μ_0 Measured by Mutual Induction

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Abstract

In this paper, we show our measurement for the permeability of free space μ_0 found by measuring the electromotive force generated in a solenoid by mutual inductance from another solenoid placed concentrically, and relating it to μ_0 through known and derived equations. We find from our measurements that $\mu_{0_{measured}} = 4.30 \cdot 10^{-12} \pm 1.38 \cdot 10^{-12}\ \left[\frac{\mathrm{H}}{\mathrm{m}}\right]$ and we compare this with the accepted value for $\mu_{0_{expected}}$ $1.26 \cdot 10^{-6} \pm 2.00 \cdot 10^{-16}\ \left[\frac{\mathrm{H}}{\mathrm{m}}\right]$ as reported by the NIST. We found that our experimental value had a percent error of 100 [%] here compared to the accepted value. .

INTRODUCTION

Mutual induction is the process by which a rate of change in current through one circuit induces a voltage in another circuit nearby. If we have two circuits, labeled n and 0 where circuit n has some changing current, and circuit 0 generates some voltage due to it, the effect can be described by the equation,

$$V_{0n} = M_n \frac{dI_n}{dt} \quad \text{(Eq. 1)}$$

where V_{0n} is the voltage across circuit 0 in units of volts [V], and I_n is the current through circuit n in units of Amperes [A], and $\frac{dI_n}{dt}$ is the rate of change of current through circuit n in units of $\left[\frac{\mathrm{A}}{\mathrm{s}}\right]$. M_n is the coefficient of mutual induction, in units of Henry [H]. In the case of two concentric solenoids where the outer solenoid is circuit n (has changing current) and the inner circuit is circuit 0 (has induced voltage across it), and the coefficient of mutual induction M_n is expressed as

$$M_n = \frac{\mu_0 N_n N_0 A_0}{L_0} \quad \text{(Eq. 2)}$$

where μ_0 is the permeability of free space that we are trying to measure, N_n is the number of turns in solenoid n, N_0 is the number of turns in solenoid 0, A_0 is the cross sectional area of solenoid 0, parallel to the loop created by a single turn of the solenoid in units of [m^2], and L is the height of the solenoid in units of [m]. By plugging (Eq. 2) into (Eq. 1) we find that

$$V_{0n} = \frac{\mu_0 N_n N_0 A_0}{L_0} \frac{dI_n}{dt} \quad \text{(Eq. 3)}$$

Recall that we are trying to find μ_0. By rearranging (Eq. 3) we find that

$$\mu_0 = \frac{V_{0n} L_0}{N_n N_0 A_0 \frac{dI_n}{dt}} \quad \text{(Eq. 4)}$$

Due to Ohm's Law which says that in a circuit,

$$V_n = I_n R_n \quad \text{(Eq. 5)}$$

where V_n is voltage across a circuit in volts [V], I_n is current through the circuit in amperes [A] and R_n is the resistance through the circuit in ohms [Ω]. This leads us to,

$$I_n = \frac{V_n}{R_n} \quad \text{(Eq. 6)}$$

and

$$\frac{dI_n}{dt} = \frac{dV_n}{dt}\frac{1}{R_n} \quad \text{(Eq. 7)}$$

Plugging (Eq. 5), (Eq. 6) and (Eq. 7) into (Eq. 4) we find that,

$$\mu_0 = \frac{I_{0n} R_0 L_0}{N_n N_0 A_0 \frac{1}{R_n}\frac{dV_n}{dt}} \quad \text{(Eq. 8)}$$

Note that while V_{0n} and I_{0n} have the subscript $0n$ as it describes voltage and current induced in solenoid 0 by solenoid n, that R_0 despite being related to V_{0n} and I_{0n} by Ohm's law, only has subscript 0 for being a circuit element of solenoid 0, NOT a variable that is affected or induced by circuit n. Using (Eq. 8) we can find the permeability of free space μ_0 by creating two concentric solenoids, each with some known turns N_0 and N_n as well as resistance R_0 and R_n [Ω] respectively, and applying to the outer solenoid, solenoid n, some known rate of change in voltage [$\frac{dV_n}{dt}$], and measuring the current that is generated in the inner solenoid, solenoid 0 that is I_{0n} [A]. Note that (Eq. 8) is not the simplest form of the theoretical equation, but as we will see later in the Methods section, (Eq. 4) does not contain the variables we need to find μ_0 in this experiment.

