Algorithmic Verification of Linear Dynamical Systems

Toghrul Karimov

Saarland University and MPI-SWS

PhD Defence

Linear dynamical systems

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- $M \in \mathbb{Q}^{d \times d}$ is the *update matrix*
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Verification of LDS

Find algorithms that

1. take (M, s) and a property φ, and
2. decide whether $\langle s, Ms, M^2 s, M^3 s, \ldots \rangle$ satisfies φ.

Example of LDS: linear loops

initialise x_1, x_2

while $\neg \mathrm{P}(x_1, x_2)$**:**

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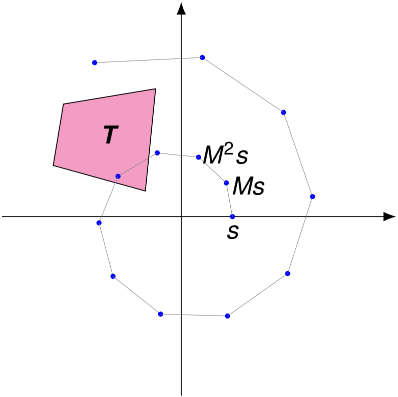

s initial value of (x_1, x_2), $\boldsymbol{T} = \{(x_1, x_2) : P(x_1, x_2)\}$

$(x_1, x_2) = M^n s$ after n iterations

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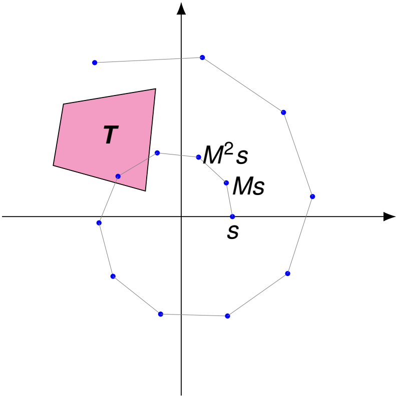

$$\boxed{\text{Loop terminates} \Leftrightarrow \langle s, Ms, M^2 s, \ldots \rangle \text{ reaches } \boldsymbol{T}}$$

The Reachability Problem

A subset of $\mathbb{R}^d$ is *semialgebraic* if it can be defined using inequalities of the form $p(x_1, \ldots, x_d) > 0$ and $p(x_1, \ldots, x_d) = 0$ for $p \in \mathbb{Q}[x_1, \ldots, x_d]$

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Reachability Problem in dimension d

Given $M \in \mathbb{Q}^{d \times d}$, $s \in \mathbb{Q}^d$ and a semialgebraic target T, decide whether $\langle s, Ms, M^2 s, \ldots, \rangle$ reaches T

Generalising reachability

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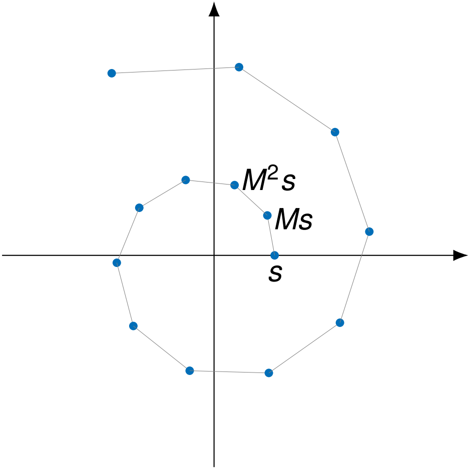
Model-Checking Problem in dimension d

Given $M \in \mathbb{Q}^{d \times d}$, $s \in \mathbb{Q}^d$, semialgebraic $T_1, \ldots, T_m$ and a property φ over $T_1, \ldots, T_m$, decide whether $\langle s, Ms, M^2 s, \ldots \rangle$ satisfies φ

Generalising reachability, cont'd

Consider (M, s) and T_1, T_2, T_3



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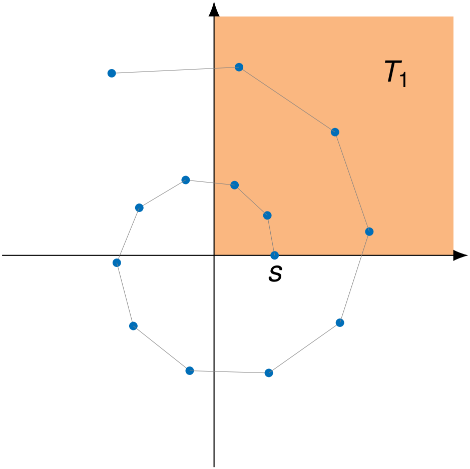

