1. A One-Dimensional Boundary-Value Problem

differential equation:

$$u_{,xx} + f = 0 \quad \text{in } \Omega$$

- comma stands for differentiation: $u_{,x} = \dfrac{\partial u}{\partial x}$.

- the domain here is fixed to be $\Omega = (0,1)$

  textbook uses $]0,1[$ for open set

- f, in fact, can be very non-smooth; but is restricted to be C^0
  u, on the other hand, has to be $C^2(\bar{\Omega})$.

- $n_{sd} = 1$ # of spatial dimensions.

Remark: It makes more physical sense by writing the eqn. as

$$\begin{cases} \dfrac{d}{dx}\sigma(x) = f(x) & \leftarrow \text{a balance law} \\ \sigma(x) = -\kappa(x)\dfrac{du}{dx} & \leftarrow \text{a constitutive law} \end{cases}$$

e.g.

physical prob.	balance law.	u,	flux σ,	material modulus κ,	source f	Constitutive law
deformation of an elastic bar	linear momentum	disp.	stress,	Young's modulus	body force	Hooke's law

heat conduction in a bar/rod | energy | temp. | heat flux | thermal conductivity | radiation | Fourier's law.

For more examples, see BCO-book, Figure 2.1.

We need boundary conditions, data on the boundary of Ω, to complete the specification of the problem.

denoted as $\partial\Omega$

1. on $\partial\Omega$, we may specify the value of u:

$$u = g \quad \text{on } \partial\Omega$$

e.g. prescribe the temperature.

<u>Dirichlet BC.</u>

2. on $\partial\Omega$, we may specify the value of $-u_{,x}$

$$\left(\sigma = -\varkappa \frac{\partial u}{\partial x} = \right) -u_{,x} = h \quad \text{on } \partial\Omega$$

e.g. prescribe the heat flux.

<u>Neumann BC</u>

In our discussion, we consider both:

$$u(1) = g$$

$$-u_{,x}(0) = h.$$

Now, the equation is uniquelly solvable.

$$
(S)\begin{cases}
\text{Given } f : \bar{\Omega} \to \mathbb{R} \text{ and constants } g \text{ \& } h. \text{ find } u : \bar{\Omega} \to \mathbb{R} \\
\text{such that} \\
\qquad u_{,xx} + f = 0 \\
\qquad u(1) = g \\
\qquad -u_{,x}(0) = h
\end{cases}
$$

↑

Strong-form problem.

Sobolev spaces of functions

$$H^k = H^k(\Omega) := \{ w : w \in L_2,\ w_{,x} \in L_2, \cdots, \underbrace{w_{,x\dots x}}_{k\text{-times}} \in L_2 \}$$

$$L_2 = L_2(\Omega) := \left\{ w : \int_\Omega w^2 \, dx = \int_0^1 w^2 \, dx < \infty \right\}$$

trial solution space $\mathcal{S} := \{ u : u \in H^1,\ u(1) = g \}$

test function, weighting function, variations space

$$\mathcal{V} := \{ w : w \in H^1,\ w(1) = 0 \}$$

$$
(W)\begin{cases}
\text{Given } f \in L_2,\ g \in \mathbb{R},\ h \in \mathbb{R}, \text{ find } u \in \mathcal{S} \text{ such that} \\
\qquad \int_0^1 w_{,x} u_{,x} \, dx = \int_0^1 w f \, dx + w(0) h. \\
\text{for all } w \in \mathcal{V}
\end{cases}
$$

- Equivalence between (S) and (W).

> Integration-by-parts
>
> $$\int_0^1 w(x)\, u_{,x}(x)\, dx = w(1)u(1) - w(0)u(0) - \int_0^1 w_{,x}(x)\, u(x)\, dx$$

Proposition a: Let u be a solution of (S), then u is a solution of (W).

$$(S) \Rightarrow (W).$$

Proof:

$$0 = \int_0^1 -w u_{,xx} - wf \, dx$$

$$= \int_0^1 w_{,x} u_{,x} - wf \, dx - w u_{,x}\Big|_0^1$$

$$= \int_0^1 w_{,x} u_{,x} - wf \, dx - wh.$$

we pick $w \in \mathcal{V}$.

$u(0) = g$ and $u \in C^2 \Rightarrow u \in \mathcal{S}$.

then u is the solution of (W).

$\blacksquare$

Fundamental Lemma of the calculus of variations.
If $f \in C(\Omega)$ satisfies $\int_{\Omega} fg \, dx = 0$ for all smooth functions g with $g(0) = g(1) = 0$, then $f = 0$.

Proof: We may choose $g = \phi f$, $\phi > 0$, is smooth, and

$\phi(0) = \phi(1) = 0$.

Then: $$0 = \int_0^1 gf \, dx = \int_0^1 \phi f^2 \, dx.$$

Since $\phi > 0$ and can be arbitrary, $f = 0$ in Ω.

∎

Proposition 6: Let u be a solution of (W), then u is a solution of (S).

Proof: First, if $u \in \mathcal{S}$, then $u(1) = g$.

$$\int_0^1 w_{,x} u_{,x} \, dx = \int_0^1 wf \, dx + w(0) h.$$

$$\Rightarrow \int_0^1 w(u_{,xx} + f) \, dx + w(0)\left[u_{,x}(0) + h\right] \quad (*)$$

(i) Let $w = \phi(u_{,xx} + f)$ with $\phi > 0$, smooth, $\phi(0) = \phi(1) = 0$

$$\Rightarrow \int_0^1 \phi (u_{,xx} + f)^2 dx = 0$$

$$\Rightarrow \quad u_{,xx} + f = 0.$$

(ii) Now we have $\quad 0 = w(0)\left[u_{,x}(0) + h\right].$

$w \in \mathcal{V}$ places no restriction on $w(0)$.

we have $\quad 0 = u_{,x}(0) + h.$

$\blacksquare$

Remark: In (W), the BC $u(1) = g$ is embeded in the definition of the trial solution space $\mathcal{S}$. We call this type of BC the essential boundary condition.

The BC $-u_{,x}(0) = h$ is satisfied in the variational equation. We call it the natural boundary condition.

Remark: (*) in the above proof is the Euler-Lagrange equations of the weak problem.

Notation: $a(w, u) = \int_0^1 w_{,x} u_{,x} dx$

$(w, f) = \int_0^1 w f \, dx$

symmetric bilinear forms.