Example 4.2: Strong Duality fails!

Background:

1. Strong Duality usually holds for (most) convex problems.
2. Slater's condition says that if there exists an $x \in D$ which is **strictly feasible** ($\exists x \in D,\ f_i(x) < 0,\ \forall i$) for inequality constraints, it is a sufficient condition for strong duality.

Problem:

$$\begin{aligned} \underset{x}{\text{minimize}} \quad & e^{-x} \\ \text{subject to} \quad & \frac{x^2}{y^3} \leq 0; \\ & y > 0. \end{aligned}$$

- Assumption: $\lim_{x\to\infty, y\to\infty} \frac{x^2}{y^3} = 0$ or y goes to ∞ faster than x.

Question:

- Is it a convex problem? It it is, find the optimal value.
- Slater's condition is satisfied or not?
- Derive Lagrange dual problem, and find optimal solution λ^* and the dual optimal value d^*. Can you point out the optimal duality gap?

Hints:

- Using assumption to find the optimal value of dual problem.

Answer 4.2:

1. The objective function e^{-x} is convex, since $(e^{-x})'' = (-e^{-x})' = e^{-x} > 0$, $\forall x \in R$. The feasible set can be reduced to $\{x = 0, y > 0\}$ and also a convex set. The optimal value of primal problem is achieved when $x = 0$, $p^* = e^{-0} = 1$.
2. If (x, y) is feasible, then $x^2 \leq 0$, hence $x = 0$. In other words, the feasible region is $\{(0, y)|y > 0\}$. Slater's condition says there exists a (x, y) strictly feasible point such that $\frac{x^2}{y^3} < 0$, but it is impossible. So Slater's fails in this case.

3. We have Lagrange dual function as

$$L(\lambda, (x, y)) = e^{-x} + \lambda \frac{x^2}{y^3} \tag{41}$$

Then

$$g(\lambda) = inf_{(x,y):y>0} e^{-x} + \lambda \frac{x^2}{y^3} = 0 \tag{42}$$

For all x, $e^{-x} \rightarrow 0$ when $x \rightarrow \infty$ is the min value of first term.

The second term achieves min value zero when $x \rightarrow \infty$ and $y \rightarrow \infty$ because of the assumption $\lim_{x \rightarrow \infty, y \rightarrow \infty} \frac{x^2}{y^3} = 0$

Hence, dual problem becomes

$$\begin{array}{ll} \underset{\lambda}{\text{maximize}} & 0 \\ \text{subject to} & \lambda \geq 0. \end{array}$$

Therefore, $d^* = 0$. Duality gap is $p^* - d^* = 1$.