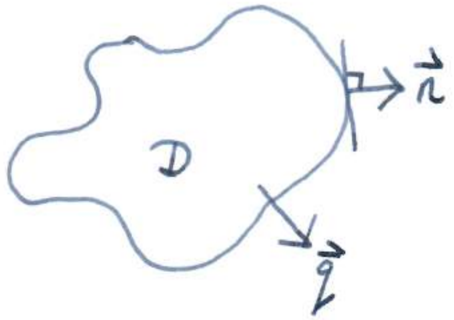
Two- and Three-dimensional problems: Heat equation.

Derivation of the model problem



u : temperature

ρ : density

c : specific heat capacity

f : heat source / radiation

ι : internal energy

$\mathcal{D}$ is arbitrarily chosen and $\mathcal{D} \subset \Omega \subset \mathbb{R}^{n_{sd}}$.

$$\int_{\mathcal{D}} \iota\Big|_{t=t_2} - \iota\Big|_{t=t_1} \, dx = \int_{t_1}^{t_2}\int_{\partial\mathcal{D}} -\vec{q}\cdot\vec{n} \, da \, dt + \int_{t_1}^{t_2}\int_{\mathcal{D}} f \, dx \, dt$$

$$\boxed{\begin{array}{l} \text{Divergence theorem: } \quad f \in C^1 \\ \displaystyle \int_{\partial\Omega} f n_i \, da = \int_{\Omega} f_{,i} \, dx. \end{array}}$$

$f_{,i} = \dfrac{\partial f}{\partial x_i}$

$n_{sd} = 2$

$\{x_i\} = \begin{Bmatrix} x \\ y \end{Bmatrix}$

$n_{sd} = 3$

$\{x_i\} = \begin{Bmatrix} x \\ y \\ z \end{Bmatrix}$

$$\int_{t_1}^{t_2}\int_{\mathcal{D}} \frac{\partial \iota}{\partial t} \, dx \, dt = \int_{t_1}^{t_2}\int_{\mathcal{D}} -\nabla\cdot\vec{q} + f \, dx \, dt$$

$$\Rightarrow \quad \frac{\partial \iota}{\partial t} = -\nabla\cdot\vec{q} + f$$

Remark: We write $\nabla \cdot \vec{q}$ as $q_{i,i}$ in index notation.

Constitutive relation:

$$c = \rho c u$$

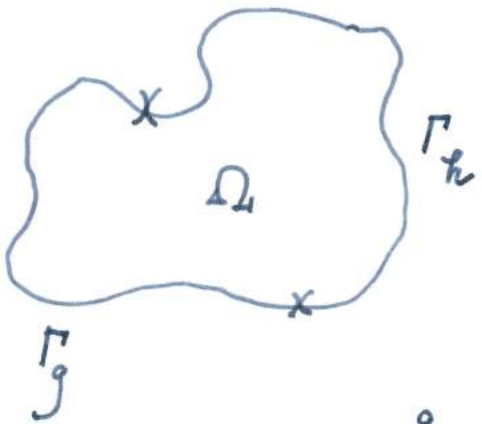
$$\vec{q} = - \bar{\bar{\kappa}} \nabla u \quad \text{or} \quad q_i = -\kappa_{ij} u_{,j} \quad \text{(Fourier's law)}$$

κ is the thermal conductivity, and it is symmetric, positive definite.

Remark: If κ_{ij}'s are constants, we say the material is homogeneous.

If $\kappa_{ij}(x) = \kappa(x) \delta_{ij}$, we say the material is isotropic.

Strong-form problem:



$$\begin{cases} \rho c u_{,t} + q_{i,i} = f & \text{in } \Omega \times (0,T) \\ u = g & \text{on } \Gamma_g \times (0,T) \\ -q_i n_i = h & \text{on } \Gamma_h \times (0,T) \\ u(x,0) = u_0(x) & \text{in } \Omega. \end{cases}$$

g: prescribed boundary temperature

h: " " heat flux

Integration by parts:

Let $f, g : \bar{\Omega} \to \mathbb{R}$ be C^1, then

$$\int_\Omega f_{,i}\, g \, d\Omega = -\int_\Omega f g_{,i} \, d\Omega + \int_\Gamma f g \, n_i \, d\Gamma.$$

Proof:

$$\int_\Omega (fg)_{,i} \, d\Omega = \int_\Gamma f g \, n_i \, d\Gamma$$

$$\parallel$$

$$\int_\Omega f_{,i}\, g + f g_{,i} \, d\Omega.$$

Trial solution space $\mathcal{S} := \{ u : u \in H^1(\Omega),\ u = g \text{ on } \Gamma_g \}$

