Mathematical Analysis

We need to generalize our notion on differentiability.

Consider a function $f(x) = \begin{cases} x & x \geqslant 0 \\ 0 & x < 0 \end{cases}$

then $f \in C^{\infty}$ almost everywhere except at $x = 0$.

$$H(x) = f'(x) = \begin{cases} 1 & x \geqslant 0 \\ 0 & x < 0 \end{cases}$$

The value of f' at $x=0$ does not create a particular problem.

$$H'(x) = f''(x) = \begin{cases} 0 & x \neq 0 \\ +\infty & x = 0 \end{cases}$$

in some sense.

Solution: we do not view functions as mappings from number to number. Instead, we view them as operators acting on other functions.

Consider

$$\int_{-\infty}^{+\infty} w\, u_{,x}\, dx = -\int_{-\infty}^{+\infty} w_{,x}\, u\, dx + \lim_{x \to +\infty} w(x)\, u(x) - \lim_{x \to -\infty} w(x)\, u(x)$$

if $w \in C^1$, we may define the derivative from the RHS.

in particular, if $w \in C^1$ and $\lim_{x \to \infty} w(x) = 0$,

we may define $u_{,x}$ as $\int_{-\infty}^{+\infty} w\, u_{,x}\, dx = -\int_{-\infty}^{+\infty} w_{,x}\, u\, dx$

Now $\displaystyle\int_{-\infty}^{+\infty} w\, H_{,x}\, dx = -\int_{-\infty}^{+\infty} w_{,x}\, H\, dx$

$$= -\int_{0}^{+\infty} w_{,x}\, dx$$

$$= w(0)$$

We call $H_{,x}$ the <u>Dirac delta</u> $\delta_0(x)$. ← a generalized function.

Let $\delta_y(x) := \delta_0(x-y)$, we have

$$\int w\, \delta_y(x)\, dx = \int w(z+y)\, \delta_0(z)\, dz = w(y).$$

The Green's function problem corresponding to (S):

Find a function g such that

$$g_{,xx} + \delta_y = 0$$

$$g(1) = 0$$

$$-g_{,x}(0) = 0 .$$

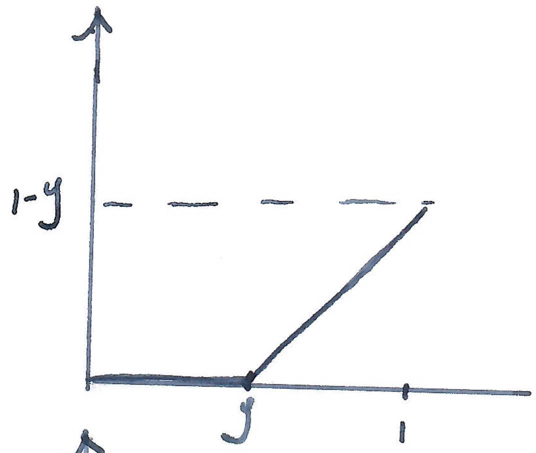
Now, from the eqn. we have

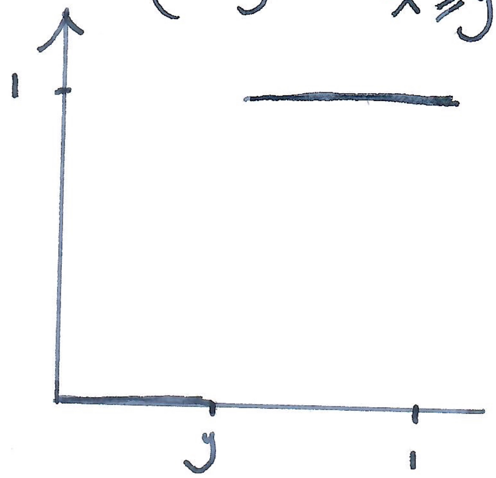
$$g_{,x} = C_1 - H_y = C_1 - H(x-y)$$

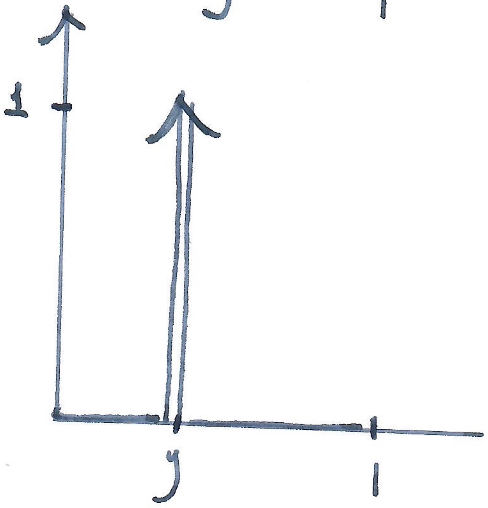
$$H(x-y) = \begin{cases} x-y & x \geq y \\ 0 & x \leq y \end{cases}$$

$$g = C_1 x + C_2 - \langle x - y \rangle$$

$$\langle x-y \rangle = \begin{cases} 0 & x \leq y \\ x-y & x \geq y . \end{cases}$$







