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Criteria for the Recurrence or Transience of Stochastic Process. I

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I. INTRODUCTION

Let $\{X_n\}$ be the random variables and $[P_{ij}]$ the transition-probability matrix of an irreducible Markov chain. Several general conditions for the *recurrence* or the *transience* of the chain are known. One well-known criterion for recurrence is the divergence of $\sum_n P_{ii}^{(n)}$ for some state i; others involve the existence and properties of solutions to systems of infinitely many linear equations [3]. These criteria are difficult to apply unless the matrix $[P_{ij}]$ has a rather special form. On the other hand, there are convenient conditions for recurrence which apply to important special classes of processes arising in queueing, random walks, etc. The purpose of this paper is to develop criteria of intermediate generality which are convenient to use if the transition probabilities (of first order only) are known, but contain many special cases of interest.

The nature of the criteria we will obtain can be illustrated by an example: suppose that $\{X_n\}$ is a random walk; that is, $\{X_n\}$ is a Markov chain with the non-negative integers as states such that if $X_n = k$, then

$$X_{n+1} = \begin{cases} k+1 & \text{with probability } p_k > 0 \\ (k-1)^+ & \text{with probability } q_k = 1 - p_k. \end{cases} \tag{1.1}$$

(The notation x^+ means max $(0, x)$.) Suppose further that for large k it is possible to write

$$p_k = \frac{1}{2}\left[1 + \frac{\beta}{k} + 0(k^{-1-\delta})\right] \tag{1.2}$$

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for some β and some $\delta > 0$. Then according to a theorem of T. E. Harris [5], the random walk is recurrent if $\beta \leqslant 1/2$, and transient if $\beta > 1/2$. If we put

$$\mu(k) = E[X_{n+1} - X_n | X_n = k], \qquad v(k) = E[(X_{n+1} - X_n)^2 | X_n = k] \quad (1.3)$$

(so that $v(k) = 1$ for $k > 0$ in this case), we see that the criterion for recurrence may be expressed in the form

$$\mu(x) \leqslant \frac{v(x)}{2x} + 0(x^{-1-\delta}) \qquad (1.4)$$

for $x = 1, 2, \ldots$. The method of proof used by Harris and also the methods of Hodges and Rosenblatt [6] and Karlin and McGregor [7] do not apply if the form of the transition law (1.1) is appreciably modified. The main theorems of the present paper, however, assert that for a wide variety of stochastic processes on the positive real line, (1.4) continues to be a sufficient condition for recurrence, while the validity for large x of

$$\mu(x) \geqslant \frac{\theta v(x)}{2x} \qquad \text{for some} \qquad \theta > 1 \qquad (1.5)$$

is sufficient to insure that $X_n \to \infty$ with probability one (transience).

It is worth reflecting on the fact that only the two moments of (1.3) appear in the recurrence criterion. This suggests that some sort of "invariance principle" is operating, which might apply to other properties of the process as well as recurrence or transience. There seem to be interesting possibilities here for further investigation.

The principal tool used in the proof of our recurrence and transience conditions is the semi-martingale convergence theorem. This theorem does not apply too directly, and the second section is devoted to adapting it into (for us) more useful forms. The main theorem is then given in Section III; a simplified version is proved there, but the rather tedious calculations needed for the more general result are relegated to an appendix. As an example, we consider "random walks" with unequal steps to left or right, and the results even in this simple case appear to be new.

In view of the use of martingale theory, it is perhaps not surprising that the processes considered need not be Markovian. This is convenient for extending the criteria to multivariate Markov processes, for the one-dimensional theorems can be applied to the radial component of an s-dimensional process, which, of course, is usually not Markovian. This idea is worked out to a certain extent in Section IV. Some insight is gained in this way into Pólya's theorem on the recurrence of spatially homogeneous random walks in s-dimensions and its elegant generalization by Chung and Fuchs [1]. Another application yields a result analogous to that of Harris mentioned above for certain random walks in s-dimensions with non-constant transition probabilities. This provides an extension of some previous work of Gillis [9].