Test function space $\mathcal{V} := \{ w : w \in H^1(\Omega),\ w = 0 \text{ on } \Gamma_g \}$

$$(W) \begin{cases} \text{Given } f, g, h, \text{ find } u \in \mathcal{S} \text{ such that} \\ -\displaystyle\int_\Omega w_{,i}\, q_i \, d\Omega = \int_\Omega w f \, d\Omega + \int_{\Gamma_h} w h \, d\Gamma \\ \text{for } \forall\, w \in \mathcal{V}. \end{cases}$$

Theorem: Assuming all functions involved are smooth enough to justify manipulations, then a solution of (S) is a solution of (W) and vice versa.

Proof: $(S) \Rightarrow (W)$

$$
\begin{aligned}
0 &= \int_{\Omega} (q_{i,i} - f)\, w \, d\Omega \\
&= -\int_{\Omega} q_i w_{,i} \, d\Omega + \int_{\Gamma} w\, q_i n_i \, d\Gamma - \int_{\Omega} w f \, d\Omega \\
&= -\int_{\Omega} q_i w_i - \int_{\Gamma} w h_i \, d\Gamma - \int_{\Omega} w f \, d\Omega .
\end{aligned}
$$

$(W) \Rightarrow (S)$

$$
\begin{aligned}
0 &= \int_{\Omega} w_{,i} q_i + w f \, d\Omega + \int_{\Gamma_h} w h \, d\Gamma \\
&= \int_{\Omega} w \underbrace{(-q_{i,i} + f)}_{\alpha} d\Omega + \int_{\Gamma_h} w \underbrace{(n_i q_i + h)}_{\beta} d\Gamma
\end{aligned}
$$

Step 1: pick $w = \alpha \phi$ with ϕ satisfying

$\phi > 0$ in Ω ; $\phi = 0$ on Γ ; ϕ is smooth.

then $0 = \int_{\Omega} \phi \alpha^2 \, d\Omega$,

which implies $\alpha = 0$.

Step 2: pick $w = \beta \psi$,

$\psi > 0$ on Γ_h ; $\psi = 0$ on Γ_g ; ψ is smooth.

then $0 = \int_{\Omega} \beta^2 \psi \, d\Gamma$

$\Rightarrow \beta = 0$. ∎

Notation:
$$a(w,u) = \int_{\Omega} w_{,i}\, \kappa_{ij}\, u_{,j}\, d\Omega$$

$$(w,f) = \int_{\Omega} w f \, d\Omega$$

$$(w,h)_{\Gamma} = \int_{\Gamma} w h \, d\Gamma$$

$$\Rightarrow \quad a(w,u) = (w,f) + (w,h)_{\Gamma}.$$

Index-free notation: Consider the case of $n_{sd} = 2$.

∇: gradient operator

$$\nabla u = \{u_{,i}\} = \begin{Bmatrix} u_{,1} \\ u_{,2} \end{Bmatrix}$$

isotropic case

$$\kappa = [\kappa_{ij}] = \begin{bmatrix} \kappa_{11} & \kappa_{12} \\ \kappa_{21} & \kappa_{22} \end{bmatrix} = \kappa \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

scalar

then
$$w_{,i}\, \kappa_{ij}\, u_{,j} = (\nabla w)^T \kappa \, \nabla u$$

and
$$a(w,u) = \int_{\Omega} (\nabla w)^T \kappa \, \nabla u \, d\Omega$$

Galerkin approximation: $\mathcal{S}^h \subset \mathcal{S}$, $\mathcal{V}^h \subset \mathcal{V}$.

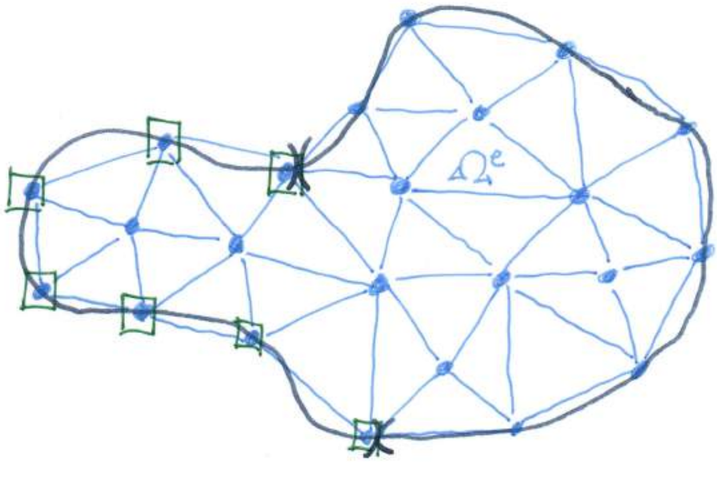
We assume $u^h \in \mathcal{S}^h$ adopts the decomposition,

$$u^h = v^h + g^h, \quad v^h \in \mathcal{V}^h$$

and g^h satisfy (or approximately satisfy)

