Quantum Optics

lecture course

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1 Secondary quantization

1.1 Introduction

Second quantization starts with an expansion of a scalar or vector field (or wave functions) in a basis consisting of a complete set of functions. These expansion functions depend on the coordinates of a single particle. The expansion coefficients have been promoted from ordinary numbers to operators, creation and annihilation operators. A creation operator creates a particle in the corresponding basis function and an annihilation operator annihilates a particle in this function.

1.2 System for A

Let us write Maxwell equations in vacuum without any charge in the system (in CGS units):

$$\begin{cases} \operatorname{rot}\mathbf{E} = -\dfrac{1}{c}\dfrac{\partial \mathbf{H}}{\partial t}, & (1a) \\ \operatorname{rot}\mathbf{H} = \dfrac{1}{c}\dfrac{\partial \mathbf{E}}{\partial t}, & (1b) \\ \operatorname{div}\mathbf{E} = 0, & (1c) \\ \operatorname{div}\mathbf{H} = 0. & (1d) \end{cases}$$

It's more convenient to work with potentials but not the fields itself. If we know $\mathbf{A}$ и φ, we know the field

$$\mathbf{E} = -\frac{1}{c}\frac{\partial \mathbf{A}}{\partial t} - \nabla\varphi, \tag{2}$$

$$\mathbf{H} = \operatorname{rot}\mathbf{A}. \tag{3}$$

It's easy to construct two (generally speaking infinitely many) different potential which can lead to the same EM fields. So we can put one arbitrary condition for $\mathbf{A}$ and φ. Let us use Lorentz gauge:

$$\operatorname{div}\mathbf{A} = 0. \tag{4}$$

Let us obtain an equation for $\mathbf{A}$. Substitution of field to the (1b) gives us

$$\operatorname{rot}\operatorname{rot}\mathbf{A} = -\frac{1}{c^2}\frac{\partial^2 \mathbf{A}}{\partial t^2} - \frac{1}{c}\nabla\frac{\partial\varphi}{\partial t} \tag{5}$$

and since

$$\operatorname{rot}\operatorname{rot}\mathbf{A} = \underbrace{\operatorname{grad}\operatorname{div}\mathbf{A}}_{\hookrightarrow=0} - \operatorname{div}\operatorname{grad}\mathbf{A} = -\Delta\mathbf{A} \tag{6}$$

then

$$\Delta\mathbf{A} - \frac{1}{c^2}\frac{\partial^2\mathbf{A}}{\partial t^2} = \frac{1}{c}\nabla\frac{\partial\varphi}{\partial t}. \tag{7}$$

If we do $\nabla\cdot$(2) then we get

$$\underbrace{\operatorname{div}\mathbf{E}}_{\hookrightarrow=0} = -\underbrace{\frac{1}{c}\frac{\partial}{\partial t}\operatorname{div}\mathbf{A}}_{\hookrightarrow=0} - \Delta\varphi \quad\to\quad \Delta\varphi = 0 \quad\to\quad \nabla\varphi = 0. \tag{8}$$

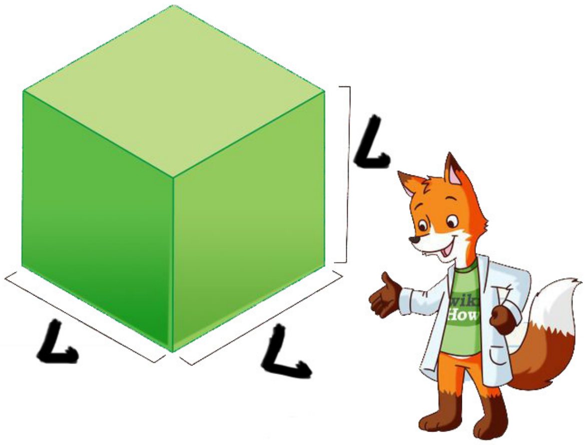

Рис. 1: Formulation of the problem

So we have system for $\mathbf{A}$:

$$\begin{cases} \Delta \mathbf{A} - \dfrac{1}{c^2}\dfrac{\partial^2 \mathbf{A}}{\partial t^2} = 0, \\ \\ \operatorname{div} \mathbf{A} = 0. \end{cases} \tag{9}$$

Remark: in deriving this system we put $\rho = 0$ and $\mathbf{j} = 0$.