Three remarks: first, in view of another recent theorem of Harris [4], in many cases by proving recurrence of a Markov process we are also proving the existence of an "invariant measure." It is hoped that a more specific discussion of the properties of this measure and the related problem of the existence of passage-time moments can be given in a future publication. Secondly, a theorem on the recurrence of real Markov processes has also been obtained in [8] by a quite different method. However, all the results of [8] are generalized in Section III of this paper. Finally, we point out that an idea similar in principle to the main one used here was employed in a different context by Doob [11].

II. Preliminaries

The probability space underlying the stochastic processes to be considered will be denoted by $(\Omega, \mathscr{F}, P)$ where Ω is a set, $\mathscr{F}$ a Borel field of subsets, complete with respect to P, the probability measure on $\mathscr{F}$. We recall the

Definition: Let $(Y_n, \mathscr{F}_n)$, $n = 0, 1, 2, \ldots$ be pairs consisting of real random variables Y_n and Borel fields of Ω subsets $\mathscr{F}_n$ such that Y_n is $\mathscr{F}_n$ measurable, $\mathscr{F}_{n+1} \supset \mathscr{F}_n$, $E(|Y_n|) < \infty$, and

$$E(Y_{n+1}|\mathscr{F}_n) \geqslant Y_n \qquad \text{a.s.} \tag{2.1}$$

Then $\{(Y_n, \mathscr{F}_n)\}$ is a semi-martingale.[1]

As mentioned already, we shall employ a version of the

Semi-martingale convergence theorem: *If* $\{(Y_n, \mathscr{F}_n)\}$ *is a semi-martingale, and if* $Y_n \leqslant M < \infty$ a.s. *for some* M, *all* n, *then limit* Y_n *exists* (*and is finite*) *with probability one.*

Suppose that $\{X_n\}$, $n = 0, 1, 2, \ldots$ is a real and non-negative stochastic process. We shall assume that

$$P(\limsup_{n\to\infty} X_n = \infty) = 1. \tag{2.2}$$

Then if there exists an $r < \infty$ such that

$$P(\liminf_{n\to\infty} X_n \leqslant r) = 1 \tag{2.3}$$

[1] [2, Chapter VII] is a general reference for martingale theory.

we shall call the process *recurrent*, while if

$$P(X_n \to \infty) = 1 \tag{2.4}$$

the process $\{X_n\}$ is *transient*. (Notice that in case the X_n form an irreducible Markov chain on the integers $0, 1, 2, \ldots$ (2.2) is automatic, and our use of the terms transient and recurrent reduces to the usual one. In this situation and many other (recurrent) cases, one can take $r = 0$ in (2.3).) In addition to (2.2), we assume from now on that $E(X_0) < \infty$.

THEOREM 2.1. *If there exists $M < \infty$ such that*

$$E(X_{n+1}|X_n = x, X_{n-1} = x_1, \ldots, X_0 = x_n) \leqslant x \qquad \text{a.s.} \tag{2.5}$$

whenever $x \geqslant M$ and for all n, then $\{X_n\}$ is recurrent (with $r \leqslant M$).

PROOF. It suffices to show that for any k,

$$Pr(X_{n+k} \leqslant M \quad \text{for some} \quad n = 0,1,2,\ldots) = 1. \tag{2.6}$$

To this end, define a new process $\{Y_n\}$ by

$$Y_n = \begin{cases} X_{n+k} & \text{if} \quad X_l > M \quad \text{for} \qquad\qquad k \leqslant l \leqslant n+k-1, \\ 0 & \text{if} \quad X_l \leqslant M \quad \text{for some} \quad l, k \leqslant l \leqslant n+k-1. \end{cases} \tag{2.7}$$

The state 0 is thus absorbing for the Y_n process, and this absorption, if it occurs, implies that the event in (2.6) has occurred also.