METHOD

We constructed a solenoid which we called solenoid 0 with some known distance and number of turns which we will calculate later. This solenoid is the one that will have voltage induced, and stayed constant throughout the experiment. We also constructed multiple solenoids (indexed as solenoid n throughout) labeled solenoid 1 through 5 that was larger than solenoid 0 and was placed concentrically with it. These solenoids carried a constant time varying voltage which will induce voltage in solenoid 0. The number of turns in a solenoid must be an integer to be as close to the theoretical model as possible and this was achieved by twisting the two ends of the solenoids together by crossing them over and then diverting the leads in separate directions. This allowed us to lock the beginning and end of the solenoid in place at the same point, ensuring that the number of turns remained an integer. A 100 [Ω] resistor was attached in series to one end of the outer solenoid, solenoid n, where solenoid n, again, represents any of the 5 outer solenoids we used. A 1 [Ω] resistor was attached in series to the end of the inner solenoid, solenoid 0. The necessity of the resistors will be explained shortly. A function generator's positive and negative leads were attached to the two leads of our outer circuit, solenoid n's circuit, and a ramp wave was fed through it. A ramp wave, while periodic, has a constant rate of change of voltage through it, allowing our variable $\frac{dV_n}{dt}$ to be a constant, greatly simplifying our calculations and making them feasible. For our inner circuit, that containing solenoid 0, an electrometer's positive and negative leads were attached to the two ends of the circuit to measure the current that ran through it. The resistors are necessary pieces in this setup, as the function generator can control only voltage, not current and while the electrometer can measure voltage, because it is hard to measure the bounds for voltage generated along the path of a circuit, we measure current through the entire circuit instead, and use Ohm's Law to convert the current into voltage across the circuit. The necessity of resistors in our experiment explains why in the Introduction, we chose (Eq. 8) rather than (Eq. 4) for our theoretical equation. See the next page for a schematic of our experimental setup.

