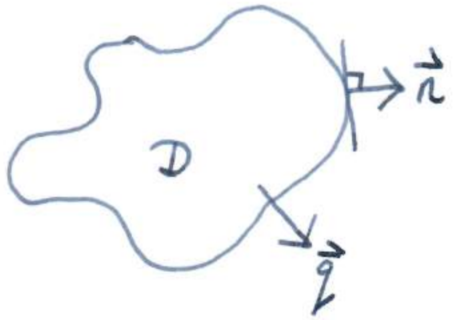
# Two- and Three-dimensional problems: Heat equation.

## Derivation of the model problem




$u$ : temperature

$\rho$ : density

$c$ : specific heat capacity

$f$ : heat source / radiation

$\ell$ : internal energy

$D$ is arbitrarily chosen and $D \subset \Omega \subset \mathbb{R}^{n_{sd}}$.

$$\int_D \ell\big|_{t=t_2} - \ell\big|_{t=t_1} \, dx = \int_{t_1}^{t_2}\int_{\partial D} -\vec{q}\cdot\vec{n} \, da \, dt + \int_{t_1}^{t_2}\int_D f \, dx \, dt$$

**Divergence theorem:** $f \in C^1$

$$\int_{\partial\Omega} f \, n_i \, da = \int_\Omega f_{,i} \, dx.$$

$f_{,i} = \dfrac{\partial f}{\partial x_i}$

$n_{sd} = 2$ $\quad \{x_i\} = \begin{Bmatrix} x \\ y \end{Bmatrix}$

$n_{sd} = 3$ $\quad \{x_i\} = \begin{Bmatrix} x \\ y \\ z \end{Bmatrix}$

$$\int_{t_1}^{t_2}\int_D \frac{\partial \ell}{\partial t} \, dx \, dt = \int_{t_1}^{t_2}\int_D -\nabla\cdot\vec{q} + f \, dx \, dt$$

$$\Rightarrow \quad \frac{\partial \ell}{\partial t} = -\nabla\cdot\vec{q} + f$$

Remark: We write $\nabla \cdot \vec{q}$ as $q_{i,i}$ in index notation.

Constitutive relation:

$$C = \rho c u$$

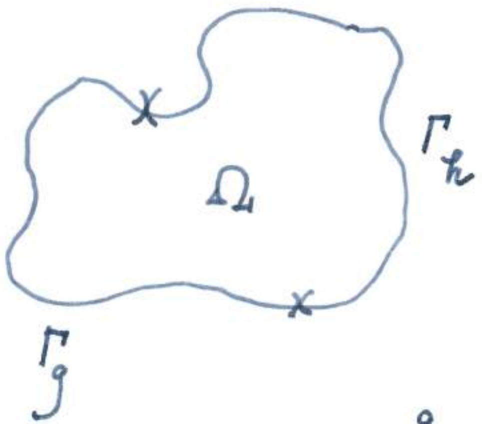
$$\vec{q} = -\bar{\bar{\kappa}} \nabla u \quad \text{or} \quad q_i = -\kappa_{ij} u_{,j} \quad (\text{Fourier's law})$$

$\kappa$ is the thermal conductivity, and it is symmetric, positive definite.

Remark: If $\kappa_{ij}$'s are constants, we say the material is homogeneous.

If $\kappa_{ij}(x) = \kappa(x)\delta_{ij}$, we say the material is isotropic.

Strong-form problem:




$$\begin{cases} \rho c u_{,t} + q_{i,i} = f & \text{in } \Omega \times (0,T) \\ u = g & \text{on } \Gamma_g \times (0,T) \\ -q_i n_i = h & \text{on } \Gamma_h \times (0,T) \\ u(x,0) = u_0(x) & \text{in } \Omega. \end{cases}$$

$g$: prescribed boundary temperature

$h$: " " heat flux

Integration by parts:

Let $f, g : \bar{\Omega} \to \mathbb{R}$ be $C^1$, then

$$\int_\Omega f_{,i}\, g \, d\Omega = -\int_\Omega f g_{,i} \, d\Omega + \int_\Gamma f g \, n_i \, d\Gamma.$$

Proof:

$$\int_\Omega (fg)_{,i} \, d\Omega = \int_\Gamma f g \, n_i \, d\Gamma$$

$$\parallel$$

$$\int_\Omega f_{,i}\, g + f g_{,i} \, d\Omega.$$

Trial solution space $\mathcal{S} := \{ u : u \in H^1(\Omega),\ u = g \text{ on } \Gamma_g \}$

Test function space $\mathcal{V} := \{ w : w \in H^1(\Omega),\ w = 0 \text{ on } \Gamma_g \}$

$$(W)\begin{cases} \text{Given } f, g, h, \text{ find } u \in \mathcal{S} \text{ such that} \\ -\int_\Omega w_{,i}\, q_i \, d\Omega = \int_\Omega w f \, d\Omega + \int_{\Gamma_h} w h \, d\Gamma \\ \text{for } \forall\, w \in \mathcal{V}. \end{cases}$$

Theorem: Assuming all functions involved are smooth enough to justify manipulations, then a solution of (S) is a solution of (W) and vice versa.