$u = g$ on Γ_g

$$(G)\begin{cases} \text{Given } f,\ g,\ h,\ \text{find } u^h = v^h + g^h \text{ such that} \\ \text{for } \forall\, w^h \in \mathcal{V}^h, \\ \quad a(w^h, v^h) = (w^h, f) + (w^h, h)_{\Gamma} \\ \qquad\qquad - a(w^h, g^h) \end{cases}$$



$\bigcup_{e=1}^{n_{el}} \Omega^e$ approximates Ω ; $\eta = \{1, 2, \dots, n_{np}\}$

↳ set of global node numbers

g-node : node at which $u^h = g$ is prescribed.

η_g : set of g-node.

complement of η_g in η : $\eta - \eta_g$ is the set of nodes at which u^h is to be determined.

$$\boxed{\dim(\eta - \eta_g) = n_{eq}}$$

$$w^h(x) = \sum_{A \in \eta - \eta_g} N_A(x)\, c_A$$

Shape function associated with node A.

$$v^h(x) = \sum_{A \in \eta - \eta_g} N_A(x)\, d_A$$

unknown temperature at node A.

$$g^h(x) = \sum_{A \in \eta_g} N_A(x)\, g_A. \qquad g_A = g(x_A)$$

then we have

$$\sum_{B \in \eta - \eta_g} a(N_A, N_B)\, d_B = (N_A, f) + (N_A, h)_\Gamma - \sum_{B \in \eta_g} a(N_A, N_B)\, g_B$$

for $A \in \eta - \eta_g$.

Now, we invoke ID array:

$$ID(A) = \begin{cases} P & \text{if } A \in \eta - \eta_g \\ 0 & \text{if } A \in \eta_g \end{cases}$$

where $1 \leq P \leq n_{eq}$.

$$\Rightarrow \quad K_{PQ} = a(N_A, N_B) \qquad P = ID(A), \; Q = ID(B)$$

$$F_P = (N_A, f) + (N_A, h)_\Gamma - a(N_A, g^h)$$

$$d_Q = d_B$$

$Q = ID(B)$

disp. vector

$$\boxed{(M) \qquad K d = F}$$

the matrix problem

Stiffness matrix

load vector / force vector.

and $$u^h(x) = \sum_{B \in \eta - \eta_g} N_B(x) d_B + g^h(x)$$

Remark: In addition to the function space approximation (i.e., $\mathcal{V}^h \subset \mathcal{V}$, $\mathcal{S}^h \subset \mathcal{S}$),

there are geometrical approximation $\left(\bigcup_{e=1}^{n_{el}} \Omega^e \approx \Omega \right)$

and BC data approximation (i.e., $g^h(x) \approx g(x)$).

<u>Local / element perspective</u>.

$$K = \sum_{e=1}^{n_{el}} K^e = \sum_{e=1}^{n_{el}} \underbrace{\int_{\Omega^e} \nabla N_A^T \kappa \, \nabla N_B \, d\Omega}_{a(N_A, N_B)^e}$$

$$F = \sum_{e=1}^{n_{el}} F^e = \sum_{e=1}^{n_{el}} \left\{ \int_{\Omega^e} N_A f \, d\Omega + \int_{\Gamma^e \cap \Gamma_h} N_A h \, d\Gamma - a(N_A, N_B)^e g_B \right\}$$

