Lesson 1 (A Brief Introduction to MATLAB, Matrix Operations and Solving Linear Systems)

Some matrix operations:

The following are some basic MATLAB commands to perform operations on matrices A and B:

| | |
|---|---|
| A(i, j) | the (i; j)-entry of A |
| A(i, :) | the $i^{th}$ row of A |
| A(:, j) | the $j^{th}$ column of A |
| A+B | matrix addition |
| A-B | matrix subtraction |
| t *A | scalar multiplication if t is a real number, i.e. tA |
| A*B | matrix multiplication. |
| A^n | raising a square matrix A to a positive integral power n, i.e. $A^n$ |
| A' | transpose of A, i.e. $A^T$ |
| inv(A) | inverse of an invertible square matrix A, i.e. $A^{-1}$ |
| zeros(n) | the n x n zero matrix, i.e. $\mathbf{0}_{nxn}$ |
| zeros(m, n) | the m x n zero matrix, i.e. $\mathbf{0}_{mxn}$ |
| eye(n) | the n x n identity matrix, i.e. $\mathbf{I}_n$ |
| rref(A | the reduced row echelon form of A |
| det(A) | the determinant of A, i.e. det(A) |

Change Precision:

All numeric computations in MATLAB are performed with about 16 decimal digits of precision. The format of the displayed output can be controlled by the following commands:

| | | |
|---|---|---|
| format short | scaled fixed point format with 5 digits (default setting) | |
| format long | scaled fixed point format with 15 digits | Scaled fixed point format (E.g. 2.00 000 000 000 000) |
| format short | floating point format with 5 digits | Scaled fixed point format (E.g. 1.0000) |
| format long | floating point format with 15 digits | |
| format rat | Approximate fractions | rational format (E.g. 1/10) |
| format | Reset precision to default format | |

**Warning:** MATLAB will approximate decimals with rational numbers when you use **format rat**. Sometimes, this may cause unexpected results. Occasionally, an asterisk * may appear when you expect the quantity to be 0.

Other controls:

| | |
|---|---|
| ans | Display value in current answer |

Questions:

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| Why $C^{-1} = [\ \mathrm{inf}\ ]_{3 \times 4}$ ? | Error message, since C is not invertible. |

Methods of solving systems of linear equations: **Ax=b** where A, b are matrices.

1) Solution for system: **inv(A)*b**

2) **A\b**

3) Most reliable method: **rref ( [ A b ] )** and solve the system for **x**

Lesson 2(The Geometrical Interpretation of Elementary Row Operations and the Computation of Powers of Square Matrices)

We use the MATLAB command ezplot which is used to plot the graph of a line or curve:

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|---|---|
| ezplot( 'E(x, y)' ) | plot points satisfying the <u>equation</u> **E(x, y)** for $-2\pi < x < 2\pi$ |
| ezplot( 'E(x, y)', [xmin, xmax] ) | plot points satisfying the <u>equation</u> **E(x, y)** for xmin < x < xmax |
| ezplot( 'f(x)' ) | plot the <u>function</u> **y=f(x)** for $-2\pi < x < 2\pi$ |
| ezplot( 'f(x)', [xmin, xmax] ) | plot the <u>function</u> **y=f(x)** for xmin < x < xmax |
| ezplot( 'x(t)', 'y(t)' ) | plot points **(x(t), y(t))** for $-2\pi < t < 2\pi$ |
| ezplot( 'x(t)', 'y(t)', [tmin, tmax]) | plot points **(x(t), y(t))** for tmin < t < tmax |

Other controls:

| | |
|---|---|
| hold on | tells MATLAB to keep all the old lines when a new ezplot command is executed |

Questions:

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|---|---|---|
| What elementary row operation has been done?<br><br>Is there any change in the graph of the new system compared with the original? | **Multiply** first **row** by ½ (1/2 $R_1$) | The first equation of the new system **gives us the same line** as that of the first equation of the (*) system.<br><br>The graph of… is the same as … |
| | Interchange rows | Although the **positions** of the 2 eqns are **interchanged**, the **2 lines are still the same** (only rep. different eqns).<br><br>Same graph. |
| | Add a multiple of a row to another. | Though the **new line** is **different** from the original second line of the system (*), it **still intersects** the first line **at the same point.**<br><br>Rmk: Refer to page 3. |