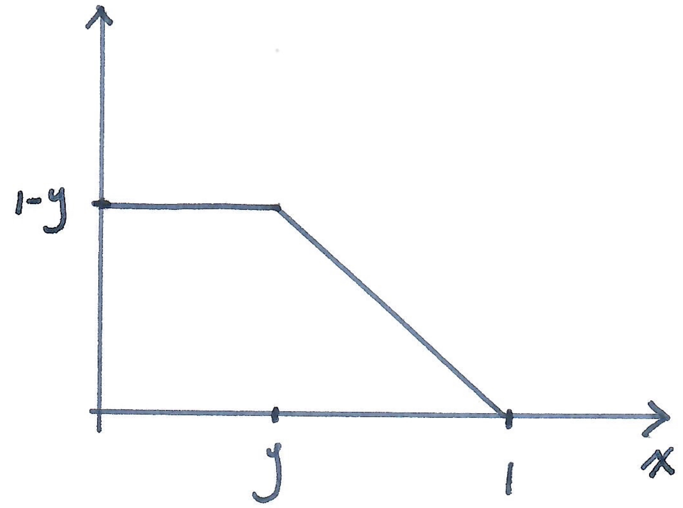
$$0 = g(1) = C_1 + C_2 - 1 + y$$

$$\Rightarrow \quad C_1 = 0, \quad C_2 = 1-y$$

$$0 = g_{,x}'(0) = C_1$$

Then we have $g = (1-y) - \langle x-y \rangle$.

$$= \begin{cases} 1-y & x \leq y \\ 1-x & x \geq y . \end{cases}$$



If $y = x_A$, we have $g \in \mathcal{V}^h$.

We may put g as a test function in (W) & (G) as

$$
\begin{aligned}
& a(g, u) = (g, f) \\
- \quad & a(g, u^h) = (g, f) \\
\hline
& a(g, u - u^h) = 0
\end{aligned}
$$

Notice that the Green's function problem has a weak-form statement:

$$a(w, g) = (w, \delta_y) = w(y)$$

then $u(x_A) - u^h(x_A) = a(u - u^h, g) = 0.$

Remark: This is known as the <u>super convergence</u> phenomena.

And it is only restricted to 1D.

More on the derivative accuracy

Theorem: Assume u is continuously differentiable, then there exists at least one point in (x_A, x_{A+1}) at which

$$u^h_{,x}(x) = \frac{u^h(x_{A+1}) - u^h(x_A)}{h_A} \quad \text{for } x \in (x_A, x_{A+1})$$

is exact.

Proof: Due to the mean value theorem, there is a point $c \in (x_A, x_{A+1})$ such that

$$\frac{u(x_{A+1}) - u(x_A)}{h_A} = u^{*}_{,x}(c)$$

Since $u(x_A) = u^h(x_A)$, we have

$$u^h_{,x}(x) = u_{,x}(c).$$

$\longrightarrow$ derivative is exact at c.

$\square$

Remark: It is hard to determine the location of c without knowing the form of the exact solution $u(x)$.

Let $e_{,x}(\alpha) := u^h_{,x}(\alpha) - u_{,x}(\alpha)$

$$= \frac{u^h_{\cancel{,x}}(x_{A+1}) - u^h(x_A)}{h_A} - u_{,x}(\alpha)$$

be the error of the derivative at $\alpha \in [x_A, x_{A+1}]$.

Lemma: Assume $u \in C^3$, then

$$e_{,x}(\alpha) = \left(\frac{x_{A+1}+x_A}{2} - \alpha\right) u_{,xx}(\alpha)$$
$$+ \frac{1}{6h_A}\left[(x_{A+1}-\alpha)^3 u_{,xxx}(c_1) - (x_A-\alpha)^3 u_{,xxx}(c_2)\right]$$

where c_1 & c_2 are in $[x_A, x_{A+1}]$.

Proof: Expand $u(x_{A+1})$ & $u(x_A)$ about $\alpha \in [x_A, x_{A+1}]$:

$$u(x_{A+1}) = u(\alpha) + (x_{A+1}-\alpha) u_{,x}(\alpha) + \frac{1}{2}(x_{A+1}-\alpha)^2 u_{,xx}(\alpha)$$
$$+ \frac{1}{6}(x_{A+1}-\alpha)^3 u_{,xxx}(c_1) \qquad c_1 \in [\alpha, x_{A+1}];$$

$$u(x_A) = u(\alpha) + (x_A-\alpha) u_{,x}(\alpha) + \frac{1}{2}(x_A-\alpha)^2 u_{,xx}(\alpha)$$
$$+ \frac{1}{6}(x_A-\alpha)^3 u_{,xxx}(c_2) \qquad c_2 \in [x_A, \alpha].$$

$$\frac{u(x_{A+1}) - u(x_A)}{h_A} = u_{,x}(\alpha) + \left(\frac{x_{A+1}+x_A}{2} - \alpha\right) u_{,xx}(\alpha)$$
$$+ \frac{1}{6h_A}\left[(x_{A+1}-\alpha)^3 u_{,xxx}(c_1) - (x_A-\alpha)^3 u_{,xxx}(c_2)\right].$$

$\square$

$$e_{,x}(x_A) = \frac{h_A}{2} u_{,xx}(x_A) + \frac{h_A^2}{6} u_{,xxx}(c_1) = O(h_A^1)$$

order of convergence
order of accuracy

Let $x_{A+\frac{1}{2}} = \frac{1}{2}(x_{A+1} + x_A)$ (i.e., the midpoint), then

$$e_{,x}(x_{A+\frac{1}{2}}) = O(h_A^2).$$

Remark: The derivative is second-order accurate at the mid-point. This corresponds to the "Barlow stress points" in linear elastic rod theory.

Remark: If the exact solution is quadratic, the derivative is exact at the mid-point.