Examples for "Financial Derivatives with C++"

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The issue time for all options coincides with the initial time. The maturities, barrier, and exercise times are strictly greater than the initial time.

Standard put

$K:$ the strike.

$T:$ the maturity.

The payoff of the option at the maturity is given by

$$V(T) = \max(K - S(T), 0),$$

where $S(T)$ is the price of the stock at T.

Algorithm. The event times are

$$\{t_0, T\},$$

where t_0 is the initial time. We have that

$$\begin{aligned} X(T) &= \max(K - S(T), 0), \\ X(t_0) &= \mathcal{R}_{t_0}(X(T)). \end{aligned}$$

At the end, we return $X(t_0)$.

Call on forward price

K : the strike.

T : the maturity of the call option.

U : the delivery time of the forward contract.

The payoff of the option at the maturity is given by

$$X(T) = \max(F(T, U) - K, 0),$$

where $F(T, U)$ is the forward price computed at T for delivery at U.
Algorithm. The event times are

$$\{t_0, T\},$$

where t_0 is the initial time. We have that

$$\begin{aligned} X(T) &= \max(F(T, U) - K, 0), \\ X(t_0) &= \mathcal{R}_{t_0}(X(T)). \end{aligned}$$

At the end, we return $X(t_0)$.

Clique option

T : the maturity.

$(t_m)_{m=1,\ldots,M}$: the averaging times, $t_M < T$.

$K:$ the strike.

The payoff of the option at maturity is given by

$$V(T) = \frac{1}{M} \sum_{m=1}^{M} \max(S(t_m) - K, 0),$$

where $S(t)$ is the spot price at t.

Algorithm. The event times are

$$\{t_0, \underbrace{(t_m)_{m=1,\dots,M}}_{\text{averaging times}}\},$$

where t_0 is the initial time. We divide the algorithm into 3 steps. We shall multiply on $1/M$ at the end.

Step 1 (Boundary condition).

$$\underbrace{X(t_M)}_{>t_M} = 0.$$

Step 2 (Loop).

$$\underbrace{t_0}_{\text{end}} \longleftarrow \underbrace{t_M}_{\text{begin}},$$
$$\underbrace{X(t_m)}_{>t_m} \longleftarrow \underbrace{X(t_{m+1})}_{>t_{m+1}},$$

where $\underbrace{X(t_{m+1})}_{>t_{m+1}}$ is the value to continue (the future calls paid at T):

$$\underbrace{X(t_{m+1})}_{>t_{m+1}} = \mathcal{R}_{t_{m+1},T} \left(\sum_{i=m+2}^{M} \max(S(t_i) - K, 0) \right).$$

The value of the current call paid at T is given by

$$Y(t_{m+1}) = B(t_{m+1}, T) \max(S(t_{m+1}) - K, 0),$$

where $B(s,t)$ is the discount factor computed at s for maturity t. We have that

$$\underbrace{X(t_{m+1})}_{>t_m} = \underbrace{X(t_{m+1})}_{>t_{m+1}} + Y(t_{m+1}),$$
$$\underbrace{X(t_m)}_{>t_m} = \mathcal{R}_{t_m}(\underbrace{X(t_{m+1})}_{>t_m}).$$

Step 3 (After the loop)*.* We return $\frac{1}{M}\underbrace{X(t_0)}_{>t_0}$.

American put

$K:$ the strike.

$(t_m)_{m=1,\ldots,M}:$ the exercise times.

A holder of the option can exercise it at any time t_m. In this case, he receives intrinsic value

$$V(t_m) = \max(K - S(t_m), 0),$$

where $S(t_m)$ is the price of the stock at time t_m.

Algorithm. The event times are

$$\{t_0, (t_m)_{m=1,\ldots,M}\},$$

where t_0 is the initial time. We divide the algorithm into 3 steps.

Step 1 (Boundary condition)*.*

$$X(t_M) = \underbrace{X(t_M)}_{>t_M} = 0.$$

Step 2 (Loop)*.* We enter the loop at t_M (included) and exit at t_0 (not included):

$$\underbrace{t_0}_{\text{end}} \longleftarrow \underbrace{t_M}_{\text{begin}}.$$

We consider the iteration:

$$\underbrace{X(t_m)}_{?} \longleftarrow \underbrace{X(t_{m+1})}_{\text{known}},$$

where

$$X(t_{m+1}) = \underbrace{X(t_{m+1})}_{>t_{m+1}}$$

is the value to continue (exercises will be made after t_{m+1}). We have that

$$\underbrace{X(t_{m+1})}_{>t_m} = \max(\underbrace{X(t_{m+1})}_{>t_{m+1}}, K - S(t_{m+1}))$$

and then that

$$\underbrace{X(t_m)}_{>t_m} = \mathcal{R}_{t_m}(\underbrace{X(t_{m+1})}_{>t_m})$$

Step 3 (After the loop). We return $X(t_0) = \underbrace{X(t_0)}_{>t_0}$.