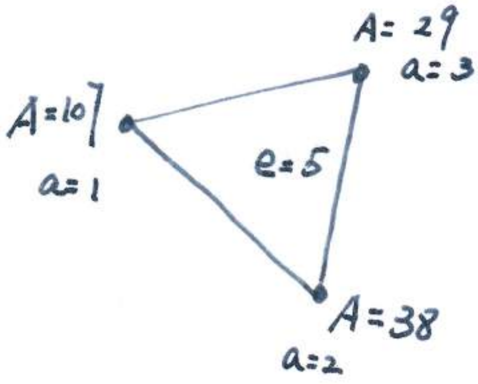
$\Gamma^e \cap \Gamma_h \rightarrow \Gamma_h^e$

element stiffness & element force vector

$$k^e = [k^e_{ab}] \qquad k^e_{ab} = \int_{\Omega^e} \nabla N_a^T \kappa \, \nabla N_b \, d\Omega$$

$$f^e = [f^e_a] \qquad f^e_a = \int_{\Omega^e} N_a f \, d\Omega + \int_{\Gamma_h^e} N_a h \, d\Gamma - \sum_{b=1}^{n_{en}} k^e_{ab} g^e_b$$

$1 \leqslant a, b \leqslant n_{en}$



$$A = IEN(a, e)$$

local node number: a; global node number: A; element number: e

Again, $LM(a,e) = ID(IEN(a,e))$.

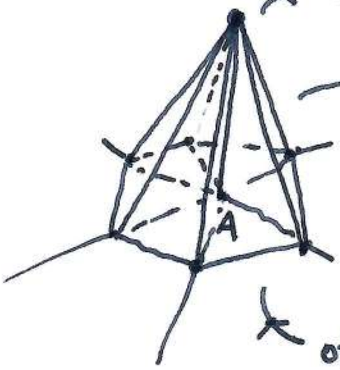
Assembly: $K = \underset{e=1}{\overset{n_{el}}{\mathbf{A}}} k^e \qquad F = \underset{e=1}{\overset{n_{el}}{\mathbf{A}}} f^e$.

Towards computer implementation for 2D problems

element technology & quadrature rules

Global view:

nodal value of N_A at x_A is 1



N_A varies linearly from 1 to 0.

