Seminar 5
Stochastic Volatility Models

Vega Institute

Problem 1 🧠

Let $S_t^{(\mu)}$ be the process from CEV model, i.e. $dS_t^{(\mu)} = \mu S_t dt + \sigma S_t^\gamma dB_t$, where $\gamma \neq 1$. Show that

$$S_t^{(\mu)} \overset{d}{=} S_{\tau(t)}^{(0)} e^{\mu t},$$

where $\tau(t) = \dfrac{e^{2\mu(\gamma-1)t} - 1}{2\mu(\gamma - 1)}$.

Problem 2 🧠

Let S_t be the process from CEV model with zero drift, i.e. $dS_t = \sigma S_t^\gamma dB_t$, $\gamma \neq 1$ and $X_t = \dfrac{S_t^{2(1-\gamma)}}{\sigma^2(1-\gamma)^2}$.

1. Show that X_t satisfies the equation
$$dX_t = \delta dt + 2\sqrt{X_t} dB_t$$
and find δ. What can you say about the existence and uniqueness of the solution of this equation? Does your answer depend on δ, if the process is considered only until reaching zero?

2. Show that if $S_0 > 0$, then if $\gamma \in (0, 1)$, S_t reaches zero with positive probability and if $\gamma \geq 1$, the S_t is strictly positive process.

3. In case of $\gamma \in (0, 1)$, we put $S_t = 0$ after reaching the zero level. Why do we do that? Why we can not reflect it back?

Problem 3 💅

Derive the Hagan formula for the implied volatility in CEV model

$$\hat{\sigma} = \frac{\sigma}{\bar{s}^{1-\gamma}} \left(1 + \frac{(1-\gamma)(2+\gamma)}{24} \left(\frac{s-K}{\bar{s}} \right)^2 + \frac{(1-\gamma)^2}{24} \frac{\sigma^2 T}{\bar{s}^{2(1-\gamma)}} + \dots \right),$$

where $\bar{s} = \frac{1}{2}(s+K)$. Investigate the convexity and slope of this curve.