Figure I. Circuit Schematic of The Experimental Setup

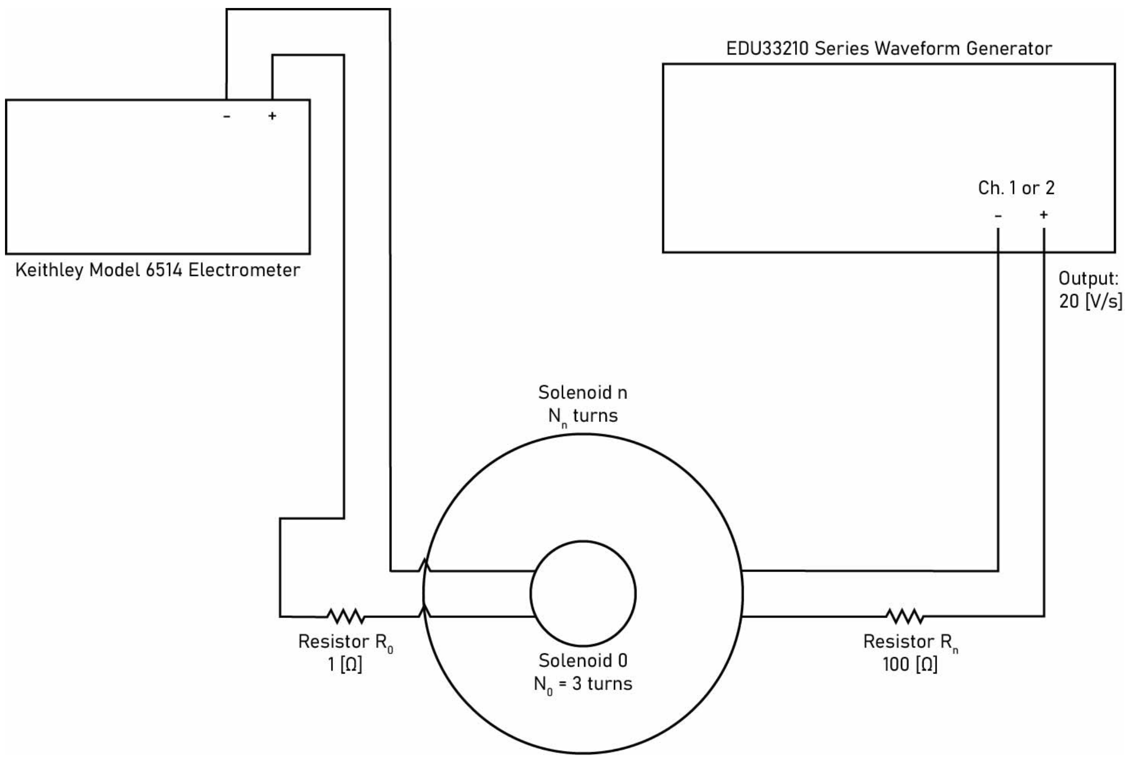

Figure I. The circuit schematic of our experimental setup is shown here. All circuit components are labeled. We did not distinguish between which channel of the waveform generator the outer solenoid n was connected to, as we assumed the output to be identical.

Construction Details

The coils are made of copper wire with non-conductive varnish over it. This allowed for the construction of a solenoid without the wires of successive turns touching each other. As the varnish is non-conductive, we sanded the leads of the solenoids down when attaching resistors and the function generator's as well as electrometer's probes so that current can flow between them. The geometry of the coils were created by simply wrapping the copper wire around a roll of blue tape. The two solenoids were taped to a cardboard sheet to fix their locations relative to each other. In an ideal solenoid, the magnetic field within it should be uniform everywhere, meaning that theoretically, the magnetic field of solenoid n is the same everywhere within it. Therefore the location of solenoid 0 relative to solenoid n should not matter as long as solenoid 0 is within solenoid n, but as our solenoids are not theoretically ideal, by fixing the location of our 2 solenoids relative to each other, we mitigate any effects that may occur from non-perfect geometry. The resistors were attached to one end of the solenoids in series by simply wrapping the leads of the resistors around the much thicker lead of the solenoid. This allowed us to attach and remove the resistors to and from our circuits easily to attach to another circuit while allowing the connection to be firm enough such that when attached current was flowing between the connection at all times. The electrometer and function generator's leads were an alligator clip and a retractable probe. Both were attached by simply clipping the leads of the circuit on, be it the end of a solenoid or a resistor. See the figure below for a photo of the apparatus.

