ISSN 1063-7729, Astronomy Reports, 2015, Vol. 59, No. 1, pp. 72–87. ©Pleiades Publishing, Ltd., 2015.
Original Russian Text ©A.A. Zlenko, 2015, published in Astronomicheskii Zhurnal, 2015, Vol. 92, No. 1, pp. 80–96.

A Celestial-Mechanical Model for the Tidal Evolution of the Earth-Moon System Treated as a Double Planet

A. A. Zlenko*

Moscow Automobile and Roadway State Technical University, Moscow, Russia
Received May 6, 2014; in final form, May 21, 2014

Abstract—A celestial-mechanical model for the motion of two viscoelastic spheres in the gravitational field of a massive point is considered, treating them as a double planet. The spheres move along quasicircular orbits in a single plane, with their rotational axes perpendicular to this plane. The deformation of the spheres is described using the classical theory of small deformations. A Kelvin–Voigt model is adopted for the viscous forces. A system of evoutionary equations is obtained and applied to analyze the joint translational–rotational tidal evolution of the Earth and Moon in the gravitational field of the Sun. This system has been numerically integrated several billion years into the past and into the future. The results are compared with the predictions of other theories, paleontological data, and astronomical observations.

DOI:10.1134/S1063772915010096

1. INTRODUCTION

The theory of tides originated with work by Newton and Laplace. The main achievements in this area were collected, systematized, and analyzed by Darwin [1], and further developed by MacDonald [2], who studied the evolution of the Earth–Moon system without including the influence of the Sun. Goldreich [3] used the method of MacDonald to investigate the oblateness of the Earth and the influence of solar tides, but neglecting the ellipticity of the lunar orbit, and correctly averaged the equations of motion using three time scales.

The method of MacDonald was then used in various other studies. Beletskii [4] investigated the tidal evolution of the inclinations and rotations of celestial bodies. Webb [5] studied the evolution of the Earth–Moon system based on the ocean tides and compared his results with the model of Goldreich [3]. Krasinsky [6] combined the methods of MacDonald and Goldreich to reconstruct a dynamical history of the Earth–Moon system. Touma andWisdom [7] developed various models for tidal phenomena in detail. It was shown that the evolution of the Earth–Moon system based on the models of Darwin–Mignard and Darwin–Cowley–Goldreich is essentially equivalent to that predicted by the model of Goldreich.

Efroimsky and Lainey [8] considered the effective dissipation function Q, which is proportional to the tidal frequency to the power α. They studied the tidal evolution of the Martian moon Phobos for α = 0.2, 0.3, 0.4. Note that $\alpha = 0$ in the model of MacDonald and α = -1 in the model of Mignard [9, 10]. The main distinguishing property of the approach proposed by Ferraz-Mello et al. [11] is that, in contrast to many studies based on the theory of Darwin, different coefficients are introduced for the harmonics of the tidal wave, instead of one Love number. A critical analysis of the mathematical formulas in the above theories describing the tidal moments, slowing of planetary rotation, and the delay angle, as well as the accuracy and range of applicability of the theories and connections with rheological models, are considered by Efroimsky and Williams [12] and Efroimsky and Makarov [13]. Note that the qualitative conclusions derived for the simpler MacDonald theory essentially remain correct [12].

The subsequent development of tidal theories is concerned with the creation of rheological models. Churkin [14–16] established a generalized theory of the Love number and applied it to the rheological models of Guk, Maxwell, Voigt, and others. His theory for the rotation of the inelastic Earth was applied to a Voigt model for the Earth's interior, and numerical estimates of rheological corrections to the precession, nutation, and axial rotation of the Earth were obtained. Efroimsky [17] introduced complex Love numbers as a function of the tidal frequency to study tides in the case of a rotational–orbital resonance between a planet and one of its satellites. Vil'ke [18] developed a method for separating motions and averaging in systems with an infinite number of degrees of freedom, aimed at studying the motions of deformable bodies using a classical linear

*E-mail: zalaf121@mail.ru

elasticity theory for small deformations and a Kelvin– Voigt model for the viscous forces. This method was used to investigate the evolution of the orbital and rotational motions of a viscoelastic planet in a central Newtonian force field [19, 20]. The model for a celestial body of Markov and Minyaev [21] includes an isotropic, viscoelastic layer and a rigid core. A qualitative analysis of the motion of the moons of Mars is given, and the model parameters were refined based on the observations of the secular acceleration of Phobos. Vil'ke and Shatina [22] studied the tidal evolution of the motion of the Earth–Moon system in the gravitational field of the Sun, treating the Moon as a point mass. Let us now turn to our model describing a double planet [23–25].