We use the MATLAB command ezsurf which is used to draw the graph of a surface:

ezsurf( 'f(x, y)' )

draw the graph of the function z=f(x, y) for $-2\pi < x < 2\pi$ and $-2\pi < y < 2\pi$

ezsurf( 'f(x, y)', [xmin, xmax], [ymin, ymax] )

draw the graph of the function z=f(x, y) for xmin < x < xmax and ymin < y < ymax

ezsurf( 'x(s, t)', 'y(s, t), 'z(s, t)' )

> draw the surface with points (x(s, t), y(s, t), z(s, t)) for $-2\pi < s < 2\pi$ and $-2\pi < t < 2\pi$

ezsurf( 'x(s, t)', 'y(s, t), 'z(s, t)', [smin, smax], [tmin, tmax] )

> draw the surface with points (x(s, t), y(s, t), z(s, t)) for smin < s < smax and tmin < t < tmax

Other controls:

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|---|---|
| clf | tells MATLAB to clear all the graphical figures in the graphic window |

*** Rewrite the equations in terms of a variable!

E.g) x +2y -z = 5 -> z= -5 +x +2y -> ezsurf( '-5 +x +2*y ' )

Questions:

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|---|---|---|
| What elementary row operation has been done?<br><br>Is there any change in the graph of the new system compared with the original? | **Multiply** first **row** by ½ (1/2 $R_1$) | The first equation of the new system **gives us the same plane** as that of the first equation of the (*) system.<br><br>The graph of... is the same as ... |
| | Interchange rows | Although the **positions** of the 2 **eqns** are **interchanged**, the **2 planes are still the same** (only rep. different eqns).<br><br>Same graph. |
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| | Add a multiple of a row to another. | Though the **new plane** is **different** from the original second plane of the system (*), it **still intersects** the first plane **on the same <u>line</u>** (solutions of the system remains the same).<br><br>Rmk: Refer to page 5. |

For Activity 3, for finding limit of a matrix $C^n$ as n tends to ∞, refer to worksheet.

Reference of eigenvalues and computation of powers of square matrices.

>> Enter matrix C

>>format short

>> [P, D] = eig(C) : to return two matrices P and D where $P^{-1}CP = D$

## Write down: $Cn = PD^nP^{-1}$

## Find $\lim_{n\to\infty} D^n$ (*)

## if $a^2 + b^2 < 1$, then $\lim_{n\to\infty} (a+bi)^n = 0$

## Find $\lim_{n\to\infty} C^n$ = $P(*)P^{-1}$

>> P*diag( [ a b c] )*inv(P)

## You can use the MATLAB command " diag ( [a b c] ) " to enter a 3 x 3 diagonal matrix with diagonal entries a, b, c. Also use " format long " here.

Lesson 3 (Vector Spaces)

Questions:

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| Determine whether a vector **v** is a linear combination of a given set of vectors. | Activity 1 (pg 1 -2) |
| | 1) Input vectors as column matrix forms<br>2) Input A = [u1 u2 u3… ]<br>3) Apply rref ( [A v] )<br>4) System A**x** = **v** consistent? Any zero row? If consistent, linear combination. Else, not. |
| Check whether a given set of vectors is linearly independent. | Activity 2 (pg 2 – 3) |
| | 1) Input vectors as **column** matrices and **z**, a zero column matrix<br>2) Form matrix A = [u1 u2 u3…]<br>3) Apply rref ( [A z] )<br>4) Homogeneous system A**x** = **0** has<br>- non-pivot columns in rref (non-trival soln): linearly dependent<br>- only pivot columns (only trivial soln): linearly independent |
| How to identify redundant vectors in a given set of vectors. | Activity 2 (pg 4 – 5) |
| | 1) Input vectors as column matrices.<br>2) Form matrix A = [ u1 u2 u3…]<br>3) Apply rref ( A )<br>4) Identify non-pivot columns in rre-form<br>5) Redundant vectors: columns in A corresponding to the non-pivot columns identified in the rre-form |
| How to write the redundant vectors as linear combination of the non-redundant vectors. | Activity 2 (pg 4 – 5) |
| | 1) Observe the entries on the non-pivot columns<br>2) Refer to worksheet ** |
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| How to show span(S) ⊆ span(T) given sets of vectors S and T?<br><br>How to show span(S)=span(T)?<br><br>How to show span(S) is not a subset of span(T)? i.e span(S) ⊈ span (T) | Activity 3: Linear spans (pg 5 – 7) |
| | For example,<br>1) Let S = { u1, u2, u3 }<br>and T= { v1, v2, v3, v4 }<br>2) Form matrices A = [ u1 u2 u3 ] for set S<br>and B = [ v1 v2 v3 v4 ] for set T<br>whether vectors are in column form<br><br>To show span(S) ⊆ span(T) :<br>3) Apply rref ( [B A] )<br>4) Shown if 3 **systems** are **consistent**, i.e. system B**x** = $\mathbf{u}_i$ is consistent for all i = 1, 2, 3<br>(Tell from rre-form:<br>no zero row for rref (B)/<br>all non-pivot columns for columns corr. to vectors in set T in the rref ( [B A]/ look at the last few augmented columns in the rref ) |
| How to find a basis for a dimension of a subspace V=span(S) for a given set S of vectors? | Activity 4: Bases and Dimensions (pg 7 – 8) |
| | 1) Input vectors as column matrices.<br>2) Form matrix A = [ u1 u2 u3…]<br>3) Apply rref (A)<br>4) Identify pivot columns in rref.<br>5) Columns in A corr. to pivot columns in rref gives **largest set of linearly independent vectors** in S (that spans V).<br>6) Basis for V = set of these vectors<br>7) Dimension of V, dim (V) = # vectors in basis |
| How to show that a given set T is a basis for a subspace V=span(S) for a given set S? | Activity 4: (pg 8) |
| | 1) Check that T is **linearly independent**<br>2) Show that span (T) = span (S), i.e. **T spans V**<br>3) If both 1) and 2) are true, T is a basis for V. |

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| ** How to express a vector **v** in V=span(S) as a linear combination of a given basis T of V? | Activity 4: (pg 9) |
| | 1) Input vectors (v1 v2 v3…) in basis of T and vector v as column vectors.<br>2) Form matrix B = [ v1 v2 v3… v]<br>3) Apply rref ( [B **v**] )<br>4) Similar to writing redundant vectors as linear combinations of non-redundant vectors in a set, write **v** as a linear combination of vectors (of the basis T)<br>i.e. **v** = a**v1** + b**v2** + c**v3** +…<br>5) Using 4), write down the coordinate vector $[\mathbf{v}]_T$ of **v** wrt. basis T<br>i.e. $[\mathbf{v}]_T = [a\ b\ c\ldots]^T$<br>** in **column** form |
| Is set S a basis for $R^n$? (where n is given) | Activity 5: (pg 10) |
| | 1) Is S linearly independent?<br>2) Is the # of vectors in S,<br>$\|S\| = \dim(R^n)$, where $\dim(R^n) = n$?<br>3) If true for 1) and 2), S a basis for $R^n$. |

Lesson 4 (Row space, Column space, Nullspace, Dot product and Orthogonal sets)

Questions:

