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Introduction to the Arbitrage Theory

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Arbitrage Theory for Frictionless Markets

1.1 Models without Friction

1.1.1 DMW Theorem

The classical result by Dalang–Morton–Willinger, usually abbreviated as DMW and sometimes referred to as the Fundamental Theory of Asset (or Arbitrage) Pricing (FTAP) for the discrete finite time model of a frictionless financial market, says:

there is no arbitrage if and only if there is an equivalent martingale measure.

This formulation is due to Harrison–Pliska who established it for a model with finite number of states of the nature, i.e., for finite Ω. Retrospectively, one can insinuate that in this case it is mainly a "linguistic" exercise: the result expressed in geometric language was known a long time ago as the Stiemke lemma. This is, to large extent, true. But the remarkable fact is that, contrarily to its predecessors, exactly this formulation of a no-arbitrage criterion, involving an important probability concept, a martingale measure, opens a way to numerous generalizations of great theoretical and practical value.

Loosely speaking, the result can be viewed as a partial converse to the assertion that one cannot win (in finite time) by betting on a martingale: if one cannot win betting on a process, the latter is a martingale with respect to an equivalent martingale measure.

We start our presentation here with a detailed analysis of the Dalang–Morton–Willinger theorem. The assertion in italics is, in fact, a *grande public* formulation which hides a profound difference between these two results and authors of advanced textbooks prefer to give a longer list of NA criteria. We follow this tradition.

The model is given by a complete probability space $(\Omega, \mathcal{F}, P)$ with a discrete-time filtration $\mathbf{F} = (\mathcal{F}_t)_{t=0,1,\ldots,T}$ and an adapted d-dimensional price process $S = (S_t)$ with the constant first component. It is convenient to assume that $\mathcal{F}_0$ is trivial and $\mathcal{F}_T = \mathcal{F}$.

The set of "results" R_T (obtained from zero starting value) consists of the terminal values of discrete-time integrals

$$H \cdot S_T := \sum_{t=1}^{T} H_t \Delta S_t$$

where $\Delta S_t := S_t - S_{t-1}$ and H runs the linear space $\mathcal{P}$ of predictable processes, i.e., $H_t \in L^0(\mathbf{R}^d, \mathcal{F}_{t-1})$ (the first component of S plays no role here because $\Delta S_t^1 = 0$).

The common terminology: H is a (portfolio) *strategy* while $H \cdot S$ is called *value process*. The larger set $A_T := R_T - L^0_+$ can be interpreted as the set of *hedgeable claims*; it is the set of random variables $H \cdot S_T - h$ where the r.v. $h \geq 0$.

By definition, the *NA property* of the model means that $R_T \cap L^0_+ = \{0\}$ (or, equivalently, $A_T \cap L^0_+ = \{0\}$). We prefer to use from the very beginning these mathematically convenient definitions in terms of intersections of certain sets rather than a popular form like this: the property $H \cdot S_T \geq 0$ implies that $H \cdot S_T = 0$.

Theorem 1.1.1 *The following properties are equivalent:*

(a) $A_T \cap L^0_+ = \{0\}$ *(NA condition);*
(b) $A_T \cap L^0_+ = \{0\}$ *and* $A_T = \bar{A}_T$ *(closure in* L^0*);*
(c) $\bar{A}_T \cap L^0_+ = \{0\}$*;*
(d) *there is a strictly positive process* $\rho \in \mathcal{M}$ *such that* $\rho S \in \mathcal{M}$*;*
(e) *there is a bounded strictly positive process* $\rho \in \mathcal{M}$ *such that* $\rho S \in \mathcal{M}$*.*

As usual, $\mathcal{M}$ is the space of martingales (if necessary, we shall use also more complicated notations showing the probability, time range, etc).

Of course, the last two properties are usually formulated as:

(d') *there is a probability* $\tilde{P} \sim P$ *such that* $S \in \mathcal{M}(\tilde{P})$*;*
(e') *there is a probability* $\tilde{P} \sim P$ *with* $d\tilde{P}/dP \in L^\infty$ *such that* $S \in \mathcal{M}(\tilde{P})$*.*

However, the chosen versions have more direct analogs in the model with transaction costs. Their equivalences are obvious due to the following elementary fact about martingales with respect to a probability measure $\tilde{P} \ll P$ with the density ρ_T:

$S \in \mathcal{M}(\tilde{P})$ *if and only if* $\rho S \in \mathcal{M}(P)$ *where* $\rho_t = E(\rho_T | \mathcal{F}_t)$, $t \leq T$.

Collecting conditions in the single theorem is useful because one can see clearly that in numerous generalizations and ramifications certain properties remain equivalent (of course, appropriately modified) but others do not. Note also that in the case of finite Ω the set A_T is always closed. Indeed, it is the arithmetic sum of a linear space and the polyhedral cone $-L^0_+$ in the finite dimensional linear space L^0. Thus, it is a polyhedral cone. So, we have no difference between the first three properties while the last two coincides trivially. The situation is completely different for arbitrary Ω. Though the linear space R_T is always closed (we show this later), the set A_T may be not closed even for $T = 1$ and a countable Ω. To see this, let $\mathcal{F}_0$ be trivial and take $\mathcal{F}_1 = \sigma\{\xi\}$, $\Delta S_1 = \xi$, where ξ is a strictly positive finite random variable such that $P(\xi < \varepsilon) > 0$ whatever is $\varepsilon > 0$. The set $A_1 = \mathbf{R}\xi - L^0_+$ does not contain any strictly positive constant but each constant $c > 0$ belong to its closure $\bar{A}_1$ because $(n\xi) \wedge c \to c$ as $n \to \infty$.

One can add to the already long list several other equivalent conditions:

(f) *there is a strictly positive process* $\rho \in \mathcal{M}$ *such that* $\rho S \in \mathcal{M}_{loc}$*;*

(f') *there is a probability* $\tilde{P} \sim P$ *such that* $S \in \mathcal{M}_{loc}(\tilde{P})$*;*

(g) $\{\eta \Delta S_t : \ \eta \in L^0(\mathcal{F}_{t-1})\} \cap L^0_+ = \{0\}$ *for all* $t \le T$ *(NA for 1-step models).*

With other conditions already established, the above addendum poses no problems. Indeed, (f') is obviously implied by (e'). On the other hand, if $S \in \mathcal{M}_{loc}(\tilde{P})$ then $\tilde{H} \cdot S \in \mathcal{M}(\tilde{P})$ with $\tilde{H}_t := 1/(1 + \tilde{E}(|\Delta S_t||\mathcal{F}_{t-1}))$. So, we know that NA holds for the process $\tilde{H} \cdot S$, hence, it holds also for S as both processes have the same set of hedgeable claims, i.e. (f') implies a property from the "main" list of equivalent conditions. Suppose now that the implication $(g) \Rightarrow (a)$ fails. Take the smallest $t \le T$ such that $A_t \cap L^0_+ \ne \{0\}$ (the set of such dates is non-empty: it contains, at least, T). We have a strategy $H = (H_s)_{s \le T}$ such that $H \cdot S_t \ge 0$ and $P(H \cdot S_t > 0) > 0$. Due to the choice of t either the set $\Gamma' := \{H \cdot S_{t-1} < 0\}$ is of strictly positive probability (and (g) is violated by $\eta := I_{\Gamma'} H_t$), or the set $\Gamma'' := \{H \cdot S_{t-1} = 0\}$ is of full measure (and (g) is violated by $\eta := I_{\Gamma''} H_t$). A contradiction.

Remark. The NA property for the class of all strategies, as defined above, is equivalent to the NA property in the narrower class of bounded strategies H. Indeed, if there is an arbitrage opportunity, then, in virtue of the condition (g), there is an arbitrage opportunity η for a certain one-step model. Clearly, $\eta I_{\{|\eta| \le n\}}$ when n is sufficiently large will be an arbitrage opportunity for this one step model. Note that the presence of (g) in the list of equivalent conditions is crucial in this reasoning.

Similarly, NA is equivalent to the absence of arbitrage in the class of so-called *admissible* strategies for which the value processes are bounded from below by constants (depending on the strategy). Moreover, if H is an arbitrage opportunity generating the value process $V = H \cdot S$, one can find another arbitrage opportunity $\tilde{H}$ such that the value process $\tilde{H} \cdot S \ge 0$. To see this, we consider the sets $\Gamma_t := \{H \cdot S_t < 0\}$ and the last instant r for which the probability of such a set is strictly positive; $0 < r < T$ since H is an arbitrage opportunity. Let us check that the strategy $\tilde{H} := I_{\Gamma_r} I_{]r,T]} H$ has the claimed property. Indeed, the process $\tilde{V} := \tilde{H} \cdot S$ is zero for all $t \le r$ and remains zero outside the set Γ_r until T. On the set Γ_r the increments $\Delta \tilde{V}_t = \Delta V_t$ for $t \ge r + 1$ and, hence the trajectories of $\tilde{V}$ are the trajectories of V shifted upwards on the value $-V_r > 0$.

Before the proof of Theorem 1.1.1 we give in the following subsection several elementary results which will be useful also to obtain NA criteria in models with transaction costs.

1.1.2 Auxiliary Results: Measurable Subsequences and the Kreps–Yan Theorem

Lemma 1.1.2 *Let $\eta^n \in L^0(\mathbf{R}^d)$ be such that $\underline{\eta} := \liminf |\eta^n| < \infty$. Then there are $\tilde{\eta}^k \in L^0(\mathbf{R}^d)$ such that for all ω the sequence of $\tilde{\eta}^k(\omega)$ is a convergent subsequence of the sequence of $\eta^n(\omega)$.*

Proof. Define the random variables $\tau_k := \inf\{n > \tau_{k-1} : \ ||\eta^n| - \underline{\eta}| \le k^{-1}\}$ starting with $\tau_0 := 0$. Then $\tilde{\eta}_0^k := \eta^{\tau_k}$ is in $L^0(\mathbf{R}^d)$ and $\sup_k |\tilde{\eta}_0^k| < \infty$. Working further with the sequence of $\tilde{\eta}_0^n$ we construct, applying the above procedure to the first component and its $\liminf$, a sequence of $\tilde{\eta}_1^k$ with convergent first component and such that for all ω the sequence of $\tilde{\eta}_1^k(\omega)$ is a subsequence of the sequence of $\tilde{\eta}_0^n(\omega)$. Passing on each step to the newly created sequence of random variables and to the next component we arrive to a sequence with the desired properties.

Remark. The claim can be formulated as follows: there exists a (strictly) increasing sequence of integer-valued random variables σ_k such that η^{σ_k} converges a.s.

Lemma 1.1.3 *Let $\mathcal{G} = \{\Gamma_\alpha\}$ be a family of measurable sets such that any non-null set Γ has a non-null intersection with an element of $\mathcal{G}$. Then there is an at most countable subfamily of sets $\{\Gamma_{\alpha_i}\}$ which union is of full measure.*

Proof. Suppose that $\mathcal{G}$ is closed under countable unions. Then $\sup_\alpha P(\Gamma_\alpha)$ is attained on some $\tilde{\Gamma} \in \mathcal{G}$. The subfamily consisting of single $\tilde{\Gamma}$ gives the answer. Indeed, $P(\tilde{\Gamma}) = 1$: otherwise we could enlarge the supremum by adding a set from $\mathcal{G}$ having a non-null intersection with $\tilde{\Gamma}^c$. The general case follows by considering a family formed by countable unions of sets from $\mathcal{G}$.

The following result is referred to as the Kreps–Yan theorem. It holds for arbitrary $p \in [1, \infty]$, $p^{-1} + q^{-1} = 1$, but the cases $p = 1$ and $p = \infty$ are the most important. Recall that for $p \ne \infty$ the norm closure of a convex set in L^p coincides with the closure in $\sigma\{L^p, L^q\}$.

Theorem 1.1.4 *Let $\mathcal{C}$ be a convex cone in L^p closed in $\sigma\{L^p, L^q\}$, containing $-L^p_+$ and such that $\mathcal{C} \cap L^p_+ = \{0\}$. Then there is $\tilde{P} \sim P$ with $d\tilde{P}/dP \in L^q$ such that $\tilde{E}\xi \le 0$ for all $\xi \in \mathcal{C}$.*

Proof. By the Hahn–Banach theorem any non-zero $x \in L^p_+ := L^p(\mathbf{R}_+, \mathcal{F})$ can be separated from $\mathcal{C}$: there is a $z_x \in L^q$ such that $Ez_x x > 0$ and $Ez_x \xi \le 0$ for all $\xi \in \mathcal{C}$. Since $\mathcal{C} \supseteq -L^p_+$, the latter property yields that $z_x \ge 0$; we may assume $||z_x||_q = 1$. Let us consider a family $\mathcal{G} := \{z_x > 0\}$. As any non-null set Γ has the non-null intersection with the set $\{z_x > 0\}$, $x = I_\Gamma$, the family

$\mathcal{G}$ contains a countable subfamily of sets (say, corresponding to a sequence $\{x_i\}$) which union is of full measure. Thus, $z := \sum 2^{-i} z_{x_i} > 0$ and we can take $\tilde{P} := zP$.

1.1.3 Proof of the DMW Theorem

Some implications, namely, $(b) \Rightarrow (a)$, $(b) \Rightarrow (c)$, and $(e) \Rightarrow (d)$ are trivial. The implication $(d) \Rightarrow (a)$ is easy. Indeed, let $\xi \in A_T \cap L^0_+$, i.e. $0 \le \xi \le H \cdot S_T$. Since the conditional expectation with respect to the martingale measure $\tilde{E}(H_t \Delta S_t | \mathcal{F}_{t-1}) = 0$, we obtain by consecutive conditioning that $\tilde{E} H \cdot S_T = 0$. Thus, $\xi = 0$. To complete the proof, it remains to verify that $(c) \Rightarrow (e)$ and $(a) \Rightarrow (b)$.

$(c) \Rightarrow (e)$ Notice that for any random variable η there is an equivalent probability P' with bounded density such that $\eta \in L^1(P')$ (e.g., one can take $P' = Ce^{-|\eta|}P$). Property (c) (as well as (a) and (b)) is invariant under equivalent change of probability. This consideration allows us to assume that all S_t are integrable. The convex set $A^1_T := \bar{A}_T \cap L^1$ is closed in L^1. Since $A^1_T \cap L^1_+ = \{0\}$, Theorem 1.1.4 ensures the existence of $\tilde{P} \sim P$ with bounded density and such that $\tilde{E}\xi \le 0$ for all $\xi \in A^1_T$, in particular, for $\xi = \pm H_t \Delta S_t$ where H_t is bounded and $\mathcal{F}_{t-1}$-measurable. Thus, $\tilde{E}(\Delta S_t | \mathcal{F}_{t-1}) = 0$.

$(a) \Rightarrow (b)$ Lemma 1.1.2 allows us to establish the closedness of A_T by simple recursive arguments even without assuming that the σ-algebra $\mathcal{F}_0$ is trivial (of course, this does not add any generality but helps to start the induction in the time variable).

Let us consider the case $T = 1$. Let $H^n_1 \Delta S_1 - r^n \to \zeta$ a.s., where H^n_1 is $\mathcal{F}_0$-measurable and $r^n \in L^0_+$. The closedness of A_1 means that $\zeta = H_1 \Delta S_1 - r$ for some $\mathcal{F}_0$-measurable H_1 and $r \in L^0_+$. To show this we represent each H^n_1 as a column vector and write the whole sequence of these column vectors as the infinite matrix

$$\mathbf{H}_1 := \begin{bmatrix} H_1^{11} & H_1^{21} & \dots & \dots & H_1^{n1} & \dots \\ H_1^{12} & H_1^{22} & \dots & \dots & H_1^{n2} & \dots \\ \dots & \dots & \dots & \dots & \dots & \dots \\ H_1^{1d} & H_1^{2d} & \dots & \dots & H_1^{nd} & \dots \end{bmatrix}.$$

If the matrix is zero, there is nothing to prove. Suppose that the assertion holds when this (random) matrix has, for each ω, at least m zero lines. We show that the claim holds true also when $\mathbf{H}_1$ has at least $m-1$ zero lines.

It is sufficient to find $\mathcal{F}_0$-measurable random variables $\tilde{H}^k_1$ convergent a.s. and $\tilde{r}^k \in L^0_+$ such that $\tilde{H}^k_1 \Delta S_1 - \tilde{r}^k \to \zeta$ a.s.

Let $\Omega_i \in \mathcal{F}_0$ form a finite partition of Ω. An important (though obvious) observation: we may argue on each Ω_i separately as on an autonomous measure space (considering the restrictions of random variables and traces of σ-algebras).

Let $\underline{H}_1 := \liminf |H_1^n|$. On $\Omega_1 := \{\underline{H}_1 < \infty\}$ we take, using Lemma 1.1.2, $\mathcal{F}_0$-measurable $\tilde{H}_1^k$ such that $\tilde{H}_1^k(\omega)$ is a convergent subsequence of $H_1^n(\omega)$ for every ω; $\tilde{r}^k$ are defined correspondingly. Thus, if Ω_1 is of full measure, the goal is achieved.

On $\Omega_2 := \{\underline{H}_1 = \infty\}$ we put $G_1^n := H_1^n/|H_1^n|$ and $h_1^n := r_1^n/|H_1^n|$. Clearly, $G_1^n \Delta S_1 - h_1^n \to 0$ a.s. By Lemma 1.1.2 we find $\mathcal{F}_0$-measurable $\tilde{G}_1^k$ such that $\tilde{G}_1^k(\omega)$ is a convergent subsequence of $G_1^n(\omega)$ for every ω. Denoting the limit by $\tilde{G}_1$, we obtain that $\tilde{G}_1 \Delta S_1 = \tilde{h}_1$ where $\tilde{h}_1$ is non-negative, hence, in virtue of (a), $\tilde{G}_1 \Delta S_1 = 0$.

As $\tilde{G}_1(\omega) \neq 0$, there exists a partition of Ω_2 into d disjoint subsets $\Omega_2^i \in \mathcal{F}_0$ such that $\tilde{G}_1^i \neq 0$ on Ω_2^i. Define $\bar{H}_1^n := H_1^n - \beta^n \tilde{G}_1$ where $\beta^n := H_1^{ni}/\tilde{G}_1^i$ on Ω_2^i. Then $\bar{H}_1^n \Delta S_1 = H_1^n \Delta S_1$ on Ω_2. The matrix $\bar{\mathbf{H}}_1$ has, for each $\omega \in \Omega_2$, at least m zero lines: our operations did not affect the zero lines of $\mathbf{H}_1$ and a new one has appeared, namely, the ith one on Ω_2^i. We conclude by the induction hypothesis.

To establish the induction step in the time variable we suppose that the claim is true for $(T-1)$-step models. Let $\sum_{t=1}^T H_t^n \Delta S_t - r^n \to \zeta$ a.s., where H_t^n are $\mathcal{F}_{t-1}$-measurable and $r^n \in L_+^0$. As at the first step, we work with the matrix $\mathbf{H}_1$ using exactly the same reasoning.

On Ω_1 we take an increasing sequence of $\mathcal{F}_0$-random variables τ_k such that $H^k := H_1^{\tau_k}$ converges to H_1. Thus, $\sum_{t=2}^T H_t^{\tau_k} \Delta S_t - r^{\tau_k}$ converges as $k \to \infty$ and we have a reduction to a $(T-1)$-step model.

On Ω_2 we use again the same induction in m, the number of zero lines of $\mathbf{H}_1$. The only modification is that the identical operations (passage to subsequences, normalization by H_1^n, etc.) should be performed simultaneously over all other matrices $\mathbf{H}_2$, ..., $\mathbf{H}_T$.

Remark 1. Exactly the same arguments as were used in the proof of the implication $(a) \Rightarrow (b)$ lead to the following assertion referred to as the Stricker lemma:

The set of results R_T is closed.

This property holds irrelevantly of the NA-condition. Indeed, the latter was used only to check that the nonnegative limit $\tilde{h}_1$ is, in fact, equal to zero. But this holds automatically if we start the arguments with $r_n = 0$.

Remark 2. The DMW theorem contains as a corollary the assertion that in the discrete-time setting with finite horizon any local martingale is a martingale with respect to a measure $\tilde{P} \sim P$ with bounded density. Moreover, this measure can be chosen in such a way that a given random variable ξ will be $\tilde{P}$-integrable. At the end of this chapter we show that even in the model with infinite horizon the local martingale is a martingale with respect to an equivalent probability measure.

1.1.4 Fast Proof of the DMW Theorem

Our detailed formulation of the DMW theorem together with its proof is intended to prepare the reader to the arguments developed for models with transactions costs. However, a short and elementary proof of the "main" equivalence $(a) \Leftrightarrow (e)$, a proof which can be used in introductory courses for mathematical students, is of separate interest. We give one here combining an optimization approach due to Chris Rogers with Lemma 1.1.2 on measurable subsequences. It is based on the one-step result the first condition of which is just an alternative reformulation of the NA-property.

Proposition 1.1.5 *Let $\xi \in L^0(\mathbf{R}^d)$ and let $\mathcal{G}$ be a sub-σ-algebra of $\mathcal{F}$. Then the following conditions are equivalent:*

(i) for any $\alpha \in L^0(\mathbf{R}^d, \mathcal{G})$ the inequality $\alpha\xi \geq 0$ holds as the equality;

(ii) there exists a bounded random variable $\varrho > 0$ such that $E\varrho|\xi| < \infty$ and $E(\varrho\xi|\mathcal{G}) = 0$.

Proof. One needs arguments only for the "difficult" implication $(i) \Rightarrow (ii)$.

First, examine the case where $\mathcal{G}$ is trivial. Let us consider the function $f(a) = Ee^{a\xi - |\xi|^2}$, $a \in \mathbf{R}^d$. If it attains its minimum at some point a_*, the problem is solved with $\rho = e^{a_*\xi - |\xi|^2}$, since at this point the derivative of f is zero: $E\xi e^{a_*\xi - |\xi|^2} = 0$. One can check that the condition (i) excludes the possibility that the minimum is not attained — we do a verification below.

Let us turn to the general case. A dimension reduction argument allows us to work assuming that the relation $\alpha\xi = 0$ with $\alpha \in L^0(\mathbf{R}^d, \mathcal{G})$ holds only if $\alpha = 0$ (when $\mathcal{G}$ is trivial this is just the linear independence of the components of ξ as elements of L^0). Let $Q(\omega, dx)$ be the regular conditional distribution of ξ with respect to $\mathcal{G}$. Define the function

$$f(\omega, a) := \int e^{ax - |x|^2} Q(\omega, dx)$$

continuous in a and $\mathcal{G}$-measurable in ω. Introduce the $\mathcal{G}$-measurable random variable $f_*(\omega) = \inf_a f(\omega, a)$ and consider in the product space $\Omega \times \mathbf{R}^d$ the sets $\{(\omega, a) : f(\omega, a) < f_*(\omega) + 1/n\}$ with non-empty open ω-sections $\Gamma_n(\omega)$. Let α_n be a $\mathcal{G}$-measurable random variable with $\alpha_n(\omega) \in \Gamma_n(\omega)$. Such α_n can be constructed easily, without appealing to a measurable selection theorem, e.g., one can take $\alpha_n(\omega) := q_{\theta(n)}$ where

$$\theta(n) := \min\{k : \; f(\omega, q_k) < f_*(\omega) + 1/n\}$$

where $\{q_n\}$ is an arbitrary countable dense subset in $\mathbf{R}^d$. Let us consider the set $\Omega_0 := \{\liminf |\alpha_n| < \infty\}$ with its complement Ω_1. Using Lemma 1.1.2 we may assume that on Ω_1 the sequence $\tilde{\alpha}_n := \alpha_n/|\alpha_n|$ converges to some β with $|\beta| = 1$ and, by the Fatou lemma,

$$\int e^{\lim|\alpha_n(\omega)|\beta(\omega)x-|x|^2} I_{\{\beta(\omega)x\neq 0\}} Q(\omega,dx)$$
$$\leq \liminf \int e^{\alpha_n(\omega)x-|x|^2} I_{\{\beta(\omega)x\neq 0\}} Q(\omega,dx) \leq f_*(\omega).$$

Necessarily, $Q(\omega,\{x:\ \beta(\omega)x>0\})=0$ implying that $\beta\xi\leq 0$ (a.s.) and, therefore, in virtue of (i), we have that $\beta\xi=0$. Due to our provision this equality holds only if $\beta=0$ and, hence, Ω_1 is a null set which does not matter. Again by Lemma 1.1.2 we may assume that on the set Ω_0 of full measure the sequence $\alpha_n(\omega)$ converges to some $\alpha_*(\omega)$. Clearly, $f(\omega,a)$ attains its minimum at $\alpha_*(\omega)$ and we conclude with $\varrho:=e^{\alpha_*\xi-|\xi|^2}/c(\alpha_*)$ where the function $c(a):=\sup_x(1+|x|)e^{ax-|x|^2}$. $\square$

The "difficult" implication $(a)\Rightarrow(e)$ follows from the above proposition by backward induction. We claim that for each $t=0,1,...,T-1$ there is a bounded random variable $\rho_t^T>0$ such that $E\rho_t^T|\Delta S_u|<\infty$ and $E\rho_t^T\Delta S_u=0$ for $u=t+1,...,T$. Since (a) implies the NA-property for each one-step model, the existence of ρ_{T-1}^T follows from the above proposition with $\xi=\Delta S_T$ and $\mathcal{G}=\mathcal{F}_{T-1}$. Suppose that we have found already ρ_t^T. Putting $\xi=E(\rho_t^T|\mathcal{F}_{t-1})\Delta S_{t-1}$ and $\mathcal{G}=\mathcal{F}_{t-2}$, we find bounded $\mathcal{F}_{t-1}$-measurable $\varrho_{t-1}>0$ such that $E(\varrho_{t-1}E(\rho_t^T|\mathcal{F}_{t-1})|\Delta S_{t-1}|)<\infty$ and $E(\varrho_{t-1}E(\rho_t^T|\mathcal{F}_{t-1})\Delta S_{t-1})=0$. It is clear that ρ_{t-1}^T meets the requirements. The property (e) of the DMW theorem holds with $\rho_t:=E(\rho_0^T|\mathcal{F}_t)$.

1.1.5 *NA* and Conditional Distributions of Price Increments

As was shown by Jacod and Shiryaev, the long list of conditions equivalent to the NA-property can be completed by the following one involving the regular conditional distributions $Q_t(\omega,dx)$ of the price increments ΔS_t knowing $\mathcal{F}_{t-1}$:

(h) $0\in \operatorname{ri}\operatorname{conv}\operatorname{supp} Q_t(\omega,dx)$ *a.s. for all t=1,..., T.*

Recall that $Q_t(\omega,\Gamma)$ is an $\mathcal{F}_{t-1}$-measurable random variable in ω and a measure in Γ such that $P(\Delta S_t\in\Gamma|\mathcal{F}_{t-1})=Q_t(\omega,\Gamma)$ (a.s.) for each Borel set Γ in $\mathbf{R}^d$. The topological support of the measure $Q_t(\omega,dx)$ is the intersection of all closed sets which complements are null sets for this measure. The abbreviation "ri" denotes the relative interior of a convex set, i.e. the interior in the relative topology of the smallest affine subspace containing it.

Comparing (h) and (g) we see that their equivalence follows from the next one-step result complementing Proposition 1.1.5.

Proposition 1.1.6 *Let $\xi\in L^0(\mathbf{R}^d)$ and let $Q(\omega,dx)$ be the regular conditional distribution of ξ with respect to a σ-algebra $\mathcal{G}\subseteq\mathcal{F}$. Then the NA-property (or the equivalent property (ii)) holds if and only if the following condition is satisfied:*

(iii) $0\in \operatorname{ri}\operatorname{conv}\operatorname{supp} Q(\omega,dx)$ *a.s.*

Proof. $(ii) \Rightarrow (iii)$ Consider the case where $\mathcal{G}$ is trivial. If the origin does not belong to $A := \operatorname{ri}\operatorname{conv}\operatorname{supp} Q(dx)$ then there exists $a \in \mathbf{R}^d$ such that the latter set lays in the closed half-space $\{x: \ ax \geq 0\}$ but not in the subspace $\{x: \ ax = 0\}$ (to see this, apply the separation theorem in the linear subspace of minimal dimension containing A and extend the separating functional to a functional on the whole $\mathbf{R}^d$ vanishing on the orthogonal complement). So, $Q(x: \ ax > 0) > 0$ and for any strictly positive bounded random variable ρ measurable with respect to $\sigma\{\xi\}$, i.e. of the form $\rho = r(\xi)$ with a Borel function r, we have

$$E\rho a\xi = \int r(x) a x I_{\{x: \ ax>0\}} Q(dx) > 0$$

in contradiction with (ii).

In the general case we consider the set $\Gamma := \{(\omega, a): \ Q(\omega, \{ax > 0\}) > 0\}$ which is measurable with respect to the product σ-algebra $\mathcal{G} \otimes \mathcal{B}^d$; let $\Gamma(\omega)$ be its ω-sections. If (iii) fails, then, as it was just shown, the projection $\mathrm{Pr}_\Omega \Gamma$ of Γ on Ω is non-null. Due to the measurable selection theorem there exists an $\mathcal{G}$-measurable $\mathbf{R}^d$-valued random variable α such that $\alpha(\omega) \in \Gamma(\omega)$ for almost all ω from $\mathrm{Pr}_\Omega \Gamma$. Now, take an arbitrary bounded strictly positive function $r(\omega, x)$ measurable with respect to $\sigma\{\mathcal{G}, \xi\} \otimes \mathcal{B}^d$ and put $\rho(\omega) := r(\omega, \xi(\omega))$. Then

$$E(\rho\alpha\xi|\mathcal{G}) = \int r(\omega, x)\alpha(\omega) x I_{\{x: \ \alpha(\omega)x>0\}} Q(\omega, dx) > 0 \quad \text{on } \mathrm{Pr}_\Omega \Gamma.$$

It is easy to see that this is a contradiction with (ii).

$(iii) \Rightarrow (ii)$ Again, let us consider first the case of trivial $\mathcal{G}$. Let L be the affine subspace of minimal dimension containing the set

$$A := \operatorname{ri}\operatorname{conv}\operatorname{supp} Q(dx).$$

The assumption $0 \in A$ implies that the function

$$f(a) := \int e^{ax-|x|^2} Q(dx) = \int e^{ax-|x|^2} I_L(x) Q(dx)$$

attains its minimum at some point a_*: otherwise, we could find, as in the proof of Proposition 1.1.5, a vector β such $\{x: \ \beta x > 0\} \cap L$ is a Q-null set. But this means that the origin is not in the relative interior of the convex hull of $\operatorname{supp} Q(dx)$. In the general case, we can find a $\mathcal{G}$-measurable random variable α_* such that $\alpha_*(\omega)$ is a minimizer of $f(\omega, a)$ and conclude in the same way as in Proposition 1.1.5.

1.1.6 Comment on Absolute Continuous Martingale Measures

One may ask whether the existence of an absolute continuous martingale measure can be related with a certain no-arbitrage property. Indeed, in the case of finite number of states of the nature we have this criterion:

Proposition 1.1.7 *Suppose that Ω is finite. Then the following conditions are equivalent:*

(a) $R_T \cap L^0(\mathbf{R}_+ \setminus \{0\}) = \emptyset$;
(b) there is a probability measure $\tilde{P} \ll P$ such that $S \in \mathcal{M}(\tilde{P})$.

Here the implication $(b) \Rightarrow (a)$ is obvious while the converse follows easily from the finite-dimensional separation theorem applied to the disjoint convex sets $A_T \setminus \{0\}$ and $L^0(\mathbf{R}_+ \setminus \{0\})$: any separating functional after normalization is a density of probability measure with the needed property. The condition (a) means that there is no "universal" arbitrage strategy H, that is such that $H \cdot S_T > 0$ (a.s.).

Unfortunately, the above proposition cannot be extended to the case of arbitrary Ω.

Example. Let us consider a one-period model with two risky assets whose price increments ΔS_1^1, ΔS_1^2 are random variable defined on a countable probability space $\Omega = \{\omega_i\}_{i \geq 0}$ with all $P(\{\omega_i\}) > 0$. The initial σ-algebra is trivial. Let $\Delta S_1^1(\omega_0) = 1$, $\Delta S_1^1(\omega_i) = -i$, $i \geq 1$. Let $\Delta S_1^2(\omega_0) = 0$, $\Delta S_1^2(\omega_i) = 1$, $i \geq 1$. Apparently, the equalities $E_Q \Delta S_1^1 = 0$ and $E_Q \Delta S_1^2 = 0$ are incompatible and, hence, there are no martingale measures. On the other hand, let $(H^1, H^2) \in \mathbf{R}^2$ be a "universal" arbitrage strategy. Then necessary $H^1 > 0$ and we get a contradiction since in such a case the countable system of inequalities $-iH^1 + H^2 > 0$, $i \geq 1$, is incompatible whatever is H^2.

1.1.7 Complete Markets and Replicable Contingent Claims

As we observed, the set of results R_T is always closed in L^0. It is an easy exercise to deduce from this property that the set $\mathbf{R} + R_T$ is also closed. We use this remark in the proof of the following

Proposition 1.1.8 *Suppose that the set $\mathcal{Q}^e$ of equivalent martingale measures is non-empty. Then the following conditions are equivalent:*

(a) $\mathcal{Q}^e$ is a singleton;
(b) $\mathbf{R} + R_T = L^0$.

Proof. $(a) \Rightarrow (b)$ We may assume without loss of generality that P is a martingale measure. Suppose that there is $\xi \in L^0$ which is not in the closed subspace $\mathbf{R} + R_T \subseteq L^0$. It follows that the random variables $\xi^n := \xi I_{\{|\xi| \leq n\}}$ are not in this subspace for all $n \geq N$. Applying the separation theorem one can find η with $|\eta| \leq 1/2$ such that $E\eta\zeta = 0$ for the elements ζ from the closed subspace $(\mathbf{R} + R_T) \cap L^1$ of L^1 but $E\eta\xi^N > 0$. Put $Q = (1 + \eta)P$. Then

$E_Q H \cdot S_T = 0$ whatever is a bounded predictable process H. This means that Q is an equivalent martingale measure different from P contradicting (a).

$(b) \Rightarrow (a)$ Take $\Gamma \in \mathcal{F}_T$. Then $I_\Gamma = c_\Gamma + H_\Gamma \cdot S_T$ where c_Γ is a constant. It follows that $Q(\Gamma) = c_\Gamma$ whatever is a martingale measure Q, i.e. the latter is unique.

The property (b), in financial literature referred to as the *market completeness*, means that any contingent claim can be replicated, that is represented as the terminal value of a self-financing portfolio starting from a certain initial endowment. The above statement, asserting that an arbitrage-free market is complete if and only if there is only one equivalent martingale measure sometimes is called the second fundamental theorem of asset pricing.

The closedness of the subspace $\mathbf{R}+R_T$ leads the next assertion concerning replicable claims on incomplete markets. In its formulation Q_l and Q_l^e denote the sets of absolute continuous and equivalent local martingale measures.

Proposition 1.1.9 *Suppose that $\mathcal{Q}^e \neq \emptyset$. Let a random variable $\xi \geq 0$ be such that $a = \sup_{Q \in \mathcal{Q}_l^e} E_Q \xi < \infty$ and the supremum is attained on some measure Q^*. Then $\xi = a + H \cdot S_T$ for some predictable process (and, hence, the function $Q \mapsto E_Q \xi$ is constant on the set Q_l^e).*

Proof. Supposing that the statement fails, we apply the Hahn–Banach theorem and separate ξ and the subspace $(\mathbf{R}+R_T) \cap L^1(Q^*)$ in $L^1(Q^*)$, that is we find $\eta \in L^\infty$ such that $E_{Q^*}\xi\eta > 0$ and $E_{Q^*}\eta\zeta = 0$ for all ζ from the subspace. In particular, $E_{Q^*}\eta = 0$ and $E_{Q^*} H \cdot S_T \eta = 0$ whatever is a predictable process H such that $H \cdot S_T$ is integrable; in particular, the last equality holds for $H = I_{[0,\tau_n]}$ where τ_n is a localizing sequence for S. Normalizing, we may assume that $|\eta| \leq 1/2$. It follows that the measure $\tilde{Q} = (1+\eta)Q^*$ is an element of $\mathcal{Q}_l^e$ and $E_{\tilde{Q}}\xi = a + E_{Q^*}\xi\eta > a$ in an apparent contradiction with the definition of Q^*.

1.1.8 DMW Theorem with Restricted Information

Let us consider the following setting which is only slightly different from the classical one. Namely, assume that we are given a filtration $\mathbf{G} = (\mathcal{G}_t)_{t \leq T}$ with $\mathcal{G}_t \subseteq \mathcal{F}_t$. Suppose that the strategies are now predictable with respect to this, smaller filtration (i.e., $H_t \in L^0(\mathcal{G}_{t-1})$), a situation which may happen when the portfolios are revised on the basis of restricted information, e.g., due to a delay. Again we may define the sets R_T, A_T and give a definition of the arbitrage which, in these symbols, looks exactly as (a) above and we can list the corresponding necessary and sufficient conditions.

To this aim we define the $\mathbf{G}$-optional projection X^o of an integrable process X by putting $X_t^o := E(X_t|\mathcal{G}_t)$, $t \leq T$.

Theorem 1.1.10 *The following properties are equivalent:*

(a) $A_T \cap L_+^0 = \{0\}$ *(NA condition);*

(*b*) $A_T \cap L^0_+ = \{0\}$ *and* $A_T = \bar{A}_T$*;*
(*c*) $\bar{A}_T \cap L^0_+ = \{0\}$*;*
(*d*) *there is a strictly positive process* $\rho \in \mathcal{M}$ *with* $(\rho S)^o \in \mathcal{M}(\mathbf{G})$*;*
(*e*) *there is a bounded strictly positive process* $\rho \in \mathcal{M}$ *with* $(\rho S)^o \in \mathcal{M}(\mathbf{G})$.

The symbol $\mathcal{M}(\mathbf{G})$ stands here for the set of $\mathbf{G}$-martingales and we presume tacitly in the last two conditions that $E\rho_t|S_t| < \infty$. Clearly, these conditions can be formulated in terms of existence of an equivalent probability $\tilde{P}$ such that $\tilde{E}(S_{t+1}|\mathcal{G}_t) = \tilde{E}(S_t|\mathcal{G}_t)$ for all $t \le T-1$.

We left to the reader as an (easy) exercise to inspect that the arguments of the previous section goes well for this theorem.

Remark. Curiously, this result, rather natural and important for practical applications, was established only recently. It happens, all numerous proofs, except one suggested in [57] and reproduced above in Subsection 1.1.3, in their most essential part concerning the construction of equivalent martingale measures given the NA-property, are based on the reduction to the one-step case with $T = 1$. Of course, (a) implies (g) (i.e., the NA-property for all one-step models). A clever argument in the Dalang–Morton–Willinger paper permits to assemble a required martingale density from martingale densities for one-step models. However, in model with restricted information the property (g) drops out from the list of equivalent conditions.

Example. Consider the model where $T = 2$, $\mathcal{G}_0 = \mathcal{G}_1 = \{\emptyset, \Omega\}$ but there is $A \in \mathcal{F}_2$ such that $0 < P(A) < 1$. Put

$$\Delta S_1 := I_A - \frac{1}{2} I_{A^c}, \quad \Delta S_2 := -\frac{1}{2} I_A + I_{A^c}.$$

There is no arbitrage at each of two steps but the constant process with $H_1 = H_2 = 1$ is an arbitrage strategy for the two-step model.

1.1.9 Hedging Theorem for European-Type Options

One of the most fundamental though simple ideas of mathematical finance is the arbitrage pricing of contingent claims.

A *contingent claim* or an *option* is a random variable ξ which can be interpreted as a pay-off of the option seller to the option buyer. For a European-type option the payment is made at the terminal (maturity) date T and may depend on the whole history up to T. What is a "fair" price for such a contract payed at time zero? Apparently, and this is the basic principle, the option price should be such that neither of two parties has arbitrage opportunities, i.e., riskless profits.

Let us define the set

$$\Gamma := \Gamma(\xi) := \{x : \exists H \in \mathcal{P} \text{ such that } x + H \cdot S_T \ge \xi\}.$$

Clearly, if not empty, it is a semi-infinite interval (maybe, coinciding with the whole line). A priori, it can be either of the form $[\bar{x}, \infty[$ or $]\bar{x}, \infty[$.

The theorem below ensures, in particular, that $\bar{x} \in \Gamma$. If the contracted price of the option, say, x is strictly larger than $\bar{x}$, then the sellar has a non-risky profit by pocketing $x - \bar{x}$, and running a self-financing portfolio process in the underlying assets $\bar{x} + H \cdot S$ which terminal value dominates the terminal pay-off (so, selling the portfolio at the date T covers the liability).

Similarly, suppose that the right extremity $\underline{x}$ of the semi-infinite interval

$$-\Gamma(-\xi) = \{x : \exists H \in \mathcal{P} \text{ such that } -x + H \cdot S_T \geq -\xi\}$$

belongs to this interval. If x is strictly less than $\underline{x}$, then the option buyer will have an arbitrage opportunity. Indeed, in this case there exists a strategy H such that $-\underline{x} + H \cdot S_T \geq -\xi$. Thus, borrowing x at $t = 0$ to buy the option, the agent runs a portfolio $-x + H \cdot S$ which has a terminal value is larger than $\underline{x} - x - \xi$. Therefore, after exercising the option, the agent will have a non-risky profit $\underline{x} - x$.

These arguments show that "fair" prices lie in the interval $[\underline{x}, \bar{x}]$.

Remark 1. Note that it is tacitly assumed that the agent (option seller) may have a short position in option: for the discrete-time model it is an innocent assumption but it is questionable for continuous-time models where the admissibility means that unbounded short positions even in the underlying are not allowed.

In the case when the contingent claim is redundant, that is of the form $\xi = x + H^\xi \cdot S_T$, we have necessarily that $x = \underline{x} = \bar{x}$ is the no-arbitrage price of the option. Indeed, let us consider the hedging portfolio process $\bar{x} + H \cdot S$ for ξ. The absence of arbitrage implies that its terminal value must coincide with ξ and, in virtue of the "law of one price" (also due to NA, see the Remark below) $x = \bar{x}$ and, by symmetry, $x = \underline{x}$. The same NA arguments show that if ξ is non-redundant, the hedging portfolio starting from $\bar{x}$ is an arbitrage opportunity. Thus, the range of no arbitrage prices is either a singleton or an open interval $]\underline{x}, \bar{x}[$.

Remark 2. The law of one price (L1P) is the property asserting that the equality $x + H \cdot S_T = x' + H' \cdot S_T$ implies the equality $x = x'$. The NA-property is a sufficient condition for L1P that follows from the DMW theorem: the latter ensure that there is a measure under which the process $(H - H') \cdot S$ is a martingale. One may ask what is a necessary and sufficient condition for L1P. The answer is the following:

L1P holds if and only if there is a bounded martingale Z with $EZ_T = 1$ and $Z_0 > 0$ such that the process ZS is a martingale.

In this formulation we do not suppose that the σ-algebra $\mathcal{F}_0$ is trivial. Notice that L1P means that $R_T \cap L^0_+(\mathcal{F}_0) = \{0\}$ where, as we already mentioned, the linear space R_T is closed. We hope that with this remark the proof of the non-trivial "only if" part will be for the reader an easy exercise.

Now we present the theorem giving a "dual" description of the set of initial capitals Γ from which one can super-replicate (hedge) the contingent claim ξ.

Notations. Let $\mathcal{Q}$ (resp. $\mathcal{Q}^e$) be the set of all measures $Q \ll P$ (resp. $Q \sim P$) such that S is a martingale with respect to Q. We add to these notations the subscript l to denote larger sets of measures $\mathcal{Q}_l$ and $\mathcal{Q}_l^e$ for which S is only a local martingale. We shall denote by $\mathcal{Z}$, $\mathcal{Z}^e$, $\mathcal{Z}_l$, ... the density processes for measures from the corresponding sets.

