Porous media equations to visualize subsurface water flows

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1. Introduction A considerable part of the Fertile Crescent is not rich in surface water resources but depends on groundwater reserves [1]. Scientific understanding of water movement in aquifers is a prerequisite for establishing sustainable strategies for developing and managing groundwater resources, especially when conflicting stakeholders are involved. Conventional mathematical models of subsurface water flows include the Boussinesq groundwater equation and the Richards equation, which are nonlinear partial differential equations categorized as porous media equations (PMEs). This study overviews the astonishing properties of PMEs in their application to visualizing subsurface water flows.

2. Materials and Method The conservation law of water mass in a subsurface domain is written as

$$\frac{\partial \theta}{\partial t} + \nabla \cdot \mathbf{V} = S \tag{1}$$

where θ is the volumetric water content, t is the time, ∇ is the three-dimensional (3-D) (x-y-z) del operator, $\mathbf{V}$ is the Darcy velocity, and S is the source term. Assuming the Darcy-Buckingham law, (1) is rewritten as the Richards equation

$$\frac{\partial \theta}{\partial t} = \nabla \cdot K(h)\left(\nabla h + \nabla z\right) + S \tag{2}$$

where h is the matric potential, and K is the variably saturated hydraulic conductivity tensor depending on h. In an unconfined aquifer, integrating (2) in the saturated zone from the impermeable bottom z_b to the water table η (= $h + z$) results in the Boussinesq equation

$$\frac{\partial}{\partial t}\left(S_S \eta\right) = \nabla_{\mathrm{H}} \cdot \left(\eta - z_b\right) K_S \nabla_{\mathrm{H}} \eta + R \tag{3}$$

where S_S is the specific storage, ∇_{H} is the horizontal two-dimensional (2-D) (x-y) del operator, K_S is the saturated hydraulic conductivity tensor, and R is the recharge rate. Under appropriate assumptions such as [2], each of the two model equations (1) and (2) is interpreted as the standard PME

$$\frac{\partial u}{\partial t} = \Delta u^m \tag{4}$$

where u is the density as a generic unknown spatio-temporal function, Δ is the Laplace operator, and m is an exponent. For instance, normalizing (3) in a homogenous level aquifer without recharge yields (4) with $u = \eta$ and $m = 2$ in the 2-D space.

3. Results and Discussion One of the most notable features of PMEs is the degeneration of the "diffusion coefficient," implying finite propagation speeds of disturbances and loss of smoothness in the solutions [3]. In the 2-D space, the standard PME (4) with m = 1, the heat equation, has a C^∞ strong solution $u = \exp\left(-(x^2+y^2)/4t\right)/t$, while the standard PME (4) with m = 2 has a Lipschitz continuous weak solution $u = \max\left(0, 1/\sqrt{t} - (x^2+y^2)/16t\right)$ with bounded support for any t [4]. The theorem in [3] asserts more critical regularity results for $m > 1$, indicating the significance of visualizing subsurface water flow.

4. Conclusions It is inferred that the Richards equation and the Boussinesq groundwater equation have properties similar to those of the standard PME. Thus, the discussions based on PMEs' properties will contribute to the strategic development and management of groundwater resources in the real world.

References

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