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Local stability analysis of a stochastic evolutionary financial market model with a risk-free asset

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Abstract This paper introduces and analyzes an evolutionary model of a financial market with a risk-free asset. Focus is on the study of local stability of the wealth dynamics through the application of recent results on the linearization and stability of random dynamical systems (Evstigneev et al. Proc Am Math Soc 139:1061–1072, 2011). Conditions are derived for the linearization of the model at an equilibrium state which ensure local convergence of sample paths to this equilibrium. The paper also shows that the concept of local stability is closely related to the notion of evolutionary stability. A locally evolutionarily stable investment strategy in the evolutionary model with a risk-free asset is derived, extending previous research. The method illustrated here is applicable for the analysis of manifold economic and financial dynamic models involving randomness.

Keywords Evolutionary finance · Risk-free asset · Local stability · Linearization · Random dynamical systems

JEL Classification G11 · G12

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1 Introduction

This paper offers new results on the local stability of evolutionary models of financial markets. It applies recent results on random dynamical systems [12] to stock market models featuring a risk-free asset. Stability here is meant as local (almost sure) sample-path wise convergence.[1] We aim to make the presentation accessible to researchers in economics and finance who want to study local stability of equilibria in stochastic dynamic models.

The model extends Evstigneev et al. [8] by allowing investors to trade an asset which is free of price risk, i.e., there is a money-market account with (potentially random) interest rate. Unlike in the original model where all prices are endogenous, the price of the risk-free asset is exogenously fixed. This implies that there is, in general, no simple equation describing the dynamics of aggregate wealth. This dynamics will depend on the total investment in the risk-free asset and, thus, becomes a function of the investment strategies.

Evolutionary finance provides an alternative way of thinking about financial markets. At its core it is a Darwinian view of markets, promoting the concepts of selection and survival over those of utility and consumption, see, e.g., the surveys Evstigneev et al. [10,11]. This class of evolutionary model are also more amenable to numerical analysis and empirical investigation. Several characteristics are common to most of these models which (a) build only on observables; (b) do not assume rational expectations; (c) require short-run market equilibrium only; (d) focus on the dynamics of prices and investors' wealth; (e) study performance (such as average returns) or survival. The analysis of evolutionary finance models is—similarly to agent-based models[2]—mainly concerned with the dynamics of asset prices and of the wealth of investors. Both are realized, observable quantities rather than theoretical constructs such as risk sharing or ex-post judgements on investor's asset allocation decisions. In line with this focus on dynamics, the ultimate success of an investor is measured by the performance such as average returns (in the short- and medium term) or survival (as a long term measure).

Our approach dispenses with the classical assumption on the perfect foresight of the economic agents. It is instructive to recall Laffont [19, p. 85]'s comment "In the Radner equilibrium, price expectations are assumed to be exact for all agents. The agents do not necessarily agree on the probabilities of different states of nature, but they expect the same prices. The reader may be surprised by this assumption of perfect foresight. It should be viewed as a necessary methodological step. We must first understand how the economy performs with incomplete markets in the best case where expectations are correct." Market selection in the Radner setting is studied, e.g., in Blume and Easley [4,5].

In our model, short-run market equilibrium is implemented through a price system that meets all investors' asset-allocation targets. Investors fix percentages which determine the amount of money to be allocated to the purchase of each asset (available budget times the percentage). Prices are then defined through market clearing.[3] The idea of equilibrium processes

[1] More precisely, the existence of a forward-invariant random neighborhood of the equilibrium state such that sample paths with initial values in that neighborhood converge almost surely to the equilibrium state.

[2] Agent-based models that can be studied with the method illustrated in our paper can be found in Chiarella et al. [6,7] and Hommes and Wagener [17] as well as Anufriev and Dindo [2] who use a 'deterministic skeleton' approach.

[3] The approach to equilibrium borrows ideas from the Shapley–Shubik market game [23] who refer to these budget shares as 'fiscal rules.' In our work we refer to these as an investor's strategy or portfolio rule. Expressing investment decisions through percentages is a common approach in asset allocation practice and theory. Many institutional (and also private) investors define their investment choice in this form. Pension funds typically would rethink their asset allocation on a regular basis (e.g. quarterly) and submit percentages to an investment

moving at different speeds, with prices adjusting quickly and asset allocation targets more slowly, has its roots in Samuelson [21] who formalizes the Marshallian 'moving equilibrium method.'

The present paper is related to the model by Schenk-Hoppé [15] with only one risky asset, two investors and without consumption. They obtain global (rather than just local) convergence result in this simpler setting. There is also a growing literature on empirical applications of evolutionary finance models, e.g., Hens et al. [13,14,16], and new approaches addressing the specification of investors' strategies. Lensberg and Schenk-Hoppé [20] apply genetic programming to evolve strategies over time through natural selection which forces investors to progressively improve their skills through imitation and trial-and-error.

The paper is organized as follows. Section 2 presents the model, Sect. 3.1 derives the dynamics of investors' asset holdings and wealth, Sect. 3.2 proves existence of short-run equilibrium, and Sect. 3.3 derives a representation of the dynamics as a random dynamical system. Section 4 presents in detail the conditions for local stability and provides examples.

2 Model

We consider a market in which a risk-free and several risky assets are traded at discrete points in time $t = 0, 1, \ldots$. The assets $k = 1, \ldots, K$ have risky dividends and prices. The total supply (volume) of asset k in period t is $V_{t,k}$. Each unit then trades at price $p_{t,k}$. Next period each unit pays $D_{t+1,k}$. The asset prices are determined *endogenously* through short-run equilibrium of supply and demand. The price of the risk-free asset $k = 0$ is *exogenous* (the absence of price risk) but its payoff $D_{t+1,0}$ can be random. Holdings of the risk-free asset are the same as balances in a bank account with a random net interest rate $\beta_{t+1} := D_{t+1,0}$. The price of this asset is used as numeraire, with all market values expressed in terms of cash.

There are $N \geq 1$ *investors* (*traders*) acting in the market. Each investor $i = 1, \ldots, N$ has an initial cash endowment $w_0^i > 0$. A *portfolio* of investor i at date $t = 0, 1, \ldots$ is specified by a vector $x_t^i = (x_{t,0}^i, x_{t,1}^i, \ldots, x_{t,K}^i) \in \mathbb{R}_+^{K+1}$, where $x_{t,0}^i$ is the amount in the investor's bank account and $x_{t,k}^i$ $(k = 1, \ldots, K)$ is the number of units of asset k held by the investor at time t. We do not allow short-selling.[4]

The market is influenced by random factors modeled in terms of an exogenous stochastic process $s_1, s_2, \ldots$, where s_t is a random element in a finite measurable space $(S, \mathcal{S})$. The event s_t is interpreted as the state of the world at date t. Asset prices $p_{t,k}$ and investors' portfolios x_t^i depend in general on the history

$$s^t := (s_1, \ldots, s_t)$$

of this process up to date t. The space of sample paths (s_t) is endowed with the product sigma-algebra.

An *investment (trading) strategy* of each investor i at date $t \geq 0$ is characterized by a vector of *investment proportions* $(\lambda_{t,0}^i, \lambda_{t,1}^i, \ldots, \lambda_{t,K}^i)$, $\lambda_{t,k}^i = \lambda_{t,k}^i(s^t)$, according to which the

Footnote 3 continued
team. This team is often internal for large pension funds. Professional financial advisors often ask private investors to choose percentages over different asset classes rather than individual assets. The implementation of such an investment strategy amounts to a rebalancing of the funds invested to maintain the specified percentages when asset prices change. In the presence of transaction costs, the optimal frequency to rebalance one's portfolio is a non-trivial task, see, e.g., Kuhn and Luenberger [18].