Figure II. Photo of the Experimental Setup

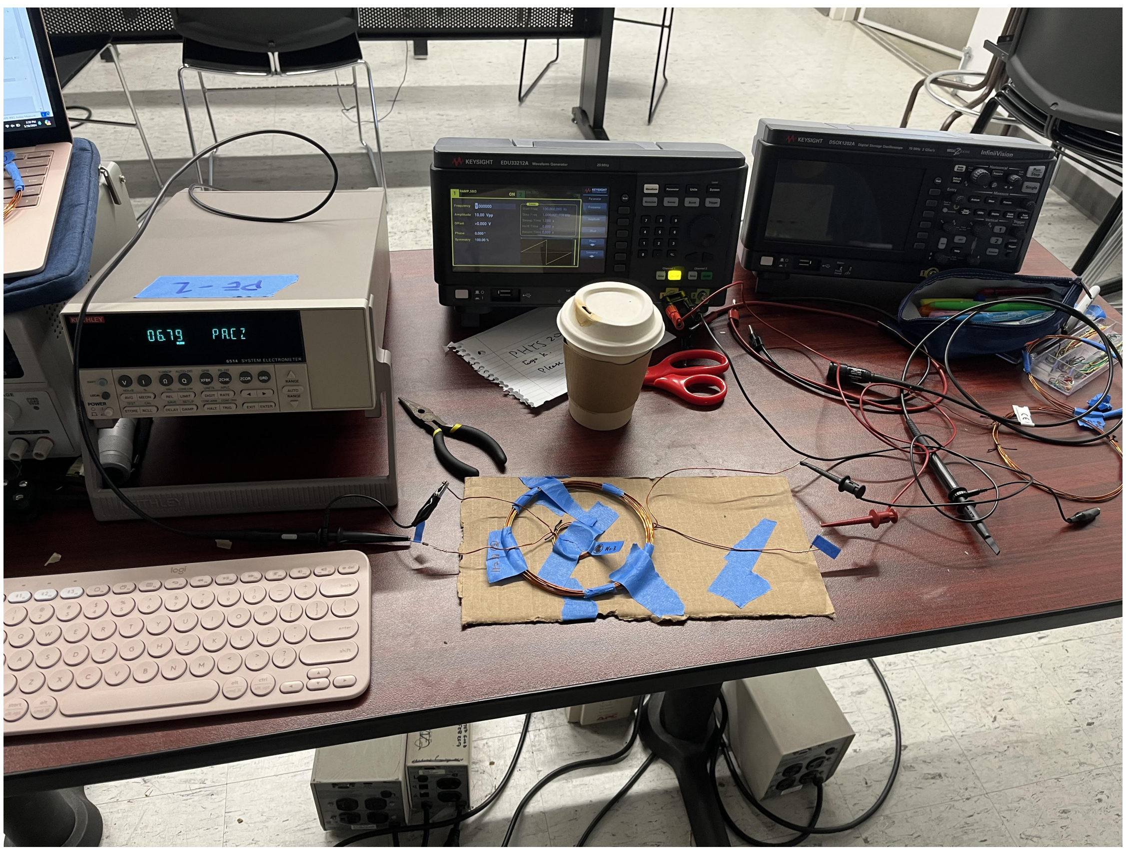

Figure II. A photo of our experimental setup is shown here. Note the waveform generator (rear center), the electrometer (left), and the solenoids (front center). Also note how the ends of the solenoids are twisted to ensure there are an integer number of turns in the solenoid, and how the solenoids are taped down to a cardboard sheet as explained in "Construction Details" above. The resistors, though barely visible, are attached to one lead of each solenoid. Solenoids and resistors were labeled with a strip of blue tape with an identifier taped on to them.

Measurement Details

Recall that for this experiment to find μ_0 we need to find 8 variables, specifically

$$I_{0n}, R_0, L_0, N_n, N_0, A_0, R_n, \frac{dV_n}{dt}$$

per

$$\mu_0 = \frac{I_{0n} R_0 L_0}{N_n N_0 A_0 \frac{1}{R_n} \frac{dV_n}{dt}} \quad \text{(Eq. 8)}$$

In our experiment, while we tested 5 different outer solenoids, the inner solenoid, solenoid 0 remained unchanged throughout the experiment. Throughout the experiment, We also used the same resistors for both the inner and outer solenoids, and we also used the same ramp function for the function generator. This means that throughout our experiment, the following variables,

$$N_0, R_0, R_n, \frac{dV_n}{dt}, L_0, A_0$$

remained unchanged while

$$N_0, I_{0n}$$

differed for each solenoid as these are variables related to or affected by the outer solenoid. Of the 8 variables, 5 have no error. These are R_0, and R_n which are given and known values, $\frac{dV_n}{dt}$, which is given by the function generator, and N_0 and N_n which are integer values. While they may not mathematically be perfect integers due to the imperfections in our solenoid, there is no easy way to quantify a precise value for N, therefore we will assume they are integers with no error. The other 3 variables, A_0, L_0, and I_{0n} are measured values and will have an error. Errors of a set of measurements are found by a root mean square error (RMSE) where the RMSE of a set of values is defined as follows.

