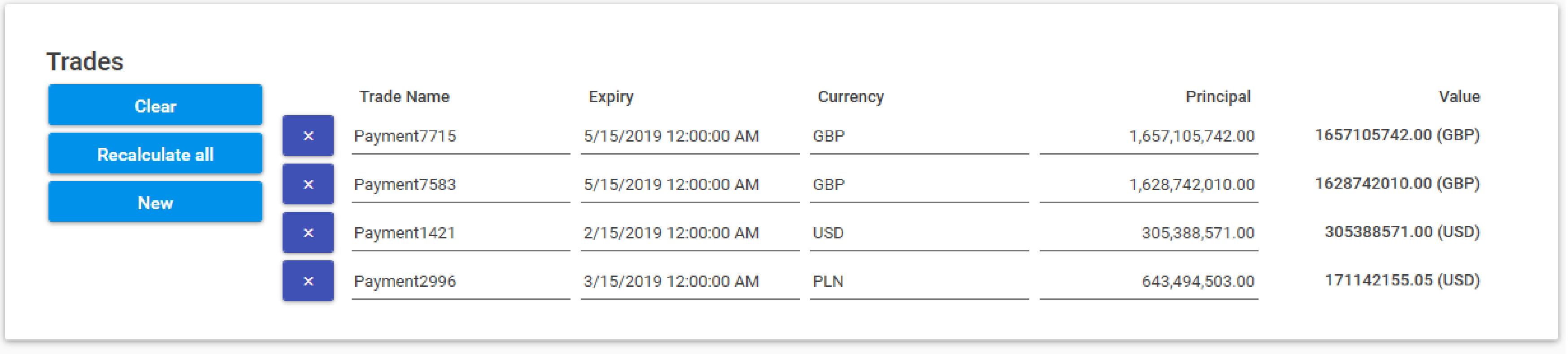
How to Simulate Random Stock Price in Black-Scholes Model

Quantitative Strategies
May 2020

Goal

- To be able to handle more complex trade types than Payments



Example: European Call Option

- Option is a financial instrument which gives a holder (buyer) a right, but not an obligation, to buy stock at predefined level (strike) and at particular moment in time (maturity).

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  – Scenario 1: Stock price turns out to be 120.
  Option holder buys a stock at 100 and immediately sells in on the exchange for 120. She gains
  $120-100=20$.
  – Scenario 2: Stock price ends up at 80.
  Option holder chooses not to exercise her option because it is cheaper to buy on the exchange.
  Her gain is 0.

Different trade data

- We have Expiry and Currency
- We need also Strike, but do not need Principal

Trades

Clear

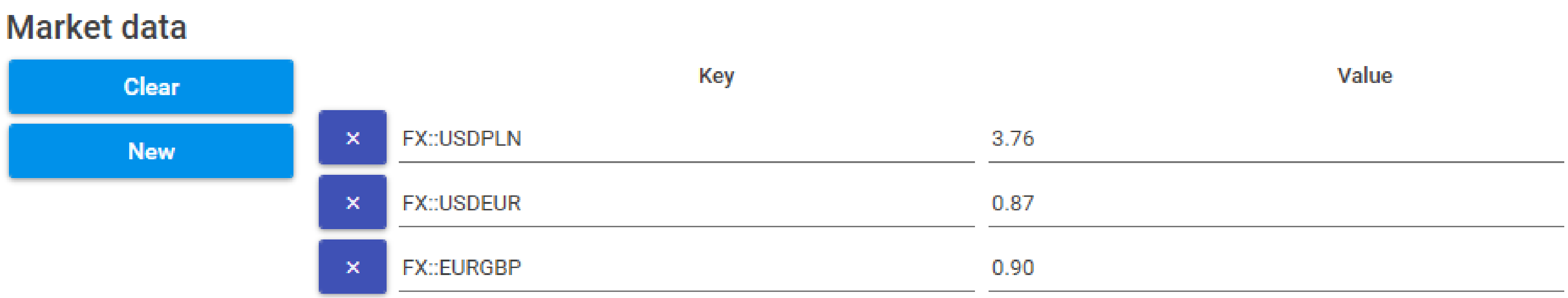
Recalculate all

New

	Trade Name	Expiry	Currency	Principal	Value
×	Payment7715	5/15/2019 12:00:00 AM	GBP	1,657,105,742.00	1657105742.00 (GBP)
×	Payment7583	5/15/2019 12:00:00 AM	GBP	1,628,742,010.00	1628742010.00 (GBP)
×	Payment1421	2/15/2019 12:00:00 AM	USD	305,388,571.00	305388571.00 (USD)
×	Payment2996	3/15/2019 12:00:00 AM	PLN	643,494,503.00	171142155.05 (USD)