4 In the absence of perfect foresight, this would entail a bankruptcy risk. Although a well-defined dynamics can be specified, these situations lend themselves to computational rather than analytical work.

budget is distributed between the assets and the bank account. Vectors $(\lambda^i_{t,0}, \lambda^i_{t,1}, \ldots, \lambda^i_{t,K})$ belong to the unit simplex

$$\Delta := \{(a_0, \ldots, a_K) \geq 0 : a_0 + \ldots + a_K = 1\}.$$

At each date $t+1 = 1, 2, \ldots$ the dividend $D_{t+1,k} \geq 0$ paid by one unit of asset k depends on the history of states of the world and the aggregate wealth of investors, $\bar{w}_t = w^1_t + \ldots + w^N_t$, where w^i_t is the wealth of investor i at time t. We model dividends as

$$D_{t+1,k} = d_{t+1,k}\, \bar{w}_t \tag{1}$$

where the functions $d_{t+1,k} = d_{t+1,k}(s^{t+1})$ are assumed to satisfy

$$\sum_{k=1}^{K} d_{t+1,k} > 0. \tag{2}$$

This condition means that at each date and in each random situation at least one risky asset pays a strictly positive dividend. The money market pays $D_{t+1,0} = \beta_{t+1} \geq 0$ per unit, β_{t+1} the random net interest rate.

Define

$$p_t = (1, p_{t,1}, \ldots, p_{t,K}),$$

where $p_{t,k}$ are the asset prices. The prices $p_{t,1}, \ldots, p_{t,K}$ will be determined endogenously. The price of the bond is set to $p_{t,0} = 1$ for all dates t.

The scalar product

$$\langle p_t, x^i_t \rangle = \sum_{k=0}^{K} p_{t,k} x^i_{t,k}$$

expresses the value of the investor i's portfolio x^i_t at date t in terms of the asset prices $p_{t,k}$.

At date $t = 0$ the investors have initial endowments $w^i_0 > 0$ $(i = 1, 2, \ldots, N)$ in their bank accounts. These are their budgets at date 0. Investor i's total budget at date $t \geq 1$ is $B^i_t := \langle D_t + p_t, x^i_{t-1} \rangle$, where

$$D_t := (D_{t,0}, \ldots, D_{t,K}),\ \ D_{t,0} = \beta_t.$$

The budget consists of two parts: the dividends $\langle D_t, x^i_{t-1} \rangle$ paid by the portfolio x^i_{t-1} (including interest) and the market value $\langle p_t, x^i_{t-1} \rangle$ of the portfolio x^i_{t-1} expressed in terms of the vector of today's prices p_t. The prices $p_{t,k}, k = 1, \ldots, K$ are defined below in terms of equilibrium between supply and demand.

Not all wealth is reinvested. Investors may have to pay taxes and/or spend on consumption. This expenditure is expressed as a rate $0 \leq \tau_{t,k}(s^t) < 1, k = 0, 1, \ldots, K$, which can vary across assets but is the same for all the traders.[5] The reinvestment rate in asset k is therefore $\alpha_{t,k} := 1 - \tau_{t,k}(s^t)$. We assume that the functions $\alpha_{t,k}$ satisfy

$$\alpha_{t,k} < V_{t,k}/V_{t-1,k} \quad \text{for } k = 1, \ldots, K. \tag{3}$$

This condition is crucial for existence of equilibrium. It holds, in particular, when the total number $V_{t,k}$ of each asset k does not decrease, i.e., when the right-hand side of (3) is not less than one. But the assumption also allows for a situation with decreasing $V_{t,k}$, as long as it does

[5] In practice different asset classes are often taxed differently. The assumption of uniformity across traders facilitates the analysis.

not decrease faster than $\alpha_{t,k}$. The condition arises because of the equilibrium nature of the model which requires that the inflow of cash (from dividends and interest) is compensated by expenditures either on taxes, consumption or new assets. There is no condition on the growth rate of asset supply for the risk-free asset (whose price is constant) because asset $k = 0$ is used as a numeraire and its supply is infinitely elastic.

3 Equilibrium

3.1 Dynamic equilibrium

The random dynamics of prices and investors' portfolios and wealth is derived as a dynamic equilibrium in the model described above. Suppose the strategies $\lambda^1, \ldots, \lambda^N$ of all the investors, $\lambda^i = (\lambda_t^i(s^t))_{t,s^t}$, and their initial endowments $w_0^1, \ldots, w_0^N$ are given. The dynamic equilibrium is defined recursively by moving from time t to time $t+1$, starting at the initial time $t = 0$. The dynamics is random because it depends on the most current realization of the random component s_{t+1} which, in particular, determines the actual dividend payments.

At each date $t = 0, 1, \ldots$ each investor i possess wealth w_t^i and has selected some investment proportions $(\lambda_{t,0}^i, \lambda_{t,1}^i, \ldots, \lambda_{t,K}^i) \in \Delta$. The amount of cash invested in asset k by trader i is $\alpha_{t,k}\lambda_{t,k}^i w_t^i$ and the total amount invested in asset k is $\alpha_{t,k} \sum_{i=1}^N \lambda_{t,k}^i w_t^i$. The amount deposited with i's bank account is $\alpha_{t,0}\lambda_{t,0}^i w_t^i$ and the total amount kept by the investors in the bank is $\alpha_{t,0} \sum_{i=1}^N \lambda_{t,0}^i w_t^i$.

It is assumed that the market is always in equilibrium (asset supply is equal to asset demand), which makes it possible to determine the equilibrium price $p_{t,k}$ of each asset $k = 1, \ldots, K$ from the equations

$$p_{t,k} V_{t,k} = \alpha_{t,k} \sum_{i=1}^{N} \lambda_{t,k}^i w_t^i, \;\; k = 1, \ldots, K. \tag{4}$$

On the left-hand side of (4) we have the total value $p_{t,k} V_{t,k}$ of risky asset k. The right-hand side represents the total wealth invested in asset k by all the investors. Equilibrium implies the equality in (4). The price of the risk-free asset (bank account) is exogenous and set to $p_{t,0} = 1$; there is no market clearing condition for this asset.

The investment proportions $(\lambda_{t,0}^i, \ldots, \lambda_{t,K}^i)$ chosen by the traders $i = 1, \ldots, N$ at date t also determine their portfolios $(x_{t,0}^i, \ldots, x_{t,K}^i)$ at date t by the formula

$$x_{t,k}^i = \frac{\alpha_{t,k}\lambda_{t,k}^i w_t^i}{p_{t,k}}, \;\; k = 0, 1, \ldots, K. \tag{5}$$

Here, $x_{t,0}^i = \alpha_{t,0}\lambda_{t,0}^i w_t^i$ specifies the amount held in investor i's bank account. Formula (5) states that the current market value $p_{t,k} x_{t,k}^i$ of the kth position of the portfolio x_t^i of investor i is equal to the taxed fraction $\lambda_{t,k}^i$ of the i's investment budget w_t^i.

The wealth w_t^i of traders $i = 1, 2, \ldots, N$ are defined recursively. At the initial time $t = 0$, the wealth w_0^i of all the investors are given constants. But at each time $t = 1, 2, \ldots$, the wealth is given by

$$w_t^i = \sum_{k=0}^{K} (D_{t,k} + p_{t,k}) x_{t-1,k}^i. \tag{6}$$

For the model to be well-defined, one needs to prove that the system of Eqs. 4–6 possesses a unique, strictly positive price process $(p_{t,1}, \dots, p_{t,K})$ (recall that $p_{t,0} = 1$). The next section gives conditions ensuring its existence and uniqueness.

