

# A Celestial-Mechanical Model for the Tidal Evolution of the Earth-Moon System Treated as a Double Planet

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**Abstract**—A celestial-mechanical model for the motion of two viscoelastic spheres in the gravitational field of a massive point is considered, treating them as a double planet. The spheres move along quasicircular orbits in a single plane, with their rotational axes perpendicular to this plane. The deformation of the spheres is described using the classical theory of small deformations. A Kelvin–Voigt model is adopted for the viscous forces. A system of evolutionary equations is obtained and applied to analyze the joint translational–rotational tidal evolution of the Earth and Moon in the gravitational field of the Sun. This system has been numerically integrated several billion years into the past and into the future. The results are compared with the predictions of other theories, paleontological data, and astronomical observations.

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## 1. INTRODUCTION

The theory of tides originated with work by Newton and Laplace. The main achievements in this area were collected, systematized, and analyzed by Darwin [1], and further developed by MacDonald [2], who studied the evolution of the Earth–Moon system without including the influence of the Sun. Goldreich [3] used the method of MacDonald to investigate the oblateness of the Earth and the influence of solar tides, but neglecting the ellipticity of the lunar orbit, and correctly averaged the equations of motion using three time scales.

The method of MacDonald was then used in various other studies. Beletskii [4] investigated the tidal evolution of the inclinations and rotations of celestial bodies. Webb [5] studied the evolution of the Earth–Moon system based on the ocean tides and compared his results with the model of Goldreich [3]. Krasinsky [6] combined the methods of MacDonald and Goldreich to reconstruct a dynamical history of the Earth–Moon system. Touma and Wisdom [7] developed various models for tidal phenomena in detail. It was shown that the evolution of the Earth–Moon system based on the models of Darwin–Mignard and Darwin–Cowley–Goldreich is essentially equivalent to that predicted by the model of Goldreich.

Efroimsky and Lainey [8] considered the effective dissipation function  $Q$ , which is proportional to the tidal frequency to the power  $\alpha$ . They studied the tidal evolution of the Martian moon Phobos for  $\alpha = 0.2, 0.3, 0.4$ . Note that  $\alpha = 0$  in the model of

MacDonald and  $\alpha = -1$  in the model of Mignard [9, 10]. The main distinguishing property of the approach proposed by Ferraz-Mello et al. [11] is that, in contrast to many studies based on the theory of Darwin, different coefficients are introduced for the harmonics of the tidal wave, instead of one Love number. A critical analysis of the mathematical formulas in the above theories describing the tidal moments, slowing of planetary rotation, and the delay angle, as well as the accuracy and range of applicability of the theories and connections with rheological models, are considered by Efroimsky and Williams [12] and Efroimsky and Makarov [13]. Note that the qualitative conclusions derived for the simpler MacDonald theory essentially remain correct [12].

The subsequent development of tidal theories is concerned with the creation of rheological models. Churkin [14–16] established a generalized theory of the Love number and applied it to the rheological models of Guk, Maxwell, Voigt, and others. His theory for the rotation of the inelastic Earth was applied to a Voigt model for the Earth’s interior, and numerical estimates of rheological corrections to the precession, nutation, and axial rotation of the Earth were obtained. Efroimsky [17] introduced complex Love numbers as a function of the tidal frequency to study tides in the case of a rotational–orbital resonance between a planet and one of its satellites. Vil’ke [18] developed a method for separating motions and averaging in systems with an infinite number of degrees of freedom, aimed at studying the motions of deformable bodies using a classical linear

elasticity theory for small deformations and a Kelvin–Voigt model for the viscous forces. This method was used to investigate the evolution of the orbital and rotational motions of a viscoelastic planet in a central Newtonian force field [19, 20]. The model for a celestial body of Markov and Minyaev [21] includes an isotropic, viscoelastic layer and a rigid core. A qualitative analysis of the motion of the moons of Mars is given, and the model parameters were refined based on the observations of the secular acceleration of Phobos. Vil’ke and Shatina [22] studied the tidal evolution of the motion of the Earth–Moon system in the gravitational field of the Sun, treating the Moon as a point mass. Let us now turn to our model describing a double planet [23–25].

## 2. MATHEMATICAL MODEL FOR THE MOTION OF TWO VISCOELASTIC SPHERES IN THE GRAVITATIONAL FIELD OF A FIXED CENTRAL BODY

### 2.1. Formulation of the Problem

In the unperturbed motion, the barycenter C of the two uniform rigid spheres  $O_1$  and  $O_2$  with masses  $m_1$  and  $m_2$  moves in a circular, Keplerian orbit in a fixed plane in the gravitational field of a stationary massive point mass  $M$ . The spheres  $O_1$  and  $O_2$ , in turn, move in circular Keplerian orbits about the barycenter C in the plane of its motion. The spheres rotate with specified constant angular speeds about axes passing through their centers of mass perpendicular to the plane of their orbital motion. All four motions are independent of each other. This formulation of the problem is possible because we have made the assumptions

$$m_2 \ll m_1 \ll M; r_{i0} \ll R_2 \ll R_1, \quad (1)$$

where  $r_{i0}$  ( $i = 1, 2$ ) are the radii of the spheres,  $R_1$  is the distance from the gravitating center to the barycenter, and  $R_2$  is the distance between the centers of mass of  $O_1$  and  $O_2$  (we will further identify the names of the spheres with their centers of mass). These assumptions are satisfied, for example, by the Sun–Earth–Moon system.

In the perturbed motion, we treat the spheres as uniform, isotropic, viscoelastic bodies. Perturbations arise due to the deformation of the bodies in response to the centrifugal and gravitational forces. Since we are studying evolutionary motions, we assume that the centers of mass of the spheres move along quasicircular

orbits.