1.3 Formulation of the problem

Let us consider a cube with length of the edge L (fig 1). Boundary conditions are periodic with the period L. System is considered to be conservative. The main idea is to solve system (9) and then just find the limit $L \to \infty$.

Solution of (9) may be written as the sum of all eigen solutions (each of it is a plane wave) in respect that boundary conditions are periodic:

$$\mathbf{A}(\mathbf{r}, t) = \sum_{\mathbf{k}} \mathbf{A}_{\mathbf{k}}(t) e^{i\mathbf{k}\mathbf{r}}, \qquad k_\alpha = \frac{2\pi n_\alpha}{L}, \quad \alpha = x, y, z. \tag{10}$$

Where $\mathbf{n}$ — a unit vector. As waves are plane then we should write

$$\mathbf{A}_{\mathbf{k}}(t) = \mathbf{e}_{\mathbf{k}} e^{-i\omega_{\mathbf{k}} t} + \mathbf{e}^*_{-\mathbf{k}} e^{i\omega_{\mathbf{k}} t}, \tag{11}$$

where $\omega_{\mathbf{k}} = ck = c\sqrt{k_x^2 + k_y^2 + k_z^2}$. Let us make a remark: $\mathbf{A}_{\mathbf{k}} \in \mathbb{R}$. The Lorentz gauge leads to the fact that *waves are transverse*:

$$\operatorname{div} \mathbf{A} = 0 \quad \to \quad \sum \mathbf{k}\mathbf{A}_{\mathbf{k}} e^{i\mathbf{k}\mathbf{r}} = 0 \quad \Longleftrightarrow \quad \boxed{\mathbf{A}_{\mathbf{k}}(t) \cdot \mathbf{k} = 0.} \tag{12}$$

Consider a wave with wave vector $\mathbf{k}$. After that let us make right-handed orthonormal basis from three vectors $(\mathbf{e}_{\mathbf{k}1}; \mathbf{e}_{\mathbf{k}2}; \mathbf{k}/k)$. It means that

$$\begin{gathered} \mathbf{k} \cdot \mathbf{e}_{\mathbf{k}s} = 0, \qquad [\mathbf{e}_{\mathbf{k1}} \times \mathbf{e}_{\mathbf{k2}}] = \mathbf{k}/k, \\ \\ \mathbf{e}_{\mathbf{k}s} \cdot \mathbf{e}_{\mathbf{k}s'} = \delta_{ss'}, \qquad \mathbf{c}_{\mathbf{k}} = \textstyle\sum_s c_{\mathbf{k}s} \mathbf{e}_{\mathbf{k}s}. \end{gathered} \tag{13}$$

After that we can rewrite decomposition of $\mathbf{A}$ as

$$\begin{aligned} \mathbf{A} = \sum_{\mathbf{k},s} \tilde{A}_{\mathbf{k}} \left(c_{\mathbf{k}s} \mathbf{e}_{\mathbf{k}s} e^{-i\omega_{\mathbf{k}} t} + c^*_{-\mathbf{k}s} \mathbf{e}^*_{-\mathbf{k}s} e^{i\omega_{\mathbf{k}} t} \right) \cdot e^{i\mathbf{k}\mathbf{r}} = \\ = /\text{inverse 2nd sum: } (-k) \to k/ = \sum_{\mathbf{k},s} \tilde{A}_{\mathbf{k}} \left(u_{\mathbf{k}s}(t) \mathbf{e}_{\mathbf{k}s} e^{i\mathbf{k}\mathbf{r}} + u^*_{\mathbf{k}s}(t) \mathbf{e}^*_{\mathbf{k}s} e^{-i\mathbf{k}\mathbf{r}} \right), \end{aligned} \tag{14}$$

where $u_{\mathbf{k}s}(t) = c_{\mathbf{k}s} e^{-i\omega_{\mathbf{k}} t}$. Now we can write fields

$$\mathbf{E} = \frac{1}{c}\frac{\partial \mathbf{A}}{\partial t} = \frac{i}{c}\sum_{\mathbf{k},s} \tilde{A}_{\mathbf{k}}\omega_{\mathbf{k}} \left(u_{\mathbf{k}s}(t)\mathbf{e}_{\mathbf{k}s} e^{i\mathbf{kr}} - u^*_{\mathbf{k}s}(t)\mathbf{e}^*_{\mathbf{k}s} e^{-i\mathbf{kr}} \right), \tag{15}$$