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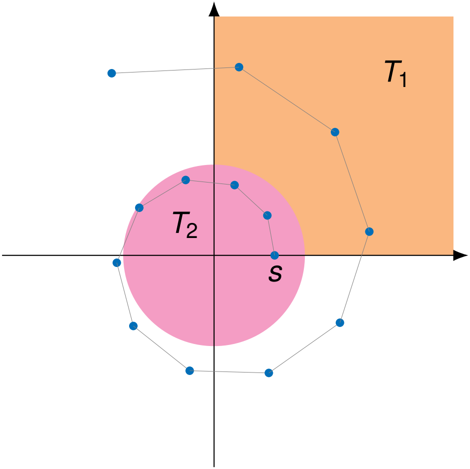

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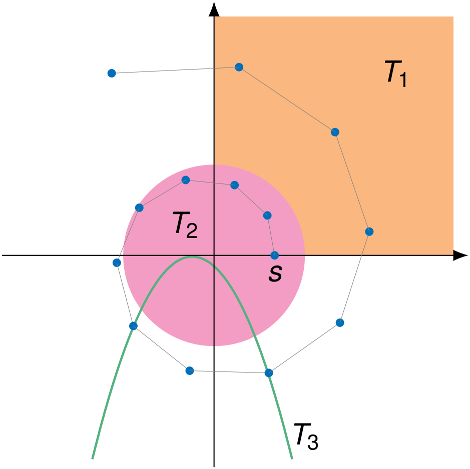
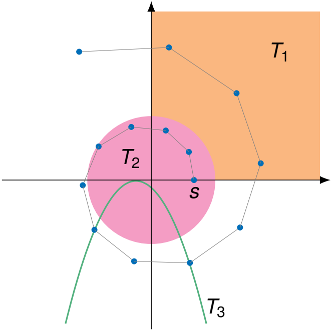


Generalising reachability, cont'd

Define *characteristic word* α of (M, s) wrt T_1, T_2, T_3



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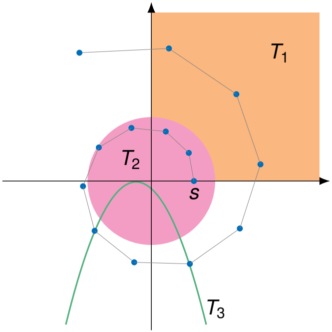

$$\alpha = \{T_1, T_2\}\,\{T_1, T_2\}\,\{T_1, T_2\}\,\{T_2\}\,\{T_2\}\,\emptyset\,\{T_3\}\,\emptyset\,\{T_3\}\cdots$$

$\alpha(n) \subseteq \{T_1, T_2, T_3\}$ records T_i such that $M^n s \in T_i$

Generalising reachability, cont'd

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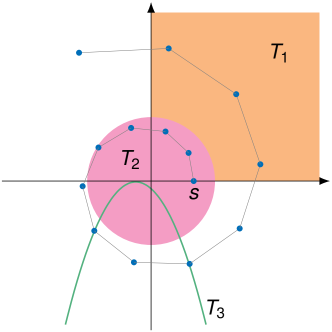

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$$\boxed{\text{Verifying } (M, s) \text{ wrt } T_1, T_2, T_3 \Leftrightarrow \text{verifying } \alpha}$$

Generalising reachability, final

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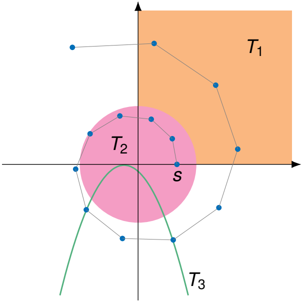


Model-Checking Problem in dimension d

Given $M \in \mathbb{Q}^{d\times d}$, $s \in \mathbb{Q}^d$, semialgebraic $T_1, \ldots, T_m$ and an automaton $\mathcal{A}$, decide whether $\mathcal{A}$ accepts the characteristic word α

This thesis

Model-Checking Problem in dimension d

Given $M \in \mathbb{Q}^{d \times d}$, $s \in \mathbb{Q}^d$, semialgebraic $T_1, \ldots, T_m$ and an automaton $\mathcal{A}$, decide whether $\mathcal{A}$ accepts the characteristic word α

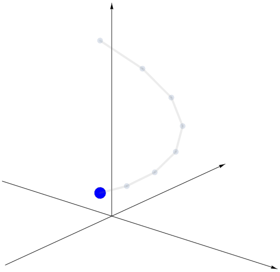
Previously, decidability was only known for reachability-type properties and restricted M

Contributions:

- A new framework in which decidability of various fragments of the MCP can be shown
- Extending our decidability results in any significant way requires number-theoretic breakthroughs

Skolem Problem

Given (M, s) and a hyperplane H, decide whether $M^n s \in H$ for some n



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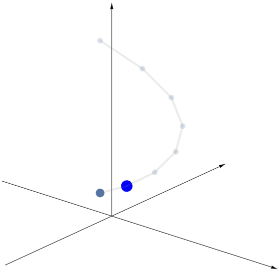

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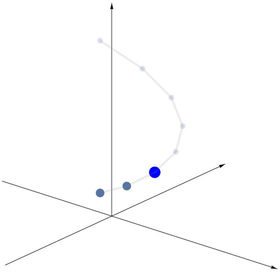

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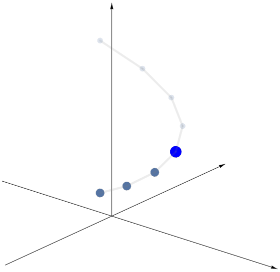

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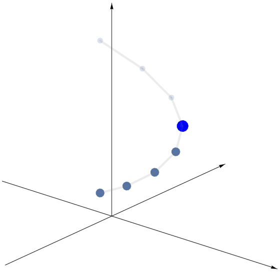

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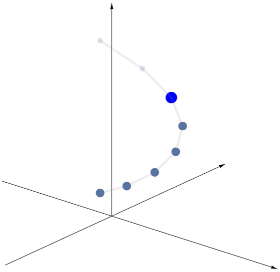

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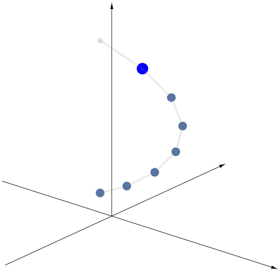

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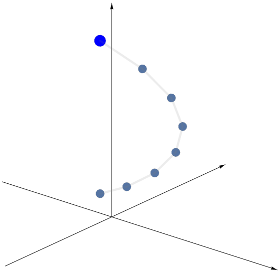

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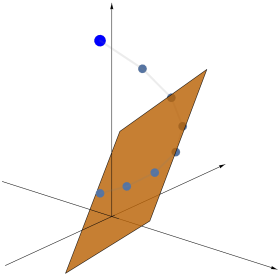

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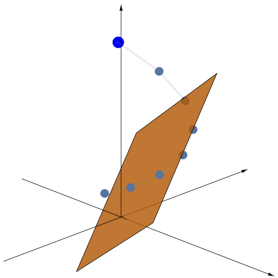

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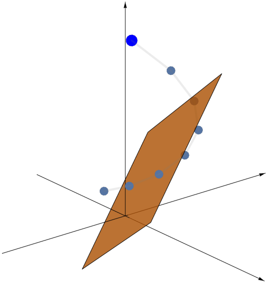

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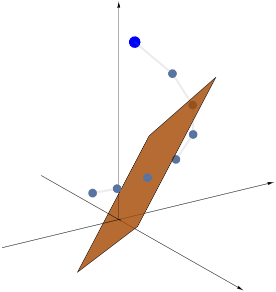

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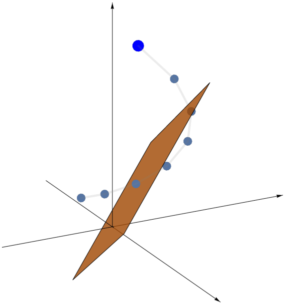

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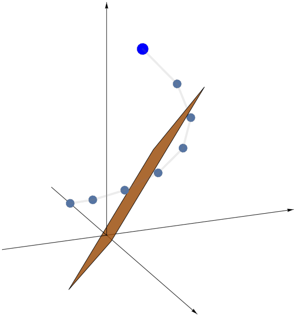

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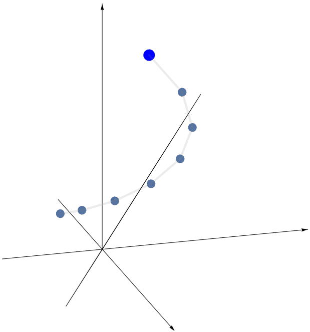

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Skolem Problem

Given (M, s) and a hyperplane H,
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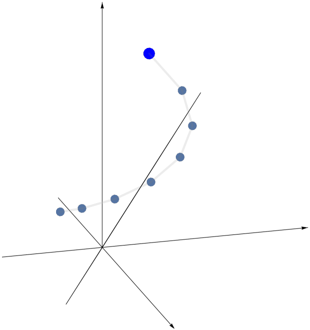
Decidable in dimension $d \leq 4$,
famously open in dimension $d = 5$