otherwise it is zero

Element view

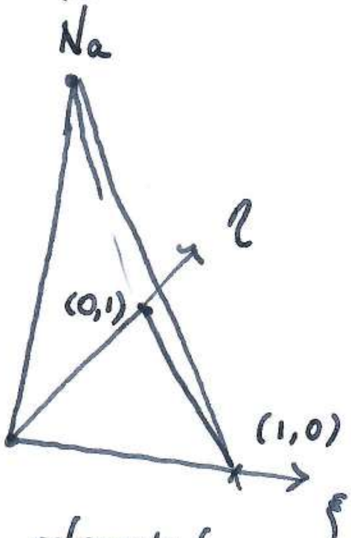
physical element

$\vec{x} = \vec{x}(\vec{\xi})$

or

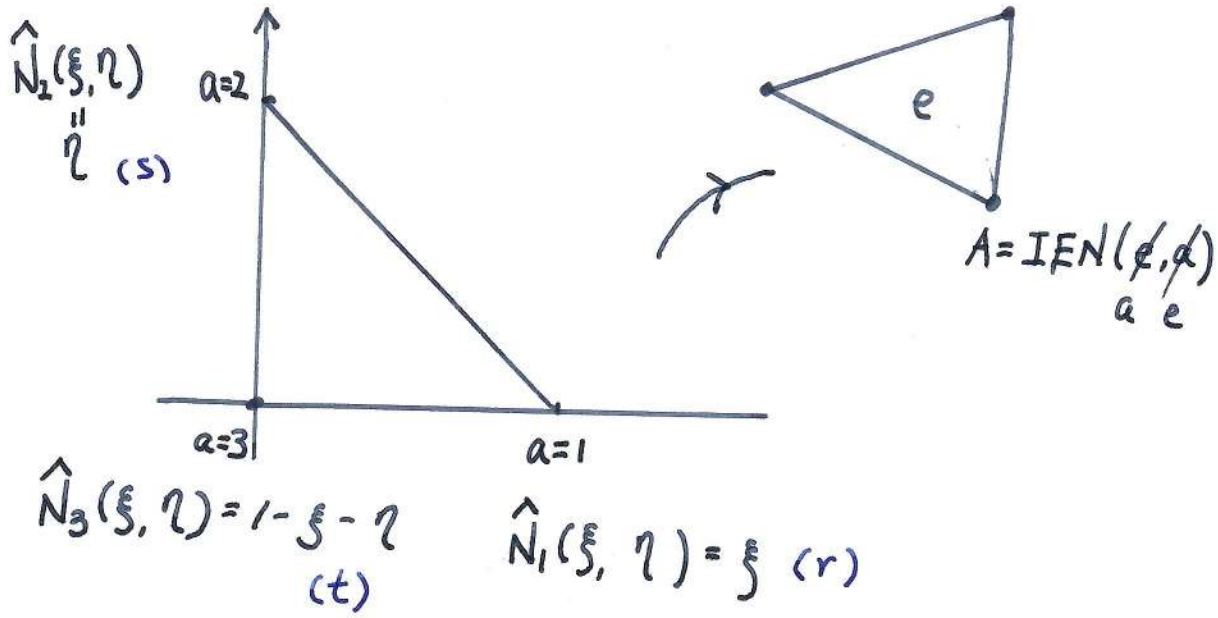
$x = x(\xi, \eta)$

$y = y(\xi, \eta)$



referential element.

$\hat{N}_a(\xi, \eta)$ is the only thing that needs to be provided, as $\vec{x}(\vec{\xi})$ mapping can be constructed through the isoparametric way.



The mapping is then

$$
\begin{aligned}
x(\xi,\eta) &= x_1^e \hat{N}_1(\xi,\eta) + x_2^e \hat{N}_2(\xi,\eta) + x_3^e \hat{N}_3(\xi,\eta) \\
&= x_1^e \xi + x_2^e \eta + x_3^e (1-\xi-\eta) \\
&= x_3^e + (x_1^e - x_3^e)\xi + (x_2^e - x_3^e)\eta
\end{aligned}
$$

$$
y(\xi,\eta) = y_3^e + (y_1^e - y_3^e)\xi + (y_2^e - y_3^e)\eta
$$

Chain rule:

$$
N_{A,x} = \hat{N}_{a,\xi}\, \underline{\underline{\xi_{,x}}} + \hat{N}_{a,\eta}\, \underline{\underline{\eta_{,x}}}
$$

$$
N_{A,y} = \hat{N}_{a,\xi}\, \underline{\underline{\xi_{,y}}} + \hat{N}_{a,\eta}\, \underline{\underline{\eta_{,y}}}
$$

Notice that
$$
\begin{bmatrix} \xi_{,x} & \xi_{,y} \\ \eta_{,x} & \eta_{,y} \end{bmatrix}
\begin{bmatrix} x_{,\xi} & x_{,\eta} \\ y_{,\xi} & y_{,\eta} \end{bmatrix}
= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}
$$

We have an explicit formula for the terms $\xi_{,x}$, $\xi_{,y}$, $\eta_{,x}$ $\eta_{,y}$:

$$\begin{bmatrix} \xi_{,x} & \xi_{,y} \\ \eta_{,x} & \eta_{,y} \end{bmatrix} = \begin{bmatrix} x_{,\xi} & x_{,\eta} \\ y_{,\xi} & y_{,\eta} \end{bmatrix}^{-1}$$

$$= \frac{1}{j} \begin{bmatrix} y_{,\eta} & -x_{,\eta} \\ -y_{,\xi} & x_{,\xi} \end{bmatrix}$$

$$j = x_{,\xi} y_{,\eta} - x_{,\eta} y_{,\xi} .$$

$$\Rightarrow \quad N_{a,x} = \frac{1}{j} \left[\hat{N}_{a,\xi} y_{,\eta} - \hat{N}_{a,\eta} \cancel{x_{,\eta}} y_{,\xi} \right]$$

$$N_{a,y} = \frac{1}{j} \left[-\hat{N}_{a,\xi} x_{,\eta} + \hat{N}_{a,\eta} x_{,\xi} \right]$$

Quadrature rules :

- G.R. Cowper "Gaussian quadrature formulas for triangles" IJNME 7, 1973: 405-408.

