Algorithmic Verification of Linear Dynamical Systems

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A decision problem is decidable: there exists an algorithm that halts on all inputs and outputs the correct answer

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Linear dynamical systems

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Verification of LDS

Find algorithms that

1. take (M, s) and a property φ, and
2. decide whether $\langle s, Ms, M^2 s, M^3 s, \ldots \rangle$ satisfies φ

Example of LDS: linear loops

initialise x_1, x_2

while $\neg \mathrm{P}(x_1, x_2)$**:**

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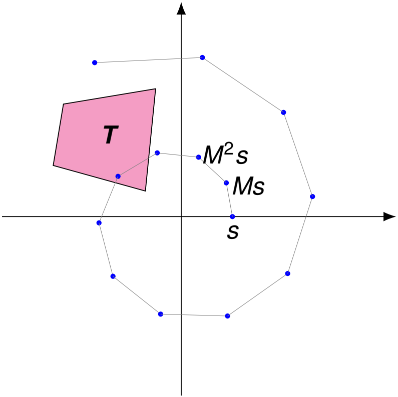

$\boldsymbol{T} = \{(x_1, x_2) \colon P(x_1, x_2)\}$, s is the initial value of (x_1, x_2)

$(x_1, x_2) = M^n s$ after n iterations

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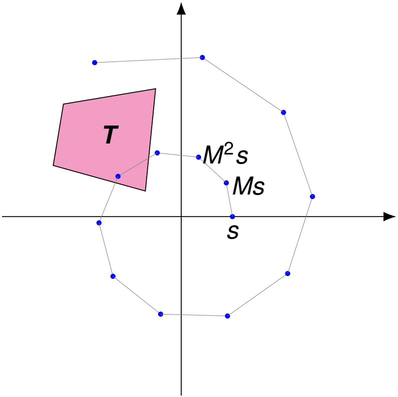

$\boxed{\text{Loop terminates} \Leftrightarrow \langle s, Ms, M^2 s, \ldots \rangle \text{ reaches } \boldsymbol{T}}$

Linear loops, cont'd

Termination problem for linear loops $\equiv$

Reachability Problem

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Branching in the loop update $\Rightarrow$ termination undecidable

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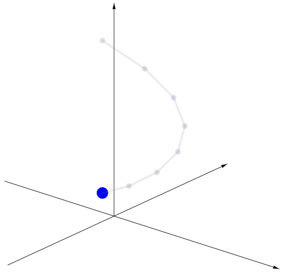

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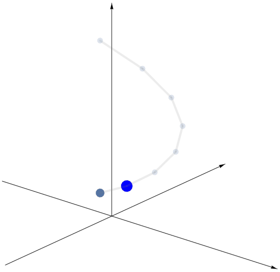

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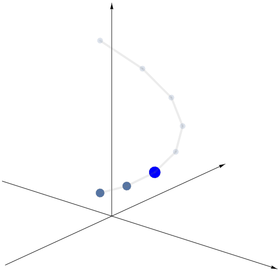

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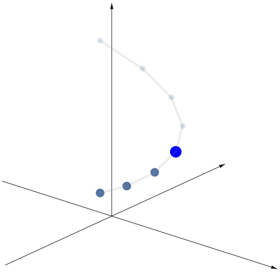

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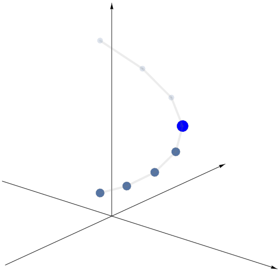

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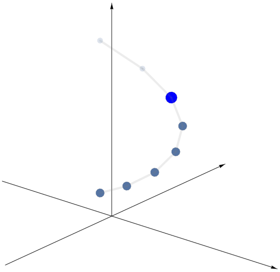

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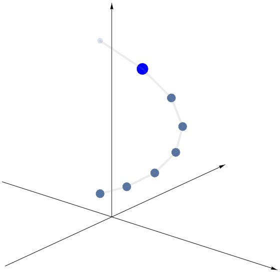