LEMMA. *Let $\mathscr{F}_n$ be the Borel field of Ω sets generated by $\{X_i, i \leqslant n + k\}$. Then $\{(-Y_n, \mathscr{F}_n)\}$ is a semi-martingale.*

PROOF. It is obvious that $\mathscr{F}_{n+1} \supset \mathscr{F}_n$, and almost as immediate that $-Y_n$ is $\mathscr{F}_n$ measurable, since $-Y_n$ is a Borel function of $X_k, \ldots, X_{k+n}$ which are all $\mathscr{F}_n$ measurable by definition of $\mathscr{F}_n$. To check (2.1), we use the fact that with probability one,

$$E(-Y_{n+1}|\mathscr{F}_n) = \begin{cases} E(-X_{k+n+1}|\mathscr{F}_n) & \text{if} \quad Y_n > M, \\ 0 & \text{if} \quad Y_n \leqslant M. \end{cases}$$

To see this, note that $\{Y_n > M\} \in \mathscr{F}_n$, so that the function $f(\omega)$ on the right-hand side above is $\mathscr{F}_n$ measurable. It is then necessary to verify that

$$\int_A f(\omega)dP = \int_A -Y_{n+1}(\omega)dP$$

whenever A is in $\mathscr{J}_n$; this may be done by considering separately the integral over $A \cap \{Y_n > M\}$ and over $A \cap \{Y_n \leqslant M\}$ and using (2.7). (Note that if $Y_n \leqslant M$, Y_{n+1} must be 0.) But now on the set $\{Y_n > M\}$ we have $X_{n+k} > M$, so

$$E(-Y_{n+1}|\mathscr{J}_n) \geqslant X_{n+k} = -Y_n \qquad \text{a.s.}$$

by (2.5); clearly if $Y_n \leqslant M$,

$$E(-Y_{n+1}|\mathscr{J}_n) = 0 \geqslant -Y_n \qquad \text{a.s.,}$$

so the $\{-Y_n\}$ process is a semi-martingale.

Returning to the proof of Theorem 1, we have from the semi-martingale convergence theorem that $P(\lim Y_n \text{ exists}) = 1$, since the $-Y_n$ are bounded above by 0. But if the limit is greater than M, we must have X_n converging to the same value; this is a.s. not the case by (2.2). Moreover, it is impossible for $0 < Y_n \leqslant M$ to hold for more than one value of n; therefore, with probability one we have $Y_n \to 0$. But $Y_n \to 0$ implies that $Y_n = 0$ for some value of n (and all larger values), and this in turn implies the truth of (2.6).

Remark. Since $\{-Y_n\}$ is a semi-martingale, $E(Y_n)$ is a decreasing sequence. It follows from Fatou's lemma that

$$Pr(\limsup_{n \to \infty} Y_n < \infty) = 1,$$

and with the aid of slight additional assumptions about the process, (2.3) can be deduced. Thus the semi-martingale convergence theorem has been used in a rather superficial way; it seems, however, to play a more essential role in the next result.

Theorem 2.1 will be useful in proving recurrence, and the next result for transience. Suppose that $\{X_n\}$ is a real stochastic process such that

$$\begin{gathered} 0 \leqslant X_n < T < \infty, \\ P(\limsup_{n \to \infty} X_n = T) = 1. \end{gathered} \tag{2.2'}$$

Theorem 2.2. *Suppose there exists $M < T$ such that*

$$E(X_{n+1}|X_n = x, X_{n-1} = x_1, \ldots, X_0 = x_n) \geqslant x \qquad \text{a.s.} \tag{2.8}$$

whenever $x \geqslant M$ and for all n. Then

$$P(\lim_{n \to \infty} X_n = T) = 1. \tag{2.9}$$

Proof. We define the auxiliary process

$$Y_n = \max(X_n, M). \tag{2.10}$$

Then with probability one,

$$E(Y_{n+1}|\mathscr{F}_n) = E(\max(X_{n+1}, M)|\mathscr{F}_n) \geqslant \max(E(X_{n+1}|\mathscr{F}_n), M),$$

where now $\mathscr{F}_n$ is the Borel field generated by $\{X_i, i \leqslant n\}$. Thus on the Ω set $\{X_n < M\}$,

$$E(Y_{n+1}|\mathscr{F}_n) \geqslant M = Y_n \qquad \text{a.s.},$$

while if $X_n \geqslant M$, we have

$$E(X_{n+1}|\mathscr{F}_n) \geqslant X_n = Y_n \qquad \text{a.s.}$$

by (2.8). Therefore, $E(Y_{n+1}|\mathscr{F}_n) \geqslant Y_n$ for almost all ω, so that $\{(Y_n, \mathscr{F}_n)\}$ is a semi-martingale. But then $\lim Y_n$ exists a.s. by the convergence theorem; from (2.2′) and (2.10) the limit must be T, and (2.9) follows.

