A}_{\mathbf{k}}(t) = \mathbf{c}_{\mathbf{k}} e^{-i\omega_{\mathbf{k}} t} + \mathbf{c}^*_{-\mathbf{k}} e^{i\omega_{\mathbf{k}} t}, \tag{11}$$

where $\omega_{\mathbf{k}} = ck = c\sqrt{k_x^2 + k_y^2 + k_z^2}$. Let us make a remark: $\mathbf{A}_{\mathbf{k}} \in \mathbb{R}$, so

$$\mathbf{A}_{\mathbf{k}}(t) = \mathbf{A}^*_{\mathbf{k}}(t) = \mathbf{c}_{-\mathbf{k}} e^{-i\omega_{\mathbf{k}} t} + \mathbf{c}^*_{\mathbf{k}} e^{i\omega_{\mathbf{k}} t} = \mathbf{A}_{-\mathbf{k}}(t), \tag{12}$$

. The Lorentz gauge leads to the fact that *waves are transverse*:

$$\operatorname{div} \mathbf{A} = 0 \quad \to \quad \sum \mathbf{k}\mathbf{A}_{\mathbf{k}} e^{i\mathbf{k}\mathbf{r}} = 0 \quad \Longleftrightarrow \quad \boxed{\mathbf{A}_{\mathbf{k}}(t) \cdot \mathbf{k} = 0.} \tag{13}$$

Consider a wave with wave vector $\mathbf{k}$. According to Maxwell equations, there are two independent polarizations, so we introduce two transverse polarization vectors $\mathbf{e}_{\mathbf{k}1}; \mathbf{e}_{\mathbf{k}2}$. Three vectors $(\mathbf{e}_{\mathbf{k}1}; \mathbf{e}_{\mathbf{k}2}; \mathbf{k}/k)$ form a right-handed orthonormal basis which implies:

$$\begin{aligned} &\mathbf{k} \cdot \mathbf{e}_{\mathbf{k}s} = 0, \qquad [\mathbf{e}_{\mathbf{k1}} \times \mathbf{e}_{\mathbf{k2}}] = \mathbf{k}/k, \\ &\mathbf{e}_{\mathbf{k}s} \cdot \mathbf{e}_{\mathbf{k}s'} = \delta_{ss'}, \qquad \mathbf{c}_{\mathbf{k}} = \textstyle\sum_s c_{\mathbf{k}s} \mathbf{e}_{\mathbf{k}s}. \end{aligned} \tag{14}$$

After that we can rewrite decomposition of $\mathbf{A}$ as

$$\mathbf{A} = \sum_{\mathbf{k},s} \tilde{A}_{\mathbf{k}} \left(c_{\mathbf{k}s} \mathbf{e}_{\mathbf{k}s} e^{-i\omega_{\mathbf{k}} t} + c^*_{-\mathbf{k}s} \mathbf{e}^*_{-\mathbf{k}s} e^{i\omega_{\mathbf{k}} t} \right) \cdot e^{i\mathbf{kr}} =$$
$$= /\text{inverse 2nd sum using (12): } (-k) \to k/ = \sum_{\mathbf{k},s} \tilde{A}_{\mathbf{k}} \left(u_{\mathbf{k}s}(t) \mathbf{e}_{\mathbf{k}s} e^{i\mathbf{kr}} + u^*_{\mathbf{k}s}(t) \mathbf{e}^*_{\mathbf{k}s} e^{-i\mathbf{kr}} \right), \tag{15}$$

where $u_{\mathbf{k}s}(t) = c_{\mathbf{k}s} e^{-i\omega_{\mathbf{k}} t}$. Now we can write fields

$$\mathbf{E} = -\frac{1}{c}\frac{\partial \mathbf{A}}{\partial t} = \frac{i}{c} \sum_{\mathbf{k},s} \tilde{A}_{\mathbf{k}} \omega_{\mathbf{k}} \left(u_{\mathbf{k}s}(t) \mathbf{e}_{\mathbf{k}s} e^{i\mathbf{kr}} - u^*_{\mathbf{k}s}(t) \mathbf{e}^*_{\mathbf{k}s} e^{-i\mathbf{kr}} \right), \tag{16}$$