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Skolem Problem, cont'd

A sequence $(u_n)_{n\in\mathbb{N}}$ is a *linear recurrence sequence* if there exist $d > 0$ and constants $a_1, \ldots, a_d$ such that

$$u_{n+d} = a_1 u_{n+d-1} + \ldots + a_d u_n$$

for all n

E.g. $u_{n+3} = 3u_{n+2} + u_{n+1} - 2u_n$

Skolem Problem, LRS version

Given an LRS $(u_n)_{n\in\mathbb{N}}$, decide whether $u_n = 0$ for some n

Skolem Problem, final

Skolem Problem in dimension d

Given $M \in \mathbb{Q}^{d \times d}$, $s \in \mathbb{Q}^d$, and a hyperplane H, decide whether there exists n such that $M^n s \in H$

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Consequently, the Reachability and Model-Checking problems are Skolem-hard in dimension $d = 5$

Positivity Problem

Positivity Problem in dimension d

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Decidable for $d \leq 5$
Resolution for $d = 6 \Rightarrow$ breakthroughs in Diophantine approximation

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The Reachability and Model-Checking problems are Positivity-hard in dimension $d = 6$

Recall: Reachability Problem allows *semialgebraic* targets.

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Using tricks we can reduce the Skolem Problem in dimension 5 and the Positivity Problem in dimension 6 to the Reachability Problem in dimension 4

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Main result #1

The Model-Checking Problem is decidable in dimension $d \leq 3$. In dimension $d = 4$, already the Reachability Problem is both Skolem-hard and Positivity-hard

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I will focus on:

MCP is decidable if $d \leq 3$

Given $d \leq 3$, $M \in \mathbb{Q}^{d \times d}$, $s \in \mathbb{Q}^d$, semialgebraic $T_1, \ldots, T_m$, and an automaton $\mathcal{A}$, it is decidable whether $\mathcal{A}$ accepts the characteristic word α of (M, s) wrt $T_1, \ldots, T_m$

Model-Checking Problem with $d \leq 3$

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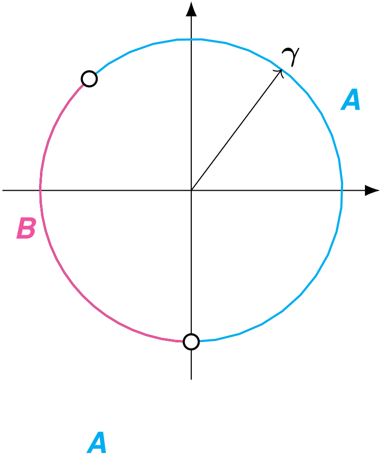
Main idea: α has a suffix that is a *toric word*

Toric words

A toric word is the coding of a rotation on a k-dimensional torus with respect to open sets



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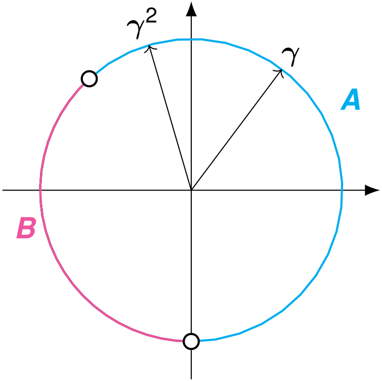

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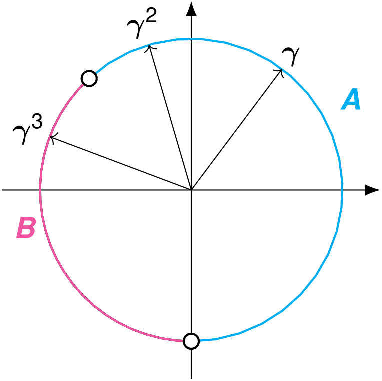

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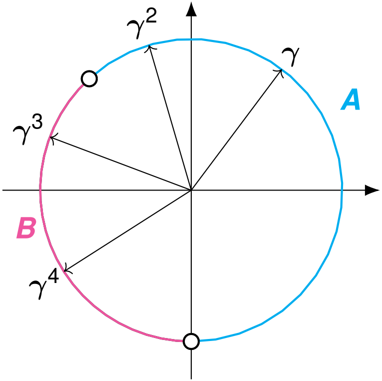

