What's Decidable about Discrete Linear Dynamical Systems?

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Abstract. We survey the state of the art on the algorithmic analysis of discrete linear dynamical systems, and outline a number of research directions.

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1 Introduction

Dynamical systems are a fundamental modelling paradigm in many branches of science, and have been the subject of extensive research for many decades. A *(discrete) linear dynamical system (LDS)* in ambient space $\mathbb{R}^d$ is given by a square $d \times d$ matrix M with rational entries, together with a starting point $x \in \mathbb{Q}^d$. The *orbit* of (M, x) is the infinite trajectory $\mathcal{O}(M, x) := \langle x, Mx, M^2x, \ldots \rangle$. An example of a four-dimensional LDS is given in Figure 1. Our main focus in the present paper is on delineating the class of assertions on the orbits of LDS that can be algorithmically decided.

$$x \stackrel{\mathsf{def}}{=} \begin{pmatrix} 1 \\ -1 \\ 2 \\ 0 \end{pmatrix} \qquad M \stackrel{\mathsf{def}}{=} \begin{pmatrix} 3 & 2 & 0 & -5 \\ 0 & 1 & 0 & 3 \\ 0 & 4 & 3 & 13 \\ 3 & 11 & 6 & 24 \end{pmatrix}$$

Fig. 1. A four-dimensional discrete linear dynamical system.

One of the most natural and fundamental computational questions concerning linear dynamical systems is the *Point-to-Point Reachability Problem*, also known as the *Kannan-Lipton Orbit Problem*: given a d-dimensional LDS (M, x) together with a point target $y \in \mathbb{Q}^d$, does the orbit of the LDS ever hit the target? This question was answered affirmatively in the 1980s in the seminal work of Kannan and Lipton [29,30]. In fact, Kannan and Lipton showed that this problem is solvable in polynomial time, answering an earlier open problem of Harrison from the 1960s on reachability for linear sequential machines [27].

Interestingly, a secondary motivation of Kannan and Lipton was to propose an approach to attack the well-known *Skolem Problem*, which had itself been famously open since the 1930s (and remains unsolved to this day); phrased in the language of linear dynamical systems, the Skolem Problem asks whether it is decidable, given (M, x) as above, together with a $(d-1)$-dimensional subspace H of $\mathbb{R}^d$, to determine if the orbit of (M, x) ever hits H. This problem is known to be decidable in dimensions $d \leq 4$, and is otherwise open—for a more detailed discussion on the topic, we refer the reader to [39]. Kannan and Lipton suggested that, in ambient space $\mathbb{R}^d$ of arbitrary dimension, the problem of hitting a low-dimensional subspace might be decidable. Indeed, this was eventually substantiated by Chonev *et al.* for linear subspaces of dimension at most 3 [18,20]. Nevertheless, the connection to longstanding and deep open mathematical problems (such as the Skolem Problem) suggests that a general algorithmic theory of linear dynamical systems is likely to be quite complex and subtle, assuming it can be achieved at all!

Subsequent research focussed on the decidability of hitting targets of increasing complexity, such as half-spaces [26,33,37,36,38], polytopes [42,19,5], and semialgebraic sets [6,7]. It is also worth noting that discrete linear dynamical systems can equivalently be viewed as linear (or affine) simple, branching-free while loops, where reachability corresponds to loop termination. There is a voluminous literature on the topic, albeit largely focussing on heuristics and semi-algorithms (via spectral methods or the synthesis of ranking functions), rather than exact decidability results. Relevant papers include [43,16,11,12,15,17,21,40,41,28,13,10]. Several of these approaches have moreover been implemented in software verification tools, such as Microsoft's Terminator [23,24].

In recent years, motivated in part by verification problems for stochastic systems and linear loops, researchers have begun investigating more sophisticated specification formalisms than mere reachability: for example, the paper [1] studies approximate LTL model checking of Markov chains (which themselves can be viewed as particular kinds of linear dynamical systems), whereas [32] focuses on LTL model checking of low-dimensional linear dynamical systems with semialgebraic predicates.[3] In [4], the authors solve the semialgebraic model-checking problem for diagonalisable linear dynamical systems in arbitrary dimension against prefix-independent MSO[4] properties, whereas [31] investigates semialgebraic MSO model checking of linear dynamical systems in which the dimensions of predicates are constrained. To illustrate this last approach, recall the dynamical system (M, x) from Figure 1, and consider the following three

[3] Semialgebraic predicates are Boolean combinations of polynomial equalities and inequalities.