Additional market data

- FX Spot is not enough
- For Options we need e.g. price of the underlying stock



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- To understand the behavior of such portfolio we need to understand it for both cash and stock.
- Time value of money is the basic concept in finance. It is the principle that a certain currency amount of money today has a different buying power (value) than the same currency amount of money in the future (e.g. interest rates, inflation).

Time Value of Money (Discount Factor)

We will assume constant continuously compounded interest rates with intensity r.
This means that USD 1 today is worth

$$e^{rT}$$

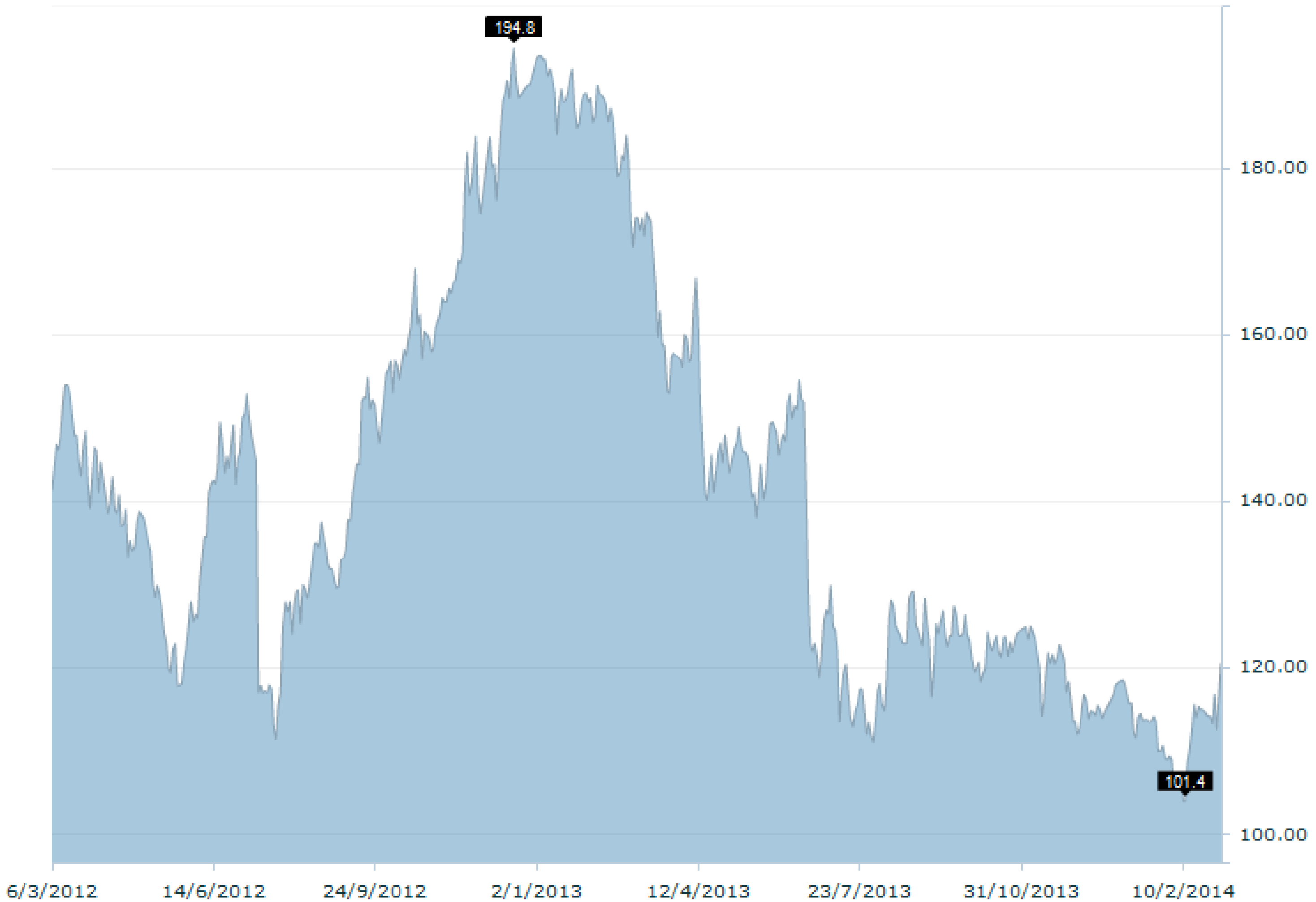
At time T ($T = 1$ denotes one year).
This is equivalent to the statement that USD 1 at time T is worth today:

$$e^{-rT}$$

Stock Price as a Random Variable (1/3)

The behavior of the quoted prices of stock is far from being predictable. In below figures we can see behavior of WIG30 (Warsaw stock index) and KGHM over the period February 2012 to February 2014.

Stock Price as a Random Variable (2/3)



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  of an experiment's outcomes.
- Random variables are often designated by letters and can be classified as discrete, which are variables that have specific values,
  or continuous, which are variables that can have any values within a continuous range.

Binomial Distribution

Definition

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$$f(k; n, p) = P(X = k) = \binom{n}{k} p^k (1-p)^{n-k}$$

For $k = 0, 1, 2, \ldots, n$ where

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For $k = 0,1,2, \ \dots, n$ where

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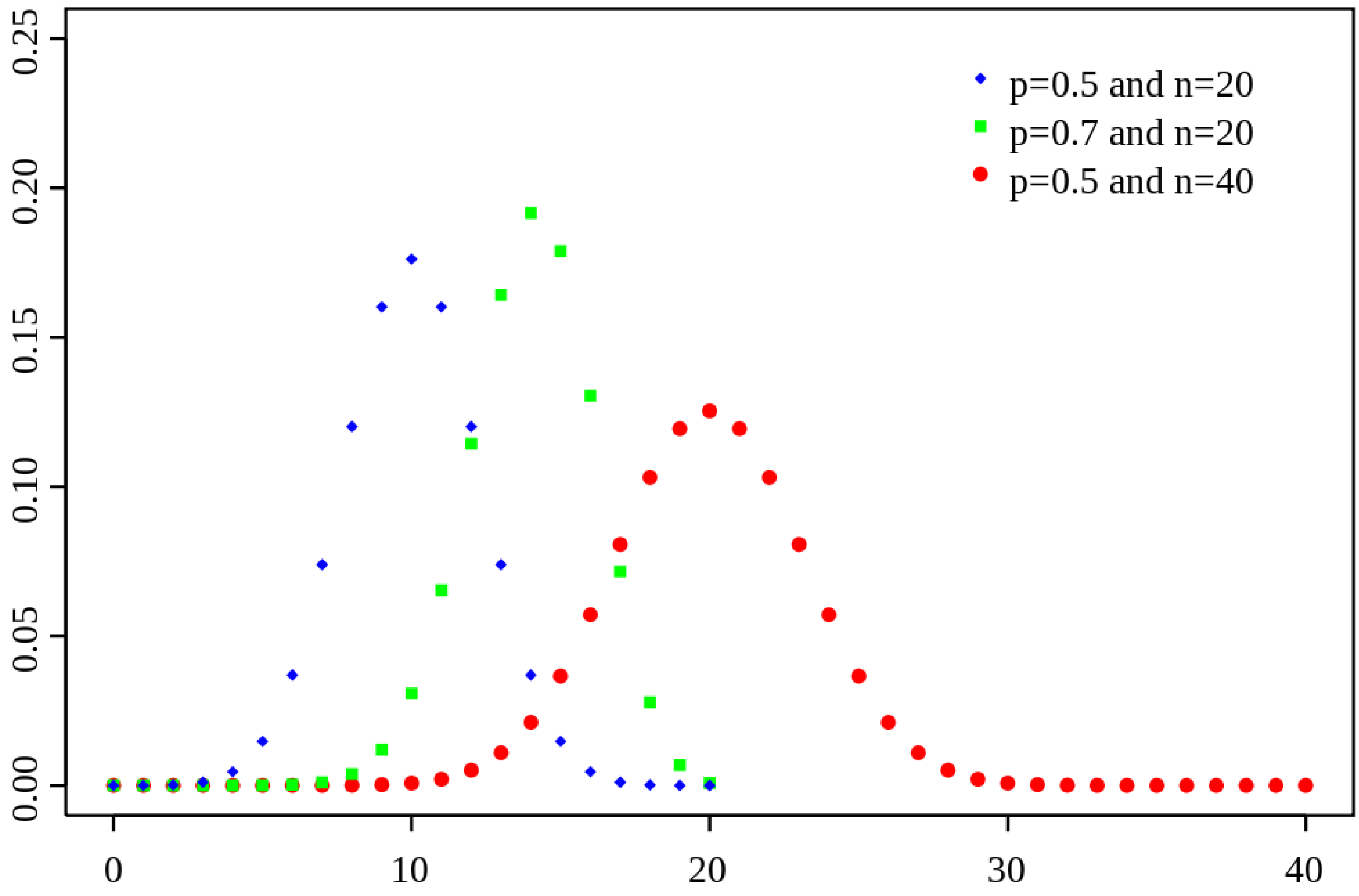
Moreover mean and variance are equal respectively