American call on forward

$K:$ the forward price.

$\delta t:$ the time to maturity of the forward contract as an year fraction.

$(t_m)_{m=1,\ldots,M}:$ the exercise times.

The option can be exercised at any time t_m. In this case, its holder enters a long position in the forward contract with forward price K and maturity $t_m + \delta t$.

Algorithm. The event times have the form:

$$\{t_0, \underbrace{(t_m)_{m=1,\ldots,M}}_{\text{exercise times}}\},$$

where t_0 is the initial time. We divide the algorithm into 3 steps. We denote by $S(t)$ the price of the stock at time t.

Step 1 (Boundary condition).

$$X(t_M) = \underbrace{X(t_M)}_{>t_M} = 0.$$

Step 2 (Loop). We enter the loop at t_M (included) and exit at t_0 (not included):

$$\underbrace{t_0}_{\text{end}} \longleftarrow \underbrace{t_M}_{\text{begin}}.$$

We consider the iteration:

$$\underbrace{X(t_m)}_{?} \longleftarrow \underbrace{X(t_{m+1})}_{\text{known}},$$

where

$$X(t_{m+1}) = \underbrace{X(t_{m+1})}_{>t_{m+1}}$$

is the value to continue (exercises will be made after t_{m+1}).

Being exercised, the option yields payoff

$$Y(t_{m+1} + \delta t) = S(t_{m+1} + \delta t) - K.$$

at time $t_{m+1} + \delta t$. Let $F(s, t)$ be the market forward price in the contract issued at s for maturity t. Since, it costs nothing to enter forward at the market forward price,

$$0 = \mathcal{R}_{t_{m+1}}(S(t_{m+1} + \delta t) - F(t_{m+1}, t_{m+1} + \delta t)).$$

The intrinsic value of the option is given by

$$\begin{aligned}
Y(t_{m+1}) &= \mathcal{R}_{t_{m+1}}\left(Y(t_{m+1} + \delta t)\right) \\
&= \mathcal{R}_{t_{m+1}}\left(S(t_{m+1} + \delta t) - K\right) \\
&= \mathcal{R}_{t_{m+1}}\left(S(t_{m+1}) - F(t_{m+1}, t_{m+1} + \delta t)\right) \\
&\quad + \mathcal{R}_{t_{m+1}}\left(F(t_{m+1}, t_{m+1} + \delta t) - K\right) \\
&= \mathcal{R}_{t_{m+1}}\left(F(t_{m+1}, t_{m+1} + \delta t) - K\right) \\
&= B(t_{m+1}, t_{m+1} + \delta t)(F(t_{m+1}, t_{m+1} + \delta t) - K),
\end{aligned}$$

where $B(s,t)$ is the discount factor computed at s for maturity t. We have that

$$\underbrace{X(t_{m+1})}_{>t_m} = \max(\underbrace{X(t_{m+1})}_{>t_{m+1}}, Y(t_{m+1})),$$
$$\underbrace{X(t_m)}_{>t_m} = \mathcal{R}_{t_m}(\underbrace{X(t_{m+1})}_{>t_m}).$$

Step 3 (After the loop). We return $X(t_0) = \underbrace{X(t_0)}_{>t_0}$.

Swing option

K : the strike.

$(t_n)_{n=1,\dots,N}$: the exercise times.

M : the maximal number of exercises, $M \leq N$.

A holder of the option is given the right to purchase M stocks at price K per share. The transactions take place at exercise times. Only *one* stock can be bought at a particular exercise time, that is, to get n stocks the holder should use n *different* exercise times. Such options are actively traded on energy markets.

Algorithm. The event times are

$$\{t_0, (t_n)_{n=1,\dots,N}\},$$

where t_0 is the initial time. We divide the algorithm into 3 steps.

Step 1 (Boundary condition).

$$X_m(t_N) = \underbrace{X_m(t_N)}_{>t_N} = 0, \quad m = 0, 1, \dots, M-1.$$

Step 2 (Loop). We enter the loop at t_N (included) and exit at t_0 (not included):

$$\underbrace{t_0}_{\text{end}} \longleftarrow \underbrace{t_N}_{\text{begin}}.$$

We consider the iteration:

$$\underbrace{X(t_n)}_{?} \longleftarrow \underbrace{X(t_{n+1})}_{\text{known}},$$

where

$$X(t_{n+1}) = (X_m(t_{n+1}))_{m=0,\dots,M-1}$$

and

$$X_m(t_{n+1}) = \underbrace{X_m(t_{n+1})}_{>t_{n+1}}$$

is swing's value if m exercises were made before and at t_{n+1}. We have that

$$\begin{aligned}\underbrace{X_m(t_{n+1})}_{>t_n} &= \max(\underbrace{X_m(t_{n+1})}_{>t_{n+1}}, \underbrace{X_{m+1}(t_{n+1})}_{>t_{n+1}} + S(t_{n+1}) - K),\\ & m = 0, 1, \dots, M-2,\\ \underbrace{X_{M-1}(t_{n+1})}_{>t_n} &= \max(\underbrace{X_{M-1}(t_{n+1})}_{>t_{n+1}}, S(t_{n+1}) - K),\end{aligned}$$

and then that

$$\underbrace{X_m(t_n)}_{>t_n} = \mathcal{R}_{t_n}(\underbrace{X_m(t_{n+1})}_{>t_n}), \quad m = 0, 1, \dots, M-1.$$

Step 3 (After the loop). We return $X_0(t_0) = \underbrace{X_0(t_0)}_{>t_0}$.

