

Linear Dynamical Systems with Weight Functions

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Abstract

In discrete-time linear dynamical systems (LDSs), a linear map is repeatedly applied to an initial vector yielding a sequence of vectors called the orbit of the system. A weight function assigning weights to the points in the orbit can be used to model quantitative aspects, such as resource consumption, of a system modelled by an LDS. This paper addresses the problems of how to compute the mean payoff, the total accumulated weight, and the discounted accumulated weight of the orbit under continuous weight functions as well as polynomial weight functions as a special case. Additionally, weight functions that are definable in an o-minimal extension of the theory of the reals with exponentiation, which can be shown to be piecewise continuous, are considered. In particular, good ergodic properties of o-minimal weight functions, instrumental to the computation of the mean payoff, are established. Besides general LDSs, the special cases of stochastic LDSs and LDSs with bounded orbits are addressed. Finally, the problem of deciding whether an energy constraint is satisfied by the weighted orbit, i.e., whether the accumulated weight never drops below a given bound, is analysed.

Keywords: linear dynamical systems, formal verification, linear recurrence sequences, Markov chains

Related version

This paper is an extension of the conference paper [1]. The main new contribution of this paper is that we identify *o-minimal* weight functions as a suitable, extremely general framework that allows us to express time averages

for (arbitrary) linear dynamical systems as integrals. In [1], existence of integral representations was only shown for *bounded* linear dynamical systems, which is a much simpler result. Finally, in this version the presentation of the technical parts has been improved and all proofs omitted in the conference version have been included.

1. Introduction

Dynamical systems, which describe how a system’s state evolves over time, serve as a fundamental modeling framework across numerous disciplines. A *discrete-time linear dynamical system* (LDS) in an ambient space \mathbb{R}^d begins at an initial point $q \in \mathbb{R}^d$. Its evolution follows a linear update rule governed by a matrix $M \in \mathbb{R}^{d \times d}$, which is applied iteratively to the system’s current state at each time step. This process generates the *orbit* (q, Mq, M^2q, \dots) . The study of LDSs is particularly significant because they represent one of the simplest classes of dynamical systems while still presenting numerous intricate challenges. Moreover, linearization plays a crucial role in control theory and engineering, where complex systems are frequently approximated using linear models (see, e.g., [2, 3]), making it an essential tool for solving many real-world problems.

Algorithmic problems concerning LDSs form a lively area of research in computer science. Reachability problems of LDSs are formidably hard: the Skolem Problem (hyperplane reachability) has been open for a hundred years, and the Positivity Problem (halfspace reachability) is at least as hard as certain long-standing open problems in Diophantine approximation whose solution would amount to a mathematical earthquake. Decidability is known in low dimensions only, and is obtained through a combination of arguments from number theory and Diophantine approximation [4, 5, 6, 7]. However, it has recently been discovered that “robust” versions of many classical open problems of linear dynamical systems are decidable [8, 9]. For example, for arbitrary $M \in \mathbb{Q}^{d \times d}$, $q \in \mathbb{Q}^d$, and semialgebraic T , it is decidable whether for every $\varepsilon > 0$ there exists q' in the ε -ball around q such that the orbit of q' under M reaches T .

The key new tool is o-minimality, which is a concept originating in logic and model theory, that is defined by the following property: Any subset of \mathbb{R}^d that is definable over an o-minimal extension of the real field with exponentiation has finitely many connected components. Although it can

Table 1: Overview of the results.

	LDS type	weight function	algorithmic result	
mean payoff	arbitrary	polynomial	computable	(Thm. 13)
	bounded orbit	continuous	integral representation computable	(Thm. 17)
	arbitrary	o-minimal	integral representation computable	(Thm. 18)
	stochastic, irreducible	continuous	computable with poly-many evaluations of the weight function	(Thm. 22)
	stochastic, reducible	continuous	computable with exp-many evaluations of the weight function	(Thm. 24)
	total/discounted weight	arbitrary	computable	(Thm. 26)
satisfaction of energy constraints	arbitrary	polynomial	decidable in dimension 3	(Thm. 32)
	stochastic	linear	Positivity-hard	(Thm. 33)
	dimension 4	polynomial	Diophantine-hard	(Thm. 34)

be seen as a simple property at first, the consequences of o-minimality are drastic: for example, o-minimality has recently been applied in a spectacular fashion to the problem of counting rational points in a variety, which is the cornerstone problem of Diophantine geometry [10].

In this paper, we address quantitative verification questions arising when dynamical systems are equipped with a weight function. To the best of our knowledge, such quantitative verification tasks on weighted LDSs have not been investigated in the literature. We consider *continuous* weight functions $w: \mathbb{R}^d \rightarrow \mathbb{R}$ assigning a weight to each state in the ambient space. Such weight functions can be used to model various quantitative aspects of a system, such as resource or energy consumption, rewards or utilities, or execution time for example. Given a weight function w , we obtain a sequence of weights of the states in the orbit $(w(q), w(Mq), w(M^2q), \dots)$. The goal of this paper is to provide algorithmic answers to the following typical questions arising for weighted systems:

- a) What is the *mean payoff*, i.e., the average weight collected per step?
- b) What is the *total accumulated weight* of the orbit and what is the so-called *discounted accumulated weight*, where weights obtained after k

time steps are discounted with a factor λ^k for a given $\lambda \in (0, 1)$?

- c) Is there an $n \in \mathbb{N}$ such that the sum of weights obtained in the first n steps lies below a given bound? This problem is referred to as *satisfaction of an energy constraint* because it corresponds to determining whether a system ever runs out of energy when weights model the energy used or gained per step.

Example 1. Assume that a scheduler assigns tasks to d different processors P_1, \dots, P_d and that the load of the processors at different time steps can be modeled as an LDS with matrix $M \in \mathbb{Q}^{d \times d}$ and orbit $(M^k q)_{k \in \mathbb{N}}$ for a $q \in \mathbb{Q}^d$. Further, assume for each processor P_i there is an optimal load μ_i under which it works most efficiently. To evaluate the scheduler, we want to know how closely the real loads in the long-run match the ideal loads. As a measure for how well a vector x matches the vector μ of ideal loads, we use the average squared distance

$$\delta_\mu(x) = \frac{1}{d} \sum_{i=1}^d (x_i - \mu_i)^2.$$

To see how well the scheduler manages to get close to optimal loads in the long-run after a possible initialization phase, we consider the mean payoff of the orbit with respect to the weight function δ_μ , i.e.,

$$\lim_{\ell \rightarrow \infty} \frac{1}{\ell} \sum_{k=0}^{\ell-1} \delta_\mu(M^k q).$$

If, on the other hand, we know that the orbit will tend to the optimal loads for $k \rightarrow \infty$, we might instead also want to measure the total deviation $\sum_{k=0}^{\infty} \delta_\mu(M^k q)$. If this value is small, the orbit converges to the optimal loads rather quickly without large deviations initially.

In order to obtain algorithmic results, we consider different combinations of classes of LDSs and classes of weight functions. Namely, besides arbitrary rational LDSs, we consider also LDSs with bounded orbit and *stochastic LDSs*. Stochastic LDSs occur in the context of the verification of probabilistic systems: For a finite-state Markov chain, the sequence of distributions over the state space naturally forms an LDS. The initial distribution can be written as a vector $\iota_{init} \in [0, 1]^d$. Afterwards, the transition probability matrix P can be repeatedly applied to obtain the distribution $P^k \iota_{init}$ over states

after k steps. In contrast to the path semantics where a probability measure over infinite paths in a Markov chain is defined, the view of a Markov chain as an LDS is also called the *distribution transformer semantics* of Markov chains (see, e.g., [11]).

For the weight functions, on the one hand, we consider general continuous functions. Of course, for algorithmic results, we have to make additional assumptions on the computability or approximability of these functions. Furthermore, we consider the subclass of polynomial weight functions with rational coefficients. On the other hand, we consider weight functions that are definable in an o-minimal structure. These functions can be shown to be piecewise continuous, and include the vast family of weight functions that can be defined using arithmetic and exponentiation in the sense of first-order logic. We show that the use of such weight functions results in a good limiting behavior of the weights of the orbit of an LDS allowing us to treat the mean payoff for arbitrary LDSs with such weight functions.

Contribution. Our contributions are as follows (see also Table 1).

- a) Mean payoff: For rational LDSs equipped with a polynomial weight function, we show that it is decidable whether the mean payoff exists, in which case it is rational and computable (section 3.1).

We then show how to decide whether the orbit of a rational LDS is bounded. If the orbit of a rational LDS is bounded, we show how to compute the set of accumulation points of the orbit and how to obtain a representation of the mean payoff as an integral using this set. This integral can be approximated to arbitrary precision for any weight function w that is sufficiently well-behaved (section 3.2).

Next, we consider LDSs with o-minimal weight functions. We show that the orbits of linear dynamical systems equipped with an o-minimal weight function—under some mild restrictions—is ergodic in the sense of time average being equal to space average (section 3.3).

Finally, we consider stochastic LDSs, which constitute a special case of LDSs with bounded orbits. Here, the orbit only has finitely many accumulation points. We show that in case the transition matrix is irreducible, one can compute polynomially many rational points in polynomial time such that the mean payoff is the arithmetic mean of the weight function evaluated at these points. In the reducible case, on

the other hand, exponentially many such rational points have to be computed (section 3.4).

- b) Total and discounted accumulated weights: For rational LDSs and polynomial weight functions, we prove that the total as well as the discounted accumulated weight of the orbit is computable and rational if finite (section 4.1).
- c) Satisfaction of energy constraints: First we prove that it is decidable whether an energy constraint is satisfied by an orbit under a polynomial weight function for LDSs of dimension $d = 3$. We furthermore provide two different hardness results regarding possible extensions of this decidability result: At $d = 4$, the problem is hard with respect to certain open decision problems in Diophantine approximation that are at the moment wide open. Further, also restricting to stochastic LDSs and linear weight functions, does not lead to decidability in general: we show that the energy satisfaction problem is at least as hard as the Positivity Problem for linear recurrence sequences in this case. The decidability status of the Positivity Problem is open, and it is known from [6] that its resolution would amount to major mathematical breakthroughs (sections 4.2–4.4).

Related work. Verification problems for linear dynamical systems have been extensively studied for decades, starting with the question about the decidability of the Skolem [4, 5] and Positivity [6, 7] problems at low orders, which are special cases of the reachability problem for LDSs. Decidable cases of the more general Model-Checking Problem for LDSs have been studied in [12, 13]. In addition, decidability results for parametric LDSs [14] as well as various notions of robust verification [15, 8] have been obtained. See [16] for a survey of what is decidable about discrete-time linear dynamical systems.

There is very little related work on LDSs with weight functions. Closest to our work is the work by Kelmendi [17]. There, it is shown that the *natural density* (which is a notion of frequency) of visits of a rational LDS in a semialgebraic set always exists and is approximable to arbitrary precision. A consequence of this result is that the mean payoff of a rational LDS with respect to a “semialgebraic step function”, which takes a partition of the ambient space \mathbb{R}^d into finitely many semialgebraic sets S_1, \dots, S_k and assigns a rational weight w_i to the points in S_i , can be approximated to arbitrary precision. This result is orthogonal to our results.

When it comes to Markov chains viewed as LDSs under the distribution transformer semantics, it is known that Skolem and Positivity-hardness results for general LDSs persist [18]. Vahanwala has recently shown [19] that this is the case even for ergodic Markov chains. In [11], Markov chains under the distribution transformer semantics are treated approximatively – in contrast to our work – by discretising the probability value space $[0, 1]$ into a finite set of intervals and the problem to decide whether an approximation of the trajectory obtained in this way satisfies a property is studied.

Several connections between o-minimality and dynamical (or, more generally, cyber-physical) systems have been established in the literature. In [20], Lafferriere et al. show that *o-minimal hybrid systems* always admit a finite bisimulation. In [21], Miller gives a dichotomy result concerning tameness of expansions of certain o-minimal structures with trajectories of linear dynamical systems.

2. Preliminaries

We write \mathbb{T} for $[0, 1)$, \mathbf{i} for the imaginary number, and $\mathbf{0}$ for a vector of all zeros whose dimension will be clear from the context. We then have that $e^{i2\pi\mathbb{T}} = \{z \in \mathbb{C} : |z| = 1\}$. We denote by $\{x\}$ the fractional part of $x \in \mathbb{R}$, and by e_i the i th standard basis vector of \mathbb{R}^d , where d will be clear from the context.

2.1. Linear dynamical systems

A (discrete-time) *linear dynamical system* (M, q) of dimension $d > 0$ consists of an update matrix $M \in \mathbb{R}^{d \times d}$ and an initial vector $q \in \mathbb{R}^d$. If the entries of M and q are rational, we say that the LDS is *rational*. The *orbit* of (M, q) is the sequence $(M^k q)_{k \in \mathbb{N}}$. We say that the orbit of (M, q) is bounded if it is bounded as a set under the Euclidean metric. An LDS is called *stochastic* if the matrix M and the initial vector q have only non-negative entries and the entries of each column of M as well as the entries of q sum up to 1. In this case we refer to the matrix M as stochastic as well.¹

¹In order to keep the notation in line with the notation for general LDSs, we deviate from the standard convention that rows of stochastic matrices sum up to 1 and that stochastic matrices are applied to distributions by multiplication from the right.

