Hydrological Application of Porous Media Equations and Their Links with Stochastic Processes
（多孔質媒体方程式の水文学的応用および確率過程との関連）

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1. INTRODUCTION

Scientific understanding of water movements is indispensable for establishing efficient water management, especially in arid and semi-arid regions. Several hydrological mathematical models such as the Boussinesq equation and the Richards equation are categorized as the porous media equations (PMEs) (Vazquez, 2007). In this sense, the discussion based on the PME has great importance in managing water movements in an effective way with mathematical rigor.

Therefore, this study overviews some important properties of the PMEs by referring to Gilding (1982) and Aronson (1969). Moreover, some stochastic approaches studied by several authors including Jourdain (2000) and Da Prato and Röckner (2004) are also examples of probabilistic representation of the PMEs.

2. SUBSURFACE WATER FLOWS

2.1 Richards equation

To begin with, we assume the following two equations (i) and (ii) for saturated-unsaturated subsurface water flows.

(i) The conservation law of water:

$$\frac{\partial \theta}{\partial t} + \nabla \cdot \boldsymbol{V} = 0 \tag{1}$$

where $\theta(t, \boldsymbol{x})$ is the volumetric water content, ∇ is the 3-dimensional del operator, and $\boldsymbol{V}$ is the water flux, which can be considered as the virtual velocity.

(ii) The Buckingham-Darcy's law (extension of the Darcy's law to unsaturated flows):

$$\boldsymbol{V} = -K(h)(\nabla h + \nabla z) \tag{2}$$

where $h(\theta)$ is the matric potential, z is the gravitational potential, and $K(h)$ denotes hydraulic conductivity. Combining these two equations (1) and (2) leads to the Richards equation

$$\frac{\partial \theta}{\partial t} = \nabla \cdot \big(K(h)(\nabla h + \nabla z)\big). \tag{3}$$

In the context of (3), the unsaturated subsurface water is regarded as a compressive fluid with nonlinearity due to the dependency of *K* on *h*.

2.2 Boussinesq equation

In an unconfined aquifer, we introduce the Dupuit's assumption that the flows are dominantly horizontal with hydrostatic pressure distribution. Then, integrating the conservation law of the saturated water zone with the Dupuit's assumption, the Boussinesq equation for subsurface water flows in unconfined aquifers is obtained as:

$$\frac{\partial}{\partial t}(S\eta) = \nabla_H \cdot (\eta - z_b) k \nabla_H \eta. \tag{4}$$

where S is the specific storage, k is the saturated hydraulic conductivity, η is the elevation of the water table, z_b is the elevation of the impermeable bottom, and ∇_H denotes horizontal 2-dimensional del operator.

3. PROPERTIES OF POROUS MEDIA EQUATIONS

Under the appropriate assumptions, these equations (3) and (4) reduce to the PME

$$\frac{\partial u}{\partial t} = \Delta u^m \tag{5}$$

where the density u is the spatio-temporal function. This equation is transformed into the following form

$$\frac{\partial u}{\partial t} = \nabla \cdot (m u^{m-1} \nabla u), \tag{6}$$

where if we consider mu^{m-1} as the diffusion coefficient, the equation is parabolic for $u > 0$, but it degenerates for $u = 0$.

The PME (5) is associated with the Cauchy problem

$$\frac{\partial u}{\partial t} = \Delta u^m, u(x, 0) = u_0(x). \tag{7}$$

where $u_0(x)$ is a given initial u of sufficient regularity. Regularity properties of the solution to the Cauchy problem (7) have been studied since Oleinik *et al*. (1958), who showed that it has a weak solution u and u^m is Lipschitz continuous under some assumptions. Then, Aronson (1969) found that the weak solution is Hölder continuous. Here, a function $f: \mathbb{R}^n \to \mathbb{R}^n$ is called Hölder continuous if there exists a constant $C \geq 0$ such that, for all x and y in $\mathbb{R}^n$,

$$|f(x) - f(y)| \leq C|x - y|^\sigma \tag{8}$$

where σ is the exponent of the Hölder condition. If $\sigma = 1$, this function means Lipschitz continuous, which is differentiable almost everywhere.

Gilding (1982) analyzed the solutions of the PME (5) obtained via the separation of variables, as well as all the known special solutions. The PME (5) is transformed into the ordinary differential equation by admissible similarity transformations, for which the generality of the reducing is preserved. Finally, by solving the ordinary equation, the five classes of its nontrivial similarity solutions are discovered.