Barrier up-or-down-and-out option

$U :$ the upper barrier.

$L :$ the lower barrier.

$(t_m)_{m=1,\dots,M} :$ the barrier times.

$N :$ the notional.

The payoff of the option at maturity (last barrier time t_M) is given by notional amount N if the stock price stays between the lower and upper barriers for all barrier times. Otherwise, the option expires worthless.

Algorithm. The event times are

$$\{t_0, (t_m)_{m=1,\dots,M}\},$$

where t_0 is the initial time. We divide the algorithm into 3 steps.

Step 1 (Boundary condition).

$$X(t_M) = \underbrace{X(t_M)}_{>t_M} = N.$$

Step 2 (Loop). We enter the loop at t_M (included) and exit at t_0 (not included):

$$\underbrace{t_0}_{\text{end}} \longleftarrow \underbrace{t_M}_{\text{begin}}.$$

We consider the iteration:

$$\underbrace{X(t_m)}_{?} \longleftarrow \underbrace{X(t_{m+1})}_{\text{known}},$$

where

$$X(t_{m+1}) = \underbrace{X(t_{m+1})}_{>t_{m+1}}$$

is the value to continue (no barriers were crossed before and at t_{m+1}). We have that

$$\underbrace{X(t_{m+1})}_{>t_m} = \underbrace{X(t_{m+1})}_{>t_{m+1}} 1_{\{S(t_{m+1})>L\}} 1_{\{U>S(t_{m+1})\}},$$

where $S(t)$ is the price of the stock at t, and then that

$$\underbrace{X(t_m)}_{>t_m} = \mathcal{R}_{t_m}(\underbrace{X(t_{m+1})}_{>t_m}).$$

Step 3 (After the loop). We return $X(t_0) = \underbrace{X(t_0)}_{>t_0}$.

Down-and-out american call

$L:$ the lower barrier.

$(u_i)_{i=1,\dots,N_1}:$ the barrier times.

$K:$ the strike.

$(v_i)_{i=1,\dots,N_2}:$ the exercise times, $v_{N_2} > u_{N_1}$.

The option behaves as the american call option with strike K and exercise times (v_i) until the first barrier time when the stock price hits lower barrier L. At this exit time the option is canceled.

Algorithm. The event times are

$$\{t_0, (t_m)_{m=1,\dots,M}\},$$

where t_0 is the initial time and $(t_m)_{m=1,\dots,M}$ is the sorted (strictly increasing) union of the barrier and exercise times. We divide the algorithm into 3 steps.

Step 1 (Boundary condition).

$$X(t_M) = \underbrace{X(t_M)}_{>t_M,>t_M} = 0.$$

Step 2 (Loop). We enter the loop at t_M (included) and exit at t_0 (not included):

$$\underbrace{t_0}_{\text{end}} \longleftarrow \underbrace{t_M}_{\text{begin}}.$$

We consider the iteration:

$$\underbrace{X(t_m)}_{?} \longleftarrow \underbrace{X(t_{m+1})}_{\text{known}},$$

where

$$X(t_{m+1}) = \underbrace{X(t_{m+1})}_{>t_{m+1},>t_{m+1}}$$

is the value to continue (no barriers were crossed and exercises made before and at t_{m+1}). If t_{m+1} is both exercise and barrier time, then the barrier event takes precedence. There are two possibilities:

1. If t_{m+1} is an exercise time, then

$$\underbrace{X(t_{m+1})}_{>t_{m+1},>t_m} = \max(\underbrace{X(t_{m+1})}_{>t_{m+1},>t_{m+1}}, S(t_{m+1}) - K),$$

   where $S(t)$ is the price of the stock at t.

2. If t_{m+1} is a barrier time, then

$$\underbrace{X(t_{m+1})}_{>t_m,>t_m} = \underbrace{X(t_{m+1})}_{>t_{m+1},>t_m} 1_{\{S(t_{m+1})>L\}}.$$

Finally, we have that

$$\underbrace{X(t_m)}_{>t_m,>t_m} = \mathcal{R}_{t_m}(\underbrace{X(t_{m+1})}_{>t_m,>t_m}),$$

Step 3 (After the loop). We return $X(t_0) = \underbrace{X(t_0)}_{>t_0,>t_0}$.