Theorem 1.1.11 *Suppose that $\mathcal{Q}^e \neq \emptyset$. Let ξ be a bounded from below random variable such that $E_Q|\xi| < \infty$ for every $Q \in \mathcal{Q}^e$. Then*

$$\Gamma = \{x : \ x \geq E\rho_T\xi \text{ for all } \rho \in \mathcal{Z}^e\}. \tag{1.1.1}$$

In other words, $\bar{x} = \sup_{Q\in\mathcal{Q}^e} E_Q\xi$ and $\Gamma = [\bar{x}, \infty]$. An obvious corollary of this theorem (applied to the set $\Gamma(-\xi)$) is the assertion that $\underline{x} = \inf_{Q\in\mathcal{Q}^e} E_Q\xi$.

The direct proof of this result is not difficult but we obtain it from two fundamental facts of own interest. The first one usually is referred to as the *optional decomposition theorem* which will be discussed in Subsection 1.1.12.

Theorem 1.1.12 *Suppose that $\mathcal{Q}^e \neq \emptyset$. Let $X = (X_t)$ be a bounded from below process which is a supermartingale with respect to each probability measure $Q \in \mathcal{Q}^e$. Then there exist a strategy H and an increasing process A such that $X = X_0 + H \cdot S - A$.*

Proposition 1.1.13 *Suppose that $Q^e \neq \emptyset$. Let ξ be a bounded from below random variable such that $\sup_{Q\in\mathcal{Q}^e} E_Q|\xi| < \infty$. Then the process X with*

$$X_t = \operatorname{ess\,sup}_{Q\in\mathcal{Q}^e} E_Q(\xi|\mathcal{F}_t)$$

is a supermartingale with respect to every $Q \in \mathcal{Q}^e$.

For the proof of this result we send the reader to Appendix (Proposition A.3.7).

Proof of Theorem 1.1.11. The inclusion $\Gamma \subseteq [\bar{x}, \infty[$ is obvious: if $x + H\cdot S_T \geq \xi$ then $x \geq E_Q\xi$ for every $Q \in \mathcal{Q}^e$. To show the opposite inclusion we may suppose that $\sup_{Q\in\mathcal{Q}} E_Q|\xi| < \infty$ (otherwise both sets are empty). Applying the optional decomposition theorem we get that $X = \bar{x} + H \cdot S - A$. Since $\bar{x} + H \cdot S_T \geq X_T = \xi$, the result follows.

1.1.10 Stochastic Discounting Factors

In this subsection to discuss financial aspects of the hedging theorem and give an interpretation of densities of martingale measures as stochastic discounting factors.

Let us consider a "practical example" where the option seller promised to deliver at the expiration date T a "basket" of d assets, namely, η^i units of the i-th asset which price process is S^i is positive. Since the market is frictionless, this is same as to deliver ηS_T units of the numéraire, i.e. to make a payment $\xi = \eta S_T$. The hedging theorem asserts that the set of initial capitals allowing to super-replicate ξ can be described in terms of prices. Namely, if the NA-property holds, one can hedge the pay-off from the initial capital x if and only if x dominates the expectation of "stochastically discounted" pay-off $\rho_T \xi = \eta S_T^\rho$ whatever is a martingale density ρ. In other words, the comparison should be done not by computing the "value" of the basket using the "true" price process but replacing the latter by a "consistent price system" $S^\rho = \rho S$ obtained by multiplying the "true" price process by the *stochastic discounting factor* ρ. The word "consistent" here reflects the fact that S_t^ρ is determined by S_T^ρ via the martingale property: $S_t^\rho = E(S_T^\rho|\mathcal{F}_T)$.

1.1.11 Hedging Theorem for American-Type Options

In the American-type option the buyer has the right to exercise at any date before T on the basis of the available information flow, so the exercise date τ is a stopping time; the buyer gets the amount Y_τ, the value at τ of an adapted process Y. The description of the pay-off process $Y = (Y_t)$ is a clause of the contract (as well as the final maturity date T).

By analogy with the case of European options we define the set of initial capitals starting from which one can run a self-financing portfolio which values dominate the eventual pay-off on the considered time-interval:

$$\varGamma := \varGamma(Y) := \{x : \exists H \in \mathcal{P} \text{ such that } x + H \cdot S \geq Y\}.$$

Theorem 1.1.14 *Suppose that $\mathcal{Q}^e \neq \emptyset$. Let $Y = (Y_t)$ be an adapted process bounded from below and such that $E_Q|Y_t| < \infty$ for every $Q \in \mathcal{Q}^e$ and $t \leq T$. Then*

$$\varGamma = \{x : \; x \geq E\rho_\tau Y_\tau \textit{ for all } \rho \in \mathcal{Z}^e \textit{ and all stopping times } \tau \leq T\}. \quad (1.1.2)$$

The proof of this result based on application of the optional decomposition is exactly the same as of Theorem 1.1.11. The only difference is that now we take as X the process

$$X_t = \operatorname{ess\,sup}_{Q \in \mathcal{Q}^e, \tau \in \mathcal{T}_t} E_Q(Y_\tau|\mathcal{F}_t),$$

where $\mathcal{T}_t$ is the set of stopping times with values in the set $\{t, t+1, ..., T\}$. Under the assumption $\sup_{Q \in \mathcal{Q}^e} E_Q|Y_t| < \infty$ for each t, the process X is a supermartingale with respect to every $Q \in \mathcal{Q}^e$, see Proposition A.3.8 in Appendix.

1.1.12 Optional Decomposition Theorem

We give here a slightly different formulation.

Theorem 1.1.15 *Suppose that $\mathcal{Q}_l^e \neq \emptyset$. Let $X = (X_t)$ be a process which is a generalized supermartingale with respect to each measure $Q \in \mathcal{Q}_l^e$. Then there are a strategy H and an increasing process A such that $X = X_0 + H \cdot S - A$.*

Proof. We start from a one-step version of the result.

Lemma 1.1.16 *Let $\mathcal{G}$ be a sub-σ-algebra of $\mathcal{F}$ and let ξ and η be random variables with values in $\mathbf{R}$ and $\mathbf{R}^d$ and for which $E(|\xi|+|\eta||\mathcal{G})<\infty$. Assume that $E(\alpha\xi|\mathcal{G})\le 0$ whatever is a random variable $\alpha>0$ with $E(\alpha|\mathcal{G})=1$ such that $E(\alpha\eta|\mathcal{G})=0$ and $E(\alpha|\xi||\mathcal{G})<\infty$, $E(\alpha|\eta||\mathcal{G})<\infty$. Suppose that such α does exist. Then there is $\lambda\in L^0(\mathbf{R}^d,\mathcal{G})$ such that $\xi-\lambda\eta\le 0$.*

Proof. First, we suppose without loss of generality that ξ and η are integrable (we may argue with $\tilde{\xi}:=\xi/(1+E(|\xi|+|\eta||\mathcal{G}))$ and $\tilde{\eta}:=\eta/(1+E(|\xi|+|\eta||\mathcal{G}))$). Define the set $A:=\{\lambda\eta:\ \lambda\in L^0(\mathbf{R}^d,\mathcal{G})\}-L^0_+$. By the DMW theorem it is closed in probability. Thus, the convex set $A^1:=A\cap L^1$ is closed in L^1. If the assertion of the lemma fails, $\xi\notin A^1$. Therefore, in virtue of the Hahn–Banach separation theorem, there is $\alpha\in L^\infty$ such that

$$E\alpha\xi>\sup_{\zeta\in A^1}E\alpha\zeta.$$

Necessary, $\alpha\ge 0$: if not, the right-hand side of the above inequality will be infinite. By the same reason $E\alpha\lambda\eta=0$ whatever is $\lambda\in L^\infty(\mathbf{R}^d,\mathcal{G})$. Hence, $E(\alpha\eta|\mathcal{G})=0$ and supremum is equal to zero. That is $E\alpha\xi>0$. But this is incompatible with the inequality $E(\alpha\xi|\mathcal{G})\le 0$ we should have for such α.

With the lemma the proof of the theorem is easy. Indeed, let $\rho\in\mathcal{Z}^e_l$. Consider the obvious identity $\rho_t=\alpha_1...\alpha_t$ where $\alpha_k:=\rho_k/\rho_{k-1}$. The martingale properties of ρ means that $E(\alpha_t|\mathcal{F}_{t-1})=1$. On the other hand, due to the coincidence of classes of local and generalized martingales, $\rho\in\mathcal{Z}^e_l$ if and only if $E(\alpha_t|\Delta S_t||\mathcal{F}_{t-1})<\infty$ and $E(\alpha_t\Delta S_t|\mathcal{F}_{t-1})=0$ for all $t\le T$. Thus, by Lemma 1.1.16, there is $H_t\in L^0(\mathbf{R}^d|\mathcal{F}_{t-1})$ such that $\Delta X_t-H_t\Delta S_t\le 0$. Denoting the right-hand side by $-\Delta A_t$ and putting $A_0=0$ we obtain the desired decomposition.

Remark. Let us return to the setting of Lemma 1.1.16 assuming that the σ-algebra $\mathcal{G}$ is trivial. Consider the maximization problem $E\alpha\xi\to\max$ under two equality constraints $E\alpha=1$ and $E\alpha\eta=0$, the constraint $\alpha>0$ (a.s.) and "admissibility" assumptions on α to ensure the needed integrability. The hypothesis of the lemma says that the value of this problem does not exceed zero. It is not difficult to prove that there is a Lagrange multiplier λ "removing" the second equality constraint. For the new maximization problem we also have that $E\alpha(\xi-\lambda\eta)\le 0$ for all α satisfying the remaining constraints. Clearly, this is possible only if $\xi-\lambda\eta\le 0$.

One may expect that these arguments can be extended for the general case, with conditional expectations. This is still easy for finite or countable Ω. This strategy of proof is feasible for arbitrary Ω but one needs to look for a $\mathcal{G}$-measurable version of Lagrange multipliers by applying a delicate measurable selection result, requiring, in turn, specific preparations. However, this approach (inspired by the original proof of the DMW) works well also for continuous-time models, see [27]. We use it for an analysis of the structure of the set of equivalent martingale measures in the next subsection.

1.1.13 Martingale Measures with Bounded Densities

The following useful result gives, in particular, the positive answer to the question whether the set $\mathcal{Q}^e$ is norm-dense in $\mathcal{Q}^e_l$ (that is whether $\mathcal{Z}^e$ is dense in $\mathcal{Z}^e_l$ in L^1-norm). Indeed, in virtue of the DMW theorem $\mathcal{Q}^e_l \neq \emptyset$ if and only if $\mathcal{Q}^e \neq \emptyset$. It remains to take as the reference measure an arbitrary element of the latter set and apply the theorem below. This theorem happens to be useful to get a similar property for the discrete time model with infinite horizon which will be discussed in the next section.

Theorem 1.1.17 *Let $P \in \mathcal{Q}^e_l$. Then the set $\{Q \in \mathcal{Q}^e,\ dQ/dP \in L^\infty\}$ is norm-dense in $\mathcal{Q}^e_l$.*

Proof. It contains three steps. The first one is a simple lemma on the approximations of positive functions on the probability space $(\mathbf{R}^m, \mathcal{B}^n, \mu)$ by positive functions from $C(\bar{\mathbf{R}}^m)$ where $\bar{\mathbf{R}}^m$ is the one-point compactification of $\mathbf{R}^m$.

Lemma 1.1.18 *Let $\phi : \mathbf{R}^m \to \mathbf{R}^l$ be a measurable mapping with $|\phi| \in L^1(\mu)$. Put $U := \{g \in L^1(\mu) :\ g > 0,\ g|\phi| \in L^1(\mu)\}$ and $U_C := U \cap C(\bar{\mathbf{R}}^m)$. Then for any $f \in U$ and $\varepsilon > 0$ there is $f^\varepsilon \in U_C$ such that $||f - f^\varepsilon||_{L^1(\mu)} < \varepsilon$ and*

$$E_\mu \phi f = E_\mu \phi f^\varepsilon. \tag{1.1.3}$$

Proof. Let $\mathcal{O}_\varepsilon(f)$ be an open ball in L^1_μ of radius ε with center at f. Define the convex sets $G := U \cap \mathcal{O}_\varepsilon(f)$ and $G_C := U_C \cap \mathcal{O}_\varepsilon(f)$ and consider the affine mapping $\Phi : G \to \mathbf{R}^l$ with $\Phi(g) = E_\mu(f-g)\phi$. We need to show that $0 \in \Phi(G_C)$. Notice that U_C is a dense subset of U and, therefore, G_C is dense in G in L^1_μ. It follows that $\Phi(G_C)$ is dense in $\Phi(G)$. The convexity of these sets implies that $\mathrm{ri}\,\Phi(G_C) = \mathrm{ri}\,\Phi(G)$ and to complete the proof it is sufficient to check that $0 \in \mathrm{ri}\,\Phi(G)$. To this aim we observe first that without loss of generality we may consider the case where $f\phi^i$, $i = 1, ..., l$, are linearly independent elements of L^1_μ. Suppose that $0 \notin \mathrm{ri}\,\Phi(G)$. Let us consider the smallest hyperplane H containing $\Phi(G)$. Since $0 \in \Phi(G)$, it is a subspace. By the separation theorem, there is a non-trivial linear functional y on H such that $yx \geq 0$ for all $x \in \Phi(G)$. Extending y to a linear functional on the whole $\mathbf{R}^l$ we may rewrite this as: $E_\mu(f-g)y\phi \geq 0$ whatever is $g \in G$. Using functions of the form $g = f \pm \delta f I_\Gamma$ where Γ is a measurable set and $\delta \in]0,1[$ is such that $g \in \mathcal{O}_\varepsilon(f)$, we get from here that $E_\mu I_\Gamma f y\phi = 0$ for any Γ. Hence, $yf\phi = 0$ in contradiction with the assumed linear independence of components.

With this preparatory result we can easily prove the claim for the one-period model.

Lemma 1.1.19 *Let $\mathcal{G}$ be a (complete) sub-σ-algebra of $\mathcal{F}$ and let α and η be random variables taking values, respectively, in $\mathbf{R}_+ \setminus \{0\}$ and $\mathbf{R}^d$ such that $E((1+\alpha)|\eta||\mathcal{G}) < \infty$. Assume that $E(\alpha|\mathcal{G}) = 1$, $E(\eta|\mathcal{G}) = 0$, and $E(\alpha\eta|\mathcal{G}) = 0$. Then there are bounded random variables $\alpha^n > 0$ converging to α a.s. and such that $E(\alpha^n|\mathcal{G}) = 1$, $E(\alpha^n\eta|\mathcal{G}) = 0$.*

Proof. Let $\mu(dx,\omega)$ be a regular conditional distribution of the random vector (α,η) knowing $\mathcal{G}$. Define on $\mathbf{R}^{d+1}$ the functions $f(x) := x^1$ and $\phi(x) := (1, x^2, ..., x^{d+1})$. Writing the conditional expectations as the integrals with respect to conditional distribution, we express properties of α as follows: $E_{\mu(.,\omega)} f\phi = e_1$ (the first orth in $\mathbf{R}^{d+1}$) for all ω except a null-set. The set

$$\Gamma^n := \{(\omega, g) \in \Omega \times C(\bar{\mathbf{R}}^{d+1}) : \; g > 0, \; E_{\mu(.,\omega)} g\phi = e_1, \; E_{\mu(.,\omega)}|f - g| < 1/n\}$$

is $\mathcal{G} \otimes \mathcal{B}(C(\bar{\mathbf{R}}^{d+1}))$-measurable and, according to the previous lemma, it has the projection on Ω of full measure. By the classical measurable selection theorem Γ^n admits a $\mathcal{G}$-measurable selector $f^n : \Omega \to C(\bar{\mathbf{R}}^{d+1})$. The function of two variables $f^n(\omega, x)$, being $\mathcal{G}$-measurable in ω and continuous in x, is $\mathcal{G} \otimes \mathcal{B}^{d+1}$-measurable. The random variables $\tilde{\alpha}^n = f^n(\omega, (\alpha(\omega), \eta(\omega)))$ converge to α in L^1, hence, in probability. Let us define the bounded random variables $\tilde{\alpha}^{n,k}(\omega) := \tilde{f}^{n,k}(\omega, (\alpha(\omega), \eta(\omega)))$ where

$$\tilde{f}^{n,k}(\omega, x) = f^n(\omega, x) I_{\{||f^n(\omega,.)|| \le k\}} + I_{\{||f^n(\omega,.)|| > k\}},$$

$||.||$ is a uniform norm in x. Since $E_{\mu(.,\omega)} g\phi = e_1$, we have the equalities $E(\tilde{\alpha}^{n,k}|\mathcal{G}) = 1$, $E(\tilde{\alpha}^{n,k}\eta|\mathcal{G}) = 0$.

Obviously, $\tilde{\alpha}^{n,k}$ converge to $\tilde{\alpha}^n$, in probability. The convergence in probability is a convergence in a metric space and, therefore, one can take a subsequence k_n such that $\alpha^n := \tilde{\alpha}^{n,k_n}$ converge to α in probability. But then there is a subsequence of α^n convergent to α a.s.

The third, concluding step, is also simple. Note first that we may replace the reference measure by any other from Q_l^e with a bounded density. According to the DMW theorem between such measures there are measures from Q^e and so we may assume without loss of generality that already $P \in Q^e$.

Again we use the multiplicative representation of the density $\rho_T = dQ/dP$, namely, $\rho_T = \alpha_1 ... \alpha_T$ with $\alpha_t := \rho_t/\rho_{t-1}$. The property $\rho \in \mathcal{Z}_l^e$ holds if and only if $E(\alpha_t|\mathcal{F}_{t-1}) = 1$, $E(\alpha_t|\Delta S_t||\mathcal{F}_{t-1}) < \infty$ and $E(\alpha_t \Delta S_t|\mathcal{F}_{t-1}) = 0$ for all $t \le T$. Applying the preceding lemma we define the measure $P^n := \rho_T^n P \in Q^e$ with bounded density $\rho_T^n := \alpha_1^n ... \alpha_T^n$ convergent to ρ_T a.s. But by the Scheffe theorem we have here also convergence in L^1. □

Remark. Theorem 1.1.17 has several obvious corollaries. E.g., if $P \in Q_l^e$ then the set of $Q \in \mathcal{Q}_l^e$ with bounded densities $\rho_T = dQ/dP$ and $\rho_T^{-1} = dP/dQ$ is dense in $\mathcal{Q}_l^e$. This facts is easily seen by considering the convex combinations $Q^n = (1-1/n)Q + (1/n)P$ and letting n tends to infinity. Noticing that $\mathcal{Q}_l^e$ is dense in the set $\mathcal{Q}_l$ (by the similar consideration), one can strengthen further the claim in another direction, etc.

It is not difficult to check that the set of local martingale measures with finite entropy (i.e. with $E\rho_T \ln \rho_T < \infty$), if non-empty, is also dense in Q_l^e. We explain the idea by establish a more general result which has applications in portfolio optimization problems.

Let $\varphi :]0, \infty[\to \mathbf{R}$ be a measurable function bounded from below and let

$$\mathcal{Q}^e_\varphi := \{Q \in \mathcal{Q}^e : \ E\varphi(dQ/dP) < \infty\}.$$

Proposition 1.1.20 *If the set $\mathcal{Q}^e_\varphi \neq \emptyset$, then it is dense in $\mathcal{Q}^e$ in the following two cases:*

(a) for every $c \geq 1$ there exist constants $r_1(c), r_2(c) \geq 0$ such that

$$\varphi(\lambda y) \leq r_1(c)\varphi(y) + r_2(c)(y+1), \quad y \in]0,\infty[, \ \lambda \in [c^{-1}, c]; \tag{1.1.4}$$

(b) φ is convex and $\mathcal{Q}^e_\varphi = \mathcal{Q}^e_{\varphi_\lambda}$ for any $\lambda > 0$ where $\varphi_\lambda(y) := \varphi(\lambda y)$.

Proof. (a) Let $\tilde P \in \mathcal{Q}^e_\varphi$. Take an arbitrary measure $Q \in \mathcal{Q}^e$. By the above theorem and the accompanying remark there exists a sequence $Q^n \in \mathcal{Q}^e$ convergent to Q with the densities $dQ^n/d\tilde P$ taking values in intervals $[c_n^{-1}, c_n]$. We have:

$$E\varphi\left(\frac{dQ^n}{dP}\right) = E\varphi\left(\frac{dQ^n}{d\tilde P}\frac{d\tilde P}{dP}\right) \leq r_1(c_n)E\varphi\left(\frac{d\tilde P}{dP}\right) + 2r_2(c_n) < \infty.$$

Hence, $Q^n \in \mathcal{Q}^e_\varphi$ and the result follows.

(b) We may assume without loss of generality that $\varphi \geq 0$ (by adding a constant) and repeat the same arguments modifying only the last step. Clearly, $dQ^n/d\tilde P = \alpha_n c_n^{-1} + (1-\alpha_n)c_n$ where α_n is a random variable taking values in $[0,1]$. By convexity,

$$\begin{aligned} E\varphi\left(\frac{dQ^n}{d\tilde P}\frac{d\tilde P}{dP}\right) &\leq E\left[\alpha_n\varphi\left(c_n^{-1}\frac{d\tilde P}{dP}\right) + (1-\alpha_n)\varphi\left(c_n\frac{d\tilde P}{dP}\right)\right] \\ &\leq E\varphi\left(c_n^{-1}\frac{d\tilde P}{dP}\right) + E\varphi\left(c_n\frac{d\tilde P}{dP}\right) < \infty \end{aligned}$$

in virtue of assumption and we conclude as before.

Note that for convex φ the condition (a) implies (b). The latter hypothesis entangles properties of φ and $\mathcal{Q}^e$.

In financial applications, typically, $\varphi(y) = y^p$, $p > 0$, or $\varphi(y) = y\ln y$. In particular, if non-empty, the set $\mathcal{Q}^e_{y\ln y}$ of martingale measures with finite entropy is dense in the set of equivalent martingale measures $\mathcal{Q}^e$.

More generally, let $u: \ \mathbf{R} \to \mathbf{R}$ be an increasing **concave** differentiable function and let u^* be its Fenchel dual (which is, by definition, the Fenchel dual of the **convex** function $-u(-.)$), i.e. the convex function,

$$u^*(y) = \sup_x (u(x) - xy).$$

For example, the dual of the exponential utility function $u(x) = 1 - e^{-x}$ is the function $u^*(y) = y\ln y - y + 1$, $y \geq 0$, and $u^*(y) = \infty$, $y < 0$.

Suppose that u has a "reasonable" asymptotic elasticity, i.e.

$$AE_+(u) := \limsup_{x\to\infty} \frac{xu'(x)}{u(x)} < 1, \qquad AE_-(u) := \liminf_{x\to-\infty} \frac{xu'(x)}{u(x)} > 1.$$

It can be shown that the function $\varphi = u^*$ satisfies the growth condition (a) of Proposition 1.1.20.

1.1.14 Utility Maximization and Convex Duality

In this subsection we explain the importance of the set of equivalent martingale measure in the problem of portfolio optimization. Namely, we consider the simplest model with finite number of state of the nature where the investor maximizes the mean value of an exponential utility function of the terminal value of his portfolio. Applying the classical Fenchel theorem we show that the dual problem involves martingale measures.

So, Ω is finite and hence, the space L^0 can be identified with a finite-dimensional Euclidean space. As usual, R_T is the set of random variables $H \cdot S_T$ and $\mathcal{Z}_T^e$ (respectively, $\mathcal{Z}^e$) is the set of densities (respectively, density processes) of equivalent martingale measures.

It is **supposed** in the following discussion that $\mathcal{Z}_T^e \neq \emptyset$.

We are interested in the portfolio optimization problem the value of which is

$$J^o := \sup_{\eta\in R_T} E(1 - e^{-\eta}). \tag{1.1.5}$$

It will be studied in jointly with a minimization problem $EZ_T \ln Z_T \to \min$ over the set of equivalent martingale densities $\mathcal{Z}_T^e$; let its value be

$$\underline{J} := \inf_{\xi\in\mathcal{Z}_T^e} E\xi \ln \xi. \tag{1.1.6}$$

The latter problem, by abuse of the language, sometimes is referred to as the "dual" one. As we shall see below, this terminology deviates from the standard one of the convex analysis.

The continuous function $\xi \to E\xi \ln \xi$ attains its minimum $\underline{J}$ on the compact set $\mathcal{Z}_T^a$ which is a closure of $\mathcal{Z}_T^e$. Due to strict convexity the minimizer ξ^o is unique. The derivative of the function $\varphi(x) = x \ln x$ (with $\varphi(0) = 0$) at zero is $-\infty$ and this property implies that ξ^o is strictly positive. Indeed, let us take an arbitrary point ξ of the set $\mathcal{Z}_T^e$ (assumed nonempty) and consider on $[0, 1]$ the function $F_t = Ef_t$ where $f_t := \varphi(t\xi + (1-t)\xi^o)$. As F attains its minimum at $t = 0$, the derivative $F_0' \geq 0$. But $F_0' = Ef_0'$. Since $f_0' = \varphi'(0)\xi = -\infty$ on the set $\{\xi^o = 0\}$ the probability of the latter is zero.

The measure $P^o = \xi^o$ is called the *entropy minimal* equivalent martingale measure.

Proposition 1.1.21 $J^o = 1 - e^{-\underline{J}}$.

This result is a direct consequence of the fundamental Fenchel theorem. We recall the simplest version of its formulation (in its traditional form, for convex functions).

Let X be a Hilbert space and let $f : X \to \mathbf{R} \cup \{\infty\}$, $g : X \to \mathbf{R} \cup \{\infty\}$ be two convex lower semicontinuous functions not identically equal to infinity, i.e. $\operatorname{dom} f \neq \emptyset$ and $\operatorname{dom} g \neq \emptyset$.

Let us consider two minimization problem: the primal

$$f(\eta) + g(\eta) \to \min \quad \text{on } X \tag{1.1.7}$$

and the dual

$$f^*(-\xi) + g^*(\xi) \to \min \quad \text{on } X^*(= X). \tag{1.1.8}$$

We denote their values $v := \inf_x [f(x) + g(x)]$ and $v_* := \inf_y [f^*(-y) + g^*(y)]$.

Suppose that

$$\operatorname{dom} f \cap \operatorname{dom} g \neq \emptyset, \qquad (-\operatorname{dom} f^*) \cap \operatorname{dom} g^* \neq \emptyset;$$

we rewrite these conditions, to relate them with those in the formulation of theorem below, as

$$0 \in \operatorname{dom} f - \operatorname{dom} g, \qquad 0 \in \operatorname{dom} f^* + \operatorname{dom} g^*.$$

They ensure that $v < \infty$ and $v_* < \infty$.

Note that always $v + v_* \geq 0$ because by the Fenchel inequality

$$f(\eta) + g(\eta) + f^*(-\xi) + g^*(\xi) \geq (-\eta, \xi) + (\eta, \xi) = 0.$$

The following result is a particular case of the Fenchel theorem.

Theorem 1.1.22 (a) *Let* $0 \in \operatorname{int}(\operatorname{dom} f - \operatorname{dom} g)$. *Then the dual problem (1.1.8) has a solution and* $v + v_* = 0$.

(b) *Let* $0 \in \operatorname{int}(\operatorname{dom} g^* + \operatorname{dom} f^*)$. *Then the primal problem (1.1.7) has a solution and* $v + v_* = 0$.

Let consider the minimization problem

$$f(\eta) + g(\eta) \to \min \quad \text{on } L^0 \tag{1.1.9}$$

where $f(\eta) := E(e^\eta - 1)$ and $g = \delta_{R_T}$, the indicator function (in the sense of convex analysis) which is equal to zero on R_T and infinity on the complement. Clearly, f^* is calculated via the dual to the convex function $e^x - 1$, namely, $f^*(-\xi) = E(\xi \ln \xi - \xi + 1)\delta_{[0,\infty[}(\xi)$ and $g^* = \delta_{R_T^\circ}$. In our case the polar R_T° is just $R_T^\perp$, the subspace orthogonal to R_T. The conditions of the Fenchel theorem, part (a), are obviously fulfilled. Thus, J^o coincides with the (attained) value of the dual problem

$$f^*(-\xi) + g^*(\xi) \to \min \quad \text{on } L^0,$$

i.e. J^o is equal to the minimum of $f^*(-\xi)$ on the set $R_T^\perp \cap L^0_+ = \mathbf{R}_+ \mathcal{Z}_T^a$. Since

$$\inf_{\xi \in \mathcal{Z}_T^a} \inf_{t \geq 0} E(t\xi \ln t\xi - t\xi + 1) = \inf_{\xi \in \mathcal{Z}_T^a} (1 - e^{-E\xi \ln \xi}) = 1 - e^{-\underline{J}},$$

we get the result.

Remark. If $\mathcal{Z}^e \neq \emptyset$, the hypothesis of Fenchel theorem, part (b), holds, ensuring the existence in the primal problem. In the case where $\mathcal{Z}^e = \emptyset$ there is an arbitrage strategy H^a with $\eta^a := H^a \cdot S_T \geq 0$ and $\eta^a \neq 0$. Clearly, for any $\eta \in R_T$ the value of the functional in (1.1.5) at $\eta^a + \eta$ is strictly larger than at η.

Proposition 1.1.23 *Let H^o be the optimal strategy for the problem of portfolio optimization, Then the random variable*

$$\xi^o := e^{-H^o \cdot S_T} / E e^{-H^o \cdot S_T} \tag{1.1.10}$$

is the density of the minimal entropy equivalent martingale measure P^o.

Proof. The right-hand side of (1.1.10) is the density of a martingale measure. Indeed, for any strategy H the function

$$f_H(\lambda) = 1 - E e^{-H^o \cdot S_T + \lambda H \cdot S_T}$$

attains its maximum at $\lambda = 0$ and, therefore, $f'_H(0) = 0$, i.e.

$$E(H \cdot S_T) e^{-H^o \cdot S_T} = 0$$

implying the claimed property. Using it we can easily verify that

$$1 - e^{-E\xi^o \ln \xi^o} = 1 - E e^{-H^o \cdot S_T}.$$

Accordingly to Proposition 1.1.21 this equality may hold only if ξ^o is the solution of the problem of the entropy minimization.

1.2 Discrete-Time Infinite-Horizon Model

The aim of this section is to present relations between the absence arbitrage and the existence of equivalent martingale measure for the model where $\mathbf{R}^d$-valued process price process $S = (S_t)_{t=0,1,\ldots}$ defined on some filtered space $(\Omega, \mathcal{F}, \mathbf{F} = (\mathcal{F})_{t=0,1,\ldots})$. We assume that the initial σ-algebra is trivial. In the first subsection we discuss some purely probabilistic questions. In particular, we show that if S admits an equivalent local martingale measure, then it admits an equivalent martingale measure. Moreover, the latter can be chosen to ensure the integrability of an arbitrary adapted process fixed in advance. Afterwards we introduce some substitutes for the no-arbitrage property and proves necessary and sufficient conditions for them.

1.2.1 Martingale Measures in Infinite-Horizon Model

We consider the discrete-time infinite-horizon model with an $\mathbf{R}^d$-valued process $S = (S_t)_{t\geq 0}$ and introduce, for $p \geq 1$, the set $\mathcal{Q}^{e,p}$ of probability measures $Q \sim P$ such that S is a Q-martingale and $S_t \in L^p(Q)$ for all $t \geq 0$. We shall use also the standard notation $S_t^* := \sup_{s\leq t} |S_t|$.

Theorem 1.2.1 *Let $S \in \mathcal{M}_{loc}(P)$. Then there exists a probability measure $\tilde{P} \sim P$ such that $S \in \mathcal{M}(\tilde{P})$.*

In the case of finite time-horizon this assertion is direct corollary of the DMW theorem (and the measure $\tilde{P}_T \sim P$ even can be chosen with the bounded density $d\tilde{P}_T/dP$). For the infinite time horizon we get it from the following much more general assertion.

Theorem 1.2.2 *Let S be a local martingale, $Y = (Y_t)$ be an adapted process dominating S^*. Let $\varepsilon > 0$. Then there exists a measure $P' \sim P$ such that S is a P'-martingale, $Y_t \in L^1(P')$ for every t and $||P' - P|| \leq \varepsilon$.*

As an obvious corollary we have:

Theorem 1.2.3 *The set $\mathcal{Q}^{e,p}$ is dense in the set $\mathcal{Q}_l^e$.*

Theorem 1.2.2 is a generalization of Theorem 1.1.17. It is interesting that the reference to the latter constitutes the essential ingredients of the proof.

Lemma 1.2.4 *Let $S = (S_t)_{t\leq T}$ be a local martingale in $\mathbf{R}^d$ and let $\xi \in L_+^0$. Then for any $\varepsilon > 0$ there is a probability measure $P^\varepsilon \sim P$ with the density Z_T^ε such that $S = (S_t)_{t\leq T}$ is a martingale with respect to P^ε, $E|Z_T^\varepsilon - 1| < \varepsilon$, and $Z_T^\varepsilon(1+\xi)$ is bounded.*

Proof. We introduce the probability measure $P^1 = ce^{-\xi}P$. Since the NA-property holds for P, it holds for P^1. By the DMW theorem there is $P^2 \sim P^1$ with $dP^2/dP^1 \in L^\infty$ such that $S \in \mathcal{M}(P^2)$. Applying Theorem 1.1.17 with P^2 as the reference measure, we obtain that there exists a measure $P^\varepsilon \sim P^2$ with $dP^\varepsilon/dP^2 \in L^\infty$ such that $||P^\varepsilon - P|| < \varepsilon$ and $S \in \mathcal{M}(P^\varepsilon)$. The measure P^ε meets the requirements. □

Proof of Theorem 1.2.2. We may suppose that $\varepsilon < 1$. Take a sequence $\varepsilon_n > 0$ such that $\sum \varepsilon_n < \varepsilon/3$. We define recursively an auxiliary sequence of probability measures $P^n \sim P$ with bounded $\mathcal{F}_n$-measurable densities dP^n/dP. Let denote ζ^k the density process of P^k with respect to P, that is $\zeta_t^k = E(dP^k/dP|\mathcal{F}_t)$. Put

$$Z_t^n := \zeta_t^1 ... \zeta_t^n, \qquad \bar{Z}_t^n := E(Z_\infty^n|\mathcal{F}_t).$$

The construction will ensure the following properties:

(a) $\zeta_n^n(1+Y_n) \leq c_n$ for some constant c_n;

(b) $E(\bar{Z}_t^n S_t|\mathcal{F}_{t-1}) = \bar{Z}_{t-1}^n S_{t-1}$ for all t, i.e. $\bar{Z}^n S$ is a martingale;

(*c*) $||P^n - P|| \le \tilde{\varepsilon}_n := \varepsilon_n/(1+c_0...c_{n-1})$.

Using Lemma 1.2.4 we define a probability measure $P^1 \sim P$ with $\mathcal{F}_1$-measurable density dP^1/dP such that $\zeta_1^1(1+Y_1) \le c_1$ for some constant c_1, the process $(S_t)_{t\le 1}$ is a P^1-martingale and $||P^1 - P|| \le \varepsilon_1$. Note that the whole process S remains a local martingale with respect to P^1.

Suppose that the measures P^k for $k \le n-1$ are already constructed. Applying Lemma 1.2.4 to the $(d+1)$-dimensional martingale $(\bar{Z}_t^{n-1}S_t, \bar{Z}_t^{n-1})_{t\le n}$ we find a measure $P^n \sim P$ with $\mathcal{F}_n$-measurable density such that the properties (*a*), (*c*) hold and $(\bar{Z}_t^{n-1}S_t, \bar{Z}_t^{n-1})_{t\le n}$ is a P^n-martingale. The latter property means that $(\bar{Z}_t^{n-1}\zeta_t^n S_t, \bar{Z}_t^{n-1}\zeta_t^n)_{t\le n}$ is a martingale. The martingale $(\bar{Z}_t^{n-1}\zeta_t^n)$, having at the date $t=n$ the value $Z_n^{n-1}\zeta_n^n = Z_n^n = Z_\infty^n$, coincides with $(\bar{Z}_t^n)$. Thus, $\bar{Z}^n S$ is a martingale and the property (*b*) holds.

By virtue of our construction

$$E\sum|\zeta_\infty^n - 1| \le \sum \varepsilon_n < \infty,$$

hence, $\sum|\zeta_\infty^n - 1| < \infty$ a.s. Thus, Z_∞^n converges almost surely to some finite random variable $Z_\infty > 0$. Moreover, the convergence holds also in L^1 because

$$E|Z_\infty^n - Z_\infty^{n-1}| = EZ_\infty^{n-1}|\zeta_\infty^n - 1| \le c_0...c_{n-1}E|\zeta_\infty^n - 1| \le \varepsilon_n$$

and $\sum \varepsilon_n$ is finite. Also

$$E|Z_\infty - 1| \le \sum E|Z_\infty^n - Z_\infty^{n-1}| \le \sum \varepsilon_n < \varepsilon/3.$$

It remains to check that the probability measure $P' = (Z_\infty/EZ_\infty)P$ meets the requirements. Since $EZ_\infty \ge 1 - \varepsilon/3 \ge 2/3$, we have that

$$||P' - P|| = E|Z_\infty/EZ_\infty - 1| \le \frac{2E|Z_\infty - 1|}{EZ_\infty} \le \varepsilon.$$

It is easy to check that for each fixed t the sequence $Z_\infty^n Y_t$, $n = t, t+1, ...,$ is fundamental in L^1. Indeed, we use again the property (*a*) and (*c*):

$$\begin{aligned} E|Z_\infty^n Y_t - Z_\infty^{n-1}Y_t| &\le E|\zeta_n^n - 1|\zeta_1^1...\zeta_t^t Y_t \zeta_{t+1}^{t+1}...\zeta_{n-1}^{n-1} \\ &\le c_0....c_{n-1}E|\zeta_n^n - 1| \le \varepsilon_n. \end{aligned}$$

It follows that $Z_\infty^n Y_t$ and $Z_\infty^n S_t$ converges in L^1 to integrable random variables. Thus, $Z_\infty Y_t \in L^1$ and, in virtue of (*b*),

$$E(Z_\infty \Delta S_t|\mathcal{F}_{t-1}) = 0.$$

i.e. P' is a martingale measure. □

1.2.2 No Free Lunch for Models with Infinite Time Horizon

Infinite-horizon discrete-time market models based on the price process (S_t), $t = 0, 1, ...,$ pose new interesting mathematical problems related with the so-called doubling strategies or the St.-Petersburg game. It is well-known that if S is a symmetric random walk on integers, hence, a martingale, the strategy $H_t = 2^t I_{\{t \le \tau\}}$ where $\tau := \inf\{t \ge 1 : \ \Delta S_t = 1\}$ looks as an arbitrage opportunity: $H \cdot S_\infty = 1$. This strategy vanishes after the stopping time τ which is finite but not bounded. So, certain restrictions on strategies are needed to exclude a such one. A satisfactory criterion relating the existence of an equivalent martingale measure with a strengthened no-arbitrage property can be obtained by assuming that there is no trading after some bounded stopping time where the bound depend on the strategy. Using the concepts and notations developed above we can formalize this easily.

Let R_∞ be the union of all sets R_T, $T \in \mathbf{N}$, and let $A_\infty := R_\infty - L^0_+$.

The infinite-horizon model has the NA-*property* if $R_\infty \cap L^0_+ = \{0\}$ (or, equivalently, $A_\infty \cap L^0_+ = \{0\}$). In general, NA is weaker than the EMM-property claiming the existence of a probability measure $\tilde{P} \sim P$ such that S is a $\tilde{P}$-martingale. The simplest reinforcing of NA is the NFL-property ("no-free-lunch") suggested by Kreps: $\bar{C}^w_\infty \cap L^\infty_+ = \{0\}$ where $\bar{C}^w_\infty$ is the closure of the set $C_\infty := A_\infty \cap L^\infty$ in the topology $\sigma(L^\infty, L^1)$ (i.e. the weak* closure).

Theorem 1.2.5 *The following properties are equivalent:*

(a) $\bar{C}^w_\infty \cap L^\infty_+ = \{0\}$ *(NFL);*

(b) *there exists* $\tilde{P} \sim P$ *such that* $S \in \mathcal{M}_{loc}(\tilde{P})$*;*

(c) *there exists* $\tilde{P} \sim P$ *such that* $S \in \mathcal{M}(\tilde{P})$*.*

Proof. The Kreps–Yan theorem says that the NFL-property holds if and only if there exists $P' \sim P$ such that $E'\xi \le 0$ for all $\xi \in \bar{C}^w_\infty$. This P' can be called *separating measure* since its density is a functional from L^1 which separates $\bar{C}^w_\infty$ and L^∞_+. Of course, a local martingale measure $\tilde{P}$ is a separating one. Indeed, if $H \cdot S_T$ is bounded from below then the process $(H \cdot S_t)_{t \le T}$ is a $\tilde{P}$-martingale. Hence, for any bounded from below random variable $\xi = H \cdot S_T - h$ where $h \in L^0_+$ we have the inequality $\tilde{E}\xi \le 0$. It follows that this inequality holds for any $\xi \in \bar{C}^w_\infty$. This gives us the implication $(b) \Rightarrow (a)$. The implication $(a) \Rightarrow (b)$, more difficult, follows from Theorem 1.2.6 below ensuring that amongst the equivalent separating measures there is a local martingale measure. The equivalence $(b) \Leftrightarrow (c)$ follows from Theorem 1.2.2. $\square$

Theorem 1.2.6 *Any neighborhood of a separating measure contains an equivalent probability measure P' under which S is a local martingale.*

Proof. We assume without loss of generality that the reference measure P is separating. Fix $\varepsilon > 0$ and a sequence of numbers $\varepsilon_s > 0$ such that $\sum_{s \ge 1} \varepsilon_s < \varepsilon$.

The theorem will be proven if, for each $s \geq 1$, we can find an $\mathcal{F}_s$-random variable $\alpha_s > 0$ with the following properties:

(*i*) $E(\alpha_s|\mathcal{F}_{s-1}) = 1$;

(*ii*) $E(|1-\alpha_s||\mathcal{F}_{s-1}) \leq \varepsilon_s$;

(*iii*) $E(\alpha_s|\Delta S_s||\mathcal{F}_{s-1}) < \infty$, $E(\alpha_s \Delta S_s|\mathcal{F}_{s-1}) = 0$.

Indeed, let us consider the process $Z_t := \alpha_1 ... \alpha_t$, $t \geq 1$, $Z_0 = 1$, which is a martingale in virtue of (*i*). In virtue of (*ii*)

$$E|\Delta Z_s| = EZ_{s-1}E(|\alpha_s - 1||\mathcal{F}_{s-1}) \leq \varepsilon_s.$$

The martingale Z, being dominated by the integrable random variable, namely, by $1 + \sum |\Delta Z_s|$, is uniformly integrable. Also $E\sum|\alpha_s - 1| < \varepsilon$. Therefore, $\sum|\alpha_s - 1| < \infty$ a.s. and the infinite product $Z_\infty > 0$ a.s. Thus, the probability measure $\tilde{P} = Z_\infty P$ is equivalent to P. In virtue of (*iii*) the process S is a generalized martingale under $\tilde{P}$, i.e. belongs to the class coinciding with $\mathcal{M}_{loc}(\tilde{P})$. Moreover,

$$E|Z_\infty - 1| \leq E\sum_{s\geq 1}|\Delta Z_s| < \varepsilon.$$

Let $H_s \in L^0(\mathbf{R}^d, \mathcal{F}_{s-1})$ be such that the random variable $H_s\Delta S_s$ is bounded from below. Then $(H_s\Delta S_s)\wedge n$, being an element of C_∞, has a negative expectation – we assumed that P is a separating measure. By the Fatou lemma $EH_s\Delta S_s \leq 0$. In the proposition below we show that this ensures the existence of α_s with the required properties. □

So, we need the following one-step result.

Proposition 1.2.7 *Let $\mathcal{G}$ be a sub-σ-algebra of $\mathcal{F}$. Suppose that $\eta \in L^0(\mathbf{R}^d)$ is such that $E\gamma\eta \leq 0$ for any $\gamma \in L^0(\mathbf{R}^d, \mathcal{G})$ for which $\gamma\eta$ is bounded from below. Let $\varepsilon > 0$. Then there is a strictly positive random variable α such that $E(\alpha|\mathcal{G}) = 1$, $E(|1-\alpha||\mathcal{G}) \leq \varepsilon$, $E(\alpha|\eta||\mathcal{G}) < \infty$, and $E(\alpha\eta|\mathcal{G}) = 0$.*

Proof. Let $\mu(dx, \omega)$ be the regular conditional distribution of η with respect to $\mathcal{G}$. We consider in the space $\Omega \times C(\bar{\mathbf{R}}^d)$ the $\mathcal{G} \otimes \mathcal{B}(C(\bar{\mathbf{R}}^d))$-measurable set Γ defined as the intersection of the sets

$$\{(\omega, g): \ g > 0, \ E_{\mu(.,\omega)}g = 1, \ E_{\mu(.,\omega)}|1-g| \leq \varepsilon\}$$

and

$$\{(\omega, g): \ E_{\mu(.,\omega)}g|x| < \infty, \ E_{\mu(.,\omega)}gx = 0\}.$$

If the projection of Γ on Ω is of full measure we apply the measurable selection theorem, take an arbitrary $\mathcal{G}$-measurable selector $f: \Omega \to C(\bar{\mathbf{R}}^d)$ and conclude by putting $\alpha(\omega) = f(\omega, \eta(\omega))$.

