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A Celestial-Mechanical Model for the Tidal Evolution of the Earth–Moon System Treated as a Double Planet

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Abstract—A celestial-mechanical model for the motion of two viscoelastic spheres in the gravitational field of a massive point is considered, treating them as a double planet. The spheres move along quasicircular orbits in a single plane, with their rotational axes perpendicular to this plane. The deformation of the spheres is described using the classical theory of small deformations. A Kelvin–Voigt model is adopted for the viscous forces. A system of evoutionary equations is obtained and applied to analyze the joint translational–rotational tidal evolution of the Earth and Moon in the gravitational field of the Sun. This system has been numerically integrated several billion years into the past and into the future. The results are compared with the predictions of other theories, paleontological data, and astronomical observations.

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1. INTRODUCTION

The theory of tides originated with work by Newton and Laplace. The main achievements in this area were collected, systematized, and analyzed by Darwin [1], and further developed by MacDonald [2], who studied the evolution of the Earth–Moon system without including the influence of the Sun. Goldreich [3] used the method of MacDonald to investigate the oblateness of the Earth and the influence of solar tides, but neglecting the ellipticity of the lunar orbit, and correctly averaged the equations of motion using three time scales. The method of MacDonald was then used in various other studies. Beletskii [4] investigated the tidal evolution of the inclinations and rotations of celestial bodies. Webb [5] studied the evolution of the Earth–Moon system based on the ocean tides and compared his results with the model of Goldreich [3]. Krasinsky [6] combined the methods of MacDonald and Goldreich to reconstruct a dynamical history of the Earth–Moon system. Touma andWisdom [7] developed various models for tidal phenomena in detail. It was shown that the evolution of the Earth–Moon system based on the models of Darwin–Mignard and Darwin–Cowley–Goldreich is essentially equivalent to that

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predicted by the model of Goldreich. Efroimsky and Lainey [8] considered the effective dissipation function Q, which is proportional to the tidal frequency to the power α . They studied the tidal evolution of the Martian moon Phobos for α = 0.2, 0.3, 0.4. Note that α = 0 in the model of MacDonald and α = -1 in the model of Mignard [9, 10]. The main distinguishing property of the approach proposed by Ferraz-Mello et al. [11] is that, in contrast to many studies based on the theory of Darwin, different coefficients are introduced for the harmonics of the tidal wave, instead of one Love number. A critical analysis of the mathematical formulas in the above theories describing the tidal moments, slowing of planetary rotation, and the delay angle, as well as the accuracy and range of applicability of the theories and connections with rheological models, are considered by Efroimsky and Williams [12] and Efroimsky and Makarov [13]. Note that the qualitative conclusions derived for the simpler MacDonald theory essentially remain correct [12]. The subsequent development of tidal theories is concerned with the creation of rheological models. Churkin [14–16] established a generalized theory of the Love number and applied it to the rheological models of Guk, Maxwell, Voigt, and others. His theory for the rotation of the inelastic Earth was applied to a Voigt model for the Earth's interior, and numerical estimates of rheological corrections to the precession, nutation, and axial rotation of the Earth were obtained. Efroimsky [17] introduced complex Love numbers as a function of the tidal frequency to study tides in the case of a rotational–orbital resonance between a planet and one of its satellites. Vil'ke [18] developed a method for separating motions and averaging in systems with an infinite number of degrees of freedom, aimed at studying the motions of deformable bodies using a classical linear elasticity theory for small deformations and a Kelvin– Voigt model for the viscous forces. This method was used to investigate the evolution of the orbital and rotational motions of a viscoelastic planet in a central Newtonian force field [19, 20]. The model for a celestial body of Markov and Minyaev [21] includes an isotropic, viscoelastic layer and a rigid core. A qualitative analysis of the motion of the moons of Mars is given, and the model parameters were refined based on the observations of the secular acceleration of Phobos. Vil'ke and Shatina [22] studied the tidal evolution of the motion of the Earth–Moon system in the gravitational field of the Sun, treating the Moon as a point mass. Let us now turn to our model describing a double planet [23–25].