[4] Monadic Second-Order Logic (MSO) is a highly expressive specification formalism that subsumes the vast majority of temporal logics employed in the field of automated verification, such as Linear Temporal Logic (LTL).

semialgebraic predicates:

$$P_1(x_1, x_2, x_3, x_4) \stackrel{\text{def}}{=} x_1 + x_2 + x_3 - x_4 = 0 \wedge (x_1^3 = x_2^2 \vee x_4 \geq 3x_1^2 + x_2)$$
$$P_2(x_1, x_2, x_3, x_4) \stackrel{\text{def}}{=} x_1 + x_2 + 2x_3 - 2x_4 = 0 \wedge x_1^3 + x_3^2 + x_3 > x_4$$
$$P_3(x_1, x_2, x_3, x_4) \stackrel{\text{def}}{=} x_1^4 - x_2^2 = 3 \wedge 2x_3^2 = x_4 \wedge x_1^2 - 2x_2^3 = 4x_3 \,.$$

Recall that the ambient space is $\mathbb{R}^4$. We identify the above predicates with the corresponding subsets of $\mathbb{R}^4$, and wish to express assertions about the orbit of (M, x) as it traces a trajectory through $\mathbb{R}^4$. For example (in LTL notation),

$$\mathbf{G}(P_1 \Rightarrow \mathbf{F}\neg P_2) \wedge \mathbf{F}(P_3 \vee \neg P_1)\,,$$

asserts that whenever the orbit visits P_1, then it must eventually subsequently visit the complement of P_2, and moreover that the orbit will eventually either visits P_3 or the complement of P_1. The reader will probably agree that whether or not the above assertion holds for our LDS (M, x) is not immediately obvious to determine (even, arguably, in principle). Nevertheless, this example falls within the scope of [31], as the semialgebraic predicates P_1, P_2, and P_3 are *admissible*, i.e., they are each either contained in some three-dimensional subspace (this is the case for P_1 and P_2), or have intrinsic dimension at most 1 (this is the case of P_3, which is 'string-like', or a curve, as a subset of $\mathbb{R}^4$). Naturally, we shall return to these notions in due course, and articulate the relevant results in full details.

The rest of the paper is organised as follows. In Section 2, we present a summary of the state of the art as regards reachability and model checking for discrete linear dynamical systems from a singleton starting set. We paint what is essentially a complete picture of the landscape, in each situation either establishing decidability, or hardness with respect to longstanding open mathematical problems. Section 3 examines questions of robustness through the notion of pseudo-orbit. In Section 4, we discuss the algorithmic synthesis of inductive invariants for linear dynamical systems, and Section 5 examines the situation in which orbits originate from an initial set rather than a single point. Finally, Section 6 concludes with a brief summary and a glimpse of several research directions.

2 Model Checking

Given an LDS (M, x) and an MSO formula φ over semialgebraic predicates $\mathcal{T} = \{T_1, \ldots, T_m\}$, the model-checking problem is to determine whether the orbit (more precisely, the characteristic word $\alpha \in \left(2^{\mathcal{T}}\right)^{\omega}$ of the orbit $\mathcal{O}(M, x)$ with respect to $\mathcal{T}$) satisfies φ. Reachability problems for LDS constitute special cases of the model-checking problem, and already the questions of determining whether a given orbit reaches a hyperplane (Skolem Problem) or a halfspace (the *Positivity Problem* [35]) are longstanding open problems in number theory couched in the parlance of linear dynamical systems. Recent research has, however, succeeded in uncovering several important decidable subclasses of the model-checking problem and demarcating the boundary between what is decidable and what is hard with respect to open mathematical problems.