Let Δ_ω be the image of the convex set

$$\{g \in C(\bar{\mathbf{R}}^d): \ g > 0, \ E_{\mu(.,\omega)}g = 1, \ E_{\mu(.,\omega)}g|x| < \infty, \ E_{\mu(.,\omega)}|1-g| \leq \varepsilon\}$$

under the linear mapping $\Psi_\omega := g \mapsto E_{\mu(.,\omega)}gx$. The full projection property means that for almost all ω the set Δ_ω contains the origin.

Let us consider first the case $d = 1$ where Δ_ω is just an interval. Define the $\mathcal{G}$-measurable random variables

$$\zeta'(\omega) = \inf\{t : \ \mu(]-\infty, t], \omega) > 0\}, \qquad \zeta''(\omega) = \sup\{t : \ \mu(]-\infty, t], \omega) < 1\}.$$

The random variables $I_A I_{\{-n\le\xi'\}}\eta$, where $A \in \mathcal{G}$, being bounded from below, have negative expectations. Hence, $I_{\{-\infty<\xi'\}}E(\eta|\mathcal{G}) \le 0$. This implies that

$$I_{\{-\infty<\xi'\}}E(\eta^+|\mathcal{G}) \le I_{\{-\infty<\xi'\}}E(\eta^-|\mathcal{G}) < \infty.$$

Therefore, $\Psi(1) \le 0$ on the set $\{-\infty < \xi'\}$ and, by symmetry, $\Psi(1) \ge 0$ on the set $\{\xi'' < \infty\}$ (a.s.). Thus, on the intersection on these sets, $\Psi(1) = 0$. It follows from the elementary lemma below that the interval $\Delta_\omega \supseteq [0,\infty[$ for almost all $\omega \in \{-\infty < \xi',\ \xi'' = \infty\}$. By symmetry, the interval $\Delta_\omega \supseteq]-\infty, 0]$ for almost all $\omega \in \{-\infty = \xi',\ \xi'' < \infty\}$. □

In the following assertion ω is fixed and omitted in notations.

Lemma 1.2.8 *If $\xi'' = \infty$, then Δ is unbounded from above.*

Proof. Fix $\varepsilon \in]0,1]$ and $a > 0$ such that $\mu(\{a\}) = 0$. Consider the subset $W_{\gamma,a}$ formed by the continuous functions g such that $g(a) = 1$, $xg(x) \to 0$ as $x \to \pm\infty$, and $E_\mu g I_{[a,\infty[} = \varepsilon$. Note that $\sup_{g\in W_{\varepsilon,a}} E_\mu xgI_{[a,\infty[} = \infty$. Indeed, as the support of μ is unbounded, we can find a continuous function g_0 with a compact support contained in the interval $]a,\infty[$ such that $E_\mu g_0 < \varepsilon$ while the value $E_\mu xg$ is arbitrarily large. Adding to g_0 the function $e^{-\lambda|x-a|}$ with an appropriately chosen parameter λ we obtain a function $g \in W_{\varepsilon,a}$ with $E_\mu xgI_{[a,\infty[} \ge E_\mu g_0$.

Take $a > 0$ such that $\mu(\{a\}) = 0$ and $\mu(\{x : \ |x| \ge a\}) \le \delta/2$. Take $f = e^{-\lambda|x+a|}$ and choose the parameter λ to ensure that

$$\varepsilon := \mu(\{x : \ |x| \ge a\}) - E_\mu f I_{]-\infty,a]} > 0.$$

By above, for any $N > 0$ we can find $f_N \in W_{\varepsilon,a}$ such that $E_\mu x f_N I_{[a,\infty[} \ge N$. The assertion became obvious since for function

$$g_N := fI_{]-\infty,-a[} + I_{]-a,a[} + g_N I_{]a,\infty[}$$

we have $\Psi(g_N) \ge N$. The lemma is proven. □

For the case of $d > 1$ and ω for which $0 \notin \Delta_\omega$ there is, in virtue of the Hahn–Banach theorem, $l(\omega) \in \mathbf{R}^d$ such that $|l(\omega)| = 1$ and $l(\omega)x > 0$ for all $x \in \Delta_\omega$. Put $l(\omega) = 0$ if $0 \in \Delta_\omega$. Using a measurable version of the Hahn–Banach theorem one can choose the separating functionals in such a way that the function $\omega \mapsto l(\omega)$ is $\mathcal{G}$-measurable. Applying the above reasoning to the scalar random variable $\eta^l := l\eta$ we find a function $f^l(\omega, y)$ on $\Omega \times \mathbf{R}$ which is $\mathcal{G}$-measurable in ω and continuous in x. Denoting by $\mu^l(dy, .)$ the regular conditional distribution of η^l with respect to $\mathcal{G}$, we get by the change of variable, that

$$l(\omega)\int_{\mathbf{R}^d} x f^l(\omega, l(\omega)x)\mu(dx,\omega) = \int_{\mathbf{R}} y f^l(\omega, y)\mu^l(dy,\omega) = 0.$$

Thus, $l = 0$ (a.s.) and the required property holds. □

Remark. Let P be a probability measure under which S is a local martingale and let H be a strategy such that the process $H \cdot S$ is bounded from below. Then this process is a true martingale converging at infinity to a random variable $H \cdot S_\infty$ almost surely. By the Fatou lemma $H \cdot S_t \geq E(H \cdot S_\infty|\mathcal{F}_t)$. Therefore, for this strategy $H \cdot S \geq 0$ if and only if $H \cdot S_\infty \geq 0$. These considerations show that there is a hope to get conditions for the existence of an equivalent local martingale measure based on strategies of such type. This is done in the next section.

In the following model the NA-property is fulfilled but there is no equivalent separating measure. Namely, $R_\infty \subset L^\infty$, $R_\infty \cap L^\infty_+ = \{0\}$, but $\bar{C}^w_\infty = L^\infty$!

Example. Let $\Omega = \mathbf{N}$, $P(\{2k-1\}) = P(\{2k\}) = 2^{-k-1}$, $k \geq 1$, and let $\mathcal{F}_t := \sigma\{\{1\},,,,\{2t\}\}$. Put $S_0 := 0$,

$$\Delta S_k = 2^{5k} I_{\{2k-1\}} + 2^{2k} I_{\{2k\}} - 2^{-k} I_{\{2k+1,\ldots\}}.$$

Since $\mathcal{F}_T$ is finite, the random variables $H \cdot S_T$ are bounded and $R_\infty \subseteq L^\infty$.

Let $0 \leq \xi \leq 1$. Then $S_T \wedge \xi \in C_\infty$ and $S_T \wedge \xi \to \xi$ in probability as T. Hence, $\xi \in \bar{C}^w_\infty$. It follows that $\bar{C}^w_\infty = L^\infty$.

Let $\eta \neq 0$ be a random variable from $R_\infty \cap L^\infty_+$, i.e. of the form $H \cdot S_T$. Let k be the first integer for which at least one of the values $\eta(2k-1)$ or $\eta(2k)$ is strictly positive. Inspecting sequentially the increments $H_t \Delta S_t$, we deduce that $H_t = 0$ for $t < k$ while the $\mathcal{F}_{k-1}$-measurable random variable $H_k(j)$ is equal to $a > 0$ for $j \geq k-1$. It follows that $H_k(j)\Delta S_k(j) \leq -ae^{-k}$ for $j \geq 2k+1$. The negative values at elementary events $2k+1$ and $2k+2$ can be compensated only if $H_{k+1}(j) \geq a2^{-k}$ for $j \geq 2k+1$. Continuing this inspection we arrive to the last increment, $H_T \Delta S_T$ which negative values on elementary events $2T+1, \ldots$ cannot be compensated.

More surprisingly, in this example the closure of R_∞ in the L^1-norm intersects L^∞_+ only at zero. This can be shown by a similar sequential inspection of $\lim_n H^n_t \Delta S_t$. To ensure the positivity of η, these random variables should take such large positive values at the elementary events with odd numbers larger than k that the L^1-norm of η would be infinite in an apparent contradiction.

1.2.3 No Free Lunch with Vanishing Risk

The NFL-condition can be criticized because the weak* closure has no good financial interpretation[1]. Fortunately, it can be replaced by the more attractive $NFLVR$-property.

[1] Of course, the definition of weak* closure involving only halfspaces is even simpler than of the norm closure. The intuition, though, appeals to the "interior" de-

To describe the latter we introduce the class of *admissible* strategies H whose value processes $H \cdot S$ are bounded from below (by constants depending on H) and converge a.s. to finite limits. Denoting R_{ad} the set of random variables $H \cdot S_\infty$, we define the sets $A_{ad} := R_{ad} - L^0_+$ and $C_{ad} := A_{ad} \cap L^\infty$.

We say that the process S has the $NFLVR$*-property* (no free lunch with vanishing risk) if $\bar{C}_{ad} \cap L^\infty_+ = \{0\}$ where $\bar{C}_{ad}$ is the norm-closure of C_{ad}. "Financial" motivation of the terminology is based on the alternative description: $NFLVR$*-property* holds if and only if P-$\lim \xi_n = 0$ for every sequence $\xi_n \in C_{ad}$ such that $||\xi_n^-||_{L^\infty} \to 0$, see Lemma 1.2.11.

Though the sets A and A_{ad} may be not related by an inclusion, the property $A_{ad} \cap L^0_+ = \{0\}$ ensures the property $A_\infty \cap L^0_+ = \{0\}$. Indeed, the former implies that for any finite T there is no arbitrage in the class of strategies with the value processes $(H \cdot S_t)_{t \le T}$ bounded from below. As we know, this is equivalent to the absence of arbitrage in the class of all strategies and, hence, to the existence of an equivalent martingale measure on $\mathcal{F}_T$. It follows that the property $A_{ad} \cap L^0_+ = \{0\}$ implies that the bound $H \cdot S_T \ge c$ propagates backwards and $C_\infty \subseteq C_{ad}$.

Theorem 1.2.9 *$NFLVR$ holds if and only if there is $P' \sim P$ such that $S \in \mathcal{M}_{loc}(P')$.*

Proof. It is easy to see (using the Fatou lemma) that a local martingale measure (and even a separating measure for R_∞) separates $\bar{C}_{ad}$ and L^∞_+. So, the implication "if" is obvious. On the other hand, the condition $\bar{C}^w_{ad} \cap L^\infty_+ = \{0\}$, ensuring that $C_\infty \subseteq C_{ad}$, implies the NFL-property and the needed measure P' does exist in virtue of Theorem 1.2.5. But according to Theorem 1.2.10 below such a condition holds because under $NFLVR$ the set $\bar{C}^w_{ad}$ coincides with $\bar{C}_{ad}$. □

Theorem 1.2.10 *Suppose that $\bar{C}_{ad} \cap L^\infty = \{0\}$. Then $C_{ad} = \bar{C}^w_{ad}$.*

Before the proof we establish some simple facts from functional analysis. Let $\lfloor \eta, \infty \lfloor$ be the set of $\xi \in L^0$ such that $\xi \ge \eta$.

Lemma 1.2.11 *Let C be a convex cone in L^∞ containing $-L^\infty_+$. Then the following properties are equivalent:*

(*a*) $\bar{C} \cap L^\infty_+ = \{0\}$;

(*b*) P-$\lim \xi_n = 0$ *for every sequence* $\xi_n \in C$ *such that* $||\xi_n^-||_{L^\infty} \to 0$;

(*c*) *the set* $C \cap \lfloor -1, \infty \lfloor$ *is bounded in probability.*

Proof. $(a) \Rightarrow (b)$ If the assertion fails, one can find a sequence $\xi_n \in C$ such that $\xi_n \ge -1/n$ and $P(\xi_n > \varepsilon) \ge \varepsilon$ for some $\varepsilon > 0$. Since $\xi_n \wedge 1 \in C$ we may assume that $\xi_n \le 1$. By the von Weizsäcker theorem there are random variables of the form $\bar{\xi}_k = k^{-1} \sum_{i=1}^k \xi_{n_i}$ (thus, elements of C) convergent to

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scription, in terms of limits. In general, the weak* sequential closure lays strictly between the norm-closure and weak* closure. To get all points of the latter as limits one needs to consider convergence along the nets which is, indeed, not intuitive.

a certain random variable ξ a.s. Note that the negative parts of $\bar{\xi}_k$ converges to zero in L^∞. On the other hand, ξ is not zero. Indeed, ξ is also the limit of $k^{-1}\sum_{i=1}^k \tilde{\xi}_{n_i}$ where $\tilde{\xi}_n := \xi_n + 1/n \ge 0$ and $P(\tilde{\xi}_n > \varepsilon) \ge \varepsilon$. It is easy to see that

$$Ee^{-\tilde{\xi}_n} \le P(\tilde{\xi}_n \le \varepsilon) + e^{-\varepsilon}P(\tilde{\xi}_n > \varepsilon) \le 1 - \varepsilon + e^{-\varepsilon}\varepsilon < 1.$$

Due to convexity of the exponential the same bound holds for the convex combinations of $\tilde{\xi}_n$, thus, for the limit ξ. So, $\beta := P(\xi > 0) > 0$. By the Egorov theorem, there is a measurable set Γ with $P(\Gamma) > 1 - \beta/2$ on which the convergence $\bar{\xi}_n \to \xi$ is uniform. But then the sequence $\bar{\xi}_n^+ I_\Gamma - \bar{\xi}_n^-$ of elements of C converges in L^∞ to a non-zero random variable $\xi I_\Gamma \ge 0$ in contradiction with (a).

$(b) \Rightarrow (c)$ If the set $C \cap \lfloor -1, \infty \lfloor$ is unbounded in probability, then it contains a sequence of random variables $\xi_n^0 \ge -1$ such that $\lim P(\xi_n^0 \ge n) > 0$. But then the sequence $\xi_n := \xi_n^0/n$ violates the condition (b).

$(c) \Rightarrow (a)$ If (a) fails to be true, there exists a sequence $\xi_n \in C$ and a non-zero $\xi \in L_+^\infty$ such that $||\xi - \xi_n||_{L^\infty} \le 1/n$. It follows that $||\xi_n^-||_{L^\infty} \le 1/n$. Then the random variables $n\xi_n$ belongs to $C \cap \lfloor -1, \infty \lfloor$ and form a sequence divergent to infinity on the set $\{\xi > 0\}$ and, therefore, is not bounded in probability. □

The next lemma, comparatively with the previous one, requires a specific structure of the cone C. We use the notation $\bar{K}^P$ for the closure of K in L^0.

Lemma 1.2.12 *Let $C = (K - L_+^0) \cap L^\infty$ where K is a cone, $K \subseteq \lfloor -1, \infty \lfloor$. Suppose that K is bounded in probability. Let ξ_n be a sequence in $C \cap \lfloor -1, \infty \lfloor$ convergent to ξ a.s. Then the set $\bar{K}^P \cap \lfloor \xi, \infty \lfloor$ is non-empty and contains a maximal element η_0.*

Proof. In virtue of the assumed structure of the set C there are $\eta_n \in K$ such that $\eta_n \ge \xi_n$. Applying the von Weizsäcker theorem we find a subsequence such that $\bar{\eta}_k := k^{-1}\sum_{i=1}^k \eta_{n_i}$ converge a.s. to to some $\bar{\eta} \ge \xi$. Since K is bounded in probability, so is the set $\bar{K}^P$. Thus, $\bar{\eta}$ is finite and belongs to $\bar{K}^P \cap \lfloor \xi, \infty \lfloor \ne \emptyset$. It remains to recall that any non-empty closed bounded subset of L^0 has a maximal element with respect to the natural partial ordering (each linearly ordered subset $\{\zeta_\alpha\}$ has as a majorant $\operatorname{ess\,sup}_\alpha \zeta_\alpha < \infty$ and the existence of the maximal element holds by the Zorn lemma). □

Lemma 1.2.13 *Let $C_{ad} \cap L_+^\infty = \{0\}$. If H is an admissible integrand, then $H \cdot S_\infty \ge -1$ if and only if the process $H \cdot S \ge -1$.*

Proof. Suppose that H is admissible and $H \cdot S_\infty \ge -1$ but there is u such that $P(\Gamma_u) > 0$ where $\Gamma_u := \{H \cdot S_u < -1\}$. Then the strategy $HI_{[u,\infty[}I_{\Gamma_u}$ is admissible, the random variable $HI_{]u,\infty[}I_{\Gamma_u} \cdot S_\infty \ge 0$ and strictly positive on Γ_u. This is a contradiction with the assumption of the lemma. □

Proof of Theorem 1.2.10. According to the Krein–Šmulian theorem a convex set is closed in $\sigma\{L^\infty, L^1\}$ if an only if its intersection with every ball of L^∞ is closed in probability. Obviously, the last condition follows if the set is *Fatou-closed*, that is if it contains the limit of any bounded from below sequence of its elements convergent almost surely. So, let ξ_n be a sequence in C_{ad} convergent to ξ a.s. and such that all $\xi_n \geq -c$. It is sufficient to argue with $c = 1$. We apply Lemma 1.2.12 with $K = R_{ad} \cap \lfloor -1, \infty \lfloor$ which is bounded in probability by virtue of Lemma 1.2.11. The theorem will be proven if we show that a maximal element η_0 in $\bar{K}^P \cap \lfloor \xi, \infty \lfloor \neq \emptyset$ belongs to K. So, we have a sequence $V^n := H^n \cdot S \geq -1$ with $V^n_\infty \to \eta$ a.s. We claim that $\sup_t |V^n_t - V^m_t| \to 0$ in probability as $n, m \to \infty$. If this not true than $P((\sup_t (V^{i_k}_t - V^{j_k}_t)^+ > \varepsilon) \geq \varepsilon$ with some $\varepsilon > 0$ and $i_k, j_k \to \infty$. For $T_k := \inf\{t : \ V^{i_k}_t - V^{j_k}_t > \alpha\}$ we have $P(T_k < \infty) \geq \varepsilon$. Let us consider the process

$$\tilde{V}^k := (I_{[0,T_k]} H^{i_k} + I_{]T_k,T]} H^{j_k}) \cdot S$$

which is an element of $K = R_{ad} \cap \lfloor -1, \infty \lfloor$. Note that

$$\tilde{V}^k_\infty = V^{i_k}_\infty I_{\{T_k = \infty\}} + V^{j_k}_\infty I_{\{T_k < \infty\}} + \xi_k$$

where $\xi_k := (V^{i_k}_{T_k} - V^{j_k}_{T_k}) I_{\{T_k < \infty\}} \geq 0$ and $P(\xi_k \geq \varepsilon) \geq \varepsilon$. Using the von Weizsäcker theorem in a same way as in the proof of Lemma 1.2.11, we find a sequence $\bar{V}^k \in K$ such that $\bar{V}^k_\infty \to \eta_0 + \xi$ where $\xi \in L^0$ and $\xi \neq 0$. This contradicts the maximality of η_0.

Taking a subsequence we may assume that $\sup_t |V^n_t - V^m_t| \to 0$ a.s. Thus, there is a process V which is a uniform limit of V^n (a.s.). Obviously, $V \geq -1$ and the limit V_∞ exists and is finite. Since $\Delta V^n_t = H^n_t \Delta S_t$ converges to ΔV_t and, as R_T is closed, $\Delta V_t = H_t \Delta S_t$. □

1.2.4 Example: "Retiring" Process

Here we present an example where a martingale measure can be constructed in a rather straightforward way. We shall use the result later, in the study of models with transaction costs.

Let $S = (S_t)_{t \geq 0}$ be an $\mathbf{R}^d$-valued discrete-time adapted process. Put $\xi_t = \Delta S_t$, $\Gamma_t := \{\xi_t = 0\}$.

Proposition 1.2.14 *Suppose that the following conditions hold:*

(i) for each finite T the process $(S_t)_{t \leq T}$ has the NA-property;

(ii) $I_{\Gamma_t} \uparrow 1$ a.s.;

(iii) $E(I_{\Gamma_t} | \mathcal{F}_{t-1}) > 0$ a.s. on Γ^c_{t-1} for each $t \geq 1$.

Then there exists a probability $Q \sim P$ such that S is a Q-martingale bounded in $L^2(Q)$ (hence, uniformly integrable with respect to Q).

Proof. By the DMW theorem condition (i) is equivalent to the NA-property for each one-step model: the relation $\gamma \xi_t \geq 0$ with $\gamma \in L^0(\mathbf{R}^d, \mathcal{F}_{t-1})$ may hold only if $\gamma \xi_t = 0$. The same theorem asserts that each ξ_t admits an equivalent martingale measure which can be chosen to ensure the integrability of any fixed finite random variable, e.g., $|\xi_t|^2$. In terms of densities this means that there are $\mathcal{F}_t$-measurable random variables $\bar{\alpha}_t > 0$ such that $E(\bar{\alpha}_t \xi_t|\mathcal{F}_{t-1}) = 0$ and $c_t := E(\bar{\alpha}_t|\xi_t|^2|\mathcal{F}_{t-1}) < \infty$. Normalizing, we can add to this also the property $E(\bar{\alpha}_t|\mathcal{F}_{t-1}) = 1$.

We define a $\mathcal{F}_t$-measurable random variable $\alpha_t > 0$ by the formula

$$\alpha_t = I_{\Gamma_{t-1}} + \left[\frac{(1-\delta_t)I_{\Gamma_t}}{E(I_{\Gamma_t}|\mathcal{F}_{t-1})} + \frac{\delta_t \bar{\alpha}_t I_{\Gamma_t^c}}{E(\bar{\alpha}_t I_{\Gamma_t^c}|\mathcal{F}_{t-1})}\right] I_{\Gamma_{t-1}^c \cap A_t} + I_{\Gamma_{t-1}^c \cap A_t^c},$$

where $A_t := \{E(\bar{\alpha}_t I_{\Gamma_t^c}|\mathcal{F}_{t-1}) > 0\}$ and $\delta_t := 2^{-t}E(\bar{\alpha}_t I_{\Gamma_t^c}|\mathcal{F}_{t-1})/(1+c_t)$. Clearly, $E(\alpha_t|\mathcal{F}_{t-1}) = 1$.

Noting that $\bar{\alpha}_t I_{\Gamma_t^c} I_{A_t^c} = 0$ (a.s.), we obtain that $E(\alpha_t \xi_t^2|\mathcal{F}_{t-1}) \le 2^{-t}$ and $E(\alpha_t \xi_t|\mathcal{F}_{t-1}) = 0$.

The process $Z_t := \alpha_1 ... \alpha_t$ is a martingale which converges (stationarily) a.s. to a random variable $Z_\infty > 0$ with $EZ_\infty \le 1$. Recalling that $I_{\Gamma_t} \uparrow 1$ (a.s.) and using the identity $Z_\infty I_{\Gamma_t} = Z_t I_{\Gamma_t}$, we obtain that

$$EZ_\infty = E \lim_t Z_\infty I_{\Gamma_t} = \lim_t EZ_\infty I_{\Gamma_t} = \lim_t EZ_t I_{\Gamma_t} = 1 - \lim_t EZ_t I_{\Gamma_t^c}.$$

It follows that $EZ_\infty = 1$ (i.e. (Z_t) is uniformly integrable martingale). Indeed, $E(\alpha_k I_{\Gamma_k^c}|\mathcal{F}_{k-1} \le 2^{-k}$ and, hence,

$$EI_{\Gamma_t^c} Z_t = E\prod_{k\le t} \alpha_k I_{\Gamma_k^c} \le \prod_{k \le t} 2^{-k} \to 0.$$

Thus, $Q := Z_\infty P$ is a probability measure under which S is a martingale. At last,

$$E_Q S_t^2 = \sum_{k\le t} EZ_k \xi_k^2 \le \sum_{k \le t} 2^{-k} \le 1,$$

i.e. S_t belongs to the unit ball of $L^2(Q)$. □

Remark 1. The condition (iii) cannot be omitted. Indeed, let S be the symmetric random walk starting from zero and stopped at the moment when it hits unit. It is a martingale and the condition (ii) holds. Since $S_\infty = 1$ a.s., the process S cannot be a uniformly integrable martingale with respect to a measure Q equivalent to P.

Remark 2. Fix $f : \mathbf{R}^d \to \mathbf{R}_+$. A minor modification of the arguments leads to a martingale measure Q for which $E_Q \sup_t f(S_t) < \infty$. Indeed, let (η_t) be an adapted process with $\eta_t = \eta_t I_{\Gamma_t^c} \ge 0$. As above, we can find α_t with the extra property $E(\alpha_t f(S_t)|\mathcal{F}_{t-1}) \le 2^{-t}$ implying that $E\sum_t \eta_t < \infty$. It remains to take $\eta_t = f(S_t)I_{\Gamma_t^c}$ and note that $\sup_t f(S_t) \le \sum_t \eta_t$.

1.2.5 The Delbaen–Schachemayer Theory in Continuous Time

This book is addressed to the reader from whom we do not expect a knowledge of stochastic calculus, beyond standard textbooks. Luckily, the theory of markets with transaction costs, in current state of art, does not require such a knowledge, in a surprising contrast to the classical continuous-time NA-theory initiated by Kreps and largely developed in a series of papers by Delbaen and Schachermayer collected in [24]. However, it seems to be useful to provide a short abstract of the main results of the latter which will serve as a background for a discussion explaining this difference.

In the classical continuous-time theory we are given a set $\mathcal{X}$ of scalar semimartingales X on a compact interval $[0,T]$ interpreted as value processes; the elements of $R_T := \{X_T : \ X \in \mathcal{X}\}$ are the investor's "results"; the NA-property means that $R_T \cap L^0_+ = \{0\}$. Typically, $\mathcal{X}$ is the set of stochastic integrals $H \cdot S$ where S is a fixed d-dimensional semimartingale (interpreted as the price processes of risky assets) and H is a d-dimensional predictable process for which the integral is defined and is bounded from below by a constant depending on H. The condition on H ("admissibility") rules out the doubling strategies. The experience with discrete-time models gives a hint that martingale densities can be obtained by a suitable separation theorem. Put $C_T := (R_T - L^0_+) \cap L^\infty$ (the set of bounded contingent claims hedgeable from zero initial endowment) and introduce the "no free lunch condition" (NFL): $\bar{C}^w_T \cap L^\infty_+ = \{0\}$ where $\bar{C}^w_T$ is a closure of C_T in the weak* topology, i.e. $\sigma\{L^\infty, L^1\}$. The Kreps–Yan theorem Th. 1.1.4 says that NFL holds if and only if there exists an equivalent "separating" measure $P' \sim P$ such that $E'\xi \le 0$ for all ξ from $\bar{C}^w_T$ (or R_T). It is easy to see that in the model with the bounded (resp., locally bounded) price process S the latter is a martingale (resp. local martingale).

The above result established by Kreps in the context of financial modelling ("FTAP") was completed by Delbaen and Schachermayer by a number of important observations for the model based on the price process S. We indicate here only a few.

First, they observed that in the Kreps theorem the condition NFL can be replaced by a visibly weaker (but, in fact, equivalent) condition "no free lunch condition with vanishing risk" $(NFLVR)$: $\bar{C}_T \cap L^\infty_+ = \{0\}$ where $\bar{C}_T$ is the norm-closure of C_T in L^∞. The reason for this is in the following simply formulated (but difficult to prove) result from stochastic calculus:

Theorem 1.2.15 *Let the $NFLVR$-condition be fulfilled. Then $C_T = \bar{C}^w_T$.*

This result which is a generalization of Theorem 1.2.9 can be formulated in a more abstract way, for a convex set $\mathcal{X}$ of bounded from below semimartingales which satisfies some closedness and concatenation properties.

Second, they establish that in any neighborhood of a separating measure P' there exists an equivalent probability measure $\tilde{P}$ (also a separating one) such that the semimartingale S with respect to $\tilde{P}$ is a σ-martingale (i.e. for some predictable integrants G^i with values in $]0,1]$ the processes $G^i \cdot S^i$, $i = 1, ..., d$, are $\tilde{P}$-martingales). The situation for the continuous-time is rather different even with respect to infinite-horizon discrete-time models: one cannot claim the existence of an equivalent local martingale measure! The reason for this is clear: in discrete time there is no difference between local martingales and σ-martingales (which are just generalized martingales).

As we shall see further, for the model with transaction costs the portfolio processes are vector-valued and their dynamics can be described using only the Lebesgue integrals. In the case of zero transaction cost one can make a reduction to scalar wealth processes $H \cdot S$ but the resulting H are (vector-valued) processes of **bounded variation** and not arbitrary integrands which is, apparently, an additional complication. In a general case the problem of no-arbitrage criteria has also other particularities arising even in the discrete-time framework.

2

Arbitrage Theory under Transaction Costs

2.1 Models with Transaction Costs

2.1.1 Basic Model

We describe here a financial framework leading to a "standard" discrete-time model with proportional transaction costs with complete information.

Suppose that the agent portfolio contains d assets which we prefer to interpret as currencies. Their quotes are given in units of a certain *numéraire* which may not be a traded security. At time t the quotes are expressed by the vector of prices $S_t = (S_t^1, \dots, S_t^d)$; its components are strictly positive.

The agent's positions can be described either by the vector of "physical" quantities $\widehat{V}_t = (\widehat{V}_t^1, \dots, \widehat{V}_t^d)$ or by the vector $V_t = (V_t^1, \dots, V_t^d)$ of values invested in each asset; they are related as follows:

$$\widehat{V}_t^i = V_t^i / S_t^i, \quad i \le d.$$

This formula suggests the notation $\widehat{V}_t = V_t / S_t$. More formally, introducing the diagonal operator

$$\phi_t : (x^1, ..., x^d) \mapsto (x^1/S_t^1, ..., x^d/S_t^d), \tag{2.1.1}$$

we may write that

$$\widehat{V}_t = \phi_t V_t.$$

In the considered market any asset can be exchanged to any other. At time t, the increase of the value of ith position in one unit of the numéraire by changing the value of jth position requires diminishing the value of the latter in $1 + \lambda_t^{ji}$ units of the numéraire. The matrix of transaction cost coefficients $\Lambda_t = (\lambda_t^{ij})$ has non-negative entries and the zero diagonal.

In the dynamical multiperiod setting $S = (S_t)$ is an adapted process as well as $\Lambda = (\Lambda_t)$; it is convenient to choose the scales to have $S_0^i = 1$ for all i and assume as a convention that $S_{-1}^i = 1$.

The portfolio evolution can be described by the initial condition $V_{-1} = v$ (the endowments of the agent when entering the market) and the increments at dates $t \geq 0$:

$$\Delta V_t^i = \widehat{V}_{t-1}^i \Delta S_t^i + \Delta B_t^i, \tag{2.1.2}$$

with

$$\Delta B_t^i := \sum_{j=1}^d \Delta L_t^{ji} - \sum_{j=1}^d (1+\lambda_t^{ij})\Delta L_t^{ij}, \tag{2.1.3}$$

where $\Delta L_t^{ji} \in L^0(\mathbf{R}_+, \mathcal{F}_t)$ represents the net amount transferred from the position j to the position i at the date t. The first term in the right-hand side of (2.1.2) is due to the price increments while the second corresponds to the agent's own actions at the date t (made after the instant when the new prices were announced). These actions are charged by the amount

$$-\sum_{i=1}^d \Delta B_t^i = \sum_{i=1}^d \sum_{j=1}^d \lambda_t^{ij} \Delta L_t^{ij}$$

(broker's fees, taxes, etc.) diminishing the nominal portfolio value.

One can interpret the matrix (ΔL_t^{ij}) as the investor orders immediately executed by the trader: the entry (i,j) means "increase the jth position on ΔL_t^{ij} units of the numéraire in exchange with the i-th position; the transaction costs indicate that for this the trader has to decrease the value of the latter on $(1+\lambda_t^{ij})\Delta L_t^{ij}$ units of the numéraire. The latter quantities also can be interpreted as orders.

Note that in the present setting the orders "to increase" ("to get") and "to decrease" ("to send") are related by a simple $\mathcal{F}_t$-measurable bijection $(\Delta L_t^{ij}) \mapsto (1+\lambda_t^{ij})\Delta L_t^{ij}$. So, only one of them is needed to describe the portfolio evolution. In the setting where the information available to the investor is given by a smaller filtration, the control of the portfolio can be done using orders of both types and not only in units of the numéraire but also in physical units. We shall discuss this issue in Section 2.5.

With every $\mathbf{M}_+^d$-valued process $L = (L_t)$ (in our notations $\mathbf{M}_+^d$ stands for the set of matrices with positive elements) and any initial endowment $v \in \mathbf{R}^d$ we associate, by the formula (2.1.2), a value process $V = (V_t)$, $t = 0, ..., T$. The terminal values of such processes form the set of "results" R_T^v.

Notice that, in "reasonable" orders, we can expect that $\Delta L_t^{ij} \Delta L_t^{ji} = 0$, i.e. there are no bidirectional fund transfers. However, it is convenient not to exclude "unreasonable orders": their inclusion has no effect on the results we are interested in, namely, on no-arbitrage criteria and hedging theorems. Similarly, one may assume the "free disposal" of assets, that is enlarge the class of strategies by extracting from B^i increasing adapted processes (in the notations we develop below this means that $\Delta B_t \in -L^0(K_t, \mathcal{F}_t)$ rather than $\Delta B_t \in -L^0(M_t, \mathcal{F}_t)$).

Finishing with the modelling issues we look now for an appropriate mathematical setting.

Observe that the relation (2.1.2) is, in fact, a linear controlled difference equation (a vector one) of a very simple structure with the components connected only via controls:

$$\Delta V_t^i = V_{t-1}^i \Delta Y_t^i + \Delta B_t^i, \quad V_{-1}^i = v^i, \tag{2.1.4}$$

where

$$\Delta Y_t^i = \frac{\Delta S_t^i}{S_{t-1}^i}, \quad Y_0^i = 1, \tag{2.1.5}$$

with B given by (2.1.3). Since the dynamics are driven by the d-dimensional process B, we can diminish the dimension of the phase space of controls and choose B as the control strategy. Indeed, any $\Delta L_t \in L^0(\mathbf{M}_+^d, \mathcal{F}_t)$ defines the $\mathcal{F}_t$-measurable random variable ΔB_t with values in the set $-M_t$ where

$$M_t := \Big\{ x \in \mathbf{R}^d : \ \exists\, a \in \mathbf{M}_+^d \text{ such that } x^i = \sum_{j=1}^d [(1+\lambda_t^{ij})a^{ij} - a^{ji}], \ i \le d \Big\}.$$

Vice versa, a simple measurable selection arguments show that any portfolio increment $\Delta B_t \in L^0(-M_t, \mathcal{F}_t)$ is generated by a certain (in general, not unique) order $\Delta L_t \in L^0(\mathbf{M}_+^d, \mathcal{F}_t)$. Note that this does work only in the case of full information: if the investor's actions are measurable with respect to a smaller filtration such a reduction is impossible.

We shall denote by $\mathcal{B}$ the set of control strategies, i.e. of the processes $B = (B_t)$ with $\Delta B_t \in -M_t$ (i.e., more formally, with $\Delta B_t \in L^0(-M_t, \mathcal{F}_t)$).

It is useful to look at the dynamics of the portfolio in "physical units". It is given by a simpler formula

$$\Delta \widehat{V}_t^i = \frac{\Delta B_t^i}{S_t^i}, \quad V_{-1}^i = v^i, \qquad i \le d, \tag{2.1.6}$$

which can be written also as

$$\Delta \widehat{V}_t = \widehat{\Delta B}_t, \qquad -\widehat{\Delta B}_t \in \widehat{M}_t := \phi_t M_t.$$

Financially, it is absolutely obvious (the increments of positions now are due to fund transfers only and do not depend on price movements) but this can be also checked formally. A closer look reveals that the formula is a half-step to solve the linear nonhomogeneous equation: the second half-step yield the solution via the discrete analog of the Cauchy formula (with S, the solution of the linear equation (2.1.5) playing the role of the "exponential" of Y):

$$V_t^i = S_t^i \widehat{V}_t^i = S_t^i \left(v^i + \sum_{s=0}^t \frac{\Delta B_s^i}{S_s^i} \right).$$

These trivial observations underlie the whole development of the discrete-time theory.

An important concept in the above setting is the *solvency cone* K_t (depending, in general, of ω). It is defined as the set of vectors $x \in \mathbf{R}^d$ for which one can find a matrix $a \in \mathbf{M}_+^d$ such

$$x^i + \sum_{j=1}^{d}[a^{ji} - (1+\lambda_t^{ij})a^{ij}] \geq 0, \ \ i \leq d.$$

In other words, K_t is the set of portfolios (denominated in units of the numéraire) which can be converted at time t, paying the transactions costs, to portfolios without short positions (i.e. without debts in any asset). Clearly, $K_t = M_t + \mathbf{R}_+^d$ and $\widehat{K}_t = \widehat{M}_t + \mathbf{R}_+^d$ the solvency cone when the accounting of assets (e.g., currencies) is done in terms of physical units.

In this model the *contingent claim* ξ is just a d-dimensional random variable. To *hedge* this contingent claim means to find a portfolio process V such that $V_T - \xi \in K_T$ (a.s.). Denoting by the symbol $\geq_T$ the partial ordering in $\mathbf{R}^d$ associated with the cone K_T (as usual: $x \geq_T 0$ means that $x \in K_T$) we may write this as the "inequality" $V_T \geq_T \xi$.

It is easy to see that the claim ξ is hedgeable if and only if there exists a portfolio process such that $V_T \geq \xi$ componentwise, i.e. in the sense of the partial ordering generated by the (smaller) cone $\mathbf{R}_+^d$. Indeed, if $V_T \geq_T \xi$, then $V_T - \xi \in K_T$ and, therefore, $V_T - \xi = \eta + \rho$ where $\eta \in L^0(M_T, \mathcal{F}_T)$ and $\rho \in L^0(\mathbf{R}_+^d, \mathcal{F}_T)$. Replacing the last transaction ΔB_T by $\Delta B_T - \eta$ we obtain a new value process which terminal value is $\xi + \rho$ and, hence, dominates ξ componentwise.

Introducing the model, we formulate two fundamental questions:

Problem 1. What is the analog of FTAP?

Problem 2. What is the analog of the hedging theorem?

Of course, the answers to these questions necessitates not only to define appropriate concepts of absence of arbitrage but also to find an analog to the notion of equivalent martingale measure or martingale density. As we shall see further, this can be easily done by placing the model in an adequate mathematical framework.

Turning back to the description of the cone M_t we observe that it is an image of the polyhedral cone $\mathbf{M}_+^d$ (or even of the smaller one $\tilde{\mathbf{M}}_+^d$ of matrices with positive entries and the zero diagonal) in the space $\mathbf{M}^d$ of $d \times d$ matrices under the linear mapping $\Psi : \mathbf{M}^d \to \mathbf{R}^d$ with

$$[\Psi((a^{ij}))]^i := \sum_{j=1}^{d}[(1+\lambda_t^{ij})a^{ij} - a^{ji}].$$

Thus, M_t is also a **polyhedral cone**. The cone $\tilde{\mathbf{M}}_+^d$ (which can be identified with $\mathbf{R}_+^{d(d-1)}$) is generated by $d \times (d-1)$ elements which are matrices with all zero entries except a single one equal to unit. The image of this set of generators is a generating set for M_t. Therefore, M_t is a (random) polyhedral cone, namely,

$$M_t = \text{cone}\,\{(1+\lambda_t^{ij})e_i - e_j,\ 1 \le i,j \le d\}.$$

Its dual positive cone

$$M_t^* := \{w :\ wx \ge 0\ \forall\, x \in M_t\} = \{w :\ (1+\lambda_t^{ij})w^i - w^j \ge 0,\ 1 \le i,j \le d\}.$$

Similarly, the cone K_t is the image of $\tilde{\mathbf{M}}_+^d \otimes \mathbf{R}_+^d$ under a linear mapping,

$$K_t = \text{cone}\,\{(1+\lambda_t^{ij})e_i - e_j,\ e_i,\ 1 \le i,j \le d\},$$

and its positive dual is

$$K_t^* = M_t^* \cap \mathbf{R}_+^d = \{w \in \mathbf{R}_+^d :\ (1+\lambda_t^{ij})w^i - w^j \ge 0,\ 1 \le i,j \le d\}.$$

It is an easy exercise to check that

$$\widehat{K}_t = \phi_t K_t = \text{cone}\,\{\pi_t^{ij} e_i - e_j,\ e_i,\ 1 \le i,j \le d\},$$

where

$$\pi_t^{ij} := (1+\lambda_t^{ij})S_t^j/S_t^i.$$

Note also that if there is a non-zero transaction costs coefficient λ_t^{ij}, then all basis orths e_i belong to $M_t = K_t$.

The above consideration shows that our “basic” model is nothing but a linear difference equation with additive control subjected to polyhedral cone constraints. The posed questions are related with properties of attainability sets of such equations and, of course, they can be addressed to more general ones. Linear equations can be solved and this make the analysis relatively easy, especially, in the case of finite Ω.

The solvency cone K_t can be generated by many matrices Λ_t. In theoretical analysis it is convenient to consider the matrix with minimal absolute norm $\sum_{ij} \lambda_t^{ij}$. Note that for this matrix

$$1+\lambda_t^{ij} \le (1+\lambda_t^{ik})(1+\lambda_t^{kj}), \qquad \forall\, i,j,k. \tag{2.1.7}$$

Indeed, if we have an opposite inequality then both vector $(1+\lambda_t^{ij})e_i - e_j$ and $(1+\tilde{\lambda}_t^{ij})e_i - e_j$ with $\tilde{\lambda}_t^{ij} := (1+\lambda_t^{ik})(1+\lambda_t^{kj}) - 1$ are conic combinations of the other generating vectors of K_t. Thus, replacing λ^{ij} by $\tilde{\lambda}^{ij}$ we obtain the same cone K_t diminishing the norm of Λ_t.

The financial interpretation of (2.1.7) is obvious: an “intelligent” investor will first try all possible chains of transfers from the ith position to the position j and act accordingly to a cheapest one, i.e. replacing effectively a given matrix of transaction costs by that with the minimal norm. However, in practice it is not always done (the real situation is more complicated than the model considered: the proportional transaction costs is already a simplification).

The linear space $K_t^0 := K_t \cap (-K_t)$ also has a simple financial interpretation. It is composed by the positions which can be converted to zero without paying transaction costs and vice versa. Indeed, let $x \in K_t \cap (-K_t)$. According to definition,

$$x^i = \sum_{j=1}^d [(1+\lambda_t^{ij})a^{ij} - a^{ji}] + h^i,$$
$$-x^i = \sum_{j=1}^d [(1+\lambda_t^{ij})\tilde{a}^{ij} - \tilde{a}^{ji}] + \tilde{h}^i.$$

Summing up, we get that

$$\sum_{i=1}^d \sum_{j=1}^d \lambda_t^{ij}(a^{ij} + \tilde{a}^{ij}) + \sum_{i=1}^d (h^i + \tilde{h}^i) = 0.$$

It follows that all summands here are zero and this leads to the claimed property.

Before going further we give in the next subsections a survey of other approaches or parameterizations which lead, essentially, to the same model with proportional transaction costs or its particular case. However, all this variety can be studied in a framework of the geometrical formalism we develop in this chapter.

2.1.2 Variants

1. Alternative parameterizations. In the literature one can find various specifications for transaction costs coefficients. To explain the situation, let us put $\Delta \tilde{L}_t^{ij} := (1+\lambda_t^{ij})\Delta L^{ij}$. The increment of value of the i-th position due to the agent's action can be written as

$$\Delta B_t^i = \sum_{j=1}^d \mu_t^{ji} \Delta \tilde{L}_t^{ji} - \sum_{j=1}^d \Delta \tilde{L}_t^{ij},$$

where $\mu_t^{ji} := 1/(1+\lambda_t^{ji})$ take value in the interval $]0,1]$. The matrix (μ_t^{ij}) can be specified as the matrix of the transaction costs coefficients.

