Hydrological Application of Porous Media Equations and Their Links with Stochastic Processes

（多孔質媒体方程式の水文学的応用および
確率過程との関連）

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1. INTRODUCTION

Understanding water movements under the ground is indispensable for efficient water management, especially in arid and semi-arid regions (Stevanovic and Iurkiewicz, 2008). Under appropriate assumptions, the theory results in the fundamental equations, and then we verify their validity by confirming the expected features of its solution. No matter how prevalent it is, the theory is incomplete unless its factual foundation is developed, including the well-posedness of the problems to be solved (Okamoto, 2023). In terms of hydrology, there are several fundamental equations such as the Richards equation (Richards, 1931) and the Boussinesq equation (Boussinesq, 1904), both of which are categorized as the porous media equations (PMEs) (Vazquez, 2007). In this sense, the discussion based on the PME has great importance in managing water movements practically with mathematical rigor.

The Richards equation governs the dynamics of fluids in the vadose zone and is derived by considering the water and the air as one compressive fluid, namely, two-phase flow. For detailed explanations of the law of the multiphase flows, see Bear (1972). Although some analytical solutions have already been proposed (Brutsaert, 2005), these solutions are limited to simple conditions. Due to the difficulty of solving the Richards equation, plenty of attempts have been made to solve it through various numerical methods (Keita et al., 2021). These methods only adopt the drainage process or absorption processes because of the hysteresis effect resulting from the character of the contact angle and the ink-bottle effect (Bear, 1972). In this context, van Duijn et al. (2018) investigated the model, including the hysteresis effect. Pop and Schweizer (2011) focused on the strange behavior of its solution around the value 0, called degeneracy, by considering the Richards equation as a doubly degenerate equation. Due to this peculiar property, more rigorous mathematical treatments are indispensable regarding water management in drylands. Cao and Pop (2016) showed the existence and uniqueness of the weak solution of the Richards equation by regarding it as a Pseudo-parabolic equation.

On the other hand, in an unconfined aquifer, the groundwater flow is described by solving the Boussinesq equation, adopting Dupuit's assumption (Dupuit, 1863). The Boussinesq equation can be regarded as the second-order PME. In this sense, Kacimov and Šimůnek (2021) introduced a special solution of the PME directly into the Boussinesq equation and then compared the analytical solution with the conventional results of HYDRUS 2-D modeling.

This study overviews some essential properties of the PMEs by referring to Gilding (1982) and Aronson (1969). Moreover, some stochastic approaches studied by several authors including Jourdain (2000) and Da Prato and Röckner (2004) are also examples of probabilistic representation of the PMEs. Those stochastic concepts are expected to become the basis of constructing effective water management strategies.

2. FORMULATION OF SUBSURFACE WATER FLOWS

2.1 Richards equation

We introduce the conservation law of water

$$\frac{\partial \theta}{\partial t} + \nabla \cdot \boldsymbol{V_1}, \tag{1}$$

where $\theta(t, x)$ is the volumetric water content, ∇ is the 3-dimensional del operator, and $\boldsymbol{V_1}$ is the water flux, which can be considered as the virtual velocity.

Darcy's law describes the fluid motion in saturated flow through porous media as

$$\boldsymbol{V_1} = -K_s \nabla H, \tag{2}$$

where K_s is the hydraulic conductivity, and H is the piezometric head. In addition, ∇H denotes the spatial change of the piezometric head, which is called the hydraulic gradient. The equation (2) was first discovered by Darcy (1856) through a series of experiments and can also be derived from the momentum conservation equation without the advection term and the viscosity term. Buckingham (1907) extended Darcy's law in saturated flow to the one in unsaturated flow, called Buckingham-Darcy's law

$$\boldsymbol{V_1} = -K(h)(\nabla h + \nabla z), \tag{3}$$

where $h(\theta)$ is the matric potential, z is the gravitational potential, and $K(h)$ denotes the hydraulic conductivity of unsaturated flow.

These laws of momentum conservation (2) and (3) are valid when the Reynolds number for the elementary channels of the porous medium is proper, which is often determined by the length of the particle (Bear, 1972). In other words, Darcy's law can be applied to the laminar flow, where the viscous forces are so dominant that the Reynolds number is less than some value between 1 and 10. On the other hand, Darcy's law deviates from the flow in nature at the low hydraulic gradient, which is equal to the Reynolds number here. Moreover, Darcy's law is defined for a homogeneous isotropic medium and is extended to an anisotropic medium by regarding the hydraulic conductivity as a second-order tensor. Here, a medium is considered homogeneous when its property is independent of the space and is called an isotropic one if its feature is identical in each direction.

Combining these equations (1) and (3) leads to the Richards equation

$$\frac{\partial\theta}{\partial t} = \nabla \cdot \big(K(h)(\nabla h + \nabla z)\big). \tag{4}$$

In the context of (4), the unsaturated subsurface water is regarded as a compressive fluid with nonlinearity due to the dependency of K on h. As the gravity term on the right-hand side of the equation (4) makes it more difficult to solve analytically, we neglect the last term. This situation corresponds to the horizontal infiltration, namely sorption, and allows us to obtain a nonlinear diffusion equation satisfied by $D(\theta) = K\frac{\partial h}{\partial \theta}$

$$\frac{\partial\theta}{\partial t} = \frac{\partial}{\partial x}\left(D(\theta)\frac{\partial\theta}{\partial x}\right), \tag{5}$$

where x denotes the horizontal distance. Here, although we assume that the relationship between water content and matric potential is described as a single-valued soil water retention curve, in nature, the relation differs between draining and wetting processes.