3.2 Existence of short-run equilibrium

Short-run equilibrium corresponds to the existence of a price system such that the market for each asset $k = 1, \dots, K$ clears in each period in time. Portfolios as defined in (5) further require that $p_{t,k} > 0$, or equivalently, that the aggregate demand for each asset (under the equilibrium prices) is strictly positive. Finally, measurability of all the variables needs to be assured.

- **(A)** There is one investor, say i, with $w_0^i > 0$ and $\lambda_{t,k}^i > 0$ for $k = 1, \dots, K$.
- **(B)** The following functions of s^t are assumed to by measurable: investment strategies λ_t^i, asset supply $V_{t,k}$, tax/consumption rates $\tau_{t,k}$, dividend rates $d_{t,k}$ and interest rate β_t.

Proposition 1 *Under assumption (A) there is a unique vector $p_t = (p_{t,1}, \dots, p_{t,K})$ with $p_{t,k} > 0$ for all $k = 1, \dots, K$ such that*

$$p_{t,k} V_{t,k} = \alpha_{t,k} \sum_{i=1}^{N} \lambda_{t,k}^i \sum_{m=0}^{K} (D_{t,m} + p_{t,m}) x_{t-1,m}^i, \;\; k = 1, \dots, K \tag{7}$$

(where $p_{t,0} = 1$). The solution is a measurable function if, in addition, assumption (B) holds.

Proof We adapt the contraction argument applied in Evstigneev et al. [9] to the present case. As the price of the risk-free asset is set to $p_{t,0} = 1$ by definition, only prices $p_{t,k}$ of assets $k = 1, \dots, K$ need to be considered.

Fix some s^t and consider the operator transforming a vector $p = (p_1, \dots, p_K) \in R_+^K$ into the vector $q = (q_1, \dots, q_K) \in R_+^K$ with coordinates

$$q_k = V_{t,k}^{-1} \alpha_{t,k} \sum_{i=1}^{N} \lambda_{t,k}^i \langle D_t + \tilde{p}, x_{t-1}^i \rangle, \;\; k = 1, \dots, K,$$

where

$$\tilde{p} = (1, p_1, \dots, p_K).$$

This operator is contracting in the norm $||p||_V := \sum_{k=1}^K |p_k| V_{t-1,k}$. Indeed, by virtue of (3) we have

$$\alpha := \max_{k=1,\dots,K} \{\alpha_{t,k} V_{t-1,k} V_{t,k}^{-1}\} < 1,$$

and so

$$\begin{aligned} ||q - q'||_V &= \sum_{k=1}^{K} |q_k - q_k'| V_{t-1,k} \\ &\le \sum_{k=1}^{K} V_{t-1,k} V_{t,k}^{-1} \alpha_{t,k} \sum_{i=1}^{N} \lambda_{t,k}^i |\langle \tilde{p} - \tilde{p}', x_{t-1}^i \rangle| \le \alpha \sum_{i=1}^{N} \sum_{k=1}^{K} \lambda_{t,k}^i |\langle \tilde{p} - \tilde{p}', x_{t-1}^i \rangle| \end{aligned}$$

$$\leq \alpha \sum_{i=1}^{N} |\langle \tilde{p} - \tilde{p}', x_{t-1}^i \rangle| \leq \alpha \sum_{i=1}^{N} \sum_{m=1}^{K} |p_m - p_m'| x_{t-1,m}^i$$
$$= \alpha \sum_{m=1}^{K} |p_m - p_m'| \sum_{i=1}^{N} x_{t-1,m}^i = \alpha \sum_{m=1}^{K} |p_m - p_m'| V_{t-1,m} = \alpha ||p - p'||_V,$$

where the last but one equality follows from (5). (Note that $\tilde{p}_0 - \tilde{p}_0' = 0$.) By using the contraction principle, we obtain the existence and uniqueness of the solution to (7). Starting the iteration with $p = (1, 0, \ldots, 0)$, one finds that the solution must be non-negative.

Under assumption (A) the solution to (7) has all components strictly positive at each period in time t and for each history s^t. This can be seen as follows. Let p_0 be the non-negative solution to (7) with time $t = 0$. Then (A) implies that there is an index i such that $x_{0,m}^i > 0$ for $m = 1, \ldots, K$. Since $D_{1,m} > 0$ for at least one m, $\lambda_{1,k}^i$ for $k = 1, \ldots, K$ and $\alpha_{1,k} > 0$ for $k = 1, \ldots, K$, one has that the right-hand side of (7), and thus the left-hand side is strictly positive. Therefore $p_{1,k} > 0$ for all k. This, in turn, implies $x_{1,m}^i > 0$ for $m = 1, \ldots, K$, which allows to apply the same argument recursively.

Measurability of the (unique) solution to (7) follows from the fact that it can be expressed, under assumption (B), as the pointwise limit of measurable functions (e.g. when starting the iteration with any constant vector). □

3.3 Wealth dynamics

The random dynamics of the investors' wealth is obtained by combining the dynamic equilibrium relations (Sect. 3.1) and the existence and uniqueness results on short-run equilibrium (Sect. 3.2). This section presents the resulting dynamics and derives an explicit representation of the dynamics, which takes on the form of a random map on the space of investors' wealths.

From (4) and (5) we get

$$p_{t,k} = \alpha_{t,k} V_{t,k}^{-1} \textstyle\sum_{i=1}^{N} \lambda_{t,k}^i w_t^i = \alpha_{t,k} \frac{\langle \lambda_{t,k}, w_t \rangle}{V_{t,k}}, \; k = 1, \ldots, K; \tag{8}$$
$$x_{t,k}^i = \frac{V_{t,k} \lambda_{t,k}^i w_t^i}{\langle \lambda_{t,k}, w_t \rangle}, \; k = 1, \ldots, K; \tag{9}$$

where $t \geq 0$, $w_t := (w_t^1, \ldots, w_t^N)$ and $\lambda_{t,k} := (\lambda_{t,k}^1, \ldots, \lambda_{t,k}^N)$, $k = 0, \ldots, K$.

The relations $p_{t,0} = 1$ and $x_{t,0}^i = \alpha_{t,0} \lambda_{t,0}^i w_t^i$ (with $t \geq 1$) can be written in the same form by setting

$$V_{t,0} := \alpha_{t,0} \langle \lambda_{t,0}, w_t \rangle, \tag{10}$$

which gives

$$p_{t,0} = \alpha_{t,0} \frac{\langle \lambda_{t,0}, w_t \rangle}{V_{t,0}} = 1$$

and

$$x_{t,0}^i = \frac{V_{t,0} \lambda_{t,0}^i w_t^i}{\langle \lambda_{t,0}, w_t \rangle} = \alpha_{t,0} \lambda_{t,0}^i w_t^i.$$

So formulas (8) and (9) for the prices and portfolios are valid for all $k = 0, 1, 2, \ldots K$.

Consequently, we have

$$\begin{aligned} w_{t+1}^i &= \sum_{k=0}^{K}(p_{t+1,k} + D_{t+1,k})x_{t,k}^i \\ &= \sum_{k=0}^{K}(\alpha_{t+1,k}\frac{\langle\lambda_{t+1,k}, w_{t+1}\rangle}{V_{t+1,k}} + D_{t+1,k})\frac{V_{t,k}\lambda_{t,k}^i w_t^i}{\langle\lambda_{t,k}, w_t\rangle} \\ &= \sum_{k=0}^{K}(\alpha_{t+1,k}\frac{\langle\lambda_{t+1,k}, w_{t+1}\rangle V_{t,k}}{V_{t+1,k}} + D_{t+1,k}V_{t,k})\frac{\lambda_{t,k}^i w_t^i}{\langle\lambda_{t,k}, w_t\rangle}. \end{aligned}$$

Using the notation

$$\rho_{t+1,k} = \alpha_{t+1,k}V_{t,k}/V_{t+1,k},$$

we have

$$w_{t+1}^i = \sum_{k=0}^{K}\left[\rho_{t+1,k}\langle\lambda_{t+1,k}, w_{t+1}\rangle + D_{t+1,k}V_{t,k}\right]\frac{\lambda_{t,k}^i w_t^i}{\langle\lambda_{t,k}, w_t\rangle}. \tag{11}$$

The equation expresses the investor's wealth as an aggregate of his position in each asset multiplied by the sum of asset re-sale price (adjusted for re-investment and dilution due to the changes in the number of outstanding shares) and the total dividend payment of the asset.