To describe the motion, we specify an inertial coordinate frame OXY Z fixed to the gravitating center O, with the spheres moving in the OXY plane. We specify Koenig coordinate systems  $O_i X_i Y_i$  with the points  $O_i$  ( $i = 1, 2$ ). The position of the barycenter C in the OXY Z system is specified by the vector  $R_1 = OC(R_1 \cos \lambda_1, R_1 \sin \lambda_1, 0)$ , where  $|R_1| = R_1$ , and  $\lambda_1$  is the angle between  $R_1$  and the OX axis. The position of  $O_2$  relative to  $O_1$  in the  $O_1 X_1 Y_1 Z_1$  frame is specified by the vector  $R_2 = O_1 O_2(R_2 \cos \lambda_2, R_2 \sin \lambda_2, 0)$ , where  $|R_2| = R_2$ , and  $\lambda_2$  is the angle between  $R_2$  and the  $OX_1$  axis. The deformed state of the bodies is described by the classical theory of elasticity for small deformations. We adopted a Kelvin–Voigt model for the viscous forces, with the dissipation function  $D_i[\dot{u}_i]$  proportional to the elastic-force function  $W_i[\dot{u}_i]$ , with the coefficient of proportionality  $X_i$  (the viscosity coefficient):  $D_i[\dot{u}_i] = X_i W_i[\dot{u}_i]$ , (2) where  $u_i(r_i, t)$  is the shift in the points of the body  $O_i$  due to the deformations,  $\dot{u}_i = du_i/dt$  (here and below, a dot above a quantity denotes a time derivative), and  $r_i$  is the radius vector of the points in a sphere relative to the center  $O_i$  in the undeformed state.

The rotating spheres are associated with their own coordinate systems  $O_i X_{ii} Y_{ii} Z_{ii}$ , where the  $O_i Z_{ii}$  axis is perpendicular to the orbital plane (the  $O_i Z_i$  and  $O_i Z_{ii}$  axes coincide). The positions of the points in the viscoelastic sphere  $O_i$  in the OXY Z coordinate system are determined by the vector field  $\zeta_i(r_i, t) = O\vec{O}_i + \Gamma_i(\varphi_i)(r_i + u_i(r_i, t))$ , where

$$\Gamma_i(\varphi_i(t)) = \begin{pmatrix} \cos \varphi_i & -\sin \varphi_i & 0 \\ \sin \varphi_i & \cos \varphi_i & 0 \\ 0 & 0 & 1 \end{pmatrix}. \quad (4)$$

Here,  $\Gamma_i$  is the orthogonal operator for the translation from the Koenig coordinates  $O_i X_i Y_i Z_i$  to the coordinates  $O_i X_{ii} Y_{ii} Z_{ii}$  and  $\varphi_i$  is the rotation of the  $O_i X_{ii} Y_{ii} Z_{ii}$  system about the  $O_i Z_{ii}$  axis ( $\varphi_i$  is the angle between the  $O_i X_i$  and  $O_i X_{ii}$  axes). In order to uniquely determine the positions of the centers of mass of the spheres  $O_i$  in the  $O_i X_{ii} Y_{ii} Z_{ii}$  coordinate systems as the spheres move by  $\zeta_i(r_i, t)$ , we imposed the following conditions (relations) on this motion:

$$\int_{V_i} u_i dv_i = 0, \quad \int_{V_i} \text{curl} u_i dv_i = 0$$

$$(dv_i = dx_{ii} dy_{ii} dz_{ii}),$$

where  $V_i = \{|r_i| < r_{i0}\}$  is the region occupied by  $O_i$  in the undeformed state.

The functional for the kinetic energy of the system has the form  $T = \frac{1}{2}m\dot{R}_1^2 + \frac{m_1m_2}{2m}\dot{R}_2^2 + \frac{1}{2}\sum_{i=1}^2 J_i[u_i]\dot{\varphi}_i^2 + 2G_i\dot{\varphi}_i + T_{0i}$ ,

where  $J_i[u_i] = \int_{V_i} [e_3 \times (r_i + u_i)]^2 \rho_i dv_i$ ,  $G_i = \int_{V_i} [e_3 \times (r_i + u_i), \dot{u}_i] \rho_i dv_i$ ,  $T_{0i} = \int_{V_i} (\dot{u}_i^2) \rho_i dv_i$ ,  $e_3$  is the unit vector

for the  $O_i Z_{ii}$  axis perpendicular to the OXY plane, and  $\rho_i$  is the density of the sphere  $O_i$ .

The potential energy associated with the gravitational interactions is given by

$$\Pi = \Pi_1 + \Pi_2 + \Pi_3,$$

$$\text{where } \Pi_1 = -f \int_{V_i} \{(R_1 - m_2/m)R_2$$

$$+ \Gamma_1(r_1 + u_1)]^2\}^{-1/2} \rho_1 dv_1$$

is the energy associated with the interaction between the gravitating center and the viscoelastic body  $O_1$ ,

$$\Pi_2 = -f \int_{V_2} \{(R_1 + m_1/m)R_2$$

$$+ \Gamma_2(r_2 + u_2)]^2\}^{-1/2} \rho_2 dv_2$$

is the energy associated with the interaction between the gravitating center and the viscoelastic body  $O_2$ ,

$$\Pi_3 = -f \int_{V_i} \int_{V_2} \{(\Gamma_2 + \Gamma_2(r_2 + u_2)$$

$$- \Gamma_1(r_1 + u_1)]^2\}^{1/2} \rho_1 \rho_2 dv_1 dv_2$$

is the energy associated with the deformable spheres  $O_1$  and  $O_2$ ,  $f = GM$ ,  $G$  is the gravitational constant, and  $m = m_1 + m_2$ . Taking into account the condition (1) and neglecting terms of order  $(R_2/R_1)^3(m_2/m)^3$  and higher order in smallness in the potential energy, and leaving only terms that are linear  $u_i$ , we obtain