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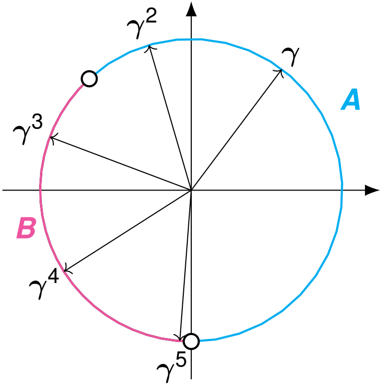

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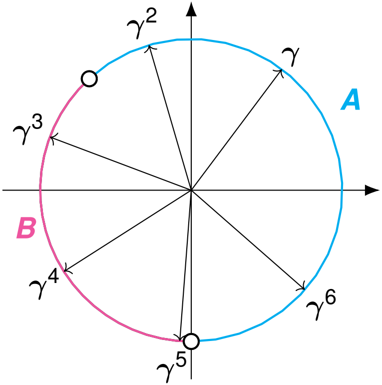

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Toric words, formally

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We are coding the trajectory $(\gamma_1^n, \dots, \gamma_k^n)_{n \in \mathbb{N}}$ with respect to $\{S_a \colon a \in \Sigma\}$

What is so good about toric words?

Well-studied in symbolic dynamics and combinatorics on words

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Always *almost-periodic* and in cases arising from the MCP, *effectively almost-periodic*

Almost-periodic words

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Almost-periodic words

$$a\overbrace{bbbaabababababba}^{\le D}\underbrace{bbbababbababa}_{\le D}\overbrace{bbbabababba}^{\le D}\underbrace{bbbabababa}_{\le D}bbb\cdots$$

Semënov's theorem

Given effectively almost-periodic α and an automaton $\mathcal{A}$, it is decidable whether $\mathcal{A}$ accepts α

Decidability of the MCP in dimension $d \leq 3$

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Given $d \leq 3$, $M \in \mathbb{Q}^{d \times d}$, $s \in \mathbb{Q}^d$, semialgebraic $T_1, \ldots, T_m$, and an automaton $\mathcal{A}$, it is decidable whether $\mathcal{A}$ accepts the characteristic word α of (M, s) wrt $T_1, \ldots, T_m$

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Proof. There exists effectively computable N such that $\beta := \alpha[N, \infty)$ is toric: there exists $\gamma \in \mathbb{T}$ and S_a for every $a \subseteq \{T_1, \ldots, T_m\}$ such that

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$$\begin{aligned} M^n s \in T &\Leftrightarrow P^{-1} C^n \Gamma^n P s \in T \\ &\Leftrightarrow \Gamma^n (Ps) \in C^{-n} P(T) \\ &\Leftrightarrow (\gamma_1^n, \ldots, \gamma_d^n) \in T_n \end{aligned}$$

where $T_n = f(C^{-n})$ for a “reasonable” function f.

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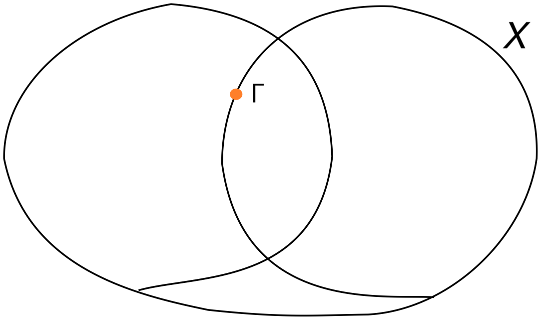

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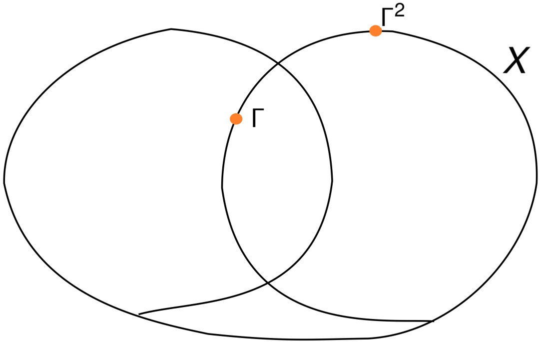

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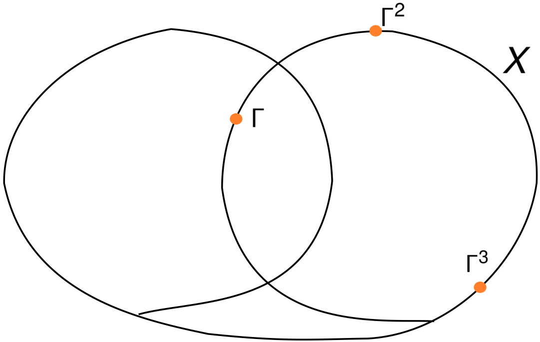