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| Find basis for the **row space** of a matrix. | Activity 1: (pg 1) |
| | 1) Input matrix A<br>2) Apply rref (A)<br>3) From rref of A, get basis for row space of A<br>i.e. **rows with leading entries/ pivot points/all rows that are non-zero** are vectors in the basis |
| Find basis for the **column space** of a matrix. | Activity 1: (pg 2) |
| | 1) Input matrix A<br>2) Apply rref (A)<br>3) Identify pivot columns of rref (A)<br>4) Basis for column space of A is the **corresponding columns** in A itself<br>** write vectors in **column** vector **form** |
| Find the rank of a matrix.<br>(Rank of a matrix = dimension of the row space/column space of that matrix) | Activity 1: (pg 2) |
| | 1) Rank (A) = # vectors in basis of row/column space of A<br>OR<br>2) Apply command rank(A) |
| Checking for invertibility of a matrix C. | Activity 1: (pg 3) |
| | True if:<br>1) det (C) $\neq$ 0<br>2) rref (C) is an identity matrix<br>3) C: n x n matrix and Rank (C) = n |
| Find a basis for a linear span where span(S) = V vector space | Activity 2: (pg 3 - 4) |
| | Similar to above:<br>1) Form a matrix D using vectors in S as COLUMNS<br>2) *Observe that V = col (D)<br>3) Basis for V same as finding basis for col (D) |
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| Extend a linearly independent set to a basis for $R^n$. | Activity 2: (pg 4) |
| | *** Refer to worksheet |
| | For example:<br>1) Basis S' has 3 vectors and need to extend it to a basis for $R^5$.<br>Need 2 more vectors.<br>2) Form a matrix E using the vectors in S' as **ROWS.**<br>3) Apply rref (E) and observe which columns do not have leading entries (i.e. non-pivot columns)<br>E.g.<br><br>4) Complete the rref (E) by adding rows to make it into a 5 x 5 matrix such that every column has a leading entry.<br>E.g.<br><br>5) These 2 row (vectors) extend the set S' to a basis for $R^5$. |
| Find basis for the nullspace of a matrix. | Activity 3: (pg 4 - 5)<br>*** Refer to worksheet |
| | 1) Input matrix A<br>2) Use the rref (A) to find the general solution **(by hand)** of the system $A\mathbf{x} = \mathbf{0}$<br>3) Derive a basis for the solution space and hence a basis for the nullspace of A<br>*** WRITE IN COLUMN FORM<br>OR<br>4) Type N = null(A,'r')<br>using command null where columns of N gives a basis for nullspace of A |
| | |

| Pre-multiply A to N. What is the product? Give an explanation to the answer. | **Activity 3: (pg 5)<br>Where A is a matrix and N = null(A,'r') |
|---|---|
| | 1) >>A*N<br>2) The product is a zero matrix<br>3) Reason: (k) columns n1, n2…nk of N are vectors in nullspace<br>so An1=0, An2=0,… Ank=0<br>so matrix multiplication of<br>AN= (An1 An2… Ank)=0 |

Controls:

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|---|---|---|
| Perform <u>dot product</u> on 2 vectors u and v | u · v | 1) Input u and v as ROW matrices.<br>2) >> **u*v'** OR **v*u'**<br>NOT: u*v nor u'*v |
| Find the norm of a vector u | Norm (u) | 1) Input u as row matrix<br>2) >> sqrt(u*u')<br>OR >> norm(u) |

Questions:

| Check that a set of vectors (a, b, c) is orthogonal/orthonormal. | Activity 5: (pg 6 - 7) |
|---|---|
| | 1) Form matrix F using vectors a,b,c, as **ROWS.**<br>2) Pre-multiply F to $F^T$ : $FF^T$<br>>> F*F'<br>3) If the product is a **diagonal** matrix, the set is **orthogonal.**<br>4) ALSO If the product is an **identity** matrix, the set is **orthonormal.**<br>5) OTHERWISE, it is neither. |
| Can you explain why? | For: "If the product is a diagonal matrix, the set is orthogonal."<br><br>For: "If the product is an identity matrix, the set is orthonormal." |
| | |

| Find an orthonormal basis for a subspace V=span(S), given set S = {**w1, w2, w3**…}. | Activity 5: (pg 6 - 7) |
|---|---|
| | 1) Form matrix G using vectors **w1, w2, w3**… as **COLUMNS** (not rows).<br>2) Check that S is a basis for vector space V: apply rref(G)<br>3) Check that S is not orthogonal by multiplying $\mathbf{G^TG}$<br>>> G'*G<br>(order different from $FF^T$ above as vectors are arranged as columns in G)<br>4) Obtain an orthonormal basis for V:<br>>> Q = orth(G)<br>5) **Columns** of Q give an orthonormal basis for V<br>6) Verify that basis obtained is orthonormal: $Q^TQ$<br>>> Q'*Q<br>Where it is if the matrix is an identity matrix. |