$$\Pi = f m / R_1 - G m_1 m_2 / R_2 + \Pi_\rho \quad (8)$$

$$\Pi_\rho = \sum_{k=1}^2 \sum_{i=1}^2 f_{ki} / R_k^3 \int_{V_i} [r_i u_i$$

$$- 3(\xi_{ki}, r_i)(\xi_{ki}, u_i)] \rho_i dv_i,$$

$$f_{1i} = f, f_{2i} = G m_{3-i},$$

$$\xi_{ki} = \tau_{ki} [\cos(\lambda_k - \phi_i), \sin(\lambda_k - \phi_i), 0],$$

$$\tau_{21} = -1, \tau_{ki} = 1 \quad (k \neq 2, i \neq 1).$$

Following [22], we introduced canonical Poincaré variables  $\lambda_k, \Lambda_k$  ( $k = 1, 2$ ) to describe the motion of the barycenter and centers of mass  $O_i$ :  $\Lambda_1 = m(f R_1)^{1/2}$ ,  $\Lambda_2 = m_r(f_0 R_2)^{1/2}$ , (10)

where  $m_r = m_1 m_2 / m$ ,  $f_0 = Gm$ .

To describe the rotational motion of the bodies, we used the Andoyer canonical variables  $\varphi_i, I_i$  ( $i = 1, 2$ ):

$$I_i = J_i[u_i]\dot{\varphi}_i + G_i, \quad (11)$$

where  $J_i[u_i]$  and  $G_i$  is defined in (6).

The equation of motion was written in the form of the Routh equations

$$\dot{\Lambda}_k = -\frac{\partial \mathfrak{R}}{\partial \lambda_k}, \dot{\lambda}_k = \frac{\partial \mathfrak{R}}{\partial \Lambda_k}, \dot{I}_i = -\frac{\partial \mathfrak{R}}{\partial \varphi_i}, \quad (12)$$

$\dot{\varphi}_i = -\frac{\partial \mathfrak{R}}{\partial I_i}, \frac{d}{dt} \nabla_{u_i} \mathfrak{R} - \nabla_{u_i} \mathfrak{R} - \nabla_{u_i} D_i = 0$ . Here,  $\mathfrak{R}$  is the Routh function, which has the form

$$\mathfrak{R} = -\frac{f^2 m^3}{2 \Lambda_1^2} - \frac{f_0^2 m_r^3}{2 \Lambda_2^2} \quad (13)$$

$$+ \sum_{i=1}^2 \left\{ \frac{I_i^2}{2 A_i} - \frac{I_i^2}{2 A_i^2} J_{i1}[u_i] \right.$$

$$\left. - \frac{I_i}{A_i} (e_3, \int_{V_i} (r \times \dot{u}_i) dv) + W_i[u_i] \right\} + \Pi_\rho$$

where  $A_i = 0.4 m_i r_{i0}^2$  is the moment of inertia of the undeformed sphere  $O_i$ ,

$$J_{i1}[u_i] = 2 \int_{V_i} [(r_i, u_i) - (e_3, r_i)(e_3, u_i)] \rho_i dv_i, \quad (14)$$

and an expression for  $\Pi_\rho$  is given by (9).

The equations of motion admit the integral of the angular momentum. When the Routh function (13) is written out in detail, it can be shown that the angular variables  $\lambda_k$  and  $\varphi_i$  appear in this function in the combination  $\psi_{ki} = \lambda_k - \varphi_i$ . It follows that

$$\dot{I}_i = -\frac{\partial \mathfrak{R}}{\partial \varphi_i} = -\sum_{k=1}^2 \frac{\partial \mathfrak{R}}{\partial \psi_{ki}} \cdot \frac{\partial \psi_{ki}}{\partial \varphi_i} = \sum_{k=1}^2 \frac{\partial \mathfrak{R}}{\partial \psi_{ki}}, \quad (15)$$

$$\dot{\Lambda}_k = -\frac{\partial \mathfrak{R}}{\partial \lambda_k} = -\sum_{i=1}^2 \frac{\partial \mathfrak{R}}{\partial \psi_{ki}} \cdot \frac{\partial \psi_{ki}}{\partial \lambda_k} = -\sum_{i=1}^2 \frac{\partial \mathfrak{R}}{\partial \psi_{ki}},$$

$$\dot{I}_1 + \dot{I}_2 + \dot{\Lambda}_1 + \dot{\Lambda}_2 = \sum_{i=1}^2 \sum_{k=1}^2 \frac{\partial \mathfrak{R}}{\partial \psi_{ki}} - \sum_{k=1}^2 \sum_{i=1}^2 \frac{\partial \mathfrak{R}}{\partial \psi_{ki}} = 0,$$

$$I_1 + I_2 + \Lambda_1 + \Lambda_2 = K_0 = \text{const.}$$

*the Displacements of Points in the Bodies due to their Deformation*

The system (12) cannot be integrated in explicit form, since this is a quite complex system of differential equations. Therefore, we applied the method for separating motions in systems with an infinite number of degrees of freedom [18]. Since we assumed the rigidity of the elastic spheres  $O_i$  were high, we introduced the small parameters  $\varepsilon_i$ , proportional to the ratio of the squares of the angular velocity of rotation of a sphere at the initial time and of the lowest frequency for the intrinsic elastic vibrations of the sphere. The displacements  $u_i$  are small, and can be represented as a series in powers of  $\varepsilon_i$ :

$$u_i(r_i, t) = \varepsilon_i u_{i1}(r_i, t) + \varepsilon_i^2 u_{i2}(r_i, t) + \dots, \quad (16)$$

$$\varepsilon_i = \rho_i r_{i0}^2 \varphi_i^2(0) / E_i, \quad (17)$$

where  $E_i$  is the Young's modulus for the body  $O_i$ .