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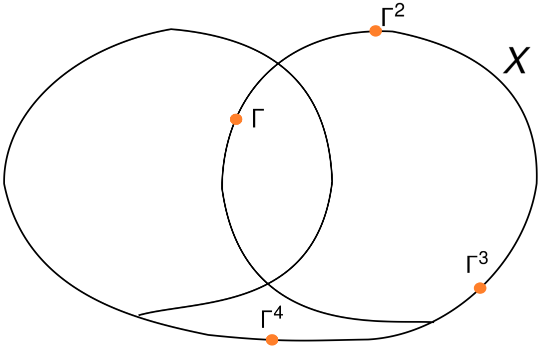

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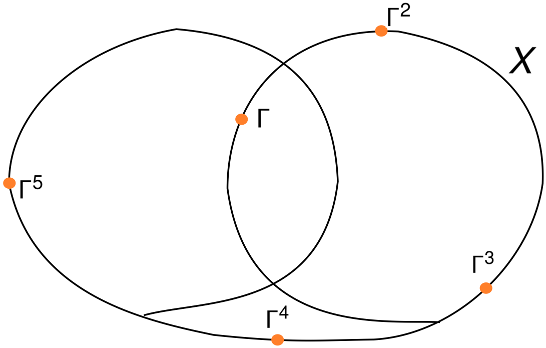

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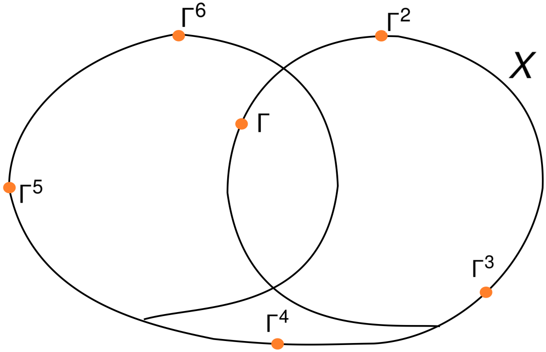

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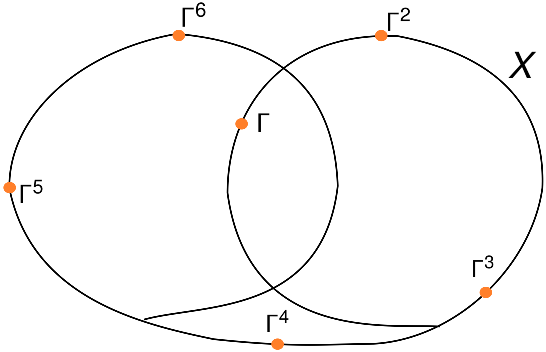

What about $(T_n)_{n\in\mathbb{N}}$?

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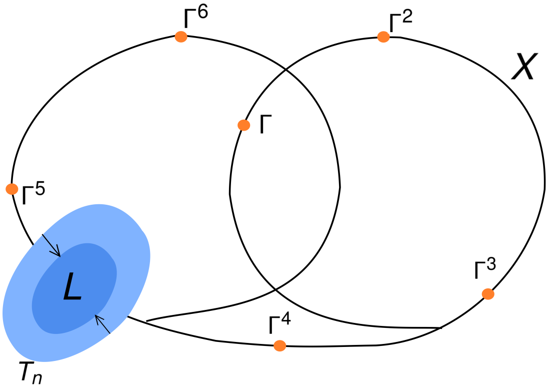
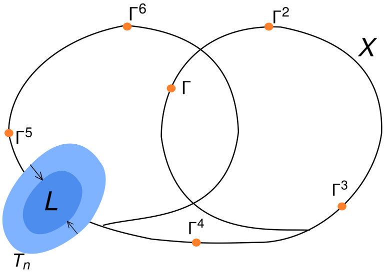


Toricity in LDS, final

Recall: $M^n s \in T \Leftrightarrow \Gamma^n \in T_n$



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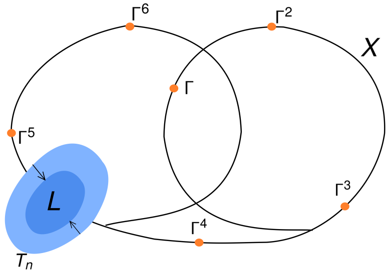

