From Dynamical Systems to Decidability of Logical Theories

Toghrul Karimov

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(I work with Valérie Berthé and Florian Luca)

First-order logical theories

Mathematical structure:

$$\mathcal{S} = \langle \underbrace{\mathbb{Z}}_{\text{domain}} ; \underbrace{0, 1}_{\text{constants}} , \underbrace{+}_{\text{functions}} , \underbrace{<, \text{isAPowerOfTwo}}_{\text{predicates}} \rangle$$

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Predicates of one variable $\equiv$ subsets of $\mathbb{N}$:

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The first-order theory of $\mathcal{S}$: the set of all sentences that are 1) true in $\mathcal{S}$ and 2) constructed using the constants, functions, predicates, quantifiers $\forall, \exists$ over the elements of the domain, logical operations $\neg, \wedge, \vee$, and equality

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- $\forall x \, \exists y : \big(y > x \wedge y \in 2^{\mathbb{N}} \wedge \exists z : y = z + z + z + 1\big)$ ✓

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The first-order theory of $\mathcal{S}$ is *decidable* if there exists an algorithm that takes a sentence and decides whether it belongs to the theory

Linear numeration systems and FO decidability

Theorem

Let U be a Pisot numeration system with an irreducible characteristic polynomial. The structure $\langle \mathbb{N}; 0, 1, <, +, U \rangle$ is automatic and has a decidable first-order theory.

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Proof. Every sentence can be equivalently written as

$$Qx_1 \cdots Qx_m \colon \varphi(x_1, \ldots, x_m)$$

where $Q_i \in \{\exists, \forall\}$ and φ is a Boolean combination of $x_i \in U$, $x_i \notin U$, $x_i = \text{constant}$, $x_i < x_j$, and $x_i = x_j + x_k$.

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Theorem (Semënov 1979)

Let U be a Perron numeration system with an irreducible characteristic polynomial. The structure $\langle \mathbb{N}; 0, 1, <, +, U \rangle$ has a decidable first-order theory.

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The proof is by quantifier elimination. For non-Perron U and Perron U with a reducible characteristic polynomial, decidability remains a major open problem.

How to prove undecidability of the FO theory?

Theorem (Gödel 1931)

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- $\mathcal{S} = \langle \mathbb{N}; 0, 1, <, +, \{n^2 \colon n \geq 0\} \rangle$. We have that $a \cdot b = c$ if and only if there exist pairs of consecutive squares $(A, A'), (B, B'), (C, C')$ such that

$$A - A' = 2a + 1 \ \wedge \ B - B' = 2b + 1 \ \wedge \ C - C' = 2c + 1 \ \wedge \ C' - A' - B' = 2c.$$

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- $\mathcal{S} = \langle \mathbb{N}; 0, 1, <, +, V_2, V_3 \rangle$ where $V_p(a, b) \Leftrightarrow b = p^{v_p(a)}$.

A modern method for proving undecidability

Theorem (Schulz, 2022)

The structure $\langle \mathbb{N}; 0, 1, <, +, 2^{\mathbb{N}}, 3^{\mathbb{N}} \rangle$ **does not** define multiplication

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Step 1: construct a map from $3^{\mathbb{N}} \times 3^{\mathbb{N}} \times \mathbb{N} \times \mathbb{N} \to \mathbb{N}^*$ that is onto

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Step 2: Assume the FO theory is decidable. Given a two-counter machine $\mathcal{M}$, write a formula Φ saying "there exist $3^a, 3^b, t, r$ such that the corresponding finite sequence is a halting run of $\mathcal{M}$." Check whether Φ is true.

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Given x, define $\alpha(x), \beta(x)$ by $2^{\alpha(x)} \leq x < 2^{\alpha(x)+1}$ and $\beta(x) = \alpha(x - 2^{\alpha}(x))$. E.g. $\langle 37 \rangle_2 = 100101$, $\alpha(37) = 5$, and $\beta(37) = 2$.

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Lemma

The map $(a, b, t, r) \to (\beta(3^n) - t)_{n=a}^{b} \cap [0, r]$ is onto $\mathbb{N}^*$

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Step 2: Given a two-counter machine $\mathcal{M}$, write a formula $\Phi \equiv$ "there exist $3^a, 3^b, t, r$ such that the corresponding sequence is a halting run of $\mathcal{M}$." Check whether Φ is true.

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A two-counter machine is a program with two variables c_1, c_2 and instructions

- $c_i = c_i + 1$,
- $\text{JUMP TO } \ell$,
- $\text{IF } c_i > 0 \text{ THEN } c_i = c_i - 1 \text{ ELSE JUMP TO } \ell$,
- HALT.

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The run of $\mathcal{M}$ is the instruction number and the values of c_1, c_2 after each step. It is undecidable whether a given two-counter machine has a finite run.