Recall that a semialgebraic target $T \subseteq \mathbb{R}^d$ is called *admissible* if it is either contained in a three-dimensional subspace of $\mathbb{R}^d$, or has intrinsic dimension at most one.[5] The focus on target sets of this type has origins in the results of [18,32,6,9]. A common theme is that for admissible targets, the proofs that establish how to decide reachability also provide us with a means of representing, in a finite manner, all the time steps at which the orbit is in a particular target set T. The authors of [31] show how to combine these representations (one for each target) to obtain structural information about the characteristic word α that is sufficient for determining whether a deterministic automaton $\mathcal{A}$ accepts α, which leads to the following.

Theorem 1. *Let (M, x) be an LDS, $\mathcal{T} = \{T_1, \ldots, T_m\}$ be a set of admissible predicates and φ be an MSO formula. It is decidable whether the characteristic word α of the orbit $\mathcal{O}(M, x)$ with respect to $\mathcal{T}$ satisfies φ.*

It is worth noting that his delineation of the decidable fragment of the model-checking problem is tight as trying to expand the definition of admissible predicates for either of the two types of targets runs into open problems already for φ that describe mere reachability properties. In particular, the Skolem Problem in dimension 5 is open and can be encoded (i) as a reachability problem with a four-dimensional LDS and a three-dimensional affine subspace [18] (that is, in general, not contained in a three-dimensional linear subspace) and (ii) as a reachability problem with a target of intrinsic dimension two [9].

To sidestep these obstacles, in [4] the authors restrict φ to formulas that define *prefix-independent* properties. Similarly to liveness specifications, a property is prefix-independent if the infinite words that satisfy it are closed under the operations of insertion and deletion of finitely many letters. Such properties can be used to describe asymptotic behaviours ("does the orbit enter T_1 infinitely often?") but not reachability. The main theorem of [4] in this direction is the following.

Theorem 2. *Let (M, x) be an LDS with diagonalisable M and φ be a prefix-independent MSO formula over $\mathcal{T}$. It is decidable whether the characteristic word α of the orbit $\mathcal{O}(M, x)$ satisfies φ.*

Note in the above that the semialgebraic sets are entirely unrestricted (in particular, not required to be admissible). However, the restrictions to prefix-independent formulas and diagonalisable systems both again turn out to be crucial. Since the Skolem Problem is open for diagonalisable systems (in dimensions $d \geq 5$), the model-checking problem for diagonalisable LDS is Skolem-hard

[5] The intrinsic dimension of a semialgebraic set is formally defined via cell decomposition; intuitively, one-dimensional semialgebraic sets can be viewed as 'strings' or 'curves', whereas zero-dimensional semialgebraic sets are finite collections of singleton points.

already for four-dimensional systems and affine subspace targets, as discussed earlier. On the other hand, if we allow non-diagonalisable systems, then the problem of determining whether the orbit of an LDS is eventually trapped in a semialgebraic target T (known as the *Ultimate Positivity Problem*, corresponding to the prefix-independent formula $\varphi = \mathbf{F}\,\mathbf{G}\,T$) is hard with respect to certain longstanding open problems in Diophantine approximation [35].

Taken together, Theorems 1 and 2 therefore not only subsume—to the best of our knowledge—all existing results regarding model-checking and reachability problems for discrete linear dynamical systems, but moreover paint an essentially complete picture of what is (even in principle) feasible, barring major breakthroughs in mathematics.

3 Pseudo-Reachability and Robustness

In this section we discuss decision problems about pseudo-orbits that are related to robustness of computation. Given a LDS (M, x), recall that the orbit of x under M is the sequence $\langle x, Mx, M^2x, \dots\rangle$. We say that the sequence $\langle x_n : n \in \mathbb{N}\rangle$ is an *ε-pseudo-orbit* of x under M if $x_0 = x$ and $x_{n+1} = Mx_n + d_n$ for some perturbation d_n with $||d_n|| < \epsilon$. The *pseudo-orbit* of x under M is then defined as the set of points that are reachable from x via an ε-pseudo-orbit for every $\epsilon > 0$. This notion of an (ε-)pseudo-orbit, introduced by Anosov [8], Bowen [14] and Conley [22], is an important conceptual tool in dynamical systems. From the computational perspective, an ε-pseudo-orbit can be viewed as a trajectory after a rounding error of magnitude at most ϵ is applied at each step.

