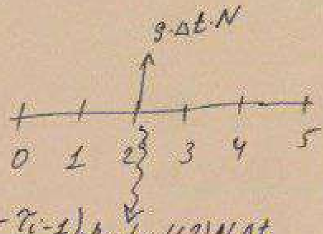
10.06.2022 матком. пробный экзамен

① $L_0(\tau_i;\tau_{i+1})\Big|_{i=0}^{5} = \frac{1}{100+i}$

а) найти swap rate для $\tau_2 \dots \tau_5$.



$$\sum_{i=2}^{5} S\cdot \underset{1}{\underset{\shortparallel}{\Delta t}}\cdot \cancel{N}\cdot B_0(\tau_i) = \sum_{i=2}^{5} \cancel{N} \underbrace{L_{\tau_{i-1}}(\tau_{i-1},\tau_i)\cdot(\tau_i-\tau_{i-1})\cdot B_0(\tau_i)}_{}$$

$$L_1(1,2)N\cdot\Delta t$$

$$\shortparallel\; \underbrace{L_0(\tau_{i-1},\tau_i)\cdot(\tau_i-\tau_{i-1})\cdot B_0(\tau_i)}$$

$$\shortparallel\; \underbrace{B_0(\tau_{i-1}) - B_0(\tau_i)}$$

$$\shortparallel\; B_0(1) - B_0(5)$$

$$\Rightarrow \boxed{S = \frac{B_0(1)-B_0(5)}{\sum_{i=2}^{5} B_0(\tau_i)} = \frac{B_0(1)-B_0(5)}{B_0(2)+B_0(3)+B_0(4)+B_0(5)}}$$

но:

$$L_t(\tau,\tau') = \frac{B_t(\tau)-B_t(\tau')}{(\tau'-\tau)\cdot B_t(\tau')}$$

$$L_0(\tau_{i-1},\tau_i) = \frac{B_0(\tau_{i-1})-B_0(\tau_i)}{(\tau_i-\tau_{i-1})B_0(\tau_i)} = \frac{B_0(\tau_{i-1})}{B_0(\tau_i)} - 1.$$

$$\Rightarrow \frac{B_0(\tau_{i-1})}{B_0(\tau_i)} = 1 + L_0(\tau_{i-1},\tau_i)$$

$$\Rightarrow \boxed{B_0(\tau_i) = \frac{B_0(\tau_{i-1})}{1+L_0(\tau_{i-1},\tau_i)}}$$

$$L_0(0,1) = \frac{\overset{=1}{B_0(0)}}{B_0(1)} - 1 \Rightarrow B_0(1) = \frac{B_0(0)}{1+L_0(0,1)} = \frac{1}{1+\frac{1}{100}} = \frac{1}{101/100} = \frac{100}{101}$$

$$L_0(1,2) = \frac{B_0(1)}{B_0(2)} - 1 = \cancel{100/101} \Rightarrow B_0(2) = \frac{B_0(1)}{1+\frac{1}{101}} = \frac{100/101}{102/101} = \frac{100}{102}$$

$$\cancel{L_0(2,3)} \quad B_0(3) = \frac{B_0(2)}{1+\frac{1}{102}} = \frac{100/102}{103/102} = \frac{100}{103}$$

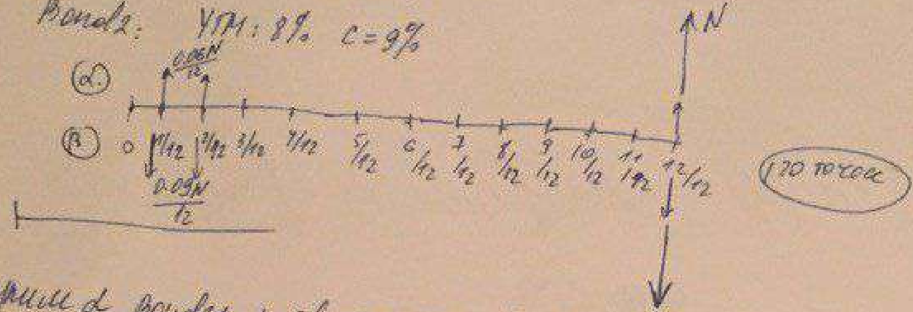
$$B_0(4) = \frac{100}{104}$$

$$B_0(5) = \frac{100}{105}$$

$$\Rightarrow S = \frac{B_0(1)-B_0(5)}{\sum_{i=2}^{5} B_0(\tau_i)} = \frac{\frac{100}{101}-\frac{100}{105}}{\frac{100}{102}+\frac{100}{103}+\frac{100}{104}+\frac{100}{105}} = \boxed{\frac{4/101\cdot 105}{\frac{1}{102}+\frac{1}{103}+\frac{1}{104}+\frac{1}{105}}}$$