This partial differential equation (5) can be reduced into the ordinary differential equation under certain conditions by applying the Boltzmann's transformation discussed in Chapter 3. Assuming that the space variable x and the time variable t consist of one variable ϕ

$$\phi = xt^{-\frac{1}{2}}, \tag{6}$$

where $\phi = \phi(\theta)$, the equation (5) is transformed into the ordinary differential equation

$$\frac{d}{d\phi}\left(D(\theta)\frac{d\theta}{d\phi}\right) + \frac{\phi}{2}\frac{d\theta}{d\phi} = 0, \tag{7}$$

since

$$\frac{\partial\theta}{\partial t}=\frac{d\theta}{d\phi}\frac{\partial\phi}{\partial t}=-\frac{1}{2}xt^{-\frac{3}{2}}\frac{d\theta}{d\phi}$$
$$\frac{\partial\theta}{\partial x}=\frac{d\theta}{d\phi}\frac{\partial\phi}{\partial x}=t^{-\frac{1}{2}}\frac{d\theta}{d\phi} \tag{8}$$
$$\frac{\partial}{\partial x}\left(D(\theta)\frac{\partial\theta}{\partial x}\right)=t^{-\frac{1}{2}}\frac{d}{d\phi}\frac{\partial\phi}{\partial x}\left(D(\theta)\frac{d\theta}{d\phi}\right)=t^{-1}\frac{d}{d\phi}\left(D(\theta)\frac{d\theta}{d\phi}\right).$$

Boltzmann's transformation is applicable if the initial and the boundary conditions are prescribed with functions of ϕ, that is, constant initial volumetric water content for $x>0$ and constant volumetric water content for $x=0$.

2.2 Boussinesq groundwater equation

In a saturated unconfined aquifer, we introduce Dupuit's assumption that the flows are dominantly horizontal with hydrostatic pressure distribution when the capillary zone is relatively thin compared to its horizontal space. This assumption provides a new perspective on Darcy's law governing the groundwater movement:

$$\boldsymbol{V_2}=-K_s\nabla_s H\simeq -K_s\nabla_s\eta\simeq -K_s\nabla_H\eta, \tag{9}$$

where $\boldsymbol{V_2}$ is the water flux, H is the piezometric head, η denotes the water table, ∇_s is the 2-dimensional del operator with respect to the plane of the dominant flows, and ∇_H is the 2-dimensional del operator with respect to the general Cartesian coordinate system xy.

Let us consider the continuity equation for the saturated groundwater flow

$$\nabla\cdot\boldsymbol{V_2}=0. \tag{10}$$

Integrating the equation (10) from the impermeable layer z_b to the phreatic surface η,

$$\int_{z_b}^{\eta}(\nabla\cdot\boldsymbol{V_2})\,dz=\int_{z_b}^{\eta}\left(\frac{\partial V_x}{\partial x}+\frac{\partial V_y}{\partial y}+\frac{\partial V_z}{\partial z}\right)dz=0. \tag{11}$$

By the Leibniz integral rule, each part of the equation (11) can be calculated like

$$\int_{z_b}^{\eta}\left(\frac{\partial V_x}{\partial x}+\frac{\partial V_y}{\partial y}\right)dz=\frac{\partial}{\partial x}\int_{z_b}^{\eta}V_x\,dz-V_x(\eta)\frac{\partial \eta}{\partial x}+V_x(z_b)\frac{\partial z_b}{\partial x}$$
$$+\frac{\partial}{\partial y}\int_{z_b}^{\eta}V_y\,dz-V_y(\eta)\frac{\partial \eta}{\partial y}+V_y(z_b)\frac{\partial z_b}{\partial y}, \tag{12}$$

and

$$\int_{z_b}^{\eta}\frac{\partial V_z}{\partial z}dz=V_z(\eta)-V_z(z_b). \tag{13}$$

Considering the water table, we obtain

$$\frac{V_z(\eta)}{S}\Delta t=\frac{\partial \eta}{\partial t}\Delta t+\frac{\partial \eta}{\partial x}\frac{V_x(\eta)}{S}\Delta t+\frac{\partial \eta}{\partial y}\frac{V_y(\eta)}{S}\Delta t, \tag{14}$$

where S is the specific yield or the effective porosity, with which the virtual velocity $\boldsymbol{V_2}$ is replaced by the true velocity in porous media ($\boldsymbol{V_2}/S$) because $\partial\eta/\partial t$ is the actual velocity of the surface. In the same way, concerning the impermeable bottom surface, the true velocity in z-axis direction is written as

$$\frac{V_z(z_b)}{S}\Delta t=\frac{\partial z_b}{\partial x}\frac{V_x(z_b)}{S}\Delta t+\frac{\partial z_b}{\partial y}\frac{V_y(z_b)}{S}\Delta t. \tag{15}$$

Substituting the equations (14) and (15) into the equation (13) leads to the following equation

$$\int_{z_b}^{\eta}\frac{\partial V_z}{\partial z}dz=S\frac{\partial \eta}{\partial t}+\frac{\partial \eta}{\partial x}V_x(\eta)+\frac{\partial \eta}{\partial y}V_y(\eta)-\frac{\partial z_b}{\partial x}\,V_x(z_b)-\frac{\partial z_b}{\partial y}\,V_y(z_b). \tag{16}$$

By combining these two equations (12) and (16) with the differential form, the equation of continuity (10) is finally rewritten as

$$S\frac{\partial \eta}{\partial t}+\frac{\partial}{\partial x}\int_{z_b}^{\eta}V_x\,dz+\frac{\partial}{\partial y}\int_{z_b}^{\eta}V_y\,dz=0, \tag{17}$$

which is like the shallow water equation. Integrating the motion equation (9) and the continuity equation (17), ones obtain the Boussinesq equation

$$S\frac{\partial \eta}{\partial t} = K_s \nabla_H \cdot (\eta - z_b)\nabla_H \eta, \tag{18}$$

in which η depends on the time t, and therefore, the equation (18) has nonlinearity.

In the context of mathematics, the Boussinesq equation (18) has been studied in an idealized situation, where the boundary conditions represent a certain symmetry that enables self-similar analysis. For instance, with the boundary conditions

$$\begin{aligned} &\eta(0,t) = D > 0 \\ &\eta(\infty, t) = 0 \qquad\qquad\qquad , \\ &\eta(x,0) = 0, \qquad for\ x > 0 \end{aligned} \tag{19}$$

we assume flows in one space variable for the sake of simplicity. These conditions represent the infiltration that keeps the water table constant $\eta = D$ at $x = 0$ for all times when its initial data is a point source.