If  $\varepsilon_i = 0$ , then  $u_i(r_i, t) = 0$ , and the equations of the unperturbed motion follow from (12):

$$\dot{\Lambda}_1 = \dot{\Lambda}_2 = \dot{I}_1 = \dot{I}_2 = 0,$$

$$\dot{\lambda}_1 = \omega_1, \dot{\lambda}_2 = \omega_2, \dot{\varphi}_1 = \omega_3, \dot{\varphi}_2 = \omega_4$$

where

$$\omega_1 = \frac{f^2 m^3}{\Lambda_1^3}, \omega_2 = \frac{f_0^2 m_r^3}{\Lambda_2^3} \quad (19)$$

$$\omega_3 = \frac{I_1}{A_1}, \omega_4 = \frac{I_2}{A_2}.$$

In this case, the center of mass of the two bodies C moves along a circular orbit about the fixed center O with a constant angular velocity  $\omega_1$ , the bodies  $O_i$  move along circular orbits about their center of mass C with a constant angular velocity  $\omega_2$ , and the bodies  $O_1$  and  $O_2$  rotate on their axes with constant angular velocities  $\omega_3$  and  $\omega_4$ , normal to their orbital plane passing through their centers of mass.

It can be shown that, after the intrinsic vibrations of the viscoelastic spheres have died away, including only the first term  $\varepsilon_i u_{i1}$  in the expansion of  $u_i(r_i, t)$  in powers of  $\varepsilon_i$  in (16), the last equations in (12) reduce to the two relations

$$\nabla_{u_i} W_i[\varepsilon_i u_{i1} + \chi_i \varepsilon_i u_{i1}] = \rho_i \{ \omega_{2+i}^2 [r_i - (e_3, r_i) e_3]$$

$$+ \sum_{k=1}^2 (f_{ki} / R_k^3) [3(\xi_{ki}, r_i) \xi_{ki} - r_i] \} \quad (i = 1, 2),$$

where

$$\nabla_{u_i} W_i[\varepsilon_i u] = \frac{\rho_i r_{i0}^2 \varphi_i^2(0)}{2(1+v_i)} \quad (21)$$

$$\times \left( \frac{1}{1-2v_i} \nabla \text{div} u + \Delta u \right),$$

and  $v_i$  is the Poisson coefficient for the matter in the sphere  $O_i$ .

Equation (20) can be written in the form

$$\varepsilon_i \nabla_{u_i} W_i[u_{i1} + \chi_i u_{i1}] \quad (22)$$

$$= \rho_i [\omega_{2+i}^2 (2r_i/3 + B_o r_i) + \sum_{k=1}^2 3(f_{ki}/R_k^3) B_{ki} r_i],$$

where

$$B_0 = \begin{pmatrix} 1/3 & 0 & 0 \\ 0 & 1/3 & 0 \\ 0 & 0 & -2/3 \end{pmatrix},$$

$$B_{ki} = \frac{1}{6} \begin{pmatrix} 3\cos 2\psi_{ki} + 1 & 3\sin 2\psi_{ki} & 0 \\ 3\sin 2\psi_{ki} & -3\cos 2\psi_{ki} + 1 & 0 \\ 0 & 0 & -2 \end{pmatrix}.$$

All the quantities in the right-hand side of (22) are calculated for the unperturbed motion.

Taking into account the fact that the stresses on the surfaces of the deformable bodies  $O_i$  are zero (i.e., the boundary conditions for the functions  $u_{i1}(r_i, t)$  have the form  $\sigma_{in} = 0$ ), with accuracy to within firstorder terms in the small quantity  $\chi_i$ , the solution of (22) has the form

$$\begin{aligned} u_i(r_i, t) &\approx \varepsilon_i u_{i1} = u_{i11} + u_{i12} + u_{i13}, \\ u_{i11} &= \rho_i / E_i \omega_{2+i}^2 [-2/3(d_{i1} r_i^2 + d_{i2} r_{i0}^2) \\ &+ a_{i1}(B_o r_i, r_i) + (a_{i1} r_i^2 + a_{i2} r_{i0}^2) B_0] r_i, \\ u_{i12} &= 3\rho_i / E_i \sum_{k=1}^2 f_{ki} R_k^{-3} [a_{i1}(B_{ki} r_i, r_i) r_i \\ &+ (a_{i2} r_i^2 + a_{i3} r_{i0}^2) B_{ki} r_i], \\ u_{i13} &= -3\rho_i / E_i \sum_{k=1}^2 f_{ki} R_k^{-3} \chi_i (\omega_k - \omega_{2+i}) \\ &\times [a_{i1} (\frac{\partial B_{ki}}{\partial \psi_{ki}} r_i, r_i) r_i + (a_{i2} r_i^2 + a_{i3} r_{i0}^2) \frac{\partial B_{ki}}{\partial \psi_{ki}} r_i], \end{aligned} \quad (23)$$

where

$$\begin{aligned} d_{i1} &= \frac{(1+v_i)(1-2v_i)}{10(1-v_i)}, \\ d_{i2} &= -\frac{(3-v_i)(1-2v_i)}{10(1-v_i)}, \\ a_{i1} &= -\frac{1+v_i}{5v_i+7}, \quad a_{i2} = -\frac{(1+v_i)(2+v_i)}{5v_i+7}. \end{aligned}$$

The structure of  $u_i(r_i, t)$  is such that the first term  $u_{i11}$  describes axially symmetrical, elastic deformation of the sphere  $O_i$ , which is compressed by the action of the centrifugal force associated with the rotation about the  $O_i Z_{ii}$  axis passing through the center of mass. The second term  $u_{i12}$  characterizes the deformation of the body  $O_i$  due to the external gravitational fields of the two other bodies. These fields also give rise to gravitational tides, given by the third term  $u_{i13}$ , which contains the viscosity coefficient  $\chi_i$  and influences the evolution of the motion.