(3) Bond 1: YTM: 7% $C=6\%$

Bond 2: YTM: 8% $C=9\%$



купим α Bonds 1 и продадим β Bonds 2.

$$\Rightarrow \begin{cases} \alpha \cdot \frac{0.06N}{12} = \beta \cdot \frac{0.09 \cdot N}{12} \Rightarrow 2\alpha = 3\beta \Rightarrow \alpha = \frac{3\beta}{2} \\ FV = (\alpha - \beta)N = 0.5\beta \end{cases}$$

$$PV = \underset{\frac{3\beta}{2}}{\alpha} \cdot PV_1 - \beta \cdot PV_2$$

$$\Rightarrow \frac{PV}{FV} = \frac{\frac{3\beta}{2}PV_1 - \beta \cdot PV_2}{0.5\beta} = 3PV_1 - 2PV_2, \text{ где } PV_1 = N \cdot \left(e^{-\frac{0.07}{12}}\right)^{120} + \sum_{i=1}^{120} \left(\frac{0.06}{12} \cdot N\right) \cdot \left(e^{-\frac{0.07}{12}}\right)^i$$

$$PV_2 = N \cdot \left(e^{-\frac{0.08}{12}}\right)^{120} + \sum_{i=1}^{120} \left(\frac{0.09}{12} \cdot N\right) \cdot \left(e^{-\frac{0.08}{12}}\right)^i$$

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(4) $\begin{cases} dr_t = \theta dt + \tilde{\sigma} dW_t \\ r_0 = r_0^* \end{cases}$ $\qquad B_t = B_0 \cdot e$

$\Rightarrow r_t = r_0^* + \theta t + \tilde{\sigma} W_t$

a) $\begin{cases} df_t(\tau) = \alpha(t,\tau)dt + \sigma_t(\tau)dW_t \\ f_0(\tau) = f_0^*(\tau) \end{cases}$

ну $B_t(\tau) = E_t^Q\left[e^{-\int_t^\tau r_s ds}\right]$

$$-\int_t^\tau r_s ds = -r_0^*(\tau - t) - \theta \cdot \frac{(\tau-t)^2}{2} - \tilde{\sigma} \underbrace{\int_t^\tau W_s ds}_{W_\tau \tau - W_t \cdot t - \int_t^\tau s dW_s} =$$

$$B_t(\tau) = e^{-\theta \cdot \int_t^\tau (\tau - u)du + \frac{\sigma^2}{2}\frac{(\tau-t)^3}{3} - (\tau - t)r_t} \quad - \text{см. стр. 50}$$

$$\theta \frac{(t-\tau)^2}{2}\Big|_t^\tau = -\theta\frac{(t-\tau)^2}{2}$$

$$B_t(\tau) = e^{-\theta\frac{(t-\tau)^2}{2} + \frac{\sigma^2}{2}\frac{(\tau-t)^3}{3} - (\tau-t)r_t}$$

$$f(t,x) = e^{-\theta\frac{(t-\tau)^2}{2} + \frac{\sigma^2}{2}\frac{(\tau-t)^3}{3} - (\tau-t)x}$$

$$dB_t(\tau) = \left(-\theta(t-\tau) - \frac{\sigma^2}{2}(\tau-t)^2 + r_t\right)B_t(\tau)dt - (\tau-t) \cdot B_t(\tau)\underset{\theta dt + \tilde{\sigma} dW_t}{dr_t} + \frac{(\tilde{\sigma})^2}{2}(\tau-t)^2 \cdot B_t(\tau)dt =$$

$$= B_t(\tau)\left(-\theta(t-\tau) - \frac{\sigma^2}{2}(\tau-t)^2 + r_t + \frac{(\tilde{\sigma})^2}{2}(\tau-t)^2 - (\tau-t)\theta\right)dt - (\tau-t)B_t(\tau)\tilde{\sigma}dW_t$$

б) Найти $y_0(\tau_k)$; $k=1\ldots5$.