4. STOCHASTIC PROCESSES LINKED WITH POROUS MEDIA EQUATIONS

Jourdain (2000) provided probabilistic representation of the PME (5) by associating it with a nonlinear martingale problem and utilizing a stochastic particle method. The PME (5) can be transformed into:

$$\frac{\partial v}{\partial t} = \frac{\partial^2}{\partial x^2}\left(m\big(H * v(t,\cdot)(x)\big)^{m-1} v(t,\cdot)\right), \qquad (9)$$

where $v = \partial_x u$ and H is Heaviside function which is derived from its discontinuous solution. This equation (9) can be interpreted as the Fokker-Plank equation, that is the limit law of an interacting particle system. In this sense, when the initial condition is the cumulative distribution function, the PME (5) is associated with the nonlinear martingale problem (MP): A probability measure P on $(\mathbb{R}^{[0,\infty]}, \mathcal{B})$ solves the martingale problem starting at p if

i, $P_0 = p$,

ii. $$M_t = \phi(X_t) - \phi(X_0) - q\int_0^t \big(H * P_s(X_s)\big)^{m-1} \phi''(X_s)\, ds, \qquad (10)$$

is a P-martingale,

iii. P_t does not weight point,

where X_t is the canonical process. Jourdain (2000) proved that if P solves the MP (10), then $(s,x) \to H * P_s(x)$ is a weak solution of the PME (5) when the initial condition is $H * p(x)$. The solution of this nonlinear martingale problem is equivalent to the distribution of the process X_t. Concerning this stochastic process, one can obtain the propagation of chaos result of the interacting particle system

$$X_t^{i,n} = X_0^i + \int_0^t m\left(H * \mu_s^n\big(X_s^{i,n}\big)\right)^{\frac{m-1}{2}} dB_s^i, \qquad (11)$$

where $X_t^{i,n}, 1 \le i \le n$ denotes the position of i-th particle at time t, B^i are Brownian motions, and $\mu^n = (1/n)\Sigma_{j=1}^n \delta_{X^{j,n}}$. Furthermore, reordering the particle system (11) to create increasing sequence, Jourdain (2000) defined another stochastic process as

$$Y_t^{i,n} = \sup_{|A|=n-i+1} \inf_{j\in A} X_t^{j,n}, \qquad (12)$$

where $A \subset \{1,\cdots,n\}$, and $|A|$ denotes the number of elements in A. The particle systems $(Y_t^{1,n},\cdots,Y_t^{n,n})$ converges to $\tilde{P}$ which is no longer the solution of the MP, but its related function $(s,x) \to H * \widetilde{P_s}$ is a weak solution of the PME.

Da Prato and Röckner (2004) studied a stochastic PME with noise and viscosity term

$$X_t = X_0 + \int_0^t (\alpha\Delta X_t + \Delta X_t^m)\, dt + \int_0^t \sqrt{c}\, dB_t, \qquad (13)$$

where $\alpha \ge 0$, c is a positive function, and B_t is a Brownian motion at time t. Initially, the Kolmogorov equation corresponding to the stochastic differential equation (13) is solved by utilizing C_0-semigroup approach, which allows the users to regard the partial differential equation as the ordinary differential equation under suitable conditions. Then, it is shown that a strong Markov process obtained from its solution solves the MP corresponding to (13). By Itô formula,

$$\begin{aligned}\phi(X_t) &= \phi(X_0) \\ &+ \int_0^t \left((\alpha\Delta X_t + \Delta X_t^m)\frac{\partial\phi}{\partial x} + \frac{c}{2}\frac{\partial^2\phi}{\partial x^2}\right) ds \\ &+ \int_0^t \sqrt{c}\,\frac{\partial\phi}{\partial x}\, dB_s, \qquad (14)\end{aligned}$$

In this equation (14), the second term of the right hand side represents the Kolmogorov operator

$$A\phi(x) = (\alpha\Delta X_t + \Delta X_t^m)\frac{\partial\phi}{\partial x} + \frac{1}{2}\sum_{\mathrm{k}=1}^{\infty} c\,\frac{\partial^2\phi}{\partial x^2}, \qquad (15)$$

which is associated with the following Kolmogorov equation

$$\frac{dw}{dt} = Aw, \qquad (16)$$

where w is an invariant measure. Analyzing this equation (16) makes it possible to construct a strong Markov process satisfying the corresponding MP.

5. CONCLUSION

The Richards equation and the Boussinesq equation are regarded as the PMEs in suitable conditions, and therefore, it is proper to consider specific hydrological phenomena in a mathematical framework associated with the PMEs. It is desirable to extend these models to the stochastic PMEs to deal with hydrological phenomena with uncertainty. The methodologies of stochastic control such as Hamilton-Jacobi-Bellman equations are required as well in the context of practical water management problems.

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