2. MATHEMATICAL MODEL FOR THE MOTION OF TWO VISCOELASTIC SPHERES IN THE GRAVITATIONAL FIELD OF A FIXED CENTRAL BODY

2.1. Formulation of the Problem

In the unperturbed motion, the barycenter C of the two uniform rigid spheres O_1 and O_2 with masses m1 and m2 moves in a circular, Keplerian orbit in a fixed plane in the gravitational field of a stationary massive point mass M. The spheres O_1 and O_2, in turn, move in circular Keplerian orbits about the barycenter C in the plane of its motion. The spheres rotate with specified constant angular speeds about axes passing through their centers of mass perpendicular to the plane of their orbital motion. All four motions are independent of each other. This formulation of the problem is possible because we have made the assumptions

$m2 \ll m1 \ll M; r_{i0} \ll R_2 \ll R_1, (1)$

where r_{i0} (i = 1, 2) are the radii of the spheres, R_1 is the distance from the gravitating center to the barycenter, and R_2 is the distance between the centers of mass of O_1 and O_2 (we will further identify the names of the spheres with their centers of mass). These assumptions are satisfied, for example, by the Sun–Earth–Moon system.

In the perturbed motion, we treat the spheres as uniform, isotropic, viscoelastic bodies. Perturbations arise due to the deformation of the bodies in response to the centrifugal and gravitational forces. Since we are studying evolutionary motions, we assume that the centers of mass of the spheres move along quasicircular orbits.

To describe the motion, we specify an inertial coordinate frame OXY Z fixed to the gravitating center O, with the spheres moving in the OXY plane. We specify Koenig coordinate systems $O_iX_iY_i$ with the points O_i (i = 1, 2). The position of the barycenter C in the OXY Z system is specified by the vector $R_1 = \vec{OC}(R_1 \cos\lambda_1, R_1 \sin\lambda_1,\ 0)$, where $|R_1| = R_1$, and λ_1 is the angle between R_1 and the OX axis. The position of O_2 relative to O_1 in the $O_1X_1Y_1Z_1$ frame is specified by the vector $R_2 = O_1O_2(R_2 \cos\lambda_2, R_2 \sin\lambda_2, 0)$, where $|R_2| = R_2$, and λ_2 is the angle between R_2 and the OX_1 axis. The deformed state of the bodies is described by the classical theory of elasticity for small deformations. We adopted a Kelvin–Voigt model for the viscous forces, with the dissipation function $D_i[\dot{u}_i]$ proportional to the elastic-force function $W_i[\dot{u}_i]$, with the coefficient of proportionality X_i (the viscosity coefficient): $D_i[\dot{u}_i] = X_iW_i[\dot{u}_i]$, (2) where $u_i(r_i, t)$ is the shift in the points of the body O_i due to the deformations, $\dot{u}_i = du_i/dt$ (here and below, a dot above a quantity denotes a time derivative), and r_i is the radius vector of the points in a sphere relative to the center O_i in the undeformed state. // The rotating spheres are associated with their own coordinate systems $O_iX_{ii}Y_{ii}Z_{ii}$, where the O_iZ_{ii} axis is perpendicular to the orbital plane (the O_iZ_i and O_iZ_{ii} axes coincide). The positions of the points in the viscoelastic sphere O_i in the OXY Z coordinate system are determined by the vector field $\zeta_i(r_i, t) = \vec{OO_i} + \Gamma_i(\varphi_i)(r_i + u_i(r_i, t))$, where $\Gamma_i(\varphi_i(t)) = \begin{pmatrix} \cos\varphi_i & -\sin\varphi_i & 0 \\ \sin\varphi_i & \cos\varphi_i & 0 \\ 0 & 0 & 1 \end{pmatrix}$.(4) Here,$\Gamma_i$ is the orthogonal operator for the translation from the Koenig coordinates $O_iX_iY_iZ_i$ to the coordinates $O_iX_{ii}Y_{ii}Z_{ii}$ and φ_i is the rotation of the $O_iX_{ii}Y_{ii}Z_{ii}$ system about the O_iZ_{ii} axis (φ_i is the angle between the O_iX_i and O_iX_{ii} axes). In order to uniquely determine the positions of the centers of mass of the spheres O_i in the $O_iX_{ii}Y_{ii}Z_{ii}$ coordinate systems as the spheres move by $\zeta_i(r_i, t)$, we imposed the following conditions (relations) on this motion:

$$\int_{v_i} u_i dv_i$$

$= 0$,

$$\int_{v_i} curl u_i dv_i$$

$= 0$

$(dv_i = dx_{ii}dy_{ii}dz_{ii})$, where $V_i = \{|r_i| < r_{i0}\}$ is the region occupied by Oi in the undeformed state.

The functional for the kinetic energy of the system has the form $T = \frac{1}{2}mR_1^2 + \frac{m1m2}{2m}R_2^2 + \frac{1}{2}\sum_{i=1}^{2} J_i[u_i]\dot{\varphi}_i^2 + 2G_i\dot{\varphi}_i + T_{0i})$, where $J_i[u_i]=$

$$\int_{v_i}[e_3 \times (r_i + u_i)]^2 \rho_i dv_i$$

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