$$\mathbf{H} = \operatorname{rot}\mathbf{A} = i\sum_{\mathbf{k},s} \tilde{A}_{\mathbf{k}} \left(u_{\mathbf{k}s} \left[\mathbf{k}\times\mathbf{e}_{\mathbf{k}s}\right] e^{i\mathbf{kr}} - u^*_{\mathbf{k}s} \left[\mathbf{k}\times\mathbf{e}^*_{\mathbf{k}s}\right] e^{-i\mathbf{kr}} \right). \tag{16}$$

The energy of EM field

$$\mathscr{H} = \frac{1}{8\pi}\int \left(H^2 + E^2\right) dV. \tag{17}$$

Remark: there is no averaging over time!

To make next calculations easier, let us notice that

$$\int\limits_{L^3} e^{i(\mathbf{k}-\mathbf{k}')\mathbf{r}} dV = L^3 \delta_{\mathbf{k}\mathbf{k}'}. \tag{18}$$

This feature vanish the $\sum_{\mathbf{k}}$. Besides, it's convinient to notice that

$$\mathbf{e}^*_{\mathbf{k}s}\cdot\mathbf{e}_{\mathbf{k}s'} = \delta_{ss'} \qquad \rightarrow \qquad \left[\mathbf{k}\times\mathbf{e}^*_{\mathbf{k}s}\right]\cdot\left[\mathbf{k}\times\mathbf{e}_{\mathbf{k}s'}\right] = k^2\delta_{ss'}. \tag{19}$$

Then we get

$$\mathscr{H} = \frac{L^3}{8\pi}\sum_{\mathbf{k},s}\tilde{A}^2_{\mathbf{k}}\left(\underbrace{\frac{\omega^2_{\mathbf{k}}}{c^2}\left|u_{\mathbf{k}s}\right|^2}_{\hookrightarrow E^2} + \underbrace{k^2\left|u_{\mathbf{k}s}\right|^2}_{\hookrightarrow H^2}\right), \qquad k^2 = \frac{\omega^2_{\mathbf{k}}}{c^2} \quad \text{(for each mode!)} \tag{20}$$

$$\mathscr{H} = \frac{L^3}{4\pi}\sum_{\mathbf{k},s}\tilde{A}^2_{\mathbf{k}} k^2 \left|u_{\mathbf{k}s}\right|^2. \tag{21}$$

Segregation if real and imaginary part of mode can be done by introducing new variables

$$q_{\mathbf{k}s}(t) = u_{\mathbf{k}s}(t) + u^*_{\mathbf{k}s}(t), \tag{22}$$

$$p_{\mathbf{k}s}(t) = -i\omega_{\mathbf{k}}\left(u_{\mathbf{k}s}(t) - u^*_{\mathbf{k}s}(t)\right). \tag{23}$$

It's obvious that

$$u_{\mathbf{k}s}(t) = \frac{1}{2}q_{\mathbf{k}s}(t) - \frac{1}{2i\omega}p_{\mathbf{k}s}(t) \quad \rightarrow \quad \left|u_{\mathbf{k}s}\right|^2 = \frac{1}{4\omega^2_{\mathbf{k}}}\left(p^2_{\mathbf{k}s} + \omega^2_{\mathbf{k}} q^2_{\mathbf{k}s}\right). \tag{24}$$

The Hamiltonian function will be as follows

$$\mathscr{H} = \frac{L^3}{8\pi c}\sum_{\mathbf{k},s}\frac{\tilde{A}^2_{\mathbf{k}}}{2}\left(p^2_{\mathbf{k}s} + \omega^2_{\mathbf{k}} q^2_{\mathbf{k}s}\right). \tag{25}$$

Let us boldly put $\tilde{A}_{\mathbf{k}} = \sqrt{8\pi c^2/L^3}$, then finally

$$\mathscr{H} = \sum_{\mathbf{k},s}\left(\frac{p^2_{\mathbf{k}s}}{2} + \frac{\omega^2_{\mathbf{k}} q^2_{\mathbf{k}s}}{2}\right). \tag{26}$$