The system of Eq. 11 can be written in more compact vector notation as:

$$\left[\mathrm{Id} - X_t\Delta\rho_{t+1}\Lambda_{t+1}\right]w_{t+1} = X_t\Delta V_t D_{t+1} + (1+\beta_{t+1})\Delta\lambda_{t,0}\,w_t \tag{12}$$

where $X_t \in R^{N\times K}$ is the matrix of all the investors' period-t portfolio holdings in all assets with endogenous prices (the ith row is given by $(x_{t,1}^i, \dots, x_{t,K}^i)$) with

$$(X_t)_{ik} = \frac{\lambda_{t,k}^i w_t^i}{\langle\lambda_{t,k}, w_t\rangle}, i = 1, \dots, N,\ k = 1, \dots, K,$$

and the supply of cash is $V_{t,0} = \alpha_{t,0}\langle\lambda_{t,0}, w_t\rangle$ by definition (10). The matrix $\Lambda_{t+1} \in R^{K\times N}$ collects the period-$t+1$ investment strategies where column i is given by investor i's investment proportions for the assets with endogenous prices, i.e. the transposed of the vector $(\lambda_{t+1,1}^i, \dots, \lambda_{t+1,K}^i)$:

$$(\Lambda_{t+1})_{ki} = \lambda_{t+1,k}^i.$$

For every vector $y = (y_1, \dots, y_m)$, we denote by Δy the matrix with entries y_k ($k = 1, \dots, m$) on the diagonal and zeros otherwise.

The system of Eq. 12 is equivalent to

$$\left[\mathrm{Id} - X_t\Delta\rho_{t+1}\Lambda_{t+1}\right]w_{t+1} = \left[X_t\Delta V_t D_{t+1} + (1+\beta_{t+1})\Delta\lambda_{t,0}\,w_t\right]. \tag{13}$$

The matrix $\mathrm{Id} - X_t\Delta\rho_{t+1}\Lambda_{t+1}$ is invertible if $\max_k \rho_{t+1,k} < 1$ because this condition ensures that the diagonal of this matrix is column-dominant. One therefore obtains an equivalent representation in explicit form:

$$w_{t+1} = \left[\mathrm{Id} - X_t\Delta\rho_{t+1}\Lambda_{t+1}\right]^{-1}\left[X_t\Delta V_t D_{t+1} + (1+\beta_{t+1})\Delta\lambda_{t,0}\,w_t\right]. \tag{14}$$

The interpretation of (14) is straightforward. The wealth of all the investors in period $t+1$ is determined by their individual dividend and interest income, multiplied by a matrix representing the price changes in the assets with endogenous prices.

The system (14) also covers the case where the resale price of all assets (except the bank account) is zero. In that case assets purchased at time t are only claims to a (random) payoff at time $t+1$ but they do not have a resale value (like a lottery ticket after the draw). Setting $\rho_{t+1,k} = 0, k = 1, \ldots, K$, the left-most matrix on the right-hand side of (14) becomes the identity matrix. The interpretation is that assets are re-issued in each period in time, cf. Evstigneev et al. [9].

The dynamics (14) can be represented by the iteration of random maps:

$$w_{t+1} = h_{t+1}(s^{t+1}, w_t).$$

This dynamics is in general not an iteration of i.i.d. maps even if the state of the world follows an i.i.d. process because the function h depends on the history s^t.

It might be of interest to briefly discuss the case with only one investor. The dynamics (14) with $N = 1$ is equivalent to

$$w^1_{t+1} = \frac{\bar{d}_{t+1} + (1+\beta_{t+1})\lambda^1_{t,0}}{\sum_{k=1}^K \rho_{t+1,k}\lambda^1_{t+1,k}}\, w^1_t \tag{15}$$

where $\bar{d}_{t+1} = \sum_{k=1}^K V_{t,k} d_{t+1,k}$. Whether the wealth in the market on average grows, declines or does not exhibit a trend depends on the growth rate of the random coefficient of w^1_t on the right-hand side of (15). The endogenous prices of assets $k = 1, \ldots, K$ are given by $p_{t,k} = (\alpha_{t,k}/V_{t,k})\lambda^1_{t,k} w^1_t$ which implies that the random dividend yield is $D_{t+1,k}/p_{t,k} = V_{t,k} d_{t+1,k}/(\alpha_{t,k}\lambda^1_{t,k})$ and the market valuation of asset k relative to asset j is given by $p_{t,k}/p_{t,j} = (\alpha_{t,k}/\alpha_{t,j})(V_{t,j}/V_{t,k})\lambda^1_{t,k}/\lambda^1_{t,j}$. If the asset supply and the taxation/consumption rates are constant, these expressions simplify further.

4 Local stability and evolutionarily stable strategies

Our study of the local stability of the wealth dynamics focusses on the case of two investors. This simplifies the presentation without sacrificing generality (see Remark 3 below). Local stability in the two-investor case is concerned with the state of the dynamics in which one investor (say, investor 1) has no wealth, and the other investor (investor 2) has strictly positive wealth. It follows from (14) that this situation is invariant under the dynamics. Local stability of this equilibrium state implies that the dynamics starting from a state in which investor 1 is provided with a (sufficiently) small amount of wealth will asymptotically revert to the state where investor 1 possesses no wealth. The comparison of the performance of two investment strategies ($N = 2$) will be carried out by analyzing the ratio of their wealths which is described by a one-dimensional random dynamics. The case of an arbitrary number of investors is briefly discussed in a remark. It turns out that there are no essential differences between the two and the N investor case. The local stability conditions in the latter case are characterized by $N - 1$ independent conditions that each take the form of a condition in the two investor case.

On an intuitive level the concept of local stability applied here is closely related to the notion of stability in evolutionary game theory, see e.g., Weibull [25]. In that setting, one is interested in the population dynamics where individual players (typically a continuum) can follow two different strategies. Stability is defined under given a replicator dynamics which describes the frequency of the two types in the population. Given a situation in which most players follow one strategy (the incumbent strategy) and a small number of players follow the other "mutant" strategy, the latter group asymptotically becomes extinct. While evolutionary game theory considers population sizes, we study evolution *in pecunia*.

In the following, we will assume that:

(C.1) Strategies, asset payoffs and interest rate only depend on the process s^t, i.e., $\lambda_t(s^t) = \lambda(s^t)$, $d_{t,k}(s^t) = d_k(s^t)$ and $\beta_t(s^t) = \beta(s^t)$

(C.2) The process s^t is stationary and ergodic (with invariant probability measure denoted by P).

Condition (C) is necessary to apply Evstigneev et al. [12]'s results on when local stability of stochastic system can be inferred from the linearization at an equilibrium.

The process s^t can be extended to negative times without loss of generality, making it a one-sided infinite sequence [3, Appendix A]. The sample path space is then given by the sequences $(s_t)_{t=-\infty}^{\infty}$. Condition (C.2) implies that $E\xi(s^t) = E\xi(s^{t+1})$ for a random variable ξ.