$$\text{RMSE} = \sqrt{\frac{1}{n-1} \sum_{i=1}^{n} (y_i - \hat{y}_i)^2} \quad \text{(Eq. A)}$$

where y_i is the individual value of a measurement, $\hat{y}_i$ is the average value of the set of measurements, and n is the number of measurements taken. We took 5 measurements of the diameter for solenoid 0 and calculated its RMSE to find that

$$d_0 = 0.04\,[m] \quad \sigma_{d_0} = 0.0014\,[m]$$

where d_0 is the average diameter from our measurements, and σ_{d_0} is our RMSE for the diameter. The mean area of our solenoid A_0, assuming it is a perfect circle is

$$A_0 = \pi(d_0/2)^2 \quad \text{(Eq. 9)}$$

where A_0 is in units of [m^2]. The error for our diameter will propagate however per the error propagation function. The error propagation function is as follows. For some function $f(x, y, z)$, the error, σ_f is

$$\sigma_f = \sqrt{\left(\frac{\partial f}{\partial x} \cdot \sigma_x\right)^2 + \left(\frac{\partial f}{\partial y} \cdot \sigma_y\right)^2 + \left(\frac{\partial f}{\partial z} \cdot \sigma_z\right)^2} \quad \text{(Eq.B)}$$

Calculating the error of A_0 from (Eq. 9) per the error propagation function, (Eq. B) we find that

$$\sigma_{A_0} = \sqrt{(\frac{\pi}{2} d_0 \sigma_{d_0})^2} \quad \text{(Eq. 10)}$$

When calculated, the mean and error of the area of our solneoid 0 is,

$$A_0 = 0.001\,[m^2] \quad \sigma_{A_0} = 8 \cdot 10^-5\,[m^2]$$

where A_0 is the mean cross sectional area of solenoid 0 and σ_{A_0} is the error of the area. We also took 5 measurements of L_0, the height of solenoid 0, allowing us to find a mean and error for that variable as well. Using (Eq. A) to find the RMSE for L_0, we find that

$$L_0 = 0.003\,[m] \quad \sigma_{L_0} = 9 \cdot 10^-4\,[m]$$

where again, L_0 is the average height of solenoid 0 and σ_{L_0} is the error for the height. As for I_{0n}, we took 5 measurements of current at separate times for each solenoid n, making sure to perform a Zero Check on the electrometer between every measurement to ensure that an accurate and precise measurement was being made. Due to the instability in the electrometer readings, we only recorded up to the 100th picoampere [pA]. We used these 5 measurements for each solenoid to find a mean and error (per the RMSE (Eq. A)) for I_{0n}. By repeating this process for 5 different outer solenoid n's, that is solenoids 1 through 5, we ended up with 5 datasets, each with the 8 variables which theoretically, when plugged into (Eq. 8) would give us the same value of μ_0. The 5 datasets, each with their 8 variables are shown in the table below. Note that many of the variables are the same as noted earlier.

Table I. Variables and their Values for Each Solenoid / Dataset

Solenoid n	N_0	N_n	R_0 [Ω]	R_n [Ω]	$\frac{dV_n}{dt}$ [$\frac{V}{s}$]	I_{0n} [A]	$\sigma_{I_{0n}}$ [A]	L_0 [m]	σ_{L_0} [m]	A_0 [m^2]	σ_{A_0} [m^2]
1	3	4	1	100	20	$1.89 \cdot 10^{-12}$	$7.43 \cdot 10^{-14}$	$2.6 \cdot 10^{-3}$	$9 \cdot 10^{-4}$	$1.03 \cdot 10^{-3}$	$8.38 \cdot 10^{-5}$
2	3	5	1	100	20	$2.33 \cdot 10^{-12}$	$1.29 \cdot 10^{-13}$	$2.6 \cdot 10^{-3}$	$9 \cdot 10^{-4}$	$1.03 \cdot 10^{-3}$	$8.38 \cdot 10^{-5}$
3	3	7	1	100	20	$7.88 \cdot 10^{-12}$	$3.77 \cdot 10^{-13}$	$2.6 \cdot 10^{-3}$	$9 \cdot 10^{-4}$	$1.03 \cdot 10^{-3}$	$8.38 \cdot 10^{-5}$
4	3	9	1	100	20	$8.01 \cdot 10^{-12}$	$3.64 \cdot 10^{-13}$	$2.6 \cdot 10^{-3}$	$9 \cdot 10^{-4}$	$1.03 \cdot 10^{-3}$	$8.38 \cdot 10^{-5}$
5	3	11	1	100	20	$7.58 \cdot 10^{-12}$	$6.72 \cdot 10^{-13}$	$2.6 \cdot 10^{-3}$	$9 \cdot 10^{-4}$	$1.03 \cdot 10^{-3}$	$8.38 \cdot 10^{-5}$