$$E(X) = np$$
$$Var(X) = E(X-EX)^2 = n(p-p^2)$$

Binomial Distribution

Probability Mass Function



Standard Normal Distribution 1

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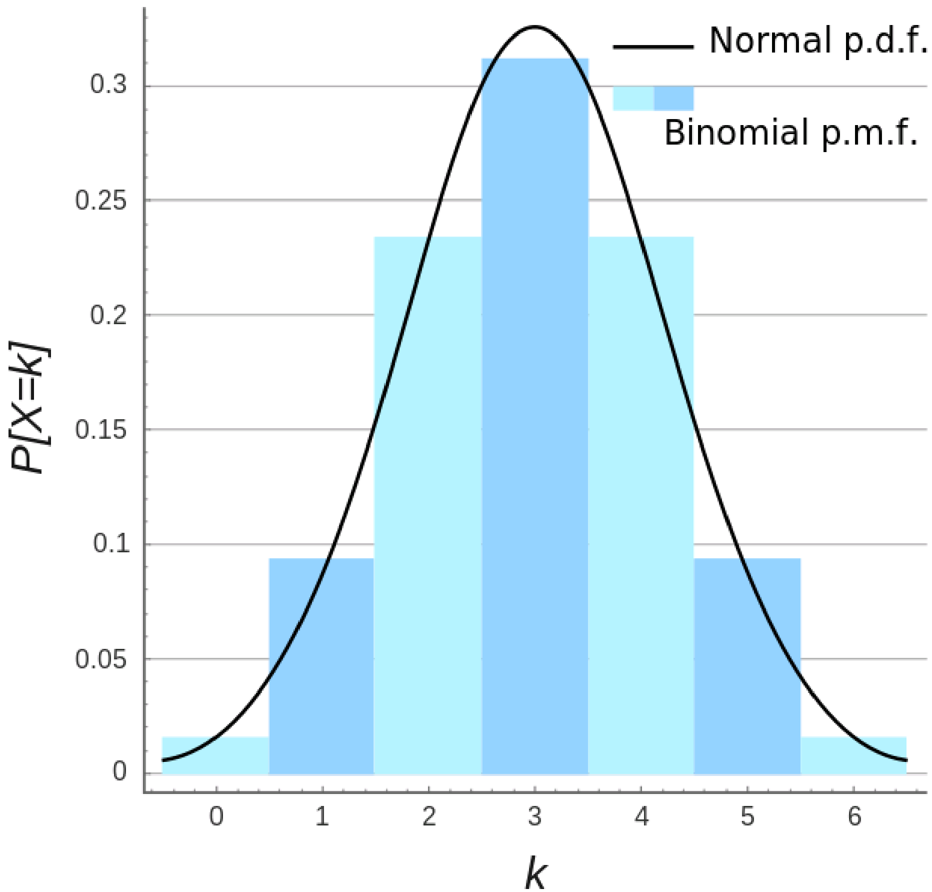
- Normal distributions are extremely important in statistics and are often used in the natural and social sciences for real-valued random variables whose distributions are not known.