2.2. Algebraic numbers

A number $\alpha \in \mathbb{C}$ is *algebraic* if there exists a polynomial $p \in \mathbb{Q}[x]$ such that $p(\alpha) = 0$. Algebraic numbers form a subfield of \mathbb{C} denoted by $\overline{\mathbb{Q}}$. The minimal polynomial of $\alpha \in \overline{\mathbb{Q}}$ is the (unique) monic polynomial $p \in \mathbb{Q}[x]$ of the smallest degree such that $p(\alpha) = 0$. The *degree* of α , denoted by $\deg(\alpha)$, is the degree of the minimal polynomial of α . For each $\alpha \in \overline{\mathbb{Q}}$ there exists a unique polynomial $P_\alpha = \sum_{i=0}^d a_i X^i \in \mathbb{Z}[x]$ with $d = \deg(\alpha)$, called the *defining polynomial* of α , such that $P_\alpha(\alpha) = 0$ and $\gcd(a_0, \dots, a_d) = 1$. The polynomial P_α and the minimal polynomial of α have identical roots, and are *square-free*, i.e., all of their roots appear with multiplicity one. The (*naive*) *height* of α , denoted by $H(\alpha)$, is equal to $\max_{0 \leq i \leq d} |a_i|$. We represent an algebraic number α in computer memory by its defining polynomial P_α and sufficiently precise rational approximations of $\text{Re}(\alpha), \text{Im}(\alpha)$ to distinguish α from other roots of P_α . We denote by $\|\alpha\|$ the bit length of a representation of $\alpha \in \overline{\mathbb{Q}}$. We can perform arithmetic operations on any fixed number of algebraic numbers (e.g., multiplying 3 given numbers) represented in this way in polynomial time [22, Chap. 4] [23, Chap. 1.5.4]. However, no polynomial-time algorithm exists if we do not fix the number of algebraic numbers in the input. For example, suppose we are given the first d primes numbers $\alpha_1, \dots, \alpha_d$. Then $\alpha := \sqrt{\alpha_1} + \dots + \sqrt{\alpha_d}$ has degree 2^d and hence any representation of α has size exponential in $\|\alpha_1\| + \dots + \|\alpha_d\|$. In practice, fast algorithms for most problems in algebraic number theory involve a *lattice reduction* step using, for example, the Lenstra-Lenstra-Lovász (LLL) algorithm [22, Sec. 2.6].

2.3. Linear recurrence sequences

A sequence $(u_n)_{n \in \mathbb{N}}$ is a *linear recurrence sequence* over a ring $R \subseteq \mathbb{C}$ if there exists a positive integer d and a *recurrence relation* $(a_0, \dots, a_{d-1}) \in R^d$ such that $u_{n+d} = \sum_{i=0}^{d-1} a_i u_{n+i}$ for all n . The *order* of $(u_n)_{n \in \mathbb{N}}$ is the smallest positive integer d such that $(u_n)_{n \in \mathbb{N}}$ satisfies a recurrence relation in R^d . In this work, we will mostly encounter linear recurrence sequences over \mathbb{Q} , called *rational LRSs*. Examples of rational LRSs include the Fibonacci sequence, $u_n = p(n)$ for $p \in \mathbb{Q}[x]$, and $u_n = \cos(n\theta)$ where $\theta \in \{\arg(\lambda) : \lambda \in \mathbb{Q}(\mathbf{i})\}$. We refer the reader to the books by Everest et al. [24] and Kauers & Paule [25] for a detailed discussion of linear recurrence sequences.

An LRS $(u_n)_{n \in \mathbb{N}}$ that is not eventually zero satisfies a unique minimal recurrence relation $u_{n+d} = \sum_{i=0}^{d-1} a_i u_{n+i}$ such that $d > 0$ and $a_0 \neq 0$. Writing

$A = [a_1 \ \cdots \ a_{d-1}]$ and $q = [u_0 \ \cdots \ u_{d-1}]^\top$, the matrix

$$C := \begin{bmatrix} \mathbf{0} & I_{d-1} \\ a_0 & A \end{bmatrix} = \begin{bmatrix} 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \\ a_0 & a_1 & \cdots & a_{d-1} \end{bmatrix} \in R^{d \times d}$$

is called the *companion matrix* of $(u_n)_{n \in \mathbb{N}}$. We have that

$$C^n q = (u_n, \dots, u_{n-d+1})$$

and $u_n = e_1^\top C^n q$ for all $n \in \mathbb{N}$, where e_i denotes the i th standard basis vector. As $a_0 \neq 0$, the matrix C is invertible and does not have zero as an eigenvalue.

The *characteristic polynomial* of $(u_n)_{n \in \mathbb{N}}$ is $p(x) = x^d - \sum_{i=0}^{d-1} a_i x^i$. Note that p is identical to the characteristic polynomial $\det(xI - C)$ of the companion matrix C . The *eigenvalues* (also called the *characteristic roots*) of $(u_n)_{n \in \mathbb{N}}$ are the d (possibly non-distinct) roots $\lambda_1, \dots, \lambda_d$ of the characteristic polynomial p . An LRS is

- *simple* (or *diagonalisable*) if its characteristic polynomial does not have a repeated root, and
- *non-degenerate* if (i) all real eigenvalues are non-negative, and (ii) for every pair of distinct eigenvalues λ_1, λ_2 , the ratio λ_1/λ_2 is not a root of unity.

For an LRS $(u_n)_{n \in \mathbb{N}}$ there exists effectively computable R such that for every $0 \leq r < R$, the sequence $u_n^{(r)} = u_{nR+r}$ is non-degenerate [24, Section 1.1.9]. If $(u_n)_{n \in \mathbb{N}}, (v_n)_{n \in \mathbb{N}}$ are LRSs over a field R , and $\circ \in \{+, -, \cdot\}$, then $w_n = u_n \circ v_n$ also defines an LRS over R [25, Theorem 4.2]. Moreover, if $(u_n)_{n \in \mathbb{N}}$ and $(v_n)_{n \in \mathbb{N}}$ are both simple, then so is $(w_n)_{n \in \mathbb{N}}$.

The exponential polynomial representation of an LRS. Every LRS $(u_n)_{n \in \mathbb{N}}$ of order $d > 0$ over $\overline{\mathbb{Q}}$ can be written in the form

$$u_n = \sum_{j=1}^m p_j(n) \lambda_j^n \tag{1}$$

where (i) $m \geq 1$, $\lambda_1, \dots, \lambda_m$ are the distinct non-zero characteristic roots of $(u_n)_{n \in \mathbb{N}}$, and (ii) each p_i is a non-zero polynomial with algebraic coefficients;

see [24, Chapter 1]. Whenever these conditions on m , λ_i and p_i are met, we say that the right-hand side is in the *exponential polynomial form*. Every LRS $(u_n)_{n \in \mathbb{N}}$ that is not eventually zero has a unique representation of the form (1) where the right-hand side is in the exponential polynomial form. Moreover, the right-hand side of (1) cannot be identically zero assuming (i-ii). This is a folklore result, but we provide a proof for completeness.

Lemma 1. *Let $u_n = \sum_{i=1}^m p_i(n) \lambda_i^n$ where $m \geq 1$, $\lambda_1, \dots, \lambda_m \in \overline{\mathbb{Q}}$ are non-zero and pairwise distinct, and each $p_i \in \overline{\mathbb{Q}}[x]$ is non-zero. Then, the sequence $(u_n)_{n \in \mathbb{N}}$ is not identically zero. Specifically, there exists $0 \leq n < d$, where $d = \sum_{i=1}^m (\deg(p_i) + 1)$, such that $u_n \neq 0$.*

Proof. Suppose $\deg(p_k) \geq 1$ for some $1 \leq k \leq m$. Consider the sequence $v_n = u_{n+1} - \lambda_k u_n$. It will be of the form

$$v_n = \sum_{i \in I} q_i(n) \lambda_i^n$$

where $I \subseteq \{1, \dots, m\}$ with $k \in I$, $\deg(q_k) < \deg(p_k)$, and for all $i \in I$, q_i is not identically zero with $\deg(q_i) \leq \deg(p_i)$. Observe that if $(u_n)_{n \in \mathbb{N}}$ is identically zero, then so is $(v_n)_{n \in \mathbb{N}}$. Moreover, if v_n is non-zero, then either u_n or u_{n+1} is non-zero. Repeating the process of constructing v_n from u_n at most $\sum_{i=1}^m \deg(p_i)$ times, we obtain

$$w_n = \sum_{i=1}^m c_i \lambda_i^n$$

that is identically zero if u_n is identically zero, where each c_i is an algebraic number and at least one c_i is non-zero.

It remains to argue that w_n cannot be identically zero. Consider the system of equations

$$\sum_{i=1}^m x_i \lambda_i^n = 0 \quad \text{for } 0 \leq n < m.$$

We can write it as $Mx = \mathbf{0}$, where $x = (x_1, \dots, x_m)$ and M is a Vandermonde matrix with $\det(M) = \prod_{i \neq j} (\lambda_i - \lambda_j)$. Since $\lambda_1, \dots, \lambda_m$ are distinct by assumption, M is invertible and $Mx = 0$ if and only if $x = \mathbf{0}$. Since $c \neq \mathbf{0}$, it follows that $w_n \neq 0$ for some $0 \leq n < m$. Hence there exists $n' \leq n + \sum_{i=1}^m \deg(p_i) = n + (d - m) < d$ such that $u_{n'} \neq 0$. \square

We can also characterise the exponential polynomial representations of real-valued LRS.

Lemma 2. *Let $(u_n)_{n \in \mathbb{N}}$ be as in the statement of Lemma 1. If $u_n \in \mathbb{R}$ for all $n \in \mathbb{N}$, then for every $1 \leq i \leq m$ there exists j with $1 \leq j \leq m$ such that $p_j(n) = \overline{p_i}(n)$ and $\lambda_j = \overline{\lambda_i}$.*

Proof. We have $\overline{u_n} = \sum_{j=1}^m \overline{p_j}(n) \overline{\lambda_j}^n$. Moreover, $u_n = \overline{u_n}$ since $u_n \in \mathbb{R}$ for all n . The result then follows from the uniqueness of the exponential polynomial representation. \square

Throughout this work we will encounter sequences of the form $u_n = p(M^n q)$ where p is a polynomial with rational coefficients and q is a vector with rational entries. Since

$$p(M^n q) = p(e_1^\top M^n q, \dots, e_d^\top M^n q),$$

each $u_n^{(k)} = e_k^\top M^n q$ is an LRS over \mathbb{Q} (this can be seen, e.g., by applying the Cayley-Hamilton theorem), and LRS over \mathbb{Q} are closed under addition and multiplication, the sequence $(p(M^n q))_{n \in \mathbb{N}}$ is itself an LRS over \mathbb{Q} .

The following is one of the most fundamental results about growth of linear recurrence sequences. It is a “folk theorem” that follows directly from the deep result of Evertse on lower bounds on sums of S -units [26]; See also [27, Theorem 2] for a recent write-up of the proof.

Theorem 3. *Let $u_n = \sum_{i=1}^{\ell} p_i(n) \lambda_i^n$ be non-degenerate, where the right-hand side is in the exponential polynomial form. Further let $\rho = \max_i |\lambda_i|$. For every $\varepsilon > 0$, we have that*

$$u_n \neq 0 \Rightarrow |u_n| > (\rho - \varepsilon)^n$$

for all sufficiently large n .

Corollary 4. *Let $(u_n)_{n \in \mathbb{N}}$ be a non-zero LRS with the exponential polynomial solution $u_n = \sum_{j=1}^m p_j(n) \lambda_j^n$. Suppose $|\lambda_1| = \dots = |\lambda_r| > |\lambda_{r+1}|, \dots, |\lambda_m|$. Then $|u_n|$ is unbounded if and only if either $|\lambda_1| > 1$, or $|\lambda_1| = 1$ and $\deg(p_j) \geq 1$ for some $1 \leq j \leq r$.*

Decision problems of linear recurrence sequences. Sign patterns of LRSs have been studied for a long time. Two prominent open problems in this area are the *Skolem Problem* and the *Positivity Problem*. The Skolem Problem is to find an algorithm that, given an LRS u_n , decides if the set $Z = \{n : u_n = 0\}$ is non-empty. The most well-known result in this direction is the celebrated Skolem-Mahler-Lech theorem, which (non-constructively) shows that Z is a union of a finite set and finitely many arithmetic progressions. In particular, it shows that a non-degenerate $(u_n)_{n \in \mathbb{N}}$ can have only finitely many zeros. The Positivity Problem, on the other hand, asks to find an algorithm that determines if $u_n \geq 0$ for all n . It is known to subsume the Skolem Problem as well as certain long-standing open problems in Diophantine approximation [23, Chapter 2].

2.4. Markov chains

A finite-state *discrete-time Markov chain* (DTMC) M is a tuple (S, P, ι) , where S is a finite set of states, $P : S \times S \rightarrow [0, 1]$ is the transition probability function satisfying $\sum_{s' \in S} P_{ss'} = 1$ for all $s \in S$ and $\iota : S \rightarrow [0, 1]$ is the initial distribution, such that $\sum_{s \in S} \iota(s) = 1$. For algorithmic problems, all transition probabilities are assumed to be rational. A finite path ρ in M is a finite sequence $s_0 s_1 \dots s_n$ of states such that $P(s_i, s_{i+1}) > 0$ for all $0 \leq i \leq n - 1$. We say that a state t is reachable from s if there is a finite path from s to t . If all states are reachable from all other states, we say that M is *irreducible*; otherwise, we say it is *reducible*. A set $B \subseteq S$ of states is called a bottom strongly connected component (BSCC) if it is strongly connected, i.e., all states in B are reachable from all other states in B and if there are no outgoing transitions, i.e., $P(s, t) > 0$ and $s \in B$ implies $t \in B$.

Without loss of generality we identify S with $\{1, \dots, d\}$ for $d = |S|$. Then, overloading notation, we consider $P \in \mathbb{R}^{d \times d}$ as a matrix with $P_{ij} = P(j, i)$ for $i, j \leq d$. Likewise, we consider ι to be a (column) vector in \mathbb{R}^d . Then, the sequence of distributions over states after k steps is given by $P^k \iota$, which forms a stochastic LDS. We also write $P_{ij}^{(k)}$ for $(P^k)_{ij}$, which is the probability to move from state j to i in exactly k steps. Further, we say that the matrix P is irreducible if the underlying Markov chain is irreducible. The *period* d_i of a state i is $d_i = \gcd\{m \geq 1 : P_{ii}^{(m)} > 0\}$. If $d_i = 1$, then we call the state i *aperiodic*. A Markov chain (and its transition matrix) is aperiodic if and only if all of its states are aperiodic. The period of a Markov chain M (and, equivalently, of its transition matrix P) is the least common multiple of the periods of the states of M .

A vector $\pi \in \mathbb{R}^d$ is called a stationary distribution of the Markov chain if a) π is a distribution, i.e., $\pi_j \geq 0$ for all j with $1 \leq j \leq d$, and $\sum_{j=1}^d \pi_j = 1$, and b) π is stationary, i.e., $\pi = P\pi$. For aperiodic Markov chains, it is known that the sequence $(P^k\pi)_{k \in \mathbb{N}}$ of probability distributions over states converges to a stationary distribution π , which can be computed in polynomial time (see [28, 29]).