2. MATHEMATICAL MODEL FOR THE MOTION OF TWO VISCOELASTIC SPHERES IN THE GRAVITATIONAL FIELD OF A FIXED CENTRAL BODY

2.1. Formulation of the Problem

In the unperturbed motion, the barycenter C of the two uniform rigid spheres O_1 and O_2 with masses m1 and m2 moves in a circular, Keplerian orbit in a fixed plane in the gravitational field of a stationary massive point mass M. The spheres O_1 and O_2, in turn, move in circular Keplerian orbits about the barycenter C in the plane of its motion. The spheres rotate with specified constant angular speeds about axes passing through their centers of mass perpendicular to the plane of their orbital motion. All four motions are independent of each other. This formulation of the problem is possible because we have made the assumptions

$$m2 \ll m1 \ll M; r_{i0} \ll R_2 \ll R_1, \quad (1)$$

where r_{i0} (i = 1, 2) are the radii of the spheres, R_1 is the distance from the gravitating center to the barycenter, and R_2 is the distance between the centers of mass of O_1 and O_2 (we will further identify the names of the spheres with their centers of mass). These assumptions are satisfied, for example, by the Sun–Earth–Moon system.

In the perturbed motion, we treat the spheres as uniform, isotropic, viscoelastic bodies. Perturbations arise due to the deformation of the bodies in response to the centrifugal and gravitational forces. Since we are studying evolutionary motions, we assume that the centers of mass of the spheres move along quasicircular orbits.

To describe the motion, we specify an inertial coordinate frame OXY Z fixed to the gravitating center O, with the spheres moving in the OXY plane. We specify Koenig coordinate systems $O_iX_iY_i$ with the points O_i (i = 1, 2). The position of the barycenter C in the OXY Z system is specified by the vector $R_1 = \vec{OC}(R_1 \cos\lambda_1, R_1 \sin\lambda_1, 0)$, where $|R_1| = R_1$, and λ_1 is the angle between R_1 and the OX axis. The position of O_2 relative to O_1 in the $O_1X_1Y_1Z_1$ frame is specified by the vector $R_2 = O_1O_2(R_2 \cos\lambda_2, R_2 \sin\lambda_2, 0)$, where $|R_2| = R_2$, and λ_2 is the angle between R_2 and the OX_1 axis. The deformed state of the bodies is described by the classical theory of elasticity for small deformations. We adopted a Kelvin–Voigt model for the viscous forces, with the dissipation function $D_i[\dot{u}_i]$ proportional to the elastic-force function $W_i[\dot{u}_i]$, with the coefficient of proportionality X_i (the viscosity coefficient): $D_i[\dot{u}_i] = X_iW_i[\dot{u}_i]$, (2) where $u_i(r_i, t)$ is the shift in the points of the body O_i due to the deformations, $\dot{u}_i = du_i/dt$ (here and below, a dot above a quantity denotes a time derivative), and r_i is the radius vector of the points in a sphere relative to the center O_i in the undeformed state.