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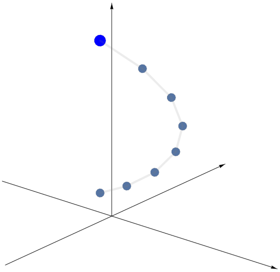

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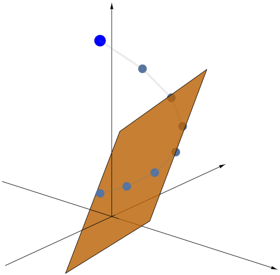

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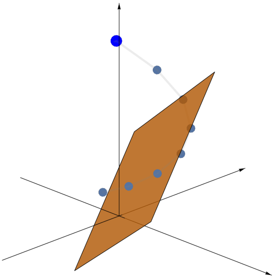

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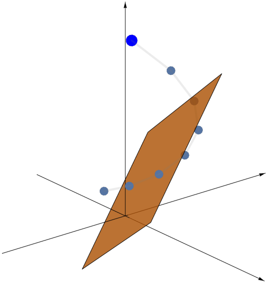

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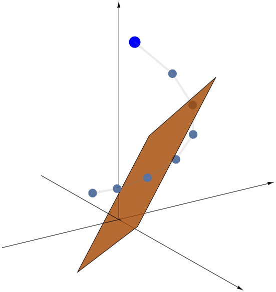

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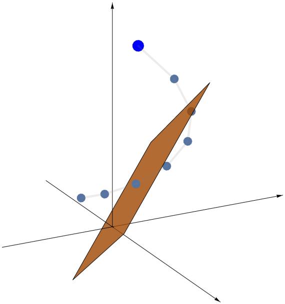

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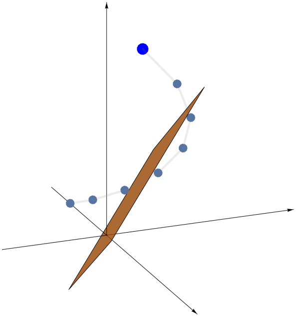

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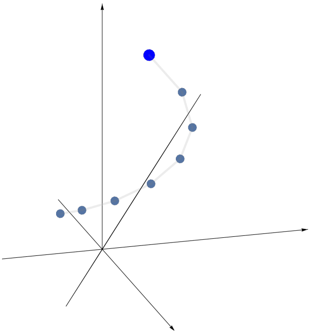

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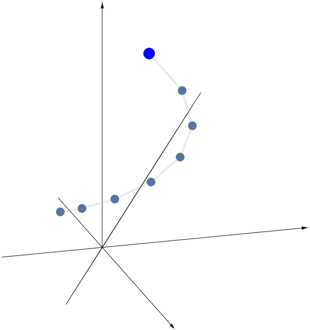
Decidable in dimension $d \leq 4$, famously open in dimension $d = 5$



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Examples: $u_n = 3u_{n-1} + u_{n-2} - 2u_{n-3}$ and

$$u_n = u_{n-1} + u_{n-2} = \frac{1}{\sqrt{5}}\left(\frac{1+\sqrt{5}}{2}\right)^n + \frac{1}{\sqrt{5}}\left(\frac{1-\sqrt{5}}{2}\right)^n$$

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Skolem Problem, LRS version

Given an LRS $(u_n)_{n\in\mathbb{N}}$, decide whether $u_n = 0$ for some n

Our contributions

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Call a set T *tame* if it can be obtained through finitely many set operations from semialgebraic sets that either (i) have intrinsic dimension 1, or (ii) are contained in a 3-dimensional subspace

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Theorem

The Reachability Problem is decidable for tame T

Uses classical number-theoretic tools: Baker's theorem, its p-adic version, heights of algebraic numbers

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Model-Checking Problem

Given $M \in \mathbb{Q}^{d \times d}$, $s \in \mathbb{Q}^d$, semialgebraic $T_1, \ldots, T_m$ and an ω-regular property φ over $T_1, \ldots, T_m$, decide whether the orbit $\langle s, Ms, M^2 s, \ldots \rangle$ satisfies φ

