Home work 2
Stochastic Volatility Models

Vega Institute

Problem 1 🧠 ⚰️

Find the solution of $dX_t = \dfrac{1}{X_t}dt + dW_t, X_0 = 1$ and prove its $\exists!$. *(Hint: Bessel process)*

Problem 2 🧠

Let $u(t, x)$ be the solution of Black-Scholes PDE:

$$\begin{cases} \dfrac{\partial u}{\partial t} + rx\dfrac{\partial u}{\partial x} + \dfrac{1}{2}\sigma^2 x^2 \dfrac{\partial^2 u}{\partial x^2} = ru \\ u(T, x) = (x - K)^+ \end{cases}$$

Prove that $u(t, x)$ can be represented as $u(t, x) = \dfrac{e^{-r(T-t)}}{\sqrt{2\pi(T-t)}} \int_{\mathbb{R}} (xe^{(r-\frac{\sigma^2}{2})(T-t)+\sigma y} - K)^+ e^{-\frac{y^2}{2(T-t)}} dy$ and derive an analytical formula:

$$u(t, x) = xN(d_+) - e^{-r(T-t)}KN(d_-)$$
$$d_- = \frac{\ln\left(\frac{x}{K}\right) + (r - \frac{1}{2}\sigma^2)(T-t)}{\sigma\sqrt{T-t}}, d_+ = d_- + \sigma\sqrt{T-t}$$

Hint 1: calculate the integral directly as in calculus.

Hint 2: calculate the integral using Girsanov theorem (if $\xi \sim N(0, 1)$ in P, then $\xi - a \sim N(0, 1)$ in $e^{a\xi - \frac{1}{2}a^2}P$), check that the answers are the same.

Problem 3 🧠

Derive the formula for call and put options in the Bachelier model, where $dS_t = \mu dt + \sigma dW_t, S_0 = s_0$ and $dB_t = rB_t dt, B_0 = 1$.

Problem 4 🧠

Provide an interpretation of $N(d_+)$ and $N(d_-)$ in Black-Scholes formula.

Problem 5 🧠

Prove that in Black-Scholes model for call and put options

$$\frac{\partial C}{\partial \sigma} = \frac{\partial P}{\partial \sigma} = SN'(d_-)\sqrt{T-t}.$$

Problem 6 🧠

Provide an example of strategy π_t, which is not admissible and leads to arbitrage (see lecture 4).

Problem 7 (extra) 💅

Prove put-call parity $C - P = S(0) - Ke^{-rT}$ for a non-dividend stock S and European options using non-arbitrage principle (do not assume any model for asset price dynamics). Using this parity prove the following general properties:

a) $max(0, S(0){-}Ke^{-rT}) \leq C < S(0)$

b) $max(0, -S(0) + Ke^{-rT}) \leq P < Ke^{-rT}$

c) C is decreasing in K, P is increasing in K

d) $K' < K'' \Rightarrow C(K'){-}C(K'') < e^{-rT}(K'' - K')$,

e) $K' < K'' \Rightarrow P(K''){-}P(K') < e^{-rT}(K'' - K')$,

f) C and P are convex functions of K

f) C is increasing in S, P is decreasing in S

g) $S' < S'' \Rightarrow C(S''){-}P(S') < S'' - S'$

g) $S' < S'' \Rightarrow P(S'){-}P(S'') < S'' - S'$

h) C and P are convex functions of S