The rotating spheres are associated with their own coordinate systems $O_iX_{ii}Y_{ii}Z_{ii}$, where the O_iZ_{ii} axis is perpendicular to the orbital plane (the O_iZ_i and O_iZ_{ii} axes coincide). The positions of the points in the viscoelastic sphere O_i in the OXY Z coordinate system are determined by the vector field $\zeta_i(r_i, t) = \vec{OO}_i + \Gamma_i(\varphi_i)(r_i + u_i(r_i, t))$, where $\Gamma_i(\varphi_i(t)) = \begin{pmatrix} \cos\varphi_i & -\sin\varphi_i & 0 \\ \sin\varphi_i & \cos\varphi_i & 0 \\ 0 & 0 & 1 \end{pmatrix}$.(4) Here,$\Gamma_i$ is the orthogonal operator for the translation from the Koenig coordinates $O_iX_iY_iZ_i$ to the coordinates $O_iX_{ii}Y_{ii}Z_{ii}$ and φ_i is the rotation of the $O_iX_{ii}Y_{ii}Z_{ii}$ system about the O_iZ_{ii} axis (φ_i is the angle between the O_iX_i and O_iX_{ii} axes). In order to uniquely determine the positions of the centers of mass of the spheres O_i in the $O_iX_{ii}Y_{ii}Z_{ii}$ coordinate systems as the spheres move by $\zeta_i(r_i, t)$, we imposed the following conditions (relations) on this motion:

$$\int_{V_i} u_i dv_i = 0, \int_{V_i} curl u_i dv_i = 0$$

$$(dv_i = dx_{ii}dy_{ii}dz_{ii}),$$

where $V_i = \{|r_i| < r_{i0}\}$ is the region occupied by Oi in the undeformed state.

The functional for the kinetic energy of the system has the form $T = \frac{1}{2}m\dot{R}_1^2 + \frac{m1m2}{2m}\dot{R}_2^2 + \frac{1}{2}\sum_{i=1}^{2} J_i[u_i]\dot{\varphi}_i^2 + 2G_i\dot{\varphi}_i + T_{0i})$,
where $J_i[u_i] = \int\limits_{V_i}[e_3 \times (r_i + u_i)]^2\rho_i dv_i$, $G_i = \int\limits_{V_i}[e_3 \times (r_i + u_i), \dot{u_i}]\rho_i dv_i$, $T_{0i} = \int\limits_{V_i}(\dot{u_i}^2)\rho_i dv_i$, e_3 is the unit vector for the O_iZ_{ii} axis perpendicular to the OXY plane, and ρ_i is the density of the sphere O_i.
The potential energy associated with the gravitational interactions is given by
$\Pi = \Pi_1 + \Pi_2 + \Pi_3$,
where $\Pi_1 = -f\int\limits_{V_i}\{(R_1 - m_2/m)R_2$
$+\Gamma_1(r_1 + u_1)]^2\}^{-1/2}\rho_1 dv_1$
is the energy associated with the interaction between the gravitating center and the viscoelatic body O_1,
$\Pi_2 = -f\int\limits_{V_2}\{(R_1 + m_1/m)R_2$
$+\Gamma_2(r_2 + u_2)]^2\}^{-1/2}\rho_2 dv_2$
is the energy associated with the interaction between the gravitating center and the viscoelatic body O_2,
$\Pi_3 = -f\int\limits_{V_i}\int\limits_{V_2}\{(R_2 + \Gamma_2(r_2 + u_2)$
$-\Gamma_1(r_1 + u_1)]^2\}^{1/2}\rho_1\rho_2 dv_1 dv_2$
is the energy associated with the deformable spheres O_1 and O_2, f = GM, G is the gravitational constant, and $m = m_1 + m_2$. Taking into account the condition (1) and neglecting terms of order $(R_2/R_1)^3(m_2/m)^3$ and higher order in smallness in the potential energy, and leaving only terms that are linear u_i, we obtain
$\Pi = fm/R_1 - Gm_1m_2/R_2 + \Pi_\rho$ (8)
$\Pi_\rho = \sum\limits_{k=1}^{2}\sum\limits_{i=1}^{2} f_{ki}/{R_k}^3\int\limits_{V_i}[r_i u_i$ (9)
$-3(\xi_{ki}, r_i)(\xi_{ki}, u_i)]\rho_i dv_i$,
$f_{1i} = f$, $f_{2i} = Gm_{3-i}$,
$\xi_{ki} = \tau_{ki}[\cos(\lambda_k - \phi_i), \sin(\lambda_k - \phi_i), 0]$,
$\tau_{21} = -1$, $\tau_{ki} = 1$ $(k \neq 2, i \neq 1)$.
Following [22], we introduced canonical Poincarė variables λ_k, Λ_k (k = 1, 2 to describe the motion of the barycenter and centers of mass O_i: $\Lambda_1 = m(fR_1)^{1/2}$, $\Lambda_2 = m_r(f_0R_2)^{1/2}$, (10)
where $m_r = m_1m_2/m, f_0 = Gm$.
To describe the rotational motion of the bodies, we used the Andoyer canonical variables φ_i, I_i (i = 1, 2):
$I_i = J_i[u_i]\dot{\varphi}_i + G_i$, (11)
where $J_i[u_i]$ and G_i is defined in (6).
The equation of motion was written in the form of the Routh equations
$\dot{\Lambda_k} = -\frac{\partial\Re}{\partial\lambda_k}$, $\dot{\lambda_k} = \frac{\partial\Re}{\partial\Lambda_k}$, $\dot{I_i} = -\frac{\partial\Re}{\partial\varphi_i}$, (12)