Under these conditions, as well as section 2.1, the self-similar solution can be defined by presenting one variable depending on the space x and the time t

$$\phi = xt^{-\frac{1}{2}}. \tag{20}$$

This transformation (20) allows the Boussinesq equation in 1-dimension to be reduced into the following ordinary differential equation

$$K_s \frac{d}{d\phi}\left((\eta - z_b)\frac{d\eta}{d\phi}\right) + S\frac{\phi}{2}\frac{d\eta}{d\phi} = 0. \tag{21}$$

This ODE was numerically studied by Plubarinova-Kochina (1948), who identified its solution with a finite propagation speed.

3. PROPERTIES OF THE POROUS MEDIA EQUATION

3.1 Relationship between the porous media equation and the heat equation

The heat equation

$$\frac{\partial u}{\partial t} = \Delta u \tag{22}$$

is one of the classical linear partial differential equations that has been studied since Fourier (1822) and is also called the heat equation. This parabolic equation (22) has some important properties such as self-similarity and smoothing effect, both of which are analogous, in a generalized sense, to those of the porous media equation

$$\frac{\partial u}{\partial t} = \Delta u^m, \qquad m > 1. \tag{23}$$

The nonlinear partial differential equation (23) is derived from the conservation law for compressive fluid and Darcy's law in the case of unsaturated water flows, which is discussed in the previous Chapter. Also, detailed analyses of the porous media equation have been conducted for decades, and Vazquez (2007) reviewed their fruitful results.

Furthermore, equation (23) is transformed into

$$\frac{\partial u}{\partial t} = \nabla \cdot (mu^{m-1}\nabla u), \qquad m > 1. \tag{24}$$

where if we consider mu^{m-1} as the diffusion coefficient, the equation is parabolic for $u > 0$. On the other hand, it degenerates for $u = 0$ because the diffusion coefficient vanishes, and therefore, the disturbance propagates at a finite speed. For example, the discontinuous boundary condition becomes a smooth function in the next moment, the disturbance reaching infinite.

The study of the porous media equation has been generally developed as the extension of the linear heat equation to the nonlinear heat equation. For this reason, comparing its characteristics with those of heat equation is meaningful in obtaining a better understanding.

When solving the Cauchy problems related to the heat equation (22), it is efficient to apply the fundamental solution

$$H(x,t) = (4\pi t)^{-\frac{n}{2}} exp\left(-\frac{|x|^2}{4t}\right) \tag{25}$$

where n is the dimension, which is described in Fig. 1 when $n = 2$ and $t = 5$. The heat kernel (25) is derived from the following Cauchy problem

$$\frac{\partial H}{\partial t} = \Delta H, \qquad H(x,0) = \delta(x) \tag{26}$$

where δ is the Dirac's delta function, which satisfies

$$\int_{-\infty}^{\infty} \delta(x)\varphi(x)dx = \varphi(0), \qquad \delta(x) = 0 \quad (x \neq 0) \tag{27}$$

where $\varphi \in C^0$. In other words, this situation describes the density distribution of particles having a total mass of 1 and starting at $x = 0$, as time proceeds. Additionally, the source-type solution (25) also provides a probabilistic view of the Brownian motion: the density distribution equals the Gaussian distribution.

Note that: the solution (25) reduces to a particular shape at an arbitrary time by the appropriate scaling, which equals a power-law relationship between two variables. In other words, "*the spatial distributions of the characteristics of the phenomenon vary with time while remaining geometrically similar* (*self-similar*)" (Barenblatt, 2012). Thus, this type of solutions is called a self-similar solution or a similarity solution in the study of partial differential equations.

Considering the porous media equation, the above notion corresponds to the ZKB solution

$$U(x,t) = t^{-\alpha}\left(b - \frac{m-1}{2m}\beta\frac{|x|^2}{t^{2\beta}}\right)_+^{\frac{1}{m-1}} \tag{28}$$

where b is an arbitrary constant, and

$$\alpha = \frac{n}{n(m-1)+2}, \qquad \beta = \frac{\alpha}{n}. \tag{29}$$

This U is plotted in Fig. 2 with $b = 0, m = 2, n = 2,$ and $t = 10$. The special solution (28) is also obtained by taking initial data as the Dirac's delta function, and therefore, it is called the source-type solution. Note that the fundamental solution (25) can express all solutions of the various Cauchy problems on the heat equation due to its linearity; conversely, the ZKB solution (28) cannot do that.