2.5. Kronecker's theorem

We write \mathbb{T} for the interval $[0, 1)$ and $\{x\}$ for the fractional part of $x \in \mathbb{R}$, i.e., $\{x\} = x - \lfloor x \rfloor$. Let $\lambda_1, \dots, \lambda_\ell \in \mathbb{C}$ with $\lambda_i = e^{i2\pi\theta_i}$ for all i , where $\theta_i \in \mathbb{T}$. A *multiplicative relation* of $\lambda_1, \dots, \lambda_\ell$ is a tuple $(k_1, \dots, k_\ell) \in \mathbb{Z}^\ell$ such that $\lambda_1^{k_1} \cdots \lambda_\ell^{k_\ell} = 1$. We write $G(\lambda_1, \dots, \lambda_\ell)$ for the set of all multiplicative relations of $(\lambda_1, \dots, \lambda_\ell)$, which is a free abelian group. If this group consists only of the identity element, we say that $\lambda_1, \dots, \lambda_\ell$ are *multiplicatively independent*. By a deep result of Masser [30], there exists a fixed polynomial p such that G has a basis B of at most ℓ elements such that for each $v \in B$, $\|v\|_\infty < p(\|\lambda_1\| + \dots + \|\lambda_\ell\|)^\ell$. Hence a basis of G can be computed in polynomial space (given $\lambda_1, \dots, \lambda_\ell$) by simply enumerating all possible bases satisfying Masser's bound. Note that $\lambda_1, \dots, \lambda_\ell$ are multiplicatively independent if and only if $1, \theta_1, \dots, \theta_\ell$ are linearly independent over \mathbb{Q} .

We are now ready to state Kronecker's theorem in Diophantine approximation and its well-known consequences for linear recurrence sequences.

Theorem 5 (Kronecker). *Let $\theta_1, \dots, \theta_\ell \in \mathbb{T}$, $\boldsymbol{\theta}_n = (\{n\theta_1\}, \dots, \{n\theta_\ell\})$ for all n , and $G(e^{i2\pi\theta_1}, \dots, e^{i2\pi\theta_\ell}) = \{(n_1, \dots, n_\ell) \in \mathbb{Z}^\ell : n_1\theta_1 + \dots + n_\ell\theta_\ell \in \mathbb{Z}\}$. The sequence $(\boldsymbol{\theta}_n)_{n \in \mathbb{N}}$ is dense in*

$$\{(x_1, \dots, x_\ell) \in \mathbb{T}^\ell \mid k_1x_1 + \dots + k_\ell x_\ell \in \mathbb{Z} \text{ for all } (k_1, \dots, k_\ell) \in G(e^{i2\pi\theta_1}, \dots, e^{i2\pi\theta_\ell})\}.$$

A noteworthy special case is the following, using that linear independence of $1, \theta_1, \dots, \theta_\ell$ over \mathbb{Q} implies that $G(e^{i2\pi\theta_1}, \dots, e^{i2\pi\theta_\ell}) = \{(0, \dots, 0)\}$.

Corollary 6. *If $1, \theta_1, \dots, \theta_\ell$ are linearly independent over \mathbb{Q} , then $(\boldsymbol{\theta}_n)_{n \in \mathbb{N}}$ is dense in \mathbb{T}^ℓ .*

Proof. From the linear independence of $1, \theta_1, \dots, \theta_\ell$ over \mathbb{Q} , it follows that if

$$n_1\theta_1 + \dots + n_\ell\theta_\ell \in \mathbb{Z} \quad \text{for some } (n_1, \dots, n_\ell) \in \mathbb{Z}^\ell,$$

then necessarily $n_1 = \dots = n_\ell = 0$. Hence,

$$G(e^{i2\pi\theta_1}, \dots, e^{i2\pi\theta_\ell}) = \{(0, \dots, 0)\}.$$

So we immediately get

$$\{(x_1, \dots, x_\ell) \in \mathbb{T}^\ell \mid k_1 x_1 + \dots + k_\ell x_\ell \in \mathbb{Z} \text{ for all } k \in G(e^{i2\pi\theta_1}, \dots, e^{i2\pi\theta_\ell})\} = \mathbb{T}^\ell,$$

since the only condition corresponds to $k = (0, \dots, 0)$.

Therefore, by Kronecker's theorem,

$$(\boldsymbol{\theta}_n)_{n \in \mathbb{N}} = (\{n\theta_1\}, \dots, \{n\theta_\ell\})_{n \in \mathbb{N}}$$

is dense in \mathbb{T}^ℓ . □

A further useful reformulation states that $(\gamma_1^n, \dots, \gamma_\ell^n)_{n \in \mathbb{N}}$ for $\gamma_1, \dots, \gamma_\ell \in e^{i2\pi\mathbb{T}}$ is dense in the set of all vectors of complex numbers on the unit circle that have at least the same multiplicative relations as $\gamma_1, \dots, \gamma_\ell$:

Corollary 7 (See [31, Prop. 3.5]). *Let $\gamma_1, \dots, \gamma_\ell \in e^{i2\pi\mathbb{T}}$. The sequence $(\gamma_1^n, \dots, \gamma_\ell^n)_{n \in \mathbb{N}}$ is dense in*

$$\{(z_1, \dots, z_\ell) \in (e^{i2\pi\mathbb{T}})^\ell \mid G(\gamma_1, \dots, \gamma_\ell) \subseteq G(z_1, \dots, z_\ell)\}.$$

Proof. Let $\gamma_i = e^{i2\pi\theta_i}$ with $\theta_i \in \mathbb{T}$ for all i . Define $S = \{(\{n\theta_1\}, \dots, \{n\theta_\ell\}) \mid n \in \mathbb{N}\}$ and $H = \{\mathbf{x} \in \mathbb{T}^\ell \mid \mathbf{k} \cdot \mathbf{x} \in \mathbb{Z} \text{ for all } \mathbf{k} \in G(\gamma_1, \dots, \gamma_\ell)\}$. By Theorem 5, S is dense in H . Now consider the continuous map $f : \mathbb{T}^\ell \rightarrow (e^{i2\pi\mathbb{T}})^\ell$ given by $f(x_1, \dots, x_\ell) = (e^{i2\pi x_1}, \dots, e^{i2\pi x_\ell})$. Then $f(S) = \{(\gamma_1^n, \dots, \gamma_\ell^n) \mid n \in \mathbb{N}\}$ and $f(H) = \{(z_1, \dots, z_\ell) \mid G(\gamma_1, \dots, \gamma_\ell) \subseteq G(z_1, \dots, z_\ell)\}$. Since f is continuous, $(\gamma_1^n, \dots, \gamma_\ell^n)$ is dense in the claimed set. □

Finally, we can apply the theorem to LRS. Namely, any LRS of the form $u_n = \sum_{i=1}^\ell c_i \lambda_i^n$ where $\lambda_i, c_i \in \mathbb{C}$ and $|\lambda_i| = 1$ for all i is either 0 or can be bounded away from 0 infinitely often:

Corollary 8. *Let $u_n = \sum_{i=1}^\ell c_i \lambda_i^n$ be an LRS that is not identically zero, where $\lambda_i, c_i \in \mathbb{C}$ and $|\lambda_i| = 1$ for all i . There exists $c > 0$ such that $|u_n| > c$ for infinitely many n .*

Proof. Let k be such that $u_k \neq 0$. By Corollary 7, the sequence $(\lambda_1^n, \dots, \lambda_\ell^n)_{n \in \mathbb{N}}$ is dense in

$$\{(z_1, \dots, z_\ell) \in (e^{2\pi i \mathbb{T}})^\ell \mid G(\lambda_1, \dots, \lambda_\ell) \subseteq G(z_1, \dots, z_\ell)\}.$$

In particular, since $G(\lambda_1, \dots, \lambda_\ell) \subseteq G(\lambda_1^k, \dots, \lambda_\ell^k)$ for every $k \in \mathbb{N}$, it follows that in every neighborhood of $(\lambda_1^k, \dots, \lambda_\ell^k)$ there exist infinitely many elements of the sequence $(\lambda_1^n, \dots, \lambda_\ell^n)_{n \in \mathbb{N}}$. Finally, since the function $(z_1, \dots, z_\ell) \mapsto \sum_{i=1}^\ell c_i z_i$ is continuous and $u_k \neq 0$, there exists $\varepsilon > 0$ such that $|u_n| > \frac{|u_k|}{2} > 0$ for infinitely many n , as claimed. \square

2.6. Logical theories and o-minimality

A *structure* \mathbb{M} consists of a universe U , constants $c_1, \dots, c_k \in U$, predicates P_1, \dots, P_l where each $P_i \subseteq U^{\mu(i)}$ for some $\mu(i) \geq 1$, and functions f_1, \dots, f_m where each f_i has the type $f_i: U^{\delta(i)} \rightarrow U$ for some $\delta(i) \geq 1$. By the *language* of the structure \mathbb{M} , written $\mathcal{L}_{\mathbb{M}}$, we mean the set of all well-formed first-order formulas constructed from symbols denoting the constants c_1, \dots, c_k , predicates P_1, \dots, P_l , and functions f_1, \dots, f_m , as well as variables and the symbols $\forall, \exists, \wedge, \vee, \neg, =$. A *theory* is simply a set of sentences, i.e. formulas without free variables. The theory of the structure \mathbb{M} , written $\text{Th}(\mathbb{M})$, is the set of all sentences in the language of \mathbb{M} that are true in \mathbb{M} . A theory \mathcal{T} is *decidable* if there exists an algorithm that takes a sentence φ and decides whether $\varphi \in \mathcal{T}$. Below is a list of structures relevant to this work.

- (a) Let $\mathbb{R}_0 = \langle \mathbb{R}; 0, 1, <, +, \cdot \rangle$, which is the ring of real numbers. Observe that using the constants 0, 1 and the addition, we can obtain any constant $c \in \mathbb{N}$. Hence every atomic formula in \mathcal{L}_{or} with k free variables is equivalent to $p(x_1, \dots, x_k) \sim 0$, where p is a polynomial with integer coefficients and \sim is either $>$ or the equality. By the Tarski-Seidenberg theorem, $\text{Th}(\mathbb{R}_0)$ admits quantifier elimination and is decidable.
- (b) Let $\mathbb{R}_{\text{exp}} = \langle \mathbb{R}; 0, 1, <, +, \cdot, \exp \rangle$, the real numbers augmented with the exponentiation function. The theory of this structure is decidable assuming Schanuel's conjecture, a unifying conjecture in transcendental number theory.
- (c) Finally, let $\mathbb{R}_{\text{exp, bt}}$ denote the structure $\langle \mathbb{R}; 0, 1, <, +, \cdot, \exp, \widetilde{\cos}, \widetilde{\sin} \rangle$, which is \mathbb{R}_{exp} augmented with *bounded trigonometric functions*: For $x \in [0, 2\pi]$, $\widetilde{\cos}(x) = \cos(x)$ and $\widetilde{\sin}(x) = \sin(x)$. For all other values of x , we have $\widetilde{\cos}(x) = \widetilde{\sin}(x) = 0$.

We say that a structure \mathbb{S} *expands* \mathbb{M} if \mathbb{S} and \mathbb{M} have the same universe and every constant, function, and relation of \mathbb{M} is also present in \mathbb{S} . We will only need structures expanding \mathbb{R}_0 . A set $X \subseteq U^d$ is *definable* in a structure \mathbb{M} if there exist $k \geq 0$, a formula φ in the language of \mathbb{M} with $d+k$ free variables, and $a_1, \dots, a_k \in U$ such that for all $x_1, \dots, x_d \in U$, $\varphi(x_1, \dots, x_d, a_1, \dots, a_k)$ is true if and only if $(x_1, \dots, x_d) \in X$. We say that X is *definable in \mathbb{M} without parameters* if we can take $k = 0$ above. Similarly, a function is definable (without parameters) in \mathbb{M} if its graph is definable (without parameters) in \mathbb{M} .

A structure \mathbb{M} expanding \mathbb{R}_0 is *o-minimal* if every set definable in \mathbb{M} has finitely many connected components. The structures \mathbb{R}_0 , \mathbb{R}_{exp} , and $\mathbb{R}_{\text{exp}, \text{bt}}$ are all o-minimal [32]. All o-minimal structures admit cell decomposition [33, §4.2]. In particular, every function definable in an o-minimal structure is piecewise continuous and every subset of $[0, 1]^m$ definable in an o-minimal structure is Jordan-measurable.

2.7. Complexity of operations on linear dynamical systems

By the result of Cai [34], given $M \in \mathbb{Q}^{d \times d}$, in polynomial time we can compute $P, J \in \overline{\mathbb{Q}}^{d \times d}$ in polynomial time such that $M = P^{-1}JP$ and J is in Jordan form. In particular, we can compute the eigenvalues $\lambda_1, \dots, \lambda_d$ of M in polynomial time. However, this is where the good news ends: recall from Sec. 2.2 that evaluating a sequence of arithmetic operations on algebraic numbers, unlike the case with integers, cannot be done in polynomial time. A consequence of this is that, after we have computed P, J as above, if we want to compute (J, Pq) (whose orbit is the image of the orbit of the LDS (M, q) under the Jordan change of basis), we need exponential time. We will need the following result [23, Lem. 2.2.4].

Lemma 9. *Given $M \in \mathbb{Q}^{d \times d}$ and $c, q \in \mathbb{Q}^d$, in time polynomial in $(\|M\| + \|q\| + \|c\|)^d$ we can compute the exponential polynomial representation of the linear recurrence sequence $u_n = c^\top M^n q$.*

However, sometimes we do not need to compute the full exponential polynomial representation, as illustrated below.