For simplicity of presentation we assume that:

(D) $V_{t,k} = 1$ $(k \geq 1)$ and $\alpha_{t+1,k} = \alpha$ $(k \geq 0)$, with $0 < \alpha < 1$.

The first condition says that the firm neither issues new shares nor carries out buy-backs. The second assumption says that all asset classes are taxed at the same rate and that consumption is uniform across assets.

4.1 Dynamics of wealth ratio in the case of two investors

The dynamics of the ratio of the two investors' wealth can be derived from the system (14). This ratio compares the wealth of one investor relative to that of the other. Let $N = 2$, and define the ratio of the investors' wealth as

$$z_t := w_t^1 / w_t^2.$$

The ratio is well-defined if investor 2 is fully diversified, i.e., $\min_k \lambda_{t,k}^2(s^t) > 0$ for all t, s^t, and he has strictly positive initial wealth, $w_0^2 > 0$. This is condition (A), assuming that $i = 2$ (if needed after relabeling of the investors). Therefore the process $w_t = (w_t^1, w_t^2)$ is well-defined with $w_t^1 \geq 0$ and $w_t^2 > 0$.

We now demonstrate that the random dynamic of the process z_t is one-dimensional. The map governing this dynamic is derived from (14) as follows. The notation used in the following is:

$$\lambda_t^i = (\lambda_{t,1}^i, \ldots, \lambda_{t,K}^i)^T,\ x_t^i = (x_{t,1}^i, \ldots, x_{t,K}^i),\ d_t = (d_{t,1}, \ldots, d_{t,K})^T,$$
$$X_t = (x_{t,k}^i)_{ik}, \text{ and } \Lambda_t = (\lambda_{t,k}^i)_{ki}$$

with $i = 1, 2$ and $k = 1, \ldots, K$. The money market account holdings are determined by $(1 - \sum_{k=1}^K \lambda_{t,k}^i) w_t^i$.

One has

$$\left[\mathrm{Id} - \alpha X_t \Lambda_{t+1}\right]^{-1} = \frac{1}{\det(\mathrm{Id} - \alpha X_t \Lambda_{t+1})} \begin{bmatrix} 1 - \alpha\langle x_t^2, \lambda_{t+1}^2\rangle & \alpha\langle x_t^1, \lambda_{t+1}^2\rangle \\ \alpha\langle x_t^2, \lambda_{t+1}^1\rangle & 1 - \alpha\langle x_t^1, \lambda_{t+1}^1\rangle \end{bmatrix}$$

with

$$\det(\mathrm{Id} - \alpha X_t \Lambda_{t+1}) = (1 - \alpha\langle x_t^1, \lambda_{t+1}^1\rangle)(1 - \alpha\langle x_t^2, \lambda_{t+1}^2\rangle) - \alpha^2\langle x_t^1, \lambda_{t+1}^2\rangle\langle x_t^2, \lambda_{t+1}^1\rangle.$$

The portfolios of the two investors (in the assets with endogenous prices) can be written as functions of z_t:

$$x_k^1(z_t, \lambda_t) = \frac{\lambda_{t,k}^1 z_t}{\lambda_{t,k}^1 z_t + \lambda_{t,k}^2}, \quad \text{and} \quad x_k^2(z_t, \lambda_t) = \frac{\lambda_{t,k}^2}{\lambda_{t,k}^1 z_t + \lambda_{t,k}^2}.$$

One further has

$$D_{t+1,k} = d_{t+1,k}(s^{t+1})(w_t^1 + w_t^2) = d_{t+1,k}(s^{t+1})(z_t + 1)w_t^2$$

for $k = 1, \ldots, K$, and

$$w_t = \begin{pmatrix} w_t^1 \\ w_t^2 \end{pmatrix} = w_t^2 \begin{pmatrix} z_t \\ 1 \end{pmatrix}.$$

The above yields

$$X_t D_{t+1} + (1 + \beta_{t+1})\Delta\lambda_{t,0}\, w_t = \left[(z_t + 1)X_t d_{t+1} + (1 + \beta_{t+1})\Delta\lambda_{t,0} \begin{pmatrix} z_t \\ 1 \end{pmatrix}\right] w_t^2$$

where $\Delta(y_1, \ldots, y_K)$ is the matrix with entries y_k on the diagonal and zeros otherwise.

These considerations show that the ratio $z_{t+1} = w_{t+1}^1/w_{t+1}^2$, where (w_{t+1}^1, w_{t+1}^2) is uniquely defined by (14) for a given $z_t \geq 0$ and a $w_t^2 > 0$, is independent of w_t^2.

Combining the above, one obtains (after some lengthy but elementary mathematical operations) the dynamic of the process z_t:

$$z_{t+1} = \frac{\begin{array}{c}[1 - \alpha\langle x_t^2, \lambda_{t+1}^2\rangle][(z_t + 1)\langle x_t^1, d_{t+1}\rangle + (1 + \beta_{t+1})\lambda_{t,0}^1 z_t] \\ +\alpha\langle x_t^1, \lambda_{t+1}^2\rangle[(z_t + 1)\langle x_t^2, d_{t+1}\rangle + (1 + \beta_{t+1})\lambda_{t,0}^2]\end{array}}{\begin{array}{c}\alpha\langle x_t^2, \lambda_{t+1}^1\rangle[(z_t + 1)\langle x_t^1, d_{t+1}\rangle + (1 + \beta_{t+1})\lambda_{t,0}^1 z_t] \\ +[1 - \alpha\langle x_t^1, \lambda_{t+1}^1\rangle][(z_t + 1)\langle x_t^2, d_{t+1}\rangle + (1 + \beta_{t+1})\lambda_{t,0}^2]\end{array}}. \tag{16}$$

Equation 16 shows that the ratio of the two investor's wealth can indeed be described as a one dimensional stochastic system. The right-hand side can be interpreted as follows. The ratio of the investor 1's wealth relative to that of the other investor 2 is driven by a comparison of the investor's dividend and interest income, adjusted for the impact of the change in the endogenous asset prices. The representation makes use of the fact that the units of an asset not owned by investor 2 must be in the possession of investor 1.

The stochastic Eq. 16 defines a random dynamical system in discrete time on the space R_+ of non-negative numbers. That is, for a given state $z_t \in R_+$, the right-hand side of (16) defines the state of the system, $z_{t+1} \in R_+$, at the next point in time $t + 1$. This new state depends on the realization of the exogenous shock s_{t+1} and time. In this way, (16) generates a random path from a given initial state $z_0 \geq 0$. For the general theory of random dynamical system the mathematically-minded reader is referred to Arnold [3]. A survey of this theory within the economic context is provided, e.g., in [22].

Under assumptions (A)–(D), we can express the Eq. 16 as

$$z_{t+1} = f(s^{t+1}, z_t) \tag{17}$$

where the right-hand side is a function of z_t and the process s^{t+1} only.

Observe that the situation in which investor 2 owns all the wealth (and investor 1 owns nothing) is a steady state of (17). If $w_0^1 = 0$ and $w_0^2 > 0$, then $z_0 = w_0^1/w_0^2 = 0$. Indeed this is a steady state because

$$f(s^{t+1}, 0) = 0$$

by the definition of the function f in (16).

Considering the ratio of the investors' wealth $z_t = w_t^1/w_t^2$ needs a little more explanation because the following asymptotic property $z_t \to 0$ does not necessarily imply $w_t^1 \to 0$. The convergence $z_t \to 0$ implies that investor 1's wealth asymptotically becomes small relative to that of investor 2, i.e., the wealth of investor 1 diminishes relative to that of investor 2. This definition does not rule out that the wealth of both investors can grow over time but it says that the wealth of investor 2 grows faster than that of investor 1. Indeed, given a positive net interest rate, wealth can grow without bound. For instance if the investor places an amount of his wealth in the bank account and reinvests all interest income.