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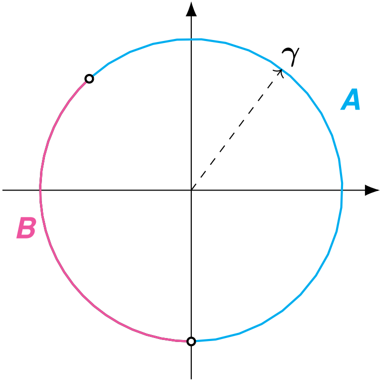
Theorem

The Model-Checking Problem is decidable for tame targets

Main idea: the sequence $(\mathbb{1}(M^n s \in T))_n$ is *toric*

Toric words

A toric word is the coding of an orbit of a rotation on a k-dimensional torus with respect to open sets



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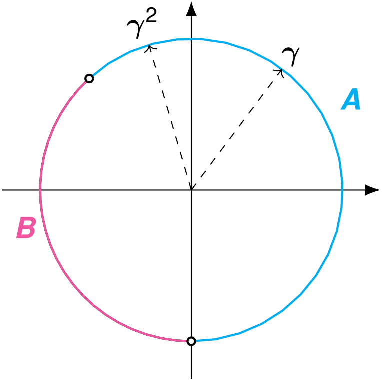

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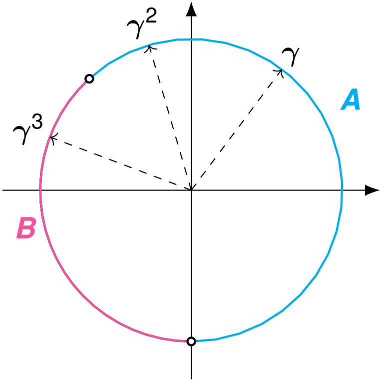

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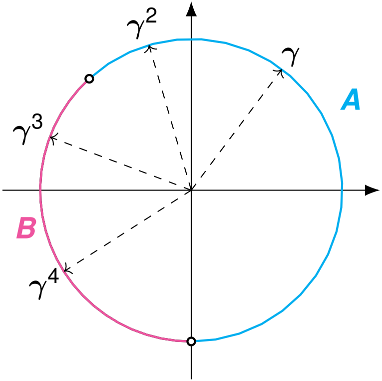

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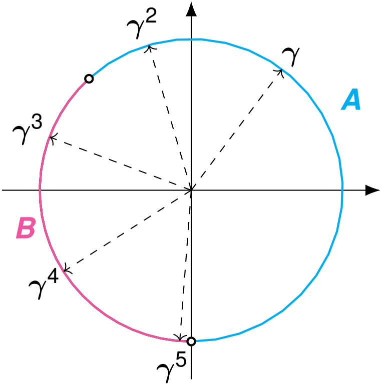

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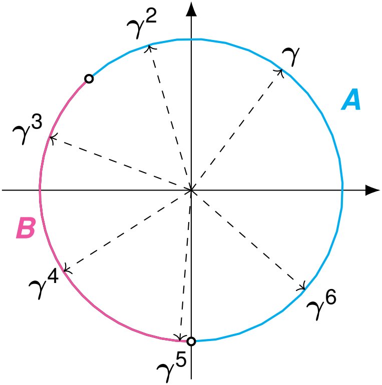

$A\ A\ B\ B\ B\ A \cdots$

Automaton on toric words

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Theorem

Given a toric word u and an automaton $\mathcal{A}$, it is decidable whether $\mathcal{A}$ accepts u

Boundaries of decidability

Theorem

For even the simplest non-tame targets, (un)decidability of the Reachability Problem would entail major breakthroughs either in Diophantine approximation, or regarding the Skolem Problem

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Hartmanis-Stearns Conjecture

Any number whose nth binary digit can be computed in time $O(n)$ must be either rational or transcendental

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> "Wir müssen wissen, wir werden wissen."
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> — David Hilbert, 1930

Verification of LDS via o-minimality

O-minimality of $\langle \mathbb{R}; <, +, \cdot, \exp(\cdot) \rangle \quad \Rightarrow \quad$ every first-order definable subset of $\mathbb{R}^k$ has finitely many connected components

Robust Safety for LDS is decidable

Given (M, s) and T, it is decidable whether there exists $\varepsilon > 0$ such that $(M^n \cdot B(s, \varepsilon))_n$ avoids T