$B_0(t)=e^{-y_0(t)\cdot t}$

$\Rightarrow \ln B_0(t)=-y_0(t)\cdot t$

$\Rightarrow y_0(\tau)=-\dfrac{\ln B_0(\tau)}{\tau}$

Но $B_0(\tau_k)=\dfrac{100}{100+k}$

$\Rightarrow \boxed{y_0(\tau_k)=-\dfrac{\ln\left(\frac{100}{100+k}\right)}{k}}$

в) $L_t(\tau_i;\tau_{i+1})=?$; $t=3$, $i=4$

т.е. $L_3(4;5)=?$

$\Rightarrow L_3(4,5)=L_0(4,5)=\boxed{\dfrac{1}{104}}$

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2. $\begin{cases} dr_t=\theta dt+\sigma^r dW_t \\ r_0=r_0^* \end{cases}$ $\qquad B_t(\tau)=E^Q\left[e^{-\int_t^\tau r_s ds}\right]$

а) $r_t=r_0^*+\theta t+\sigma^r W_t$

б) $B_0^*(\tau)=E^Q\left[e^{-\int_0^\tau r_s ds}\right]=e^{-\int_0^T f_0(s)ds}$

Ну $-\int_0^\tau r_s ds=-\int_0^\tau (r_0^*+\theta s+\sigma^r W_s)ds=-r_0^*\tau-\theta\cdot\frac{s^2}{2}\Big|_0^\tau-\sigma^r\underbrace{\int_0^\tau W_s ds}=-r_0^*\tau-\theta\frac{\tau^2}{2}-\sigma^r\int_0^\tau(\tau-s)dW_s$

$\underbrace{\int_0^\tau W_s ds}=\tau\cdot W_\tau-\int_0^\tau s\,dW_s=$

$=\int_0^\tau(\tau-s)dW_s$

$\Rightarrow \mu_\xi=-r_0^*\tau-\dfrac{\theta\cdot\tau^2}{2}$

$\sigma_\xi^2=(\sigma^r)^2\int_0^\tau(\tau-s)^2ds=(\sigma^r)^2\cdot\dfrac{(s-\tau)^3}{3}\Big|_0^\tau=(\sigma^r)^2\cdot\dfrac{\tau^3}{3}$

$\Rightarrow -\int_0^\tau r_s ds\sim N(\mu_\xi;\sigma_\xi)$

$\Rightarrow E^Q\left[e^{-\int_0^\tau r_s ds}\right]=Ee^{\xi}=e^{\mu_\xi+\frac{\sigma_\xi^2}{2}}=\boxed{e^{-r_0^*\tau-\frac{\theta\cdot\tau^2}{2}+\frac{(\sigma^r)^2\cdot\tau^3}{6}}}$

c) $f_0^*(\tau)=-\dfrac{\partial}{\partial\tau}\ln(B_0^*(\tau))=-\dfrac{\partial}{\partial\tau}\left(-r_0^*\tau-\dfrac{\theta\tau^2}{2}+\dfrac{(\sigma^r)^2\tau^3}{6}\right)=\boxed{r_0^*+\theta\tau-\dfrac{(\sigma^r)^2\cdot\tau^2}{2}}$

$$\Rightarrow \boxed{dB_t(\tau) = B_t(\tau) r_t dt - (\tau - t) B_t(\tau)\, \sigma^r dW_t}$$

$$\cancel{B_t(\tau) = e^{-\int_t^\tau f_t(s)ds}}$$

$$d X_t(\tau) = -\underbrace{f_t(t)}_{r_t} dt + \int_t^\tau d f_t(s) ds = -r_t dt + \left(\int_t^\tau \alpha_t(s) ds\right) dt + \left(\int_t^\tau \sigma_t(s) ds\right) dW_t$$

$$= -r_t dt$$

Знаем:

Если дано $df_t(\tau) = \alpha_t(\tau)dt + \sigma_t(\tau) dW_t$,

то $\boxed{dB_t(\tau) = B_t^{(\tau)} r_t dt - B_t^{(\tau)} \left(\int_t^\tau \sigma_t(s) ds\right) dW_t}$

$$\Rightarrow \begin{cases} \alpha_t(\tau) = r_t \\ \int_t^\tau \sigma_t(s) ds = (\tau - t)\cdot \sigma^r \end{cases}$$

$$\Rightarrow \sigma_t(\tau) = \sigma^r$$

Ответ: $\alpha_t(\tau) = r_t$
$\sigma_t(\tau) = \sigma^r$.

б) $dS_t = S_t(r_t dt + \sigma^S dW_S)$

$$dQ^S = \underbrace{\frac{S_T \cdot B_0}{B_T \cdot S_0}}_{L_t} dQ$$

$$L_t = \frac{S_t \cdot B_0}{B_t \cdot S_0} \quad \begin{cases} dS_t = S_t(r_t dt + \sigma^S dW_t) \\ dB_t = r_t B_t dt \end{cases} \Rightarrow d\left(\frac{S_t}{B_t}\right) = \left(\frac{S_t}{B_t}\right) \sigma_t^S dW_t$$

$$\Rightarrow dL_t = \sigma^S L_t dW_t$$

$$= -\mu_S = \sigma^S$$

$$\Rightarrow \tilde{W}_t^S = W_t + \int_0^t \mu_s ds = W_t - \sigma^S \cdot t$$

$$\Rightarrow W_t = \tilde{W}_t^S + \sigma^S t$$

$$\Rightarrow dW_t = d\tilde{W}_t^S + \sigma^S dt$$

$$\Rightarrow dr_t = \theta dt + \sigma^r dW_t = \theta dt + \sigma^r (d\tilde{W}_t^S + \sigma^S dt) = (\theta + \sigma^r \sigma^S) dt + \sigma^r d\tilde{W}_t^S$$

$$r_t = r_0^* + (\theta + \sigma^r \sigma^S) t + \sigma^r \tilde{W}_t^S$$

$$\Rightarrow \boxed{E^{Q^S}[r_t] = r_0^* + (\theta + \sigma^r \sigma^S) t}$$

(5.) $B_0^*(\tau) = \frac{1}{1+c\tau}$

$\sigma_t(\tau) = \sigma\sqrt{\tau - t}$ - в НУМ

а) $\alpha_t(\tau) = \sigma_t(\tau)\int_t^\tau \sigma_t(s)ds = \sigma^2\sqrt{\tau-t}\int_t^\tau \sqrt{s-t}\,ds = \sigma^2\sqrt{\tau-t}\cdot(s-t)^{3/2}\cdot\frac{2}{3}\Big|_t^\tau = \boxed{\frac{2}{3}\sigma^2(\tau-t)^2}$

б) $f_0^*(\tau) = ?$

$\text{т.к. } B_0^*(\tau) = e^{-\int_0^\tau f_0(s)ds}$

$\Rightarrow -\ln B_0^*(\tau) = \int_0^\tau f_0(s)ds$

$\Rightarrow f_0^*(\tau) = -\frac{\partial}{\partial\tau}\ln B_0^*(\tau) = -\frac{\partial}{\partial\tau}\ln\left(\frac{1}{1+c\tau}\right) = \frac{\partial}{\partial\tau}\ln(1+c\tau) = \boxed{\frac{c}{1+c\tau}}$

с) найти $\sigma_t^B(\tau)$ для $dB_t(\tau) = B_t(\tau)(r_t dt + \sigma_t^B(\tau)dW_t)$

~~т.к. $B_t(\tau) = e^{-\int_t^\tau f_t(s)ds}$~~

$\text{т.к. } \sigma_t^B(\tau) = -\int_t^\tau \sigma_t(s)ds = -\int_t^\tau \sigma\sqrt{s-t}\,ds = -\sigma\cdot\frac{2}{3}(s-t)^{3/2}\Big|_t^\tau = \boxed{-\frac{2}{3}\sigma(\tau-t)^{3/2}}$