$\dot{\varphi_i} = -\frac{\partial\Re}{\partial I_i}$, $\frac{d}{dt}\nabla_{\dot{u_i}}\Re - \nabla_{u_i}\Re - \nabla_{\dot{u_i}}D_i = 0$. Here, $\Re$ is the Routh function, which has the form

$$\Re = -\frac{f^2m^3}{2{\Lambda_1}^2} - \frac{{f_0}^2{m_r}^3}{2{\Lambda_2}^2} \quad (13)$$
$$+\sum_{i=1}^{2}\{\frac{{I_i}^2}{2A_i} - \frac{{I_i}^2}{2{A_i}^2}J_{i1}[u_i]$$
$$-\frac{I_i}{A_i}(e_3, \int\limits_{V_i}(r \times \dot{u_i})dv) + W_i[u_i]\} + \Pi_\rho$$

where $A_i = 0.4m_ir_{i0}^2$ is the moment of inertia of the undeformed sphere O_i,
$J_{i1}[u_i] = 2\int\limits_{V_i}[(r_i, u_i) - (e_3, r_i)(e_3, u_i)]\rho_i dv_i$,(14)
and an expression for Π_ρ is give by (9).

The equations of motion admit the integral of the angularmomentum. When the Routh function (13) is written out in detail, it can be shown that the angular variables λ_k and φ_i appear in this function in the combination $\psi_{ki} = \lambda_k - \varphi_i$. It follows that

$$\dot{I_i} = -\frac{\partial\Re}{\partial\varphi_i} = -\sum_{k=1}^{2}\frac{\partial\Re}{\partial\psi_{ki}}\cdot\frac{\partial\psi_{ki}}{\partial\varphi_i} = \sum_{k=1}^{2}\frac{\partial\Re}{\partial\psi_{ki}}, \quad (15)$$
$$\dot{\Lambda_k} = -\frac{\partial\Re}{\partial\lambda_k} = -\sum_{i=1}^{2}\frac{\partial\Re}{\partial\psi_{ki}}\cdot\frac{\partial\psi_{ki}}{\partial\Lambda_k} = -\sum_{i=1}^{2}\frac{\partial\Re}{\partial\psi_{ki}},$$
$$\dot{I_1} + \dot{I_2} + \dot{\Lambda_1} + \dot{\Lambda_2}$$
$$= \sum_{i=1}^{2}\sum_{k=1}^{2}\frac{\partial\Re}{\partial\psi_{ki}} - \sum_{k=1}^{2}\sum_{i=1}^{2}\frac{\partial\Re}{\partial\psi_{ki}} = 0,$$
$$I_1 + I_2 + \Lambda_1 + t\Lambda_2 = K_0 = const.$$