4.2 Sufficient conditions ensuring local stability

We now present the conditions ensuring local stability of the state $z = 0$ (in which investor 2 owns all the wealth). The sufficiency of these conditions follows from the recent results presented in Evstigneev et al. [12]. We apply their Theorem 1, but in other applications it might be convenient to rather use the sufficient conditions in Theorem 2 (together with and Remark 3).

In the following it is assumed that

$$E \ln \min_{k=0,\dots,K} \lambda_{t,k}^2 > -\infty; \tag{18}$$

$$E \ln^+ \beta_{t+1} < \infty; \text{ and } E \ln^+ d_{t+1,k} < \infty \text{ for } k = 1, \dots, K. \tag{19}$$

Recall the dividend payment per unit of asset k is given by $D_{t+1,k} = d_{t+1,k}\bar{w}_t$ with the aggregate wealth defined as $\bar{w}_t = \sum_{i=1}^N w_t^i$.

These two integrability assumptions can be interpreted as conditions ensuring that everything that happens in the model at an exponential rate is caused by the wealth dynamics rather than by changes in the 'ingredients' (investment strategies and dividend and interest payments). Note that, for instance, condition (18) is stronger than the assumption $\lambda_{t,k}^2 > 0$ for all t, s^t which was needed to ensure that the model is well-defined. Log-integrability conditions of the above type are common in stochastic dynamic models.

Proposition 2 *The steady state $z = 0$ of* (17) *is locally stable, if*

$$E \ln f'(s^{t+1}, 0) < 0 \tag{20}$$

with $f'(s^{t+1}, 0)$ denoting the derivative of the right-hand side of (17) *evaluated at $z = 0$.*

Local stability of a stochastic dynamical system is defined here as follows. There exists a neighborhood $U(s^0)$ (a random set) of the steady state $z = 0$ such that for almost all sample paths (s_t) the following holds: For each initial value $z_0 \in U(s^0)$, the sample path $z_t \to 0$. Under condition (20), the convergence is exponentially fast with constant (i.e., non-random) rate given by $E \ln f'(s^{t+1}, 0)$ (which is independent of t by virtue of stationarity). The condition (20) is analogue to those in deterministic dynamical systems where local stability can be verified by studying the derivative at the steady state. In stochastic dynamic models, the condition can be interpreted as ensuring that the dynamics is locally contracting *on average*, see Evstigneev et al. [12].

The economic interpretation of local stability in the present model is that if $z = 0$ is locally stable, then the 'incumbent' investment strategy λ^2 is locally stable against the 'mutant' strategy λ^1.

The derivative of $f(s^{t+1}, z)$ at $z = 0$ (which defines the linearization of the system (17) at the steady state $z = 0$) can be found after some elementary but lengthy calculations as:

$$f'(s^{t+1}, 0) = \frac{[1 - \alpha(1 - \lambda^2_{t+1,0})][\langle \eta_t, d_{t+1}\rangle + (1 + \beta_{t+1})\lambda^1_{t,0}]}{\bar{d}_{t+1} + (1 + \beta_{t+1})\lambda^2_{t,0}} + \alpha\langle \eta_t, \lambda^2_{t+1}\rangle \tag{21}$$

where

$$\eta_{t,k} = \frac{\lambda^1_{t,k}}{\lambda^2_{t,k}},\ k = 1, \ldots, K, \tag{22}$$

and $d_{t+1} = (d_{t+1,1}, \ldots, d_{t+1,K})$. Note that the derivative $f'(s^{t+1}, 0) > 0$.

Proof of Proposition 2 The proof is an the application Theorem 1 in Evstigneev et al. [12] which requires to verify the conditions (B1) and (B2) defined in Sect. 2 of their paper.

First note that the dynamics (17) is also well-defined for a larger set values of z_t than $[0, \infty)$ because the right-hand side of this equation actually makes sense for negative values of z_t as well—provided they are not too small. This property is useful because it ensures that the derivative of the dynamics at $z = 0$ can be understood in the usual sense rather than as a directional derivative: by extending the dynamics (17) to the space $X = (-\infty, \infty)$, as we do in the next paragraph, the point $z = 0$ becomes an interior point.

The extension can be done as follows: Note that the right-hand side of (17) is well-defined even for negative $z_t \geq -\varepsilon(s^t)$, as long as $\lambda^2_{t,k} - \varepsilon(s^t)\lambda^1_{t,k} > 0$ (which ensures that the portfolios are well-defined) and the denominator is strictly positive. The first condition is satisfied for each $\varepsilon(s^t) < \min_k(\lambda^2_{t,k}/\lambda^1_{t,k})$. Existence of an $\varepsilon(s^t) > 0$ such that the second condition holds follows from the fact that the denominator is continuous in z_t and that it is larger than $(1 - \alpha)(1 + \beta_{t+1})\lambda^2_{t,0} > 0$ for $z_t = 0$. Let us assume a suitable $\varepsilon(s^t) > 0$ is chosen and fixed. Define the dynamics by (17) for all $z_t > -\varepsilon(s^t)$ and by -1 (or any other constant) for $z_t \leq -\varepsilon(s^t)$. The stochastic dynamics is then well-defined on the set $X = (-\infty, \infty)$, and the (random but trivial) set $X(s^t) = [0, \infty) \subset X$ is invariant under the dynamics.

Condition (B1) in Evstigneev et al. [12] requires that there exist random variables $L(s^t)$ and $\delta(s^t)$ with $E|\ln L| < \infty$ and $E|\ln \delta| < \infty$ (by stationarity the expected value is independent of t) such that

$$|f(s^t, z) - f(s^t, 0)| \leq L(s^t)|z - 0|$$

for all $z \in X(s^t)$ with $0 \leq z \leq \delta(s^t)$. This condition can be interpreted as local Lipschitz continuity with a log-integrable Lipschitz 'constant.'

Fix any constant $\delta \in (0, 1)$ and let $\delta(s^t) \equiv \delta$. Since $f(s^t, 0) = 0$ and $f(s^t, z) \geq 0$ for $z \geq 0$, the above inequality is equivalent to

$$\frac{f(s^t, z)}{z} \leq L(s^t)$$

for all $z > 0$. Indeed, for every $z_t > 0$, one has

$$\frac{f(s^t, z_t)}{z_t} \leq \frac{(z_t + 1)\langle \zeta^1_t, d_{t+1}\rangle + (1 + \beta_{t+1})\lambda^1_{t,0}}{(1 - \alpha)(1 + \beta_{t+1})\lambda^2_{t,0}} + \frac{\alpha\langle \zeta^1_t, \lambda^1_{t+1}\rangle}{1 - \alpha\langle x^1_t, \lambda^1_{t+1}\rangle}$$

where

$$\zeta^1_{t,k} = \frac{x^1_{t,k}}{z_t} = \frac{\lambda^1_{t,k}}{\lambda^1_{t,k} z_t + \lambda^2_{t,k}}.$$

Since $\zeta^1_{t,k} \leq M(s^t) := \max_k(\lambda^1_{t,k}/\lambda^2_{t,k})$ and $\beta_{t+1} \geq 0$ one has

$$\frac{f(s^t, z_t)}{z_t} \leq 2\frac{M(s^t)\bar{d}_{t+1} + 1 + \beta_{t+1}}{(1-\alpha)\lambda^2_{t,0}} + \frac{\alpha M(s^t)}{1-\alpha}$$

where it is used that $(z_t + 1)\langle \zeta^1_t, d_{t+1}\rangle \leq 2M(s^t)\bar{d}_{t+1}$ for $0 \leq z_t \leq \delta < 1$ and that $\langle \zeta^1_t, \lambda^1_{t+1}\rangle \leq M(s^t)$. The right-hand side of the last inequality is not less than one and it therefore suffices to verify that

$$E \ln^+ \left[2\frac{M(s^t)\bar{d}_{t+1} + 1 + \beta_{t+1}}{(1-\alpha)\lambda^2_{t,0}} + \frac{\alpha M(s^t)}{1-\alpha} \right] < \infty.$$

But this property follows from the above integrability assumptions (18) and (19).