Historically, the theory of models with transaction costs was initiated having in mind the interpretation of the stock market with an obvious *numéraire*: cash or bank account. In models with a traded numéraire, i.e. a non-risky asset, a mixture of both specifications is frequent. For example, for the two-asset model the dynamics quite often is written as:

$$\Delta V_t^1 = (1-\mu_t)\Delta M_t - (1+\lambda_t)\Delta L_t,$$
$$\Delta V_t^2 = V_{t-1}^2 \Delta Y_t^2 + \Delta L_t - \Delta M_t,$$

where $\Delta L_t \geq 0$ and $M_t \geq 0$ are $\mathcal{F}_t$-measurable random variables.

2. A model of stock market. In this model it is assumed that all transactions pass through the money: so the orders are either ”buy a stock”, or “sell a stock”. At time t they can be represented by $\mathcal{F}_t$-measurable random vectors $(\Delta L_t^2, ..., \Delta L_t^d)$ and $(\Delta M_t^2, ..., \Delta M_t^d)$.

The corresponding d-asset dynamics is given by the system

$$\Delta V_t^1 = \sum_{j=2}^{d}(1-\mu_t^j)\Delta M_t^j - \sum_{j=2}^{d}(1+\lambda_t^j)\Delta L_t^j,$$
$$\Delta V_t^i = V_{t-1}^i \Delta Y_t^i + \Delta L_t^i - \Delta M_t^i, \quad i = 2, ..., d.$$

The generators of the cone M_t in $\mathbf{R}^d$ are the vectors $-(1+\lambda_t^j)e_1 + e_j$, $(1-\mu_t^j)e_1 - e_j$, $j = 2, ..., d$. The solvency cone K_t is generating by this set augmented by the vector e_1. It is not difficult to see that it can be described as follows:

$$K_t = \Big\{ x \in \mathbf{R}^d : \; x^1 + \sum_{j=2}^{d}[(1-\mu_t^j)x^j I_{\{x^j>0\}} - (1+\lambda_t^j)x^j I_{\{x^j<0\}}] \geq 0 \Big\}.$$

Comparing this with the model of currency market (given by a matrix of transaction costs coefficients) we notice that it can be imbedded into the former by choosing sufficiently large transaction costs coefficients. Of course, this leads to a larger set of value processes but such a procedure has no effect on the arbitrage properties.

3. Modelling in physical units. Let us consider a kind of a barter market where we are given not quotes of assets in terms of a numéraire but only the matrix $\Pi = (\pi^{ij})$ (depending on t and ω) which generic entry $\pi^{ij} > 0$ represents a number of units of the ith asset needed to get in exchange one unit of the jth asset (of course, $\pi^{ii} = 1$). In the literature it is usually assumed that $\pi^{ij} \leq \pi^{ik}\pi^{kj}$, i.e., the direct exchange is better than two consecutive ones. As was explained above, this is not a loss of generality: an “intelligent” agent will act not according to a given matrix of exchange rates $\tilde{\Pi}$ but to a corrected one, with

$$\pi^{ij} = \min\{\tilde{\pi}^{ii_1} ... \tilde{\pi}^{i_n j}\}.$$

Now the solvency region, i.e. the set of $y \in \mathbf{R}^d$ for which one can find $c \in \mathbf{M}_+^d$ such that

$$y^i \geq \sum_{j=1}^{d}[\pi_t^{ij}(\omega)c^{ij} - c^{ji}], \quad i \leq d,$$

is cone $\{e_i,\ \pi^{ij}e_i - e_j,\ 1 \le i,j \le d\}$.

In dynamics, $\varPi = (\varPi_t)$ is an adapted process sometimes called in the literature the bid-ask process.

Obviously, the market model with the price quotes S_t and the transaction costs coefficients $\varLambda_t$ can be reformulated in terms of $\varPi$: in this case we have

$$\pi_t^{ij} = (1+\lambda_t^{ij})S_t^j/S_t^i, \qquad 1 \le i,j \le d.$$

Reciprocally, one can introduce in the barter market "money" and generate a price process S and a matrix $\varLambda$ of transaction costs coefficients. Indeed, take an arbitrary $S_t \in L^0(\widehat{K}_t^* \setminus \{0\}, \mathcal{F}_t)$ and note that the components of this process are strictly positive. Put

$$\lambda_t^{ij} := \pi_t^{ij} S_t^i/S_t^j - 1.$$

Clearly, $\lambda_t^{ij} \ge 0$ because by duality $S_t(\pi_t^{ij}e_i - e_j) \ge 0$. It remains to recall that the solvency cone in terms of physical units for the model described by S and $\varLambda_t = (\lambda_t^{ij})$ coincides with $\widehat{K}_t$.

Thus, modelling via $\varPi$ is nothing but just another parametrization of the model considered in the first subsection. The former has certain advantages: first, it follows the tradition already established in the financial literature, second, suits better to study portfolio optimization problems and, third, allows for a rather straightforward generalizations for the continuous-time setting.

4. Models where the transactions charge the bank account.

In this case the dynamics is given as follows:

$$\Delta V_t^1 = \sum_{j=2}^{d}(\Delta L_t^{j1} - \Delta L_t^{1j}) - \sum_{i,j=1}^{d} \gamma_t^{ij}\Delta L_t^{ij},$$

$$\Delta V_t^i = \widehat{V}_{t-1}^i \Delta S_t^i + \sum_{j=1}^{d}\Delta L_t^{ji} - \sum_{j=1}^{d}\Delta L_t^{ij}, \quad i = 2, ..., d,$$

where $\gamma_t^{ij} \in [0,1[$, $\gamma^{ii} = 0$. For this model, again linear and with polyhedral cone constraints on the controls, the solvency cone is always a polyhedral one:

$$K_t = \text{cone}\{\gamma^{ij}e_1 + e_i,\ (1+\gamma^{1i})e_1 - e_i,\ (-1+\gamma^{j1})e_1 + e_j,\ e_i,\ 1 \le i,j \le d\}.$$

5. Models with a price spread. Usually such a model is designed for stock markets, i.e. transactions are only buying or selling shares according to two price processes $\bar{S}$ and $\underline{S}$ where $\bar{S}^j \ge \underline{S}^j > 0$, $j = 2, ..., d$. Clearly, it can be given in terms of a single price or quote process and transaction cost coefficients. E.g., one can put $S_t := (\bar{S}_t + \underline{S}_t)$ and define $\lambda_t^j := \bar{S}_t^j/S_t^j - 1$, $\mu_t^j := 1 - \underline{S}_t^j/S_t^j$. The absence of arbitrage opportunities means that $R_T \cap L_+^0 = \{0\}$ where the "results" here are terminal values of the money component of the portfolio processes (in our terminology this will correspond to the NA^w-property).

Historically, the first criterion of absence of arbitrage was obtained for a model described in terms of bid and ask prices. The Jouini–Kallal theorem claims (under some conditions) that there is no-arbitrage if and only if there exist a probability measure $\tilde{P} \sim P$ and an $\mathbf{R}^{d-1}$-valued $\tilde{P}$-martingale $\tilde{S}$ such that $\underline{S}_t^j \le \tilde{S}_t^i \le \bar{S}_t^i$, $i = 2, ..., d$. In the case where $\underline{S} = \bar{S}$ this assertion coincides with the DMW theorem.

As we shall see further, for the model with finite Ω (or in the case where $d = 2$) the NA^w-property is equivalent to the existence of the martingale Z with strictly positive components which evolves in the duals to the solvency cones in physical units, i.e. such that $Z_t \in L^0(\widehat{K}_t^* \backslash \{0\}, \mathcal{F}_t)$. The latter property is more general and for specific models can be rewritten in the formulation due to Jouini and Kallal.

Let us consider the basic model assuming in addition that $S^1 \equiv 1$, i.e. the first asset is the numéraire ("money"), and for all i and j

$$(1 + \lambda^{i1})(1 + \lambda^{1j}) \le 1 + \lambda^{ij}.$$

This means that the direct exchanges are more expensive than those via money; they can be excluded at all (as it is usually done in stock market models). The cone K^* consists of all $w \in \mathbf{R}_+^d$ satisfying the inequalities

$$\frac{1}{1 + \lambda^{i1}} w^1 \le w^i \le (1 + \lambda^{1i}) w^1, \qquad i > 1.$$

Indeed, other inequalities defining K^* hold automatically: for any pair i, j we have:

$$w^j \le (1 + \lambda^{1j}) w^1 \le (1 + \lambda^{i1})(1 + \lambda^{1j}) w^i \le (1 + \lambda^{ij}) w^i.$$

Let $Z_t \in L^0(\widehat{K}_t^* \setminus \{0\}, \mathcal{F}_t)$ be a martingale. Normalizing, we can assume that $EZ_T^1 = 1$ and define the probability measure $\tilde{P} = Z_T^1 P$. The condition that Z evolves in $\widehat{K}^*$ reads as

$$\frac{1}{1 + \lambda^{i1}} Z^1 \le \frac{Z^i}{S^i} \le (1 + \lambda^{1i}) Z^1, \qquad i > 1.$$

Introducing the selling and buying prices

$$\underline{S}^i := \frac{1}{1 + \lambda^{i1}} S^i, \qquad \overline{S}^i := (1 + \lambda^{1i}) S^i,$$

we obtain that the process $\tilde{S} := Z/Z^1$ is a martingale with respect to $\tilde{P}$ and

$$\underline{S}^i \le \tilde{S}^i \le \overline{S}^i, \quad i > 1.$$

Thus, for models with this particular structure of the solvency cones the NA-criteria can be written in the formulation suggested by Jouini and Kallal.

2.1.3 No-arbitrage problem: NA^w for finite Ω

Considering our basic model described above we define the *strict arbitrage opportunity* as a strategy $B \in \mathcal{B}$ such that the terminal value V_T of the portfolio process $V = V^B$ given by (2.1.2) with $V_{0-} = 0$ belongs to $L^0(\mathbf{R}^d_+)$ but is not equal to zero. We shall say that a model has the *weak no-arbitrage property* (in symbols: NA^w) if it does not admit strict arbitrage opportunities. Denoting by R^0_T the set of terminal values of portfolios with zero initial endowments, we may rewrite the definition of NA^w in a conformity with the previous section: $R^0_T \cap L^0(\mathbf{R}^d_+) = \{0\}$ or, equivalently, $\widehat{R}^0_T \cap L^0(\mathbf{R}^d_+) = \{0\}$ where $\widehat{R}^0_T = \phi_T R^0_T$ is the set of attainable results in physical units.

We define also the set A^0_T of hedgeable claims $A^0_T := R^0_T - L^0(K_T, \mathcal{F}_T)$.

Let denote by $\mathcal{M}^T_0(\widehat{K}^* \setminus \{0\})$ the set of martingales $Z = (Z_t)_{t \le T}$ such that $Z_t \in L^0(\widehat{K}^*_t \setminus \{0\})$ for all t.

In the literature the elements of $\mathcal{M}(\widehat{K}^* \setminus \{0\})$ are sometimes referred to as *consistent price systems*. In the model with finite number of states of the nature the existence of the latter is equivalent to the absence of strict arbitrage opportunities.

Theorem 2.1.1 *Suppose that Ω is finite. Then the following conditions are equivalent:*

(*a*) $R^0_T \cap L^0(\mathbf{R}^d_+) = \{0\}$ *(i.e. NA^w);*

(*b*) $A^0_T \cap L^0(\mathbf{R}^d_+) = \{0\}$*;*

(*c*) $\mathcal{M}^T_0(\widehat{K}^* \setminus \{0\}) \ne \emptyset$.

Proof. Conditions (a) and (b) are obviously equivalent. Let us check the equivalence of (a) and (c). Without loss of generality we may assume that all elementary events of $\Omega = \{\omega_1, ..., \omega_N\}$ have strictly positive probabilities. Thus, the space of d-dimensional random variables can be identified with the Euclidean space of dimension $d \times N$ with the scalar product $(\xi, \eta) := E\xi\eta$. Note that if $G = \operatorname{cone}\{\xi_1, ..., \xi_m\}$ is a random cone generated by $\mathcal{F}_t$-measurable $\mathbf{R}^d$-valued random variables, then $L^0(G, \mathcal{F}_t)$ is a polyhedral cone generated by the random variables $\xi_i I_{\Gamma_j}$ where Γ_j are the atoms of the σ-algebra $\mathcal{F}_t$. Thus, being a sum of polyhedral cones, $\widehat{R}^0_T$ is also a polyhedral cone and so is $L^0(\mathbf{R}^d_+)$; the positive dual of the latter coincides with the primal one. Applying the Stiemke lemma as it is formulated in Lemma A.1.1 of the Appendix, we obtain that the condition (a) holds if and only if there exists a d-dimensional random variable η in the intersection of $(-\widehat{R}^{0*}_T)$ with the interior of $L^0(\mathbf{R}^d_+)$. This means that $E\xi\eta \le 0$ for all $\xi \in \widehat{R}^0_T$ and all components of η are strictly positive. It remains to notice that the martingale $Z_t = E(\eta|\mathcal{F}_t)$ belongs to $\mathcal{M}^T_0(\widehat{K}^* \setminus \{0\})$. Indeed, for any $\zeta \in L^0(\widehat{K}_t, \mathcal{F}_t) \subseteq -\widehat{R}^0_T + L^0(\mathbf{R}^d_+)$ we have that

$$EZ_t\zeta = E\eta\zeta \ge 0.$$

This property obviously means that $Z_t \in L^0(\widehat{K}^*_t, \mathcal{F}_t)$.

The above statement contains as a particular case the Harrison–Pliska theorem. To see this, suppose that $\Lambda = 0$ and the first asset is the numéraire, i.e. $\Delta S^1_t = 0$. Let $\bar{V}_t = \sum_{i \le d} V^i_t$. Summing up the relations (2.1.2) we get that the dynamics of $\bar{V}$ is given by

$$\Delta \bar{V}_t = \sum_{i=1}^{d} \widehat{V}^i_{t-1} \Delta S^i_t = H_t \Delta S_t$$

where $H_t \in L^0(\mathbf{R}^d, \mathcal{F}_{t-1})$. There is a single linear relation for the components $\widehat{V}^i_{t-1}$, namely, $\bar{V}_{t-1} = \bar{V}_t$, but it is of no importance since $\Delta S^1_t = 0$. Thus, the set of attainable random variables $\bar{V}_T$ is exactly the same as R_T in our model of frictionless market and the classical NA-condition $R_T \cap L^0_+ = \{0\}$ is equivalent to the NA^w-condition.

On the other hand, in the case $\Lambda = 0$ the cone $\widehat{K}^*_t = \mathbf{R}_+ S_t$ and, hence, the property $Z_t \in L^0(\widehat{K}^*_t, \mathcal{F}_t)$ means simply that $Z_t = \rho_t S_t$ for some $\rho_t \ge 0$. Thus, $Z \in \mathcal{M}^T_0(\widehat{K}^* \setminus \{0\})$ if and only if there exists a strictly positive martingale $\rho = (\rho_t)$ such that ρS is a martingale; normalizing, we may assume always that $E\rho_t = 1$.

It is worth noticing that the NA^w-condition can be reformulated in many various ways. In particular, below one can replace R^0_T and K_T by $\widehat{R}^0_T$ and $\widehat{K}_T$.

Proposition 2.1.2 *The following conditions are equivalent:*
(*a*) $R^0_T \cap L^0(K_T, \mathcal{F}_T) \subseteq L^0(\partial K_T, \mathcal{F}_T)$;
(*b*) $R^0_T \cap L^0(\mathbf{R}^d_+, \mathcal{F}_T) = \{0\}$;

Proof. The implication $(a) \Rightarrow (b)$ holds because $\mathbf{R}^d_+ \cap \partial K_T = \{0\}$. To prove the implication $(b) \Rightarrow (a)$ we notice that if $V^B_T \in L^0(K_T, \mathcal{F}_T)$ where $B \in \mathcal{B}$ then there exists $B' \in \mathcal{B}$ such that $V^{B'}_T \in L^0(\mathbf{R}^d_+, \mathcal{F}_T)$ and $V^{B'}_T(\omega) \ne 0$ on the set $V^B_T(\omega) \notin \partial K_T(\omega)$. To construct such B', it is sufficient to modify only ΔB_T by combining the last transfer with the liquidation of the negative positions.

As we shall see later, a straightforward generalization of Theorem 2.1.1 for arbitrary Ω, unlike the case of Harrison–Pliska and DWW theorems, fails to be true. Thus, one can try to find other definitions of arbitrage opportunities permitting to extend NA-criteria beyond models with finite number of states of the nature. The alternative definitions of the NA^w excluding the final liquidation gives a hint to alternative approaches.

2.1.4 No-Arbitrage Problem: NA^s for Finite Ω

Suppose that at time t investors portfolio is a subject of an audit. Auditors are not interested in real transaction costs needed to liquidate negative positions. Summing up the black and red figures (of course, latter with minus) they may obtain a strictly positive value. From their point of view, the investor has an arbitrage. Judging this as an exaggeration, we consider now a minor modification of the NA^w-property more acceptable from the point of view of the investor.

We say that a strategy B realized a *weak arbitrage opportunity* at time $t \le T$ if $V_t^B \in K_t$ but $P(V_t^B \notin K_t^0) > 0$ where $K_t^0 := K_t \cap (-K_t)$. Respectively, the absence of a such one at time t is referred to as a *strict no arbitrage* property NA_t^s:

$$R_t^0 \cap L^0(K_t, \mathcal{F}_t) \subseteq L^0(K_t^0, \mathcal{F}_t),$$

or, equivalently, in the realm of physical values:

$$\widehat{R}_t^0 \cap L^0(\widehat{K}_t, \mathcal{F}_t) \subseteq L^0(\widehat{K}_t^0, \mathcal{F}_t).$$

We use the notation NA^s when NA_t^s holds for every $t \le T$. Clearly, in the no-friction case this definition coincides with the classical one.

Theorem 2.1.3 *The following conditions are equivalent:*
(a) $R_T^0 \cap L^0(K_T, \mathcal{F}_T) \subseteq L^0(K_T^0, \mathcal{F}_T)$ *(i.e. NA_T^s holds);*
(b) $A_T^0 \cap L^0(K_T, \mathcal{F}_T) \subseteq L^0(K_T^0, \mathcal{F}_T)$*;*
(c) *there exists* $Z^{(T)} \in \mathcal{M}_0^T(\widehat{K}^* \setminus \{0\})$ *such that* $Z_T^{(T)} \in L^1(\operatorname{ri} \widehat{K}_T^*, \mathcal{F}_T)$.

Proof. The equivalence of (a) and (b) is obvious. The proof of the equivalence (a) $\Leftrightarrow$ (c) differs from that of Theorem 2.1.1 only that we use now Theorem A.1.3 instead of Lemma A.1.1.

As a direct corollary we obtain:

Theorem 2.1.4 *The following conditions are equivalent:*
(a) $R_t^0 \cap L^0(K_t, \mathcal{F}_t) \subseteq L^0(K_t^0, \mathcal{F}_t)$ *for all t (i.e. NA^s holds);*
(b) $A_t^0 \cap L^0(K_t, \mathcal{F}_t) \subseteq L^0(K_t^0, \mathcal{F}_t)$ *for all t;*
(c) *for each $t \le T$ there exists a process* $Z^{(t)} \in \mathcal{M}_0^t(\widehat{K}^* \setminus \{0\})$ *such that* $Z_t^{(t)} \in L^1(\operatorname{ri} \widehat{K}_t^*, \mathcal{F}_t)$.

Note that NA_T^s does not imply NA_t^s for $t < T$. In other words, a weak arbitrage opportunities may disappear next day.

Example. Let us consider the (deterministic) one-period two-asset model with $(S_0^1, S_0^2) = (1,1)$ and $(S_1^1, S_2^2) = (1,2)$. Assume that the entries of Λ are equal to zero except $\lambda^{12} = \lambda$. The vectors $(1,1)$ and $(1, 1+\lambda)$ are generators of K^*. Clearly, transfers at $T = 1$ cannot increase the value, so the only strategy to be inspected is with $\Delta B_0 = (-(1+\lambda), 1)$ (the transfers are $\Delta L_0^{12} = 1$, $\Delta L_0^{21} = 0$) and $\Delta B_1 = (0,0)$. So, $V_1^B = (-(1+\lambda), 2)$. For $\lambda \in [0,1[$ we have $V_1^B \in \operatorname{int} K$, i.e. B is a strict arbitrage opportunity, for $\lambda = 1$ the model satisfies NA_1^w condition but the strategy B is a weak arbitrage opportunity, and if $\lambda > 1$ the model enjoys NA_1^s property.

We can extend in time this model by assuming that at the second period S_2^2 takes values ε and $1/\varepsilon$, say, with probabilities $1/2$. For $\lambda = 1$ this model satisfies NA_2^s when the parameter $\varepsilon > 2$ (i.e. the price increment ΔS_2^2 takes a negative and a positive value).

A specific feature of the NA^s-property is that the above criterion holds true in the general case (i.e., without finiteness assumption on Ω) for important classes of models, namely, for models with constant coefficients and also for models with *efficient friction*, with $K_t^0 = \{0\}$. In both cases (c) is equivalent to the condition $\mathcal{M}_0^T(\mathrm{ri}\,\widehat{K}^*) \neq \emptyset$, see Theorem 2.2.1 and the accompanying discussion. However, without further assumptions this is not true. It happened that for a general Ω the condition $\mathcal{M}_0^T(\mathrm{ri}\,\widehat{K}^*) \neq \emptyset$ is equivalent to the so-called robust no-arbitrage property NA^r which excludes arbitrage even under better investment opportunities, i.e. when NA^w-property holds for a certain process $\tilde{\Lambda}_t = (\tilde{\lambda}_t^{ij})$ where $\tilde{\lambda}_t^{ij}(\omega) < \lambda_t^{ij}(\omega)$ if the latter coefficient is not zero. We shall analyze this in a more general geometric context.

Remark. The following example shows that even in the case of Ω consisting of two elementary events ω_1, ω_2 having equal probabilities the condition (c) does not imply that $\mathcal{M}_0^T(\mathrm{ri}\,\widehat{K}^*) \neq \emptyset$ and, hence, NA^s is weaker than NA^r. Indeed, let $\mathcal{F}_0$ be trivial, $\mathcal{F}_1 = \mathcal{F}_2 = \mathcal{F}$. Take now $\widehat{K}_0^* = \widehat{K}_2^* = \mathrm{cone}\,\{e_1 + e_2\}$ and let $\widehat{K}_1^*(\omega_1) = \mathrm{cone}\,\{e_1 + e_2, e_1 + 2e_2\}$, $\widehat{K}_1^*(\omega_2) = \mathrm{cone}\,\{e_1 + e_2, 2e_1 + e_2\}$. Clearly, one can find martingales $Z^{(t)}$, $t \le 2$, satisfying (c) but there is no one with values in $\mathrm{ri}\,\widehat{K}_t^*$, $t = 0, 1, 2$.

2.2 No-arbitrage Problem: Abstract Approach

2.2.1 NA^r- and NA^s-Properties: Formulations

We consider now the case of arbitrary Ω in an abstract framework which covers various discrete-time models of financial markets with proportional transaction costs (eventually, with additional linear inequality constraints). The development below is purely mathematical. To make the idea more transparent we prefer not to insist here on financial aspects. We use the notation G_t instead of our traditional $\widehat{K}_t$ hoping that the reader understand well that we have in mind the corresponding structure in the world of physical units.

In the present framework we are given a sequence of set-valued mappings $G = (G_t)$ called $\mathcal{C}$*-valued process* specified by a countable sequence of adapted $\mathbf{R}^d$-valued processes $X^n = (X_t^n)$ such that for every t and ω only a finite but non-zero number of $X_t^n(\omega)$ is different from zero and

$$G_t(\omega) := \mathrm{cone}\,\{X_t^n(\omega),\ n \in \mathbf{N}\},$$

i.e. $G_t(\omega)$ is a polyhedral cone generated by the finite set $\{X_t^n(\omega),\ n \in \mathbf{N}\}$.

Let G and $\tilde{G}$ be closed cones. We shall say that G is *dominated* by $\tilde{G}$ if $G \setminus G^0 \subseteq \mathrm{ri}\,\tilde{G}$ where G^0 is the linear space $G \cap (-G)$. We extend this notion in the obvious way to $\mathcal{C}$-valued or cone-valued processes. It can be formulated in terms of the dual cones because

$$G \setminus G^0 \subseteq \operatorname{ri} \tilde{G} \quad \Leftrightarrow \quad \tilde{G}^* \setminus \tilde{G}^{*0} \subseteq \operatorname{ri} G^*.$$

In particular, if G has an interior (which is always the case for financial models),

$$G \setminus G^0 \subseteq \operatorname{int} \tilde{G} \quad \Leftrightarrow \quad \tilde{G}^* \setminus \{0\} \subseteq \operatorname{ri} G^*.$$

Let G be a $\mathcal{C}$-valued process, $A_t(G) := -\sum_{s=0}^t L^0(G_s, \mathcal{F}_s)$.

Extending to the abstract setting the concepts introduced in a financial context, we say that G satisfies:

– *weak no-arbitrage property* NA^w if

$$A_t(G) \cap L^0(G_t, \mathcal{F}_t) \subseteq L^0(\partial G_t, \mathcal{F}_t) \qquad \forall\, t \le T;$$

– *strict no-arbitrage property* NA^s if

$$A_t(G) \cap L^0(G_t, \mathcal{F}_t) \subseteq L^0(G_t^0, \mathcal{F}_t) \qquad \forall\, t \le T;$$

– *robust no-arbitrage property* NA^r if G is dominated by $\tilde{G}$ satisfying NA^w.

It is an easy exercise to check that if G dominates the constant process $\mathbf{R}_+^d$ then NA^w holds if and only if $A_T(G) \cap L^0(\mathbf{R}_+^d) = \{0\}$.

The main results can be formulated as follows:

Theorem 2.2.1 *Assume that G dominates $\mathbf{R}_+^d$. Then*

$$NA^r \quad \Leftrightarrow \quad \mathcal{M}_0^T(\operatorname{ri} G^*) \ne \emptyset.$$

Theorem 2.2.2 *Assume that $L^0(G_s^0, \mathcal{F}_{s-1}) \subseteq L^0(G_{s-1}^0, \mathcal{F}_{s-1})\ \forall\, s \le T$. Then*

$$NA^s \quad \Leftrightarrow \quad \mathcal{M}_0^T(\operatorname{ri} G^*) \ne \emptyset.$$

We postpone the proof of extended versions of these theorems to the next subsections. The "easy" implication $\Leftarrow$ will be established without extra assumptions.

The hypothesis of the second theorem holds trivially when $G^0 = \{0\}$ (the efficient friction condition in financial context). More interesting, it is fulfilled also for the setting corresponding to the market model for which the subspace $K_t^0 = K_t \cap (-K_t)$ is constant over time (e.g., the transaction costs are constant) and NA^s holds. Therefore, in such a case the properties NA^r and NA^s coincide.

Indeed, let us suppose that $G_t = \phi_t K_t$ where K is a $\mathcal{C}$-valued process dominating $\mathbf{R}_+^d$,

$$\phi_t(\omega) : (x^1, ..., x^d) \mapsto (x^1/S_t^1(\omega), ..., x^d/S_t^d(\omega)),$$

and S_t^i are strictly positive $\mathcal{F}_t$-measurable random variables.

For $J \subseteq \{1, ..., d\}$ we put $\mathbf{1}_J := \sum_{i \in J} e_i$ where $\{e_i\}$ is the canonical basis in $\mathbf{R}^d$.

Proposition 2.2.3 *Suppose that there is a partition $J_1,\dots,J_l$ of $\{1,\dots,d\}$ such that $K_t^{0\perp} = \operatorname{lin}\{\mathbf{1}_{J_1}\dots,\mathbf{1}_{J_l}\}$. If G satisfies NA^w-property, then the assumption of Theorem 2.2.2 is fulfilled.*

Proof. If the claim fails, there is $\xi \in L^0(G_s^0, \mathcal{F}_{s-1})$ such that $\{\xi \notin G_{s-1}^0\}$ is a non-null set. Without loss of generality we may assume that ξ is equal to zero outside it. Necessarily, some set $\{\phi_{s-1}^{-1}\xi\mathbf{1}_{J_k} \neq 0\}$ is non-null. We may assume that $\{\phi_{s-1}^{-1}\xi\mathbf{1}_{J_k} > 0\}$, the random variable ξ is zero outside this set and, moreover, all components of ξ vanish except those corresponding to J_k. Notice that J_k is not a singleton because ϕ_s and ϕ_{s-1} are diagonal operators. Take $i_0 \in J_k$ such that $\xi^{i_0} > 0$ and consider ξ' different of ξ only by the i_0th component

$$\xi'^{i_0} := -\frac{1}{S_{s-1}^{i_0}} \sum_{i\in J_k\setminus\{i_0\}} \xi^i S_{s-1}^i.$$

Clearly, $\xi' \in L^0(G_{s-1}^0, \mathcal{F}_{s-1})$ and $\xi - \xi' = h$ where h is equal to zero except the nontrivial component $h^{i_0} \geq 0$. This violates NA^w property. □

In a specific financial context, where K_t is the solvency cone in values (generated by the matrix of transaction costs coefficients Λ_t) and S is the price process, the linear space $K_t^{0\perp}$ is always the linear span of the random vectors $\mathbf{1}_{J_1(t)},\dots,\mathbf{1}_{J_l(t)}$ where $J_i(t)$ are the classes of "equivalent" assets (i.e., the assets which can be converted one into another without transaction costs). Of course, in the case of constant transaction costs these vectors do not evolve in time. Since NA^w is weaker than NA^s, the latter implies that $\mathcal{M}_0^T(\operatorname{ri} G^*) \neq \emptyset$.

Remark. The hypothesis of Theorem 2.2.1 can be slightly relaxed by demanding that G dominates an increasing $\mathcal{C}$-valued process H such that all H_t have non-empty interiors.

As in the frictionless case, one can modify the right-hand of the equivalence in the suggested theorems by taking into account the next result which is an easy corollary of the DMW theorem.

Lemma 2.2.4 *Let $\mathcal{M}_0^T(\operatorname{ri} G^*)$ be non-empty and $\tilde{P} \sim P$. Then $\mathcal{M}_0^T(\operatorname{ri} G^*, \tilde{P})$ is non-empty as well and even contains a bounded martingale.*

Proof. Let $Z \in \mathcal{M}_0^T(\operatorname{ri} G^*)$ and let $\zeta := 1+\sup_{t\leq T}|Z_t|$. By the DMW theorem ("easy" part) the P-martingale $(1, Z)$ has the NA-property. The latter, being invariant under equivalent change of probability measure, holds also with respect to $P' := c\zeta^{-1}P$. Again by the same theorem but this time its "difficult" part, there is a bounded density process $\rho > 0$ such that ρZ is P'-martingale or, equivalently, the process $\tilde{Z}_t := E(\zeta^{-1}|\mathcal{F}_t)\rho_t Z_t$ is a P-martingale. It is bounded and, since $\operatorname{ri} G_t^*(\omega)$ are cones, belongs to $\mathcal{M}_0^T(\operatorname{ri} G^*)$. Using the same idea, we can obtain from $\tilde{Z}$ a bounded element of $\mathcal{M}_0^T(\operatorname{ri} G^*, \tilde{P})$. □

In the proof we used only that the values of $\operatorname{ri} G^*$ are cones. The extension of the lemma (as well as its version below) to arbitrary cone-valued processes is obvious.

Lemma 2.2.5 *Let $\mathcal{M}_0^T(\operatorname{ri} G^*) \neq \emptyset$, $\tilde{P} \sim P$, and $\alpha \geq 0$ be a (finite) random variable. Then $\mathcal{M}_0^T(\operatorname{ri} G^*, \tilde{P})$ contains Z such that $(1+\alpha)|Z|$ is bounded.*

Of course, the previous proof still works with ζ replaced by $\zeta + \alpha$.

Remark. The above result is extremely useful. In many cases it removes technical difficulties related with integrability and provides a great flexibility in arguments. Note also that we can fabricate a "convenient" vector-valued martingale just by multiplying the given one by a strictly positive scalar process.

Recall that for any sequence of random variables there exists an equivalent probability measure $\tilde{P}$ with bounded density such that all these random variables are integrable and, moreover, if, initially, the sequence was convergent almost surely, it will be convergent in $L^1(\tilde{P})$. Lemma 2.2.4 (more precisely, its variant with $\mathcal{M}_0^T(G^* \setminus \{0\})$) combined with this elementary fact makes almost obvious the following assertion:

Proposition 2.2.6 *Let $\mathcal{M}_0^T(G^* \setminus \{0\}) \neq \emptyset$. Then*

$$\bar{A}_0^T \cap L^0(G_T, \mathcal{F}_T) \subseteq L^0(\partial G_T, \mathcal{F}_T).$$

Proof. Suppose that the sequence $\zeta^n = \sum_{t=0}^T \xi_t^n$, where $\xi_t^n \in -L^0(G_t, \mathcal{F}_t)$, converge a.s. to some $\zeta \in L^0(G_T, \mathcal{F}_T)$. according to above remark, there is no loss of generality to assume, by a suitable choice of the reference measure, that $\xi_t^n \in -L^1(G_t, \mathcal{F}_t)$ and $\zeta^n \to \zeta$ in L^1. Take a bounded $Z_t \in \mathcal{M}_0^T(G^* \setminus \{0\})$. Then

$$EZ_T\zeta^n = \sum_{t=0}^T EZ_t\xi_t^n \leq 0.$$

On the other hand, $EZ_T\zeta \geq 0$ and this inequality is strict if $\zeta \in \operatorname{int} G_T$ with positive probability. This implies that $\zeta \in L^0(\partial G_T, \mathcal{F}_T)$.

2.2.2 NA^r- and NA^s-Properties: Proofs

We start the proof with certain useful properties. It is convenient to formulate them in a more general framework.

Let N_s, $s = 0, 1, ..., T$, be closed convex cones in $L^0(\mathbf{R}^d, \mathcal{F}_s)$ stable under multiplication by the elements of $L^0(\mathbf{R}_+, \mathcal{F}_s)$. The last property implies that N_s are decomposable and, hence, can be represented as $L^0(G_s, \mathcal{F}_s)$ for some cone-valued process (G_s), see Appendix A.4.

Let $N_s^0 := N_s \cap (-N_s)$, $A_t := -\sum_{s=0}^t N_s$.

We introduce the following conditions:

(i) $A_T \cap N_t \subseteq N_t^0$ for every $t = 0, ..., T$;

(ii) $A_{t-1} \cap N_t \subseteq N_t^0$ for every $t = 1, ..., T$;

(iii) the relation $\sum_{s=0}^T \xi_s = 0$ with $\xi_s \in N_s$ implies that all $\xi_s \in N_s^0$.

In the case where $N_s = L^0(G_s, \mathcal{F}_s)$ the condition (i) coincides with the NA^s-property.

Lemma 2.2.7 $(iii) \Rightarrow (i)$.

Proof. Suppose that $\sum_{s=0}^{T} \zeta_s = -\eta$ where $\zeta_s \in N_s$ and $\eta \in N_t$. In virtue of (iii) we have that $\xi_t := \zeta_t + \eta$ is an element of N_t^0. Thus, $\eta = \xi_t - \zeta_t$ is in $-N_t$, i.e. $\eta \in N_t^0$.

Remark. Trivially, $(i) \Rightarrow (ii)$. In general, the implication $(ii) \Rightarrow (iii)$ may fail. However, it is easily seen that it holds if all $N_s^0 = \{0\}$, and in this case these three properties are equivalent.

The following lemma shows that the condition (iii) ensures the closedness of A_T in probability. Though, its proof is based on the same arguments as for the "difficult" implication in the DMW theorem, we spell out it here.

Lemma 2.2.8 *If (iii) holds then $A_T = \bar{A}_T$.*

Proof. We proceed by induction. For $T = 0$ there is nothing to prove. Suppose that the claim holds up to $T-1$ periods. Let $\sum_{s=0}^{T} \xi_s^n \to \xi$ a.s. where $\xi_s^n \in N_s$. The question is whether $\xi = \sum_{s=0}^{T} \xi_s$ with $\xi_s \in N_s$. If $\Omega_i \in \mathcal{F}_0$ form a partition of Ω, we may argue separately with each part as it were the whole Ω, find appropriate representations and "assemble" ξ_s from separate pieces.

The case $\Omega = \{\liminf |\xi_0^n| < \infty\}$ is simple: by Lemmas 1.1.2 we may assume that ξ_0^n converge to $\xi_0 \in N_0$ and, hence, $\sum_{s=1}^{T} \xi_s^n$ converge a.s. to a random variable ζ which is in $\sum_{s=1}^{T} N_s$ by the induction hypothesis.

In the case $\Omega = \{\liminf |\xi_0^n| = \infty\}$ we put $\tilde{\xi}_s^n := \xi_s^n / |\xi_0^n|$ (with the convention $0/0 = 0$). As $|\tilde{\xi}_0^n| \le 1$, we again may assume that $\tilde{\xi}_0^n$ converge to some $\tilde{\xi}_0 \in N_0$. Then $\sum_{s=1}^{T} \tilde{\xi}_s^n$ converge a.s. to a random variable which can be represented by the induction hypothesis as $\sum_{s=1}^{T} \tilde{\xi}_s$ where $\tilde{\xi}_s \in N_s$. Since $\xi / |\xi_0^n| \to 0$ a.s., the limit of the whole normalized sum is zero, i.e. $\sum_{s=0}^{T} \tilde{\xi}_s = 0$. By the assumption all $\tilde{\xi}_s \in N_s^0$. Since $|\tilde{\xi}_0| = 1$ there are disjoint sets $\Gamma_i \in \mathcal{F}_0$ such that $\Omega = \cup_{i=1}^{d} \Gamma_i$ and $\Gamma_i \subseteq \{\tilde{\xi}_0^i \ne 0\}$.

Put $\bar{\xi}_s^n = \sum_{i=1}^{d} (\xi_s^n + \beta^{ni} \tilde{\xi}_s) I_{\Gamma_i}$ where $\beta^n = -\xi_0^{ni} / \tilde{\xi}_0^i$. Clearly, $\bar{\xi}_s^n \in N_s$ and $\sum_{s=0}^{T} \bar{\xi}_s^n$ converge to ξ a.s. The situation is reproduced. It is instructive to represent sequences ξ_0^n and $\bar{\xi}_0^n$ as infinite dimensional matrices with d-dimensional columns. Of course, every zero line of the first matrix remains zero line of the second one. But the second matrix contains one more zero line (namely, the ith for $\omega \in \Gamma_i$). Thus, if the first matrix contains one non-zero line a.s., the proof is accomplished (all $\bar{\xi}_0^n = 0$ and we can use the induction hypothesis). If not, we repeat the whole procedure with the sequence of processes $(\bar{\xi}_s^n)$ etc.

Lemma 2.2.9 *Assume that (iii) holds. Then for any $\zeta \in N_t$, $t \le T$, there is a bounded $\mathbf{R}^d$-valued martingale Z^ζ such that:*

1) $Z_s^\zeta \xi \ge 0$ *for any $\xi \in N_s$, $s \le T$;*
2) $\zeta I_{\{Z_t^\zeta \zeta = 0\}} \in N_t^0$.

Proof. Let $A^1_T := A_T \cap L^1$ and $\mathcal{Z}_T := \{Z_T \in L^\infty(\mathbf{R}^d) : \; E\eta\xi \le 0 \;\forall\, \xi \in A^1_T\}$. With each $Z_T \in \mathcal{Z}_T$ we associate the martingale Z with $Z_s := E(Z_T|\mathcal{F}_s)$. It satisfies the condition 1) since otherwise we could find $\xi \in N_s \cap L^1$ such that the set $\Gamma := \{Z_s\xi < 0\}$ is non-null and, therefore, $EZ_T(\xi I_\Gamma) = EZ_s(\xi I_\Gamma) < 0$ contradicting the definition of $\mathcal{Z}_T$. Let $c_t := \sup_{Z_T\in\mathcal{Z}_T} P(Z_t\zeta > 0)$. There exists $Z^\zeta_T \in \mathcal{Z}_T$ such that for the corresponding martingale Z^ζ the supremum is attained, i.e. we have $c_t = P(Z^\zeta_t\zeta > 0)$. To see this, take martingales Z^n generated by $Z^n_T \in \mathcal{Z}_T$ with $P(Z^n_t\zeta > 0) \to c_t$ and put $Z^\zeta_T := \sum 2^{-n} Z^n_T/||Z^n_T||_\infty$.

If 2) fails, then, for a sufficiently large, $\zeta^a := \zeta I_{\{Z^\zeta_t\zeta=0,\ |\zeta|\le a\}}$ does not belong to N^0_t and, being in $N_t \cap L^1$, cannot not belong, in virtue of the condition (i) implied by (iii), to the convex cone A^1_T which is closed in L^1 accordingly to Lemma 2.2.8. By the Hahn–Banach theorem one can separate ζ^a and A^1_T: that is to find $\eta \in L^\infty(\mathbf{R}^d)$ such that

$$\sup_{\xi\in A^1_T} E\eta\xi < E\eta\zeta^a.$$

Since A^1_T is a cone, the supremum above is equal to zero, ensuring that $\eta \in \mathcal{Z}_T$ and $E\eta\zeta^a > 0$. The latter inequality implies that for $Z_t = E(\eta|\mathcal{F}_t)$ the product $Z_t\zeta$ (always ≥ 0) is strictly positive on a non-null set. Thus, for the martingale $\tilde Z := Z + Z^\zeta$ with the terminal value $\tilde Z_T := \eta + Z_T\zeta$ we have

$$P(\tilde Z_t\zeta > 0) > P(Z^\zeta_t\zeta > 0) = c_t.$$

This contradiction shows that 2) holds.

Lemma 2.2.10 *Assume that (iii) holds. Let Γ be a countable subset of $\cup_{s\le T} N_s$. Then there is a bounded $\mathbf{R}^d$-valued martingale Z such that for all $s \le T$ we have:*

1) $Z_s\xi \ge 0$ for any $\xi \in N_s$;

2′) $\zeta I_{\{Z_s\zeta=0\}} \in N^0_s$ whatever is $\zeta = \sum_n \alpha_n\xi^n$ with $\xi^n \in \Gamma \cap N_s$ and $\alpha_n \in L^0_+(\mathcal{F}_s)$.

Proof. One can take as Z any (countable) convex combination with strictly positive coefficients of all elements of the family $\{Z^\zeta\}_{\zeta\in\Gamma}$ with $|Z^\zeta_T| \le 1$. Taking into account that N_s is stable under multiplication on the elements of $L^0_+(\mathcal{F}_s)$, we verify consecutively that the property 2′) holds for ξ^n, $\alpha_n\xi^n$, and, at last, for ζ which are represented as series of ξ_n with $\mathcal{F}_s$-measurable positive coefficients.

In the next lemma we argue in the case where $N_t = L^0(G_t, \mathcal{F}_t)$, where (G_t) is a $\mathcal{C}$-valued process and, hence, $N^0_t = L^0(G^0_t, \mathcal{F}_t)$ and $A_t = A_t(G)$.

Lemma 2.2.11 *If (iii) holds, then $\mathcal{M}^T_0(\operatorname{ri} G^*)$ is non-empty.*

Proof. Consider the process Z from the previous lemma corresponding to the set Γ which is the union of all X^n_t, $t \le T$, $n \in \mathbf{N}$. The property 1) means that $Z \in \mathcal{M}^T_0(G^*)$. Recall also that $Z_s(\omega) \notin \operatorname{ri} G^*_s(\omega)$ if and only if there exists $x \in G_s(\omega) \setminus G_s^0(\omega)$ such that $Z_s(\omega)x = 0$. Thus, if the property $Z_s \in L^0(\operatorname{ri} G_s^*, \mathcal{F}_s)$ is violated, the $\mathcal{F}_s \otimes \mathcal{B}^d$-measurable set

$$\Delta := \{(\omega, x): \; Z_s(\omega)x = 0, \; x \in G_s(\omega) \setminus G_s^0(\omega)\}$$

has a non-null projection onto Ω. Take its $\mathcal{F}_s$-measurable selector ζ putting it zero outside the projection of Δ. Then $\zeta = \sum_n \alpha_n X_s^n$ with $\alpha_n \in L_+^0(\mathcal{F}_s)$ (for each ω this sum has only a finite number of non-zero summands). Thus, according the property $2'$), we have $\zeta I_{\{Z_s\zeta=0\}} \in G_s^0$ in an apparent contradiction with our assumption.