Lemma 10. *Given $M \in \mathbb{Q}^{d \times d}$ and $q \in \mathbb{Q}^d$, we can decide in polynomial space whether the orbit of (M, q) is bounded.*

Proof. It suffices to decide whether for every $1 \leq i \leq d$, the LRS $u_n = e_i^\top M^n q$, where e_i is the i th standard basis vector, is bounded. Compute the distinct eigenvalues $\lambda_1, \dots, \lambda_m$ of M in polynomial time using the Jordan normal form. As discussed in Sec. 2.3, we can *uniquely* write

$$u_n = \sum_{j=1}^m \sum_{k=0}^{d-1} c_{j,k} n^k \lambda_j^n.$$

By Lem. 4, we need to determine which $c_{j,k}$ are zero. Write

$$u_n = f(n, c_{1,0}, \dots, c_{m,d}) \tag{2}$$

for a polynomial f in $1+md$ variables. Because the d initial values of $(u_n)_{n \in \mathbb{N}}$ determine all of the remaining values, we have that (2) holds for all n if and only if it holds for $n = 0, \dots, d-1$. We can determine whether $c_{j,k} = 0$ as follows. Recall that the values of $c_{j,k}$ are unique. We write the polynomial-size formula

$$\exists c_{1,0}, \dots, c_{m,d} \in \mathbb{C}: c_{j,k} = 0 \wedge \bigwedge_{n=0}^{d-1} u_n = f(n, c_{1,0}, \dots, c_{m,d})$$

in the language of $\langle \mathbb{C}; 0, 1, +, \cdot \rangle$. We then check, in polynomial space, whether the formula is true using the decision procedure for the theory of algebraically closed fields [35], which is identical to the theory of $\langle \mathbb{C}; 0, 1, +, \cdot \rangle$. \square

3. Mean payoff

In this section we study the *mean payoff* of an orbit, which is the average weight collected per step in the long run. For an LDS given by $M \in \mathbb{Q}^{d \times d}$ and $q \in \mathbb{Q}^d$, and a weight function $w: \mathbb{R}^d \rightarrow \mathbb{R}$, we define the mean payoff of the orbit to be

$$\mathcal{MP}_w(M, q) := \lim_{n \rightarrow \infty} \frac{1}{n} \sum_{k=0}^{n-1} w(M^k q)$$

whenever it exists. We show how to compute $\mathcal{MP}_w(M, q)$ in three cases: when w is polynomial (section 3.1), when the orbit is bounded and w is continuous (section 3.2), and when w is bounded and o-minimal (section 3.3). Let us now establish some of the key tools that we will need.

Recall that we denote the interval $[0, 1)$ by \mathbb{T} and the fractional part of x by $\{x\}$. For $\mathbf{x} = (x_1, \dots, x_m) \in \mathbb{T}^m$, let

$$\sigma(\mathbf{x}) = (e^{i2\pi x_1}, e^{-i2\pi x_1}, \dots, e^{i2\pi x_m}, e^{-i2\pi x_m}). \quad (3)$$

We will link mean payoff to certain integrals using Weyl's equidistribution theorem from ergodic theory.

Theorem 11 ([36], see [17] for an exposition). *Let $\theta_1, \dots, \theta_m \in \mathbb{T}$ be such that $1, \theta_1, \dots, \theta_m$ are linearly independent over \mathbb{Q} , and U be a Jordan measurable subset of \mathbb{T}^m . Writing $\boldsymbol{\theta}_n = (\{n\theta_1\}, \dots, \{n\theta_m\})$, we have that*

$$\lim_{n \rightarrow \infty} \frac{|\{0 \leq k < n \mid \boldsymbol{\theta}_k \in U\}|}{n} = \mathcal{L}(U).$$

Consequently, for every (piecewise) continuous $g: \mathbb{T}^m \rightarrow \mathbb{R}$,

$$\lim_{n \rightarrow \infty} \frac{1}{n} \sum_{k=0}^{n-1} g(\boldsymbol{\theta}_k) = \int_{\mathbf{x} \in \mathbb{T}^m} g(\mathbf{x}) d\mathbf{x}$$

where $d\mathbf{x} = dx_1 \cdots dx_m$.

We will apply this theorem to orbits of linear dynamical systems in Sections 3.2 and 3.3 via the following lemma.

Lemma 12. *Let $\lambda_1, \dots, \lambda_\ell \in \overline{\mathbb{Q}}$ where $\lambda_i = \rho_i \gamma_i$ with $\rho_i > 0$, $|\gamma_i| = 1$ and $\gamma_i = e^{i2\pi\theta_i} \in \overline{\mathbb{Q}}$. We can compute $R, m > 0$ for $1 \leq i \leq m$ such that $1, \theta_1, \dots, \theta_m \in \mathbb{T}$ are linearly independent over \mathbb{Q} , and polynomials $p_{i,r}$ for $1 \leq i \leq \ell$ and $0 \leq r < R$ such that*

$$\lambda_i^{nR+r} = \rho_i^{nR+r} \cdot p_{i,r}(\sigma(\boldsymbol{\theta}_n))$$

for all i, n , where $\boldsymbol{\theta}_n = (\{n\theta_1\}, \dots, \{n\theta_m\})$.

Proof. The first step is to compute a basis of the group of multiplicative relations $G(\gamma_1, \dots, \gamma_\ell)$ as described in section 2.5. Next, select a largest multiplicatively independent subset of $\{\gamma_1, \dots, \gamma_\ell\}$. Without loss of generality we can take this to be $\{\gamma_1, \dots, \gamma_m\}$ for some $m \leq \ell$. Then for every $i > m$, we have that $\gamma_1, \dots, \gamma_m, \gamma_i$ are multiplicatively independent, i.e.

$$\gamma_1^{k_{i,1}} \cdots \gamma_m^{k_{i,m}} = \gamma_i^{k_i}$$

for some positive integer k_i and $k_{i,j} \in \mathbb{Z}$ for $1 \leq j \leq m$. Choose $R = k_{m+1} \cdots k_\ell$. Then for $1 \leq i \leq m$, $\lambda_i^{nR+r} = \rho_i^{nR+r}(\gamma_i^r \cdot \gamma_i^{nR})$, and for $i > m$,

$$\lambda_i^{nR+r} = \rho_i^{nR+r} \cdot \gamma_i^{nR+r} = \rho_i^{nR+r} \cdot \gamma_i^r \cdot \left(\gamma_1^{\frac{R \cdot k_{i,1}}{k_i}} \cdots \gamma_m^{\frac{R \cdot k_{i,m}}{k_i}} \right)^n.$$

By construction of R , we have that $\frac{R \cdot k_{i,1}}{k_i}$ is an integer for all $i > m$. \square

3.1. Polynomial weight-functions

In this section fix $M \in \mathbb{Q}^{d \times d}$, $q \in \mathbb{Q}^d$, and a polynomial $p \in \mathbb{Q}[X_1, \dots, X_d]$. We prove the following in this section.

Theorem 13. *It is decidable whether the mean payoff $\mathcal{MP}_w(M, q)$ exists, in which case it is rational and effectively computable.*

In order to analyse the mean payoff of the orbit, first recall from section 2.3 that the sequence $(p(M^n q))_{n \in \mathbb{N}}$ is an LRS over \mathbb{Q} . The following lemma states that the sequence of partial sums of the weights is also a rational LRS.

Lemma 14. *The sequence $u_n = \sum_{k=0}^n p(M^k q)$ is an LRS over \mathbb{Q} .*

Proof. As discussed in section 2.3, $w_n = p(M^n q)$ defines a rational LRS. Suppose $(w_n)_{n \in \mathbb{N}}$ satisfies a recurrence relation

$$w_{n+d} = a_0 w_n + \dots + a_{d-1} w_{n+d-1}$$

where $a_0, \dots, a_{d-1} \in \mathbb{Q}$. Then

$$u_{n+d+1} = u_{n+d} + a_{d-1}(u_{n+d} - u_{n+d-1}) + \dots + a_0(u_{n+1} - u_n)$$

and hence $(u_n)_{n \in \mathbb{N}}$ itself is a rational LRS of order at most $d+1$. \square

Computing $\mathcal{MP}_w(M, q)$ therefore boils down to computing $\lim_{n \rightarrow \infty} u_n/n$ for a rational LRS $(u_n)_{n \in \mathbb{N}}$.

Lemma 15. *Given a rational LRS $(u_n)_{n \in \mathbb{N}}$, it is decidable whether $\lim_{n \rightarrow \infty} \frac{u_n}{n}$ exists. When the limit exists, it is rational and effectively computable.*

Proof. Suppose $(u_n)_{n \in \mathbb{N}}$ is not eventually zero and write $u_n = \sum_{i=1}^m p_i(n) \lambda_i^n$ where $m > 0$, $\lambda_1, \dots, \lambda_m$ are pairwise distinct, $|\lambda_1| \geq \dots \geq |\lambda_m| > 0$, and each p_i is not identically zero. By Theorem 3, for $\epsilon > 0$ we have that $|u_n| > (|\lambda_1| - \epsilon)^n$ for infinitely many n . Hence if $|\lambda_1| > 1$, then the limit does not exist. Similarly, if $|\lambda_1| < 1$, then the limit is zero. Suppose therefore $|\lambda_1| = 1$. Let k be the largest integer such that $|\lambda_i| = 1$ for all $i \leq k$, and define $v_n = \sum_{i=1}^k p_i(n) \lambda_i^n$. It suffices to consider $\lim_{n \rightarrow \infty} v_n/n$ as $\lim_{n \rightarrow \infty} \sum_{i=k+1}^m p_i(n) \lambda_i^n = 0$.

Write $v_n = \sum_{i=0}^l n^i \sum_{j=1}^{k_i} c_{i,j} \lambda_{i,j}^n$ where $\sum_{j=1}^{k_i} c_{i,j} \lambda_{i,j}^n$ is in the exponential polynomial form for all i . If $l = 0$, then $(v_n)_{n \in \mathbb{N}}$ is bounded and $\lim_{n \rightarrow \infty} v_n/n = 0$. Assume therefore $l \geq 1$. Let

$$w_n = \sum_{j=1}^{k_l} c_{l,j} \lambda_{l,j}^n.$$

By construction, $k_l \geq 1$, and by Lemma 1, $(w_n)_{n \in \mathbb{N}}$ is not identically zero. It is shown in [37, Lemma 4] that if $\lambda_{l,j} \neq 1$ for some j (which is implied by $k_l > 1$), then there exist $a, b \in \mathbb{R}$ such that $a < b$, $w_n < a$ for infinitely many n , and $w_n > b$ for infinitely many n . Hence $\lim_{n \rightarrow \infty} v_n/n$ can exist only if $k_l = 1$ and $\lambda_{l,1} = 1$. Suppose indeed that $k_l = 1$ and $\lambda_{l,1} = 1$. Then $\lim_{n \rightarrow \infty} v_n/n$, when it exists, is equal to $\lim_{n \rightarrow \infty} \frac{n^l c_{l,1}}{n}$. Since $l \geq 1$, the latter limit exists if and only if $l = 1$, in which case it is equal to $c_{l,1}$.

To prove rationality of the limit, suppose $\lim_{n \rightarrow \infty} v_n/n$ exists, in which case it is equal to $c_{l,1}$. There must exist $1 \leq i \leq k$ such that $\lambda_i = 1$ and $p_i(n)$ is equal to either $c_{l,1}$ or $nc_{l,1}$. Recall that $\alpha \in \mathbb{C}$ is rational if and only if it is fixed by every automorphism of \mathbb{C} . Since $(u_n)_{n \in \mathbb{N}}$ takes rational values, $\sigma(u_n) = u_n$ for all $n \in \mathbb{N}$ and σ an automorphism of \mathbb{C} . Moreover, $\sigma(u_n) = \sum_{i=1}^m \sigma(p_i(n))\sigma(\lambda_i)^n$ and $\sigma(\lambda_{l,1}) = \lambda_{l,1} = 1$ for every automorphism σ . By the uniqueness of the exponential polynomial representation, $\sigma(c_{l,1} \lambda_{l,1}^n) = c_{l,1} \lambda_{l,1}^n$ for all n and σ , which implies that $c_{l,1}$ is rational. \square

Our main result (Theorem 18) now follows immediately from Lemma 15 and Lemma 14. \square

Complexity of computing the mean payoff. Recall that if the limit of $(\frac{u_n}{n})_{n \in \mathbb{N}}$ exists then it is equal to a coefficient of some p_i appearing in the exponential polynomial solution of $(u_n)_{n \in \mathbb{N}}$. Hence the complexity of computing the limit is bounded by the complexity of computing the exponential polynomial. If

the description length of $(u_n)_{n \in \mathbb{N}}$ is \mathcal{I} and its order is d , the time required to compute the exponential polynomial representation of u_n is polynomial in \mathcal{I}^d (Lem. 9). In particular, computing the mean payoff requires polynomial time if we assume the dimension d of the LDS to be fixed.

3.2. Bounded orbits

In this section, fix a continuous weight function $w: \mathbb{R}^d \rightarrow \mathbb{R}$, as well as $M \in \mathbb{Q}^{d \times d}$ and $q \in \mathbb{Q}^d$ such that $(M^n q)_{n \in \mathbb{N}}$ is bounded. (Recall from Lem. 10 that we can decide whether the orbit is bounded in polynomial space.) We will show that $\mathcal{MP}_w(M, q)$ has a simple integral representation. Our approach can be explained geometrically as follows. A bounded orbit of an LDS approaches a “limit shape”—which is the set of accumulation points of the orbit—closer and closer. This allows us to express the mean payoff in terms of an integral of the weight function over this limit shape. This integral computes the “average” value of the weight function on the limiting shape. Of course, we have to carefully ensure that we also know how “frequently” the orbit approaches different parts of the limiting shape. Let us look at an example that illustrates our approach.

Example 2. Let $w: \mathbb{R}^3 \rightarrow \mathbb{R}$ be a continuous weight function and consider the LDS

$$M = \begin{bmatrix} 3/5 & 4/5 & 0 \\ -4/5 & 3/5 & 0 \\ 0 & 0 & 1/2 \end{bmatrix} \quad \text{and} \quad q = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}.$$

The first two coordinates evolve under a repeated application of a rotation. In the complex plane, this rotation is given by multiplication with $3/5 - 4/5i$. As $3/5 - 4/5i$ is not a root of unity, the orbit never reaches a point with $(1, 0)$ in the first two coordinates again. In fact, the first two components of the orbit are dense in the unit circle in \mathbb{R}^2 . Furthermore, by, for example, Weyl’s equidistribution theorem, each sub-interval of the unit circle is visited with frequency proportional to its length. The third component is halved at every step and converges to 0. As the weight function is continuous, we can hence treat the third coordinate as equal to 0 when determining the mean payoff. More formally, the set of accumulation points of the orbit is

$$L = \{v = (v_1, v_2, v_3) \in \mathbb{R}^3 \mid v_3 = 0, |v| = 1\}$$

which can be parametrised via $g(\theta) = (\cos(2\pi\theta), \sin(2\pi\theta), 0)$, $g: \mathbb{T} \rightarrow \mathbb{R}^3$. As this parametrisation moves through the circle with constant speed reflecting

the fact that the orbit is “equally distributed” over the circle in the first two components, we can now express the mean payoff of the orbit with respect to the weight function w as

$$\mathcal{MP}_w(M, q) = \int_0^1 w(\cos(2\pi\theta), \sin(2\pi\theta), 0) d\theta.$$

We now move on to the proof. For $1 \leq i \leq d$, let $u_n^{(i)} = e_i M^n q$, where e_i is the i th standard basis vector. We will need the following lemma.