### 2.3. Simplification and Averaging of the Equations of Motion

Let us write the canonical equations (12) in more detail, taking into account the Routh function and the displacements  $u_i(r_i, t) \approx \varepsilon_i u_{i1}$ :

$$\begin{aligned} \dot{\Lambda}_k &= 3 \sum_{i=1}^2 \rho_i f_{ki} R_k^{-3} \\ &\times \int_{V_i} [(\frac{\partial \xi_{ki}}{\partial \lambda_k}, r_i) (\xi_{ki} \varepsilon_i u_{i1}) \\ &+ (\xi_{ki}, r_i) (\frac{\partial \xi_{ki}}{\partial \lambda_k}, \varepsilon_i u_{i1})] dv_i, \end{aligned} \quad (24)$$

$$\begin{aligned} \dot{\lambda}_k &= \omega_k - 3 \sum_{i=1}^2 \rho_i f_{ki} R_k^{-4} \partial R_k / \partial \Lambda_k \\ &\times \int_{V_i} [r_i \varepsilon_i u_{i1}] - 3(\xi_{ki}, r_i) (\xi_{ki}, \varepsilon_i u_{i1})] dv_i, \end{aligned} \quad (25)$$

$$\begin{aligned} \dot{I}_i &= 3\rho_i \sum_{k=1}^2 f_{ki} R_k^{-3} \\ &\times \int_{V_i} [(\frac{\partial \xi_{ki}}{\partial \varphi_i}, r_i) (\xi_{ki}, \varepsilon_i u_{i1}) \\ &+ (\xi_{ki}, r_i) (\frac{\partial \xi_{ki}}{\partial \varphi_i}, \varepsilon_i u_{i1})] dv_i, \end{aligned} \quad (26)$$

$$\begin{aligned} \dot{\varphi}_i &= \omega_{2+i} - 2\rho_i \frac{\omega_{2+i}}{A_i} \\ &\times \int_{V_i} [(r_i, \varepsilon_i u_{i1}) - (e_3, r_i) (e_3, \varepsilon_i u_{i1})] dv_i \\ &- \frac{\rho_i}{A_i} [e_3, \int_{V_i} (r \times \varepsilon_i \dot{u}_{i1}) dv_i]. \end{aligned} \quad (27)$$

Substituting the resulting variables (23) into the equations of motion (24)–(27) and calculating the necessary cumbersome integrals yields the system of equations of motion

$$\begin{aligned} \dot{\Lambda}_k &= -18 \sum_{i=1}^2 \rho_i^2 / E_i D_{i2} m_{3-i} \omega_k^2 / m \\ &\times \{ \chi_i (m_{3-i} / m)^{2k-3} \omega_k^2 (\omega_k - \omega_{2+i}) \\ &+ \omega_{3-k}^2 [(-1)^{2-k} 0.5 \sin \tau + \chi_i (\omega_{3-k} - \omega_{2+i}) \cos \tau] \}, \\ \dot{I}_i &= 18 \rho_i^2 / E_i D_{i2} \chi_i [\omega_1^4 (\omega_1 - \omega_{2+i}) \\ &+ (m_{3-i} / m)^2 \omega_2^4 (\omega_2 - \omega_{2+i}) \\ &+ m_{3-i} / m \omega_1^2 \omega_2^2 (\omega_1 + \omega_2 - 2\omega_{2+i} \cos \tau)], \end{aligned} \quad (28)$$

$$\begin{aligned} \dot{\lambda}_k &= \omega_k + 6 \sum_{i=1}^2 \rho_i^2 / E_i D_{i2} (m_{3-i} / m)^{k-1} \\ &\times \Lambda_k^{-1} \omega_k^2 \{ \omega_{2+i}^2 + 6(m_{3-i} / m)^{k-1} \omega_k^2 \\ &+ 3(m_{3-i} / m)^{2-k} \omega_{3-k}^2 [0.5 + 1.5 \cos \tau \\ &+ 3(-1)^{k-1} \chi_i (\omega_{3-k} - \omega_{2+i}) \sin \tau] \}, \end{aligned} \quad (29)$$

$$\begin{aligned} \dot{\varphi}_i &= \omega_{2+i} - 2\rho_i^2 / E_i A_i^{-1} \omega_{2+i} [D_{i2} (\omega_1^2 \\ &+ m_{3-i} / m \omega_2^2) + 2/3 (2D_{i1} + D_{i2}) \omega_{2+i}^2], \end{aligned} \quad (30)$$

where

$$\begin{aligned} D_{i1} &= \frac{8\pi r_{i0}^7 (1-2v_i)(4-3v_i)}{525(1-v_i)}, \\ D_{i2} &= \frac{4\pi r_{i0}^7 (1+v_i)(9v_i+13)}{105(5v_i+7)}, \end{aligned}$$

$\tau = 2(\lambda_2 - \lambda_1)$ . The right-hand sides of these equations do not depend on  $\varphi_i$ . Since the equations (31) depend only on  $\lambda_k$  and  $I_i$ , they can be separated from the remaining equations and integrated after the integration of

(28)–(30). If  $\omega_1 = \omega_2$ , the angular variable  $\tau$  is a rapid variable, and averaging Eqs. (28)–(30) over  $\tau$  yields the evolutionary system of equations

$$\dot{\Lambda}_k = -18 \omega_k^4 \sum_{i=1}^2 \chi_i \rho_i^2 / E_i \quad (32)$$

$$\begin{aligned} &\times D_{i2} (m_{3-i} / m)^{2k-2} (\omega_k - \omega_{2+i}), \\ \dot{I}_i &= 18 \chi_i \rho_i^2 / E_i D_{i2} [\omega_1^4 (\omega_1 - \omega_{2+i}) \\ &+ (m_{3-i} / m)^2 \omega_2^4 (\omega_2 - \omega_{2+i})], \end{aligned} \quad (33)$$

$$\dot{\lambda}_k = \omega_k + 6 \Lambda_k^{-1} \omega_k^2 \quad (34)$$

$$\times \sum_{i=1}^2 \rho_i^2 / E_i D_{i2} (\frac{m_{3-i}}{m})^{k-1}$$

$$\times [\omega_{2+i}^2 + (m_{3-i} / m)^{k-1} \omega_k^2$$

$$+ 1.5(m_{3-i} / m)^{2-k} \omega_{3-k}^2].$$

The right-hand sides of the averaged equations do not depend on  $\lambda_k$ , and Eq. (34) can be separated from (32)–(33) and integrated after the solution of the independent system (32)–(33).