Given these definitions, we can consider the reachability and model-checking problems for pseudo-orbits. A natural analogy to the Kannan-Lipton Orbit Problem is the *Pseudo-Orbit Problem*, which is to determine whether a target point y belongs to the pseudo-orbit of x under M. In [25] the authors show that, just like the Orbit Problem, the Pseudo-Orbit Problem is decidable in polynomial time. Generalising from points to sets, let us say that a target set T is pseudo-reachable if for every $\epsilon > 0$ there exists an ε-pseudo-orbit of x under M that reaches T. We can then define the *Pseudo-Skolem Problem* and the *Pseudo-Positivity Problem* to be the pseudo-reachability problems with a hyperplane and a halfspace as target sets, respectively. Surprisingly, [25] shows that both of these problems are in fact decidable!

Inspired by the decidability of the Pseudo-Skolem and the Pseudo-Positivity problems, we can consider the model-checking problem for pseudo-orbits, namely the problem of determining, given (M, x) and a formula φ, whether for every $\varepsilon > 0$, there exists an ε-pseudo-orbit that satisfies φ. After all, as discussed in the preceding section, for genuine orbits the fragments of the reachability problem and the full MSO model-checking problem that are known to be decidable (i.e., the restrictions on the targets $\mathcal{T}$ and the property φ that make the problems decidable) are essentially the same. This optimism is, however, quickly shattered by the following observation. Let H be a closed halfspace and φ be the property $\mathbf{G}\,H$ ("the trajectory always remains inside H"). Then the pseudo-orbits satisfy φ (in the sense defined above) if and only the (genuine) orbit satisfies φ. The problem of determining whether the orbit $\mathcal{O}(M,x)$ always remains in H, is however, equivalent to the problem of determining whether the orbit ever hits an open halfspace, which itself is the open Positivity Problem.

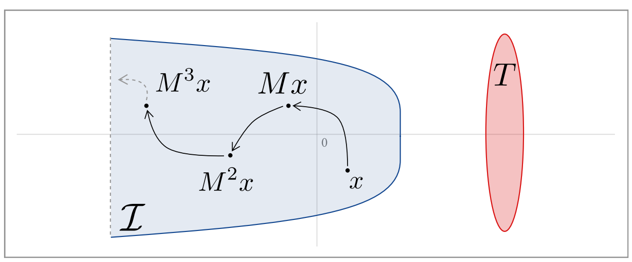
4 Invariant Generation

In the absence of fully general algorithms to decide whether the orbit of a given LDS reaches targets of arbitrary forms, much effort has been expended on over-approximation problems, particularly in constructing certificates of (non-)reachability. This splits into two broad lines of attacks: ranking functions and invariants. The former are certificates of reachability, demonstrating that progress is being made towards the target. Inductive invariants are, on the other hand, certificates of non-reachability, they demonstrate that the orbit will not reach the target by enclosing the orbit within a set that is itself disjoint from the target. We focus on the algorithmic generation of invariants in this section.

To be precise, a set $\mathcal{I} \subseteq \mathbb{R}^d$ is said to be an *inductive invariant* of (M,x) if it contains x ($x \in \mathcal{I}$), and is stable under M, that is:

$$M\mathcal{I} \stackrel{\text{def}}{=} \{My \ : \ y \in \mathcal{I}\} \subseteq \mathcal{I}.$$

Clearly there are some trivial invariants, such as $\mathbb{R}^d$ and the orbit $\mathcal{O}(M,x)$ itself. They are not particularly useful in the sense that the ambient space $\mathbb{R}^d$ is never disjoint from whatever target is under consideration, whereas for various classes of targets (such as hyperplanes or half-spaces; or more generally arbitrary semialgebraic sets) we do not in general know how to decide whether $\mathcal{O}(M,x)$ is disjoint from the target. Hence one does not seek *any* invariant, but rather an invariant that can be algorithmically established to be disjoint from the target.