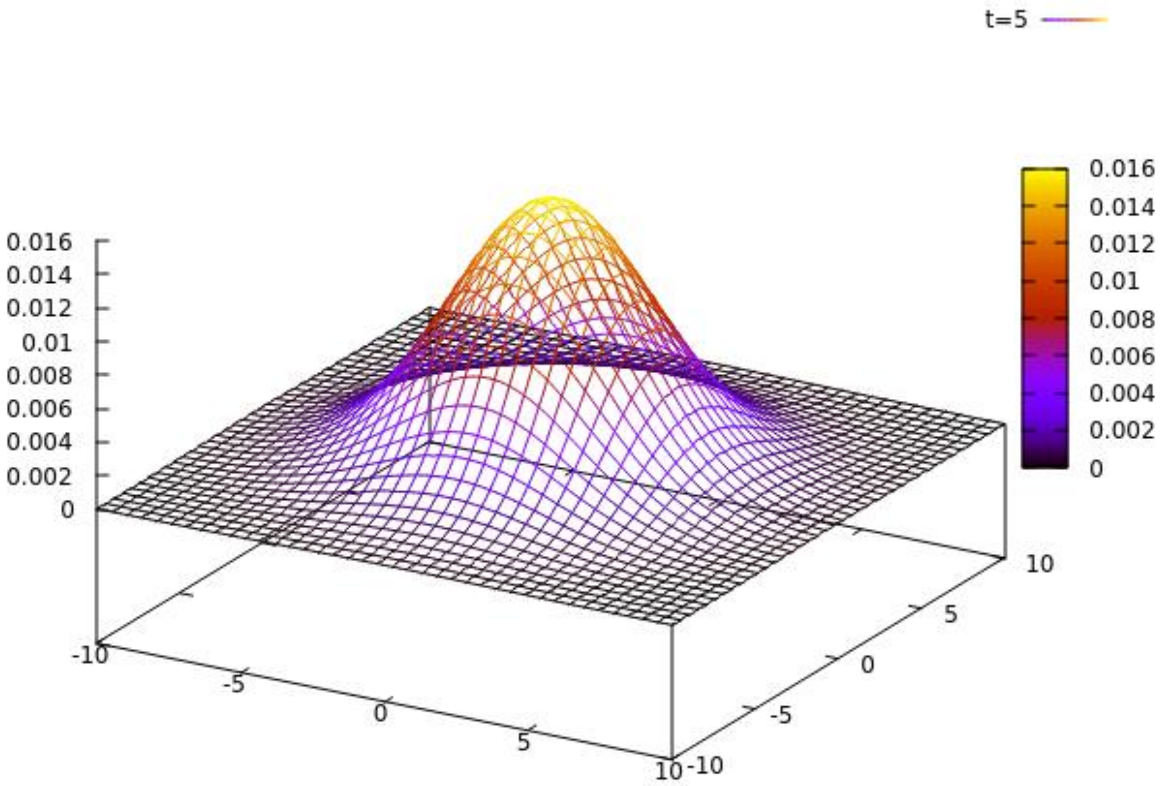

Fig. 1 The fundamental solution of the heat equation ($\boldsymbol{n = 2}$).

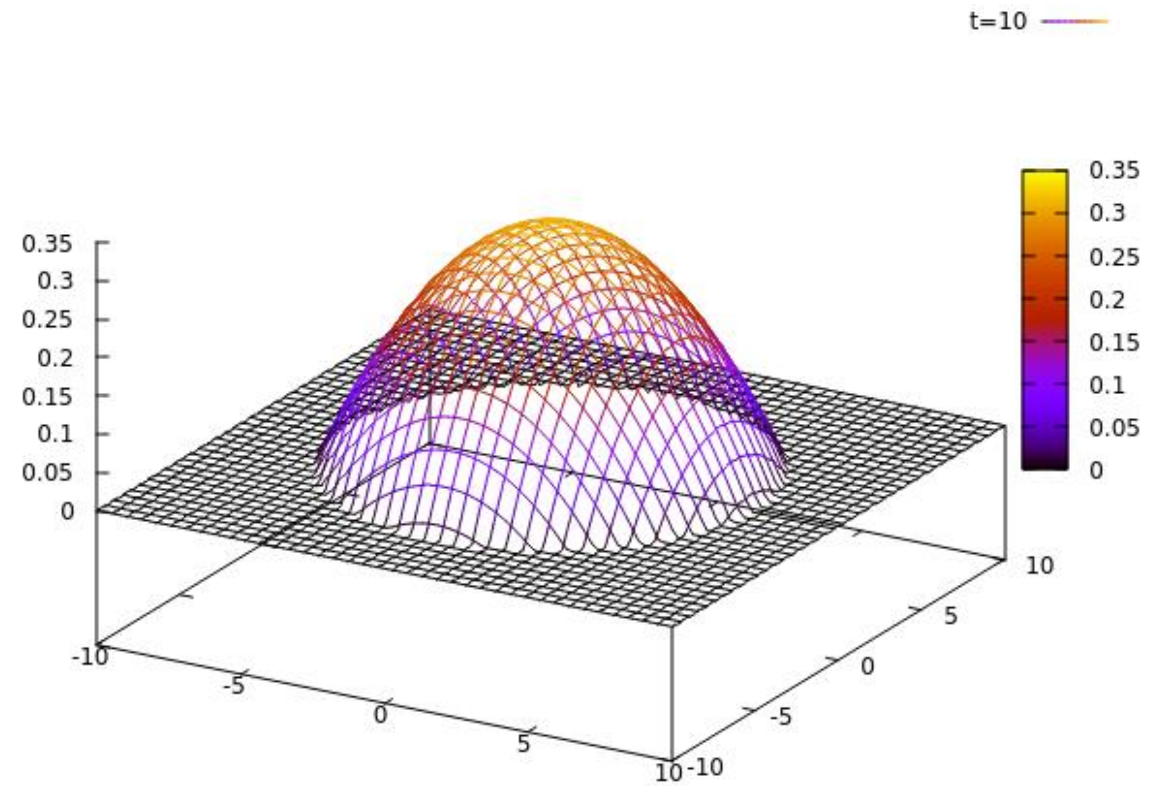

Fig. 2 The ZKB solution of the porous media equation ($\boldsymbol{b = 0}$, $\boldsymbol{m = 2}$, $\boldsymbol{n = 2}$).

Furthermore, Boltzmann's transformations, introduced by Boltzmann (1894), are helpful for solving the diffusion equation

$$\frac{\partial u}{\partial t} = \frac{\partial}{\partial x}\left(D\frac{\partial u}{\partial x}\right), \tag{30}$$

where a diffusion coefficient D is a generic function of u. This technique makes it possible to reduce the former partial differential equation into the ordinary differential equation in no more than the particular condition (Koiwa and Sprengel, 2007)). Boltzmann's transformation is also efficient in the porous media equation (23), discussed in the later section.