$$\mathbf{H} = \text{rot}\, \mathbf{A} = i \sum_{\mathbf{k},s} \tilde{A}_{\mathbf{k}} \left(u_{\mathbf{k}s} \left[\mathbf{k} \times \mathbf{e}_{\mathbf{k}s}\right] e^{i\mathbf{kr}} - u^*_{\mathbf{k}s} \left[\mathbf{k} \times \mathbf{e}^*_{\mathbf{k}s}\right] e^{-i\mathbf{kr}} \right). \tag{17}$$

The energy of EM field

$$\mathscr{H} = \frac{1}{8\pi} \int \left(\mathbf{H}^2 + \mathbf{E}^2 \right) dV. \tag{18}$$

Remark: there is no averaging over time!

To make next calculations easier, let us notice that

$$\int_{L^3} e^{i(\mathbf{k}-\mathbf{k}')\mathbf{r}} dV = L^3 \delta_{\mathbf{k}\mathbf{k}'}. \tag{19}$$

This feature vanish the $\sum_{\mathbf{k}}$. Besides, it's convenient to notice that

$$\mathbf{e}^*_{\mathbf{k}s} \cdot \mathbf{e}_{\mathbf{k}s'} = \delta_{ss'} \qquad \to \qquad \left[\mathbf{k} \times \mathbf{e}^*_{\mathbf{k}s}\right] \cdot \left[\mathbf{k} \times \mathbf{e}_{\mathbf{k}s'}\right] = k^2 \delta_{ss'}. \tag{20}$$

Then we get

$$\mathscr{H} = \frac{L^3}{8\pi} 2 \sum_{\mathbf{k},s} \tilde{A}^2_{\mathbf{k}} \left(\underbrace{\frac{\omega^2_{\mathbf{k}}}{c^2} \left|u_{\mathbf{k}s}\right|^2}_{\hookrightarrow E^2} + \underbrace{k^2 \left|u_{\mathbf{k}s}\right|^2}_{\hookrightarrow H^2} \right), \qquad k^2 = \frac{\omega^2_{\mathbf{k}}}{c^2} \ \text{(for each mode!)} \tag{21}$$

$$\mathscr{H} = \frac{L^3}{2\pi} \sum_{\mathbf{k},s} \tilde{A}^2_{\mathbf{k}} k^2 \left|u_{\mathbf{k}s}\right|^2. \tag{22}$$

Segregation if real and imaginary part of mode can be done by introducing new variables

$$q_{\mathbf{k}s}(t) = u_{\mathbf{k}s}(t) + u^*_{\mathbf{k}s}(t), \tag{23}$$
$$p_{\mathbf{k}s}(t) = -i\omega_{\mathbf{k}} \left(u_{\mathbf{k}s}(t) - u^*_{\mathbf{k}s}(t) \right). \tag{24}$$

It's obvious that

$$u_{\mathbf{k}s}(t) = \frac{1}{2} q_{\mathbf{k}s}(t) - \frac{1}{2i\omega} p_{\mathbf{k}s}(t) \quad \to \quad \left|u_{\mathbf{k}s}\right|^2 = \frac{1}{4\omega^2_{\mathbf{k}}} \left(p^2_{\mathbf{k}s} + \omega^2_{\mathbf{k}} q^2_{\mathbf{k}s} \right). \tag{25}$$

The Hamiltonian function will be as follows

$$\mathscr{H} = \frac{L^3}{4\pi c} \sum_{\mathbf{k},s} \frac{\tilde{A}^2_{\mathbf{k}}}{2} \left(p^2_{\mathbf{k}s} + \omega^2_{\mathbf{k}} q^2_{\mathbf{k}s} \right). \tag{26}$$