2.2 Finding the Displacements of Pointsin the Bodies due to their Deformation

The system (12) cannot be integrated in explicit form, since this is a quite complex system of differential equations. Therefore, we applied the method for separating motions in systems with an infinite number of degrees of freedom [18]. Since we assumed the rigidity of the elastic spheres O_i were high, we introduced the small parameters ε_i, proportional to the ratio of the squares of the angular velocity of rotation of a sphere at the initial time and of the lowest frequency for the intrinsic elastic vibrations of the sphere. The displacements u_i are small, and can be represented as a series in powers of ε_i:
$$u_i(r_i, t) = \varepsilon_i u_{i1}(r_i, t) + \varepsilon_i^2 u_{i2}(r_i, t) + ..., \quad (16)$$
$$\varepsilon_i = \rho_i r_{i0}^2\varphi_i^2(0)/E_i, \quad (17)$$
where E_i is the Young's modulus for the body O_i.
If $\varepsilon_i = 0$, then $u_i(r_i, t) = 0$, and the equations of the unperturbed motion follow from (12):
$\dot{\Lambda_1} = \dot{\Lambda_2} = \dot{I_1} = \dot{I_2} = 0$,
$\dot{\lambda_1} = \omega_1$, $\dot{\lambda_2} = \omega_2$ $\dot{\varphi_1} = \omega_3$ $\dot{\varphi_2} = \omega_4$
where
$$\omega_1 = \frac{f^2m^3}{\Lambda_1^3}, \ \omega_2 = \frac{f_0^2m_r^3}{\Lambda_2^3} \quad (19)$$
$$\omega_3 = \frac{I_1}{A_1}, \ \omega_4 = \frac{I_2}{A_2}.$$
In this case, the center of mass of the two bodies C moves along a circular orbit about the fixed center O with a constant angular velocity ω_1, the bodies O_i move along circular orbits about their center of mass C with a constant angular velocity ω_2, and the bodies O_1 and O_2 rotate on their axes with constant angular velocities ω_3 and ω_4, normal to their orbital plane passing through their centers of mass.

It can be shown that, after the intrinsic vibrations of the viscoelastic spheres have died away, including only the first term $\varepsilon_i u_{i1}$ in the expansion of $u_i(r_i, t)$ in powers of ε_i in (16), the last equations in (12) reduce to the two relations
$$\nabla_{u_i}W_i[\varepsilon_i u_{i1} + \chi_i\varepsilon_i\dot{u_{i1}}] = \rho_i\{\omega_{2+i}^2[r_i - (e_3, r_i)e_3]$$
$$+\sum_{k=1}^{2}(f_{ki}/R_k^3)[3(\xi_{ki}, r_i)\xi_{ki} - r_i]\} \text{ (i = 1, 2)},$$
where
$$\nabla_u W_i[\varepsilon_i u] = \frac{\rho_i r_{i0}^2\dot{\varphi_i^2}(0)}{2(1+v_i)} \quad (21)$$
$$\times(\frac{1}{1-2v_i}\nabla divu + \Delta u),$$
and v_i is the Poisson coefficient for the matter in the sphere O_i.
Equation (20) can be written in the form
$$\varepsilon_i\nabla_{u_i}W_i[u_{i1} + \chi_i\dot{u}_{i1}] \quad (22)$$

$= \rho_i[\omega^2_{2+i}(2r_i/3 + B_o r_i) + \sum_{k=1}^{2} 3(f_{ki}/R^3_k)B_{ki}r_i],$

where

$$B_0 = \begin{pmatrix} 1/3 & 0 & 0 \\ 0 & 1/3 & 0 \\ 0 & 0 & -2/3 \end{pmatrix},$$

$$B_{ki} = \frac{1}{6}\begin{pmatrix} 3cos2\psi_{ki} + 1 & 3sin2\psi_{ki} & 0 \\ 3sin2\psi_{ki} & -3cos2\psi_{ki} + 1 & 0 \\ 0 & 0 & -2 \end{pmatrix}.$$

All the quantities in the right-hand side of (22) are calculated for the unperturbed motion.

Taking into account the fact that the stresses on the surfaces of the deformable bodies O_i are zero (i.e., the boundary conditions for the functions $u_{i1}(r_i,t)$ have the form $\sigma_{in} = 0$), with accuracy to within firstorder terms in the small quantity χ_i, the solution of (22) has the form

$$u_i(r_i,t) \approx \varepsilon_i u_{i1} = u_{i11} + u_{i12} + u_{i13}, \quad (23)$$

$u_{i11} = \rho_i/E_i\omega^2_{2+i}[-2/3(d_{i1}r^2_i + d_{i2}r^2_{i0})$
$+a_{i1}(B_0 r_i, r_i) + (a_{i1}r^2_i + a_{i2}r^2_{i0})B_0]r_i,$