Since

$$\Lambda_1 = (f^2 m^3 \omega_1^{-1})^{1/3},$$

$$\Lambda_2 = (G^2 m_1^3 m_2^3 m^{-1} \omega_2^{-1})^{1/3},$$

$$\omega_3 = I_1 / A_1, \omega_4 = I_2 / A_2,$$

the system (32)–(33) can be written in the more “intuitive” variables  $\omega_j$  ( $j = 1-4$ ):

$$\dot{\omega}_1 = c_1 \omega_1^{16/3} [k_1 (\omega_1 - \omega_3) + k_2 (\omega_1 - \omega_4)], \quad (35)$$

$$\dot{\omega}_2 = c_2 \omega_2^{16/3} [k_1 (m_2 / m)^2 (\omega_2 - \omega_3)$$

$$+ k_2 (m_1 / m)^2 (\omega_2 - \omega_4)],$$

$$\dot{\omega}_3 = c_3 k_1 [\omega_1^4 (\omega_1 - \omega_3) + (m_2 / m)^2 (\omega_2^4) (\omega_2 - \omega_3)],$$

$$\dot{\omega}_4 = c_4 k_2 [\omega_1^4 (\omega_1 - \omega_4) + (m_1/m)^2 (\omega_2^4 (\omega_2 - \omega_4))],$$

where  $k_i = \chi_i \rho_i^2 / E_i D_{i2}$ ,  $c_1 = 54 f^{-2/3} m^{-1}$ ,  
 $c_2 = 54 G^{-2/3} m^{1/3} m_1^{-1} m_2^{-1}$ ,  
 $c_3 = 18 A_1^{-1}$ ,  $c_4 = 18 A_2^{-1}$ ,  
The system (35) has the first integral

$$3c_1^{-1} \omega_1^{-1/3} + 3c_2^{-1} \omega_2^{-1/3} + c_3^{-1} \omega_3 + c_4^{-1} \omega_4 = K_0. \quad (37)$$

Thus, we obtained the independent system of firstorder ordinary differential equations (35) to investigate the evolution of the slow variables: the angular velocity of the orbital motion of the center of mass of the binary planet (two viscoelastic bodies) about the gravitating center  $\omega_1$ , the angular velocity of the planets about their common center of mass  $\omega_2$ , and the angular rotational velocities of the two bodies  $\omega_3$  and  $\omega_4$ . The right-hand sides of these equations all contain the viscosity coefficient  $\chi_i$  through the coefficients  $k_i$ . If  $\chi_i$  is zero, there is no tidal evolution of the system.

### 3. NUMERICAL INTEGRATION OF THE EQUATIONS OF MOTION

#### 3.1. Input Data

The system (35) was numerically integrated using the MATLAB R2013a programme package. We took the Sun to be the fixed gravitating center, the first body to be the Earth, and the second body to be the Moon. We took the following data from [26] for the current epoch:

- $G = 6.67428 \times 10^{11} m^3 kg^{-1} s^{-2}$  is the gravitational constant,

- $f = GM_S = 1.32712442099 \times 10^{20} m^3/s^2$  is the heliocentric gravitational constant,

- $GM_E = 3.986004418 \times 10^{14} m^3/s^2$  is the geocentric gravitational constant,

- $M_M/M_E = 1.23000371 \times 10^{-2}$  is the ratio of the masses of the Moon  $M_M$  and the Earth  $M_E$ .

We also took the following quantities at the current epoch from [27]:

-the duration of a single orbit of the (38)

Earth-Moon system around the Sun, equal to a sidereal year,  $T_1 = 365.25636296$  d,

-the duration  $T_2$  of a single orbit of the Moon about the Earth, equal to a sidereal month and also equal to the rotational period of the Moon  $T_4$ ; i.e.,  $T_2 = T_4 = 27.3216616$  d,

-the duration of an Earth day  $T_3 = 86\,400$  s,

-the radius of the Moon  $r_{20} = 1737.4$  km

-the radius of the Earth  $r_{10} = 6371.032$  km. We used the units of measurement -for mass the solar mass  $M = M_S = 1$ , (39)

-for length  $L = 10^8$  m,

-for time  $T = 31\,558\,149.759744$  s, equal to one year.

In these new units

$$G = f = 1.321705528131173 \times 10^{11}, \quad (40)$$

$$m_1 = M_E = 3.003489616313853 \times 10^{-6},$$

$$m_2 = M_M = 3.694303371012516 \times 10^{-8},$$

$$m = 3.040432650023978 \times 10^6,$$

$$A_1 = 4.876471597254109 \times 10^9,$$

$$A_2 = 4.460588721066945 \times 10^{12},$$

where  $A_i$  are the moments of inertia (13).