Lemma 16. *For all $1 \leq i \leq d$,*

$$u_n^{(i)} = v_n^{(i)} + w_n^{(i)}$$

where $(v_n^{(i)})_{n \in \mathbb{N}}$ is a simple LRS over $\mathbb{R} \cap \overline{\mathbb{Q}}$ whose characteristic roots all have modulus one, and $(w_n^{(i)})_{n \in \mathbb{N}}$ is an LRS over $\mathbb{R} \cap \overline{\mathbb{Q}}$ whose characteristic roots all have modulus less than one, i.e. $\lim_{n \rightarrow \infty} w_n^{(i)} = 0$.

Proof. Fix $2 \leq i \leq d$, and write

$$u_n^{(i)} = \sum_{j=1}^{\ell} p_j(n) \lambda_j^n$$

where the right-hand side is in the exponential polynomial form. If $|\lambda_j| > 1$ for some j , then by Theorem 3, $|u_n^{(i)}|$ diverges to infinity, contradicting boundedness. Therefore, $|\lambda_j| \leq 1$ for all j .

First, let $J_{=1} = \{j : |\lambda_j| = 1\}$ and $J_{<1} = \{j : |\lambda_j| < 1\}$. Then, we can write

$$u_n^{(i)} = \sum_{j \in J_{=1}} p_j(n) \lambda_j^n + \sum_{j \in J_{<1}} p_j(n) \lambda_j^n$$

and define $v_n^{(i)} = \sum_{j \in J_{=1}} p_j(n) \lambda_j^n$ and $w_n^{(i)} = \sum_{j \in J_{<1}} p_j(n) \lambda_j^n$.

Now, let $m = \max_{j \in J_{=1}} \deg(p_j)$ and define $J_{=1}^{\max} = \{j \in J_{=1} : \deg(p_j) = m\}$. We will show that $m = 0$. Write

$$v_n^{(i)} = n^m \sum_{j \in J_{=1}^{\max}} c_j \lambda_j^n + \sum_{j \in J_{=1}} q_j(n) \lambda_j^n \tag{*}$$

where, for all $j \in J_{=1}$, the polynomial q_j is either the polynomial p_j without the leading monomial if $j \in J_{=1}^{\max}$ or equal to p_j if $j \notin J_{=1}$. In other words,

for all $j \in J_{=1}^{\max}$, we have $p_j = c_j n^m + q_j$, and, for all $j \in J_{=1} \setminus J_{=1}^{\max}$, we have $p_j = q_j$. So, $\deg(q_j) < m$ for all $j \in J_{=1}$.

By Corollary 8, there exists $c > 0$ such that $|\sum_{j \in J_{=1}^{\max}} c_j \lambda_j^n| > c$ for infinitely many $n \in \mathbb{N}$. It follows that if $m > 0$, then $|n^m \sum_{j \in J_{=1}^{\max}} c_j \lambda_j^n|$ becomes arbitrarily large and dominates the sum in $(*)$ infinitely often, which again contradicts the boundedness assumption. So, indeed $m = 0$. Hence, the sequences $v_n^{(i)}$ and $w_n^{(i)}$ are as claimed in the statement of the lemma. \square

Let $v_n^{(i)}, w_n^{(i)}$ be as above. We have that for all $N \in \mathbb{N}$,

$$\mathcal{MP}_w(M, q) = \lim_{n \rightarrow \infty} \frac{1}{n} \sum_{k=N}^{N+n-1} w(M^k q).$$

From the continuity of w and the fact that $\lim_{n \rightarrow \infty} w_n^{(i)} = 0$ we deduce that

$$\mathcal{MP}_w(M, q) = \lim_{n \rightarrow \infty} \frac{1}{n} \sum_{k=0}^{n-1} w(v_k^{(1)}, \dots, v_k^{(d)}).$$

Next, let $\Lambda = \{\lambda_1, \dots, \lambda_\ell\} \subset \overline{\mathbb{Q}}$ be such that $|\lambda_i| = 1$ for all i and the characteristic roots of every $(v_n^{(i)})_{n \in \mathbb{N}}$ belong to Λ . Invoking Lemma 12 compute $\theta_1, \dots, \theta_m \in \mathbb{T}$ that are linearly independent over \mathbb{Q} , $R > 0$, and the polynomials $p_{i,r}$ for $1 \leq i \leq \ell$ and $0 \leq r < R$. Write $\boldsymbol{\theta}_n$ for $(\{n\theta_1\}, \dots, \{n\theta_m\})$ as usual. We now prove the main result of this section.

Theorem 17. *We can compute a function $f: \mathbb{R}^m \rightarrow \mathbb{R}$ such that*

$$\mathcal{MP}_w(M, q) = \int_{\mathbf{x} \in \mathbb{T}^m} f(\mathbf{x}) d\mathbf{x}$$

where $d\mathbf{x} = dx_1 \cdots dx_m$.

Proof. Write $v_n^{i,r} = v_{nR+r}^{(i)}$ for $1 \leq i \leq d$ and $0 \leq r < R$. We have that

$$\mathcal{MP}_w(M, q) = \lim_{n \rightarrow \infty} \frac{1}{n} \sum_{k=0}^{n-1} w(v_k^{(1)}, \dots, v_k^{(d)}) \tag{4}$$

$$= \frac{1}{R} \sum_{r=0}^{R-1} \lim_{n \rightarrow \infty} \frac{1}{n} \sum_{k=0}^{n-1} w(v_k^{1,r}, \dots, v_k^{d,r}). \tag{5}$$

Applying Lemma 12, because $|\lambda_i| = 1$ for all $\lambda_i \in \Lambda$, we have that $\lambda_i^{nR+r} = p_{i,r}(\sigma(\boldsymbol{\theta}_n))$. Since each $(v_n^{1,r})_{n \in \mathbb{N}}$ is simple and $v_n^{1,r}$ is a $\overline{\mathbb{Q}}$ -linear combination of $\lambda_1^{nR+r}, \dots, \lambda_\ell^{nR+r}$, we have that

$$w(v_k^{1,r}, \dots, v_k^{d,r}) = w(h_{1,r} \circ \sigma(\boldsymbol{\theta}_k), \dots, h_{d,r} \circ \sigma(\boldsymbol{\theta}_k))$$

where each $h_{i,r}: \mathbb{R}^{2m} \rightarrow \mathbb{R}$ is a polynomial with algebraic coefficients. Thus for each r there exists an entrywise polynomial function h_r such that

$$w(v_k^{1,r}, \dots, v_k^{d,r}) = w \circ h_r \circ \sigma(\boldsymbol{\theta}_k)$$

for all k . Note that $h_r \circ \sigma$ and thus $w \circ h_r \circ \sigma$ are continuous. Since the weight function w is continuous by assumption, we can apply Theorem 11 to each summand of (5) to obtain

$$\begin{aligned} \mathcal{MP}_w(M, q) &= \frac{1}{R} \sum_{r=0}^{R-1} \int_{\mathbf{x} \in \mathbb{T}^m} w \circ h_r \circ \sigma(\mathbf{x}) d\mathbf{x} \\ &= \int_{\mathbf{x} \in \mathbb{T}^m} \frac{w \circ h_0 \circ \sigma(\mathbf{x}) + \dots + w \circ h_{R-1} \circ \sigma(\mathbf{x})}{R} d\mathbf{x}. \quad \square \end{aligned}$$

Approximation of the mean payoff. Although each h_i is entrywise polynomial and σ is entrywise exponential (see Eq. 3), there is no general way to evaluate the m -fold multiple integral computed above. Nevertheless, the numerical approximation of integrals is an extensively studied area (see, e.g., [38]). By linearity of integration, it suffices to approximate $\int_{\mathbf{x} \in \mathbb{T}^m} g_i(\mathbf{x}) d\mathbf{x}$, where $g_i = w \circ h_i \circ \sigma(\mathbf{x})$, for $i = 1, \dots, d$. When is g_i approximately integrable? There is no single answer, but perhaps most well-known requirement is that g_i be approximable to arbitrary precision at every $\mathbf{x} \in \mathbb{T}^m$ and be Lipschitz continuous with known upper bound for its Lipschitz constant over \mathbb{T}^m . (For more details on conditions under which the integral can be approximated to arbitrary precision, we refer the reader to [38].) The functions h_r, σ are differentiable and we can bound the modulus of the gradient of $h_r \circ \sigma$ on \mathbb{T}^m . Consequently, if the function w is approximable to arbitrary precision at every $\mathbf{x} \in \mathbb{T}^m$ and Lipschitz continuous over \mathbb{T}^m with known upper bound for its Lipschitz constant, then we can deduce the same for every g_i , from which approximability of the integral follows.

3.3. O-minimal weight functions

In this section, fix $M \in \mathbb{Q}^{d \times d}$ and $q \in \mathbb{Q}^d$, as well as a bounded weight function $w: \mathbb{R}^d \rightarrow [-b, b]$ that is definable in some o-minimal expansion \mathbb{M}

of $\mathbb{R}_{\text{exp}, \text{bt}}$. Let $\lambda_1, \dots, \lambda_d$ be the eigenvalues of M . Compute $R, \theta_1, \dots, \theta_m$ as in Lemma 12 and write $\boldsymbol{\theta}_n = (\{n\theta_1\}, \dots, \{n\theta_m\})$. Our main result is the following, which shows that o-minimality implies good ergodic properties.

Theorem 18. *We can compute a function $f: \mathbb{T}^m \rightarrow \mathbb{R}$ definable in \mathbb{M} such that*

$$\mathcal{MP}_w(M, q) = \int_{\mathbf{x} \in \mathbb{T}^m} f(\mathbf{x}) d\mathbf{x}$$

where $d\mathbf{x} = dx_1 \cdots dx_m$.

Example 3. Let $M = CD$ and $q = (0, 1) \in \mathbb{R}^2$ where $C = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$ and $D = \begin{bmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{bmatrix}$ for some θ that is not a rational multiple of π . Consider the weight function

$$w(x, y) = \left(1 + \frac{1}{|x| + |y| + 1}\right) \cdot (1 + e^{-(x^2 + y^2)}) \cdot \arctan\left(\frac{xy}{x^2 + y^2}\right).$$

Then w is definable in $\mathbb{R}_{\text{exp}, \text{bt}}$: we have that

$$\arctan(x) = y \Leftrightarrow -\frac{\pi}{2} < y < \frac{\pi}{2} \wedge \frac{\sin(y)}{\cos(y)} = x.$$

Moreover, $M^n = C^n D^n = 2^n D^n$ for all $n \geq 0$. Write $(u_n, v_n) = D^n q$: we then have that $((u_n, v_n))_{n \in \mathbb{N}}$ is dense (and equidistributed) in the unit circle by the assumption on θ . Plugging in $M^n q$ into w , we immediately obtain that

$$\begin{aligned} & \lim_{n \rightarrow \infty} \left(w(M^n q) - \arctan\left(\frac{4^n \cdot u_n v_n}{4^n \cdot u_n^2 + 4^n \cdot v_n^2}\right) \right) \\ &= \lim_{n \rightarrow \infty} \left(\left(1 + \frac{1}{2^n(|u_n| + |v_n|) + 1}\right) \cdot (1 + e^{-4^n \cdot (u_n^2 + v_n^2)}) \cdot \arctan\left(\frac{4^n \cdot u_n v_n}{4^n \cdot u_n^2 + 4^n \cdot v_n^2}\right) \right. \\ &\quad \left. - \arctan\left(\frac{4^n \cdot u_n v_n}{4^n \cdot u_n^2 + 4^n \cdot v_n^2}\right) \right) \\ &= 0. \end{aligned}$$

Combining this with the fact that $u_n^2 + v_n^2 = 1$, we obtain that

$$\mathcal{MP}_w(M, q) = \lim_{n \rightarrow \infty} \frac{1}{n} \sum_{k=0}^{n-1} \arctan(u_n v_n).$$

By the Weyl equidistribution theorem from ergodic theory (discussed below), the latter quantity is equal to the integral of the function

$$g(x, y) = \arctan(xy)$$

over the unit circle in \mathbb{R}^2 . Therefore,

$$\mathcal{MP}_w(M, q) = \int_{x \in [0, 1)} \arctan(\cos(2\pi x) \cdot \sin(2\pi x)) dx.$$

We prove Theorem 18 in the remainder of this section. For $n \in \mathbb{N}$ and $1 \leq i \leq r$, let $u_n^{(i)} = e_i^\top M^n q$ and $u_n^{i,r} = u_{nR+r}^{(i)}$. Recall that

$$\begin{aligned} \mathcal{MP}_w(M, q) &= \lim_{n \rightarrow \infty} \frac{1}{n} \sum_{k=0}^{n-1} w(u_k^{(1)}, \dots, u_k^{(d)}) \\ &= \frac{1}{R} \sum_{r=0}^{R-1} \lim_{n \rightarrow \infty} \frac{1}{n} \sum_{k=0}^{n-1} w(u_k^{1,r}, \dots, u_k^{d,r}). \end{aligned}$$

Applying Lemma 12, for all i, r we can compute a function $f_{i,r}$ that is definable in $\mathbb{R}_{\text{exp}, \text{bt}}$ such that $u_n^{i,r} = f_{i,r}(n, \{n\theta_1\}, \dots, \{n\theta_m\})$ for all n . Because \mathbb{M} extends $\mathbb{R}_{\text{exp}, \text{bt}}$ by assumption, we can therefore compute a function $g_r: \mathbb{R} \times \mathbb{T}^m \rightarrow [-b, b]$ definable in \mathbb{M} such that

$$w(u_n^{1,r}, \dots, u_n^{d,r}) = g_r(n, \boldsymbol{\theta}_n).$$

We will next show that the sequence of functions $(g_{r,n})_{n \in \mathbb{N}}$, defined by

$$g_{r,n}(x_1, \dots, x_m) = g_r(n, x_1, \dots, x_m)$$

must converge pointwise for all n .