III. Recurrence of Real Processes

It is now very easy to prove a theorem under restrictive hypotheses which illustrates how the results of Section II are applied. Let $\{X_n\}$ be a Markov process on the non-negative real line with the stationary transition probability function

$$F_x(y) = P(X_{n+1} - X_n \leqslant y | X_n = x) \tag{3.1}$$ [2]

for all n. We assume (2.2); a sufficient condition to insure this would be, for instance,

$$1 - F_x(\varepsilon) \geqslant \varepsilon \quad \text{for some} \quad \varepsilon > 0, \quad \text{all} \quad x \geqslant 0. \tag{3.2}$$

Suppose that there exists $B < \infty$ such that

$$|X_{n+1} - X_n| \leqslant B \qquad \text{a.s.} \tag{3.3}$$

for all n. Denote, as we then surely can,

$$\begin{aligned} \mu(x) &= E[X_{n+1} - X_n | X_n = x] = \int_{-B}^{B} y\, dF_x(y); \\ v(x) &= E[(X_{n+1} - X_n)^2 | X_n = x] = \int_{-B}^{B} y^2\, dF_x(y). \end{aligned} \tag{3.4}$$

[2] Of course, $F_x(y)$ need not be defined for all x but only for a set S such that $X_n \in S$ a.s. for all n. For example, for an ordinary Markov chain S might be the non-negative integers. This remark also applies to the functions defined in (3.4).

THEOREM 3.1. *Let $v(x)$ be bounded away from 0. Suppose that for all large enough x,*

$$\mu(x) \leqslant \frac{\theta v(x)}{2x} \tag{3.5}$$

for some $\theta < 1$. Then $\{X_n\}$ is recurrent. Conversely, if for all large x and a value of $\theta > 1$

$$\mu(x) \geqslant \frac{\theta v(x)}{2x}, \tag{3.6}$$

then $\{X_n\}$ is transient ($X_n \to \infty$ a.s.).

PROOF. For the first part of the theorem we will use Theorem 2.1. The hypothesis (2.5) is not satisfied by $\{X_n\}$, so consider instead $\{Y_n\}$ where

$$Y_n = \log (X_n + 1). \tag{3.7}$$

Clearly this one-to-one bicontinuous transformation of the state space preserves the Markov property and the property of being recurrent; we have only to compute

$$E[Y_{n+1} - Y_n | Y_n = \log (x+1)] = \int_{-B}^{B} [\log (x+1+y) - \log (x+1)]\, dF_x(y).$$

Expanding the integrand we obtain

$$\int_{-B}^{B} \left[\frac{y}{x+1} - \frac{y^2}{2(x+1)^2} + 0(x^{-3})\right] dF_x(y) = \frac{\mu(x)}{x+1} - \frac{v(x)}{2(x+1)^2} + 0(x^{-3}),$$

using the definition (3.4). But upon substituting (3.5), we obtain at once

$$E[Y_{n+1} - Y_n | Y_n = \log (x+1)] \leqslant 0$$

for all x (and hence all Y_n) sufficiently large. From Theorem 2.1 we have the recurrence of $\{Y_n\}$ and so that of $\{X_n\}$.

REMARKS. The choice of the log function in (3.7) is not entirely arbitrary; a little reflection shows that a concave function from $(0, \infty)$ to itself is desired, and trying x^α for $\alpha < 1$ leads to the log. It is conceivable that other transformations would give results for some forms of the function $\mu(x)$ not covered at present, but most functions with smooth behavior at ∞ are already included.