To begin with, introducing a variable ξ

$$\xi = \frac{x}{\sqrt{t}}, \tag{31}$$

the partial derivatives of ξ become:

$$\frac{\partial \xi}{\partial t} = -\frac{x}{2t^{\frac{3}{2}}} = -\frac{\xi}{2t}, \qquad \frac{\partial \xi}{\partial x} = \frac{1}{\sqrt{t}}. \tag{32}$$

Thus, the original derivatives are rewritten as:

$$\frac{\partial u}{\partial t} = \frac{\partial u}{\partial \xi}\frac{\partial \xi}{\partial t} = -\frac{\xi}{2t}\frac{\partial u}{\partial \xi}, \qquad \frac{\partial u}{\partial x} = \frac{\partial u}{\partial \xi}\frac{\partial \xi}{\partial x} = \frac{1}{\sqrt{t}}\frac{\partial u}{\partial \xi}\ . \tag{33}$$

Substituting (33) in (30) yields:

$$-\frac{\xi}{2t}\frac{\partial u}{\partial \xi} = \frac{1}{\sqrt{t}}\frac{\partial}{\partial x}\left(D\frac{\partial u}{\partial \xi}\right), \tag{34}$$

where the time variable on the right-hand side is taken outside of the partial derivative. Then, (31) allows (34) to become the partial differential equation with one variable

$$-\frac{\xi}{2}\frac{\partial u}{\partial \xi} = \frac{\partial}{\partial \xi}\left(D\frac{\partial u}{\partial \xi}\right). \tag{35}$$

Now, (35) is interpreted as the ordinary differential equation

$$-\frac{\xi}{2}\frac{du}{d\xi} = \frac{d}{d\xi}\left(D\frac{du}{d\xi}\right). \tag{36}$$

3.2 Regularity properties (Aronson (1969))

The regularity properties of flows through porous media were studied by Aronson (1969), who investigated the properties of the solution of the porous media equation (23) with a priori estimate. In other words, initially, the existence of a solution is assumed with some properties before knowing that. Subsequently, further properties of the solution were discussed, that is Hölder continuous weak solution of the Cauchy problem for the porous media equation (23).

Let us consider the Cauchy problem

$$\frac{\partial u}{\partial t} = \Delta u^m, \qquad u(x,0) = u_0(x), \qquad (x,t) \in S, \tag{37}$$

where spatio-temporal function u is in one dimension, u_0 is a given bounded, continuous, and nonnegative function, and S denotes $(-\infty, +\infty) \times (0, T]$. Here, the finite range of S denotes that the time-local type of solutions is introduced to preclude the blow-up at infinity regardless of whether it happens.

Before going to the main parts, it is necessary to provide appropriate definitions of Hölder continuity and a weak solution. A function $f: \mathbb{R}^n \to \mathbb{R}^n$ is called Hölder continuous if there exists a constant $C \geq 0$ such that, for all x and y in Euclidean space,

$$|f(x) - f(y)| \leq C|x - y|^{\sigma}. \tag{38}$$

If $\sigma = 1$, this function means Lipschitz continuous, which is differentiable almost everywhere. Next, a weak solution of the Cauchy problem (37) holds the following three conditions:

i. u is bounded, continuous, and nonnegative in $\bar{S}$,

ii. u^m possesses a bounded weak derivative with respect to x in S, and

iii. u satisfies the integral identity

$$\int\int_S \{\psi_x (u^m)_x - \psi_t u\}\, dxdt = \int_{-\infty}^{\infty} \psi(x,0) u_0(x)\, dx. \tag{39}$$

As described previously, the property of the porous media equation (23) changes at $u > 0$ and $u = 0$, or, the transition from a region where $u > 0$ to one where $u = 0$ is not smooth. Therefore, it is expected that a solution $u(x,t)$ of the Cauchy problem (37) to be a weak solution if the problem has a unique solution. Thus, it is assumed that the Cauchy problem (37) has the weak solution u, and u_0^m is Lipschitz continuous.

As a result, Aronson (1969) proved that a weak solution u of the Cauchy problem (37) is Hölder continuous with exponent $\min\{1, (m-1)^{-1}\}$. Moreover, the Hölder continuity of $\partial u^m/\partial x$ and $\partial u/\partial x$ for $m < 2$ at points where $u = 0$ was also shown. In addition, the possibility that the same holds for $u > 0$ was mentioned.

3.3 Similarity solutions (Gilding (1982))

As mentioned in section 3.1, Boltzmann's transformation could also be applied to the porous media equation. However, owing to its simple form, the equation allows a stronger similarity transformation, which was studied by Gilding (1982), who also derived some explicit solutions of the porous media equation by applying the transformations.

It is firstly assumed that the porous media equation has the following solution

$$u(x,t) = \mu(t) f(\eta), \tag{40}$$

$$\eta = \rho(t)[x + \lambda(t)], \tag{41}$$

where a similarity variable η shows that f takes the same value if $\rho(t)$, x, $\lambda(t)$ satisfy the relationship (41).

Moreover, the following discussion is considered in the domain $0 < x < \infty$, for $0 < t \le T$.

The generality of reducing the partial differential equation to the ordinary differential equation is preserved for the three types of admissible transformations (α and τ are constant).

i. For any α and $\tau \geq 0$,

$$u(x,t) = (t+\tau)^{\alpha} f(\eta), \qquad \eta = x(t+\tau)^{-\frac{[1+(m-1)\alpha]}{2}}. \tag{42}$$

ii. For any α and $\tau > T$,

$$u(x,t) = (\tau-t)^{\alpha} f(\eta), \qquad \eta = x(\tau-t)^{-\frac{[1+(m-1)\alpha]}{2}}. \tag{43}$$

iii. For any α,

$$u(x,t) = exp(\alpha t)\, f(\eta), \qquad \eta = x\, exp\left[-\frac{1}{2}(m-1)\alpha t\right]. \tag{44}$$

Substituting these admissible transformations (42)-(44) into the porous media equation, the unified the results are reduced to the ordinary differential equation with respect to η

$$(f^m)'' + p\eta f' = qf, \qquad 0 < \eta < \infty, \tag{45}$$

where p $\left(= \frac{[1+(m-1)\alpha]}{2}\right)$ and q $(= \alpha)$ are constant. Through a series of these transformations, $\mu(t)$ in (40) successfully vanishes due to the convenient structure of the original equation. Consequently, the similarity solutions are invariant under these steps.