Lemma 19. *Let $h_n: \mathbb{T}^m \rightarrow [-b, b]$ for $n \in \mathbb{N}$ be such that for all $\mathbf{x} = (x_1, \dots, x_m) \in \mathbb{T}^m$ and $y \in [-b, b]$,*

$$h_n(\mathbf{x}) = y \Leftrightarrow \varphi(n, x_1, \dots, x_m, y)$$

where φ is a fixed formula in the language of \mathcal{M} . There exists a function $h: \mathbb{T}^m \rightarrow [-b, b]$ definable in \mathbb{M} , whose representation can be computed effectively, such that for every $\mathbf{x} \in \mathbb{T}^m$,

$$\lim_{n \rightarrow \infty} h_n(\mathbf{x}) = h(\mathbf{x}).$$

Proof. Fix $\mathbf{x} \in \mathbb{T}^m$, and let z be an accumulation point of $(h_n(\mathbf{x}))_{n \in \mathbb{N}}$, whose existence is guaranteed by the Bolzano-Weierstraß theorem. Define the function $f: \mathbb{R} \rightarrow \mathbb{R}_{\geq 0}$ by

$$f(t) = d \Leftrightarrow y \in [-b, b]: \varphi(t, x_1, \dots, x_m, y) \wedge |y - z| = d$$

which, at $t \in \mathbb{N}$, measures the distance from $h_t(\mathbf{x})$ to z . Since it is definable in the o-minimal structure \mathbb{M} , it is ultimately monotonic [33, §4.1]. By construction of z , $\liminf_{t \rightarrow \infty} f(t) = 0$. It follows that $\lim_{t \rightarrow \infty} f(t) = 0$. The function h is therefore defined by

$$\begin{aligned} h(x_1, \dots, x_m) = y &\Leftrightarrow \\ \forall \varepsilon > 0. \exists t. \forall t' > t. \exists y': \varphi(t, x_1, \dots, x_m, y') \wedge |y' - y| < \varepsilon. &\quad \square \end{aligned}$$

Applying the lemma above, for $0 \leq r < R$, let $g_r: \mathbb{T}^m \rightarrow [-b, b]$ be the pointwise limit of $(g_{r,n})_{n \in \mathbb{N}}$, which is definable in \mathbb{M} and hence piecewise continuous (see section 2.6). We next show that when computing the mean payoff, we can work with the limit g_r rather than the exact sequence $(g_{r,n})_{n \in \mathbb{N}}$. This bring us closer to applying the equidistribution theorem, which requires a fixed function that is applied at all to the orbit of the dynamical system given by a translation on \mathbb{T}^m .

Lemma 20. *Let $0 \leq r < R$. We have that*

$$\lim_{n \rightarrow \infty} \frac{1}{n} \sum_{k=0}^{n-1} g_{r,k}(\boldsymbol{\theta}_k) = \lim_{n \rightarrow \infty} \frac{1}{n} \sum_{k=0}^{n-1} g_r(\boldsymbol{\theta}_k).$$

Proof. For $\varepsilon > 0$ and $n \in \mathbb{N}$, let $X_{\varepsilon,n}$ be the set of all $x \in \mathbb{T}^m$ such that for all $m \geq n$, $|g_{r,m}(x) - g_r(x)| < \varepsilon$. We have that $X_{\varepsilon,n}$ is definable in \mathbb{M} and thus (Jordan and Lebesgue) measurable (section 2.6). We denote the Jordan measure of a set $X \subseteq \mathbb{T}^m$ by $\mathcal{C}(X)$. By construction of g_r as the pointwise limit of $(g_{r,n})_{n \in \mathbb{N}}$ and the definition of $X_{\varepsilon,n}$,

$$\mathbb{T}^m = \bigcup_{n \in \mathbb{N}} X_{\varepsilon,n}$$

for all $\varepsilon > 0$. Moreover, $X_{\varepsilon,n} \subseteq X_{\varepsilon,m}$ for all $\varepsilon > 0$ and $n \leq m$. By a standard property of measures (“continuity from below”), $\mathcal{C}(X_{\varepsilon,n})$ converges monotonically to 1 as $n \rightarrow \infty$ for all $\varepsilon > 0$.

Let $\Delta > 0$. To prove our result it suffices to show that for all sufficiently large n ,

$$\left| \frac{1}{n} \sum_{k=0}^{n-1} g_{r,k}(\boldsymbol{\theta}_k) - \frac{1}{n} \sum_{k=0}^{n-1} g_r(\boldsymbol{\theta}_k) \right| < \Delta$$

which is equivalent to

$$\left| \sum_{k=0}^{n-1} (g_{r,k}(\boldsymbol{\theta}_k) - g_r(\boldsymbol{\theta}_k)) \right| < n\Delta.$$

Recall that $[-b, b]$ is the image of g_r , and choose $\mu \in (0, 1)$ and $\varepsilon, \delta > 0$ such that $(1 - \mu + \delta) \cdot 2b < \frac{\Delta}{2}$ and $(\mu + \delta) \cdot \varepsilon < \frac{\Delta}{4}$. Let N be such that $\mathcal{C}(X_{\varepsilon, N}) > \mu$, and write $X = X_{\varepsilon, N}$. For all $n \in \mathbb{N}$,

$$\begin{aligned} \sum_{k=0}^{n-1} g_{r,k}(\boldsymbol{\theta}_k) - g_r(\boldsymbol{\theta}_k) &= \\ &\sum_{k=0}^{n-1} \mathbb{1}(\boldsymbol{\theta}_k \in X)(g_{r,k}(\boldsymbol{\theta}_k) - g_r(\boldsymbol{\theta}_k)) + \sum_{k=0}^{n-1} \mathbb{1}(\boldsymbol{\theta}_k \notin X)(g_{r,k}(\boldsymbol{\theta}_k) - g_r(\boldsymbol{\theta}_k)) \end{aligned}$$

where $\mathbb{1}$ denotes the indicator function. By the Weyl equidistribution theorem, for all sufficiently large n we have that

$$\frac{1}{n} \sum_{k=0}^{n-1} \mathbb{1}(\boldsymbol{\theta}_k \in X) \in (\mu - \delta, \mu + \delta)$$

and

$$\frac{1}{n} \sum_{k=0}^{n-1} \mathbb{1}(\boldsymbol{\theta}_k \notin X) \in (1 - \mu - \delta, 1 - \mu + \delta).$$

Hence for all sufficiently large n ,

$$\left| \sum_{k=0}^{n-1} \mathbb{1}(\boldsymbol{\theta}_k \notin X)(g_{r,k}(\boldsymbol{\theta}_k) - g_r(\boldsymbol{\theta}_k)) \right| < (1 - \mu + \delta)n \cdot 2b < \frac{n\Delta}{2}.$$

Next, recall that for $k \geq N$, $|g_{r,k}(\boldsymbol{\theta}_k) - g_r(\boldsymbol{\theta}_k)| < \varepsilon$ by the construction of X . For $k < N$, we have that $|g_{r,k}(\boldsymbol{\theta}_k) - g_r(\boldsymbol{\theta}_k)| \leq 2b$. Hence for all sufficiently large n ,

$$\left| \sum_{k=0}^{n-1} \mathbb{1}(\boldsymbol{\theta}_k \in X)(g_{r,k}(\boldsymbol{\theta}_k) - g_r(\boldsymbol{\theta}_k)) \right| < n(\mu + \delta) \cdot 2\varepsilon < \frac{n\Delta}{2}.$$

It remains to apply the triangle inequality. \square

Combining the lemma above with Theorem 11 we deduce that

$$\lim_{k \rightarrow \infty} \frac{1}{n} \sum_{k=0}^{n-1} g_r(\boldsymbol{\theta}_k) = \int_{\mathbf{x} \in \mathbb{T}^m} g_r(\mathbf{x}) d\mathbf{x}.$$

Thus

$$\begin{aligned} \mathcal{MP}_w(M, q) &= \lim_{n \rightarrow \infty} \frac{1}{n} \sum_{k=0}^{n-1} w(u_k^{(1)}, \dots, u_k^{(d)}) \\ &= \frac{1}{R} \sum_{r=0}^{R-1} \lim_{n \rightarrow \infty} \frac{1}{n} \sum_{k=0}^{n-1} w(u_k^{1,r}, \dots, u_k^{d,r}) \\ &= \frac{1}{R} \sum_{r=0}^{R-1} \lim_{n \rightarrow \infty} \frac{1}{n} \sum_{k=0}^{n-1} g_r(\boldsymbol{\theta}_k) \\ &= \int_{\mathbf{x} \in \mathbb{T}^m} \frac{g_0(\mathbf{x}) + \dots + g_{R-1}(\mathbf{x})}{R} d\mathbf{x}. \end{aligned}$$

which concludes the proof of Theorem 18.

Approximating mean payoff. Recall that each $g_r: \mathbb{T}^m \rightarrow [-b, b]$ is definable in the o-minimal structure \mathbb{M} and thus piecewise continuous (section 2.6). Let us take $\mathbb{M} = \mathbb{R}_{\text{exp, bt}}$. The first-order theory of this structure is decidable assuming Schanuel's conjecture. Thus, by verifying truth of various formulas (which can be done assuming Schanuel's conjecture), we can compute an upper bound on the Lipschitz constant of each g_r and approximate its value on $\mathbf{x} \in \mathbb{T}^m$ with rational coordinates to arbitrary precision. We can therefore, similarly to section 3.2, apply generic techniques to (conditionally) approximate mean payoff to arbitrary precision.

A few comments about the use of Schanuel's conjecture in deciding the first-order theory of $\mathbb{R}_{\text{exp, bt}}$ is in order. Wilkie [39] gives an algorithm that, given any formula φ (in the language of $\mathbb{R}_{\text{exp, bt}}$) with free variables x_1, \dots, x_d , constructs an equivalent system of polynomial equalities and inequalities in $x_i, e^{x_i}, \widetilde{\cos}(x_i)$ for $1 \leq i \leq d$. This system is then conditionally solved using the approach of [40]. To see how Schanuel's conjecture might be applied, recall that $\cos(x) = \frac{1}{2}(e^{ix} + e^{-ix})$. Importantly, the method of [40] relies on

Schanuel's conjecture only for termination: whenever it terminates, it is unconditionally guaranteed to correctly decide whether φ holds in $\mathbb{R}_{\text{exp}, \text{bt}}$ or not. Unfortunately, the computational complexity of these decision procedures is not known. We refer the reader to the survey [41] for more information on applications of Schanuel's conjecture.

3.4. Stochastic linear dynamical systems

Stochastic LDSs are a special case of LDSs with a bounded orbit. We will now show that for such systems, we can compute the mean payoff of the orbit under a continuous weight function by evaluating the weight function on finitely many points. In the aperiodic case, the orbit even converges to a single point, and consequently it suffices to evaluate the weight function once.

Theorem 21. *Let $P \in \mathbb{Q}^{d \times d}$ be a stochastic, aperiodic matrix and $\iota \in \mathbb{Q}^d$ an initial distribution. Further let $w: \mathbb{R}^d \rightarrow \mathbb{R}$ be a continuous weight function. Then, $\mathcal{MP}_w(P, \iota) = w(\pi)$ where π is the stationary distribution $\lim_{n \rightarrow \infty} P^n \iota$ of P , which is computable in polynomial time.*

Proof. As described in section 2.4, we know that the orbit $(P^n \iota)_{n \in \mathbb{N}}$ converges to a stationary distribution π in this case, which can be computed in polynomial time [28, 29]. As w is continuous, $\lim_{n \rightarrow \infty} w(P^n \iota) = w(\pi)$. It follows that

$$\mathcal{MP}_w(P, \iota) = \lim_{n \rightarrow \infty} \frac{1}{n} \sum_{k=0}^{n-1} w(P^k \iota) = w\left(\lim_{n \rightarrow \infty} P^n \iota\right) = w(\pi). \quad \square$$

Hence in the aperiodic case the computation of the mean payoff reduces to evaluating the function w once on a rational point computable in polynomial time. We next address the periodic case by splitting up the orbit into subsequences.

For an irreducible and periodic Markov chain with period L , we have that P^L is aperiodic and $L \leq d$ by [42, Theorem 1.8.4]. Together with Theorem 21 this allows us to compute $\mathcal{MP}_w(P^L, P^r \iota)$, which satisfies

$$\mathcal{MP}_w(P, \iota) = \frac{1}{L} \sum_{r=0}^{L-1} \mathcal{MP}_w(P^L, P^r \iota).$$

That is, for an irreducible stochastic LDS we can divide $(P^n \iota)_{n \in \mathbb{N}}$ into L equally spaced subsequences and compute the mean payoff $\mathcal{MP}_w(P, \iota)$ as the arithmetic mean of the mean payoffs of these subsequences.

Theorem 22. Let $P \in \mathbb{Q}^{d \times d}$ be a stochastic, irreducible matrix and $\iota \in \mathbb{Q}^d$ an initial distribution. Let $w: \mathbb{R}^d \rightarrow \mathbb{R}$ be a continuous weight function. Then, we can compute points $\pi_0, \dots, \pi_{L-1} \in \mathbb{Q}^d$ in polynomial time for some $L \leq d$ such that $\mathcal{MP}_w(P, \iota) = \frac{1}{L} \sum_{i=0}^{L-1} w(\pi_i)$.

Write $\|x\|$ for the bit length of x . Since the points π_0, \dots, π_{L-1} can be computed in polynomial time, they have bit length at most polynomial in the length of the original input. Therefore, we have the following.

Corollary 23. Assume that the value $w(a)$ can be approximated in time $f_w(\|a\|, \epsilon)$ up to some precision $\epsilon \geq 0$ (where $\epsilon = 0$ corresponds to exact computation) for all rational inputs $a \in \mathbb{Q}^d$. There is a fixed polynomial p such that the mean payoff $\mathcal{MP}_w(P, \iota)$ can be approximated up to precision ϵ in time at most $d \cdot f_w(p(\|(P, \iota)\|, \epsilon) + p(\|(P, \iota)\|))$.