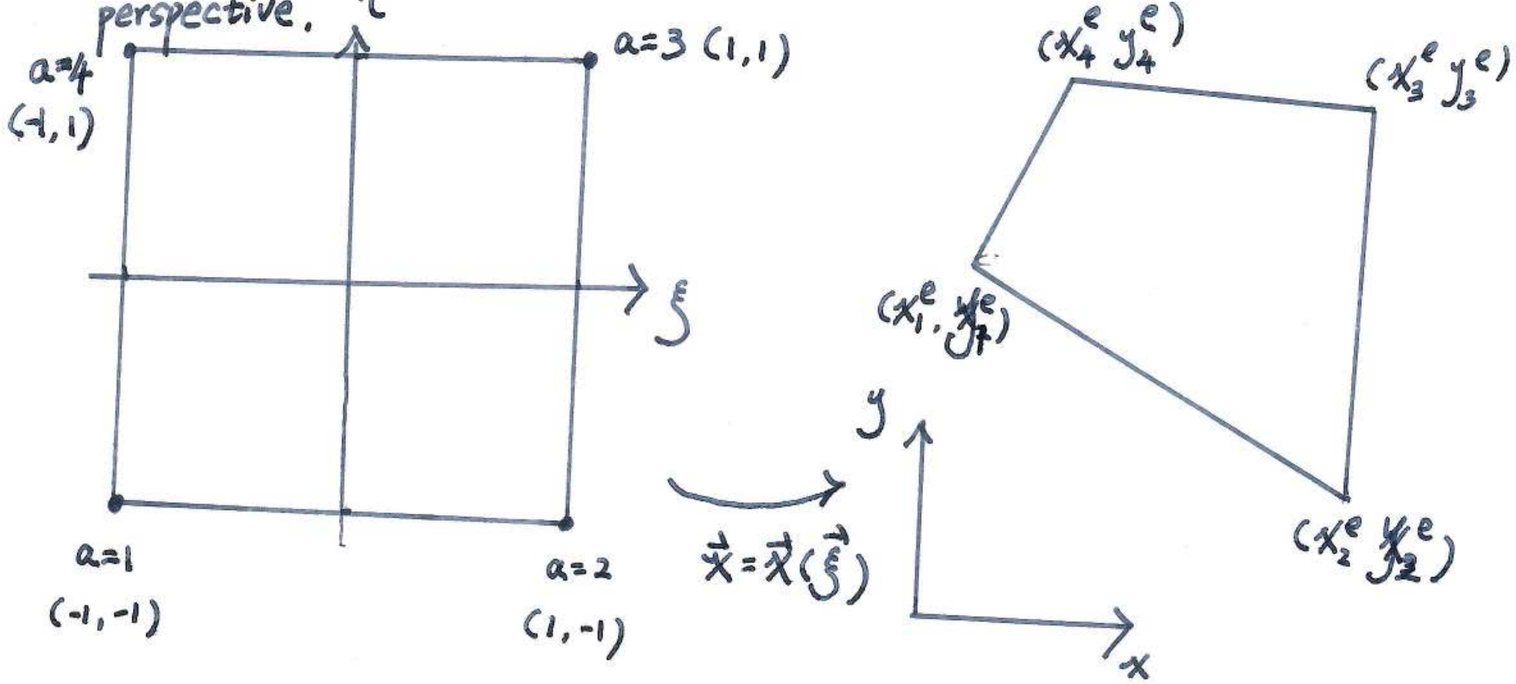
- Table 3.1.1 on page 173.

ex. 3-pt rule

w	$\hat{r}$	$\hat{s}$	$\hat{t}$
0.333333	0.666667	0.166667	0.166667
0.333333	0.166667	0.666667	0.166667
0.333333	0.166667	0.166667	0.666667

Bilinear quadrilateral element

We may define an element directly from the local . element perspective.



$$\hat{N}_a(\xi,\eta) = \frac{1}{2}(1+\xi_a\xi)\cdot\frac{1}{2}(1+\eta_a\xi) = \frac{1}{4}(1+\xi_a\xi)(1+\eta_a\eta)$$

The quadrature rule can be derived from the 1D quadrature rule by

$$\int_{-1}^{+1}\int_{-1}^{+1} g(\xi,\eta)\,d\xi\, d\eta = \int_{-1}^{+1}\sum_{l^{(1)}=1}^{n_{int}^{(1)}} W_{l^{(1)}}^{(1)}\, g(\tilde{\xi}_{l^{(1)}}^{(1)},\eta)\,d\eta$$

$$= \sum_{l^{(1)}=1}^{n_{int}^{(1)}}\sum_{l^{(2)}=1}^{n_{int}^{(2)}} W_{l^{(1)}}^{(1)} W_{l^{(2)}}^{(2)}\, g(\tilde{\xi}_{l^{(1)}}^{(1)},\tilde{\eta}_{l^{(2)}}^{(2)}).$$

e.g. the 2-pt Gaussian rule in 1D may generate the 4-pt 2D rule:

$$(\tilde{\xi},\tilde{\eta}) = \left(-\frac{1}{\sqrt{3}}, -\frac{1}{\sqrt{3}}\right), \left(\frac{1}{\sqrt{3}}, -\frac{1}{\sqrt{3}}\right), \left(\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right), \left(-\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right)$$

$$W = 1, \quad 1, \quad 1, \quad 1.$$