Table I. All the variables in (Eq. 8) are listed for datasets 1 through 5. Many variables were kept constant for each dataset, therefore many of the variables are identical throughout a column.

While theoretically, each dataset should give us the same value of μ_0, due to imperfections and variations within each dataset, each dataset will provide a different value. To find an accurate μ_0, we can perform a linear regression where μ_0 is the slope. Rearranging (Eq. 8) we find that

$$\frac{N_0 N_n}{R_0 R_n}\frac{dV_n}{dt}\mu_0 = \frac{I_{0n} L_0}{A_0} \quad \text{(Eq. 11)}.$$

In (Eq. 11), we have an equation where $\frac{N_0 N_n}{R_0 R_n}\frac{dV_n}{dt}$ is the x-axis coordinate of a data point formed from one of the datasets, and $\frac{I_{0n} L_0}{A_0}$ is the corresponding y-axis value of the data point, where if enough points are plotted, the slope should give the value of μ_0. In other words, if

$$x_n = \frac{N_0 N_n}{R_0 R_n}\frac{dV_n}{dt} \quad \text{(Eq. 12)}$$

and

$$y_n = \frac{I_{0n} L_0}{A_0} \quad \text{(Eq. 13)}$$

then

$$x_n \mu_0 = y_n \quad \text{(Eq. 14)}$$

where with enough sets of x_n and y_n, we could perform a linear regression where the slope is mu_0. Note that all our variables with error are on the right side, that is the y-value side of the equation. By making sure that all our error is on the y-axis, allows us to consider all the errors and uncertainties that we find and calculate in our experiment. From (Eq. 14) and per the error propagation function (Eq. B), we find that the error in our y-axis for each dataset of solenoid n is

$$\sigma_{y_n} = \sqrt{(\frac{L_0}{A_0}\sigma_{I_{0n}})^2 + (\frac{I_{0n}}{A_0}\sigma_{L_0})^2 + (-\frac{I_0 L_0}{A_0^2}\sigma_{A_0})^2} \quad \text{(Eq. 15)}$$

where σ_{y_n} is the error in the y-axis for one of the datasets / solenoid n.

Calculating the x_n, y_n, and σ_{y_n} for each dataset per (Eq. 12), (Eq. 13) and (Eq. 15), we found that the the values for each variable for each dataset was as follows in the table below.

Table II. Data for Linear Regression, x_n, y_n, σ_{y_n}

Solenoid n	x_n	y_n	σ_{y_n}
1	2.40	$4.77 \cdot 10^{-12}$	$1.70 \cdot 10^{-12}$
2	3.00	$5.89 \cdot 10^{-12}$	$2.10 \cdot 10^{-12}$
3	4.20	$2.00 \cdot 10^{-11}$	$7.10 \cdot 10^{-12}$
4	5.40	$2.02 \cdot 10^{-11}$	$7.22 \cdot 10^{-12}$
5	6.60	$1.91 \cdot 10^{-11}$	$6.98 \cdot 10^{-12}$

Table II. The values from Table I were grouped into two variables x_n and y_n per (Eq. 12) and (Eq. 13). As all the variables with error are in y_n only y_n has error.

ANALYSIS

When the data in Table II was plotted and a linear regression was performed, the results were as shown below.