When a Markov chain is reducible, the states can be renamed in a way such that, the matrix representation of the Markov chain will contain distinct blocks corresponding to the bottom strongly connected components (BSCCs) on the diagonal along with additional columns at the right representing states that do not belong to any BSCC:

$$\begin{bmatrix} & 0\dots0 & 0\dots0 & * & * \\ & & & & \\ 0\dots0 & & 0\dots0 & * & * \\ & & & & \\ 0\dots0 & 0\dots0 & & * & * \\ & 0\dots0 & 0\dots0 & 0\dots0 & * & * \end{bmatrix}$$

Each block representing a BSCC constitutes an irreducible Markov chain. Assume we have k blocks with periods L_1, L_2, \dots, L_k correspondingly. Let l be the least common multiple of the periods. Now we will have l subsequences of the orbit each of which will converge. The convergence of the rows in the bottom is a result of the fact that Markov chain will enter a BSCC with probability 1. So, in general, we have l subsequences of the orbit, all of which converge. We observe that $l \leq d^d$, from which the following result follows:

Theorem 24. Let $P \in \mathbb{Q}^{d \times d}$ be a stochastic matrix and $\iota \in \mathbb{Q}^d$ an initial distribution. Let $w: \mathbb{R}^n \rightarrow \mathbb{R}$ be a continuous weight function. Then, we can compute points $\pi_0, \dots, \pi_{l-1} \in \mathbb{Q}^d$ in exponential time for some $l \leq d^d$ such that $\mathcal{MP}_w(P, \iota) = \frac{1}{l} \sum_{i=0}^{l-1} w(\pi_i)$.

As the transition matrix P^l of the l subsequences as well as the initial values $P^{r\iota}$ with $0 \leq r < l$ can be computed in polynomial time by repeated squaring, each of the points π_i with $0 \leq i < l$ can be computed in polynomial time. Assuming that the value $w(a)$ can be approximated in time $f_w(\|a\|, \epsilon)$ for all rational inputs $a \in \mathbb{Q}^d$, we can hence conclude that there is again a fixed polynomial q such that the mean payoff of reducible stochastic LDSs can be approximated to precision ϵ in time bounded by $d^d \cdot f_w(q(\|(P, \iota)\|, \epsilon) + d^d \cdot q(\|(P, \iota)\|)$ analogously to Corollary 23.

4. Total (discounted) reward and satisfaction of energy constraints

In this section, we address the computation of total accumulated rewards and total discounted rewards as well as the problem to decide whether the accumulated reward ever drops below a given bound – a problem known as the satisfaction of energy constraints – for LDSs with polynomial weight functions. First, we prove that the total as well as the discounted accumulated weight of the orbit is computable and rational if finite. Afterwards, we discuss Baker’s theorem and its consequences before applying these to solve the energy constraint problem for low-dimensional LDSs. We furthermore provide two different hardness results showing that, in dimension 4, the problem is hard with respect to certain open decision problems in Diophantine approximation and that restricting to stochastic LDSs and linear weight functions also does not lead to decidability in general.

4.1. Total reward

Let $M \in \mathbb{Q}^{d \times d}$ be a matrix, $q \in \mathbb{Q}^d$ be an initial vector, and $w: \mathbb{R}^d \rightarrow \mathbb{R}$ be a polynomial weight function with rational coefficients. We define the *total reward* as

$$\text{tr}(M, q, w) := \sum_{k=0}^{\infty} w(M^k q).$$

Likewise, for a discount factor $\delta \in (0, 1) \cap \mathbb{Q}$ we define the *total discounted reward* as

$$\text{dr}(M, q, w, \delta) := \sum_{k=0}^{\infty} \delta^k \cdot w(M^k q).$$

Both of these quantities, when they exist, can be determined effectively.

Lemma 25. *For a rational LRS $(u_n)_{n \in \mathbb{N}}$, it is decidable whether $\lim_{n \rightarrow \infty} u_n$ exists, in which case the limit is rational and effectively computable.*

Proof. The sequence $v_n = nu_n$ is also a rational LRS. It remains to observe that $\lim_{n \rightarrow \infty} u_n = \lim_{n \rightarrow \infty} v_n/n$ and apply Lemma 15. \square

Theorem 26. *It is decidable whether $\sum_{k=0}^{\infty} w(M^k q)$ and $\sum_{k=0}^{\infty} \delta^k \cdot w(M^k q)$ converge, in which case their values are rational and effectively computable.*

Proof. Let $u_n = \sum_{k=0}^n w(M^k q)$. As argued in section 3.1, $(u_n)_{n \in \mathbb{N}}$ is a rational LRS, and we can apply Lemma 25. Similarly, let $v_n = \sum_{k=0}^{\infty} \delta^k \cdot w(M^k q)$. As $(\delta^n)_{n \in \mathbb{N}}$ is itself a (rational) LRS and such LRS are closed under pointwise multiplication, v_n is also a rational LRS. We again apply Lemma 25. \square

4.2. Baker's theorem and its applications

We now discuss Baker's theorem, which is the most important tool for quantitatively analysing growth of low-order linear recurrence sequences. We will use it in the analysis of energy constraints in the next section. A *linear form in logarithms* is an expression of the form

$$\Lambda = b_1 \operatorname{Log} \alpha_1 + \cdots + b_m \operatorname{Log} \alpha_m$$

where $b_i \in \mathbb{Z}$ and $\alpha_i \in \overline{\mathbb{Q}}$ for all $1 \leq i \leq m$. Here Log denotes the principal branch of the complex logarithm. The celebrated theorem of Baker places a lower bound on $|\Lambda|$ in case $\Lambda \neq 0$. Baker's theorem, as well as its p -adic analogue, play a critical role in the proof of [4] that the Skolem Problem is decidable for linear recurrence sequences of order at most 4, as well as decidability of the Positivity Problem for low-order sequences [6].

Theorem 27 (Special case of the main theorem in [43]). *Let Λ be as above, D be the degree of the field extension $\mathbb{Q}(\alpha_1, \dots, \alpha_m)/\mathbb{Q}$, and suppose $A, B \geq 3$ are such that $A > H(\alpha_i)$ and $B > |b_i|$ for all $1 \leq i \leq m$. If $\Lambda \neq 0$, then*

$$\log |\Lambda| > -(16mD)^{2(m+2)} (\log A)^m \log B.$$

A consequence of Baker's theorem is the following [7, Corollary 8].

Lemma 28. *Let $\alpha, \beta \in \overline{\mathbb{Q}}$ with $|\alpha| = 1$. For all $n \geq 2$, if $\alpha^n \neq \beta$ then $|\alpha^n - \beta| > n^{-C}$ where C is an effective constant that depends only α and β .*

If α is not a root of unity, $\alpha^n = \beta$ holds for at most one n which can be effectively bounded.

Lemma 29. *Let $\alpha, \beta \in \overline{\mathbb{Q}}$ be non-zero, and suppose α is not a root of unity. There exists effectively computable $N \in \mathbb{N}$ such that $\alpha^n \neq \beta$ for all $n > N$.*

Proof. The Weil height [44, Chapter 3.2] of α , denoted by $h(\alpha)$, is non-zero under the assumption on α and satisfies $h(\alpha^n) = nh(\alpha)$. We can therefore choose $N = \lceil h(\beta)/h(\alpha) \rceil$. \square

Combining the two lemmas above, we obtain the following.

Theorem 30. *Let $\alpha, \beta \in \overline{\mathbb{Q}}$, and suppose that $|\alpha| = 1$ and α is not a root of unity. There exist effectively computable $N, C \in \mathbb{N}$ such that for all $n > N$, $|\alpha^n - \beta| > n^{-C}$.*

The next lemma summarises the family of linear recurrence sequences to which we can apply Baker's theorem.

Lemma 31. *Let $\gamma \in \{z \in \overline{\mathbb{Q}} : |z| = 1\}$ be not a root of unity, $r_1, \dots, r_\ell \in \mathbb{R}$ be non-zero, and $u_n = \sum_{i=1}^m c_i \Lambda_i^n$ be an LRS over \mathbb{R} where $m \geq 1$, $c_i, \Lambda_i \in \overline{\mathbb{Q}}$ are non-zero for all i , and $\Lambda_1, \dots, \Lambda_m$ are pairwise distinct. Suppose each Λ_i is in the multiplicative group generated by $\{\gamma, r_1, \dots, r_\ell\}$.*

- (a) *There exists effectively computable N_1 such that $u_n \neq 0$ for all $n > N_1$.*
- (b) *For $n > N_1$, $|u_n| > L^n n^{-C}$, where $L = \max_i |\Lambda_i|$ and C is an effectively computable constant.*
- (c) *It is decidable whether $u_n \geq 0$ for all n .*

Proof. Define $\mathcal{D} = \{i : |\Lambda_i| = L\}$ and $\mathcal{R} = \{i : |\Lambda_i| < L\}$. The terms $c_i \Lambda_i^n$ with $i \in \mathcal{D}$ are called *dominant*. We have

$$u_n = \underbrace{\sum_{i \in \mathcal{D}} c_i \Lambda_i^n}_{v_n} + \underbrace{\sum_{i \in \mathcal{R}} c_i \Lambda_i^n}_{z_n}.$$

We first investigate $|v_n|$ as $n \rightarrow \infty$. Recall that each Λ_i is of the form $\gamma^{m_0} r_1^{m_1} \cdots r_\ell^{m_\ell}$, where $m_0, \dots, m_\ell \in \mathbb{Z}$. That is, for all i , $\Lambda_i = |\Lambda_i| \gamma^{k_i}$ for some $k_i \in \mathbb{Z}$. Hence we can write

$$v_n = L^n \sum_{i=-K}^K b_i \gamma^{in} \tag{6}$$

where each b_i is equal to some c_j and $b_K \neq 0$. The values of i range over $\{-K, \dots, K\}$ because v_n is real-valued and hence the summands in (6) appear in conjugate pairs. We have

$$v_n = \gamma^{-Kn} L^n \sum_{i=0}^{2K} b_{-K+i} \gamma^{in} = \gamma^{-Kn} L^n \prod_{i=0}^{2K} (\gamma^n - \alpha_i)$$

where $\alpha_0, \dots, \alpha_{2K} \in \overline{\mathbb{Q}}$ are the zeros of the polynomial $p(z) = \sum_{i=0}^{2K} b_{-K+i} z^i$. Since γ is not a root of unity, we can apply Theorem 30 to each factor $(\gamma^n - \alpha_i)$ to conclude that there exist effectively computable N_1, C such that $|v_n| > L^n n^{-C}$ for all $n > N_1$. Since $|\Lambda_i| < L$ for all $i \in \mathcal{R}$, there exists (effectively computable) N_2 such that $|v_n| > |z_n|$ for $n > N_2$. We have proven (a) and (b).

Since u_n is real-valued, by Lemma 2 for each $1 \leq i \leq m$ there exists $1 \leq j \leq m$ such that $c_j \Lambda_j = \overline{c_i \Lambda_i}$. Hence both v_n and z_n are real-valued. By the analysis above $\text{sign}(u_n) = \text{sign}(v_n)$ for $n > N_2$. Hence to check if u_n is positive we have to check whether $u_n \geq 0$ for $0 \leq n \leq N_2$ and $v_n \geq 0$ for $n > N_2$. We show how to do the latter. Since γ is not a root of unity, $(\gamma^n)_{n \in \mathbb{N}}$ is dense in $e^{i2\pi\mathbb{T}}$. Define $f(z) = z^{-K} p(z)$, noting that $v_n = L^n \cdot f(\gamma^n)$. Consider $Z := f(e^{i2\pi\mathbb{T}}) \subset \mathbb{R}$, which is compact and equal to the closure of $\{\gamma^{-Kn} L^n p(\gamma^n) \mid n \in \mathbb{N}\}$. If Z contains a negative number, then by density of $(\gamma^n)_{n \in \mathbb{N}}$ in $e^{i2\pi\mathbb{T}}$, v_n is negative for infinitely many n . Hence $u_n < 0$ for infinitely many n . Otherwise, $v_n \geq 0$ for all n and hence $u_n \geq 0$ for all $n > N_2$. This concludes the proof of (c). \square

4.3. Satisfaction of energy constraints

We next discuss *energy constraints*. We say that a series of real weights $(w_i)_{i \in \mathbb{N}}$ satisfies the energy constraint with budget B if

$$\sum_{i=0}^k w_i \geq -B$$

for all $k \in \mathbb{N}$. We will prove that for LDS (M, q) of dimension at most 3, satisfaction of energy constraints is decidable. The proof is based on the fact that three-dimensional systems are tractable thanks to Baker's theorem [45]. For higher-dimensional systems, no such tractability result is known. We will show that deciding satisfaction of energy constraints is, in general, at least as hard as the Positivity Problem, already with linear weight functions. Before

giving our decidability result, we need one final ingredient about partial sums of LRS. Let $w_n = n^l \lambda^n$ for some $l \geq 0$ and $\lambda \in \mathbb{Q}$, and $u_n = \sum_{k=0}^n w_k$. If $\lambda = 1$, then $u_n = p(n)$, where p is a polynomial of degree $l+1$ with rational coefficients. If $\lambda \neq 1$, then $u_n = q(n)\lambda^n$, where q is a polynomial of degree at most l with algebraic coefficients satisfying $q(n+1)\lambda = q(n) + n^l$. It follows that if the LRS $(w_n)_{n \in \mathbb{N}}$ has only real eigenvalues, then so does the sequence given by $u_n = \sum_{k=0}^n w_k$. Similarly, if $(w_n)_{n \in \mathbb{N}}$ is diagonalisable and does not have 1 as an eigenvalue, then the same applies to $(u_n)_{n \in \mathbb{N}}$. In fact, the eigenvalues of $(u_n)_{n \in \mathbb{N}}$ form a subset of the eigenvalues of $(w_n)_{n \in \mathbb{N}}$.