Binomial Distribution
Normal Approximation



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- Irrespective of the form of the original distribution: physical quantities that are expected to be the sum of many independent processes often have a distribution very close to the normal.
- Moreover, many results and methods can be derived analytically in explicit form when the relevant variables are normally distributed.

Cumulative Distribution Function

The cumulative distribution function of a real-valued random variable X is the function given by

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In case of continuous distribution we also define probability density function f which is equal to the first derivative of the cumulative distribution function with respect to x

$$F_X(x) = \int_{-\infty}^{x} f(s)ds$$

Standard Normal Distribution 3

The cumulative distribution function of the standard normal distribution, usually denoted with the capital Greek letter Φ, is the integral

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Then

$$\mu + \sigma X \sim N(\mu, \sigma^2)$$

Brownian Motion

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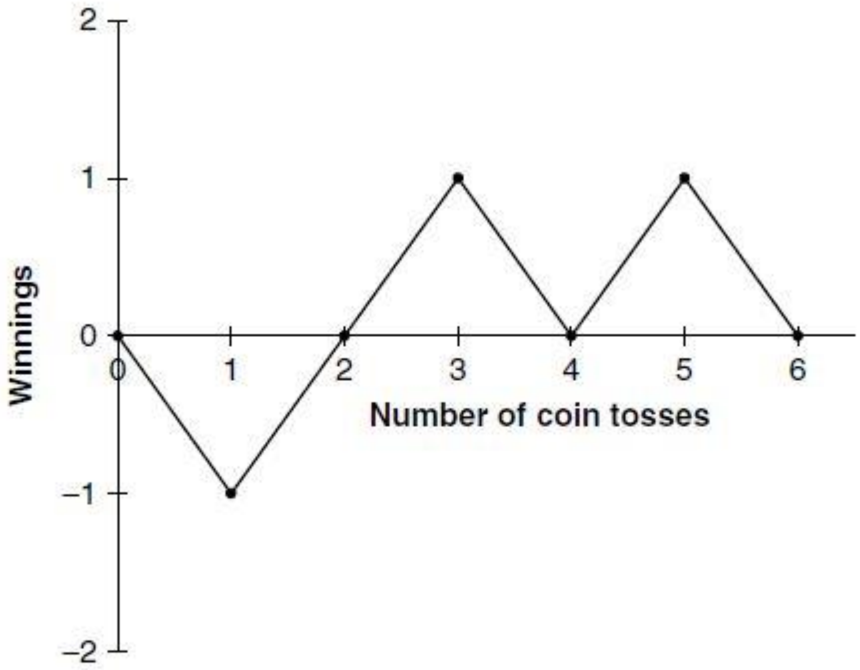
- It occurs frequently in pure and applied mathematics, economics, quantitative finance and physics.

Random Walk (1/3)

Every time you throw a head I give you USD 1, every time you throw a tail you give me USD 1. Below figure shows how much money you have after six tosses. In this experiment the sequence was THHTHT, and we finished even.