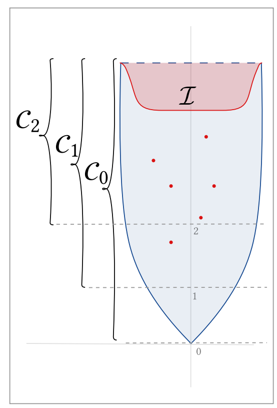
We therefore seek a sufficiently large, or expressive, class of invariants $\mathcal{F}$ which moreover exhibits favourable algorithmic properties. A natural family to consider is the collection of all semialgebraic sets, which can be defined in the first-order theory of real numbers, or equivalently as Boolean combination of polynomial inequalities. We now have:

Theorem 3 ([3]). *Given an LDS (M,x) in ambient space $\mathbb{R}^d$, together with a semialgebraic target $T \subseteq \mathbb{R}^d$, it is decidable whether there exists a semialgebraic invariant of (M,x) that is disjoint from T.*

Furthermore, the algorithm explicitly constructs the invariant when it exists, in the form of a Boolean combination of polynomial inequalities.

Theorem 3 holds for an even larger class $\mathcal{F}$, namely that of o-minimal sets. We give an informal definition. Recall the contents of Tarski's quantifier-elimination theorem, to the effect that semialgebraic subsets of $\mathbb{R}^d$ are closed under projections. Moreover, semialgebraic subsets of $\mathbb{R}$ are quite simple: they are finite unions of intervals and singletons. Other families of sets that enjoy these crucial properties exist, notably those definable in the first-order theory of the real numbers augmented with a symbol for the exponential function, an important result due to Wilkie [45]. Structures of $\mathbb{R}^d$ that are induced by such logical theories are called *o-minimal*, and an *o-minimal set* is a set that belongs to such a structure [44]. These include semialgebraic sets, as well as sets definable in the first-order theory of the reals with restricted analytic functions.

In [2], it is shown that it is decidable whether there exists an o-minimal invariant (for a given LDS (M, x)) that is disjoint from a semialgebraic target T; and moreover, when such an invariant exists, it is always possible to exhibit one that is in fact semialgebraic [3]. Once again, these results are effective, and the invariants can always be explicitly produced.



Varying the class of invariants and the class of targets gives rise to a number of natural questions that—for the most part—remain unexplored. Let us however mention a further desirable property enjoyed both by the class of o-minimal sets and that of semialgebraic sets: in either case, they admit *minimal families* of invariants, a notion which we now explain. Let $\mathcal{F}$ be a class of sets—either the class of o-minimal sets or that of semialgebraic sets. Note that in general, $\mathcal{F}$ does not possess minimal invariants: indeed, let (M, x) be an LDS whose orbit is (countably) infinite, and consider any invariant $\mathcal{I} \in \mathcal{F}$ for (M, x). Since $\mathcal{I}$ will necessarily be uncountably infinite, one can always remove finitely many points from $\mathcal{I}$ in such a way that the resulting set remains an invariant of (M, x). However, [2,3] show how to produce a sequence of (M, x)-invariants $\langle \mathcal{C}_k : k \in \mathbb{N} \rangle$, all belonging to $\mathcal{F}$, and such that $\mathcal{C}_{k+1} \subset \mathcal{C}_k$. It can moreover be shown that, given any (M, x)-invariant $\mathcal{I} \in \mathcal{F}$, it is always the case that $\mathcal{I}$ contains one of the $\mathcal{C}_k$, except for finitely many points (depicted in red on the picture) corresponding to some initial segment of the orbit of (M, x).

5 Semialgebraic Initial Sets

Up until now we have exclusively considered problems concerning the orbit $\mathcal{O}(M, x)$ of a *single* initial point x. It is natural to ask whether the algorithmic problems which we have discussed remain solvable if one instead considers an entire *set* of initial points $S \subseteq \mathbb{R}^d$. Unfortunately the answer is negative.

We sketch below the proof of the undecidability of a natural model-checking problem.

Theorem 4. *The following problem is undecidable. Given a natural number $k \in \mathbb{N}$, a semialgebraic set $S \subseteq \mathbb{R}^d$, a $d \times d$ rational matrix M, and a hyperplane H in $\mathbb{R}^d$ (having rational normal vector), determine whether there exists $x \in S$ such that the orbit generated by (M, x) hits H at least k times.*

In symbols, whether there is some $x \in S$ such that

$$|\mathcal{O}(M,x) \cap H| \geq t$$

is undecidable.