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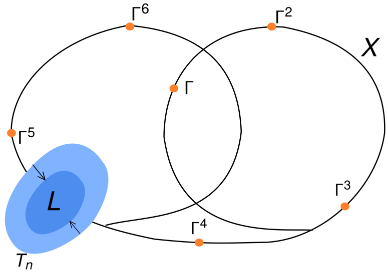

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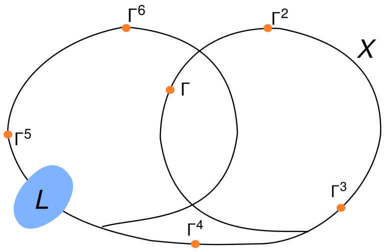

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Summary of decidability

MCP is decidable if $d \leq 3$

Given $d \leq 3$, $M \in \mathbb{Q}^{d \times d}$, $s \in \mathbb{Q}^d$, semialgebraic $T_1, \ldots, T_m$, and an automaton $\mathcal{A}$, it is decidable whether $\mathcal{A}$ accepts the characteristic word α of (M, s) wrt $T_1, \ldots, T_m$

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A set is tame if it can obtained, through set operations, from semialgebraic sets that (i) are contained in a 3D subspace, or (ii) have intrinsic dimension 1

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Given arbitrary $d > 0$, **diagonalisable** $M \in \mathbb{Q}^{d \times d}$, $s \in \mathbb{Q}^d$, semialgebraic $T_1, \ldots, T_m$, and a **prefix-independent** automaton $\mathcal{A}$, it is decidable whether $\mathcal{A}$ accepts α

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$\mathcal{A}$ prefix-independent $\Leftrightarrow$ whether α is accepted does not change through finitely many modifications

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- deciding infinite reachability in T is Diophantine-hard

Hardness

MCP is decidable for tame targets

Given arbitrary $d > 0$, $M \in \mathbb{Q}^{d \times d}$, $s \in \mathbb{Q}^d$, **tame semialgebraic** $T_1, \ldots, T_m$, and an automaton $\mathcal{A}$, it is decidable whether $\mathcal{A}$ accepts α

There exists a family of $M \in \mathbb{Q}^{4 \times 4}$, $s \in \mathbb{Q}^4$, and T of intrinsic dimension 2 such that

- deciding reachability in T is Positivity-hard
- deciding infinite reachability in T is Diophantine-hard
- the characteristic word of (M, s) wrt T is not almost-periodic and not toric

Hardness, cont'd

Prefix-independent MCP decidable

Given arbitrary $d > 0$, **diagonalisable** $M \in \mathbb{Q}^{d \times d}$, $s \in \mathbb{Q}^d$, semialgebraic $T_1, \ldots, T_m$, and a **prefix-independent** automaton $\mathcal{A}$, it is decidable whether $\mathcal{A}$ accepts α

Hardness, cont'd

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- Lifting diagonalisability $\Rightarrow$ Diophantine-hardness

Hardness, cont'd

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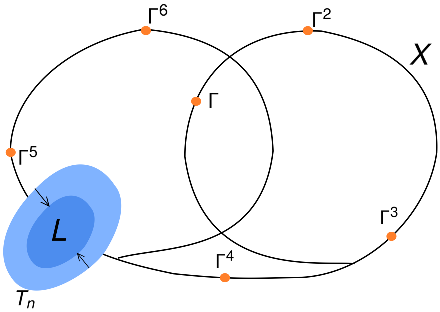
- Lifting diagonalisability $\Rightarrow$ Diophantine-hardness
- Lifting prefix-independence $\Rightarrow$ Skolem-hardness

Reachability Problem revisited

$M^n s \in T \Leftrightarrow \Gamma^n \in T_n$

$(\Gamma^n)_{n\in\mathbb{N}}$ is dense in compact $X \subseteq \mathbb{T}^d \subset \mathbb{C}^d$

$(T_n)_{n\in\mathbb{N}}$ converges to a limit shape L



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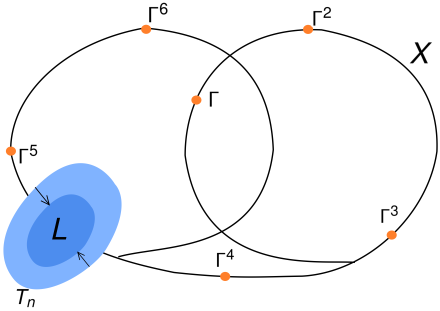

