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Question 1 Consider LIBOR rates observed today ($t = \tau_0 = 0$) for times $\tau_i = i,\ i = 0, \dots 5$ measured in years

$$L_0(\tau_i, \tau_{i+1}))_{i=0}^{5} = \frac{1}{100+i}.$$

a) Determine the at-the-money interest rate (swap rate) for an interest rate swap with dates $\tau_2,\ \dots,\ \tau_5$ $(m = 2,\ n = 5)$. (3 marks)

b) Calculate the yields $y_0(\tau_k)$ $k = 1, \dots, 5$ (continuous compounding). (3 marks)

c) Assume that the market expectation hypothesis holds exactly. Determine the LIBOR rate $L_t(\tau_i, \tau_{i+1})$ for $t = 3,\ i = 4$. (4 marks)

Question 2

Consider zero bond dynamics $(B_t(\tau))_{t\in[0,\tau]}$, $\tau \in [0, T]$ defined by the short rate model

$$dr_t = \theta dt + \sigma^r dW_t, \qquad r_0 = r_0^*$$

where the process $(W_t)_{t\in[0,T]}$ follows a Brownian motion with respect to the spot martingale measure and the parameters are given as

$$\theta, r_0^* \in \mathbb{R}, \quad \sigma^r \in]0, \infty[.$$

a) Calculate the short rate evolution $(r_t)_{t\in[0,T]}$. (3 marks)

b) Calculate the initial bond curve $B_0^*(\tau) = \mathbb{E}^{\mathbb{Q}}(e^{-\int_0^\tau r_s ds})$. (3 marks)
Hint: $\int_0^t W_s ds$ follows normal distribution with mean zero and variance $\frac{t^3}{3}$.
Hint: If N is normally distributed then $\mathbb{E}(e^N) = e^{\mathbb{E}(N)+\frac{1}{2}\mathrm{Var}(N)}$.

c) Determine the initial forward rates $f_0^*(\tau) = -\frac{\partial}{\partial\tau}\ln(B_0^*(\tau)$. (4 marks)

Question 3 (10 marks)

Consider two coupon paying bonds (Bond 1 and Bond 2) with face value 10,000 AUD paying coupons monthly at the (annual) coupon rate of 6% (Bond 1) and 9% (Bond 2). Assume that the first coupon just has been paid at $\tau = 0$, the last coupon (in addition to the face value) will be paid at $\tau = 10$ and the bonds are traded now, at $\tau = 0$ at the yields 7% (Bond 1) and 8% (Bond 2). Is it possible to determine the price of a zero bond maturing at $\tau = 10$ using no-arbitrage arguments? If yes, calculate the yield of this bond (continuous compounding).

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Question 4

Consider zero bond dynamics $(B_t(\tau))_{t\in[0,\tau]}$, $\tau \in [0,T]$ defined by the short rate model

$$dr_t = \theta dt + \sigma^r dW_t, \qquad r_0 = r_0^*$$

where the process $(W_t)_{t\in[0,T]}$ follows a Brownian motion with respect to the spot martingale measure $\mathbb{Q}$ and the parameters are given as

$$\theta, r_0^* \in \mathbb{R}, \quad \sigma^r \in]0,\infty[.$$

a) For the above short rate model, consider continuously compounded forward rates $(f_t(\tau))_{t\in[0,\tau]}$, $\tau \in [0,T]$ which follow

$$df_t(\tau) = \alpha(t,\tau)dt + \sigma_t(\tau)dW_t, \qquad f_0(\tau) = f_0^*(\tau).$$

Determine the functions $(\alpha_t(\tau))_{t\in[0,\tau]}$ and $(\sigma_t(\tau))_{t\in[0,\tau]}$. (5 marks)

b) Suppose that the price evolution $(S_t)_{t\in[0,T]}$ of a stock is given by the strong solution to the stochastic differential equation

$$dS_t = S_t(r_t dt + \sigma^S dW_t), \qquad S_0 = S_0^* \in]0,\infty[, \text{ with volatility } \sigma^S \in]0,\infty[.$$

Consider the the S-measure $\mathbb{Q}^S$ defined by

$$d\mathbb{Q}^S = \frac{S_T}{B_T}\frac{B_0}{S_0}d\mathbb{Q}$$

where $(B_t = e^{\int_0^t r_s ds})_{t\in[0,T]}$ denotes the price evolution of the standard savings account. Determine the distribution of r_T with respect to the measure $\mathbb{Q}^S$. (5 marks)

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Question 5 Consider the horizon $T = 5$ of a bond market and suppose that today's ($t = 0$) bond curve is given by

$$B_0^*(\tau) = \frac{1}{1 + c\tau} \qquad \text{for all } \tau \in [0, T] \text{ with } c \in]0, \infty[.$$

For a one-factor HJM model with deterministic forward rate volatilities

$$\sigma_t(\tau) = \sigma\sqrt{\tau - t}, \qquad 0 \le t \le \tau \le T, \ \sigma \in]0, \infty[.$$

a) Calculate the initial forward rates $(f_0^*(\tau))_{\tau \in [0,T]}$. (3 marks)

b) Calculate the drift $(\alpha_t(\tau))_{t \in [0,\tau]}$, $\tau \in [0, T]$ from the HJM drift condition. (3 marks)

c) Determine the bond volatility $(\sigma_t^B(\tau))_{t \in [0,\tau]}$ for $\tau \in [0, T]$ defined by (4 marks)

$$dB_t(\tau) = B_t(\tau)(r_t dt + \sigma_t^B(\tau) dW_t), \qquad 0 \le t \le \tau \le T.$$

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