It is worth noting that all the problems that we have discussed so far (including the Skolem and Positivity Problems) are not known to be undecidable, and are in fact conjectured to be decidable. It is therefore perhaps somewhat surprising that this natural generalisation of our setting immediately leads to undecidability.

The proof of Theorem 4 proceeds by reduction from a variant of Hilbert's tenth problem. Recall that Hilbert's tenth problem asks whether a given polynomial $P \in \mathbb{Z}[Y_1, Y_2, \dots, Y_{d-1}]$ with integer coefficients and $d-1$ variables has a root with all unknowns taking integer values. It is undecidable, as shown by Davis, Putnam, Robinson, and Matiyasevich [34].

The variant that we will reduce from asks whether the polynomial has roots with the unknowns being distinct natural numbers. It is fairly straightforward to show that this variant is also undecidable.

Let $d \in \mathbb{N}$, $d > 1$, and $P \in \mathbb{Z}[Y_1, Y_2, \dots, Y_{d-1}]$ be an arbitrary polynomial. We define the subset $S \subseteq \mathbb{R}^d$ via a formula of the first-order theory of the reals:

$$S(x_1, x_2, \dots, x_d) \overset{\text{def}}{=} \exists y_1, y_2, \dots y_{d-1} \begin{cases} 0 & = P(y_1, y_2, \dots, y_{d-1}), \\ x_1 & = (1-y_1)(1-y_2)\cdots(1-y_{d-1}), \\ x_2 & = (2-y_1)(2-y_2)\cdots(2-y_{d-1}), \\ & \vdots \\ x_d & = (d-y_1)(d-y_2)\cdots(d-y_{d-1}). \end{cases}$$

A point $x := (x_1, \dots, x_d) \in \mathbb{R}^d$ is in the set S if and only if one can find real numbers $y_1, \dots, y_{d-1}$ for which the above equations hold. The idea behind this definition comes from the fact that one can construct a $d \times d$ matrix M with rational entries such that

$$\left(M^n x\right)_1 = (n-y_1)(n-y_2)\cdots(n-y_{d-1}),$$

for all $n \in \mathbb{N}$, where $(\cdot)_1$ refers to the first entry of the vector. Admitting the existence of such a matrix, let H be the hyperplane having normal vector $(1, 0, \dots, 0)$ and going through the origin. Then clearly $\mathcal{O}(M, x)$ enters H at least $d-1$ times if and only if the reals $y_1, \dots, y_{d-1}$ are distinct natural numbers, because only then is the first entry of $M^n x$—the polynomial $(n - y_1) \cdots (n - y_{d-1})$—equal to zero.

The existence of the matrix M rests on the fact that the expression $u_n = (n - y_1) \cdots (n - y_{d-1})$ (for fixed $y_1, \ldots, y_{d-1}$) can be obtained as a linear recurrence sequence of order d, and in turn such a linear recurrence sequence can be represented as the sequence of fixed-position entries of increasing powers of a fixed $d \times d$ matrix M. In the standard construction of this matricial representation, one must in addition set $x_1 = u_1$, $x_2 = u_2$, $\ldots$, $x_d = u_d$, which is achieved through the definition of our initial semialgebraic set S.

It is worth noting that Theorem 4 holds even if k is fixed, due to the fact that Hilbert's tenth problem remains undecidable for a fixed number of variables. Furthermore, if k is fixed to be 1, then the problem becomes decidable in low dimensions, however even in the case where the ambient space has dimension 2 and $k = 2$, the problem does not seem to be trivially decidable.

6 Research Directions and Open Problems

We have presented an overview of the state of the art regarding decidability and solvability of a range of algorithmic problems for discrete linear dynamical systems, focussing on reachability, model-checking, and invariant-generation problems. Regarding model checking in particular, we have painted an essentially complete picture of what is achievable even in principle, in the sense that extending the existing results further runs up against formidable mathematical obstacles (longstanding open problems in number theory); the results presented therefore appear to lie at the very frontier of what is achievable, barring major breakthroughs in mathematics. From a technical standpoints, the tools deployed offer a pleasant mixture of number theory and mathematical logic.

Further interesting research directions nevertheless remain, particularly along the lines of parametric synthesis and analysis (see, e.g., [9]), continuous and hybrid systems, control-theoretic problems, and non-linear dynamics.

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