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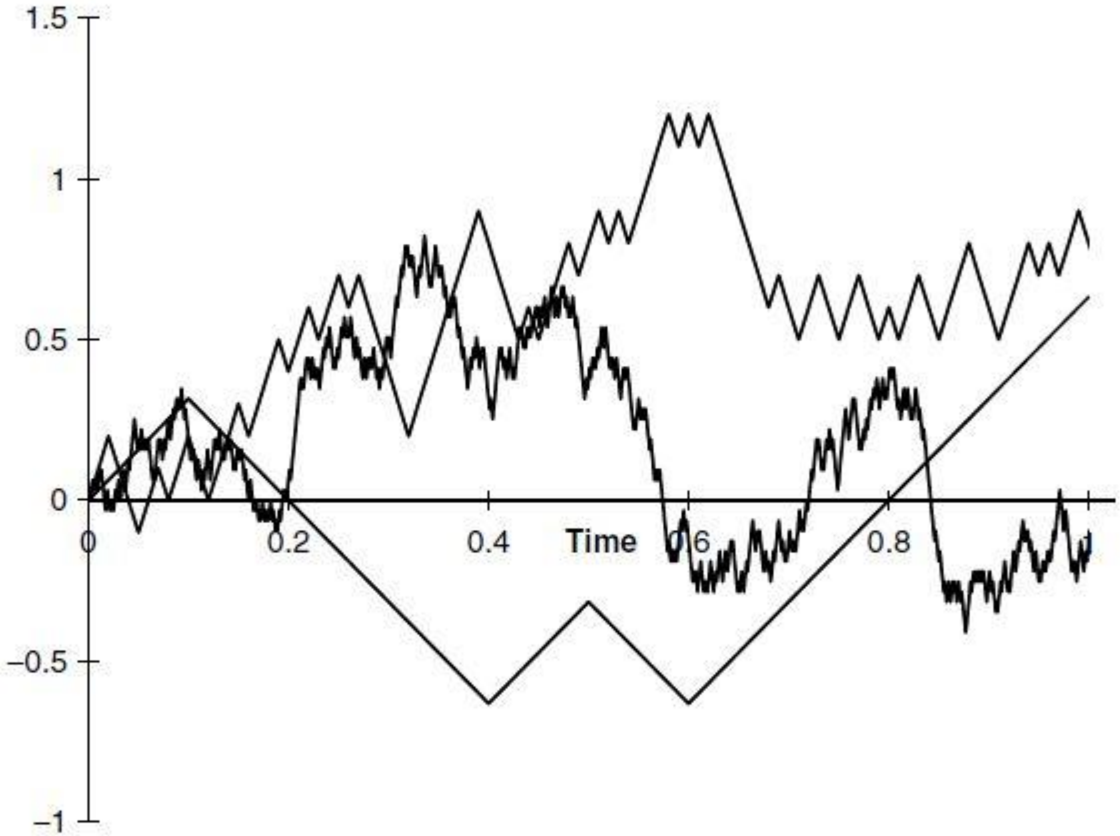

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- Second, the size of the bet will not be USD 1 but $\sqrt{\frac{t}{n}}$ where n is number of steps (tosses). In below figure we show a series of experiments, each lasting for a time 1, with increasing number of tosses per experiment.
- The limiting process for this random walk as the time steps go to zero is called **Brownian motion** or Wiener process.

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- $W_t - W_s \sim N(0, t-s)$ for $0 \leq s \leq t$.

Where $N(\mu, \sigma^2)$ denotes the normal distribution with expected value μ and variance σ^2. The condition that it has independent increments means that if $0 \leq s_1 \leq t_1 \leq s_2 \leq t_2$ then $W_{t_1} - W_{s_1}$ and $W_{t_2} - W_{s_2}$ are independent random variables.

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- W_t is nowhere differentiable but everywhere continuous

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- A GBM process shows the same kind of “roughness” in its paths as we see
  in real stock prices.
- Calculations with GBM processes are relatively easy.