We find from (36) the values

$$c_1 = 6.844923013835195 \times 10^{-1}, \quad (41)$$

$$c_2 = 2.717213270640726 \times 10^5,$$

$$c_3 = 3.691193446125189 \times 10^9,$$

$$c_4 = 4.035341773382441 \times 10^{12}.$$

The initial values for  $\omega_i$  were taken to be  $\omega_i(0) = 2\pi/T_i$  ( $i = 1-4$ ), where we took  $T_i$  from (38). It follows [in the new units (39)]:

$$\omega_1(0) = \omega_{10} = 6.283185307179586, \quad (42)$$

$$\omega_2(0) = \omega_{20} = \omega_4(0) = \omega_{40} = 83.99831045063988,$$

$$\omega_3(0) = \omega_{30} = 2.294973413104126 \times 10^3.$$

We found  $k_1$  from the second equation of (35) under the condition that the radius of the lunar orbit  $R_2$  is increasing at the rate  $\Delta R_{20} = 0.038$  m/yr at the current epoch [28]:

$$d\omega_{20} = c_2 k_1 \omega_{20}^{16/3} (m_2/m)^2 (\omega_{20} - \omega_{30}) dt.$$

Assuming a time interval  $dt = 1$  (in years) yields:

$$k_1 = d\omega_{20} / [c_2 \omega_{20}^{16/3} (m_2/m)^2 (\omega_{20} - \omega_{30})]. \quad (43)$$

Since  $R_2 = \sqrt[3]{Gm\omega_2^{-2}}$ , it follows that  $dR_2 =$

$-\frac{2}{3} \sqrt[3]{Gm\omega_2^{-5}} d\omega_2$ . Since the time interval  $dt = 1$  is very small on evolutionary scales, we have to a high degree of accuracy  $dR_2 = \Delta R_2$ . Therefore,

$$d\omega_{20} = -1.5 \sqrt[3]{G^{-1}m^{-1}\omega_{20}^5} \Delta R_{20}. \quad (44)$$

Substituting (44) into (43) yields

$$k_1 = -\frac{1}{36} \sqrt[3]{Gm^4\omega_{20}^{-11}m_1m_2^{-1}} \times \Delta R_{20} / (\omega_{20} - \omega_{30}) \quad (45)$$

Calculations using (45) together with the data from (40) and (42) give

$$k_1 = 7.661095722418093 \times 10^{-24}. \quad (46)$$

We assumed in these calculations  $\Delta R_{20} = 0.038 L^{-1}$ , where  $L$  is defined in (39).

We found  $k_2$  from the condition that  $d\omega_2/dt = d\omega_4/dt$  at the current epoch, since the orbital angular velocity of the Moon around the Earth is currently equal to the angular velocity of the Moon about its center of mass, and their two rates of variation are so close that we can take them to be equal. We find from the second and fourth equations of (35)

$$c_2 k_1 \omega_{20}^{16/3} (m_2/m)^2 (\omega_{20} - \omega_{30}) \quad (47)$$

$$= c_4 k_2 \omega_{10}^4 (\omega_{10} - \omega_{40}).$$

Equation (47) together with the expressions (36) for  $c_2$  and  $c_4$  yield

$$k_2 = 3k_1 A_2 (m_2/m_1) \quad (48)$$

$$\times \sqrt[3]{G^{-2}m^{-5}\omega_{20}^{16}} (\omega_{20} - \omega_{30}) / (\omega_{10}^4 (\omega_{10} - \omega_{40})).$$

Substituting all the required quantities from (40), (42), and (46) into (48) yields

$$k_2 = 2.546031240069387 \times 10^{-26}. \quad (49)$$

The value of the integral of the angular momentum  $K_0$  (37) with the coefficients  $c_i$  (41) and the input data (42) is equal to (in the new units)

$$K_0 = 42.75292278673111. \quad (50)$$

The adequateness of the derived coefficients, in particular  $k_1$ , and of our model at the current epoch can be tested by finding the slowing of the Earth's rotation  $\Delta T_{30}$ . The third equation of (35) yields

$$d\omega_{30} = c_3 k_1 [\omega_{10}^4 (\omega_{10} - \omega_{30}) \quad (51)$$

$$+ (m_2/m)^2 \omega_{20}^4 (\omega_{20} - \omega_{30})] dt.$$

Setting  $dt = 1$  in (51) with the data (40)-(42) and (46) yields

$$d\omega_{30} = -5.604049652131884 \times 10^{-7}. \quad (52)$$

However,  $T_3 = 2\pi/\omega_3$ . Thus

$$dT_{30} = -2\pi d\omega_{30}/\omega_{30}^2 T(s) \quad (53)$$

Substituting the value of  $d\omega_{30}$  from (52) and  $T$  from (39) into (53), we find  $dT_{30} = \Delta T_{30} \approx 0.002 \text{ s}/100 \text{ yr}$ , in good consistency with astronomical observations [29].

### 3.2. Results of Numerical Integration and Analysis

1. Integration into the Past We integrated the system (35) with the input data (42) and the coefficients from (40), (41), (46), and (49) over time into the past from zero to -5 billion years using the ode45 software, designed to solve non-rigid systems of differential

equations, with a relative error  $\text{RelTol} = 10^{-13}$  and an absolute error  $\text{AbsTol} = 10^{-15}$  (these errors did not change in the subsequent computations into the past and future). The following values of the angular variables were obtained for  $t = -5$  billion years:

$$\omega_1 = 6.283149581125962, \quad (54)$$

$$\omega_2 = 3.704491483803059,$$

$$\omega_3 = 2.862809504155766 \times 10^3,$$

$$\omega_4 = -3.740751844225381 \times 10^7,$$

The integral of the angular momentum was equal to  $K_0 = 42.75292278673117$ .

