0.2.3 Lecture 3 - 140319 - Convex Algorithms

Example 3.1: AAKB2012

Paper: *A. Akbari et al., "Energy Efficiency Contours for Broadcast Channels Using Realistic Power Models," IEEE Transactions on Wireless Communications, 2012.*

This paper formulates problems of maximizing energy efficiency (EE_T) in single-cell, single-carrier, single-antenna, multi-user system, where a Base Station broadcasts a signal x to each of the K users in the system.

Problem:

$$EE_T = A \frac{\sum_{k=1}^{K} X_k}{\alpha_0 + \sum_{k=1}^{K} \alpha_k \prod_{j=1}^{k} e^{X_j}} = A \frac{g(\mathbf{X})}{f(\mathbf{X})} \tag{28}$$

We cannot conclude about the convexity of function EE_T because it is just the **ratio** between a concave function $g(\mathbf{X})$ and a convex function $f(\mathbf{X})$. The authors used logarithm to transform into a concave function in order to formulating a original concave maximization problem.

$$ln(EE_T) = ln(A) + ln(g(\mathbf{X})) + h(\mathbf{X}) \tag{29}$$

where $h(\mathbf{X}) = -ln(f(\mathbf{X}))$.

Question:

- Prove $h(\mathbf{X})$ is a concave function, assumed number of users $K = 2$ Hint: The second-order condition will work.

Answer 3.1: A function $h(\mathbf{X})$ is concave iff the Hessian of $h(\mathbf{X})$ is a negative semi-definite matrix, according to second-order condition. This example is interesting because it introduces a concept of composition function in convexity properties (Read 3.2.4. in BOYD)

Background - Composition: Assume we have two functions $h : R^k \to R$ and $g : R^n \to R^k$, we want to find out about convexity of their composition $f = h \circ g : R^n \to R$, defined by: $f(x) = h(g(x))$. We discover the convexity property of function f by assuming h and g are twice differentiable.

The second derivative of the composition function $f = h \circ g$ is given by

$$f''(x) = h''(g(x))g'(x)^2 + h'(g(x))g''(x) \tag{30}$$

Some results in the textbook are

- f is convex if h is convex and nondecreasing, and g is convex,

- f is convex if h is convex and nonincreasing, and g is concave,
- f is concave if h is concave and nondecreasing, and g is concave,
- f is concave if h is concave and nonincreasing, and g is convex,

In this problem, function h is $ln(f)$ and g is $f(X)$. g is convex so that $g''(x) \geq 0$. h is concave so that $h''(g) \leq 0$. And h is nondecreasing so that $h' \geq 0$. This case is different from all above results. Therefore, we need to verify its convexity by another approaches.

Step 1: Calculate Hessian $H(h(\mathbf{X}))$

$$H(h(\mathbf{X})) = \begin{bmatrix} \frac{\partial^2 h(X)}{\partial X_1^2} & \frac{\partial^2 h(X)}{\partial X_1 \partial X_2} \\ \frac{\partial^2 h(X)}{\partial X_2 \partial X_1} & \frac{\partial^2 h(X)}{\partial X_2^2} \end{bmatrix}$$

$$\begin{aligned} \frac{\partial h(X)}{\partial X_1} &= -\frac{\partial f(X)/\partial X_1}{f(\mathbf{X})} \\ \frac{\partial h(X)}{\partial X_2} &= -\frac{\partial f(X)/\partial X_2}{f(\mathbf{X})} \end{aligned} \tag{31}$$

$$\begin{aligned} \frac{\partial^2 h(X)}{\partial X_1^2} &= -\frac{1}{f(X)^2}\{f(X)\frac{\partial^2 f(X)}{\partial X_1^2} - \frac{\partial f(X)}{\partial X_1}\frac{\partial f(X)}{\partial X_1}\} \\ \frac{\partial^2 h(X)}{\partial X_2^2} &= -\frac{1}{f(X)^2}\{f(X)\frac{\partial^2 f(X)}{\partial X_2^2} - \frac{\partial f(X)}{\partial X_2}\frac{\partial f(X)}{\partial X_2}\} \\ \frac{\partial^2 h(X)}{\partial X_2 X_1} = \frac{\partial^2 h(X)}{\partial X_1 X_2} &= -\frac{1}{f(X)^2}\{f(X)\frac{\partial^2 f(X)}{\partial X_1 \partial X_2} - \frac{\partial f(X)}{\partial X_1}\frac{\partial f(X)}{\partial X_2}\} \end{aligned} \tag{32}$$

In order to simplify (32), we will calculate all the derivatives of $f(X) = \alpha_0 + \sum_{k=1}^{K} \alpha_k \prod_{j=1}^{k} e^{X_j}$. With $K = 2$, we obtain $f(X) = \alpha_0 + \alpha_1 e^{X_1} + \alpha_2 e^{X_1+X_2}$.

$$\begin{aligned} \frac{\partial f(X)}{\partial X_1} &= \alpha_1 e^{X_1} + \alpha_2 e^{X_1+X_2} = f(X) - \alpha_0 \\ \frac{\partial f(X)}{\partial X_2} &= \alpha_2 e^{X_1+X_2} = f(X) - \alpha_0 - \alpha_1 e^{X_1} \\ \frac{\partial^2 f(X)}{\partial X_1^2} &= \frac{\partial f(X)}{\partial X_1} \\ \frac{\partial^2 f(X)}{\partial X_2^2} &= \frac{\partial f(X)}{\partial X_2} \\ \frac{\partial^2 f(X)}{\partial X_1 \partial X_2} &= \frac{\partial f(X)}{\partial X_2} \end{aligned} \tag{33}$$

Substitute (32) and (33) into Hessian Matrix and do simplification, we obtain

$$H(h(\mathbf{X})) = -\frac{1}{f(X)^2}\begin{bmatrix} (f(X)-\alpha_0)\alpha_0 & (f(X)-\alpha_0-\alpha_1 e^{X_1})\alpha_0 \\ (f(X)-\alpha_0-\alpha_1 e^{X_1})\alpha_0 & (f(X)-\alpha_0-\alpha_1 e^{X_1})(\alpha_0+\alpha_1 e^{X_1}) \end{bmatrix}$$

Step 2: Pick a random vector $z = (z_1, z_2)$, $z \in R^2$. We must show the condition $zH(h(\mathbf{X}))z^T \leq 0$ holds for all z

$$zH(h(\mathbf{X}))z^T = [z_1, z_2]H[z_1, z_2]^T$$
$$= -f(X)^2(z_1^2\alpha_0(\alpha_1 e^{X_1} + \alpha_2 e^{X_1+X_2}) + 2z_1z_2\alpha_0\alpha_2 e^{X_1+X_2} + z_2^2\alpha_2 e^{X_1+X_2}(\alpha_0+\alpha_1 e^{X_1}))(34)$$
$$= -f(X)^2(z_1^2\alpha_0\alpha_1 e^{X_1} + z_2^2\alpha_2\alpha_1 e^{2X_1+X_2} + \alpha_0\alpha_2 e^{X_1+X_2}(z_1+z_2)^2)$$

By definition, $\alpha_k \geq 0, \forall k \in [1, \dots, K]$ (differences between channel gains of user k^{th} and $(k+1)^{th}$), so (34) ≤ 0 (**Completed proof**)