Lemma 2.2.12 *Suppose that the cone-valued process G satisfies NA^r property. If G dominates $\mathbf{R}_+^d$, then (iii) holds.*

Proof. Let $\tilde{G}$ dominates G and $A_T(\tilde{G}) \cap L^0(\mathbf{R}_+^d, \mathcal{F}_T) = \{0\}$. Assume that in the identity $\sum_{s=0}^T \xi_s = 0$ where $\xi_s \in L^0(G_s, \mathcal{F}_s)$, a random variable ξ_t does not belong to $L^0(G_t^0, \mathcal{F}_t)$. This means that $\xi_t(\omega) \in \operatorname{int} \tilde{G}_t(\omega)$ on a set B of positive probability. It follows that there is a random variable $\epsilon \in L^0(\mathbf{R}_+^d, \mathcal{F}_t)$ strictly positive on B such that $\xi_t - \epsilon$ is still in $L^0(\tilde{G}_t, \mathcal{F}_t)$. The nontrivial random variable $\epsilon = -\sum_{s=0}^T \xi_s'$ where $\xi_s' := \xi_s$, $s \neq t$, $\xi_t' := \xi_t - \epsilon$, being in $A_T(\tilde{G}) \cap L^0(\mathbf{R}_+^d, \mathcal{F}_T)$, violates the NA^w-property of $\tilde{G}$.

Lemma 2.2.13 *Suppose that the cone-valued process G has the NA^s-property. If, in addition,*

$$L^0(G_s^0, \mathcal{F}_{s-1}) \subseteq L^0(G_{s-1}^0, \mathcal{F}_{s-1}) \qquad \forall\, s \leq T, \tag{2.1.8}$$

then the condition (iii) holds.

Proof. This can be shown by induction starting trivially and with an easy step. The equality $\sum_{s=0}^{T-1} \xi_s = -\xi_T$ implies that ξ_T is $\mathcal{F}_{T-1}$-measurable and, in virtue of NA^s-property, belongs to $L^0(G_T^0, \mathcal{F}_T)$. By the assumed inclusion ξ_T belongs also to $L^0(G_{T-1}^0, \mathcal{F}_{T-1})$ and can be combined with ξ_{T-1}, reducing the sum to $T-1$ terms which are elements of $L^0(G_s^0, \mathcal{F}_s)$ (the induction hypothesis). In particular, $\xi_{T-1} + \xi_T$ belongs to $L^0(G_{T-1}^0, \mathcal{F}_{T-1})$ as well as both summands.

The "difficult" implications ($\Rightarrow$) in Theorems 2.2.1 and 2.2.2 follow from the last three lemmas. Now we establish the "easy" implications ($\Leftarrow$).

Lemma 2.2.14 *If $\mathcal{M}_0^T(\operatorname{ri} G^*) \neq \emptyset$, then NA^s and NA^r hold.*

Proof. Let us check first the NA^s-property. To this aim we apply Lemma 2.2.7 with $N_s = L^0(G_s, \mathcal{F}_s)$, reducing the problem to a verification of (iii). So, let $\sum_{s=0}^T \xi_s = 0$ with $\xi_s \in N_s$. Choose a reference measure such that all ξ_s are integrable. By Lemma 2.2.4 there is bounded martingale $Z \in \mathcal{M}_0^T(\operatorname{ri} G^*)$. Multiplying the identity by Z_T and taking the expectation, we obtain that

$\sum_{s=0}^{T} EZ_s\xi_s = 0$. Since $Z_s\xi_s \geq 0$, this is possible only when all $Z_s\xi_s = 0$, i.e. when $\xi_s \in L^0(G_s^0, \mathcal{F}_s)$.

To verify the NA^r-property we take an arbitrary $Z \in \mathcal{M}_0^T(\mathrm{ri}\, G^*)$ and define the $\mathcal{C}$-valued process $\tilde{G}$ whose values are half-spaces $\tilde{G}_s(\omega) = (\mathbf{R}_+ Z_s(\omega))^*$. Applying the above arguments with $N_s = L^0(\tilde{G}_s, \mathcal{F}_s)$, we conclude that $\tilde{G}$ has the NA^s-property coinciding with NA^w-property because $\partial\tilde{G}_s = \tilde{G}_s^0$. Since $\tilde{G}$ dominates G, the latter has NA^r-property. □

Remark 1. Inspecting the first part of the above proof, it is not difficult to see that the condition (c) in Theorem 2.1.4 implies (iii), hence, the NA^s-property. Unlike the situation of finite Ω, the converse is not true. However, if all $G_t^0 = \{0\}$, then (2.1.8) holds trivially and NA^s implies that $\mathcal{M}_0^T(\mathrm{ri}\, G^*) \neq \emptyset$, i.e. in this particular case the latter condition, (c) and NA^s are equivalent.

Remark 2. Recently, T. Pennanen and I. Penner, [74], suggested a simple argument showing that for the cone-valued processes the NA^r-property holds if and only if $\mathcal{M}_0^T(\mathrm{ri}\, G^*) \neq \emptyset$. Indeed, it is not difficult to check that if (G_t) has the NA^r-property, then there is a cone-valued process (G'_t) dominating (G_t) still having the NA^r-property. From the above general lemmas we infer that the cone $A_T(G')$ is closed in L^0, and, using the Kreps–Yan theorem in L^1, we obtain that $\mathcal{M}_0^T(G'^* \setminus \{0\})$ is non-empty. But $G'^* \setminus \{0\} \subseteq \mathrm{ri}\, G^*$.

The condition $\mathcal{M}_0^T(\mathrm{ri}\, G^*) \neq \emptyset$ guarantees that $A_T(G)$ is closed. This topological property looks indispensable in the theory of no-arbitrage criteria. A simple example shows that NA^w does implies the closedness of $A_T(G)$ and, therefore, the extension of NA^w-criteria given in Theorem 2.1.1 are impossible. The more surprising is the result by Grigoriev asserting that for two-asset model the necessary and sufficient conditions of this theorem, except (b), hold for arbitrary Ω .

2.2.3 The Grigoriev Theorem

Throughout this subsection we shall work with a $\mathcal{C}$-valued process G dominating the constant process $\mathbf{R}_+^2$. The initial σ-algebra $\mathcal{F}_0$ is not assumed trivial.

The proof of the theorem involves models with initial dates different from zero. By this reason we shall use the notations A_0^T, A_1^T, etc. giving an idea about the considered time range.

We formulate the statement in the same way as the Dalang–Morton–Willinger theorem omitting, of course, the condition (B) (which looks exactly as (b) of the latter): being stronger than (A), it is a sufficient but not necessary for the NA^w-property.

Theorem 2.2.15 *Let $d = 2$. Then the following conditions are equivalent:*

(A) $A_0^T \cap L^0(\mathbf{R}_+^d) = \{0\}$*;*
(C) $\bar{A}_0^T \cap L^0(\mathbf{R}_+^d) = \{0\}$*;*
(D) $\mathcal{M}_0^T(G^* \setminus \{0\}) \neq \emptyset$*;*
(E) $\mathcal{M}_0^T(G^* \setminus \{0\}, \tilde{P})$ *contains a bounded process whatever is* $\tilde{P} \sim P$*.*

Proof. The equivalence of (D) and (E) was already discussed for the general setting (see Lemma 2.2.14 and the accompanying remark). The implication $(E) \Rightarrow (A)$ is obvious because a freedom in the choice of the reference measure allows us to avoid the problems with integrability and we can argue in the same way as for the case of finite Ω.

Thus, the problem poses only the implication $(A) \Rightarrow (C)$ the proof of which uses essentially the specificity of the plane where the cones $G_t(\omega)$ are just sectors containing the first orthant.

Let us define the set

$$\Gamma_1^T := \{\xi \in L^\infty(\mathbf{R}^2, \mathcal{F}_0) : \; E\xi Z_1 \le 0 \; \forall\, Z \in \mathcal{M}_1^T(G^* \setminus \{0\})\}.$$

Proposition 2.2.16 *Suppose that $(G_t)_{0\le t\le T}$ satisfies the NA^w-property. Then:*

(a) we have the inclusion $\Gamma_1^T \subseteq A_1^T$;

(b) if $\xi \in \Gamma_1^T$ is such that $E\xi Z_1 < 0$ for all $Z \in \mathcal{M}_1^T(G^* \setminus \{0\})$, then there is $\epsilon \in L^0(\mathbf{R}_+^2, \mathcal{F}_T)$, $\epsilon \ne 0$, such that $\xi + \epsilon \in A_1^T$.

The above proposition plays a crucial role. It makes easy the induction step on the length of the time interval for the implication $(A) \Rightarrow (C)$. Reciprocally, the NA^w-criterion permits to prove the induction step for this proposition. Let T_N and P_N denote the assertions of the Theorem 2.2.15 and Proposition 2.2.16 for the N-step model. Symbolically, the arguments can be chained in the following way:

1) $(T_N, P_{N+1}) \Rightarrow T_{N+1}$,

2) $(T_{N+1}, P_{N+1}) \Rightarrow P_{N+2}$.

Proof of the implication $(T_N, P_{N+1}) \Rightarrow T_{N+1}$.

So, let NA^w holds for $N+1$-step model. We are looking for a martingale $Z \in \mathcal{M}_0^{N+1}(G^* \setminus \{0\})$. By the induction hypothesis we know that there is $Z^0 \in \mathcal{M}_1^{N+1}(G^* \setminus \{0\})$. We want to show that there is an element of the latter set which can be extended one step backward as a martingale which initial values lays in $G_0^* \setminus \{0\}$.

Case 1. For any non-null set $\Gamma \in \mathcal{F}_0$ there is $Z^\Gamma \in \mathcal{M}_1^{N+1}(G^* \setminus \{0\})$ such that the intersection $\Gamma \cap \{E(Z_1^\Gamma|\mathcal{F}_0) \in G_0^*\}$ is non-null. By Lemma 1.1.3 we can find at most countable family of sets $\Delta_i := \{E(Z_1^{\Gamma_i}|\mathcal{F}_0) \in G_0^*\}$ which union is of full measure. Normalizing, we may assume that $E|Z_{N+1}^{\Gamma_i}| = 1$. It is easily seen that $Z_t := \sum_i 2^{-i} I_{\Delta_i} E(Z_{N+1}^{\Gamma_i}|\mathcal{F}_t)$ is an element of $\mathcal{M}_0^{N+1}(G^* \setminus \{0\})$.

Case 2. There is a non-null set $\Gamma \in \mathcal{F}_0$ such that $\Gamma \cap \{E(Z_1|\mathcal{F}_0) \in G_0^*\}$ is a null set whatever is $Z \in \mathcal{M}_1^{N+1}(G^* \setminus \{0\})$. The arguments below exclude this case.

Let g_i, $i = 1, 2$, be the random vectors of unit length generating the boundary rays of the sector G_0 and let

$$g := g_1 I_{H_1} + g_2 I_{H_2 \setminus H_1}$$

where $H_i := \{g_i E(Z_1^0|\mathcal{F}_0) < 0\}$. Since

$$\{E(Z_1^0|\mathcal{F}_0) \notin G_0^*\} = H_1 \cup H_2,$$

we have $gE(Z_1^0|\mathcal{F}_0) < 0$ on Γ (a.s.). More: $gE(Z_1|\mathcal{F}_0) < 0$ on Γ (a.s.) whatever is $Z \in \mathcal{M}_1^{N+1}(G^* \setminus \{0\})$. The latter assertion holds because, in the opposite case, we could find in Γ a non-null subset $\Gamma' \in \mathcal{F}_0$ and $Y \in \mathcal{M}_1^{N+1}(G^* \setminus \{0\})$ such that $gE(Y_1|\mathcal{F}_0) \geq 0$ on Γ'. It is easy to see that there is a scalar $\mathcal{F}_0$-measurable random variable $\alpha \geq 0$ such that on Γ'

$$\alpha g E(Z_1^0|\mathcal{F}_0) + gE(Y_1|\mathcal{F}_0) = 0.$$

It follows that for the process $Z := \alpha Z^0 + Y$ from $\mathcal{M}_1^{N+1}(G^* \setminus \{0\})$ we have on Γ' the equality $gp = 0$ where $p := E(Z_1|\mathcal{F}_0)$. But for the considered two-dimensional model this means that a.s. on Γ' the vector $p(\omega)$ with strictly positive components belongs to one of two boundary rays of $G_0^*(\omega)$. This is an apparent contradiction with the defining property of Γ.

Applying P_{N+1} with $\xi := gI_\Gamma$ we infer that there exists a non-trivial $\epsilon \in L^0(\mathbf{R}_+^2, \mathcal{F}_{N+1})$ such that $gI_\Gamma + \epsilon \in \widehat{A}_1^{N+1}$. Since $gI_\Gamma \in L^0(G_0, \mathcal{F}_0)$, this contradicts to the NA^w-property of G_0^{N+1}.

Proof of the implication $(T_{N+1}, P_{N+1}) \Rightarrow P_{N+2}$.

(a) We shall prove that

$$\Gamma_1^{N+2} := \{\xi \in L^\infty(\mathbf{R}^2, \mathcal{F}_0) : \ E\xi Z_1 \leq 0 \ \forall\, Z \in \mathcal{M}_1^{N+2}(G^* \setminus \{0\})\} \subseteq A_1^{N+2}$$

assuming NA^w for G_1^{N+2} and knowing already that the latter property is equivalent to the existence of $Z^0 \in \mathcal{M}_1^{N+2}(G^* \setminus \{0\})$ with $|Z_1^0| = 1$; moreover, the claim holds for all shorter time ranges. Note that the existence of Z^0 implies that in the definition of Γ_1^{N+2} we can replace the set $\mathcal{M}_1^{N+2}(G^* \setminus \{0\})$ by a larger set $\mathcal{M}_1^{N+2}(G^*)$ (due to the “regularization” $Z + \varepsilon Z^0$ with $\varepsilon \downarrow 0$).

If subsets $\Omega_i \in \mathcal{F}_1$ form a finite partition of Ω, it is sufficient to establish the result separately for each Ω_i considered as a new model, with traces of the filtration and probability on Ω_i and random variables restricted on this set (clearly, NA^w is inherited for such restrictions of the cone-valued process).

We apply this remark (frequent in proofs of NA criteria) to the subsets $\Omega_1 := \{\xi \notin -G_1\}$ and $\Omega_2 := \Omega_1^c$. For Ω_2 the claim is trivial. Therefore, we may assume without loss of generality that $\Omega_1 = \Omega$, i.e. $\xi\eta > 0$ for some $\eta \in L^0(G_1^*, \mathcal{F}_1)$, and normalizing, that $|\xi| = 1$ and $|\eta| = 1$.

Moreover, we need to investigate only the case when there exists a process $\tilde{Z} \in \mathcal{M}_2^{N+2}(G^* \setminus \{0\})$ such that $E\xi\tilde{Z}_2 > 0$ (otherwise, $\xi \in A_2^{N+2}$ by the induction hypothesis and there is nothing to prove because A_2^{N+2} is a part of A_1^{N+2}).

The intersection Θ of the set $\Theta_1 := \{\xi E(\tilde{Z}_2|\mathcal{F}_1) > 0\}$ (of positive probability) with $\Theta_2 := \{E(\tilde{Z}_2|\mathcal{F}_1) \in G_1^*\}$ is a null set. Indeed, in the opposite case we would have the inequality $E\xi Z_1 > 0$ for the process $Z \in \mathcal{M}_1^{N+2}(G^*)$ with $Z_t := I_\Theta E(\tilde{Z}_{N+2}|\mathcal{F}_t)$ which is impossible in virtue of the remark at the beginning of the proof.

Thus, neglecting a null set, we may assume that $\Theta_1 \subseteq \Theta_2^c$ and, normalizing, that $|E(\tilde{Z}_2|\mathcal{F}_1)| = 1$ on the set $\Theta_2^c = \{E(\tilde{Z}_2|\mathcal{F}_1) \notin G_1^*\}$.

The following **elementary geometric fact** is obvious:

if x_1, x_2, x_3 are unit vectors in $\mathbf{R}_+^2$ with $yx_1 \geq 0$, $yx_2 \geq 0$, $yx_3 \leq 0$, and $yx_1 \geq yx_2$ for some vector y, then x_2 is a conic combination of x_1 and x_3.

Recalling that $\xi Z_1^0 \leq 0$, we obtain from this observation that on the set $\Theta_1 \cap \{\xi\eta > \xi E(\tilde{Z}_2|\mathcal{F}_1)\}$ the random vector $E(\tilde{Z}_2|\mathcal{F}_1)$ lays between η and Z_1^0, i.e. takes values in G_1^*.

Thus, by above, $\Theta_1 \subseteq \{\xi\eta \leq \xi E(\tilde{Z}_2|\mathcal{F}_1)\}$ (a.s.). Using again the mentioned geometric fact we get that there are non-negative $\mathcal{F}_1$-measurable coefficients α and β such that $\eta = \alpha E(\tilde{Z}_2|\mathcal{F}_1) + \beta Z_1^0$ on Θ_1. It follows that $E\xi Z_1' > 0$ for the process $Z' \in \mathcal{M}_1^{N+2}(G^*)$ with $Z_1' = \eta I_{\Theta_1}$ and $Z_t' = (\alpha\tilde{Z}_2 + \beta Z_t^0)I_{\Theta_1}$ for $t \geq 2$. We get a contradiction that $\xi \in \Gamma_1^{N+2}$.

(b) Let $\xi \in \Gamma_1^{N+2}$ be such that $E\xi Z_1 < 0$ for every $Z \in \mathcal{M}_1^{N+2}(G^* \setminus \{0\})$. Without loss of generality we shall assume that $|\xi| = 1$. By above we know that $\xi \in A_1^{N+2}$, i.e. $\xi = \sum_{t=1}^{N+2} \zeta_t$ with $\zeta_t \in -L^0(G_t, \mathcal{F}_t)$.

Let us consider three possible case:

Case 1: The set $\{\xi \in -\operatorname{int} G_1\}$ is non-null. The assertion is obvious since we can increase ξ on this set.

Case 2: The set $\Gamma := \{\xi \in \operatorname{int} G_1\}$ is non-null. Then $\xi - \theta \in L^0(G_1, \mathcal{F}_1\}$ for some $\theta \in L^0(\mathbf{R}_+^2, \mathcal{F}_1)$ such that $\theta > 0$ on Γ and $\theta = 0$ outside. Since $\xi \in A_1^{N+2}$, we have that also $\theta \in A_1^{N+2}$ in violation of the assumed NA^w-property. So, this case is impossible.

Case 3: Complementary to the previous ones. The components of ξ have different signs and $\xi\eta \geq 0$ for some $\eta \in L^0(G_1^*, \mathcal{F}_1)$ with $|\eta| = 1$.

We introduce the cone-valued process $\tilde{G}_1^{N+2}$ with $\tilde{G}_1 = \mathbf{R}\xi + \mathbf{R}_+^2$ and $\tilde{G}_t = G_t$, $t \geq 2$. Let us check whether NA^w holds for $\tilde{G}_1^{N+2}$. If $\tilde{G}_1^{N+2}$ satisfies NA^w, there is a bounded $\tilde{Z} \in \mathcal{M}_1^{N+2}(\tilde{G}^* \setminus \{0\})$. Let Z be an arbitrary element from $\mathcal{M}_1^{N+2}(G^* \setminus \{0\})$. Then $\xi Z_1 \leq 0$. Recalling that $\xi\eta \geq 0$ and $\xi\tilde{Z}_1 = 0$ (by definition of $\tilde{G}_1$), we obtain using the main geometric fact that $\tilde{Z}_1$ takes values between the rays generated by Z_1 and η. It follows that $\tilde{Z}_1 \in L^\infty(G_1^*, \mathcal{F}_1)$. This means that $\tilde{Z} \in \mathcal{M}_1^{N+1}(G^* \setminus \{0\})$ but $\xi\tilde{Z}_1 = 0$ which is impossible.

Thus, NA^w does not hold for the process $\tilde{G}_1^{N+2}$ and, therefore, there exists $\epsilon \in L^0(\mathbf{R}_+^2, \mathcal{F}_{N+2})$, $\epsilon \neq 0$, such that

$$\epsilon = \sum_{t=1}^{N+2} \xi_t, \qquad \xi_t \in -L^0(\tilde{G}_t, \mathcal{F}_t).$$

By definition, $\xi_1 = -\alpha\xi - \beta$ where β takes values in $\mathbf{R}_+^2$. Modifying, if necessary, ϵ and ξ_1 (by adding β) we may assume that $\xi_1 = -\alpha\xi$ where α is a scalar $\mathcal{F}_0$-measurable random variable. Moreover, we may assume without loss of generality that all ξ_i are integrable (changing, eventually, the probability measure) and α takes values -1, 0, 1.

Since G_2^{N+2} satisfies NA^w, necessarily, $\epsilon I_{\{\alpha=0\}} = 0$. Take a bounded process $Z \in \mathcal{M}_2^{N+2}(G^* \setminus \{0\})$ (existing by the induction hypothesis). Using the martingale property of Z and the duality we have:

$$E\epsilon E(Z_1|\mathcal{F}_0)I_{\{\alpha=-1\}} - E\xi E(Z_1|\mathcal{F}_0)I_{\{\alpha=-1\}} = EI_{\{\alpha=-1\}}\sum_{t=2}^{N+2}\xi_t Z_t \le 0.$$

Since $\xi E(Z_1|\mathcal{F}_0) \le 0$, it follows that $\epsilon E(Z_1|\mathcal{F}_0)I_{\{\alpha=-1\}} \le 0$ (a.s.) and, therefore, $\epsilon I_{\{\alpha=-1\}} = 0$. Thus, $\epsilon I_{\{\alpha=1\}} \ne 0$. On the set $\{\alpha = 1\}$ we have the equality

$$\epsilon = -\xi + \sum_{t=2}^{n+2}\xi_t.$$

It follows that $\xi + \epsilon I_{\{\alpha=1\}} \in A_1^{N+2}$, which is a required property of ξ.

2.2.4 Counterexamples

Example 1. A two-asset one-period model satisfying NA^w for which A_0^1 is not closed. Let $\Omega = \mathbf{N}$, $\mathcal{F} = 2^\Omega$, $P(k) = 2^{-k}$, $\mathcal{F}_0 = \{\emptyset, \Omega\}$, $\mathcal{F}_1 = \mathcal{F}$. Take $G_0 = \mathrm{cone}\{2e_2 - e_1, e_1 - e_2\}$ and $G_1 = \mathrm{cone}\{2e_1 - e_2, e_2 - e_1\}$. The vector $e_1 + e_2$ belongs to both G_0^* and G_1^* and, hence, the constant process $Z = e_1 + e_2$ is an element of $\mathcal{M}_0^1(G^* \setminus \{0\})$. Let us check that the random variable ξ with $\xi(k) = k(e_2 - e_1)$ does not belong to the set A_0^1 but lays in the closure of the latter. Indeed, suppose that $\xi = \xi_0 + \xi_1$ where $\xi_0 \in -G_0$ and $\xi_1 \in -L^0(G_1, \mathcal{F})$. Since $Z\xi = 0$ and $Z\xi_0 \le 0$, $Z\xi_1 = 0$, we have that $Z\xi_0 = 0$. But this means that $\xi_0 = c(e_2 - e_1)$ with some $c \ge 0$. It follows that $\xi_1(k) = (k - c)(e_2 - e_1)$ and this vector cannot belongs to G_1 for $k > c$ in contradiction with the assumption. On the other hand, ξ is the limit of a sequence of random variables from A_0^1, namely, $\xi^n = \xi_0^n + \xi_1^n$ where $\xi_0^n := n(e_2 - e_1)$ and $\xi_1^n(k) := \min\{k - n, 0\}(e_2 - e_1)$.

Example 2. A three-dimensional one-period model satisfying NA^w for which $\mathcal{M}_0^1(G^* \setminus \{0\}) = \emptyset$. The probabilistic setting is the same as in the previous example. Take $G_0^* = \mathbf{R}_+\eta$, $G_1^* = \mathrm{cone}\{\eta_1, \eta_2\}$ where $\eta = (3, 1, 1)$ and $\eta_1 = (4, 1, 1)$ are deterministic vectors in $\mathbf{R}_+^3$ while η_2 is a random one with $\eta_2(k) = (2, 1, 1 + 1/k)$.

Clearly, $\mathcal{M}_0^1(G^* \setminus \{0\}) = \emptyset$ because one cannot find random variables $\alpha, \beta \ge 0$ to meet the conditions $E\alpha = E\beta = 1/2$ and $E\beta\gamma = 0$, where $\gamma(k) = 1/k$, needed to ensure the equality $EZ_1 = Z_0$.

Let $\xi_0 \in -G_0$ and $\xi_1 \in -L^0(G_1, \mathcal{F})$ be such that $\xi = \xi_0 + \xi_1$ takes values in $\mathbf{R}_+^3$. The latter condition implies that $\eta_1\xi \ge 0$. Since $\eta_1\xi_1 \le 0$, we have $\eta_1\xi_0 \ge 0$. In the same way we get that $\eta_2(k)\xi_0 \ge 0$ whatever is k. But

$$\eta_1\xi_0 + \lim_k \eta_2(k)\xi_0 = 2\eta\xi_0 \le 0$$

and, therefore, both terms in the left-hand side are zero. So, $\eta_1\xi_0 = 0$. As a result, $\eta_1\xi = \eta_1\xi_1 \le 0$. With ξ taking values in $\mathbf{R}^3_+$ this is possible only when $\xi = 0$ and NA^w holds.

Thus, a straightforward generalization of Theorem 2.2.15 for an arbitrary $\mathcal{C}$-valued process fails to be true already in dimension three. However, the above counterexample does not exclude that it holds in a narrower class of financial models.

Example 3. A four-asset two-period model satisfying NA^s for which $\mathcal{M}_0^2(G^* \setminus \{0\}) = \emptyset$. The probability space is $\Omega = \{-1, 0, 1, ...\}$, $\mathcal{F} = 2^\Omega$, $P(k) = 2^{-k-2}$, $\mathcal{F}_0 = \{\emptyset, \Omega\}$, $\mathcal{F}_1 = \mathcal{F}_2 = \mathcal{F}$. We consider a $\mathcal{C}$-valued process G corresponding to $\widehat{K}$ of the financial model parameterized by the adapted matrix-valued process $\Pi = (\Pi_t)_{t\le 2}$ depending on the parameter $a \in (1,2)$ and given as follows:

$$\Pi_0 = \begin{pmatrix} 1 & a & a & a \\ 1 & 1 & a & a \\ a & a & 1 & a \\ a & a & 1 & 1 \end{pmatrix}; \qquad \Pi_1(-1) = \begin{pmatrix} 1 & a & a & a \\ a & 1 & a & a \\ a & a & 1 & a \\ a & a & a & 1 \end{pmatrix};$$

$$\Pi_1(0) = \begin{pmatrix} 1 & 1 & 1 & 1/2 \\ 1 & 1 & 1 & 1/2 \\ 1 & 1 & 1 & 1/2 \\ 2 & 2 & 2 & 1 \end{pmatrix}; \qquad \Pi_1(k) = \begin{pmatrix} 1 & & & k \\ & 1 & k & \\ & k^{-1} & 1 & a \\ k^{-1} & & 1 & 1 \end{pmatrix}, \ k \ge 2;$$

the matrices $\Pi_1(1)$ and $\Pi_2(-1)$ are filled by units (this means the absence of transaction costs); at last, for $k \ge 0$ the non-diagonal elements of the matrix $\Pi_2(k)$ are $\pi_2^{ij}(k) = 2^{k+2}$; the empty spaces can be filled by arbitrary sufficiently large numbers (say, k) to avoid chains of transfers cheaper than already specified. Recall that the cone G_t is generated by the orths e_i and the vectors $\pi_t^{ij} e_i - e_j$.

Verification of the NA_1^s-property. If $\xi = \xi_0 + \xi_1 \in L^0(G_1, \mathcal{F}_1)$ where $\xi_0 \in -G_0$, $\xi_1 \in -L^0(G_1, \mathcal{F}_1)$, then $\xi_0 \in G_1(-1)$ which is possible only if $\xi_0 = 0$: the scalar product of any non-trivial element of $G_1(-1)$ on the vector $\mathbf{1} = (1,1,1,1)$ is strictly positive while the scalar products of the elements of $-G_0$ on the same vector are negative (i.e. the linear space orthogonal to $\mathbf{1}$ separates the cones $G_1(-1)$ and $-G_0$). Thus, $\xi = \xi_1 \in L^0(\mathcal{G}_1^0, \mathcal{F}_0)$.

Verification of the NA_2^s-property. Suppose that $\xi = \xi_0 + \xi_1 + \xi_2$ where $\xi_t \in -L^0(G_t, \mathcal{F}_t)$ and $\xi \in L^0(G_2, \mathcal{F}_2)$. Notice that $\mathbf{1}\xi(1) \ge 0$ and $\mathbf{1}\xi_t(1) \le 0$ for $t = 0,1,2$. Therefore, $\mathbf{1}\xi_t(1) = 0$. This implies, in particular, that

$$\xi_0 = \alpha(e_1 - e_2) + \beta(e_3 - e_4), \quad \alpha, \beta \ge 0.$$

Suppose that the coefficient $\beta > 0$. For the vector $w_k = (a, 1, k, ak)$ we have

$$w_k\xi_0(k) = \alpha(a-1) - \beta k(a-1) < 0$$

when k is large. This leads to a contradiction because $w_k\xi_1(k) \le 0$ (since $w_k \in G_1^*(k)$ as one can easily verify by multiplying w_k on the generators of $G_1(k)$) and $w_k\xi_2(k) \le 0$ while $w_k\xi(k) \ge 0$ (since $w_k \in G_2^*(k)$). Thus, $\beta = 0$.

For any $k \ge 1$ the scalar product of the vector $f_k = (1,1,k,k)$ with $\xi_0 = \alpha(e_1 - e_2)$ is zero; the scalar products of this vector with vectors from $G_1(k)$ are positive and strictly positive with vectors from $G_2(k)\setminus\{0\}$. It follows that $\xi_0 + \xi_1(k) + \xi_2(k) = 0$; the latter equality can be obtained also for $k = 0$ by the same arguments but with the vector $f_0 = (1,1,1,1/2)$.

Finally, we have that the vector $\xi(-1) = \xi_0(-1) + \xi_1(-1) + \xi_2(-1)$ belongs to $G_2(-1)$. The vector $\xi_0 \in G_2^0(-1)$. It follows that $\xi_1(-1)$ belongs to $G_2(-1)$ and, by assumption, to $-G_1(-1)$; these two cones are separated by the subspace orthogonal to $\mathbf{1}$, that is $G_2^0(-1)$. So, $\xi_1(-1) \in G_2^0(-1)$. With this we conclude that $\xi(-1) \in G_2^0(-1)$ and, hence, $\xi(-1) \in G_2^0(-1)$.

Thus, $A_0^2 \cap L^0(G_2, \mathcal{F}_2) \subseteq L^0(G_2, \mathcal{F}_2)$, i.e. the NA_2^s-property holds.

Verification that $\mathcal{M}_0^2(G^* \setminus \{0\}) = \emptyset$. Recall that $\mathcal{M}_0^2(G^* \setminus \{0\}) \neq \emptyset$ if and only if $\bar{A}_0^2 \cap L^0(\mathbf{R}_+^4, \mathcal{F}_2)$. So, it is sufficient to construct an appropriate sequence $\xi^n = \xi_0^n + \xi_1^n + \xi_2^n$ convergent to $\xi \in L^0(\mathbf{R}_+^4, \mathcal{F}_2)$, $\xi \neq 0$: this will inform us also that $A_0^2 \neq \bar{A}_0^2$.

We put

$$
\begin{aligned}
\xi_0^n &= N(e_1 - e_2) + (e_3 - e_4),\\
\xi_1^n(k) &= \begin{cases} 0, & k = -1,\\ -N(e_1 - e_2) - (\frac{1}{2}e_3 - e_4), & k = 0,\\ -N(e_1 - e_2) - (e_3 - e_4), & k = 1,\\ -\frac{N}{k}(ke_1 - e_4) - N(\frac{1}{k}e_3 - e_2) - (\frac{N}{k} - 1)(e_4 - e_3), & 2 \le k \le N,\\ -\frac{N}{k}(ke_1 - e_4) - N(\frac{1}{k}e_3 - e_2) - \frac{N}{k}(e_4 - e_3), & 2 \le k > N, \end{cases}\\
\xi_2^n(k) &= \begin{cases} -\xi_0^n, & k = -1,\\ 0, & k \ge 0. \end{cases}
\end{aligned}
$$

It is tedious but elementary exercise to verify that $\xi_t^n \in L^0(G_t, \mathcal{F}_t)$ and

$$
\xi^n = \xi_0^n + \xi_1^n + \xi_2^n = \begin{cases} 0, & k = -1,\\ \frac{1}{2}e_3, & k = 0,\\ 0, & 1 \le k \le N,\\ e_3 - e_4, & k > N. \end{cases}
$$

Thus, ξ^n is a required sequence.

2.2.5 A Complement: the Rásonyi Theorem

Unlike other proofs, the arguments used to establish the Grigoriev theorem do not rely upon the closedness of the set A_0^T in L^0 and a separation theorem in an infinite-dimensional space. It was shown by Rásonyi that they can be extended to get the following interesting result which can be considered as a complement to Theorem 2.2.2.

Let $G = (G_t)$ be an adapted cone-valued process, i.e. $G_t(\omega)$ are closed cones in $\mathbf{R}^d$ and the sets $\{(\omega, x) : \ x \in G_t(\omega)\}$ are in $\mathcal{F}_t \otimes \mathcal{B}(\mathbf{R}^d)$. Let

$$A_0^T := -\sum_{t=0}^{T} L^0(G_t, \mathcal{F}_t).$$

Suppose that $G_t^0 := G_t \cap (-G_t) = \{0\}$ or, equivalently, $\operatorname{int} G_t^* \neq \emptyset$, for every t (efficient friction condition in the context of financial modelling). In accordance with our previous terminology we say that the NA^s-property holds if $A_0^t \cap L^0(G_t, \mathcal{F}_t) = \{0\}$ for $t = 0, 1, ..., T$.

Theorem 2.2.17 *Suppose that $G_t + G_{t+1}$ are closed cones for $t = 0, ..., T-1$. Then $NA^s \Leftrightarrow \mathcal{M}_0^T(\operatorname{int} G^*) \neq \emptyset$.*

Note that the condition of the theorem is fulfilled when the cones G_t are polyhedral.

The proof of the result needs some prerequisites from finite-dimensional convex analysis and measurable selections we recall immediately. Further information, in particular, on expectations of set-valued mappings, can be found in the Appendix A.4. To work comfortably, we assume that all σ-algebras here are complete.

1. If B is a closed convex set with $\operatorname{int} B \neq \emptyset$, then B is a closure of $\operatorname{int} B$ and for any closed convex set A

$$\operatorname{int} A \cap B \neq \emptyset \quad \Leftrightarrow \quad \operatorname{int}(A \cap B) \neq \emptyset.$$

If also $\operatorname{int} A \neq \emptyset$, then

$$\operatorname{int} A \cap \operatorname{int} B = \operatorname{int}(A \cap B).$$

2. Let $\omega \mapsto K(\omega)$ be a measurable mapping the values of which are non-empty convex closed subsets of $\mathbf{R}^d$. Then the mapping $\omega \mapsto \operatorname{int} K(\omega)$ is also measurable.

Suppose that the values of K are closed convex cones. Let $\omega \mapsto K_1(\omega)$ be another measurable mapping which values are non-empty open subsets of $\mathbf{R}^d$ and $K \cap K_1 = \emptyset$ on a set Γ. Then K and K_1 can be separated on Γ in a measurable way, i.e. there is $\xi \in L^0(\mathbf{R}^d, \mathcal{F})$ with $\xi = \xi I_\Gamma$ such that $\sup \eta\xi \leq 0$ for every $\eta \in L^0(K, \mathcal{F})$ and $\zeta\xi > 0$ on Γ for every ζ in $L^0(K_1, \mathcal{F})$.

3. Let $\omega \mapsto C(\omega)$ be a measurable mapping which values are closed convex subsets of the unit ball $\bar{\mathcal{O}}_1(0)$ in $\mathbf{R}^d$. Let $\mathcal{G}$ be a sub-σ-algebra of $\mathcal{F}$. There exists a $\mathcal{G}$-measurable mapping, denoted $E(C|\mathcal{G})$, which values are closed convex subsets of the unit ball and such that

$$L^0(E(C|\mathcal{G}),\mathcal{G}) = \{E(\eta|\mathcal{G}) : \ \eta \in L^0(C,\mathcal{F})\}.$$

Lemma 2.2.18 *Suppose that* $0 \in C$ *and* $\operatorname{int} C \neq \emptyset$ *(a.s.). Then*

$$\begin{aligned}\{E(\eta|\mathcal{G}) : \ \eta \in L^0(\operatorname{int} C,\mathcal{F})\} &\subseteq L^0(\operatorname{int} E(C|\mathcal{G}),\mathcal{G})\} \\ &\subseteq \{E(\eta|\mathcal{G}) : \ \eta \in L^0(2\operatorname{int} C,\mathcal{F})\}.\end{aligned}$$

The first inclusion implies that $\operatorname{int} E(C|\mathcal{G}) \neq \emptyset$ *(a.s.).*

Proof. Let $\eta \in L^0(\operatorname{int} C,\mathcal{F})$. Then $\rho := 2\operatorname{dist}(\eta,\partial C)$ is a strictly positive random variable and $\eta+\rho\bar{\mathcal{O}}_1(0) \subseteq C$. Hence, $E(\eta|\mathcal{G})+E(\rho|\mathcal{G})\bar{\mathcal{O}}_1(0) \subseteq E(C|\mathcal{G})$ and $E(\eta|\mathcal{G}) \in L^0(\operatorname{int} E(C|\mathcal{G}),\mathcal{G})\}$, i.e. the first inclusion holds.

To check the second inclusion we fix an arbitrary $\vartheta_0 \in L^0(\operatorname{int} C,\mathcal{F})$ and put $\tilde{\vartheta}_0 := E(\vartheta_0|\mathcal{G})$. Let $\eta \in L^0(\operatorname{int} E(C|\mathcal{G}),\mathcal{G})$. Let ϵ denote the distance of η from the boundary of $E(C|\mathcal{G})$. It is a $\mathcal{G}$-measurable random variable taking values in $]0,1]$. We have $\eta-\epsilon\tilde{\vartheta}_0 \in E(C|\mathcal{G})$ and, by definition of the set-valued conditional expectation, $\eta-\epsilon\tilde{\vartheta}_0 = E(\vartheta_1|\mathcal{G})$ for some $\vartheta_1 \in L^0(C,\mathcal{F})$. Putting $\vartheta := \vartheta_1+\epsilon\vartheta_0$, we get that $\eta = E(\vartheta|\mathcal{G})$. Since

$$\vartheta = (1+\epsilon)\left[\frac{1}{(1+\epsilon)}\vartheta_1 + \frac{\epsilon}{(1+\epsilon)}\vartheta_0\right]$$

and the expression in the square bracket defines an element from $L^0(\operatorname{int} C)$, the random variable ϑ is a selector of the set $(1+\epsilon)\operatorname{int} C$. The latter is contained in the set $2\operatorname{int} C$ due to the assumption $0 \in C$ which we use only here. $\square$

Proof of Theorem 2.2.17. We need to establish only the "difficult" implication $(\Rightarrow)$. To this aim define by backward induction the adapted set-valued process $C = (C_t)$ with $C_T := G_T^* \cap \bar{\mathcal{O}}_1(0)$ and $C_t := E(C_{t+1}|\mathcal{F}_t) \cap G_t^*$ for $t \le T-1$.

In the case where all ($\mathcal{F}_t$-measurable) random sets $\operatorname{int} C_t \neq \emptyset$ the needed martingale is $Z = Z^T$ where Z^T is obtained by the following procedure. Take an arbitrary $Z_0^0 \in L^0(\operatorname{int} C_0,\mathcal{F}_0)$. Suppose that we constructed already $Z^t \in \mathcal{M}_0^t(\operatorname{int} C)$. In virtue of Lemma 2.2.18 for the element $Z_t^t \in L^0(\operatorname{int} C_t,\mathcal{F}_t)$ we can find $\eta \in L^0(2\operatorname{int} C_{t+1},\mathcal{F}_{t+1})$ such that $Z_t^t = E(\eta|\mathcal{F}_t)$. The process $Z^{t+1} := \frac{1}{2}(Z_0^t,...,Z_t^t,\eta)$ is an element of $\mathcal{M}_0^{t+1}(\operatorname{int} C)$.

So, the theorem will be proven if we show that NA^s implies that $\operatorname{int} C_t \neq \emptyset$ for all t. Suppose that this is not the case and there is s such that $\operatorname{int} C_t \neq \emptyset$ for every $t \ge s+1$, but $\operatorname{int} C_s = \emptyset$ on a non-null set $\Gamma_s \in \mathcal{F}_s$; note that $s \le T-1$ as $\operatorname{int} C_T \neq \emptyset$. Since $\operatorname{int} C_s = \operatorname{int}\big(E(C_{s+1}|\mathcal{F}_s) \cap G_s^*\big) = \emptyset$ on Γ_s and, by assumption $\operatorname{int} G_s^* \neq \emptyset$, one can separate $\operatorname{int} E(C_{s+1}|\mathcal{F}_0)$ and G_s^* on Γ_s in an $\mathcal{F}_s$-measurable way, that is to find $\xi \in L^0(\mathbf{R}^d,\mathcal{F}_s)$ vanishing outside Γ_s and such that $\eta\xi \le 0$ for every $\eta \in L^0(G_s^*,\mathcal{F}_s)$ (therefore, $\xi \in -L^0(G_s,\mathcal{F}_s)$) and $\zeta\xi > 0$ on Γ_s for any selector ζ of $\operatorname{int} E(C_{s+1}|\mathcal{F}_1)$. According to the lemma below this contradicts to NA^s.

Lemma 2.2.19 *Assume that* $\operatorname{int} C_t \neq \emptyset$ *for* $t \geq s+1$. *Suppose that there exists* $\xi \in A_s^T$ *such that* $\zeta\xi \geq 0$ *whatever is* $\zeta \in L^0(\operatorname{int} E(C_{s+1}|\mathcal{F}_s), \mathcal{F}_s)$ *and this inequality is strict on a non-null set (hence,* $\xi \neq 0$*). Then* NA^s *fails.*

Proof. If $s = T-1$, then ξ is a weak arbitrage opportunity already for A_{T-1}^T. Indeed, let $B := \{\xi \notin G_T\}$ be a non-null set. Then one can find $\nu \in L^0(G_T^*, \mathcal{F}_T)$ with $|\nu| \leq 1$ such that $\nu\xi < 0$ on B. Thus,

$$E(\nu I_B|\mathcal{F}_{T-1})\xi = E(\nu\xi I_B|\mathcal{F}_{T-1}) \leq 0$$

and the inequality is strict on a non-null set. This contradicts to the assumption of the lemma that the inequality $\zeta\xi \geq 0$ holds for all random vectors $\zeta \in L^0(E(C_T|\mathcal{F}_{T-1}), \mathcal{F}_{T-1})$. Hence, ξ are in G_T and NA^s fails to be true.