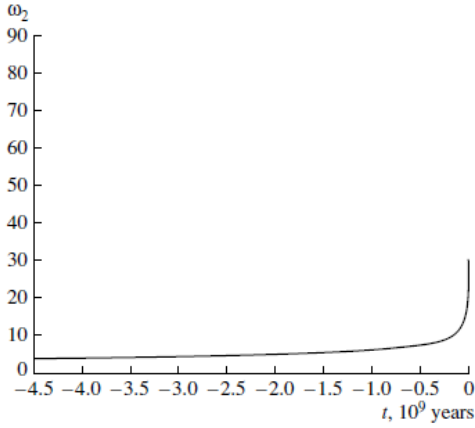


Fig. 1. Plot of the orbital angular velocity of the Moon  $\omega_2$  in the time interval from -4.5 billion years to zero.

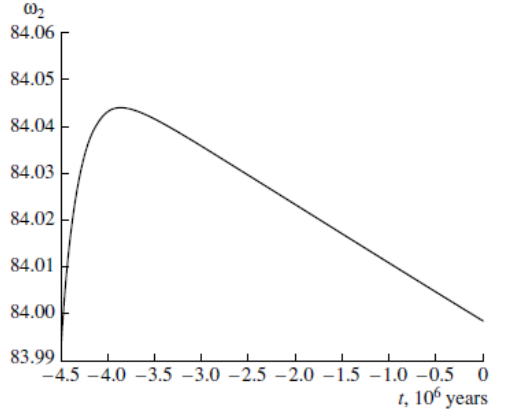


Fig. 3. Same as Fig. 1 for the time interval from -4.5 million years to zero.

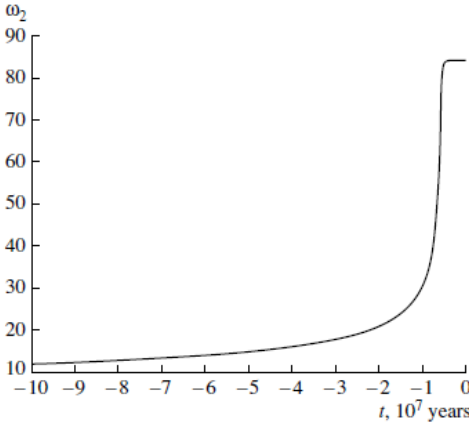


Fig. 2. Same as Fig. 1 for the time interval from -100 million years to zero.

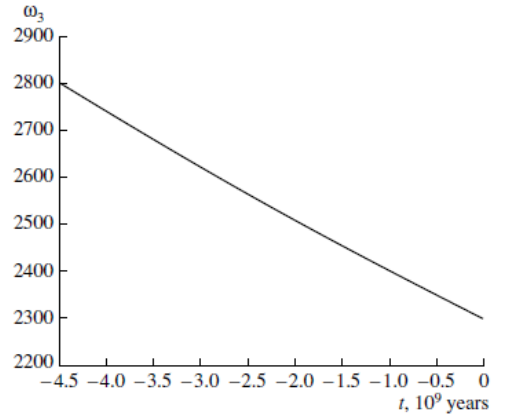


Fig. 4. Plot of the rotational angular velocity of the Earth  $\omega_3$  for the time interval from -4.5 billion years to zero.

We can see that this differs from the value of  $K_0$  at  $t = 0$  given by (50) by  $6 \times 10^{-14}$ .

As a test of the computations, the system (35) with the input data (54) was integrated from -5 billion years to zero. This yielded the following values at  $t = 0$  after this reverse computation:  $\omega_1 = 6.283185307179550$ ,

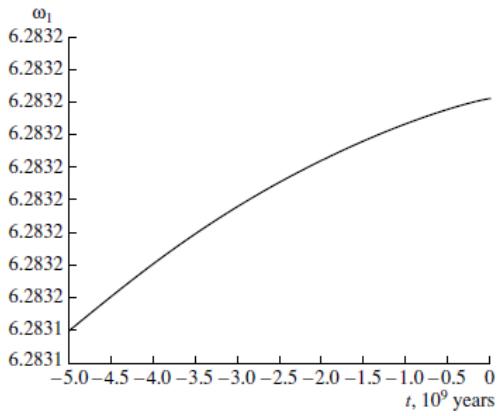
$$\omega_2 = 8.399831045067062 \times 10^1,$$

$$\omega_3 = 2.294973413103788 \times 10^3,$$

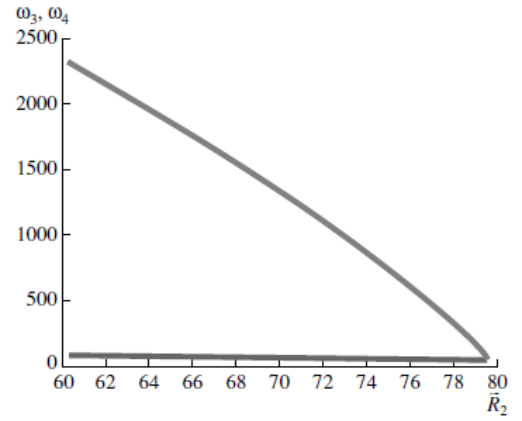
$$\omega_4 = 8.399831045066902 \times 10^1.$$

These differ from the original input data (42) in the

digits indicated in bold. The computation from zero to -5 billion years occupied less than a minute on the Samsung NP510R5E laptop computer used. Plots of the angular velocities for integration into the past in the time interval from 0 to -4.5 billion years are presented in Figs. 1-9. The horizontal axes plot the time and the vertical axes the angular variables (in the new units). The orbital angular velocity of the Moon slowly increases in the time interval from -4.5 billion years to -100 million years (Fig. 1). It begins



**Fig. 9.** Plot of the angular velocity of the Earth–Moon barycenter in its motion about the Sun  $\omega_1$  for the time interval from  $-5$  billion years to zero.



**Fig. 10.** Rotational angular velocities of the Earth  $\omega_3$  (upper) and the Moon  $\omega_4 = \omega_2$  (lower) as the Moon recedes from the Earth, as a function of the distance between the Moon and Earth (in Earth radii).