Theorem 32. *Let $M \in \mathbb{Q}^{3 \times 3}$, $q \in \mathbb{Q}^3$, $\delta \leq 1$ be a discount factor, and $w: \mathbb{R}^3 \rightarrow \mathbb{R}$ be a polynomial weight function with rational coefficients. For $B \in \mathbb{Q}_{\geq 0}$, it is decidable whether the weights $(\delta^n \cdot w(M^n q))_{n \in \mathbb{N}}$ satisfy the energy constraint with budget B .*

Proof. Let $w_n = \delta^n \cdot w(M^n q)$ and $u_n = B + \sum_{i=0}^n w(M^i q)$. We have to decide whether $u_n \geq 0$ for all n . First suppose M has only real eigenvalues. Then w_n and u_n are both LRSs with only real eigenvalues. By taking subsequences if necessary, we can assume $(u_n)_{n \in \mathbb{N}}$ is non-degenerate. Write $u_n = \sum_{i=1}^m p_i(n) \rho_i^n$ where the right-hand side is in the exponential-polynomial form. In particular, for all i , p_i is not the zero polynomial. Since $(u_n)_{n \in \mathbb{N}}$ is non-degenerate, without loss of generality we can assume $\rho_1 > \dots > \rho_m > 0$. If $p_1(n)$ is negative for sufficiently large n , then the energy constraint is not satisfied. Otherwise, we can compute N such that for all $n > N$, $u_n > 0$. It remains to check whether $u_n \geq 0$ for $0 \leq n \leq N$.

Next, suppose M has non-real eigenvalues $\lambda, \bar{\lambda}$, and a real eigenvalue ρ . Write $\gamma = \lambda/|\lambda|$ and $r = |\lambda|$. Then u_n is of the form

$$u_n = cn + \sum_{i=1}^m c_i \Lambda_i^n := cn + v_n$$

where $\Lambda_1, \dots, \Lambda_m$ are pairwise distinct and in the multiplicative group generated by r, ρ, δ, γ . Without loss of generality we can assume $c_i \neq 0$ for all i , but c may be zero. If γ is a root of unity of order $k > 0$ (i.e. $\gamma^k = 1$), then we can take subsequences $(u_n^{(0)})_{n \in \mathbb{N}}, \dots, (u_n^{(k-1)})_{n \in \mathbb{N}}$, where $u_n^{(j)} = u_{nk+j}$ for $n \in \mathbb{N}$ and $0 \leq j < k$, and each $(u_n^{(j)})_{n \in \mathbb{N}}$ has only real eigenvalues. We can then apply the analysis above. Hereafter we assume γ is not a root of unity.

Suppose $c = 0$. Then Lemma 31 (c) applies and we can decide whether $u_n \geq 0$ for all n . Next, suppose $c \neq 0$ and $L := \max_i |\Lambda_i|$ satisfies $L \leq 1$.

We can compute N_2 such that $|cn| > |v_n|$ for all $n > N_2$. Hence in this case $u_n \geq 0$ for all n if and only if $c > 0$ and $u_n > 0$ for $0 \leq n \leq N_2$. Finally, suppose $c \neq 0$ and $L > 1$. Applying Lemma 31 (b), there exists effectively computable N_3 such that $|u_n| > |cn|$ for $n > N_3$. Hence $u_n \geq 0$ for all n if and only if $u_n \geq 0$ for $0 \leq n \leq N_3$ and $\sum_{i=1}^m c_i \Lambda_i^n \geq 0$ for $n > N_3$. The latter can be decided by applying Lemma 31 (c) to the sequence $v_n = \sum_{i=1}^m c_i \Lambda_i^{N_3+n} = \sum_{i=1}^m (c_i \Lambda_i^{N_3}) \Lambda_i^n$. \square

4.4. Positivity and Diophantine hardness

Recall that the energy satisfaction problem is to decide, given a matrix $M \in \mathbb{Q}^{d \times d}$, $q \in \mathbb{Q}^d$, $B \in \mathbb{Q}$, and a polynomial p with rational coefficients, whether there exists n such that $\sum_{k=0}^n p(M^k q) < B$. This problem is at least as hard as the Positivity Problem already for stochastic linear dynamical systems and linear weight functions.

Theorem 33. *The Positivity Problem can be reduced to the energy satisfaction problem restricted to a Markov chain (M, q) and a linear function w .*

Proof. It is known from [18, 19] that the Positivity Problem for arbitrary LRS over \mathbb{Q} can be reduced to the following problem: given a Markov chain (M, q) , decide whether there exists n such that $e_1^\top M^n q \geq 1/2$, where $e_1 = (1, 0, \dots, 0) \in \mathbb{R}^d$. We reduce the latter to the energy satisfaction problem. Given a Markov chain $(M, q) \in \mathbb{Q}^{d \times d} \times \mathbb{Q}^d$, let

$$P = \begin{bmatrix} M & \mathbf{0} \\ \mathbf{0} & M \end{bmatrix}$$

and $t = (\frac{1}{2}q, \frac{1}{2}Mq) \in \mathbb{Q}^{2d}$. Observe that (P, t) is also a Markov chain. Moreover, $P^n t = (\frac{1}{2}M^n q, \frac{1}{2}M^{n+1}q)$ for all n . We choose the weight function $w(x_1, \dots, x_{2d}) = 2(x_{d+1} - x_1)$ and $B = \frac{1}{2} - e_1 \cdot q$. Then

$$w(P^n q) = e_1^\top M^{n+1} q - e_1^\top M^n q$$

and $u_n := \sum_{k=0}^n w(P^k q) \geq B$ if and only if $e_1^\top M^{n+1} q \geq \frac{1}{2}$. Hence there does not exist n such that $e_1^\top M^n q \geq \frac{1}{2}$ if and only if $e_1 \cdot q < \frac{1}{2}$ and there does not exist n such that $u_n < B$. \square

The Positivity Problem for rational linear recurrence sequences of order 6 was shown in [6] to be *Diophantine-hard*. Specifically, for $r \in \mathbb{Q}$ and $\lambda \in \mathbb{Q}(\mathbf{i})$ (that is, $\lambda = a + bi$ where $a, b \in \mathbb{Q}$) let

$$u_n^{\lambda, r} = -n + \frac{n}{2}(\lambda^n + \overline{\lambda^n}) + \frac{r\mathbf{i}}{2}(\overline{\lambda^n} - \lambda^n) = r \operatorname{Im}(\lambda^n) - n \operatorname{Re}(\lambda^n) + n.$$

If for all $r \in \mathbb{Q}$ and $\lambda \in \mathbb{Q}(i)$ we can decide whether $u_n^{\lambda,r} \geq 0$ for all $n \geq 0$, then we could compute the *Lagrange constants* of a large class of numbers, which would amount to a major mathematical breakthrough in number theory. The following theorem states that a solution to the energy satisfaction problem for rational LDS in dimension 4 with polynomial weight functions would also yield the same breakthrough.

Theorem 34. *The energy satisfaction problem for rational linear dynamical systems in dimension 4 and polynomial weight functions with rational coefficients is Diophantine-hard.*

Proof. We prove the following: If we can decide the energy satisfaction problem with $d = 4$, then we can decide for every r, λ whether $u_n^{\lambda,r} \geq 0$ for all n . Fix $r \in \mathbb{Q}$ and $\lambda = a + bi \in \mathbb{Q}(i)$. Define

$$M = \begin{bmatrix} a & -b & 1 & 0 \\ b & a & 0 & 1 \\ 0 & 0 & a & -b \\ 0 & 0 & b & a \end{bmatrix}$$

and the initial point $q = (0, 0, 0, 1)$. We have

$$M^n q = (-n \operatorname{Im}(\lambda^{n-1}), n \operatorname{Re}(\lambda^{n-1}), -\operatorname{Im}(\lambda^n), \operatorname{Re}(\lambda^n)).$$

Recall that for all $n \in \mathbb{N}$, (i) $\operatorname{Re}(\lambda^{n+1}) = a \operatorname{Re}(\lambda^n) - b \operatorname{Im}(\lambda^n)$ and (ii) $\operatorname{Im}(\lambda^{n+1}) = a \operatorname{Im}(\lambda^n) + b \operatorname{Re}(\lambda^n)$. Hence there exist polynomials p_1, p_2, p_3, p_4 with rational coefficients such that $p_1(M^n q) = n \operatorname{Im}(\lambda^n)$, $p_2(M^n q) = n \operatorname{Re}(\lambda^n)$, $p_3(M^n q) = \operatorname{Re}(\lambda^n)$ and $p_4(M^n q) = \operatorname{Im}(\lambda^n)$ for all n . Next, consider

$$w_n = u_{n+1}^{\lambda,r} - u_n^{\lambda,r} = r \operatorname{Im}(\lambda^{n+1}) - n \operatorname{Re}(\lambda^{n+1}) - r \operatorname{Im}(\lambda^n) + n \operatorname{Re}(\lambda^n) + 1.$$

Since $u_0^{\lambda,r} = 0$, we have $u_{n+1}^{\lambda,r} = \sum_{k=0}^n w_k$. Moreover, using facts (i), (ii) and the polynomials p_1, \dots, p_4 we can construct a polynomial p with rational coefficients such that $w_n = p(M^n q)$ for all n . Hence $u_n^{\lambda,r} \geq 0$ for all n if and only if the weights $(p(M^n q))_{n \in \mathbb{N}}$ satisfy the energy constraint with budget $B = 0$. \square

5. Conclusion and discussion

We have shown how to compute (or approximate) the mean-payoff and the total or discounted weight of the orbit of a rational linear dynamical system for various combinations of classes of systems as well as weight functions

(see Table 1). Remarkably, the results concerning *infinite horizon* questions (e.g. mean payoff, as opposed to satisfaction of energy constraints) do not rely on restrictions of the dimension in a stark contrast to decidability results for the Skolem [4, 5] and the Positivity [6, 7] problems, which themselves are special cases of reachability questions about the orbit of an LDS. This is in line with vast decidability of various *robust* versions of the reachability (alternatively, safety) problem of linear dynamical systems, which similarly does not depend on the dimension and is also proven using o-minimality [9]. Our results in section 3.3 can be seen as a counterpoint to these decidability results regarding robust reachability: we show that o-minimality implies strong ergodic properties.

For the question of whether an orbit of a rational LDS with a polynomial weight function satisfies an energy constraints, on the other hand, we have shown decidability for dimension 3 by utilising deep number-theoretic tool that form the cornerstone of the theory of linear recurrence sequences, most notable Baker’s theorem on linear forms in logarithms. As it is expected with all finite (but not a priori bounded) horizon problems of linear dynamical systems, we obtain Diophantine-hardness for the full problem, specifically already for systems in the ambient space \mathbb{R}^d . In fact, restricting the dynamical system to be stochastic (i.e. a Markov chain) and the weight function to be linear still results in a problem that is at least as hard as the Positivity Problem for linear recurrence sequences. This, again, is unsurprising in light of [18, 19] results that show that for the Skolem and Positivity problems, restricting the LDS to be a Markov chain does not change much.

What happens if we consider arbitrary (in particular, unbounded) LDS and weight functions w that are not o-minimal? We believe that in this case, generally speaking, one should not expect to have an integral representation for the mean payoff in the way we proved for o-minimal weight function. To see this, first of all observe that in the proof of Thm. 18, given a d -dimensional LDS (M, q) and an o-minimal weight function $w: \mathbb{R}^d \rightarrow \mathbb{R}$, we really construct an o-minimal function $\tilde{f}: \mathbb{R}^{m+d} \rightarrow \mathbb{R}$ such that

$$\mathcal{MP}_w(M, q) = \int_{\mathbf{x} \in \mathbb{T}^{m+d}} \tilde{f}(\mathbf{x}, q) d\mathbf{x}.$$

That is, our integral expression depends on the starting configuration in a controlled way; This fact crucially depends on the inability of w to have any “oscillatory” or “fractal-like” behaviour, which is guaranteed by the o-minimality assumption. (Without a reasonable dependence on the starting

point q , in our opinion, it does not even make sense to talk about whether an integral representation exists or not.) On the other hand, if we take $d = 1$, the linear map $x \mapsto 2x$, and the weight function $w(x) = 1$ if $\sin(x) \geq 0$ and $w(x) = -1$ if $\sin(x) < 0$ (which is not o-minimal), then the sequence $\frac{1}{n} \sum_{k=0}^{n-1} \sin(2^k x)$ diverges to either $+\infty$ or $-\infty$ (depending on the initial point x in an inextricable way), i.e. the mean payoff does not exist [46]. Results of this kind are proven using *discrepancy theory*. In summary, we believe that o-minimality nicely captures the class of weight functions for which the mean payoff is guaranteed to exist and can be written as an integral.

Recall that we showed the energy satisfaction problem with polynomial weights is decidable in dimension 3 (Thm. 32), and Diophantine-hard in dimension 4. Restricting the weight function to be linear and the LDS to be stochastic still results in a Diophantine-hard problem, but for LDS of dimension $d = 14$; see [19] and the proof of Thm. 33. If we additionally restrict M to be diagonalisable, then we do not have Diophantine-hardness. However, the resulting problem still subsumes the Positivity Problem for diagonalisable LRS, which is open for LRS of order $d = 9$. Inspecting the proof of Thm. 33 and invoking [19], we obtain that the corresponding energy satisfaction problem is hard in dimension $d = 20$. However, in practical verification, these hardness results may not matter: It is well-known that if we sample a polynomial at random, then under any reasonable probability distribution, with probability one it will have either a single real or two non-real dominant roots [47]. For LRS whose characteristic polynomial has at most two dominant roots, all major problems, including the Skolem Problem and the Positivity Problem, are known to be decidable. Hence whether we can decide a given instance $p, (M, q), B$, where p is a polynomial weight function and B is the budget, really depends on the number of dominant roots of the LRS $u_n = p(M^n q)$, which for typical instances might be 1 or 2.

In the future, this work can be extended in at least two ways. Firstly, one can consider *continuous-time* linear dynamical systems equipped with a weight function. For such systems only hyperplane and halfspace reachability problems (which are analogues of the Skolem and Positivity problems, respectively) are well-understood, and the number-theoretic tools used in their analyses differ significantly from the ones used in this work [48]. However, continuous-time linear dynamical systems also benefit from applications of o-minimality [9], and we believe that they also have good ergodic properties with respect to “tame” (e.g., o-minimal) weight functions. Secondly, one can partition \mathbb{R}^d into a collection of semialgebraic sets S_1, \dots, S_m and

assign a fixed reward $w(S_i)$ to each S_i . Here the reward received at time n is $\sum_{i=1}^m \mathbb{1}(M^n q \in S_i)w(S_i)$. In this setting, already for M that is a rotation in \mathbb{R}^2 , comparing the (discounted) total reward as well as the mean payoff against a given threshold appears to have deep connections to Diophantine approximation [49].

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