Reached infinitely often: L has ample intersection with X

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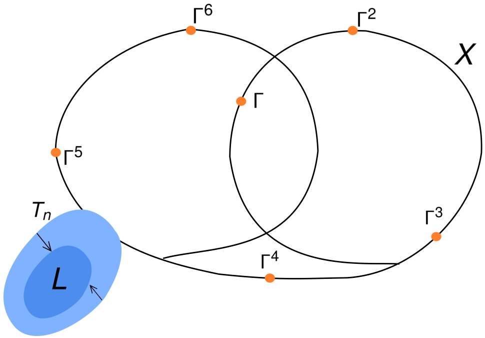

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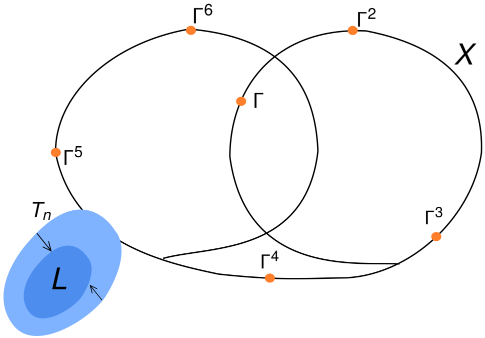

Not reachable: T_n is separated from X for large n

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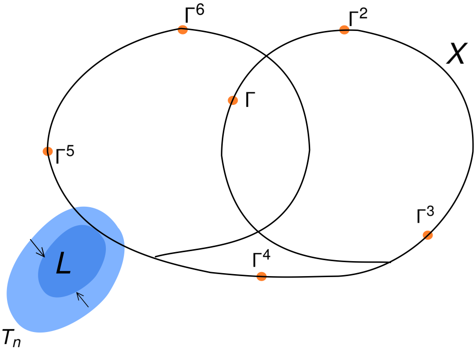

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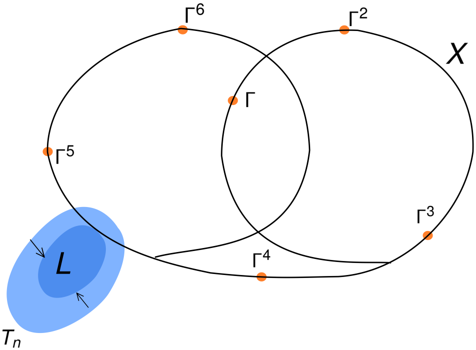
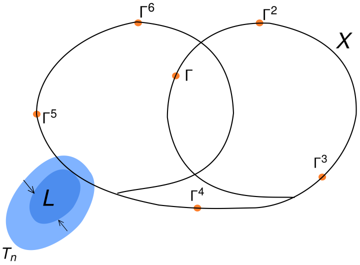


Difficult: $T_n \cap X$ disappears but T_n not separated from X

Topological reachability

Difficult case of the Reachability Problem:



Topological reachability

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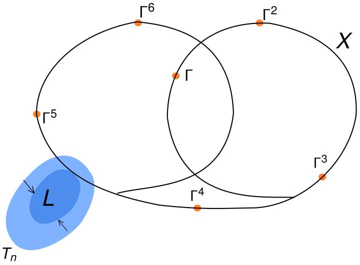

Topological Reachability Problem. Given (M, s) and target T, decide whether in every $\mathcal{B}(s, \varepsilon)$ for $\varepsilon > 0$ there exists $\widehat{s}$ such that the orbit of $(M, \widehat{s})$ reaches T

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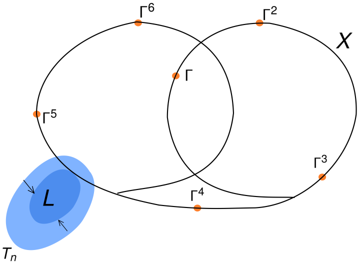

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Decidable by just looking at $L \cap X$!

Summary of decidability

MCP is decidable for tame targets

Given arbitrary $d > 0$, $M \in \mathbb{Q}^{d\times d}$, $s \in \mathbb{Q}^d$, **tame semialgebraic** $T_1, \ldots, T_m$, and an automaton $\mathcal{A}$, it is decidable whether $\mathcal{A}$ accepts α

Prefix-independent MCP is decidable

Given arbitrary $d > 0$, **diagonalisable** $M \in \mathbb{Q}^{d\times d}$, $s \in \mathbb{Q}^d$, semialgebraic $T_1, \ldots, T_m$, and a **prefix-independent** automaton $\mathcal{A}$, it is decidable whether $\mathcal{A}$ accepts α

Tame $\Leftrightarrow$ can be obtained from semialgebraic sets that (i) are contained in a 3D subspace, or (ii) have intrinsic dimension 1

Prefix-independent $\Leftrightarrow$ whether α is accepted does not change after finite modifications