Going backward, let us establish the assertion for an arbitrary s assuming that it holds for $s+1$, ..., $T-1$. Since the behavior before s does not matter, we may assume for the notational convenience that $s = 0$ and also that the cone $G_0 = -\mathbf{R}\xi$, i.e. $G_0^* = \{y : \ y\xi \leq 0\}$. Consider the partition of Ω into the following three $\mathcal{F}_1$-measurable subsets:

$$\begin{aligned}
&\Omega_1 := \{\operatorname{int} E(C_2|\mathcal{F}_1) \cap \operatorname{int} G_1^* \cap G_0^* \neq \emptyset\},\\
&\Omega_2 := \{\operatorname{int} E(C_2|\mathcal{F}_1) \cap G_1^* \cap G_0^* \neq \emptyset\} \cap \{\operatorname{int} E(C_2|\mathcal{F}_1) \cap G_1^* \cap G_0^* \subseteq \partial G_1^*\},\\
&\Omega_3 := \{\operatorname{int} E(C_2|\mathcal{F}_1) \cap G_1^* \cap G_0^* = \emptyset\}.
\end{aligned}$$

Suppose that $P(\Omega_1) > 0$. On Ω_1 the intersection of $\operatorname{int}\big(E(C_2|\mathcal{F}_1) \cap G_1^*\big)$ and $\operatorname{int} G_0^*$ is non-empty, hence $C_1 \cap \{y : \ y\xi < 0\} \neq \emptyset$. Let ν be a selector of the latter set extended by zero outside Ω_1. Then $\eta := E(\nu|\mathcal{F}_0)$ belongs to $E(C_1|\mathcal{F}_0)$ and $\eta\xi \leq 0$ with the strict inequality on a non-null set. This contradiction with the hypothesis of the lemma means that $P(\Omega_1) = 0$.

Suppose now that $P(\Omega_2) > 0$. Put $H := \partial G_0^* = \{y : \ y\xi = 0\}$. Note that the assumption of the lemma implies that (a.s.)

$$\operatorname{int} E(C_2|\mathcal{F}_1) \cap G_1^* \cap G_0^* = \operatorname{int} E(C_2|\mathcal{F}_1) \cap G_1^* \cap H.$$

Indeed, on the subset of Ω where the left-hand side is empty this identity is obvious. On the complementary subset

$$\operatorname{int} E(C_2|\mathcal{F}_1) \cap G_1^* \cap \operatorname{int} G_0^* \neq \emptyset$$

and, hence, $\operatorname{int}\big(E(C_2|\mathcal{F}_1) \cap G_1^*\big) \cap \operatorname{int} G_0^* \neq \emptyset$ implying that $C_1 \cap \operatorname{int} G_0^* \neq \emptyset$. Using this we arrive to a contradiction by the same separation argument as was done for the set Ω_1.

Arguing with convex sets in the subspace H we get easily that

$$\begin{aligned}
&\{\operatorname{int} E(C_2|\mathcal{F}_1) \cap G_1^* \cap G_0^* \neq \emptyset\} \cap \{\operatorname{int} G_1^* \cap H \neq \emptyset\}\\
&= \{(\operatorname{int} E(C_2|\mathcal{F}_1) \cap H) \cap (G_1^* \cap H) \neq \emptyset, \ \operatorname{int} G_1^* \cap H \neq \emptyset\}\\
&\subseteq \{\operatorname{int} E(C_2|\mathcal{F}_1) \cap \operatorname{int} G_1^* \cap H \neq \emptyset\} \subseteq \Omega_1.
\end{aligned}$$

It follows that $\Omega_2 \cap \{\operatorname{int} G_1^* \cap H \neq \emptyset\} = \emptyset$. So, Ω_2 is the union of two sets $\Omega_2 \cap \{G_1^* \subseteq G_0^*\}$ and $\Omega_2 \cap \{G_1^* \subseteq -G_0^*\}$. The first has zero probability because in virtue of the assumption of the lemma the inclusion $C_1 \subseteq \{y : \; y\xi \leq 0\}$ may hold only on a null-set. Thus, $P(\Omega_2) = P(\Omega_2 \cap \{G_1^* \subseteq -G_0^*\})$. This means that $\xi \in G_1$ on Ω_2. If $P(\Omega_2) = 1$ then the NA^s-property fails. It remains to consider the case $P(\Omega_2) < 1$.

So, suppose that $P(\Omega_3) > 0$. In this case we can separate $\operatorname{int} E(C_2|\mathcal{F}_1)$ and $G_1^* \cap G_0^*$ on Ω_3, i.e. find an $\mathcal{F}_1$-measurable random vector $\nu = I_{\Omega_3}\nu$ equal to zero outside Ω_3 and such that $\nu\zeta > 0$ for any $\zeta \in L^0(\operatorname{int} E(C_2|\mathcal{F}_1), \mathcal{F}_1)$ on Ω_3 and $\nu\zeta \leq 0$ for any $\zeta \in L^0(G_1^* \cap G_0^*, \mathcal{F}_1)$. Since $(G_1^* \cap G_0^*)^* = G_0 + G_1$ in virtue of the assumption on closedness of $G_0 + G_1$, the second property means that $\nu = \alpha\xi - \xi_1$ where $\alpha \in L^0(\mathbf{R}_+, \mathcal{F}_1)$ and $\xi_1 \in L^0(G_1, \mathcal{F}_1)$. If $\alpha > 0$ on Ω_3, we can divide the identity by α and claim the existence of ν with the above properties having the form $\nu = \xi - \xi_1$. In this case $\nu \in A_0^1$ and by the induction hypothesis NA^s fails. If $\tilde{\Omega}_3 := \{\alpha = 0\} \cap \Omega_3$ is a non-null set, we arrive to the same conclusion applying the induction hypothesis with $\nu = \xi_1 I_{\tilde{\Omega}_3}$. □

2.2.6 Arbitrage Opportunities of the Second Kind

On some markets satisfying the NA^w-property it may happen that an investor with an initial endowment outside the solvency cone may run a portfolio to get rid of debts for sure. As the following example shows, sometimes it is sufficient just to wait.

Example. Let us consider the two-asset model with $S_0^1 = S_0^2 = 1$ where the first asset is the numéraire, i.e. $S_1^1 = 1$, while S_1^2 takes values $1+\varepsilon$ and $1-\varepsilon > 0$ with probabilities $1/2$. The filtration is generated by S. Suppose that K_0^* is the cone generated by the vectors $(1, 2)$ and $(1, 1/2)$ and $K_1^* = \mathbf{R}_+\mathbf{1}$, i.e. there is no transaction costs at the date $T = 1$. Then $\widehat{K}_1^*$ is the ray generated by the vector S_1. The process Z with $Z_0 = (1, 1)$ and $Z_1 = S_1$ is a strictly consistent price system, so the NA^w-property holds. Let v be a point from the dual C of the cone generated by the vectors $(1, 1+\varepsilon)$ and $(1, 1-\varepsilon)$. It lays in the solvency cone $\widehat{K}_1$ a.s. But for $\varepsilon \in]0, 1/2[$ this dual C is strictly larger than the solvency cone $\widehat{K}_0 = K_0$. The investor having $v \in C \setminus K_0$ as the initial endowment became solvent at time $T = 1$ though he was not solvent at the date zero. Clearly, we can modify the model by introducing small transaction costs at time $T = 1$ to get the same conclusion for a model with efficient friction.

We say that the model G admits *arbitrage opportunities of the second kind* if there exist $s \leq T - 1$ and an $\mathcal{F}_s$-measurable d-dimensional random variable for which $\Gamma := \{\xi \notin G_s\}$ is not a null-set and such that

$$(\xi + A_s^T) \cap L^0(G_T, \mathcal{F}_T) \neq \emptyset,$$

i.e. $\xi = \xi_s + ... + \xi_T$ for some $\xi_t \in L^0(G_t, \mathcal{F}_t)$, $s \leq t \leq T$. If such ξ does exist then, in the financial context where $G = \widehat{K}$, an investor having $I_\Gamma \xi$ as the initial endowments at time s, may use the strategy $(I_\Gamma \xi_t)_{t\geq s}$ and get rid of all debts at time T.

So, the model has *no arbitrage opportunities of the second kind* (abbreviation: has the $NA2$-property) if for every date s and $\xi \in L^0(\mathbf{R}^d, \mathcal{F}_s)$ the intersection $(\xi + A_s^T) \cap L^0(G_T, \mathcal{F}_T)$ is non-empty only if $\xi \in L^0(G_s, \mathcal{F}_s)$. Alternatively, the $NA2$-property can be expressed in the following way:

$$L^0(\mathbf{R}^d, \mathcal{F}_s) \cap (-A_s^T) = L^0(G_s, \mathcal{F}_s) \qquad \forall s \leq T.$$

Note that in the original paper by Rásonyi the $NA2$-property was called the *no sure gain in liquidation value* property (NGV) or, in earlier version, *no sure profits* property with the abbreviation NSP. We use the terminology consistent with that developed in the theory of large financial markets, see [**?**].

Theorem 2.2.20 *Suppose that the efficient friction condition is fulfilled and* $\mathbf{R}_+^d \subseteq G_t$ *for all* t. *Then the following conditions are equivalent:*

(a) $NA2$;

(b) $L^0(\mathbf{R}^d, \mathcal{F}_s) \cap L^0(G_{s+1}, \mathcal{F}_{s+1}) \subseteq L^0(G_s, \mathcal{F}_s)$ *for all* $s = 0, ..., T-1$;

(c) $\operatorname{cone} \operatorname{int} E(G_{s+1}^* \cap \bar{\mathcal{O}}_1(0)|\mathcal{F}_s) \supseteq \operatorname{int} G_s^*$ *(a.s.) for all* $s = 0, ..., T-1$;

(d) *whatever is* $s \leq T-1$ *and* $\eta \in L^1(\operatorname{int} G_s^*, \mathcal{F}_s)$ *there is* $Z \in \mathcal{M}_s^T(\operatorname{int} G^*)$ *such that* $Z_s = \eta$.

Proof. $(a) \Rightarrow (b)$ It follows from the inclusions

$$L^0(G_{s+1}, \mathcal{F}_s) \subseteq L^0(G_{s+1}, \mathcal{F}_{s+1}) \subseteq -A_s^T.$$

$(b) \Rightarrow (c)$ Put $H_s := \operatorname{cone} \operatorname{int} E(G_{s+1}^* \cap \bar{\mathcal{O}}_1(0)|\mathcal{F}_s)$. Suppose that for some $t \leq T-1$ the set of $\Gamma := \{\operatorname{int} G_t^* \setminus H_t \neq \emptyset\}$ has strictly positive probability. The convex sets $H_t(\omega) \setminus \{0\}$ are open. Hence, $\Gamma = \{\operatorname{int} G_t^* \setminus \bar{H}_t \neq \emptyset\}$. Using measurable selection we can find $\mathcal{F}_t$-measurable d-dimensional random variable $\zeta = \zeta I_\Gamma$ such that $\zeta \in \operatorname{int} G_t^* \setminus \bar{H}_t$ a.s. on the set Γ. By the measurable version of the separation theorem we find a d-dimensional random variable $\tilde{\xi} = \tilde{\xi} I_\Gamma$ satisfying the following two properties:

$i)$ $\tilde{\xi}\eta \geq 0$ for all $\eta \in L^0(\bar{H}_t, \mathcal{F}_t)$;

$ii)$ $\tilde{\xi}\zeta < 0$ a.s. on Γ.

Normalizing, we may assume that $|\tilde{\xi}| = I_\Gamma$.

The property $i)$ implies that $\tilde{\xi} \in L^0(G_{t+1}, \mathcal{F}_t)$ (a.s.). Indeed, a random variable $\xi \in L^1(\mathbf{R}^d, \mathcal{F}_t)$ takes values in G_{t+1} if and only if $E\xi E(\eta|\mathcal{F}_t) \geq 0$ for every $\eta \in L^0(G_{t+1}^*, \mathcal{F}_{t+1})$ with $|\eta| \leq 1$. But the set of the conditional expectations $E(\eta|\mathcal{F}_t)$ for such η coincides with the set $L^0(E(G_{t+1}^* \cap \bar{\mathcal{O}}_1(0)|\mathcal{F}_t), \mathcal{F}_t)$.

The property $ii)$ means that $\tilde{\xi}$ does not take values in G_t on the set Γ of the strictly positive probability. The existence of such $\tilde{\xi}$ contradicts (b).

$(c) \Rightarrow (d)$ Let $Z_t \in L^1(G_t^*, \mathcal{F}_t)$. Since

$$\{(a, y) \in]0, \infty[\times \operatorname{int} E(G_{t+1}^* \cap \bar{\mathcal{O}}_1(0)|\mathcal{F}_t)(\omega) : \ ay = Z_t(\omega)\} \neq \emptyset \quad \text{a.s.},$$

we get, using measurable selection, that $Z_t = \alpha_t Y_t$ where $\alpha_t \in L^0(]0,\infty[,\mathcal{F}_t)$ and $Y_t \in L^0(\operatorname{int} E(G^*_{t+1} \cap \bar{\mathcal{O}}_1(0)|\mathcal{F}_t),\mathcal{F}_t)$. By Lemma 2.2.18 $Y_t \in E(\tilde{Z}_{t+1}|\mathcal{F}_t)$ for some $\tilde{Z}_{t+1} \in L^0(2\operatorname{int}(G^*_{t+1} \cap \bar{\mathcal{O}}_1(0)),|\mathcal{F}_{t+1})$. Put $Z_{t+1} := \alpha_t \tilde{Z}_{t+1}$. Then $E(|\tilde{Z}_{t+1}||\mathcal{F}_t)) \le 2\alpha < \infty$ and $E(\tilde{Z}_{t+1}|\mathcal{F}_t) = Z_t$. Since the process G^* evolves in $\mathbf{R}^d_+$, we have that $E(\alpha_t Z^i_{t+1}) = EZ^i_t < \infty$, i.e. Z_{t+1} is integrable. Repeating successively these arguments starting from $Z_s = \eta$ we obtain a martingale with the required property. Note that without the assumption that $\mathbf{R}^d_+ \subseteq G_t$ by this construction we could obtain only a generalized martingale.

$(d) \Rightarrow (a)$ Let us suppose that $\xi \in L^0(\mathbf{R}^d,\mathcal{F}_s)$ admits the representation $\xi = \xi_s + ... + \xi_T$ with $\xi_t \in L^0(G_t,\mathcal{F}_t)$, $s \le t \le T$ but the set $\Gamma := \{\xi \notin G_s\}$ is of strictly positive probability. Without loss of generality we may assume that ξ is bounded. For each $\omega \in \Gamma$ one can find $\eta(\omega) \in \operatorname{int} G^*_s(\omega)$ such that $\eta(\omega)\xi(\omega) < 0$. Using measurable selection we can find $\eta \in L^0(\operatorname{int} G^*_s,\mathcal{F}_s)$ such that the latter inequality holds a.s. The condition (d) ensures the existence $Z \in \mathcal{M}^T_s(\operatorname{int} G^*)$ such that $Z_s = \eta$. Then $EZ_T\xi I_\Gamma = EZ_s\xi I_\Gamma < 0$ and, in virtue of Lemma 2.3.2 (next section)

$$EZ_T \sum_{t=s}^{T} \xi_t I_\Gamma \ge 0.$$

A contradiction. □

Remark. As we could see, the criteria investigated in this chapter relate certain properties of financial models with the existence of a consistent price system, i.e. a martingale which is a selector of a set-valued process (for the considered models, in the dual to the process of solvency cones in physical units). It was observed by Rokhlin that the latter problem can be placed in a more general framework of the *martingale selection problem* which is not only of mathematical but also financial interest because it is useful also for models with liquidity constraints. The martingale selection problem suggests that there is a set-valued adapted process $G = (G_t)$ and the question is, whether there exist a probability $Q \sim P$ and a Q-martingale $Z \in \mathcal{M}(G)$. For the development in this direction we refer to the papers [90], [91], [88], [89].

2.3 Hedging of European Options

2.3.1 Hedging Theorem: Finite Ω

Let C be a $\mathbf{R}^d$-valued $\mathcal{F}_T$-measurable random variable, interpreted as a *contingent claim* of values of corresponding assets.

Our aim now is to describe the set of all initial endowments starting from which one can "super-replicate", in the the sense of the partial ordering, the contingent claim C by the terminal value of a self-financing portfolio.

The formal description of the convex set of *hedging endowments* (in values or in physical units since we use a convention that all $S_0^i = 1$) is as follows:

$$\Gamma := \{v \in \mathbf{R}^d : \ \exists\, B \in \mathcal{B} \text{ such that } \ v + V_T^B \succeq_T C\}$$

It is easy to see that

$$\Gamma = \{v \in \mathbf{R}^d : \ \widehat{C} \in v + \widehat{A}_0^T\}.$$

We introduce also the closed convex set

$$D := \left\{v \in \mathbf{R}^d : \ \sup_Z E(Z_T\widehat{C} - Z_0 v) \le 0\right\} = \bigcap_Z \{v \in \mathbf{R}^d : \ Z_0 v \ge EZ_T\widehat{C}\}$$

where Z runs the set $\mathcal{M}_0^T(\widehat{K}^* \setminus \{0\})$ assumed to be non-empty. Having in mind the approximation of Z_T by $Z_T^e := (1-\varepsilon)Z_T + \varepsilon\tilde{Z}_T$ where $\tilde{Z}$ belongs to $\mathcal{M}_0^T(\widehat{K}^* \setminus\{0\})$, we can, of course, take above the supremum over $Z \in \mathcal{M}_0^T(\widehat{K}^*)$.

Theorem 2.3.1 *Let Ω be finite and $\mathcal{M}_0^T(\widehat{K}^* \setminus \{0\}) \neq \emptyset$. Then $\Gamma = D$.*

Proof. Take $\xi = \sum_{t=0}^T \xi_t$ with $\xi_t \in -L^0(\widehat{K}_t, \mathcal{F}_t)$. For any $Z \in \mathcal{M}_0^T(\widehat{K}^* \setminus \{0\})$ we have:

$$EZ_T\widehat{C} \le EZ_T\left(v + \sum_{t=0}^T \xi_t\right) = Z_0 v + \sum_{t=0}^T EZ_t\xi_t \le Z_0 v.$$

and the "easy" inclusion $\Gamma \subseteq D$ holds.

Take now $v \notin \Gamma$. To show that $v \notin D$ it is sufficient to find $Z \in \mathcal{M}_0^T(\widehat{K}^*)$ with $Z_0 v < EZ_T\widehat{C}$. Since $\widehat{C} \notin v + \widehat{A}_0^T$ and the latter set, being a shift of a polyhedral cone, is closed, the separation theorem for the finite dimensional space implies that

$$\sup_{\xi \in v + \widehat{A}_0^T} E\eta\xi < E\eta\widehat{C} \tag{2.3.1}$$

for some d-dimensional random variable η. Define a martingale $Z_t := E(\eta|\mathcal{F}_t)$. It follows that $EZ_t\xi_t \ge 0$ for all $\xi_t \in L^0(\widehat{K}_t, \mathcal{F}_t)$ implying that $Z \in \mathcal{M}_0^T(\widehat{K}^*)$. Taking in (2.3.1) $\xi = v$ and using the martingale property, we get the desired inequality $EZ_0 v < E\eta\widehat{C}$. $\square$

Financial interpretation. We want to attract the reader's attention to the financial interpretation of the obtained result and the role of consistent price systems. The theorem asserts that a contingent claim $\widehat{C}$ (in physical units) can be super-replicated starting from an initial endowment v by a self-financing portfolio if and only if the "value" $Z_0 v$ of this initial endowment is not less than the expected "value" of the contingent claim $EZ_T\widehat{C}$, whatever is the consistent price system Z (we write "value" in quotation marks to emphasize its particular meaning in the present context). In other words, consistent price systems allow the option seller to relate benefits from possessing v at time $t = 0$ and the liabilities $\widehat{C}$ at time $t = T$ and provide information whether there is a portfolio ending up on the safe side.

2.3.2 Hedging Theorem: Discrete Time, Arbitrary Ω

Now we present a hedging result extending the Theorem 2.3.1 to the case of arbitrary Ω. There is no need to change the definition of the set of Γ of initial endowments from which one can start a portfolio process which terminal values dominate the contingent claim. However, its dual description, that is the definition of the set D requires some precautions needed to ensure the existence of expectations involved. Moreover, the techniques used requires closedness of the set of replicable claims. That is why we shall assume that the set $\mathcal{M}_0^T(\operatorname{ri} G^*)$ is non-empty.

We present the result in the abstract setting of the $\mathcal{C}$-valued process G dominating the constant process $\mathbf{R}^d_+$, the setting which, as we can indicated early, is not only more mathematically transparent but covers various financial models with proportional transaction costs besides our basic one.

So, we fix a d-dimensional random variable ζ (which correspond in financial context to $\widehat{C}$, the contingent claim expressed in physical units). Define the set

$$\Gamma = \{v \in \mathbf{R}^d : \ \zeta \in v + A_0^T\}.$$

Let $\mathcal{Z}$ be the set of martingales from $\mathcal{M}_0^T(\operatorname{ri} G^*)$ such that $E(Z_T\zeta)^- < \infty$. Put

$$D := \left\{ v \in \mathbf{R}^d : \ \sup_{Z\in\mathcal{Z}} E(Z_T\zeta - Z_0 v) \le 0 \right\}.$$

The following simple assertion is a key to understanding of a role of the integrability assumption involved in the definition of $\mathcal{Z}$.

Lemma 2.3.2 *Let Z be an $\mathbf{R}^d$-valued martingale and let $\Sigma_T := Z_T \sum_{s=0}^T \xi_s$ where $\xi_s \in L^0(\mathbf{R}^d, \mathcal{F}_s)$ are such that $Z_s\xi_s \le 0$. If $E\Sigma_T^- < \infty$, then all products $Z_s\xi_s$ are integrable, Σ_T is integrable and $E\Sigma_T \le 0$.*

Proof. For $T = 0$ there is nothing to prove. Assume that the claim is true for $T - 1$. Clearly,

$$Z_T \sum_{s=0}^{T-1} \xi_s \ge -\Sigma_T^- - Z_T\xi_T \ge -\Sigma_T^-.$$

By conditioning of the resulting inequality we get that

$$Z_{T-1}\sum_{s=0}^{T-1}\xi_s \geq -E(\Sigma_T^-|\mathcal{F}_{T-1}).$$

Since the left-hand side is Σ_{T-1}, we have the bound $\Sigma_{T-1}^- \leq E(\Sigma_T^-|\mathcal{F}_{T-1})$ implying that $E\Sigma_{T-1}^- \leq E\Sigma_T^- < \infty$. By the induction hypothesis Σ_{T-1} is integrable and $E\Sigma_{T-1} \leq 0$. We have the same properties for Σ_T because $Z_T\xi_T \leq 0$.

Theorem 2.3.3 *Suppose that $\mathcal{M}_0^T(\mathrm{ri}\, G^*) \neq \emptyset$. Then $\Gamma = D$.*

Proof. The arguments follows the same line as in the proof of Theorem 2.3.1. The inclusion $\Gamma \subseteq D$ is clear: if $\zeta = v + \sum_{s=0}^T \xi_s$ with $\xi_s \in -L^0(G_s, \mathcal{F}_s)$ then $EZ_T\zeta \leq Z_0 v$ for any martingale from $\mathcal{Z}$ in virtue of the above lemma.

To check the opposite inclusion we take a point $v \notin \Gamma$ and show that $v \notin D$. It is sufficient to find $Z \in \mathcal{Z}$ such that $Z_0 v < EZ_T\zeta$. Consider a measure $\tilde{P} \sim P$ with bounded density ρ such that $\zeta \in L^1(\tilde{P})$. The convex set $\tilde{A}^1 := A_0^T \cap L^1(\tilde{P})$ is closed and does not contain the point $\zeta - v$ and, hence, can be separated from the latter by a functional η from L^∞. This means that

$$\sup_{\xi\in\tilde{A}^1} E\rho\eta\xi < E\eta\rho(\zeta - v).$$

The bounded martingale $E(\rho\eta|\mathcal{F}_t)$ satisfies the needed inequality and belongs to $\mathcal{M}_0^T(G^*)$. Adding to it the martingale $\epsilon\tilde{Z}$ where $\tilde{Z} \in \mathcal{M}_0^T(\mathrm{ri}\, G^*)$ and taking $\varepsilon > 0$ small enough we get Z with all needed properties. □

2.4 Hedging of American Options

2.4.1 American Options: Finite Ω

We consider again the abstract setting where the model is given $\mathcal{C}$-valued process $G = (G_t)$, $t = 0, 1, ..., T$, dominating the constant process $\mathbf{R}_+^d$. Recall that a particular case of this setting serves as model of financial market with transactions costs specified in the "hat" terms, i.e. the assets are counted in physical units. The pay-off process $Y = (Y_t)$ is now $\mathbf{R}^d$-valued. Our aim is to describe the set $\Gamma = \Gamma(Y)$ of all $v \in \mathbf{R}^d$ such that there is a portfolio process $X = (X_t)$ starting from zero for which $v + X_t \geq_{G_t} Y_t$, i.e. the process $v + X$ dominates Y in the sense of partial orderings generated by G. More formally, we denote by $\mathcal{X}^0$ the set of $X = (X_t)$ with $X_{-1} = 0$ and $\Delta X_t \in -L^0(G_t, \mathcal{F}_t)$ for $t = 0, 1, ..., T$ and put

$$\Gamma := \{v \in \mathbf{R}^d :\ \exists X \in \mathcal{X}^0 \text{ such that }\ v + X_t - Y_t \in G_t,\ t = 0, 1, ..., T\}.$$

We also introduce the set $A_0^T(.)$ of hedgeable American claims consisting of all processes Y which can be dominated, in the above sense, by a portfolio process with zero initial capital. Clearly, if $Y \in A_0^T(.)$, then $Y_t \in A_0^t$ for all $t \geq 0$. Note that the adapted processes can be viewed as measurable functions on the measure space $(\tilde{\Omega}, \tilde{\mathcal{F}}, P^N)$ where $\tilde{\Omega} = \Omega \times \{0, 1, ..., T\}$, the σ-algebra $\tilde{\mathcal{F}}$ is generated by the adapted processes themselves[1] and $dP^N = dPdN$, N is the counting measure on integers (i.e. $N(\{t\}) = 1$). The averaging with respect to P^N will be denoted by E^N.

By analogy with the result available for frictionless market (Proposition 1.1.14) and having in mind just established hedging theorems for European-type options under transaction costs one may guess that, at least for the case of finite Ω ,

$$\Gamma = \{v \in \mathbf{R}^d : \ Z_0 v \geq E Z_\tau Y_\tau \ \forall Z \in \mathcal{M}(G^*), \ \tau \in \mathcal{T}\}. \tag{2.4.1}$$

Surprisingly, in general, this equality, as we show later, fails to be true. To formulate the correct result we introduce the notation

$$\bar{Z}_t := \sum_{r=t}^{T} E(Z_r | \mathcal{F}_t)$$

and define the set of adapted bounded processes

$$\mathcal{Z}(G^*, P) := \{Z : \ Z_t, \bar{Z}_t \in L^\infty(G_t^*, \mathcal{F}_t), \ t = 0, 1, ..., T\}.$$

Clearly, all bounded martingales from $\mathcal{M}(G^*, P)$ belongs to $\mathcal{Z}(G^*, P)$.

Theorem 2.4.1 *Suppose that Ω is finite. Then*

$$\Gamma = \{v \in \mathbf{R}^d : \ \bar{Z}_0 v \geq E^N Z Y \quad \forall Z \in \mathcal{Z}(G^*, P)\}. \tag{2.4.2}$$

Proof. The inclusion $\subseteq$ is easy. Indeed, let $v \in \Gamma$ and let X be a value process which dominate $Y - v$. Then for $Z \in \mathcal{Z}(G^*, P)$ we have:

$$E^N Z Y = \sum_{t=0}^{T} E Z_t Y_t \leq \sum_{t=0}^{T} E Z_t (v + X_t) = \bar{Z}_0 v + \sum_{t=0}^{T} E Z_t X_t \leq \bar{Z}_0 v$$

since

$$\sum_{t=0}^{T} E Z_t X_t = \sum_{t=0}^{T} \sum_{r=0}^{t} E Z_t \Delta X_r = \sum_{t=0}^{T} E \left(\sum_{r=t}^{T} Z_r \right) \Delta X_t = \sum_{t=0}^{T} E \bar{Z}_t \Delta X_t \leq 0$$

because ΔX_t and $\bar{Z}_t$ take values in $-G_t$ and G_t^*, respectively.

To prove the reverse inclusion we follow the usual pattern. Suppose that $v \notin \Gamma$, i.e. $Y - v$ does not belong to the closed convex cone $A_0^T(.)$ in the finite-dimensional Euclidean space $L^2(\mathbf{R}^d; \tilde{\Omega}, \tilde{F}, P^N)$. The separation theorem

[1]That is $\tilde{\mathcal{F}}$ is the discrete-time analogue of the optional σ-algebra of the general theory of processes.

provides us an element Z from this space (which is simply an adapted process) such that

$$\sup_{X \in A_0^T(.)} E^N ZX < E^N Z(Y - v). \tag{2.4.3}$$

Since $A_0^T(.)$ is a cone, the supremum is zero. Thus, for every $X \in A_0^T(.)$

$$\sum_{t=0}^T EZ_t X_t \le 0, \qquad \sum_{t=0}^T E\bar{Z}_t \Delta X_t \le 0 \tag{2.4.4}$$

(the left-hand sides of this equalities are the same according to the above calculation). It follows that $EZ_t\xi \le 0$ and $E\bar{Z}_t\xi \le 0$ whatever is $\xi \in -L^0(G_t, \mathcal{F}_t)$. Therefore, $Z \in \mathcal{Z}(G^*, P)$. As $E^N Z(Y - v) > 0$, the point v does not belong to the set in the right-hand side of (2.4.2) and we conclude. □

Remark. It is easy to see that the hedging theorem for American options remains true if we replace the counting measure by an arbitrary probability measure ν on the set $\{0, 1, ..., T\}$ with $\nu(\{t\}) > 0$ for all $t = 0, 1, ..., T$. Of course, $\bar{Z}_t$ should be replaced by

$$\bar{Z}_t^\nu = \sum_{r=t}^T E(Z_r|\mathcal{F}_t)\nu(\{r\}) = \int_{[t,T]} E(Z_r|\mathcal{F}_t)\nu(dr).$$

This gives a hint to the corresponding result in the continuous-time setting.

Financial interpretation. As we shall see below, even in a very simple discrete-time models consistent price systems form a class which is too narrow to evaluate American claims correctly. The phenomenon appears because one cannot prohibit the option buyer to toss a coin and take a decision, to exercise the option at time t or not, in dependence of the outcome. It happens that the expected "value" of an American claim is a mathematical expectation of the weighted average of "values" of assets obtained by the option holder for a variety of exercise dates. This expected "value" should be compared with the "value" of the initial endowment. The main question is what is the class of price systems which should be involved to calculate "values" to be compared. The above theorem shows that the comparison can be done with the systems for which the expected weighted average of future prices knowing the past is again a price system. The structure of such price systems is coherent with the option buyer actions. We shall call them *coherent price systems* and use the abbreviation CoPS.

2.4.2 American Options: Arbitrary Ω

Assuming a condition which guarantees the closedness of $A_0^T(.)$ (see Proposition 2.4.3 below) we can easily establish the following result which holds for an arbitrary probability space.

Theorem 2.4.2 *Suppose that NA^r-property holds. Then*

$$\Gamma = \{v \in \mathbf{R}^d : \ \bar{Z}_0 v \geq E^N ZY \ \ \forall Z \in \mathcal{Z}(G^*, P) \ \textit{with} \ E^N|ZY| < \infty\}. \quad (2.4.5)$$

Proof. The arguments to establish the inclusion $\subseteq$ remain the same as in the previous theorem. To check the opposite inclusion take an arbitrary $v \notin \Gamma$. So, $Y - v \notin A_0^T(.)$. Choose an auxiliary probability measure $\tilde{P}$ such that the density process $\rho = (\rho_t)_{t \leq T}$ is bounded and $\tilde{E}^N|Y| < \infty$. Now $Y - v \in L^1(\tilde{P}^N)$ and does not belong to the closed convex cone $A_0^T(.) \cap L^1(\tilde{P}^N)$. Let $\tilde{Z}$ denote the separating functional. The bounded process $Z := \rho\tilde{Z}$ satisfies (2.4.3) and (2.4.4) and we conclude exactly in the same way as above. □

Proposition 2.4.3 *If NA^r-property holds, then the set $A_0^T(.)$ is closed in $L^0(P^N)$.*

Proof. We know already that the claim holds for the one-step model. Arguing by induction we suppose that it is true for $T-1$. Let us consider a sequence of processes $Y^n \in A_0^T(.)$ converging to some Y. By definition, there is a sequence of portfolio processes X^n (i.e. with $\Delta X_t^n \in -G_t$) such that $X_t^n - Y_t^n \in G_t$. By the standard reduction it is sufficient to consider the following two cases.

The first, easy, case: X_0^n converges to a limit $X_0 \in L^0(-G_0, \mathcal{F}_0)$. It follows that $X_0 - Y_0 \in L^0(G_0, \mathcal{F}_0)$. The process $\tilde{Y}_t^n := Y_t^n - \Delta X_0^n$, $t \geq 1$, is an element of $A_1^T(.)$. By the induction hypothesis, the limit of the sequence of processes $\tilde{Y}^n$ belongs to $A_1^T(.)$. This means that $\tilde{X}_t - \tilde{Y}_t \in L^0(G_t, \mathcal{F}_t)$, for some portfolio process defined for $t \geq 1$. Put $X_t := \tilde{X}_t + X_0$ for $t \geq 1$. Then $X = (X_t)_{t \geq 0}$ is a portfolio process dominating Y on the whole time range and, hence, $Y \in A_0^T(.)$.

The second case: $|X_0^n|$ diverges to infinity. Using the lemma on subsequences we can assume that $X_0^n/|X_0^n|$ converges to an element $\tilde{X}_0$ from $L^0(G_0^0, \mathcal{F}_0)$ where $G_0 := G_0 \cap (-G_0)$. By the induction hypothesis the set $A_1^T(.)$ is closed. We get from here that the constant process $-\tilde{X}_0$ is an element of $A_1^T(.)$ and, hence, the zero process is an element of $A_0^T(.)$ dominated by some process $\tilde{X}$ with $|\tilde{X}_0| = 1$. The domination property means that $\tilde{X}_t = \sum_{r=0}^t \Delta\tilde{X}_r$ takes values in G_t for each $t \geq 0$. Recalling that the equality

$$\sum_{r=0}^{t-1} \Delta\tilde{X}_r + (\Delta\tilde{X}_t - \tilde{X}_t) = 0$$

is fulfilled only if the summands are elements of the corresponding linear spaces $L^0(G_r^0, \mathcal{F}_r)$ (see Lemma 2.2.12) we obtain that $\Delta\tilde{X}_t$ and $\tilde{X}_t$ belongs to $L^0(G_t^0, \mathcal{F}_t)$ for every $t \geq 0$. The existence of such a process allows us to make a step of the Gauss-type elimination algorithm to diminish the number of non-zero components of X_0^n and we accomplish the proof in the same way as in Lemma 2.2.8. □

2.4.3 Complementary Results and Comments

Now we examine the question under which condition on the market the identity $\Gamma(Y) = D(Y)$ holds for every pay-off process Y; here $D(Y)$ is the set in the right-hand side of (2.4.1).

Put

$$c_t(x) := \inf\{\lambda \in \mathbf{R} : \ \lambda e_1 - x \in G_t\}, \qquad x \in \mathbf{R}^d.$$

In financial context, $c_t(x)$ is a number of units of the first asset needed to acquire, at date t, the portfolio x; if the first asset is the numéraire, $c_t(x)$ is a *constitutional value* of x.

Proposition 2.4.4 *Let $T \geq 1$. Suppose that there is $x \in \mathbf{R}^d$ such that the following two conditions are fulfilled:*

(i) if $y - c_0(x)e_1 \in G_0^0$, then either $y - x \in G_0^0$ or $P(y - x \in G_1) < 1$;

(ii) $x - c_0(x)e_1 \notin G_0$.

Then there exists $Y = (Y_t)$ such that $\Gamma(Y) \neq D(Y)$.

Proof. The process $Y_t = c_0(x)e_1 I_{\{t=0\}} + x I_{\{t>0\}}$ has the needed property. For any $Z \in \mathcal{M}(G^*)$ and an arbitrary stopping time $\tau \leq T$ we have:

$$E\big(Z_\tau Y_\tau - Z_0 c_0(x)e_1\big) = E\big(Z_\tau x - Z_0 c_0(x)e_1\big) I_{\{\tau>0\}} = Z_0\big(x - c_0(x)e_1\big) P(\tau > 0).$$

The right-hand side being negative, $c_0(x)e_1 \in D(Y)$. If $c_0(x)e_1 \in \Gamma(Y)$, there is a portfolio process X such that $c_0(x)e_1 + X$ dominates Y. In particular, $X_0 \in G_0$. But $X_0 = \Delta X_0 \in -G_0$. According to the condition (i) we have two possibilities. The first one: $c_0(x)e_1 + X_0 - x \in G_0^0$, hence, $c_0(x)e_1 - x \in G_0^0$ and this is impossible due to (ii). The second possibility is also impossible because the domination property $c_0(x)e_1 + X_0 + \Delta X_1 - x \in G_1$ implies that $c_0(x)e_1 + X_0 - x \in G_1$ (a.s.). Therefore, $v \notin \Gamma(Y)$. □

Remark. Note that always $x - c_0(x)e_1 \in -G_0$. Thus, in the case $G_0^0 = \{0\}$ the condition (ii) holds for all x except $x = c_0(x)e_1$.

Example. Let us consider the two-dimensional model with $T = 1$, trivial filtration and

$$G_t := \{x : \ p_t x \geq 0, hx \geq 0 \ \}, \qquad t = 0, 1,$$

$p_t = (1, 1 + \lambda_t)$, $\lambda_1 > \lambda_0 \geq 0$, and $h^1 > h^2 > 0$. That is G_t are sharpened sectors containing the first quadrant; the upper boundary taken to be common. For $x = e_2$ the condition (ii) holds obviously. Let $y = c_0(e_2)e_1$, i.e. y is a projection on the x-axis of the point $y - x = c_0(e_2)e_1 - e_2$ laying on the intersection point of the lower boundary of G_0 with the line parallel to the x-axis and containing $-e_2$; note that $c_1(e_2) > 1$. The lower boundary of G_1 lays above that of G_0. Thus, $y - x \notin G_1$, the condition (i) holds and $\Gamma(Y) \neq D(Y)$.

2.5 Ramifications

2.5.1 Models with Incomplete Information

Models with transactions costs where the investor has an uncomplete (for example, delayed) information not only necessitate important changes in the description of value processes but also appropriate modifications of the main concepts. In particular, one cannot work on the level of portfolio positions, represented by a point in $\mathbf{R}^d$, but has to remain on the primary level, of the investor's decisions (orders), i.e. in the space of much higher dimension.

Example 1. Let us consider the barter market which is described by an $\mathbf{F}$-measurable conversion ("bid-ask") process $\Pi = (\pi_t^{ij})$ taking values in the set of strictly positive $d \times d$ matrices such that $\pi_t^{ij}\pi_t^{ji} \geq 1$. The entry π_t^{ij} stands for a number of units of the ith asset needed to exchange, at time t, for one unit of the jth asset. The above inequality means that exchanging one unit of the ith asset for $1/\pi_t^{ij}$ units of the jth asset with simultaneous exchange back of the latter quantity results in decreasing of the ith position.

In the case of fully informed investor, the portfolio process is generated by an $\mathbf{F}$-adapted process (η_t^{ij}) with values in the set $\mathbf{M}_+^d$ of positive $d \times d$ matrices; the entry $\eta_t^{ij} \geq 0$ is the investor's order to increase the position j on η_t^{ij} units by converting a certain number of units of the ith asset. The investor has a precise idea about this "certain number": it is $\pi_t^{ij}\eta_t^{ij}$. The situation is radically different when the information available is given by a smaller filtration $\mathbf{G}$, i.e. η_t^{ij} is only $\mathcal{G}_t$-measurable. The decrease of the i-th asset implied by such an order, being $\mathcal{F}_t$-measurable, is unknown to the investor. However, one can easily imagine a situation where the latter is willing to control the lower level of investments in some assets in his portfolio. This can be done by using the $\mathbf{G}$-adapted order process $(\tilde{\eta}_t^{ij})$ with the element $\tilde{\eta}_t^{ij}$ representing the number of units of the ith asset to be exchanged for the jth asset – the result of this transaction yields an increase of the j-th position in $\tilde{\eta}_t^{ij}/\pi_t^{ij}$ units and, in general, now this quantity is unknown to the investor at time t. Of course, orders of both types, "to get", "to send", can be used simultaneously. In other words, the investor's orders form a $\mathbf{G}$-adapted process $[(\eta_t^{ij}), (\tilde{\eta}_t^{ij})]$ taking values in the set of positive rectangular matrices $\mathbf{M}_+^{d\times 2d} = \mathbf{M}_+^d \times \mathbf{M}_+^d$. The dynamics of the portfolio processes is given by the formula

$$\Delta \widehat{V}_t = \widehat{\Delta B}_t^1 + \widehat{\Delta B}_t^2, \tag{2.5.1}$$

where the coordinates of $\widehat{\Delta B}_t^1$ and $\widehat{\Delta B}_t^2$ are

$$\widehat{\Delta B}_t^{1,i} := \sum_{j=1}^d [\eta_t^{ji} - \pi_t^{ij}\eta_t^{ij}],$$

$$\widehat{\Delta B}_t^{2,i} := \sum_{j=1}^d [\tilde{\eta}_t^{ji}/\pi_t^{ji} - \tilde{\eta}_t^{ij}].$$

Let $(e^{ij}) \in \mathbf{M}^d_+$ be a matrix with all zero entries except the entry (i,j) which is equal to unity. The union of the elementary orders $[(e^{ij}),0]$ and $[0,(e^{ji})]$ forms a basis in $\mathbf{M}^{d\times 2d}$. The execution of the order $[(e^{ij}),(e^{ji})]$ (buying a unit of the jth asset in exchange for the ith asset and then exchanging it back) leads to a certain loss in the ith position while others remain unchanged, i.e. $\Delta\widehat{V}^i_t \le 0$, $\Delta\widehat{V}^j_t = 0$, $j \neq i$. This observation will be used further, in the analysis of the NA^r property.

Example 2. Let us turn back to our basic model which is defined by a price process $S=(S_t)$ (describing the evolution of prices of units of assets in terms of some numéraire, e.g., the euro) and an $\mathbf{M}^d_+$-valued process $\Lambda = (\lambda^{ij}_t)$ of transaction costs coefficients. This model admits a formulation in terms of portfolio positions in physical units: one can introduce the matrix Π by setting

$$\pi^{ij}_t = (1+\lambda^{ij}_t)S^j_t/S^i_t, \qquad 1 \le i,j \le d.$$

In the full information case the difference between two models is only in parametrizations: one can introduce in the barter market "money" by taking as the price process S an arbitrary one evolving in the duals to the solvency cones and non-vanishing and defining λ^{ij}_t from the above relations. On the other hand, from the perspective of partial information, the setting based on price quotes is more flexible and provides a wider range of possible generalizations.

Again, assume that the investor's information is described by a smaller filtration $\mathbf{G}$ while S and Λ are $\mathbf{F}$-adapted (note that these processes may be adapted with respect to different filtrations).

In contrast to the barter market, the investor now may communicate orders of four types: in addition to the orders (η^{ij}_t) and $(\tilde{\eta}^{ij}_t)$ one can imagine also similar orders, "to get", "to send", but expressed in units of the numéraire and given by $\mathbf{G}$-adapted matrix-valued processes (α^{ij}_t) and $(\tilde{\alpha}^{ij}_t)$ with positive components. The entry α^{ij}_t is the increment of value in the position j due to diminishing the position i, while the entry $\tilde{\alpha}^{ij}_t$ is a value of the ith asset ordered to be exchanged for the jth asset.

The dynamics of value processes in such a model, in physical units, is given by the formula

$$\Delta\widehat{V}_t = \widehat{\Delta B}^1_t + \widehat{\Delta B}^2_t + \widehat{\Delta B}^3_t + \widehat{\Delta B}^4_t, \tag{2.5.2}$$

where $\widehat{\Delta B}^{3,i}_t := \Delta B^{3,i}_t/S^i_t$, $\widehat{\Delta B}^{4,i}_t := \Delta B^{4,i}_t/S^i_t$ with

$$\Delta B^{3,i}_t := \sum_{j=1}^d \alpha^{ji}_t - \sum_{j=1}^d (1+\lambda^{ij}_t)\alpha^{ij}_t,$$

$$\Delta B^{4,i}_t := \sum_{j=1}^d \frac{\tilde{\alpha}^{ji}_t}{1+\lambda^{ji}_t} - \sum_{j=1}^d \tilde{\alpha}^{ij}_t.$$

Of course, in this case the dynamics can be expressed also in values, that is in units of the numéraire (using the relation $X^i = \widehat{X}^i S^i$).