Figure III. Linear Regression of (Eq. 14)

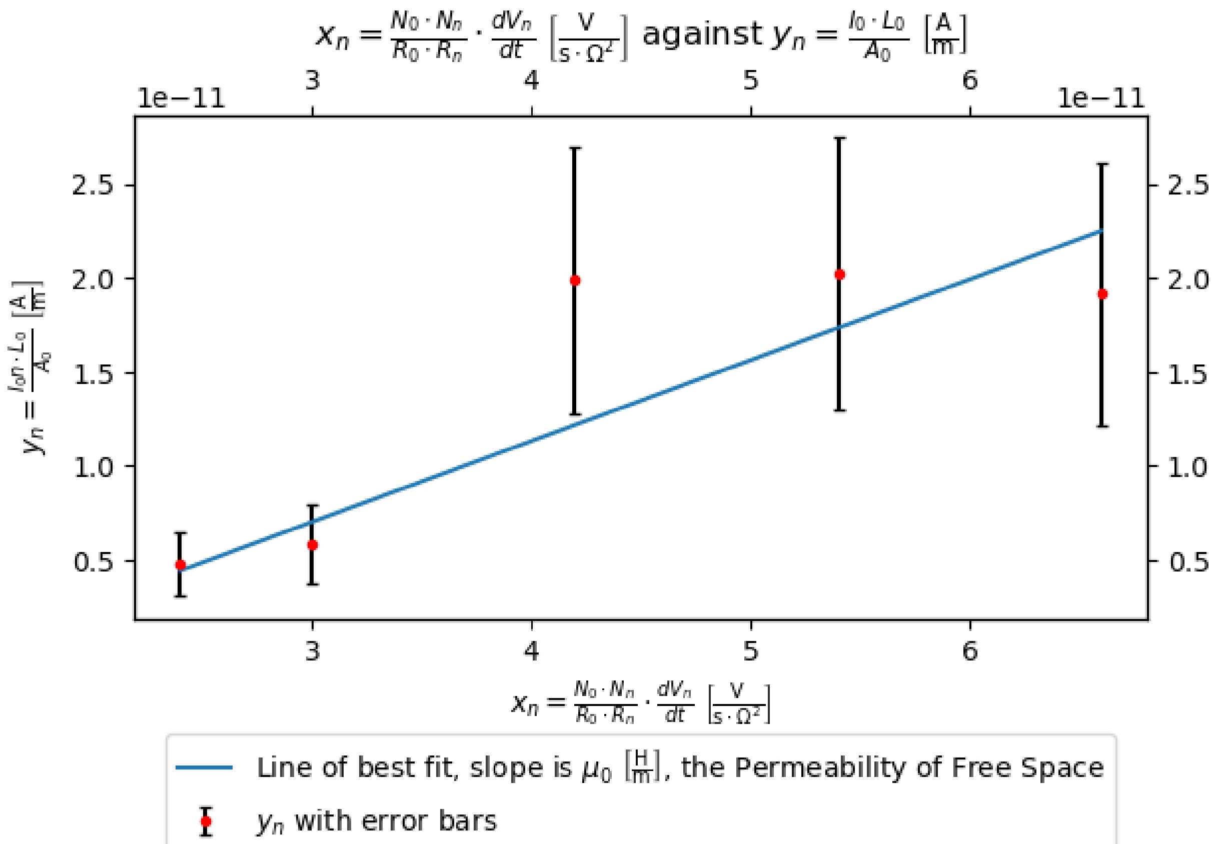

Figure III. The linear regression of (Eq. 14) is shown here. The value plotted on the x-axis is $x_n = \frac{N_0 N_n}{R_0 R_n} \frac{dV_n}{dt}$ (Eq. 12) and the corresponding y-value for the variable is $y_n = \frac{I_{0n} L_0}{A_0}$ (Eq. 13). The slope is the value we are looking for, the permeability of free space, μ_0 $\left[\frac{H}{m}\right]$.