Condition (B2) in Evstigneev et al. [12] is identical to (20) because $f'(s^{t+1}, 0) > 0$ (see (21)). Therefore their Theorem 1 implies the assertion in the Proposition. □

Remark 2 Condition (20) is in particular satisfied if

$$E_{s^t} \ln f'(s^{t+1}, 0) \leq 0 \quad \text{for } P \text{ almost all } s^t \tag{23}$$

and the inequality is strict on a set of s^t which has strictly positive probability. Here $E_{s^t} = E[\cdot \mid s^t]$ denotes the conditional expectation under the invariant measure P.

Remark 3 We briefly comment on the N investor case. Consider a market with N investment strategies, $\lambda^1, \ldots, \lambda^N$. In contrast to the above one has $N-1$ ratios $z^i_t = w^i_t/w^N_t$. One can work out the analogous steps to the above and derive sufficient conditions for the local stability of $z = (0, \ldots, 0)$ using the results in Evstigneev et al. [12]. It turns out that these $N-1$ conditions correspond to pairwise comparisons of λ^N and λ^i, $i = 1, \ldots, N-1$, each of which takes the form (20).

4.3 Evolutionarily stable strategy

In this section we derive an investment strategy $\lambda^*(s^t)$ such that the dynamics (16) whose local properties we studied in the previous section has the following property: if investor 2 follows strategy $\lambda^*(s^t)$ and investor 1 uses a different strategy $\lambda^1(s^t)$, then the state $z = 0$ is locally stable. 'Different' means $P(\lambda^*(s^t) = \lambda^1(s^t)) < 1$ (ergodicity of the process s^t implies that this probability is independent of t). We call a strategy with this property a (locally) *evolutionarily stable strategy*.

The economic interpretation of this property is that an incumbent λ^*-investor is *unbeatable*. If his 'mutant' competitor plays any different investment strategy, then the wealth ratio reverts to zero (locally). If the mutant also plays λ^*, then the wealth ratio remains equal to its initial value. A discussion of unbeatable investment strategies in a full game-theoretic evolutionary finance model is provided in in Amir et al. [1].

The method to derive conditions characterizing (and hopefully being able to fully identify) evolutionarily stable strategies is as follows. One shows that the derivative (23) is a (strictly) concave function of $\lambda^1(s^t)$ for any given investment strategy $\lambda^2(s^t)$. Therefore there is a 'best response' strategy where best response refers to the growth rate obtained by choosing $\lambda^1(s^t)$ for given $\lambda^2(s^t)$. Any evolutionarily stable strategy is characterized by the fact that this best response to the investment strategy $\lambda^2 = \lambda^*$ is λ^* itself (for all s^t). We now carry out this program and derive conditions characterizing evolutionarily stable strategies.

First, note that for a given process $\lambda^2(s^t) = (\lambda_0^2(s^t), \lambda_1^2(s^t), \ldots, \lambda_K^2(s^t))$, the map

$$\lambda^1(s^t) \mapsto E_{s^t} \ln f'(s^{t+1}, 0), \quad \Delta \to [-\infty, \infty]$$

is strictly concave for each s^t if the right-hand side of (21) is not constant in s_{t+1} on a set of strictly positive P-measure. Second, observe that for $\lambda^1(s^t) = \lambda^2(s^t)$, $f'(s^{t+1}, 0) = 1$ for all s^{t+1} and thus $E_{s^t} \ln f'(s^{t+1}, 0) = 0$.

For an interior solution (i.e., an investment strategy with all components strictly positive)) one has the conditions

$$\frac{\partial E_{s^t} \ln f'(s^{t+1}, 0)}{\partial \lambda_k^1(s^t)}\Big|_{\lambda^1(s^t)=\lambda^*(s^t)} = c, \quad k = 0, 1, \ldots, K. \tag{24}$$

with c a constant. This constant is determined by the condition $\sum_{k=0}^K \lambda_{t,k}^* = 1$ as follows: Using (21), one finds

$$E_{s^t}\left([1 - \alpha(1 - \lambda_{t+1,0}^*)]\frac{1 + \beta_{t+1}}{\bar{d}_{t+1} + (1 + \beta_{t+1})\lambda_{t,0}^*}\right) = c \tag{25}$$

and, for $k = 1, .., K$,

$$E_{s^t}\left([1 - \alpha(1 - \lambda_{t+1,0}^*)]\frac{d_{t+1,k}}{\bar{d}_{t+1} + (1 + \beta_{t+1})\lambda_{t,0}^*} + \alpha\lambda_{t+1,k}^*\right) = c\lambda_{t,k}^*. \tag{26}$$

Adding (26) over $k = 1, \ldots, K$ and (25) (after multiplying with $\lambda_{t,0}^*$) one obtains

$$c(\sum_{k=0}^K \lambda_{t,k}^*) = E_{s^t}\left([1 - \alpha(1 - \lambda_{t+1,0}^*)]\frac{\bar{d}_{t+1} + (1 + \beta_{t+1})\lambda_{t,0}^*}{\bar{d}_{t+1} + (1 + \beta_{t+1})\lambda_{t,0}^*} + \alpha \sum_{k=1}^K \lambda_{t+1,k}^*\right). \tag{27}$$

Therefore the constant $c = 1$.

The investment strategy λ_t^* is obtain by first determining $(\lambda_{t,0}^*)$ by solving

$$E_{s^t}\left([1 - \alpha(1 - \lambda_{t+1,0}^*)]\frac{1 + \beta_{t+1}}{\bar{d}_{t+1} + (1 + \beta_{t+1})\lambda_{t,0}^*}\right) = 1 \tag{28}$$

(subject to the constraint that $0 \leq \lambda_{t,0}^* \leq 1$ for all t) and, using this solution, to solve for $k = 1, .., K$

$$E_{s^t}\left([1 - \alpha(1 - \lambda_{t+1,0}^*)]\frac{d_{t+1,k}}{\bar{d}_{t+1} + (1 + \beta_{t+1})\lambda_{t,0}^*} + \alpha\lambda_{t+1,k}^*\right) = \lambda_{t,k}^*. \tag{29}$$

This procedure determines an investment strategy through the solution $\lambda^*(s^t) \in \Delta$ obtained for each s^t. By construction the strategy $\lambda^*(s^t)$ is (locally) evolutionary stable in the sense that the wealth of an investor following this strategy is locally stable in a market in which the other investor uses any strategy different to $\lambda^*(s^t)$.

It follows from (29) that $\lambda_{t,k}^* > 0$ for all $k = 1, \ldots, K$ (and from (28) that $\lambda_{t,0}^* < 1$). However, in general the holdings in the money market can be zero, i.e., it can happen that $\lambda_{t,0}^* = 0$.