Thus, in both cases the set of "results" (for portfolios with zero initial endowments) consists of the d-dimensional random variables

$$\xi=\sum_{t=0}^{T}\mathcal{L}_t\zeta_t,\qquad \zeta_t\in O_t:=L^0(\mathbf{M}_+^{d\times m},\mathcal{G}_t),\tag{2.5.3}$$

where m is either $2d$ or $4d$ and $\mathcal{L}_{\omega,t}:\mathbf{M}^{d\times m}\to\mathbf{R}^d$ are linear operators such that the mappings $\omega\mapsto\mathcal{L}_{\omega,t}$ are measurable with respect to the σ-algebra $\mathcal{F}_t$. We shall denote this set $\widehat{R}_T$ or, when needed, $\widehat{R}_T(\mathcal{L})$ to show the dependence on the defining operator-valued random process. As usual, we define the set of hedgeable claims $\widehat{A}_T(\mathcal{L}):=\widehat{R}_T(\mathcal{L})-L^0(\mathbf{R}^d_+)$.

Let us associate with the random linear operator $\mathcal{L}_t$ (acting on elements of $\mathbf{M}^{d\times m}$) the linear operator $\mathbf{L}_t$ acting on $\mathbf{M}^{d\times m}$-valued random variables, $\mathbf{L}_t:L^0(\mathbf{M}^{d\times m},\mathcal{G}_t)\to L^0(\mathbf{R}^d,\mathcal{F}_t)$, by setting $(\mathbf{L}_t\zeta)(\omega)=\mathcal{L}_{\omega,t}\zeta(\omega)$. With this notation,

$$\widehat{R}_T=\sum_{t=0}^{T}\mathbf{L}_t(O_t).$$

Sometimes, it is convenient to view $\mathbf{M}^{d\times m}$ as the set of linear operators defined by the corresponding matrices.

Unlike the case of frictionless market the set R_T, in general, is not closed even for models with complete information: see Example 1 in Subsection 2.2.4 where the set $\widehat{R}_1=\widehat{A}_1$ is not closed though the NA^w- condition is satisfied. However, similarly to the models with complete information, we have the following result.

Proposition 2.5.1 *The sets $\mathbf{L}_t(O_t)$ are closed in probability.*

Proof. The arguments being standard, we only sketch them. In a slightly more general setting, consider a sequence of random vectors $\zeta^n=\sum_{i=1}^N c_i^n g_i$ in a finite-dimensional Euclidean space where g_i are $\mathcal{G}$-measurable random vectors and $c_i^n\in L^0_+(\mathcal{G})$. Let $\mathcal{L}$ be an $\mathcal{F}$-measurable random linear operator. Knowing that the sequence $\xi^n=\mathcal{L}\zeta^n$ converges to ξ, we want to show that $\xi=\mathcal{L}\zeta$ for some $\zeta=\sum_{i=1}^N c_i g_i$. Supposing that the result holds for $N-1$ (for $N=1$ it is obvious), we extend it to N. Indeed, it is easy to see, recalling, as usual, the lemma on random subsequences, that we may assume without loss of generality that all sequences c_i^n converge to infinity and, moreover, the normalized sequences $\tilde{c}_i^n:=c_i^n/|c^n|$, where $|c^n|$ is the sum of c_i^n, converge to some $\mathcal{G}$-measurable random variables $\tilde{c}_i$. For the random vector $\tilde{\zeta}:=\sum_{i=1}^N\tilde{c}_i g_i$ we have that $\mathcal{L}\tilde{\zeta}=0$. Put $\alpha^n:=\min_i\{c_i^n/\tilde{c}_i:\ \tilde{c}_i>0\}$. Note that the random variable $\bar{c}_i^n:=c_i^n-\alpha^n\tilde{c}_i\geq 0$ and, for each ω, at least one of $\bar{c}_i^n(\omega)$ vanishes. For $\bar{\zeta}^n=\sum_{i=1}^N\bar{c}_i^n g_i$ we have that $\mathcal{L}\bar{\zeta}^n$ also tends to ξ. Considering the partition of Ω by N disjoint $\mathcal{G}$-measurable subsets Γ_i constructed from the covering of Ω by sets $\{\liminf_n\bar{c}_i^n=0\}$ and replacing on Γ_i the coefficients $\bar{c}_i^n$ by zero (without affecting the limit ξ), we obtain a reduction to the case with $N-1$ generators. □

2.5.2 No Arbitrage Criteria: Finite Ω

The definition of the NA^w-property remains the same as in the model with full information: $\widehat{R}_T \cap L^0(\mathbf{R}^d_+, \mathcal{F}_T) = \{0\}$ or $\widehat{A}_T \cap L^0(\mathbf{R}^d_+, \mathcal{F}_T) = \{0\}$.

As always, criteria in the case of finite Ω are easy to establish using the finite-dimensional separation theorem.

Proposition 2.5.2 *Let Ω be finite. The following conditions are equivalent:*

(a) NA^w;

(b) there exists $Z \in \mathcal{M}(\operatorname{int} \mathbf{R}^d_+, \mathbf{F})$ such that $E(Z_t \mathcal{L}_t \zeta | \mathcal{G}_t) \le 0$ for any $\zeta \in O_t$.

Proof. $(a) \Rightarrow (b)$ Note that $\widehat{A}_T$ is a finite-dimensional polyhedral (thus, closed) cone containing $-L^0(\mathbf{R}^d_+)$. The NA^w-property implies that non-zero elements of $L^0(\mathbf{R}^d_+)$ can be separated from $\widehat{A}_T$ in a strict sense. Using a classical argument, we construct an $\mathbf{F}$-martingale $Z = (Z_t)$ with strictly positive components such that $EZ_T\xi \le 0$ for every $\xi \in \widehat{A}_T$. Namely, we can take Z_T equal to the sum of functionals negative on $\widehat{A}_T$ and strictly positive on $e_i I_\Gamma$ with the summation index Γ running through the family of atoms of $\mathcal{F}_T$ and $i = 1, 2, ..., d$. It follows that $E(Z_t \mathcal{L}_t \zeta_t) \le 0$ for any $\zeta_t \in O_t$, implying the assertion.

$(b) \Rightarrow (a)$ This implication is obvious because for ζ admitting the representation (2.5.3) we have that

$$EZ_T\xi = \sum_{t=0}^{T} E[E(Z_t \mathcal{L}_t \zeta_t | \mathcal{G}_t)] \le 0$$

and, therefore, ξ cannot be an element of $L^0(\mathbf{R}^d_+, \mathcal{F}_T)$ other than zero.

As we know, even in the case of full information, a straightforward generalization of the above criterion to an arbitrary Ω fails to be true. To get "satisfactory" theorems one needs either to impose extra assumptions, or to modify the concept of absence of arbitrage. We investigate here an analog of the NA^r-condition starting from the simple case when Ω is finite.

First, we establish a simple lemma which holds in a "very abstract" setting where the word "premodel" instead of "model" means that we do not suggest any particular properties of $(\mathcal{L}_t)$.

Fix a subset $\mathcal{I}_t$ of O_t. The elements of $\mathcal{I}_t$ will be interpreted later, in a more specific "financial" framework, as the reversible orders.

We say that the premodel has the *NA^r-property* if the NA^w-property holds for the premodel based on an $\mathbf{F}$-adapted process $\mathcal{L}' = (\mathcal{L}'_t)$ such that

(i) $\mathcal{L}'_t\zeta \ge \mathcal{L}_t\zeta$ componentwise for every $\zeta \in O_t$;

(ii) $\mathbf{1}\mathcal{L}'_t\zeta \ne \mathbf{1}\mathcal{L}_t\zeta$ if $\zeta \in O_t \setminus \mathcal{I}_t$ (i.e. the above inequality is not identity).

Lemma 2.5.3 *Let Ω be finite. If a premodel has the NA^r-property, then there is a process $Z \in \mathcal{M}(\operatorname{int} \mathbf{R}^d_+, \mathbf{F})$ such that $E(Z_t \mathcal{L}_t \zeta | \mathcal{G}_t) \le 0$ for every $\zeta \in O_t$ and, if $\zeta \in O_t \setminus \mathcal{I}_t$,*

$$\zeta I_{\{E(Z_t\mathcal{L}_t\zeta|\mathcal{G}_t)=0\}} \in \mathcal{I}_t. \tag{2.5.4}$$

Proof. According to Proposition 2.5.2 applied to the premodel based on the process $\mathcal{L}'$ from the definition of NA^r there exists $Z \in \mathcal{M}(\operatorname{int}\mathbf{R}^d_+, \mathbf{F})$ such that $E(Z_t\mathcal{L}'_t\zeta|\mathcal{G}_t) \le 0$ for any $\zeta \in O_t$. Hence, $E(Z_t\mathcal{L}_t\zeta|\mathcal{G}_t) \le 0$ by virtue of (i). Again by (i) we have, for $\zeta \in O_t \setminus \mathcal{I}_t$, that

$$Z_t\mathcal{L}'_t\zeta I_{\{E(Z_t\mathcal{L}_t\zeta|\mathcal{G}_t)=0\}} \ge Z_t\mathcal{L}_t\zeta I_{\{E(Z_t\mathcal{L}_t\zeta|\mathcal{G}_t)=0\}}.$$

If the order $\zeta I_{\{E(Z_t\mathcal{L}_t\zeta|\mathcal{G}_t)=0\}}$ is not in $\mathcal{I}_t$, this inequality is strict on a non-null set. Thus, taking the expectation, we obtain

$$EZ_t\mathcal{L}'_t\zeta I_{\{E(Z_t\mathcal{L}_t\zeta|\mathcal{G}_t)=0\}} > 0$$

which is contradiction. □

Now we give a precise meaning to the word "model" by imposing an **assumption** on the generating process (fulfilled in both our examples) and specifying the sets $\mathcal{I}_t$.

Namely, we suppose that in $\mathbf{M}^{d\times m}$ there is a basis formed by the union of two families of vectors $\{f_i\}$ and $\{\tilde{f}_i\}$, $1 \le i \le md/2$, belonging to $\mathbf{M}^{d\times m}_+$ and such that componentwise

$$\mathcal{L}_t f_i + \mathcal{L}_t\tilde{f}_i \le 0, \tag{2.5.5}$$

while $\mathcal{I}_t$ is the cone of (matrix-valued) random variables having the form $\sum_i(\eta_i f_i + \tilde{\eta}_i\tilde{f}_i)$ with $\eta_i, \tilde{\eta}_i \in L^0_+(\mathcal{G}_t)$ and such that $\mathcal{L}_t\sum_i(\eta_i+\tilde{\eta}_i)(f_i+\tilde{f}_i) = 0$.

Note that the latter equality implies that $\mathbf{L}_t(\mathcal{I}_t) \subseteq \mathbf{L}_t(O_t)\cap(-\mathbf{L}_t(O_t))$. It is clear that the set $\mathcal{I}_t$ is stable under multiplication by elements of $L^0(\mathbf{R}_+, \mathcal{G}_t)$. This implies that the equality (2.5.4) for $\zeta \in \mathcal{I}_t$ always holds (cf. the formulations of Lemma 2.5.3 and the theorems below).

The inequality (2.5.5) means that the elementary transfers in opposite directions cannot lead to gains. The orders from $\mathcal{I}_t$, even symmetrized, do not incur losses.

For the models, in the definition of the NA^r the words "premodel" are replaced by "models", i.e. we require that the property (2.5.5) should hold also for the dominating process $\mathcal{L}'$.

Theorem 2.5.4 *Let Ω be finite. Then the following properties of the model are equivalent:*

(a) NA^r*;*

(b) *there is* $Z \in \mathcal{M}(\operatorname{int}\mathbf{R}^d_+, \mathbf{F})$ *such that* $E(Z_t\mathcal{L}_t\zeta|\mathcal{G}_t) \le 0$ *for every* $\zeta \in O_t$ *and, if* $\zeta \in O_t$*,*

$$\zeta I_{\{E(Z_t\mathcal{L}_t\zeta|\mathcal{G}_t)=0\}} \in \mathcal{I}_t.$$

Proof. To check the remaining implication $(b) \Rightarrow (a)$ we put $\mathcal{L}'_t\zeta := \mathcal{L}_t\zeta - \bar{\mathcal{L}}_t\zeta$ defining the action of $\bar{\mathcal{L}}_t$ on the element $\zeta = \sum_i(\eta_i f_i + \tilde{\eta}_i\tilde{f}_i)$ by the formula $\bar{\mathcal{L}}_t\zeta := \sum_i(\eta_i + \tilde{\eta}_i)\theta_i$ where $\theta_i = \theta_i(t)$ has the components

$$\theta_i^k := \max\left\{\frac{1}{2}[\mathcal{L}_t(f_i+\tilde{f}_i)]^k, \frac{1}{d}\frac{E(Z_t\mathcal{L}_t f_i|\mathcal{G}_t)}{E(Z_t^k|\mathcal{G}_t)}, \frac{1}{d}\frac{E(Z_t\mathcal{L}_t \tilde{f}_i|\mathcal{G}_t)}{E(Z_t^k|\mathcal{G}_t)}\right\}.$$

The values $\theta_i^k(t)$ being negative, the condition (i) holds. The inequality (2.5.5) for $\mathcal{L}_t'$ is obviously fulfilled due to the first term in the definition of $\theta_i^k(t)$. Now let ζ be an element of $O_t \setminus \mathcal{I}_t$. This means that for some k and i the set

$$\Gamma := \{(\eta_i+\tilde{\eta}_i)[\mathcal{L}_t(f_i+\tilde{f}_i)]^k < 0\} = \{(\eta_i+\tilde{\eta}_i)Z_t^k[\mathcal{L}_t(f_i+\tilde{f}_i)]^k < 0\}$$

is non-null. From elementary properties of conditional expectations it follows that $(\eta_i+\tilde{\eta}_i)E(Z_t^k[\mathcal{L}_t(f_i+\tilde{f}_i)]^k|\mathcal{G}_t) < 0$ on Γ. The property (ii) holds because on Γ both $E(Z_t\mathcal{L}_t f_i|\mathcal{G}_t)$ and $E(Z_t\mathcal{L}_t\tilde{f}_i|\mathcal{G}_t)$ are strictly negative as follows from the coincidence of sets

$$\{E(Z_t\mathcal{L}_t f_i|\mathcal{G}_t) < 0\} = \{E(Z_t\mathcal{L}_t\tilde{f}_i)|\mathcal{G}_t) < 0\} = \{E(Z_t\mathcal{L}_t(f_i+\tilde{f}_i)|\mathcal{G}_t) < 0\}$$

which can be established easily. Indeed, $f_i I_{\{E(Z_t\mathcal{L}_t f_i|\mathcal{G}_t)=0\}} \in \mathcal{I}_t$ and, by definition of $\mathcal{I}_t$,

$$I_{\{E(Z_t\mathcal{L}_t f_i|\mathcal{G}_t)=0\}}\mathcal{L}_t\tilde{f}_i = -I_{\{E(Z_t\mathcal{L}_t f_i|\mathcal{G}_t)=0\}}\mathcal{L}_t f_i.$$

Multiplying this identity by Z_t and taking the conditional expectation with respect to $\mathcal{G}_t$ we get that

$$I_{\{E(Z_t\mathcal{L}_t f_i|\mathcal{G}_t)=0\}}E(Z_t\mathcal{L}_t\tilde{f}_i|\mathcal{G}_t) = 0.$$

Similarly,

$$I_{\{E(Z_t\mathcal{L}_t \tilde{f}_i|\mathcal{G}_t)=0\}}E(Z_t\mathcal{L}_t f_i|\mathcal{G}_t) = 0.$$

These two equalities imply the coincidence of sets where the conditional expectations (always negative) are zero, i.e. the required assertion.

Finally, we check the NA^w-property of $(\mathcal{L}_t')$ using Proposition 2.5.2. For any $\zeta = \sum_i(\eta_i f_i + \tilde{\eta}_i\tilde{f}_i)$ from O_t we have:

$$\begin{aligned} E(Z_t\mathcal{L}_t'\zeta|\mathcal{G}_t) &= E(Z_t\mathcal{L}_t\zeta|\mathcal{G}_t) - E\Big(\sum_i(\eta_i+\tilde{\eta}_i)\sum_{k=1}^d Z_t^k\theta_i^k\Big|\mathcal{G}_t\Big) \\ &\le E(Z_t\mathcal{L}_t\zeta|\mathcal{G}_t) - \sum_i \eta_i E(Z_t\mathcal{L}_t f_i|\mathcal{G}_t) - \sum_i \tilde{\eta}_i E(Z_t\mathcal{L}_t\tilde{f}_i|\mathcal{G}_t) = 0. \end{aligned}$$

It follows that $EZ_T\xi \le 0$ for every $\xi \in \widehat{R}_T(\mathcal{L}') \cap L^0(\mathbf{R}_+^d)$, excluding arbitrage opportunities for the model based on $\mathcal{L}'$.

The theorem is proven. □

Remark 1. One might find it convenient to view $\mathbf{M}^{d\times m}$ as the set of linear operators defined by corresponding matrices and consider the adjoint operators $\mathcal{L}_{\omega,t}^* : \mathbf{R}^d \to (\mathbf{M}^{d\times m})^*$. This gives a certain flexibility of notations, e.g., the property "$E(Z_t\mathcal{L}_t\zeta|\mathcal{G}_t) \le 0$ for every $\zeta \in O_t$" can be formulated as "the operator $E(\mathcal{L}_t^* Z_t|\mathcal{G}_t)$ is negative" (in the sense of partial ordering induced by $\mathbf{M}_+^{d\times m}$), the inclusion $f_i \in \operatorname{Ker} E(\mathcal{L}_t^* Z_t|\mathcal{G}_t)$ can be written instead of the equality $E(Z_t\mathcal{L}_t f_i|\mathcal{G}_t) = 0$ and so on. However, the current notation has the advantage of being easier adjustable for more general situation where $\mathcal{L}_t$ is a concave positive homogeneous mapping from $\mathbf{M}_+^{d\times m}$ into $L^0(\mathbf{R}^d, \mathcal{F}_t)$.

Remark 2. The hypothesis on the structure of invertible claims may not be fulfilled for Examples 1 and 2. For the investor having access to full information, the set of all assets can be split into classes of equivalence within which one can do frictionless transfers though not necessary in one step. Our assumption means that all transfers within each class are frictionless, a hypothesis which does not lead to a loss of generality as a fully informed "intelligent" investor will not lose money making charged transfers within an equivalence class. However, in the context of restricted information it seems that such an assumption means that the information on equivalence classes is available to the investor.

2.5.3 No Arbitrage Criteria: Arbitrary Ω

In the general case the assertion of Proposition 2.5.2 fails to be true though with a suitable modification its condition (b) remains sufficient for the NA^w-property. Namely, we have:

Proposition 2.5.5 *The NA^w-property holds if there exists $Z \in \mathcal{M}(\operatorname{int}\mathbf{R}_+^d, \mathbf{F})$ such that all conditional expectations $E(|Z_t||\mathcal{L}_t f_i||\mathcal{G}_t)$ and $E(|Z_t||\mathcal{L}_t \tilde{f}_i||\mathcal{G}_t)$ are finite and $E(Z_t\mathcal{L}_t\zeta|\mathcal{G}_t) \le 0$ for any $\zeta \in O_t$.*

This result is an obvious corollary of the following technical lemma dealing with integration issues.

Lemma 2.5.6 *Suppose that $\Sigma_T = Z_T \sum_{t=0}^T \xi_t$ where $Z \in \mathcal{M}(\mathbf{R}_+^d, \mathbf{F})$ and $\xi_t \in L^0(\mathbf{R}^d, \mathcal{F}_t)$ are such that $E(|Z_t||\xi_t||\mathcal{G}_t) < \infty$ and $E(Z_t\xi_t|\mathcal{G}_t) \le 0$. Put $\bar{\Sigma}_T := E(\Sigma_T|\mathcal{G}_T)$. If $\bar{\Sigma}_T^- \in L^1$, then $\bar{\Sigma}_T \in L^1$ and $E\bar{\Sigma}_T \le 0$.*

Proof. We proceed by induction. The claim is obvious for $T = 0$. Suppose that it holds for $T-1$. Clearly,

$$Z_T \sum_{t=0}^{T-1} \xi_t = \Sigma_T - Z_T\xi_T.$$

By the martingale property $E(Z_T^i|\xi_t||\mathcal{G}_t) = E(Z_t^i|\xi_t||\mathcal{G}_t) < \infty$ implying that $E(|Z_T||\xi_t||\mathcal{G}_t) < \infty$ for any $t \le T$. Thus, $\bar{\Sigma}_T$ is well-defined and finite. Taking the conditional expectation with respect to $\mathcal{G}_T$ in the above identity we get, using the martingale property, that

$$E(\Sigma_{T-1}|\mathcal{G}_T) = E\left(Z_T \sum_{t=0}^{T-1} \xi_t \middle| \mathcal{G}_T\right) = \bar{\Sigma}_T - E(Z_T\xi_T|\mathcal{G}_T) \ge \bar{\Sigma}_T.$$

Therefore, the negative part of $E(\Sigma_{T-1}|\mathcal{G}_T)$ is dominated by the negative part of $\bar{\Sigma}_T$ which is integrable. Using Jensen's inequality we have:

$$\begin{aligned}\bar{\Sigma}_{T-1}^- &= [E(E(\Sigma_{T-1}|\mathcal{G}_T)|\mathcal{G}_{T-1})]^- \\ &\le E([E(\Sigma_{T-1}|\mathcal{G}_T)]^-|\mathcal{G}_{T-1}) \le E(\bar{\Sigma}_T^-|\mathcal{G}_{T-1}).\end{aligned}$$

Thus, $\bar{\Sigma}_{T-1}^- \in L^1$ and, by virtue of the induction hypothesis, $\bar{\Sigma}_{T-1} \in L^1$ and $E\bar{\Sigma}_{T-1} \le 0$. In the representation $\bar{\Sigma}_T = E(\bar{\Sigma}_{T-1}|\mathcal{G}_T) + E(\bar{\Sigma}_T^-|\mathcal{G}_{T-1})$ the first term is integrable and has negative expectation while the second is negative. Thus, $E\bar{\Sigma}_T \le 0$ and, automatically, $E\bar{\Sigma}_T^+ < \infty$. □

The NA^r-criterion, suitably modified, remains true without any restriction on the probability space. Of course, in its formulation one needs to take care about the existence of the involved conditional expectations. This can be done as in the next result.

Theorem 2.5.7 *The following conditions are equivalent:*

(a) *NA^r;*

(b) *there is $Z \in \mathcal{M}(\operatorname{int}\mathbf{R}^d_+, \mathbf{F})$ such that all random variables $E(Z_t\mathcal{L}_t f_i|\mathcal{G}_t)$, $E(Z_t\mathcal{L}_t\tilde{f}_i|\mathcal{G}_t)$ are finite, $E(Z_t\mathcal{L}_t\zeta|\mathcal{G}_t) \le 0$ for every $\zeta \in O_t$ and, if $\zeta \in O_t$,*

$$\zeta I_{\{E(Z_t\mathcal{L}_t\zeta|\mathcal{G}_t)=0\}} \in \mathcal{I}_t. \tag{2.5.6}$$

We have no trouble with the implication (b) ⇒ (a): an inspection of the arguments given in the case of finite Ω shows that they work well until the concluding step which now can be done just by reference to Lemma 2.5.6.

The proof of the "difficult" implication (a) ⇒ (b) follows the same line of ideas as in the case of full information.

Lemma 2.5.8 *Suppose that the equality*

$$\sum_{t=0}^{T} \mathcal{L}_t\tilde{\zeta}_t - \tilde{r} = 0 \tag{2.5.7}$$

with $\tilde{\zeta}_t \in O_t$ and $\tilde{r}_t \in L^0(\mathbf{R}^d_+)$ holds only if $\tilde{\zeta}_t \in \mathcal{I}_t$ and $\tilde{r} = 0$. Then $\widehat{A}_T$ is closed in probability.

Proof. For $T = 0$ the arguments are exactly the same as were used for Proposition 2.5.1 with obvious changes caused by the extra term describing the funds withdrawals. Namely, the difference is that for the limiting normalized order $\tilde{\zeta} := \sum_{i=1}^N \tilde{c}_i g_i$ we get the equality $\mathcal{L}\tilde{\zeta} - \tilde{r} = 0$ where $\tilde{r} \in L^0(\mathbf{R}^d_+, \mathcal{F}_T)$ is the limit of normalized funds withdrawals. By hypothesis, $\tilde{r} = 0$ and we can complete the proof using the same Gauss-type reduction procedure.

Arguing by induction, we suppose that $\widehat{A}_{T-1}$ is closed and consider the sequence of order processes $(\zeta_t^n)_{t\le T}$ such that $\sum_{t=0}^T \mathcal{L}_t\zeta_t^n - r^n \to \eta$. There is an obvious reduction to the case where at least one of "elementary" orders at time zero tends to infinity. Normalizing and using the induction hypothesis we obtain that there exists an order process $(\tilde{\zeta}_t)_{t\le T}$ with nontrivial $\tilde{\zeta}_0$ such that $\sum_{t=0}^T \mathcal{L}_t\tilde{\zeta}_t - \tilde{r} = 0$ and we can use the assumption of the lemma. It ensures that $\tilde{r} = 0$ and there are $\zeta'_t \in O_t$ such that $\mathcal{L}_t\zeta'_t = -\mathcal{L}_t\tilde{\zeta}_t$. This allows us to reduce a number of non-zero coefficients (i.e. "elementary" orders) at the initial order by putting, $\bar{\zeta}_0^n = \zeta_0^n - \alpha^n\tilde{\zeta}_0$, as in the proof of Proposition 2.5.1, and $\bar{\zeta}_t^n = \zeta_t^n + \alpha^n\zeta'_t$ for $t \ge 1$. □

Lemma 2.5.9 *The NA^r-condition implies the hypothesis of the above lemma.*

Proof. Of course, $\tilde{r} = 0$ (otherwise, $(\tilde{\zeta}_t)$ is an arbitrage opportunity, i.e. even NA^w is violated). For the process $(\mathcal{L}'_t)$, from definition of NA^r we have that componentwise

$$\sum_{t=0}^T \mathcal{L}'_t\tilde{\zeta}_t \ge \sum_{t=0}^T \mathcal{L}_t\tilde{\zeta}_t = 0$$

and $\mathbf{1}\sum_{t=0}^T \mathcal{L}'_t\tilde{\zeta}_t > 0$ with strictly positive probability if at least one of $\tilde{\zeta}_t$ does not belong to $\mathcal{I}_t$. This means that $(\tilde{\zeta}_t)$ is an arbitrage opportunity for the model based on $(\mathcal{L}'_t)$. □

Lemma 2.5.10 *Assume that the hypothesis of Lemma 2.5.8 holds. Then for any "elementary" order f and every $t \le T$ one can find a bounded process $Z = Z^{(t,f)} \in \mathcal{M}(\operatorname{int}\mathbf{R}^d_+, \mathbf{F})$ such that:*

1) *$E(|Z_s||\mathcal{L}_s g|) < \infty$ and $E(Z_s\mathcal{L}_s g|\mathcal{G}_s) \le 0$ for all $s \le T$ and all "elementary" orders g,*

2) *$fI_{\{E(Z_t\mathcal{L}_t f|\mathcal{G}_t)=0\}} \in \mathcal{I}_t$.*

Proof. We may assume without loss of generality that all portfolio increments $\mathcal{L}_s g$ corresponding to the elementary orders g are integrable (otherwise we can pass to an equivalent measure P' with the bounded density ρ, find the process Z' with the needed properties under P' and take $Z = \rho\tilde{Z}'$).

Let $\mathcal{Z}$ be the set of all bounded processes $Z \in \mathcal{M}(\mathbf{R}^d_+, \mathbf{F})$ such that $EZ_T\xi \le 0$ whenever is $\xi \in \widehat{A}^1_T := \widehat{A}_T \cap L^1$. Let

$$c_t := \sup_{Z\in\mathcal{Z}} P(E(Z_t\mathcal{L}_t f|\mathcal{G}_t) < 0). \tag{2.5.8}$$

Let Z be an element for which the supremum is attained (one can take as Z a countable convex combination of any uniformly bounded sequence along which the supremum is attained).

If 2) fails, then the random vector $\mathcal{L}_t(f+\tilde{f})I_{\{E(Z_t\mathcal{L}_t f|\mathcal{G}_t)=0\}}$ (all components of which are negative) is not zero. This implies that the element $-\mathcal{L}_t\tilde{f}I_{\{E(Z_t\mathcal{L}_t f|\mathcal{G}_t)=0\}}$ does not belong to $\widehat{A}^1_T$. Indeed, in the opposite case we would have the identity

$$\sum_{s=0}^T \mathcal{L}_s\zeta_s = -\mathcal{L}_t\tilde{f}I_{\{E(Z_t\mathcal{L}_t f|\mathcal{G}_t)=0\}}.$$

The assumption of Lemma 2.5.8 ensures that the order $\tilde{f} I_{\{E(Z_t\mathcal{L}_t f|\mathcal{G}_t)=0\}} + \zeta_t$ is in $\mathcal{I}_t$. Thus, for the symmetrized order we have that

$$\mathcal{L}_t(f+\tilde{f})I_{\{E(Z_t\mathcal{L}_t f|\mathcal{G}_t)=0\}} + \mathcal{L}_t(\zeta+\tilde{\zeta}) = 0.$$

Since the second term is also negative componentwise, both should be equal to zero and we get a contradiction.

By the Hahn–Banach theorem one can separate $\varphi := -\mathcal{L}_t \tilde{f} I_{\{E(Z_t\mathcal{L}_t f|\mathcal{G}_t)=0\}}$ and $\widehat{A}^1_T$: that is we may find $\eta \in L^\infty(\mathbf{R}^d)$ such that

$$\sup_{\xi\in\widehat{A}^1_T} E\eta\xi < E\eta\varphi.$$

Since $\widehat{A}^1_T$ is a cone containing $-L^1(\mathbf{R}^d_+)$ the supremum above is equal to zero, $\eta \in L^1(\mathbf{R}^d_+)$ and $E\eta\varphi > 0$. The latter inequality implies that for $Z^\eta_t = E(\eta|\mathcal{G}_t)$ we have $EE(Z^\eta_t\mathcal{L}_t f|\mathcal{G}_t)I_{\{E(Z_t\mathcal{L}_t f|\mathcal{G}_t)=0\}} < 0$. Therefore, for the martingale $Z' := Z + Z^\eta$ we have that

$$P\big(E(Z'_t\mathcal{L}_t f|\mathcal{G}_t) < 0\big) > P\big(E(Z_t\mathcal{L}_t f|\mathcal{G}_t) < 0\big) = c_t.$$

This contradiction shows that 2) holds.

The process Z constructed in this way may be not in $\mathcal{M}(\operatorname{int}\mathbf{R}^d_+, \mathbf{F})$. However, it can be easily "improved" to meet the latter property. To this end, fix $i \le d$ and consider, in the subset of $\mathcal{Z}$ on which the supremum c_t in (2.5.8) is attained, a process Z with maximal probability $P(Z^i_T > 0)$ (such process does exist). Then $P(\bar{Z}^i_T > 0) = 1$. Indeed, in the opposite case, the element $e_i I_{\{Z^i_T=0\}} \in L^1(\mathbf{R}^d_+)$ is not zero and, therefore, does not belong to $\widehat{A}^1_T$. So it can be separated from the latter set. The separating functional generates a martingale $Z' \in \mathcal{Z}$. Since $P(\bar{Z}_T + Z'_T > 0) > P(\bar{Z}_T > 0)$, we arrive to a contradiction with the definition of $\bar{Z}$. The set of $Z \in \mathcal{Z}$ satisfying 1) and 2) is convex and, hence, a convex combination of d processes obtained in this way for each coordinate has the required properties. $\Box$

The implication $(a) \Rightarrow (b)$ of the theorem follows from the lemmas above. Indeed, by virtue of Lemmas 2.5.9 – 2.5.10, NA^r ensures the existence of processes $Z^{(t,f)}$ satisfying 1) and 2) of Lemma 2.5.10. One can take as a required martingale Z the process $Z := \sum_{t,f} Z^{(t,f)}$ where $t = 0, 1, ..., T$ and f runs through the set of "elementary" orders. An arbitrary order $\zeta \in O_t$ is a linear combination of elementary orders with positive $\mathcal{G}_t$ measurable coefficients. The condition $E(Z_t\mathcal{L}_t\zeta|\mathcal{G}_t) \le 0$ follows from the property 1) of Lemma 2.5.10. To prove the inclusion (2.5.6) we note that $I_{\{\Sigma\xi_i=0\}} = \prod I_{\{\xi_i=0\}}$ when $\xi_i \le 0$. With this observation the required inclusion is an easy corollary of the property 2) of Lemma 2.5.10 and the stability of $\mathcal{I}_t$ under multiplication by positive $\mathcal{G}_t$-measurable random variables.

Remark. In the above proof we get from NA^r a condition which looks stronger than (b), with bounded Z and integrable random variables $|Z_t||\mathcal{L}_t f|$, but, in fact, it is equivalent to (b).

2.5.4 Hedging Theorem

Thanks to the previous development, hedging theorems in the model with partial information do not require new ideas. For the case of finite Ω the result can be formulated in our "very abstract" setting without additional assumptions on the structure of the sets $\mathcal{I}_t$.

We fix a d-dimensional random variable $\widehat{C}$, the contingent claim expressed in physical units. Define the set

$$\Gamma = \{v \in \mathbf{R}^d : \ \widehat{C} \in v + \widehat{A}_T\}.$$

Let $\mathcal{Z}$ be the set of martingales $Z \in \mathcal{M}_T(\mathbf{R}^d_+, \mathbf{F})$ such that $E(Z_t \mathcal{L}_t \zeta_t | \mathcal{G}_t) \le 0$ for every $\zeta_t \in O_t$. Put

$$D := \left\{ v \in \mathbf{R}^d : \ \sup_{Z \in \mathcal{Z}} E(Z_T \widehat{C} - Z_0 v) \le 0 \right\}.$$

Proposition 2.5.11 *Let Ω be finite and $\mathcal{Z} \ne \emptyset$. Then $\Gamma = D$.*

In this theorem the inclusion $\Gamma \subseteq D$ is obvious while the reverse inclusion is an easy exercise on the finite-dimensional separation theorem. We leave it to the reader.

In the case of general Ω we should take care about integrability and closedness of the set $\widehat{A}_T$. To this end we shall work with the model in the "narrow" sense of the preceding sections assuming the NA^r-property. Now $\mathcal{Z}$ is the set of bounded martingales $Z \in \mathcal{M}_T(\mathbf{R}^d_+, \mathbf{F})$ such that $E(Z_t \mathcal{L}_t f_i | \mathcal{G}_t)$, $E(Z_t \mathcal{L}_t \tilde{f}_i | \mathcal{G}_t)$ are finite, $E(Z_t \mathcal{L}_t \zeta | \mathcal{G}_t) \le 0$ and $E(Z_T \widehat{C})^- < \infty$. The definitions of the sets Γ and D remain the same.

Theorem 2.5.12 *Suppose that NA^r holds. Then $\Gamma = D$.*

Proof. The inclusion $\Gamma \subseteq D$ follows from the inequality

$$Z_T(\widehat{C} - v) \le Z_T \sum_{t=0}^{T} \mathcal{L}_t \zeta_t, \qquad \zeta_t \in O_t,$$

and Lemma 2.5.6.

To check the inclusion $D \subseteq \Gamma$ we take a point $v \notin \Gamma$ and show that $v \notin D$. It is sufficient to find $Z \in \mathcal{Z}$ such that $Z_0 v < E Z_T \widehat{C}$. Consider a measure $\tilde{P} \sim P$ with bounded density ρ such that $\widehat{C}$, and all $|\mathcal{L}_t||f_i|$ and $|\mathcal{L}_t||f_i|$ belong to $L^1(\tilde{P})$. Under NA^r the convex set $\tilde{A}^1 := A_0^T \cap L^1(\tilde{P})$ is closed and does not contain the point $\widehat{C} - v$. Thus, we can separate the latter by a functional η from L^∞. This means that

$$\sup_{\xi \in \tilde{A}^1} E\rho\eta\xi < E\eta\rho(\widehat{C} - v).$$

It is clear, that the bounded martingale $Z_t := E(\rho\eta | \mathcal{F}_t)$ satisfies the required properties. □

A

Appendix

A.1 Facts from Convex Analysis

By definition, a subset K in $\mathbf{R}^n$ (or in a linear space X) is a *cone* if it is convex and stable under multiplication by the non-negative constants. It defines the partial ordering:

$$x \geq_K y \quad \Leftrightarrow \quad x - y \in K;$$

in particular, $x \geq_K 0$ means that $x \in K$.

A closed cone K is *proper* if the linear space $F := K \cap (-K) = \{0\}$, i.e. if the relations $x \geq_K$ and $x \leq_K = 0$ imply that $x = 0$.

Let K be a closed cone and let $\pi : \mathbf{R}^n \to \mathbf{R}^n/F$ be the canonical mapping onto the quotient space. Then πK is a proper closed cone.

For a set C we denote by $\operatorname{cone} C$ the set of all conic combinations of elements of C. If C is convex then $\operatorname{cone} C = \cup_{\lambda \geq 0} \lambda C$.

Let K be a cone. Its *dual positive cone*

$$K^* := \{z \in \mathbf{R}^n : \; zx \geq 0 \; \forall x \in K\}$$

is closed. The *polar cone* K° is defined using the opposite inequality[1], i.e. $K^\circ = -K^*$); K is closed if and only if $K = K^{**}$.

We use the notations $\operatorname{int} K$ for the interior of K and $\operatorname{ri} K$ for the relative interior (i.e., the interior in $K - K$, the linear subspace generated by K).

Recall that in a finite dimensional Euclidean space the convex hull of a compact set is a compact.

A closed cone K in the Euclidean space $\mathbf{R}^n$ is proper if and only if there is a compact convex set C such that $0 \notin C$ and $K = \operatorname{cone} C$. One can take as C the convex hull of the intersection of K with the unit sphere $\{x \in \mathbf{R}^n : \; |x| = 1\}$.

A closed cone K is proper if and only if $\operatorname{int} K^* \neq \emptyset$.

We have

[1]In the literature one can find examples where both, K° and K^* are called dual cones.

$$\operatorname{ri} K^* = \{w : \ wx > 0 \ \forall x \in K, \ x \neq F\};$$

in particular, if K is proper then

$$\operatorname{int} K^* = \{w : \ wx > 0 \ \forall x \in K, \ x \neq 0\}.$$

By definition, the cone K is *polyhedral* if it is the intersection of a finite number of half-spaces $\{x : \ p_i x \geq 0\}$, $p_i \in \mathbf{R}^n$, $i = 1, ..., N$.

The Farkas–Minkowski–Weyl theorem:

a cone is polyhedral if and only if it is finitely generated.

Intuitively, this theorem, giving an alternative but equivalent definition of a polyhedral cone, is almost obvious but its proof is not a just straightforward exercise (unlike other previously formulated statements) and requires certain efforts. Easy to remember, it provides a freedom to chose an appropriate definition to establish needed properties. E.g., the closedness of a polyhedral cone is trivial from the initial definition. On the other hand, the property that *the (arithmetic) sum of polyhedral cones is again a polyhedral cone* is clear from the alternative definition: the union of generating sets for each cone is a generating set for the sum.

The following result is a direct generalization of the Stiemke lemma.

Lemma A.1.1 *Let K and R be closed cones in $\mathbf{R}^n$. Assume that K is proper. Then*

$$R \cap K = \{0\} \quad \Leftrightarrow \quad (-R^*) \cap \operatorname{int} K^* \neq \emptyset.$$

Proof. ($\Leftarrow$) The existence of w such that $wx \leq 0$ for all $x \in R$ and $wy > 0$ for all y in $K \setminus \{0\}$ obviously implies that R and $K \setminus \{0\}$ are disjoint.

($\Rightarrow$) Let C be a convex compact set such that $0 \notin C$ and $K = \operatorname{cone} C$. By the separation theorem (for the case where one set is closed and another is compact) there is a non-zero $z \in \mathbf{R}^n$ such that

$$\sup_{x \in R} zx < \inf_{y \in C} zy.$$

Since R is a cone, the left-hand side of this inequality is zero, hence $z \in -R^*$ and, also, $zy > 0$ for all $y \in C$. The latter property implies that $zy > 0$ for $z \in K$, $z \neq 0$, and we have $z \in \operatorname{int} K$. □

In the classical Stiemke lemma $R = \{y \in \mathbf{R}^n : \ y = Bx, \ x \in \mathbf{R}^d\}$ where B is a linear mapping and $K = \mathbf{R}^n_+$. Usually, it is formulated as the alternative:

either there is $x \in \mathbf{R}^d$ such that $Bx \geq_K 0$ and $Bx \neq 0$ or there is $y \in \mathbf{R}^n$ with strictly positive components such that $B^* y = 0$.

Let A be a convex set with nonempty interior and let U be an open set. Then $A \cap U \neq \emptyset$ if and only if $\operatorname{int} A \cap U \neq \emptyset$. Combining this fact with the Stiemke lemma we get the following assertion:

Lemma A.1.2 *Let K and R be proper closed cones in $\mathbf{R}^n$. Then*

$$R \cap K = \{0\} \quad \Leftrightarrow \quad (-\operatorname{int} R^*) \cap \operatorname{int} K^* \neq \emptyset.$$

Lemma A.1.1 can be slightly generalized.

Let π be the natural projection of $\mathbf{R}^n$ onto the quotient space $\mathbf{R}^n/F$.

Theorem A.1.3 *Let K and R be closed cones in $\mathbf{R}^n$. Assume that the cone πR is closed. Then*

$$R \cap K \subseteq F \quad \Leftrightarrow \quad (-R^*) \cap \operatorname{ri} K^* \neq \emptyset.$$

Proof. It is easy to see that $\pi(R \cap K) = \pi R \cap \pi K$ and, hence,

$$R \cap K \subseteq F \quad \Leftrightarrow \quad \pi R \cap \pi K = \{0\}.$$

By Lemma A.1.1

$$\pi R \cap \pi K = \{0\} \quad \Leftrightarrow \quad (-\pi R)^* \cap \operatorname{int}(\pi K)^* \neq \emptyset.$$

Since $(\pi R)^* = \pi^{*-1} R^*$ and $\operatorname{int}(\pi K)^* = \pi^{*-1}(\operatorname{ri} K^*)$, the condition in the right-hand side can be written as

$$\pi^{*-1}((-R^*) \cap \operatorname{ri} K^*) \neq \emptyset$$

or, equivalently,

$$(-R^*) \cap \operatorname{ri} K^* \cap \operatorname{Im} \pi^* \neq \emptyset.$$

But $\operatorname{Im} \pi^* = (K \cap (-K))^* = K^* - K^* \supseteq \operatorname{ri} K^*$ and we get the result. □

Notice that if R is polyhedral then πR is also polyhedral, hence closed.

Lemma A.1.4 *Let K_1 and K_2 be closed cones in $\mathbf{R}^n$ such that the cone $K_1^* + K_2^*$ is closed. Then $(K_1 \cap K_2)^* = K_1^* + K_2^*$.*

Proof. The inclusion $\supseteq$ follows immediately from the definition of the dual cone. To prove the converse, suppose that $y \in (K_1 \cap K_2)^*$ but does not belong to the convex closed cone $K_1^* + K_2^*$. By the separation theorem there exists x such that $xy < 0$ and $xy_i \geq 0$ for all $y_i \in K_i^*$. But the latter property means that $x \in K_i^{**} = K_i$ for $i = 1, 2$, i.e. $x \in K_1 \cap K_2$. Thus, $xy \geq 0$. A contradiction. □

Remark. If K_1 and K_2 are polyhedral cones, then $K_1^* + K_2^*$ is polyhedral and the identity of the above lemma holds. Since in the left-hand we always have a closed set, the identity fails when $K_1^* + K_2^*$ is not closed.