The converse part of Theorem 3.1 is similar; let

$$Y_n' = 1 - (X_n + 1)^{-\alpha} \qquad \text{for} \qquad \alpha > 0. \tag{3.8}$$

Again $\{Y_n'\}$ is a Markov process, and $X_n \to \infty$ if and only if $Y_n' \to 1$. But

$$E[Y_{n+1}' - Y_n' | Y_n' = 1 - (x+1)^{-\alpha}]$$
$$= - \int_{-B}^{B} [(x+1+y)^{-\alpha} - (x+1)^{-\alpha}] \, dF_x(y),$$

and expanding as before we obtain

$$\frac{\alpha\mu(x)}{(x+1)^{1+\alpha}} - \frac{\alpha(1+\alpha)\,v(x)}{2(x+1)^{2+\alpha}} + 0(x^{-3-\alpha}).$$

Substituting (3.6) and choosing $\alpha < \theta - 1$ yields

$$E[Y_{n+1}' - Y_n' | Y_n' = 1 - (x+1)^{-\alpha}] \geqslant 0$$

for all sufficiently large values of x (and so for all values of $y = Y_n'$ in some interval $M \leqslant y < 1$). Then Theorem 2.2 is applicable to $\{Y_n'\}$ and yields the desired conclusion.

EXAMPLE 1. A *generalized random walk* on the non-negative real line will mean a Markov process with the transition law

$$X_{n+1} = \begin{cases} X_n + a & \text{with probability } \varphi(X_n) \\ (X_n - b)^+ & \text{with probability } 1 - \varphi(X_n). \end{cases} \tag{3.9}$$

Here a and b are positive constants, and $\varphi(x)$ is a measurable function with values in $[0, 1]$. (In order to be sure the hypotheses of Theorem 3.1 are satisfied, we assume that $\varphi(x)$ is bounded away from 0.) It is then very easy to conclude from the theorem that $\{X_n\}$ is recurrent in case, for some $\theta < 1$,

$$\varphi(x) \leqslant \frac{b}{a+b} + \frac{\theta}{2x}\frac{ab}{a+b} \tag{3.10}$$

holds for all sufficiently large x, while the opposite inequality with a value $\theta > 1$ guarantees transience. The case $\theta = 1$ is not covered; as we shall see below, in this case recurrence holds even if the right-hand side of (3.10) is further increased by $0(x^{-1-\delta})$, $\delta > 0$. (In the special case

$a = b = 1$ this result reduces to that of Harris mentioned in the introduction.)

The next task is to generalize and sharpen Theorem 3.1. First, what if $\theta = 1$ in (3.5)? Examples show that the process $\{Y_n\}$ defined by (3.7) does not necessarily continue to satisfy the hypothesis of Theorem 2.1, but substituting the second logarithm for the first in (3.7) resolves this difficulty. Next, assumption (3.3) on bounded increments can be replaced by

$$E(|X_{n+1} - X_n|^{2+\varepsilon}|\mathscr{J}_n) \leqslant B < \infty \qquad \text{a.s.} \tag{3.11}$$

for some $\varepsilon > 0$, where $\mathscr{J}_n$ is the Borel field generated by $\{X_i, i \leqslant n\}$ as in Section II. Finally, the Markov property is not essential. We define

$$\begin{aligned} \bar{\mu}(x) &= \text{ess sup}\, E\,[X_{n+1} - X_n|X_n = x, X_{n-1}, \ldots, X_0] \\ \bar{v}(x) &= \text{ess sup}\, E\,[(X_{n+1} - X_n)^2|X_n = x, X_{n-1}, \ldots, X_0] \end{aligned} \tag{3.12}$$

where the sup is over n and over the values of $\{X_i, i \leqslant n - 1\}$. In addition, $\underline{\mu}(x)$ and $\underline{v}(x)$ are defined by replacing sup by inf in (3.12). The finiteness of $\bar{\mu}$ and $\bar{v}$ follows from (3.11), but as an additional assumption we suppose $\underline{v}(x)$ to be bounded away from 0. We then obtain a better result to be proved later:

Theorem 3.2. *Let the non-negative stochastic process $\{X_n\}$ satisfy* (2.2), (3.11), *and for all large x,*

$$\bar{\mu}(x) \leqslant \frac{\underline{v}(x)}{2x} + 0(x^{-1-\delta}), \qquad \delta > 0. \tag{3.13}$$

Then $\{X_n\}$ is recurrent. If instead for some $\theta > 1$,

$$\underline{\mu}(x) \geqslant \frac{\theta\bar{v}(x)}{2x} \tag{3.14}$$

for all large x, then $\{X_n\}$ is transient.