Powers of 2 and 3, bigger picture

Theorem (K., Luca, Nieuwveld, Ouaknine, Worrell 2025)

The existential fragment of the first-order theory of $\langle \mathbb{N}; 0, 1, <, +, 2^{\mathbb{N}}, 3^{\mathbb{N}} \rangle$ is decidable. The exists-forall-exists fragment of the full theory is undecidable.

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The $\exists$-fragment: solving systems of inequalities of the form

$$\exists x_1 \in D_1 \cdots \exists x_m \in D_m : A \cdot (x_1, \ldots, x_m) \sim \mathbf{b}$$

where D_i is one of $\mathbb{N}, 2^{\mathbb{N}}, 3^{\mathbb{N}}$, $A \in \mathbb{Z}^{k \times m}$, $\mathbf{b} \in \mathbb{Z}^{k \times 1}$, and $\sim$ consists of (in)equalities.

Proof: Baker's theorem + Diophantine approximation arguments

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Now consider $\langle \mathbb{N}; 0, 1, <, +, n \mapsto 2^n, n \mapsto 3^n \rangle$. In this structure we have access to numbers of the form $2^{2^n}, 2^{3^n}, 2^{2^n+3^n}$ etc.

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Problem

Is the first-order theory of $\langle \mathbb{N}; 0, 1, <, n \mapsto 2^n, n \mapsto 3^n \rangle$ decidable?

The Ramanujan τ function

The function $\tau\colon \mathbb{N} \to \mathbb{Z}$ is defined by the generating function $q\prod_{m=1}^{\infty}(1-q^m)^{24}$

It is multiplicative, and has the values 1, -24, 252, -1472, 4830, -6048, -16744, ...

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Theorem (Bilu, Deshouillers, Gun, Luca, 2018)

Assume $\tau(n) \neq 0$ for all $n \geq 1$. Then for any permutation $\sigma\colon \{1,\ldots,k\} \to \{1,\ldots,k\}$, there exist infinitely many m such that

$$|\tau(m+\sigma(1))| < \cdots < |\tau(m+\sigma(k))|.$$

$\Rightarrow$ we can extract arbitrary finite permutations from the τ function

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Corollary

The first-order theory of $\langle \mathbb{Z}; 0, 1, <, +, n \mapsto \tau(n) \rangle$ is undecidable.

Strong Diophantine approximation properties $\Rightarrow$ FO undecidability

Problem

Let U be a linear numeration system. Is $\langle \mathbb{N}; 0, 1, <, +, U \rangle$ automatic? Does it have a decidable first-order theory?

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Suppose U is defined by (the non-negative values of) a linear recurrence sequence with an irreducible characteristic polynomial and two non-real, non-repeated dominant roots.

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The prized structure

Consider $\langle \mathbb{N}; 0, 1, <, +, P \rangle$ where P is the set of primes.

Goldbach's conjecture: $\forall x \geq 2\colon (\exists y\colon x = y + y \Rightarrow \exists p, q \in P\colon x = p + q)$
Twin primes: $\forall x\, \exists y > x\colon y \in P \;\wedge\; y + 2 \in P$

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Dickson's conjecture

Let $a_i x_i + b_i$, $1 \leq i \leq m$ be such that no prime p divides $(a_1 x_1 + b_1) \cdots (a_m x_m + b_m)$ for all $x \in \mathbb{Z}$. Then there exist infinitely many $x \in \mathbb{Z}$ such that all $a_i x_i + b_i$ are prime.

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Theorem (Bateman, Jockusch, Woods 1993)

Assuming Dickson's conjecture, the first-order theory of $\langle \mathbb{N}; 0, 1, <, +, P \rangle$ is undecidable, but the existential fragment is decidable.

Summary

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Let U be a linear numeration system. Is $\langle \mathbb{N}; 0, 1, <, +, U \rangle$ automatic? Does it have a decidable first-order theory?

Theorem (Hieronymi and Schulz, 2022)

The first-order theory of $\langle \mathbb{N}; 0, 1, <, +, 2^{\mathbb{N}}, 3^{\mathbb{N}} \rangle$ is undecidable

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Is the first-order theory of $\langle \mathbb{N}; 0, 1, <, n \mapsto 2^n, n \mapsto 3^n \rangle$ decidable?

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How to solve $c_1 \cdot (2^{n_1}, \ldots, 2^{n_k}) + c_2 \cdot (3^{m_1}, \ldots, 3^{m_k}) + c_3 \cdot (5^{l_1}, \ldots, 5^{l_k}) = d$, where $c_1, c_2, c_3 \in \mathbb{Z}^k$ and $d \in \mathbb{Z}$ are given?

Problem

Prove anything about $\langle \mathbb{N}; 0, 1, <, +, P \rangle$ without assuming Dickson's conjecture