Finally, solving the ordinary differential equation (45) give rise to the five classes of its nontrivial similarity solutions

i. If $q < 0$ and $2p + q < 0$,

   there are no nontrivial solutions.

ii. If $q < 0$ and $2p + q = 0$,

   the equation (45) has a solution of type A for $f(0) = 0$.

iii. If $q \leq 0$ and $2p + q > 0$,

   the equation (45) has a solution of type B for any $f(0) \geq 0$ and $(f^m)'(0) > f(0)^{\frac{m+1}{2}}\beta$, where β satisfies the three conditions, such that $\beta > 0$ for $p + q < 0$, $\beta = 0$ for $p + q = 0$, and $\beta < 0$ for $p + q > 0$.

iv. If $q > 0$ and $p \geq 0$,

   the equation (45) has a solution of type A for any $f(0) > 0$ and $(f^m)'(0) = f(0)^{\frac{m+1}{2}}\beta$, where $\beta < 0$.

v. If $q > 0$ and $p < 0$ or $q = 0$ and $p \leq 0$, the equation (45) has a solution of type B for any $f(0) > 0$ and $(f^m)'(0) = f(0)^{\frac{m+1}{2}}\beta$, where β satisfies the two conditions, such that $\beta < 0$ for $q > 0$, and $\beta = 0$ for $q = 0$.

Here, the solution type A and B denote the following meaning.

A: There exists a parameter a such that $f > 0$ for $0 < \eta < a$, and $f(\eta) \equiv 0$ for $\eta \geq a$.

B: f become a bounded positive function for all $0 < \eta < \infty$.

Additionally, it has also been found that some explicit solutions of the porous media equation including the ZKB solution (28) are derived by employing these similarity transformations in Gilding (1982).

4. STOCHASTIC PROCESSES FOR A POROUS MEDIA EQUATION

4.1 Stochastic differential equation

To construct the theory of stochastic processes, it is needed to define the concept of Brownian motion, that is, a stochastic process $B_t(\omega)$ on a probability space $(\Omega, \mathcal{F}, P^x)$, starting at x $(B_0(\omega) = x)$. Here, we use the notations for an outcome ω in the sample space Ω, $\mathcal{F}$ for the σ-algebra, and P^x for the probability measure given by

$$\begin{gathered} P^x\left(B_{t_1} \in F_1, \cdots, B_{t_k} \in F_k\right) = \\ = \int_{F_1 \times \cdots \times F_k} p(t_1, x, x_1) \cdots p(t_k - t_{k-1}, x_{k-1}, x_k)\, dx_1 \cdots dx_k, \end{gathered} \tag{46}$$

where the time ordering $0 \leq t_1 \leq \cdots \leq t_k$ and the position y define the probability measure p by

$$p(t, x, y) = (2\pi t)^{-\frac{n}{2}} \cdot exp\left(-\frac{|x - y|^2}{2t}\right), \tag{47}$$

which is the probability density function of the normal distribution with mean y and variance t.

The following three points characterize the Brownian motion

i. B_t has a normal distribution,

ii. B_t has independent increments, and

iii. $B_0 = x$ for a specified x.

Assuming the following differential equation on the deterministic laws

$$\frac{dX_t}{dt} = a(t)V_t, \tag{48}$$

where X_t is the position at time t, a is the coefficient depending on t, and V_t is the velocity, the approximation theory of the real world with noise can be described by the following stochastic differential equation

$$\frac{dX_t}{dt} = a(t)V_t + \sigma\frac{dB_t}{dt}, \tag{49}$$

where X_t is considered as the random variable. In the context of (49), the noise term, the second term of the right-hand side, is assumed to be stationary independent with mean 0, and therefore, the process B_t is the Brownian motion.

More generally, a stochastic differential equation can be rewritten as

$$dX_t = b(t, X_t)dt + \sigma(t, X_t)dB_t, \tag{50}$$

where $X_t \in \mathbb{R}^n$, $b(t, X_t) \in \mathbb{R}^n$, $\sigma(t, X_t) \in \mathbb{R}^{n\times m}$, and B_t denotes the m-dimensional Brownian motion. Moreover, integrating (50) over the domain $[0, t]$ gives rise to

$$X_t = X_0 + \int_0^t b(s, X_s)\, ds + \int_0^t \sigma(s, X_s)\, dB_s, \tag{51}$$

where the last term of the right-hand side is beyond the Riemann-Stieltjes sense since the mapping $t \to B_t$ is no longer differentiable but is still continuous. Therefore, a novel concept is introduced, namely the Itô integral, which has some important properties such as the martingale and the Itô formula.

A martingale is defined by the notation of filtration, which is a family $\mathcal{M} = \{\mathcal{M}_t\}_{t\geq 0}$ such that

$$0 \leq s < t \Rightarrow \mathcal{M}_s \subset \mathcal{M}_t \subset \mathcal{F}, \tag{52}$$

where we use the σ-algebra $\mathcal{F}$ on the considered probability space $(\Omega, \mathcal{F})$. Then, a stochastic process $\{M_t\}_{t\geq 0}$ is called $\mathcal{M}_t$-martingale if

i. M_t is $\mathcal{M}_t$-measurable for all t,

ii. $E[|M_t|] < \infty$ for all t, and

iii. $E[M_s|\mathcal{M}_t] = M_t$ for all $s \geq t$.