Let us boldly put $\tilde{A}_{\mathbf{k}} = \sqrt{4\pi c^2/L^3}$, then finally

$$\mathscr{H} = \sum_{\mathbf{k},s} \left(\frac{p_{\mathbf{k}s}^2}{2} + \frac{\omega_{\mathbf{k}}^2 q_{\mathbf{k}s}^2}{2} \right). \tag{27}$$

If we can write full energy of the system like $\mathscr{H} \sim p^2/2 + \omega^2 q^2/2$, then it means this system can be quantize. So next, we *quantize fields*. First of all we need to move to operators by doing

$$\begin{cases} q_{\mathbf{k}s} \to \hat{q}_{\mathbf{k}s}, \\ p_{\mathbf{k}s} \to \hat{p}_{\mathbf{k}s}. \end{cases} \tag{28}$$

After that our Hamiltonian function will transform to Mr. Hamiltonian. Besides, coordinate and impulse operator must obey next commutation relations:

$$[\hat{q}_{\mathbf{k}s}; \hat{p}_{\mathbf{k}'s'}] = i\hbar \delta_{\mathbf{k}\mathbf{k}'}^{(3)} \delta_{ss'}, \tag{29}$$

$$[\hat{q}_{\mathbf{k}s}; \hat{q}_{\mathbf{k}'s'}] = [\hat{p}_{\mathbf{k}s}; \hat{p}_{\mathbf{k}'s'}] = 0. \tag{30}$$

Moreover, this operators must be Hermitian because $\mathscr{H}$ stands for real energy which we can measure. Therefore

$$\hat{q}_{\mathbf{k}s} = \hat{q}_{\mathbf{k}s}^{\dagger}, \qquad \hat{p}_{\mathbf{k}s} = \hat{p}_{\mathbf{k}s}^{\dagger}. \tag{31}$$

Relations (29), (30) and (31) impose conditions for $\hat{q}_{\mathbf{k}s}$ and $\hat{p}_{\mathbf{k}s}$. Hereafter we can introduce ladder operators:

$$\hat{a}_{\mathbf{k}s}(t) = \frac{1}{\sqrt{2\hbar\omega}} \left(\omega \hat{q}_{\mathbf{k}s} + i \hat{p}_{\mathbf{k}s} \right), \tag{32}$$

$$\hat{a}_{\mathbf{k}s}^{\dagger}(t) = \frac{1}{\sqrt{2\hbar\omega}} \left(\omega \hat{q}_{\mathbf{k}s} - i \hat{p}_{\mathbf{k}s} \right). \tag{33}$$

This leads to the useful representation of $\hat{q}_{\mathbf{k}s}$ and $\hat{p}_{\mathbf{k}s}$:

$$\hat{q}_{\mathbf{k}s}(t) = \sqrt{\frac{\hbar}{2\omega}} \left(\hat{a}_{\mathbf{k}s}^{\dagger} + \hat{a}_{\mathbf{k}s} \right), \tag{34}$$

$$\hat{p}_{\mathbf{k}s}(t) = i\sqrt{\frac{\hbar\omega}{2}} \left(\hat{a}_{\mathbf{k}s}^{\dagger} - \hat{a}_{\mathbf{k}s} \right). \tag{35}$$

Commutation relations can be easily derived from consideration $[\hat{q}_{\mathbf{k}s}; \hat{p}_{\mathbf{k}s}]$ in the representation of ladder operators and using (29) and (30). So we get

$$\left[\hat{a}_{\mathbf{k},s}; \hat{a}_{\mathbf{k}',s'}^{\dagger} \right] = \delta_{\mathbf{k}\mathbf{k}'}^{(3)} \delta_{ss'}. \tag{36}$$

Easy to show that Hamiltonian can be written as follows

$$\hat{\mathscr{H}} = \sum_{\mathbf{k},s} \hbar\omega_{\mathbf{k}} \left[\hat{a}_{\mathbf{k},s}^{\dagger} \hat{a}_{\mathbf{k},s} + \frac{1}{2} \right]. \tag{37}$$