Proof: $(S) \Rightarrow (W)$

$$0 = \int_{\Omega} (q_{i,i} - f)\, w \, d\Omega$$

$$= -\int_{\Omega} q_i w_{,i} \, d\Omega + \int_{\Gamma} w q_i n_i \, d\Gamma - \int_{\Omega} w f \, d\Omega$$

$$= -\int_{\Omega} q_i w_i - \int_{\Gamma} w h_i \, d\Gamma - \int_{\Omega} w f \, d\Omega .$$

$(W) \Rightarrow (S)$

$$0 = \int_{\Omega} w_{,i} q_i + w f \, d\Omega + \int_{\Gamma_h} w h \, d\Gamma$$

$$= \int_{\Omega} w \underbrace{(-q_{i,i} + f)}_{\alpha} d\Omega + \int_{\Gamma_h} w \underbrace{(n_i q_i + h)}_{\beta} d\Gamma$$

Step 1: pick $w = \alpha \phi$ with $\phi$ satisfying

$\phi > 0$ in $\Omega$ ; $\phi = 0$ on $\Gamma$ ; $\phi$ is smooth.

then $0 = \int_{\Omega} \phi \alpha^2 \, d\Omega$ ,

which implies $\alpha = 0$.

Step 2: pick $w = \beta \psi$,

$\psi > 0$ on $\Gamma_h$ ; $\psi = 0$ on $\Gamma_g$ ; $\psi$ is smooth.

then $0 = \int_{\Omega} \beta^2 \psi \, d\Gamma$

$\Rightarrow \beta = 0$. ∎

Notation: $$a(w, u) = \int_\Omega w_{,i}\, \kappa_{ij}\, u_{,j}\, d\Omega$$

$$(w, f) = \int_\Omega w f \, d\Omega$$

$$(w, h)_\Gamma = \int_\Gamma w h \, d\Gamma$$

$$\Rightarrow \quad a(w, u) = (w, f) + (w, h)_\Gamma .$$

Index-free notation: Consider the case of $n_{sd} = 2$.

$\nabla$: gradient operator

$$\nabla u = \{ u_{,i} \} = \begin{Bmatrix} u_{,1} \\ u_{,2} \end{Bmatrix}$$

isotropic case

$$\kappa = [\kappa_{ij}] = \begin{bmatrix} \kappa_{11} & \kappa_{12} \\ \kappa_{21} & \kappa_{22} \end{bmatrix} = \kappa \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

scalar

then $$w_{,i}\, \kappa_{ij}\, u_{,j} = (\nabla w)^T \kappa \, \nabla u$$

and $$a(w, u) = \int_\Omega (\nabla w)^T \kappa \, \nabla u \, d\Omega$$

Galerkin approximation: $\mathcal{S}^h \subset \mathcal{S}$ , $\mathcal{V}^h \subset \mathcal{V}$.

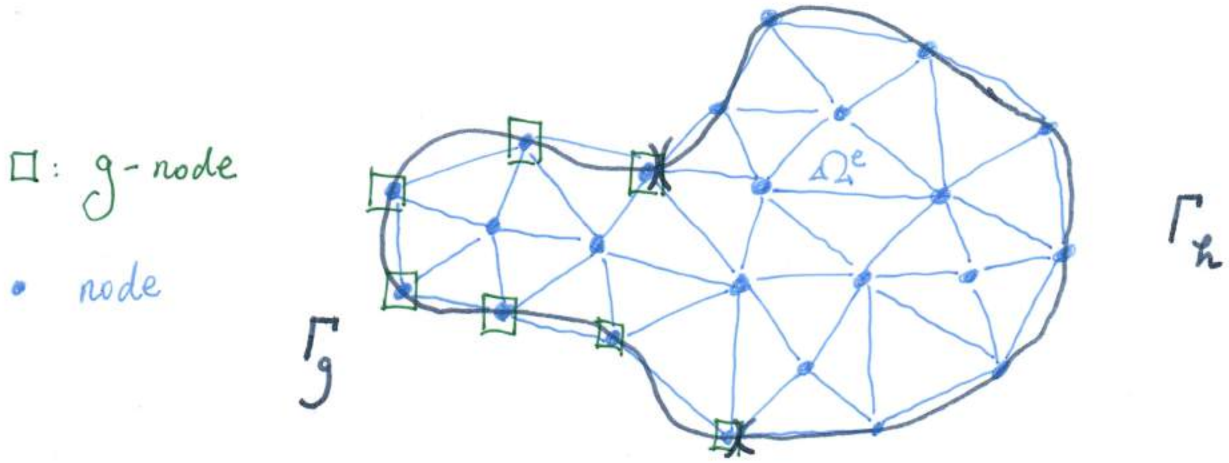
We assume $u^h \in \mathcal{S}^h$ adopts the decomposition,

$$u^h = \underset{\underset{\mathcal{V}^h}{\cap}}{v^h} + g^h$$

and $g^h$ satisfy (or approximately satisfy)