The statistical results of the linear regression were as follows.

$$\text{Slope: } 4.30 \cdot 10^{-12} \pm 1.38 \cdot 10^{-12} \quad \left[\frac{\text{H}}{\text{m}}\right]$$

Chi-squared: 1.90

P-value: 0.59

The slope is μ_0, the permeability of free space that we are trying to measure as explained in the previous "Introduction" and "Method" sections. This means that according to our statistical analysis, our measurement of μ_0 is

$$\mu_{0_{measured}} = 4.30 \cdot 10^{-12} \pm 1.38 \cdot 10^{-12} \left[\frac{\mathrm{H}}{\mathrm{m}}\right]$$

The Chi-squared value is calculated as follows.

$$\sum_n \frac{y_n - \hat{y}(x_n)}{\sigma_{y_n}} \quad \text{(Eq. C)}$$

The Chi-squared fomula is a summation of n terms of $\frac{y_n - \hat{y}(x_n)}{\sigma_{y_n}}$ where y_n is the y-value of a data point relative to its x-value, $\hat{y}(x_n)$ is the y-value of the data point according to the linear regression function relative to its x-value, and σ_{y_n} is the error of the data point. This means that theoretically, in an ideal scenario, the n terms within the summation should be close to 1, and therefore the summation should be close to n, the number of data points for our linear regression. A Chi-squared value larger than the number of data points means that either the error of the data points is too small, or that the data fits poorly into the linear regression. A Chi-squared value smaller than the number of data points means that either the error is too large, or that the data points fit seemingly too well into the linear regression. As we had 5 data points, and our Chi-squared value is somewhat smaller than that at 1.90, and not all our data points are placed accurately on the linear regression line it is likely that the error in our data was too large. The p-value ranges from between 0 and 1 and indicates how reproducible the data is, with higher values being more reproducible. As our p-value is 0.59, we can conclude that it is somewhat reproducible.

RESULTS

We found that our value for the permeability of free space μ_0 is,

$$\mu_{0_{\text{measured}}} = 4.30 \cdot 10^{-12} \pm 1.38 \cdot 10^{-12} \left[\frac{\mathrm{H}}{\mathrm{m}}\right]$$

relative to the accepted value of μ_0,

$$\mu_{0_{expected}} = 1.26 \times 10^{-6} \pm 2.00 \times 10^{-16} \left[\frac{\mathrm{H}}{\mathrm{m}}\right]$$

as reported by the NIST. A percent error shows how close a measured value is relative to the expected value, where the percent error is calculated as follows,

$$\text{Percent Error} = \frac{|\text{Measured value} - \text{Expected Value}|}{\text{Expected Value}} \cdot 100\ [\%] \quad \text{(Eq. D)}$$

This means that our percent error is

$$\text{Percent Error} = \frac{|4.30 \cdot 10^{-12} - 1.26 \times 10^{-6}|}{1.26 \times 10^{-6}} \cdot 100\ [\%] \quad \text{(Eq. 16)}$$

which when calculated is

$$\text{Percent Error} = 100\ [\%].$$

A percent error of 100 is considered very poor, and therefore we can say that our measurement is incredibly inaccurate. It is likely that many factors contributed to this inaccuracy. For one, mutual induction is dependent on changes in current in nearby circuits, and in the vicinity of the apparatus, there were always multiple electronic equipment operating. The rate of change in current on those devices may have affected our value of μ_0. In addition, the electrical supply running through the walls at high amounts of alternating current at 60 [Hz] may also have interfered with our experiment. As μ_0 is a function of rate of change of current, such high amounts of rate of change of current could easily affect the readings on the electrometer by overwhelming the comparatively minute rate of change of current in solenoid 0. If we were to do this experiment again, with enough time and resources, we could consider the use of a Faraday cage to block out the influence of external electromagnetic fields generated by external rates of change in current. We may also use pre-assembled and identical solenoids just with differing turns to reduce variability. As μ_0 is not a function of the area of the outer solenoid, there is no need to create variability in the geometry of the solenoids beyond the number of turns when creating datasets.

BIBLIOGRAPHY

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