4.4 Example with explicit solution

An explicit solution to the problem of finding a locally stable investment strategy can be given under certain conditions. Let us assume that the process s^t is Markovian, and that the aggregate dividend factor $\bar{d}(s^{t+1}) = \sum_{k=1}^{K} d_k(s^{t+1})$ and the interest rate $\beta(s^{t+1})$ are both constants denoted by $\bar{d}$ and β, respectively. Under these conditions, (28) is equivalent to

$$(1+\beta)(1-\alpha+\alpha E_{s_t}\lambda^*_{t+1,0}) = \bar{d} + (1+\beta)\lambda^*_{t,0}$$

which has the constant solution

$$\lambda^*_0 = 1 - \frac{\bar{d}}{(1+\beta)(1-\alpha)}.$$

This solution is interior (i.e., $\lambda^*_0 \in (0,1)$) if $\bar{d} < (1+\beta)(1-\alpha)$. Otherwise one obtains a 'corner solution' in which the investor does not place money in the money market. The first case has been studied in Hens and Schenk-Hoppé [15] where a global convergence result is obtained in the case of only one risky asset. Their main finding is that holding the risk-free bond does not ensure survival if the other investor only holds the risky asset. This assertion is proved under the assumption that either (a) there is no consumption or (b) that the gross return of the bond is dominated by the ratio of the dividend rate and the consumption rate in all states of the world. (The dividend rate is the total amount of dividends paid in a period divided by the total wealth of all the investors. The consumption rate is the ratio of the amount spent on consumption and the investor's wealth.) Their finding relates to Tobin [24] who argued that in the face of potential capital losses on bonds it is reasonable to hold cash as a means to transfer wealth over time. It runs out that Tobin's argument is not correct from an evolutionary perspective.

Inserting the last term in (29) gives

$$\lambda^*_{t,k} - \alpha E_{s_t}\lambda^*_{t+1,k} = E_{s_t}\left((1-\alpha\lambda^*_0)\frac{d_{t+1,k}}{\bar{d}+(1+\beta)\lambda^*_0}\right).$$

The right-hand side of this equation is equal to

$$\frac{1-\alpha(1-\lambda^*_0)}{\bar{d}+(1+\beta)\lambda^*_0} E_{s_t} d_{t+1,k} = \frac{1}{1+\beta}\, E_{s_t} d_{t+1,k}.$$

Thus, one needs to solve

$$\lambda^*_{t,k} - \alpha E_{s_t}\lambda^*_{t+1,k} = \frac{1}{1+\beta}\, E_{s_t} d_{t+1,k}$$

which has the solution

$$\lambda^*_{t,k} = \frac{1}{1+\beta}\sum_{m=1}^{\infty} \alpha^{m-1} E_{s_t} d_{t+m,k}. \tag{30}$$

This investment strategy allocates wealth across all of the available assets in proportions corresponding to the discounted expected (relative) dividend payoffs. The asset valuation implied by this strategy is a net present value in relative terms. In the model at hand, this valuation will obtain asymptotically provided the initial endowments all the other investors are small enough for the local stability to determine the dynamics. A market with this asset valuation can be interpreted as being (locally) evolutionarily stable, (cf. [8,9]).

If the other investors' initial endowments are not small enough, the linearization approach does not imply any guidance for making predictions on short and medium term price dynamics.

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References

1. Amir, R., Evstigneev, I.V., Schenk-Hoppé, K.R.: Asset Market Games of Survival: A Synthesis of Evolutionary and Dynamic Games. NCCR FinRisk Working Paper No. 505, August (2010)
2. Anufriev, M., Dindo, P.: Wealth-driven selection in a financial market with heterogeneous agents. J. Econ. Behav. Org. **73**, 327–358 (2010)
3. Arnold, L.: Random Dynamical Systems. Springer, Berlin (1998)
4. Blume, L., Easley, D.: Evolution and market behavior. J. Econ. Theory **58**, 9–40 (1992)
5. Blume, L., Easley, D. : Market Selection and Asset Pricing. In: Hens, T., Schenk-Hoppé, K.R. (eds.) Handbook of Financial Markets: Dynamics and Evolution, chap. 7, pp. 403–437. North-Holland, Amsterdam (2009)
6. Chiarella, C., Dieci, R., He, X.-Z. : Heterogeneity, Market Mechanisms, and Asset Price Dynamics. In: Hens, T., Schenk-Hoppé, K.R. (eds.) Handbook of Financial Markets: Dynamics and Evolution, Chap. 5, pp. 277–344. North-Holland, Amsterdam (2009)
7. Chiarella, C., He, X.-Z., Zheng, M.: An analysis of the effect of noise in a heterogeneous agent financial market model. J. Econ. Dyn. Control **35**, 148–162 (2011)
8. Evstigneev, I.V., Hens, T., Schenk-Hoppé, K.R.: Evolutionary stable stock markets. Econ. Theory **27**, 449–468 (2006)
9. Evstigneev, I.V., Hens, T., Schenk-Hoppé, K.R.: Globally evolutionarily stable portfolio rules. J. Econ. Theory **140**, 197–228 (2008)
10. Evstigneev, I.V., Hens, T., Schenk-Hoppé, K.R. : Evolutionary Finance. In: Hens, T., Schenk-Hoppé, K.R. (eds.) Handbook of Financial Markets: Dynamics and Evolution, Chap. 9, pp. 507–566. North-Holland, Amsterdam (2009)
11. Evstigneev, I.V., Hens, T., Schenk-Hoppé, K.R. : Survival and Evolutionary Stability of the Kelly Rule. In: MacLean, L.C., Thorp, E.O., Ziemba, W.T. (eds.) The Kelly Capital Growth Investment Criterion: Theory and Practice, Chap. 20, pp. 273–284. World Scientific, Singapore (2011)
12. Evstigneev, I.V., Pirogov, S.A., Schenk-Hoppé, K.R.: Linearization and local stability of random dynamical systems. Proc. Am. Math. Soc. **139**, 1061–1072 (2011)
13. Hens, T., Lensberg, T., Schenk-Hoppé, K.R., Woehrmann, P.: An evolutionary explanation of the value premium puzzle. J. Evol. Econ. **21**, 803–815 (2011)
14. Hens, T., Schenk-Hoppé, K.R. : Survival of the Fittest on Wall Street. In: Lehmann-Waffenschmidt, M., Ebner, A., Fornahl, D. (eds.) Proceedings of the VI Buchenbach Workshop, pp. 339–367. Metropolis, Marburg (2004)
15. Hens, T., Schenk-Hoppé, K.R.: Markets do not select for a liquidity preference as behavior towards risk. J. Econ. Dyn. Control **30**, 279–292 (2006)
16. Hens, T., Schenk-Hoppé, K.R., Stalder, M.: An application of evolutionary finance to firms listed in the Swiss market index. Swiss J. Econ. Stat. **138**, 465–488 (2002)
17. Hommes, C.H., Wagener, F.O.O. : Complex Evolutionary Systems in Behavioral Finance. In: Hens, T., Schenk-Hoppé, K.R. (eds.) Handbook of Financial Markets: Dynamics and Evolution, Chap. 4, pp. 217–276. North-Holland, Amsterdam (2009)
18. Kuhn, D., Luenberger, D.G.: Analysis of the rebalancing frequency in log-optimal portfolio selection. Quant. Financ. **10**, 221–234 (2010)
19. Laffont, J.-J.: The Economics of Uncertainty and Information. MIT Press, Cambridge (1989)
20. Lensberg, T., Schenk-Hoppé, K.R.: On the evolution of investment strategies and the Kelly rule—a Darwinian approach. Rev. Financ. **11**, 25–50 (2007)
21. Samuelson, P.A.: Foundations of Economic Analysis. Harvard University Press, Cambridge (1947)
22. Schenk-Hoppé, K.R.: Random dynamical systems in economics. Stoch. Dyna. **1**, 63–83 (2001)

23. Shapley, L.S., Shubik, M.: Trade using one commodity as a means of payment. J. Polit. Econ. **85**, 937–968 (1977)
24. Tobin, J.: Liquidity preference as behavior towards risk. Rev. Econ. Stud. **25**, 65–86 (1958)
25. Weibull, J.: Evolutionary Game Theory. MIT Press, Cambridge (1995)