Interpreting the process X_t of the form (51) as an Itô process, we can introduce the chain rule for the Itô integral, which is the Itô formula. Let $g(t,x) = \big(g_1(t,x), \cdots, g_l(t,x)\big) \in C^2([0,\infty] \times \mathbb{R}^n \to \mathbb{R}^l)$, using n-dimensional Itô process (51). Then the process

$$Y(t,\omega) = g(t,X_t) = g\big(t,X(t)\big), \tag{53}$$

is also an Itô process, whose matrix element is given by

$$dY_k = \frac{\partial g_k}{\partial t}(t,X)dt + \Sigma_i \frac{\partial g_k}{\partial x_i}(t,X)dX_i + \frac{1}{2}\Sigma_{i,j}\frac{\partial^2 g_k}{\partial x_i \partial x_j}(t,X)dX_i dX_j, \tag{54}$$

where $dB_i dB_j = \delta_{ij}dt$, $dtdt = dB_i dt = dtdB_i = 0$, using the Kronecker delta $\delta_{ij} = \begin{cases} 0 \ (i \neq j) \\ 1 \ (i = j) \end{cases}$. Subsequently, integrating (54) over $[0,t]$ results in

$$\begin{aligned} Y_k = g_k(0,X_0) + \int_0^t \left\{\frac{\partial g_k}{\partial s}(s,X_s) + \Sigma_i b_i \frac{\partial g_k}{\partial x_i}(s,X_s) + \frac{1}{2}\Sigma_{i,j}(\sigma\sigma^T)_{i,j}\frac{\partial^2 g_k}{\partial x_i \partial x_j}(s,X_s)\right\} ds \\ + \int_0^t \Sigma_i \sigma_i \frac{\partial g_k}{\partial x_i}(s,X_s)\, dB_i(s) \end{aligned} \tag{55}$$

since

$$\begin{aligned} dX_i dX_j &= \big(b_i dt + \sigma_{ip} dB_p\big)\big(b_j dt + \sigma_{jq} dB_{jq}\big) \\ &= b_i b_j (dt)^2 + \sigma_{ip} dB_p \sigma_{jq} dB_q + b_i \sigma_{jp} dt dB_p + b_j \sigma_{iq} dt dB_q \\ &= (\sigma\sigma^T)_{i,j} dt. \end{aligned} \tag{56}$$

Considering (50) as the mathematical description of the diffusion process (called an Itô diffusion), $b(t, X_t)$ is called the drift coefficient and $\sigma(t, X_t)$ is called the diffusion coefficient. An Itô diffusion has some essential characteristics, including the Markov property and the generator.

The Markov property refers to the property in which the present behavior of a particle is unaffected by the future event but is only determined by its history up to the current time. The mathematical description of the property is given by

$$E^x\left[f(X_{t+h})\middle|\mathcal{F}_t^{(m)}\right](\omega) = E^{X_t(\omega)}[f(X_h)], \tag{57}$$

where $E^x[\cdot]$ denotes the expectation of the mapping f from $\mathbb{R}^n$ to $\mathbb{R}$ starting at x $(t = 0)$, $\mathcal{F}_t^{(m)} \subset \mathcal{F}$ is the σ-algebra generated by the m-dimensional Brownian motion $\{B_{r\leq t}(\omega)\}$, namely

$$\mathcal{F}_t^{(m)} = \bigcap\{\omega; B_0(\omega) \in F_0, \cdots, B_{r\leq t}(\omega) \in F_{r\leq t}\}, \tag{58}$$

where $F_i \subset \mathbb{R}^m$ are Borel sets (that is, subsets of $\mathbb{R}^m$ except ones uncontained in any σ-algebra).

The generator of an Itô diffusion allows us to connect the diffusion with the corresponding second-order partial differential operator L defined by

$$Lf(x) = \lim_{t\to+0} \frac{E^x[f(X_t)] - f(x)}{t}. \tag{59}$$

In addition, by applying the Itô formula (54) to the Itô process (51), we can obtain

$$\begin{aligned} &E^x[f(X_s)] = \\ &\quad = f(x) + E^x\left[\int_0^\tau \left(\Sigma_i b_i \frac{\partial f}{\partial x_i}(X_s) + \frac{1}{2}\Sigma_{i,j}(\sigma\sigma^T)_{i,j} \frac{\partial^2 f}{\partial x_i \partial x_j}(X_s)\right) ds\right], \end{aligned} \tag{60}$$

where the mapping $f \in C^2(\mathbb{R}^n \to \mathbb{R})$ has compact support. Then, these definitions, (59) and (60), allow us to gain the differential form of the generator L

$$Lf(x) = \Sigma_i b_i(x) \frac{\partial f}{\partial x_i} + \frac{1}{2} \Sigma_{i,j} (\sigma\sigma^T)_{i,j}(x) \frac{\partial^2 f}{\partial x_i \partial x_j}. \tag{61}$$

4.2 Probabilistic approximation (Jourdain (2000))

Jourdain (2000) provided a probabilistic representation of the PME (23) when the initial condition is a cumulative distribution function by associating it with a nonlinear martingale problem and utilizing a stochastic particle method developed by Bossy and Talay (1996). To construct the solution of the probabilistic PME, n-particle system is first introduced with the empirical measure and its distribution. Then, in terms of the propagation of chaos (Kac (1956) introduced the notation of chaos), the distribution converges to that of the nonlinear diffusion process, which is almost surely equivalent to the solution of the PME. In addition, the reordering and symmetrizing of n-particle system is also proved to be nearly a solution of the PME. Concerning the martingale problem (MP), which corresponds to the PME of interest, the former probabilistic law solves the problem, but the latter doesnot.

To connect a macroscopic scale equation with a microscopic scale one, the idea of propagation of chaos is used by McKean (1969), who first suggests the probabilistic representation of the solution of the non-linear deterministic PDE. According to Sznitman (1991), in n-particle system, the independence (or chaos) of the velocity of each particle is destroyed at time t, but the distribution of the velocities of the first k particles is still independent of each other when n is large enough. Therefore, the initial independence keeps propagated (propagation of chaos). A further literature review can be found in Chaintron and Diez (2022).

The PME (23) can be transformed into:

$$\begin{aligned}\frac{\partial v}{\partial t} &= \frac{\partial^2}{\partial x^2}(mu^{m-1}v) \\ &= \frac{\partial^2}{\partial x^2}\Big(m\big(H * v(t,\cdot)(x)\big)^{m-1} v(t,\cdot)\Big),\end{aligned} \tag{62}$$

where $v = \partial_x u$ and H is the Heaviside function which is derived from its discontinuous solution. This equation (62) can be interpreted as the Fokker-Plank equation, that is, the limit law of a particle system. In this sense, when the initial condition is the cumulative distribution function, the PME (23) is associated with the nonlinear martingale problem:

A probability measure P on $(\mathbb{R}^{[0,\infty]}, \mathcal{B})$ solves the martingale problem starting at p if

i, $P_0 = p$,

ii. $M_t = \phi(X_t) - \phi(X_0) - q\int_0^t \left(H * P_s(X_s)\right)^{m-1} \phi''(X_s)\, ds$ is a P-martingale,

iii. P_t does not weight point,

where p is the density, $\phi \in C^2(\mathbb{R})$ is the bounded continuous function, and X_t is the canonical process, that is, the event $\omega \in \Omega$ is equal to the random variable X_t. If P solves the above MP, $(s,x) \to H * P_s(x)$ is a weak solution of the PME (23) when the initial condition is $H * p(x)$, which are proven in this paper.