$u_{i12} = 3\rho_i/E_i \sum_{k=1}^{2} f_{ki}R^{-3}_k[a_{i1}(B_{ki}r_i, r_i)r_i$
$+(a_{i2}r^2_i + a_{i3}r^2_{i0})B_{ki}r_i],$

$u_{i13} = -3\rho_i/E_i \sum_{k=1}^{2} f_{ki}R^{-3}_k \chi_i(\omega_k - \omega_{2+i})$
$\times[a_{i1}(\frac{\partial B_{ki}}{\partial\psi_{ki}}r_i, r_i)r_i + (a_{i2}r^2_i + a_{i3}r^2_{i0})\frac{\partial B_{ki}}{\partial\psi_{ki}}r_i],$

where

$d_{i1} = \frac{(1+v_i)(1-2v_i)}{10(1-v_i)},$

$d_{i2} = -\frac{(3-v_i)(1-2v_i)}{10(1-v_i)},$

$a_{i1} = -\frac{1+v_i}{5v_i+7}$, $a_{i2} = -\frac{(1+v_i)(2+v_i)}{5v_i+7}.$

The structure of $u_i(r_i,t)$ is such that the first term u_{i11} describes axially symmetrical, elastic deformation of the sphere O_i, which is compressed by the action of the centrifugal force associated with the rotation about the O_iZ_{ii} axis passing through the center of mass. The second term u_{i12} characterizes the deformation of the body O_i due to the external gravitational fields of the two other bodies. These fields also give rise to gravitational tides, given by the third term u_{i13}, which contains the viscosity coefficient χ_i and influences the evolution of the motion.

2.3. Simplification and Averagingof the Equations of Motion

Let us write the canonical equations (12) in more detail, taking into account the Routh function and the displacements $u_i(r_i,t) \approx \varepsilon u_{i1}$:

$$\dot{\Lambda}_k = 3\sum_{i=1}^{2} \rho_i f_{ki}R^{-3}_k \quad (24)$$
$$\times \int_{V_i}[(\frac{\partial\xi_{ki}}{\partial\lambda_k}, ri)(\xi_{ki}\varepsilon_i u_{i1})$$
$$+(\xi_{ki}, r_i)(\frac{\partial\xi_{ki}}{\partial\lambda_k}, \varepsilon_i u_{i1})]dv_i,$$

$$\dot{\lambda}_k = \omega_k - 3\sum_{i=1}^{2} \rho_i f_{ki}R^{-4}_k \partial R_k/\partial\Lambda_k \quad (25)$$
$$\times \int_{V_i}[r_i\varepsilon_i u_{i1}) - 3(\xi_{ki}, r_i)(\xi_{ki}, \varepsilon_i u_{i1})]dv_i,$$

$$\dot{I}_i = 3\rho_i \sum_{k=1}^{2} f_{ki}R^{-3}_k \quad (26)$$
$$\times \int_{V_i}[(\frac{\partial\xi_{ki}}{\partial\varphi_i}, r_i)(\xi_{ki}, \varepsilon_i u_{i1})$$
$$+(\xi_{ki}, r_i)(\frac{\partial\xi_{ki}}{\partial\varphi_i}, \varepsilon_i u_{i1})]dv_i,$$

$$\dot{\varphi}_i = \omega_{2+i} - 2\rho_i\frac{\omega_{2+i}}{A_i} \quad (27)$$
$$\times \int_{V_i}[(r_i, \varepsilon_i u_{i1}) - (e_3, r_i)(e_3, \varepsilon_i u_{i1})]dv_i$$
$$-\frac{\rho_i}{A_i}[e_3, \int_{V_i}(r \times \varepsilon_i \dot{u}_{i1})dv_i].$$

Subsituting the resulting variables (23) into the equations of motion (24)–(27) and calculating the necessary cumbersome integrals yields the system of equations of motion

$$\dot{\Lambda}_k = -18\sum_{k=1}^{2} \rho^2_i/E_i D_{i2}m_{3-i}\omega^2_k/m \quad (28)$$

*E-mail: zalaf121@mail.ru