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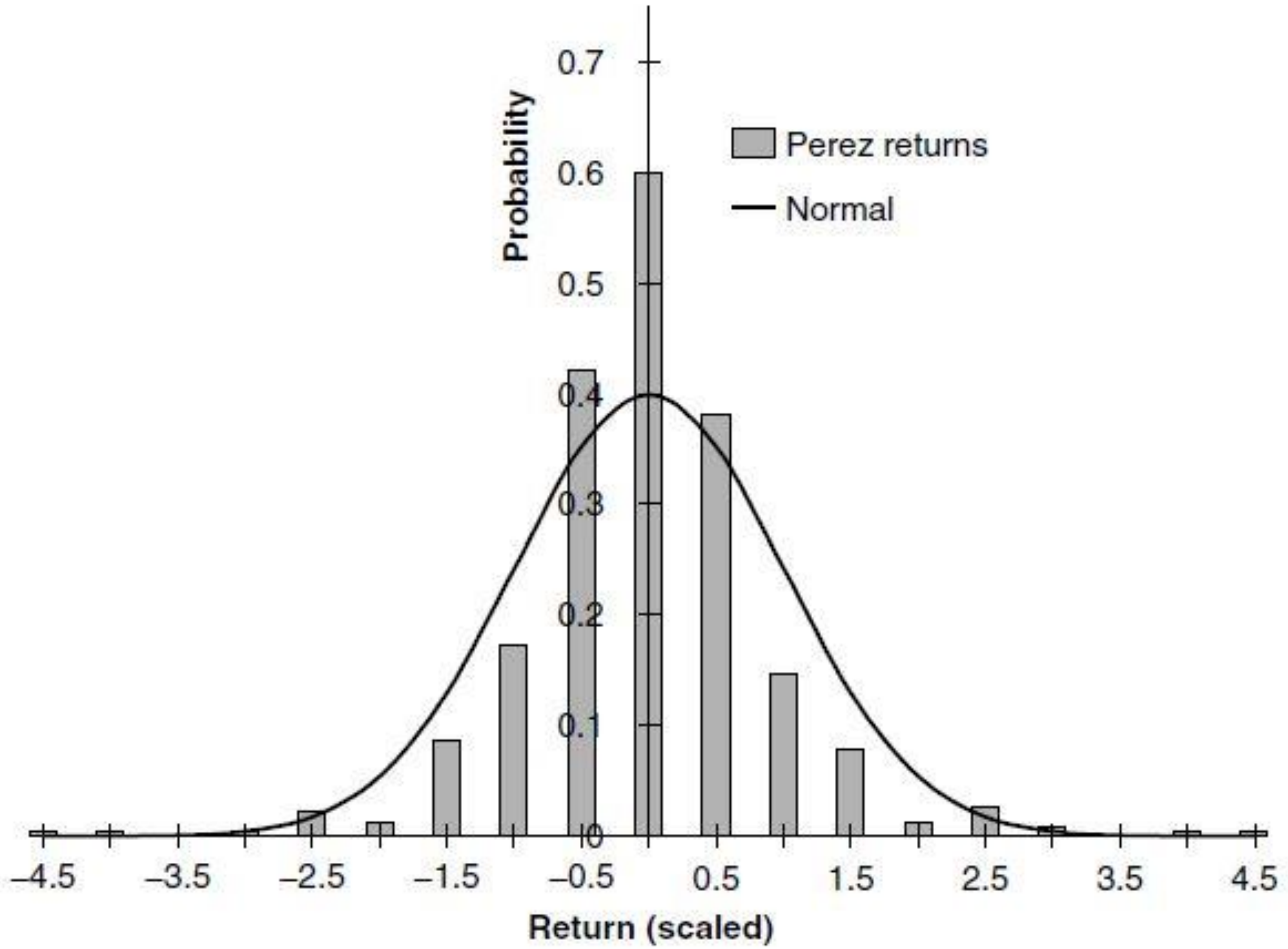
For example if we take a look at logarithm returns for Perez Companc we see that it doesn’t suit normal distribution perfectly:

Geometric Brownian Motion (4/4)



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- Box-Muller transform: draw two independent uniformly distributed random variables U_1 and U_2 from the interval $[0,1]$. Then $\mathrm{N}_1 = \sqrt{-2\ln U_1}\sin(2\pi U_2)$ and $\mathrm{N}_2 = \sqrt{-2\ln U_1}\cos(2\pi U_2)$ are two independent standard normal random variables.

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Historical Volatility (1/2)

We can calculate historical (realized) volatility of the stock looking at the logarithm of the returns

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$$\ln\frac{S_{t+\Delta t}}{S_t} = \left(r-\frac{\sigma^2}{2}\right)\Delta t+\sigma(W_{t+\Delta t}\text{-}W_t)\sim N((r-\frac{\sigma^2}{2})\Delta t, \sigma^2\Delta t)$$

Historical Volatility (2/2)

Hence we can get estimation of the volatility

$$R_i = \ln \frac{S_{(i+1)\frac{t}{n}}}{S_{i\frac{t}{n}}}$$

$$\bar{R} = \frac{1}{n} \sum_{i=1}^{n} R_i$$

$$\hat{\sigma}^2 = \frac{n}{t(n-1)} \sum_{i=1}^{n} (R_i - \bar{R})^2$$