$u = g$ on $\Gamma_g$

$$(G)\begin{cases} \text{Given } f,\ g,\ h,\ \text{find } u^h = v^h + g^h \text{ such that} \\ \text{for } \forall\, w^h \in \mathcal{V}^h, \\ \qquad a(w^h, v^h) = (w^h, f) + (w^h, h)_\Gamma \\ \qquad\qquad\qquad - a(w^h, g^h) \end{cases}$$




$\bigcup_{e=1}^{n_{el}} \Omega^e$ approximates $\Omega$ ; $\eta = \{1, 2, \cdots, n_{np}\}$

$\hookrightarrow$ set of global node numbers

g-node : node at which $u^h = g$ is prescribed.

$\eta_g$ : set of $g$-node.

complement of $\eta_g$ in $\eta$ : $\eta - \eta_g$ is the set of nodes at which $u^h$ is to be determined.

$$\boxed{\dim(\eta - \eta_g) = n_{eq}}$$

$$w^h(x) = \sum_{A \in \eta - \eta_g} N_A(x)\, c_A$$

Shape function associated with node A.

$$v^h(x) = \sum_{A \in \eta - \eta_g} N_A(x)\, d_A$$

unknown temperature at node A.

$$g^h(x) = \sum_{A \in \eta_g} N_A(x)\, g_A. \qquad g_A = g(x_A)$$

then we have

$$\sum_{B \in \eta - \eta_g} a(N_A, N_B)\, d_B = (N_A, f) + (N_A, h)_\Gamma - \sum_{B \in \eta_g} a(N_A, N_B)\, g_B$$

for $A \in \eta - \eta_g$.

Now, we invoke ID array :

$$ID(A) = \begin{cases} P & \text{if } A \in \eta - \eta_g \\ 0 & \text{if } A \in \eta_g \end{cases}$$

where $1 \leq P \leq n_{eq}$ .

$$\Rightarrow \quad K_{PQ} = a(N_A, N_B) \qquad P = ID(A), \; Q = ID(B)$$

$$F_P = (N_A, f) + (N_A, h)_{\Gamma} - a(N_A, g^h)$$

$$d_Q = d_B$$

$Q = ID(B)$

disp. vector

$$\boxed{(M) \qquad K d = F}$$

the matrix problem

Stiffness matrix

load vector / force vector.

and $$u^h_{(x)} = \sum_{B \in \eta - \eta_g} N_B^{(x)} d_B + g^h_{(x)}$$

Remark: In addition to the function space approximation (i.e., $\mathcal{V}^h \subset \mathcal{V}$, $\mathcal{S}^h \subset \mathcal{S}$ ),

there are geometrical approximation ( $\bigcup_{e=1}^{n_{el}} \Omega^e \approx \Omega$ )

and BC data approximation (i.e., $g^h(x) \approx g(x)$).

<u>Local / element perspective</u>.

$$K = \sum_{e=1}^{n_{el}} K^e = \sum_{e=1}^{n_{el}} \overbrace{\int_{\Omega^e} \nabla N_A^T \kappa \nabla N_B \, d\Omega}^{a(N_A,\, N_B)^e}$$

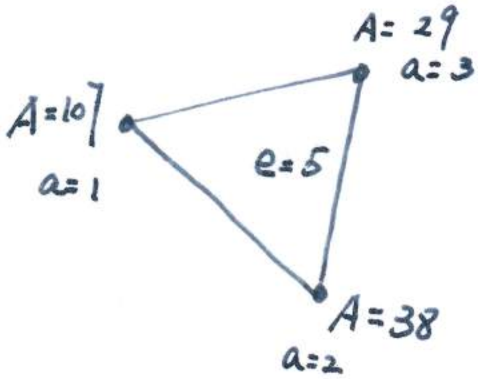
$$F = \sum_{e=1}^{n_{el}} F^e = \sum_{e=1}^{n_{el}} \left\{ \int_{\Omega^e} N_A f \, d\Omega + \int_{\Gamma^e \cap \Gamma_h} N_A h \, d\Gamma - a(N_A, N_B)^e g_B \right\} \qquad \Gamma_h^e$$

element stiffness & element force vector

$$k^e = [k^e_{ab}] \qquad k^e_{ab} = \int_{\Omega^e} \nabla N_a^T \kappa \nabla N_b \, d\Omega$$

$$f^e = [f^e_a] \qquad f^e_a = \int_{\Omega^e} N_a f \, d\Omega + \int_{\Gamma_h^e} N_a h \, d\Gamma - \sum_{b=1}^{n_{en}} k^e_{ab} g^e_b$$

$1 \leqslant a, b \leqslant n_{en}$




local node number

$$A = IEN(a, e)$$

global node number

element number

Again, $LM(a,e) = ID(IEN(a,e))$.

Assembly: $K = \mathop{\large A}_{e=1}^{n_{el}} k^e \qquad F = \mathop{\large A}_{e=1}^{n_{el}} f^e$.