Home work 1
Stochastic Volatility Models

Vega Institute

Problem 1 🧠

Prove that for any stopping times τ, σ the following properties hold:

1. $\tau + \sigma, \tau \vee \sigma, \tau + t$ are stopping times.

2. τ is $\mathcal{F}_\tau$ - measurable.

3. If stopping times $\tau_n \uparrow \tau$ a.s., then τ is also a stopping time.

Problem 2 💅

Let τ be a stopping time w.r.t. filtration $\mathcal{F}_t$. Let $\mathcal{F}_\tau := \{A \in \mathcal{F} : A \cap \{\tau \le t\} \in \mathcal{F}_t \text{ for all } t\}$. Prove that $\mathcal{F}_\tau$ is a sigma-algebra.

Problem 3 🧠 ⚰️

Provide an example of

1. a local martingale, which is not a supermartingale.

2^*. a local martingale which is not a martingale.

Problem 4 💅

Let $\Sigma = (\sigma_{ij})_{i,j=1}^n$ be a positive-definite symmetric matrix. Prove the existence of the n-dimensional Brownian motion with covariance matrix $cov(B_t^i, B_t^j) = t\sigma_{ij}$.

Problem 5 💅

Find the distribution of $\int_0^T f(t)dB_t$, where $f(t) \in L^2[0, T]$.

Problem 6 🧠

Apply Ito's formula

a) $Y_t = cos(te^{B_t})$

b) $Y_t = B_t^4$

Problem 7 🧠

Prove that the following stochastic processes are Brownian motions

a) $X_t = -B_t$

b) $X_t = \sqrt{\alpha} B_{\frac{t}{\alpha}}$

c) $X_t = B_{t+a} - B_a, a \geq 0$

Problem 8 🧠

Find EX_t and DX_t of

a) $dX_t = dt + a dB_t$

b) $dX_t = (aX_t + b)dt + dB_t$

Problem 9 🧠

Show that the processes satisfy the differential equations:

a) $X_t = X_0 e^{(\mu - \frac{\sigma^2}{2})t + \sigma B_t}$, $dX_t = \mu X_t dt + \sigma X_t dB_t$

b) $X_t = e^{-\mu t} X_0 + \sigma e^{-\mu t} \int_0^t e^{\mu s} dB_s$, $dX_t = -\mu X_t dt + \sigma dB_t$