Let X be a Hilbert space and let $f : X \to \mathbf{R} \cup \{\infty\}$. The *effective domain* $\operatorname{dom} f := \{x : \ f(x) < \infty\}$. The *dual* or *conjugate* to f is the convex function $f^*(y) := \sup_x [yx - f(x)]$. The *Fenchel inequality* is an immediate corollary of this definition: $f(x) + f^*(y) \geq yx$.

A.2 Césaro Convergence

A.2.1 Komlós Theorem

The following Komlós theorem asserting that a sequence of random variable bounded in L^1 contains a subsequence converging in Césaro sense a.s. is very useful in various applications, especially, in proofs of the existence of optimal controls.

Theorem A.2.1 *Let (ξ_n) be a sequence of random variables on $(\Omega, \mathcal{F}, P)$ bounded in L^1, i.e. with $\sup_n E|\xi_n| < \infty$. Then there exist a random variable $\xi \in L^1$ and a subsequence (ξ_{n_k}) Césaro convergent to ξ a.s., that is $k^{-1}\sum_{i=1}^k \xi_{n_i} \to \xi$ a.s. Moreover, the subsequence (ξ_n) can be chosen in such a way that any its further subsequence is also Césaro convergent to ξ a.s.*

A.2.2 Application to Convex Minimization in L^1

Let $\underline{J} := \inf_{\xi\in\mathcal{Z}} Ef(\xi)$ be the value of the minimization problem

$$Ef(\xi) \to \min \quad \text{on } \mathcal{Z} \tag{A.2.1}$$

where $\mathcal{Z}$ is a non-empty convex set of probability densities and $f : \mathbf{R}_+ \to \mathbf{R}$ is a convex (hence, continuous) function such that $f \geq -c$ and $f(x)/x \to \infty$ as $x \to \infty$.

Proposition A.2.2 *If $\mathcal{Z}$ is closed in L^1 and $\underline{J} < \infty$, then there exists $\underline{\xi} \in \mathcal{Z}$ such that $\underline{J} = Ef(\underline{\xi})$.*

Proof. Take a sequence $\xi_j \in \mathcal{Z}$ such that $Ef(\xi_j) \to \underline{J}$. Since $\xi_j \geq 0$ and $E\xi_j = 1$, in virtue of the Komlós theorem there is a subsequence j_k such that $\tilde{\xi}_n := n^{-1}\sum_{k=1}^n \xi_{j_k}$ converge a.s. to a certain $\underline{\xi} \in L^1$. Due to the Fatou lemma and convexity of f

$$Ef(\underline{\xi}) = E\lim f(\tilde{\xi}_n) \leq \liminf Ef(\tilde{\xi}_n) \leq \lim \frac{1}{n}\sum_{k=1}^n Ef(\xi_{j_k}) = \underline{J}.$$

The de la Vallée-Poussin criterion ensures that the sequence $(\tilde{\xi}_n)$ is uniformly integrable and, hence, converges also in L^1. Thus, $\underline{\xi} \in \mathcal{Z}$. □

If f is strictly convex, then, obviously, the minimizer is unique.

A.2.3 Von Weizsäcker Theorem

The next result due to von Weizsäcker is a beautiful extension of the Komlós theorem to sequences of random variables from L^0_+. It is less known and we give its proof here.

Theorem A.2.3 *Let (ξ_n) be a sequence of positive random variables. Then there exist a random variable ξ taking values in $[0,\infty]$ and a subsequence (ξ_{n_k}) such that all subsequences of the latter are Césaro converging a.s. to ξ.*

Proof. It consists of several steps.

Lemma A.2.4 *Let (ξ_n) be a sequence of positive random variables unbounded in probability. Then there exist a set B with $P(B)>0$ and a subsequence (ξ_{n_k}) such that all its further subsequences of are Césaro converging a.s. on B to infinity.*

Proof. Recall that a set G of random variable is bounded in probability if

$$\sup_{\eta\in G} P(|\eta|\geq N)\to 0, \quad N\to\infty.$$

Since in our case this property does not hold, we may assume, passing, if necessary, to a subsequence, that there is $\varepsilon>0$ such that $\sup_n P(B_n)>\varepsilon$ where $B_n:=\{\xi_n>n\}$. Applying the Komlós theorem to the sequence (I_{B_n}) we may assume that there is a random variable η such that for all subsequences

$$\frac{1}{m}\sum_{k=1}^{m} I_{B_{n_k}}\to\eta \quad \text{a.s.}$$

Clearly, $0\leq\eta\leq 1$ and, by dominated convergence, $E\eta\geq\varepsilon$. Thus, $P(B)\geq\varepsilon$ where $B:=\{\eta>0\}$. Then for every subsequence (ξ_{n_k}) and every $N>0$

$$\liminf_m \frac{1}{m}\sum_{k=1}^{m}\xi_{n_k}\geq\liminf_m \frac{1}{m}\sum_{k=1}^{m} N I_{B_{n_k}}\geq N\eta \quad \text{a.s.}$$

and, therefore, (ξ_{n_k}) is Césaro converging a.s. on B to infinity. □

The key step is to find a "maximal" set B satisfying, with a certain subsequence of the original sequence, the property declared above. The formal framework can be describe as follows. Let us consider the set $\mathcal{R}$ of pairs $(B,(\xi_{n_k}))$ where $B\in\mathcal{F}$ and all subsequences of (ξ_{n_k}) are Césaro converging a.s. on B to infinity. We introduce on $\mathcal{R}$ a partial ordering by letting $(\tilde{B},(\xi_{\tilde{n}_k}))\succeq(B,(\xi_{n_k}))$ if $\tilde{B}\supseteq B$ and $(\xi_{\tilde{n}_k})$ is a subsequence of (ξ_{n_k}).

Lemma A.2.5 *There is a pair $(B,(\xi_{n_k}))\in\mathcal{R}$ such that $P(\tilde{B})=P(B)$ for any pair $(\tilde{B},(\xi_{\tilde{n}_k}))\in\mathcal{R}$ dominating $(B,(\xi_{n_k}))$.*

Proof. Let $B_0=\emptyset$. Trivially, the pair $(B_0,(\xi_n))$ is in $\mathcal{R}$. We construct in $\mathcal{R}$ recursively a sequence $(B_l,(\xi_{n_{l,k}}))$, $l=0,1,...$, increasing with respect to the partial ordering. Suppose that it is defined yet up to a number l. Let

$$\mathcal{R}_l:=\{(\tilde{B},(\xi_{\tilde{n}_k}))\in\mathcal{R}:\ (\tilde{B},(\xi_{\tilde{n}_k}))\succeq(B_l,(\xi_{n_{l,k}}))\}$$

and let $a_l:=\sup\{P(\tilde{B})\ :\ (\tilde{B},(\xi_{\tilde{n}_k}))\in\mathcal{R}_l\}$. We take as $(B_{l+1},(\xi_{n_{l+1,k}}))$ any element of $\mathcal{R}_l$ for which $P(B_{l+1})\geq a_l-1/l$. The pair $(B,\xi_{n_{k,k}})$ where

$B = \cup B_l$ has the desired property. Indeed, suppose, on the contrary, that there is a pair $(\tilde{B}, (\xi_{\tilde{n}_k}))$ such that $(\tilde{B}, (\xi_{\tilde{n}_k})) \succeq (B, (\xi_{n_k}))$ and $P(\tilde{B}) > P(B) + 1/l$ for some $l \in \mathbf{N}$. Since $(\tilde{B}, (\xi_{\tilde{n}_k})) \in \mathcal{R}_l$ we have the inequality $a_l \geq P(\tilde{B})$. Thus, $P(B) < a_l - 1/l \leq P(B_{l+1})$ which is a contradiction. $\Box$

Now we are able to complete the proof. Choose the "maximal" pair $(B, (\xi_{n_k})$ as in the above lemma. Without loss of generality we may assume all the subsequences of (ξ_n) are Césaro converging a.s. on B to infinity and we cannot enlarge B to keep this property even passing to a subsequence. Let $A := B^c$. By Lemma A.2.4 the sequence $(\xi_n I_A)$ is bounded in probability. Combining the Komlós theorem with a diagonal procedure we may also assume that for each $N \in \mathbf{N}$ there is a random variables $\zeta_N \leq N$ such that the sequence $(\xi_n \wedge N)$ and all its subsequences are Césaro converging a.s. to ζ_N. Obviously, ζ_N are increasing to a certain limit ζ. Taking into account that

$$\liminf_m \frac{1}{m} \sum_{k=1}^m \xi_{n_k} \geq \sup_N \zeta_N = \zeta$$

we conclude that $\zeta < \infty$ a.s. on A (otherwise we could enlarge B by adding the set $A \cap \{\zeta = \infty\}$).

Put $A_m := A \cap \{\zeta \leq m\}$. Using the dominated convergence and obvious inequalities we have:

$$\liminf_n EI_{A_m}(\xi_n \wedge N) \leq \lim_n EI_{A_m} \sum_{k=1}^n (\xi_k \wedge N) = EI_{A_m}\zeta_N \leq EI_{A_m}\zeta \leq m.$$

Passing again to a suitable subsequence we may assume that

$$\liminf_n EI_{A_m}(\xi_n \wedge N) \leq m$$

for all integer N and m. Since the sequence $(\xi_n I_A)$ is bounded in probability, there are integers N_k such that $P(\xi_n I_A > N_k) \leq 2^{-k}$ for all n. Let n_k be such that $EI_{A_m}(\xi_n \wedge N_k) \leq 2m$ for all $n \geq n_k$. Let us consider the decomposition

$$I_{A_m}\xi_{n_k} = \eta'_{k,m} + \eta''_{k,m}$$

where $\eta'_{k,m} := I_{A_m}(\xi_{n_k} \wedge N_k)$ and $\eta''_{k,m} := I_{A_m} I_{\{\xi_{n_k} > N_k\}}(\xi_{n_k} - N_k)$. Notice that $P(\eta''_{k,m} > 0) \leq 2^{-k}$. By the Borel–Cantelli lemma the sequences $(\eta''_{k,m}(\omega))_{k \in \mathbf{N}}$ has only a finite number of nonzero terms for almost all ω. By construction, $E\eta'_{k,m} \leq 2m$. The Komlós theorem applied to $(\eta'_{k,m})_{k \in \mathbf{N}}$ shows that we may assume that (ξ_{n_k}) and all its subsequences are Césaro converging a.s. on A_m (to a finite limit). Using the diagonal procedure we can easily construct a subsequence for which the same property holds on the union of A_m, i.e. on the set A. This proves the theorem. $\Box$

Of course, the result remains true if the sequence (ξ_n) is bounded from below (by a constant or a finite random variable).

A.2.4 Application to Convex Minimization in L^0

Again let $\underline{J} := \inf_{\xi \in \mathcal{Z}} Ef(\xi)$ be the value of the minimization problem

$$Ef(\xi) \to \min \quad \text{on } \mathcal{Z} \tag{A.2.2}$$

where now $\mathcal{Z}$ is a non-empty convex set of positive random variables and $f : \mathbf{R}_+ \to \mathbf{R}$ is a convex function bounded from below with $f(\infty) = \infty$.

Proposition A.2.6 *If $\mathcal{Z}$ is closed in L^0 and $\underline{J} < \infty$, then there exists $\underline{\xi} \in \mathcal{Z}$ such that $\underline{J} = Ef(\underline{\xi})$.*

Proof. Take a sequence $\xi_j \in \mathcal{Z}$ such that $Ef(\xi_j) \to \underline{J}$. In virtue of the von Weizsäcker theorem there is a subsequence j_k such that $\tilde{\xi}_n := n^{-1} \sum_{k=1}^{n} \xi_{j_k}$ converge a.s. to a certain random variable $\underline{\xi}$ which may take infinite values. Due to the Fatou lemma and convexity of f we obtain, as in the proof of Proposition A.2.2, that $Ef(\underline{\xi}) \le \underline{J} < \infty$. It follows that ξ is finite and hence an element of $\mathcal{Z}$ minimizing the functional. □

A.2.5 Delbaen–Schachermayer Lemma

The above theorems are easy to memorize and apply in various situations. However, their proofs are rather lengthy. The following assertion due to Delbaen and Schachermayer can be used for the same purposes as the much more delicate von Weizsäcker's theorem. It has an important advantage - its proof is elementary and short.

For a sequence (ξ_n) we define the set $\mathcal{T}_n := \mathcal{T}_n(\xi) := \operatorname{conv}\{\xi_k,\ k \ge n\}$, the convex hull of its tail.

Lemma A.2.7 *Let (ξ_n) be a sequence of positive random variables. Then there exist a sequence $\eta_n \in \mathcal{T}_n$ and a random variable η with values in $[0, \infty]$ such that $\eta_n \to \eta$ a.s.*

Proof. The sequence $J_n := \inf_{\eta \in \mathcal{T}_n} Ee^{-\eta}$ increases to some $J \le 1$. Let us take $\eta_n \in \mathcal{T}_n$ with $Ee^{-\eta_n} \le J_n + 1/n$. It is easy to see that for any $\varepsilon > 0$ there is $\delta = \delta(\varepsilon) > 0$ such that

$$e^{-(x+y)/2} \le (e^{-x} + e^{-y})/2 - \delta I_{B_\varepsilon}(x, y)$$

where $B_\varepsilon := \{(x, y) \in \mathbf{R}_+^2 :\ |x - y| \ge \varepsilon,\ x \wedge y \le 1/\varepsilon\}$. Therefore,

$$J_{n \wedge m} \le Ee^{-(\eta_n + \eta_m/2} \le (Ee^{-\eta_n} + Ee^{-\eta_m})/2 - \delta P((\eta_n, \eta_m) \in B_\varepsilon).$$

It follows that $\lim_{m,n \to \infty} P((\eta_n, \eta_m) \in B_\varepsilon) = 0$. We infer from the inequality

$$E|e^{-\eta_n} - e^{-\eta_m}| \le \varepsilon + 2e^{-1/\varepsilon} + P((\eta_n, \eta_m) \in B_\varepsilon)$$

that $e^{-\eta_n}$ is a Cauchy sequence in L^1. It remains to recall that a sequence convergent in L^1 (hence, in L^0) contains a subsequence convergent a.s. □

Remark. The limit points of a set in L^0 bounded in probability are finite random variables. Thus, if a certain set $\mathcal{T}_n$ is bounded in probability then $\eta < \infty$.

We give a sufficient condition ensuring that the limit η is strictly positive.

Lemma A.2.8 *Let $P(\xi_n \geq \alpha) \geq \alpha > 0$ for all $n \geq n_0$. Then $P(\eta > 0) > 0$.*

Proof. If $P(\xi \geq \alpha) \geq \alpha$ then

$$Ee^{-\xi} \leq 1 - \alpha + \alpha e^{-\alpha}.$$

It follows from the Jensen inequality that the same bound (strictly less than the unit) holds for any ξ which is a convex combination of ξ_n, $n \geq n_0$. By the dominating convergence it holds also for the limit points of $\mathcal{T}_{n_0}$. But the inequality $Ee^{-\xi} < 1$ implies that $P(\xi > 0) > 0$. □

A.3 Facts from Probability

A.3.1 Essential Supremum

For any family $\{\xi_\alpha\}_{\alpha \in J}$ of scalar random variables (which may take also infinite values) there exists a random variable η with the following properties:

1) $\eta \geq \zeta_\alpha$ for all α;

2) if $\eta' \geq \zeta_\alpha$ for all α then $\eta' \geq \eta$.

Obviously, this random variable (more precisely, a class of equivalence) is unique and denoted by $\operatorname{ess\,sup}_{\alpha \in J} \zeta_\alpha$.

The proof of existence is easy. It suffices to consider the case where all ζ_α take values in a bounded interval (indeed, if $\tilde{\eta}$ is the essential supremum for the family $\{\tilde{\zeta}_\alpha\}$ with $\tilde{\zeta}_\alpha := \arctan \zeta_\alpha$, then $\tan \tilde{\eta}$ is the essential supremum for $\{\zeta_\alpha\}$). Let $a := \sup_I E\zeta_I$ where I runs through the set of finite subsets of J and ζ_I stands for $\sup_{\alpha \in I} \zeta_\alpha$. Take a sequence of I_n such that $E\zeta_{I_n} \to a$. Replacing, if necessary, I_n by $\cup_{k \leq n} I_k$ we may assume without loss of generality that $I_n \uparrow I_\infty$. Then $\zeta_{I_n} \uparrow \eta := \zeta_{I_\infty}$. Taking into account that $E\eta = a$ and using the monotone convergence, it is easy to verify that η is the essential supremum. Notice that $\eta = \sup_{\alpha \in I_\infty} \zeta_\alpha$ where I_∞ is a **countable** subset of I.

The above arguments show clearly that if the family $\{\zeta_\alpha\}$ is directed upward (i.e. for every α_1, α_2 there is α such that $\zeta_\alpha \geq \zeta_{\alpha_1} \vee \zeta_{\alpha_2}$), then one can find an **increasing** sequence ζ_{α_n} such that $\lim_n \zeta_{\alpha_n} = \operatorname{ess\,sup}_\alpha \zeta_\alpha$. An immediate consequence of this observation is

Proposition A.3.1 *Assume that the family $\{\zeta_\alpha\}$ is directed upwards and $\zeta_\alpha \geq \zeta$ where $E(|\zeta||\mathcal{G}) < \infty$. Then*

$$E(\operatorname{ess\,sup}_\alpha \zeta_\alpha | \mathcal{G}) = \operatorname{ess\,sup}_\alpha E(\zeta_\alpha | \mathcal{G}).$$

A.3.2 Generalized Martingales

The classical definition of a martingale consists of two parts: the integrability property and the property involving conditional expectations. The former can be replaced by a weaker one, namely, by the existence of the conditional expectations needed for the latter. This leads to a notion of generalized martingale which is, in a discrete time setting, coincides with that of local martingale.

Let $(\Omega, \mathcal{F}, P)$ be a probability space equipped with a discrete-time filtration $\mathbf{F} = (\mathcal{F}_t)$, $t = 0, 1, \dots$. An adapted process $X = (X_t)$ is *generalized martingale* if $E(|X_{t+1}||\mathcal{F}_t) < \infty$ and $E(X_{t+1}|\mathcal{F}_t) = X_t$ for $t = 0, 1, \dots$.

Proposition A.3.2 *Let X be an adapted process with $X_0 \in L^1$. Then X is generalized martingale if and only if X is a local martingale.*

Proof. Suppose that X is a local martingale, i.e. there is a sequence of stopping times $\tau_n \uparrow \infty$ (a.s.) such that the stopped processes $X^{\tau_n} = (X_{\tau_n \wedge t})$ are martingales. The set $\{\tau_n \geq t+1\}$ is in $\mathcal{F}_t$ and

$$EI_{\{\tau_n \geq t+1\}} E(|X_{t+1}||\mathcal{F}_t) = EI_{\{\tau_n \geq t+1\}} |X_{t+1}| = EI_{\{\tau_n \geq t+1\}} |X^{\tau_n}_{t+1}| < \infty.$$

Thus, $E(|X_{t+1}||\mathcal{F}_t)$ is finite (a.s.) on each set $\tau_n \geq t+1$ and, therefore, on their union which is of full measure. Moreover, for any $\Gamma \in \mathcal{F}_t$ we have in virtue of the martingale property of X^{τ_n}:

$$\begin{aligned} EI_{\{\tau_n \geq t+1\}} I_\Gamma E(X_{t+1}|\mathcal{F}_t) &= EI_{\{\tau_n \geq t+1\}} I_\Gamma X_{t+1} \\ &= EI_{\{\tau_n \geq t+1\}} I_\Gamma X^{\tau_n}_{t+1} = EI_{\{\tau_n \geq t+1\}} I_\Gamma X^{\tau_n}_t. \end{aligned}$$

This means $E(X_{t+1}|\mathcal{F}_t) = X_t$ on each set $\tau_n \geq t+1$, i.e. almost everywhere. Hence, X is a generalized martingale.

The proof of the converse based on the following observation: if M is a martingale and H is a predictable process then the process $H \cdot M$ (which increments are $H_t \Delta M_t$) is a local martingale. Indeed, the random variables $\tau_n := \inf\{t : \ |H_{t+1}| \geq n\}$ are stopping times increasing to infinity, the increment of the stopped process $H \cdot M^{\tau_n}$, being of the form $I_{\{\tau_n \leq t\}} H_t \Delta M_t$, is bounded by the integrable random variable $n|\Delta M_t|$ and its conditional expectation with respect to $\mathcal{F}_{t-1}$ is zero.

Represent an arbitrary generalized martingale X as $X = X_0 + H \cdot M$ with $H_t = 1 + E(|\Delta X_t||\mathcal{F}_{t-1})$ and $M = H^{-1} \cdot X$. The increment ΔM_t takes values in the interval $[-1, 1]$ and its conditional expectation with respect to $\mathcal{F}_{t-1}$ is zero. Thus, M is a martingale, $H \cdot M$ is a local martingale and so is X.

Remark. We could conclude because $X_0 \in L^1$. In general, the class of generalized martingales is larger than the class of local martingales. The reason is that the definition of generalized martingale does not require integrability of its initial value. Thus, X shifted by any $\mathcal{F}_0$-measurable random variable remains a generalized martingale while the local martingale may not. Of course, both classes coincide under assumption (frequent in the literature) that $\mathcal{F}_0$ is trivial.

Proposition A.3.3 *Let X be a local martingale with $X_0 \in L^1$. Suppose that $EX_T^- < \infty$. Then X is a martingale.*

Proof. Adding to X the martingale $E(X_T^-|\mathcal{F}_t)$ we may assume without loss of generality that $X_T \geq 0$. By above X is generalized martingale and, therefore, $X_{T-1} = E(X_{T-1}|\mathcal{F}_T) \geq 0$. It follows that the whole process $X \geq 0$. The Fatou lemma applied to the equality $EX_{\tau_n \wedge t} = EX_0$ implies that $EX_t \leq EX_0 < \infty$, i.e. $X_t \in L^1$. □

A.3.3 Equivalent Probabilities

Let $(\Omega, \mathcal{F}, P)$ be a probability space with a filtration $(\mathcal{F}_t)_{t \leq T}$ and let Q be a probability measure equivalent to P. Let $\rho = (\rho_t)$ be the corresponding density process, i.e. the martingale with $\rho_t = E(dQ/dP|\mathcal{F}_t)$.

Lemma A.3.4 *The process M is a Q-martingale (resp. local Q-martingale) $M \in \mathcal{M}$ if and only if ρM is P-martingale (resp. local martingale).*

Proof. Using the definition of the density and the martingale property of ρ we get that

$$E_Q|M_t| := E\rho_T|M_t| = E\rho_t|M_t| = E|\rho_t M_t|$$

implying simultaneous finiteness of the first and last terms in this chain of equalities. If it is the case, for any $\Gamma \in \mathcal{F}_s$, $s \leq t$,

$$E_Q M_t I_\Gamma = E\rho_T M_t I_\Gamma = E\rho_t M_t I_\Gamma.$$

If $M \in \mathcal{M}(Q)$ the left-hand side above is $E_Q M_s I_\Gamma = E\rho_T M_s I_\Gamma = E\rho_s M_s I_\Gamma$ and we obtain the martingale property of ρM. Conversely, if $\rho M \in \mathcal{M}$, the left-hand side is equal to $E\rho_T M_s I_\Gamma = E_Q M_s I_\Gamma$ and, hence, $M \in \mathcal{M}(Q)$. The extension to the local martingales is obvious. □

Lemma A.3.5 *Let $\{\xi_n\}$ be a sequence in L^0_+. Then there exists a sequence of strictly positive reals such that that $\sum_n a_n \xi_n < \infty$.*

Proof. Take $b_n > 0$ such that $P(\xi_n > b_n) \leq 1/n^2$. Then the assertion holds with $a_n := 1/(b_n n^2)$ because in virtue of the Borel–Cantelli lemma for almost all ω only a finite number of $\xi_n(\omega)$ are larger than b_n. □

Proposition A.3.6 *Let $\{\xi_n\}$ be at most countable set in L^0. Then there exists a probability measure $Q \sim P$ such that the density $\rho = dQ/dP$ is bounded and all $\xi_n \in L^1(Q)$. Moreover, if ξ_n converges a.s. to a finite random variable ξ, then one can choose Q in such a way that $E_Q|\xi_n - \xi| \to 0$.*

Proof. The assertion is obvious for a single random variable ξ: one can take $Q = \rho P$ with $\rho = c(1 + |\xi|)e^{-(1+|\xi|)}$ where c is a normalizing constant. So it holds also for a finite set. The countable case follows from the lemma above. In the case of convergent sequence one can take Q such that the finite random variable $\sup_n |\xi_n - \xi|$ is also integrable ensuring the required convergence in $L^1(Q)$. □

A.3.4 Snell Envelopes of Q-Martingales

In the usual setting of discrete-time model with finite horizon consider a set $\mathcal{Q}$ of probability measures equivalent to P denoting by $\mathcal{Z}$ be the set of corresponding density processes. We say that $\mathcal{Q}$ is *stable under concatenation* if for each elements $\rho^1, \rho^2 \in \mathcal{Z}$ and every $\Gamma \in \mathcal{F}_s$, $s \le T$, it contains also the process $\rho = (\rho_t)_{t \le T}$ with

$$\rho_t := \rho_t^1 I_{\{t \le s\}} + \rho_t^1 I_\Gamma I_{\{t>s\}} + \rho_t^2 \frac{\rho_s^1}{\rho_s^2} I_{\Gamma^c} I_{\{t>s\}}.$$

clearly, ρ, being constructed in this way from strictly positive martingales is also a martingale; the concatenation property does not depend on the choice of the reference measure.

Note that the sets Q^e (of equivalent martingale measures for some process S) and Z_l^e possess the concatenation property.

Proposition A.3.7 *Suppose that $\mathcal{Q}$ is stable under concatenation. Let $\xi \ge 0$ be a random variable such that $\sup_{Q \in \mathcal{Q}} E_Q \xi < \infty$. Then the process*

$$X_t = \operatorname{ess\,sup}_{Q \in \mathcal{Q}} E_Q(\xi | \mathcal{F}_t). \tag{A.3.1}$$

is a Q-supermartingale whatever is $Q \in \mathcal{Q}$.

Proof. Since the assertion does not depend on the choice of the reference measure, we assume that P is an element of $\mathcal{Q}$ and check the supermartingale property $E(X_s|\mathcal{F}_{s-1}) \le X_{s-1}$ with respect to this measure.

Let us consider the set $\mathcal{Z}_s$ of local densities from $\mathcal{Z}$ which are equal to the unit for $t \le s$. The process equal to the unit identically is in $\mathcal{Z}$. By the concatenation property the process $I_{\{t \le s\}} + (\rho_t/\rho_s) I_{\{t>s\}}$ belongs to $\mathcal{Z}_t$ whatever is $\rho \in \mathcal{Z}$. Expressing the conditional expectation in terms of the reference measure and the corresponding density, we obtain using this observation that

$$X_s = \operatorname{ess\,sup}_{\rho \in \mathcal{Z}_s} E(\rho_T \xi | \mathcal{F}_s).$$

Put $\zeta_\rho := E(\rho_T \xi|\mathcal{F}_s)$. By the concatenation property $\{\zeta_\rho\}_{\rho \in \mathcal{Z}_s}$ is directed upward: $\zeta_\rho = \zeta_{\rho^1} \vee \zeta_{\rho^2}$ for

$$\rho_t = I_{\{t \le s\}} + (I_\Gamma \rho_t^1 + I_{\Gamma^c} \rho_t^2) I_{\{t>s\}}$$

with $\Gamma := \{\zeta_{\rho^1} \ge \zeta_{\rho^2}\}$. Using Proposition A.3.1 and the inclusion $\mathcal{Z}_s \subseteq \mathcal{Z}_{s-1}$ we get that

$$E(X_s|\mathcal{F}_{s-1}) = \operatorname{ess\,sup}_{\rho \in \mathcal{Z}_s} E(\rho_T \xi|\mathcal{F}_{s-1}) \le \operatorname{ess\,sup}_{\rho \in \mathcal{Z}_{s-1}} E(\rho_T \xi|\mathcal{F}_{s-1})$$

and we conclude. □

The above result can be easily extended to include the classical formulation of the Snell envelope which was introduced in the theory of optimal stopping (for the case when $\mathcal{Q}$ is a singleton $\{P\}$).

Let $\mathcal{T}_t$ denote the set of stopping times with values in the set $\{t, t+1, ..., T\}$.

Proposition A.3.8 *Suppose that $\mathcal{Q}$ is stable under concatenation. Let $Y_t \geq 0$ be an adapted process such that $\sup_{Q\in\mathcal{Q}} E_Q Y_t < \infty$ for $t = 0, 1, ..., T$. Then the process*

$$X_t = \operatorname{ess\,sup}_{Q\in\mathcal{Q},\, \tau\in\mathcal{T}_t} E_Q(Y_\tau|\mathcal{F}_t). \tag{A.3.2}$$

is a Q-supermartingale whatever is $Q \in \mathcal{Q}$. Moreover, X is the smallest process which is a Q-supermartingale for all Q and which dominates Y.

Proof. The supermartingale properties is verified by the same argument as in the previous proposition. One should only check that for a fixed s the family of random variables $\zeta_\rho := E(\rho_T Y\tau|\mathcal{F}_s)$, $\rho \in \mathcal{Z}_s$, $\tau \in \mathcal{T}_s$, is directed upward. This is an easy exercise. Finally, if $\tilde{Y}$ is a Q-supermartingale for all Q then the corresponding process $\tilde{X}$ coincides with Y. The operator defined by the right-hand side of (A.3.2) is monotone, that is a larger input process results in a larger output. Combining these two properties we obtain the concluding assertion. □

A.4 Measurable Selection

Measurable spaces $(E, \mathcal{E})$ and $(E', \mathcal{E}')$ are *Borel isomorphic* if there exists a bijection $f : E \to E'$ such that $f(A) \in \mathcal{E}'$ and $f^{-1}(A') \in \mathcal{E}$ whatever are $A \in \mathcal{E}$ and $A' \in \mathcal{E}'$.

A measurable space (Borel) isomorphic to a Borel subset of Polish space (i.e. complete separable metric space) is called Borel (or Lusin, or standard measurable) space. The basic fact: any infinite Borel space is isomorphic either to $\mathbf{N}$ or $\mathbf{R}_+$. Exactly this result allows to reduce the proof of existence of regular conditional distributions to the scalar case where one can work comfortably with distribution functions and use the linear ordering of the real line. Similarly, it works in the proof of the following measurable selection theorem.

Theorem A.4.1 *Let $(\Omega, \mathcal{F}, P)$ be a complete probability space, let $(E, \mathcal{E})$ be a Borel space, and let $\Gamma \subset \Omega \times E$ be an element of the σ-algebra $\mathcal{F} \otimes \mathcal{E}$. Then the projection $Pr_\Omega \Gamma$ of Γ onto Ω is an element of $\mathcal{F}$ and there exists an E-valued random variable ξ such that $\xi(\omega) \in \Gamma_\omega$ for all non-empty ω-sections Γ_ω of Γ.*

In applications, the y-axis is usually a Polish space while the x-axis can be a σ-finite complete measurable space (this extension is obvious).

Let $(E, \mathcal{E})$ be Polish space. A set-valued mapping $\omega \mapsto \Gamma(\omega)$, $\Gamma(\omega) \subseteq E$ is *measurable* if its graph $\{(\omega, x) : \ x \in \Gamma(\omega)\}$ is measurable; a measurable selector ξ of the graph is called *measurable selector* (or simply *selector*) of the set-valued mapping; the terminology *measurable selection*, *selection* is frequent in the modern literature. The distance between two measurable set-valued mappings, as a function of ω, is a random variable.

Measurability properties of a set-valued mapping which values are sets in $(\mathbf{R}^d, \mathcal{B}^d)$ are preserved by major operations of convex analysis. In particular, if Γ is a measurable set-valued mapping, the mapping $\omega \mapsto \operatorname{conv} \Gamma(\omega)$ is also measurable, for the convex-valued Γ so is the dual mapping $\omega \mapsto \Gamma^\circ(\omega)$, the mapping $\omega \mapsto \operatorname{ri} \Gamma(\omega)$ etc.

We denote by $L^p(\Gamma)$ or, in the need, $L^p(\Gamma, \mathcal{F})$, $L^p(\Gamma, \mathcal{F}, P)$, the set of all L^p-selectors of a measurable set-valued mapping Γ. If the values of Γ are convex sets (respectively, cones) then, $L^p(\Gamma)$ is a convex subset (respectively, a cone) in L^p.

Let Γ be a measurable mapping the values of which are closed non-empty subsets of $\mathbf{R}^d$. Knowing that such a mapping admits a measurable selector, it is easy to infer that Γ admits a countable family of measurable selectors $\{\xi_n\}$ such that the values $\xi_n(\omega)$ are dense in the section $\Gamma(\omega)$ for each $\omega \in \Omega$, i.e., $\Gamma(\omega) = \overline{\{\xi_n(\omega)\}}$. This family is called *Castaing representation* of Γ. Note that if $L^0(\Gamma)$ is a closed cone, and $\{\xi_n\}$ is a Castaing representation of Γ, then the set of conic combinations of elements of $\{\xi_n\}$ with rational coefficients is also a Castaing representation of Γ. It follows that $L^0(\Gamma)$ is a closed cone in L^0 if and only if almost all values of Γ are closed cones. A similar assertion relates the convexity of values of Γ and of the set of its selectors.

The following useful lemma shows that such a Castaing representation being fixed, all other selectors can be approximated by members of this family.

Lemma A.4.2 *Let $\xi_n \in L^p$, $p \in [0, \infty[$, and $\Gamma(\omega) = \overline{\{\xi_n(\omega)\}}$. Then $L^p(\Gamma)$ is a closure in L^p of the set of random variables of the form $\sum I_{A_i}\xi_i$, where $\{A_i\}$ is a finite measurable partition of Ω.*

Proof. Let $p \in]0, \infty[$, let $\xi \in L^p$ be a measurable selector of Γ and let $\varepsilon > 0$. Starting from the sets $\{|\xi - \xi_i| < \varepsilon/2\}$, we construct a measurable countable partition $\{B_i\}$ of Ω such that $|\xi - \xi_i| < \varepsilon/2$ on B_i. Choose n such that

$$\sum_{i \geq n+1} EI_{B_i}|\xi - \xi_1|^p \leq \varepsilon/2$$

and put $A_1 := B_1 \cup (\cup_{i \geq n+1} B_i)$, $A_i := B_i$, $i = 2, ..., n$. Then

$$E\Big|\sum_{i=1}^n I_{A_i}(\xi - \xi_i)\Big|^p = \sum_{i=1}^n EI_{B_i}|\xi - \xi_i|^p + \sum_{i=n+1}^\infty EI_{B_i}|\xi - \xi_1|^p < \varepsilon.$$

The case $p = 0$ is a corollary of the previous one. Indeed, there is $\tilde{P} \sim P$ such that ξ and all ξ_n a $\tilde{P}$-integrable. Thus, we can approximate ξ in $L^1(\tilde{P})$ hence, in $L^0(\tilde{P})$. But the convergence in $\tilde{P}$-probability coincides with the convergence in P-probability. $\Box$

The next assertion gives a characterization of closed subsets in L^p which are formed by the selectors of a measurable set-valued mapping. To formulate it we use the following definition. A subset $\Xi \in L^p$ is called *decomposable* if with two its elements ξ_1, ξ_2 it contains also $\xi_1 I_A + \xi_2 I_{A^c}$ whatever is $A \in \mathcal{F}$. It is easily seen that in this case $\sum \xi_i I_{A_i}$ belongs to Ξ for every finite measurable partition of Ω and $\xi_i \in \Xi$, $i = 1, ..., n$.

Proposition A.4.3 *Let Ξ be a closed subset of $L^p(\mathbf{R}^d)$, $p \in [0,\infty[$. Then $\Xi = L^p(\Gamma)$ for some Γ which values are closed sets if and only if Ξ is decomposable.*

Proof. Only the "if" part needs arguments. Let consider the case $p \in]0,\infty[$. Let $\{x_i\}$ be a countable dense subset of $\mathbf{R}^d$ and let $a_i := \inf_{\eta\in\Xi} E|\eta - x_i|^p$. By definition, there exists $\eta_{ij} \in \Xi$ such that $E|\eta_{ij} - x_i|^p \leq a_i + 1/j$. We consider the mapping Γ which values are closures of the sets $\{\eta_{ij}(\omega)\}$. The inclusion $L^p(\Gamma) \subseteq \Xi$ follows from the lemma above and the fact that Ξ is decomposable. To prove the reverse inclusion we suppose that there is $\xi \in \Xi$ which is not an element of $L^p(\Gamma)$ and, hence, for some $\delta \in]0,1]$, the set $A := \{|\xi - \eta_{ij}| > \delta \; \forall\, i,j\}$ is of strictly positive probability. Fix i such that the set $B := A \cap \{|\xi - x_i| < \delta/3\}$ also is of strictly positive probability and put $\eta'_j := \xi I_B + \eta_{ij} I_{B^c}$. Then $\eta'_j \in \Xi$. On the set B we have

$$|x_i - \eta_{ij}| \geq |\xi - \eta_{ij}| - |\xi - x_i| \geq 2\delta/3.$$

Thus,

$$\begin{aligned} E|x_i - \eta_{ij}|^p - a_i &\geq E(|x_i - \eta_{ij}|^p - |x_i - \eta'_j|^p) \\ &= E(|x_i - \eta_{ij}|^p - |x_i - \xi|^p) I_B \\ &\geq ((2\delta/3)^p - (\delta/3)^p) P(B) \not\to 0, \quad j \to \infty, \end{aligned}$$

and this is a contradiction with the choice of η_{ij}.

For the case $p = 0$ we get the claim using the same arguments but replacing the function $|x|^p$ by the function $|x| \wedge 1$. □

Proposition A.4.4 *Let $\mathcal{G}$ be a (complete) sub-σ-algebra of $\mathcal{F}$. Let Γ be a measurable mapping which values are non-empty closed convex subsets of the unit ball in $\mathbf{R}^d$. Then there exists a $\mathcal{G}$-measurable mapping, $E(\Gamma|\mathcal{G})$, which values are non-empty convex compact subsets of the unit ball in $\mathbf{R}^d$ and the set of its $\mathcal{G}$-measurable a.s. selectors coincides with the set of $\mathcal{G}$-conditional expectations of a.s. selectors of Γ.*

For $d = 1$ the result is almost obvious. In this case $\Gamma(\omega) = [\alpha(\omega), \beta(\omega)]$ where α and β are random variables. Then $E(\Gamma|\mathcal{G}) = [E(\alpha|\mathcal{G}), E(\beta|\mathcal{G})]$ (the conditional expectations here can be chosen as the integrals with respect to regular conditional distributions). Indeed, the inequalities $\alpha \leq \xi \leq \beta$ implies the corresponding inequalities for the $\mathcal{G}$-conditional expectations. On the other hand, a $\mathcal{G}$-measurable selector $\tilde\xi$ of the newly defined mapping is of the form $\tilde\xi = \tilde\lambda E(\alpha|\mathcal{G}) + (1-\tilde\lambda) E(\beta|\mathcal{G})$ where λ is a $\mathcal{G}$-measurable random variable taking values in the interval $[0,1]$. Therefore, $\tilde\xi = E(\tilde\lambda\alpha + (1-\tilde\lambda)\beta|\mathcal{G})$.

In the general case we use the previous proposition. Indeed, let Ξ be the set of random variables $E(\eta|\mathcal{G})$ where η runs the set of measurable selectors of Γ. The boundedness and convexity of Γ ensures that Ξ is closed in L^1. Indeed, let $\xi_n := E(\eta_n|\mathcal{G})$ converges in L^1 to ξ. Since $|\eta_n| \leq 1$ we can apply the Komlós theorem and find a measurable selector η of Γ such that $\xi = E(\eta|\mathcal{G})$. Note that Ξ is decomposable on the space $(\Omega, \mathcal{G}, P)$. Thus, $\Xi = L^1(\tilde{\Gamma}) = L^\infty(\tilde{\Gamma})$ for some $\mathcal{G}$-measurable mapping $\tilde{\Gamma}$ which values are non-empty convex compact subsets of the unit ball in $\mathbf{R}^d$ and we conclude. □

A.5 Fatou-Convergence and Bipolar Theorem in L^0

Usually, bipolar theorems giving dual descriptions of convex sets are formulated for locally convex spaces. Unfortunately, the infinite-dimensional space L^0 does not belong to the latter class. Nevertheless, for some specific types of convex sets in L^0 one can give a dual description. We need a relatively simple theorem which can be easily obtained from the usual bipolar theorem in L^∞.

Let K be a measurable multifunction on Ω whose values are convex closed cones in $\mathbf{R}^d$ containing $\mathbf{R}^d_+$. We denote by L^0_b the cone in $L^0(\mathbf{R}^d)$ formed by random variables ξ such that $\xi + \kappa_\xi \mathbf{1} \in L^0(K)$ for some constant κ_ξ, i.e. elements of L^0_b are bounded from below in the sense of the partial ordering induced by K. We shall use the terminology "Fatou-convergence" in relation with this partial ordering.

Let $A \neq \emptyset$ be a convex subset in L^0_b, and $A^\infty := A \cap L^\infty$. For $\eta \in L^1(K^*)$ and $\xi \in L^0_b$ the expectation $E\xi\eta$ is well-defined ($\xi\eta \geq -\kappa_\xi|\eta|$). We put

$$f(\eta) := \sup_{\xi \in A} E\eta\xi.$$

Now we recall the following fact.

Proposition A.5.1 *Let $F \subseteq L^\infty$ be a convex set. Then*

F is weak closed $\Leftrightarrow$ $F \cap \{\xi : \ ||\xi||_\infty \leq \kappa\}$ is closed in probability for every κ.*

Proof. The classical Krein–Šmulian theorem (see, e.g., [70]) says that a convex set in the dual to a Banach space is weak* closed (i.e. closed in $\sigma\{L^\infty, L^1\}$) if and only if its intersection with every ball around the origin is weak* closed. Thus, we may assume that F is bounded. Now the dominated convergence works and the limit of a sequence convergent in probability is a limit of weak* convergent sequence. Thus, the implication $\Rightarrow$ holds. To prove the reverse, we consider F as a subset of L^2. Being closed in probability, it is closed also in L^2. But a closed convex set in a Hilbert space is weakly closed. Thus, F is closed in $\sigma(L^\infty, L^2)$ hence in $\sigma(L^\infty, L^1)$. □

As corollary we have:

Lemma A.5.2 *If A is Fatou-closed then the set A^∞ is weak* closed.*

Proof. A sequence convergent in probability contains a subsequence convergent almost surely. But $\{\xi : \ ||\xi||_\infty \leq \kappa\} + \kappa\mathbf{1} \subseteq L^0(\mathbf{R}^d_+) \subseteq L^0(K)$. Thus, a bounded sequence convergent a.s. is Fatou-convergent. Therefore, an intersection of A^∞ with balls is closed in probability and we conclude by the the above proposition. □

Theorem A.5.3 *Assume that the set A is Fatou-closed, A^∞ is dense in A with respect to the Fatou convergence, and there exists $\xi_0 \in A^\infty$ such that $\xi_0 - L^\infty(K) \subseteq A^\infty$. Then*

$$A = \{\xi \in L_b^0 : \ E\xi\eta \le f(\eta) \quad \forall \eta \in L^1(K^*)\}. \tag{A.5.1}$$

Proof. It is sufficient to verify that

$$A^\infty = \{\xi \in L^\infty : \ E\xi\eta \le f(\eta) \quad \forall \eta \in L^1(K^*)\}. \tag{A.5.2}$$

To check the only nontrivial inclusion "$\supseteq$", take $\zeta \in L^\infty \setminus A^\infty$. Since A^∞ is weak* closed, by the Hahn–Banach theorem there is $\eta \in L^1$ such that

$$\sup_{\xi \in A^\infty} E\xi\eta < E\zeta\eta. \tag{A.5.3}$$

Considering the r.v.'s of the form $\xi_0 - \xi$ with $\xi \in L^\infty(K)$, we deduce from here that

$$\inf_{\xi \in L^\infty(K)} E\xi\eta > -\infty.$$

As K is a cone, the infimum is equal to zero. By the usual measurable selection argument we deduce from this that $\eta \in L^1(K^*)$. The set A^∞ being Fatou-dense in A, the supremum in (A.5.3) can be taken over A, and the relation (A.5.3) means that ζ does not belong to the set given by the right-hand side of (A.5.2). □

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