The solution of this nonlinear martingale problem is equivalent to the distribution of the process X_t. Concerning this stochastic process, one is obtained as the propagation of chaos result of the interacting particle system

$$X_t^{i,n} = X_0^i + \int_0^t m\left(H * \mu_s^n(X_s^{i,n})\right)^{\frac{m-1}{2}} dB_s^i, \tag{63}$$

where $X_t^{i,n}, 1 \leq i \leq n$ denotes the position of the i-th particle at time t, B^i are Brownian motions, and $\mu^n = (1/n)\Sigma_{j=1}^n \delta_{X^{j,n}}$. Applying the Itô formula on a heuristic level to the process (63), we can obtain

$$\begin{aligned}\phi(X_t) = \phi(X_0) &+ \int_0^t \left(\frac{\partial \phi}{\partial s} + \frac{\partial^2 \phi}{\partial x^2} q\left(H * \mu_s^n(X_s^{i,n})\right)^{q-1}\right) ds \\ &+ \int_0^t \frac{\partial \phi}{\partial x}\sqrt{2q}\left(H * \mu_s^n(X_s^{i,n})\right)^{q-1} dB_s^i.\end{aligned} \tag{64}$$

In the context of (64), the process M_t in the MP coincides with the noise term of the right-hand side.

Furthermore, reordering the particle system (63) to create an increasing sequence, the author obtained another stochastic process

$$Y_t^{i,n} = \sup_{|A|=n-i+1} \inf_{j\in A} X_t^{j,n}, \tag{65}$$

where $|A|$ denotes the number of elements in $A \subset \{1,\cdots,n\}$. The particle systems $(Y_t^{1,n},\cdots,Y_t^{n,n})$ converges to $\widetilde{P}$ which is no longer the solution of the MP, but its related function $(s,x) \to H * \widetilde{P_s}$ is a weak solution of the PME.

4.3 Porous media equations with noise (Da Prato and Röckner (2004))

Da Prato and Röckner (2004) studied a porous media equation with noise and viscosity term

$$X_t = X_0 + \int_0^t (\alpha\Delta X_t + \Delta X_t^m)\,dt + \int_0^t \sqrt{c}\,dB_t, \tag{66}$$

where $\alpha \geq 0$, c is a positive function, and B_t is a Brownian motion at time t. In the context of (66), the viscosity term $\alpha\Delta X_t$ is introduced to make the porous media equation into a non-degenerate one. The authors regard (66) as the extension of the stochastic heat equation studied by Barbu and Da Prato (2002), who connected the equation with the two-phase stochastic Stefan problem. The problem describes the evolution of materials in phase transition, which is equal to solving the corresponding heat equations, subject to the appropriate boundary and initial conditions. The results of the research can be expanded to a stochastic porous media equation by applying the function

$$\mathcal{B}_\varepsilon(x) := \frac{x^m}{1+\varepsilon x^{m-1}} + (\alpha+\varepsilon)x, \qquad \varepsilon \in (0,1]. \tag{67}$$

Initially, the Kolmogorov equation corresponding to the stochastic differential equation (66) is solved by utilizing C_0-semigroup approach, which allows the users to regard the partial differential equation as the ordinary differential equation under suitable conditions. Then, it is shown that a strong Markov process obtained from its solution solves the MP corresponding to (66).

By Itô formula,

$$f(X_t) = f(X_0)$$

$$+\int_0^t \left((\alpha \Delta X_t + \Delta X_t^m)\frac{\partial f}{\partial x} + \frac{c}{2}\frac{\partial^2 f}{\partial x^2}\right) ds$$

$$+\int_0^t \sqrt{c}\,\frac{\partial f}{\partial x}\, dB_s, \tag{68}$$

In this equation (68), the second term of the right-hand side represents the Kolmogorov operator

$$Af(x) = (\alpha \Delta X_t + \Delta X_t^m)\frac{\partial f}{\partial x} + \frac{1}{2}\sum_{k=1}^{\infty} c\,\frac{\partial^2 f}{\partial x^2}, \tag{69}$$

which is associated with the following Kolmogorov equation

$$\frac{dv}{dt} = Av, \tag{70}$$

where v is an invariant measure. The solutions of this ODE are expected to have C_0-semigroup $T_t = e^{tA}$, which can be regarded as the generalization of the exponential function. Analyzing this equation (70) makes it possible to construct a strong Markov process satisfying the corresponding MP.

5. CONCLUSIONS

The Richards equation and the Boussinesq equation are regarded as the PMEs in suitable conditions. Therefore, it is proper to consider specific hydrological phenomena in a mathematical framework associated with the PMEs. Each model in this study was explained in the context of the mathematics of the PMEs (Pop and Schweizer, 2011; Cao and Pop, 2016; Kacimov and Šimůnek, 2021). Moreover, the theories of the PMEs were extended to those on the probabilistic spaces; that is, the related independent variables are considered as probability measures, such as random variables and distribution functions. Correspondingly, it is desirable to expand these models, including the Richards equation and the Boussinesq equation, to the stochastic PMEs to deal with hydrological phenomena with uncertainty. However, we might overlook the issues intrinsic to these models, typically the hysteresis phenomenon commonly observed in the real world. The methodologies of stochastic control such as Hamilton-Jacobi-Bellman equations, which govern the value functions and the optimal controls for controlled Markov processes, are also required in practical water management problems.

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