Example 2. An interesting class of Markov processes have the random part of each increment the same; that is, they are of the form

$$X_{n+1} = (X_n + m(X_n) + \xi_{n+1})^+, \tag{3.15}$$

where $m(x)$ is a bounded measurable function, and $\{\xi_n\}$ is a sequence of independent, identically-distributed random variables with mean 0, variance $\sigma^2 \neq 0$ and finite $2 + \varepsilon$ moment for some $\varepsilon > 0$. Then the process is recurrent if $m(x) \leqslant \sigma^2/2x + 0(x^{-1-\delta})$ and transient if $m(x) \geqslant \sigma^2\theta/2x$, $\theta > 1$; in either case the inequality is to hold for all large x.

In fact, this is immediate from the theorem if $\xi_n > -M$ a.s. for some M, since then $m(x)$ and $\mu(x)$ coincide for $x \geqslant M$, as do $v(x)$ and $\sigma^2 - m^2(x)$. Even without this condition on ξ_n, it is easy to show, using the Markov inequality and similar estimates,[3] that $\mu(x) = m(x) + 0(x^{-1-\epsilon})$ and $v(x) = \sigma^2 + 0(x^{-\epsilon})$, and the result again follows from Theorem 3.2. This example could, of course, have been discussed from Theorem 3.1, but the result would possess less precision and generality. Note that if $m(x) = 0$, the process is recurrent; this would prove a one-dimensional recurrence theorem of Chung and Fuchs [1] if it were not necessary to assume the finiteness of $E(|\xi_n|^{2+\epsilon})$.

EXAMPLE 3. Since Theorem 3.2 yields less than the full strength of Chung and Fuchs' recurrence theorem, it is natural to wonder if assumption (3.11) can be relaxed for other processes besides sums of random variables. In lieu of a complete answer to this question, we offer one counter-example: let $\{X_n\}$ be a Markov chain on the non-negative integers with the transition probabilities

$$P_{n,n+1} = 1 - \frac{\alpha}{n}, \qquad P_{n0} = \frac{\alpha}{n} \qquad \text{if} \qquad n > 0;$$

$$P_{01} = 1, \qquad P_{ij} = 0 \qquad \text{otherwise.}$$

We easily calculate that

$$\mu(n) = -\alpha + 1 - \frac{\alpha}{n}, \qquad v(n) = \alpha n + 1 - \frac{\alpha}{n}.$$

Then choosing $\alpha < 2/3$, we ought to have transience if Theorem 3.2 applied. But, in fact, it is easily verified that $\{X_n\}$ is recurrent for every $\alpha > 0$.

REMARKS. We have explicitly considered processes on half-line, and have found criteria for $+\infty$ to be absorbing or reflecting. The results extend easily to processes on the whole line: such processes (with mild conditions) are recurrent if and only if both $+\infty$ and $-\infty$ act as reflecting barriers, and this can be determined from our conditions. Another possible extension consists of changing the hypothesis in Theorem 3.2 that $v(x)$ be of the order of magnitude of a positive constant for large x; this will be discussed elsewhere.

[3] This verification is a little similar to the proof of Theorem 3.2 given below, but simpler.

IV. MULTIVARIATE MARKOV PROCESSES

In this section we do not strive for maximum generality, but merely to illustrate the possibilities of our methods. Accordingly, suppose that

$$\mathbf{X_n} = \begin{pmatrix} X_n^{(1)} \\ \vdots \\ X_n^{(s)} \end{pmatrix}, \qquad n = 0, 1, 2, \ldots, ^4$$

are random vectors forming a Markov process with the transition probability function

$$F(y_1, \ldots, y_s; \mathbf{x}) = P(X_{n+1}^{(i)} - X_n^{(i)} \leqslant y_i, \quad i = 1, \ldots, s \,|\, \mathbf{X_n} = \mathbf{x}) \qquad (4.1)$$

independent of n. For simplicity in the proofs we make the assumption that for some $B < \infty$

$$|X_{n+1}^{(i)} - X_n^{(i)}| \leqslant B \qquad \text{a.s.} \qquad i = 1, \ldots, s, \qquad (4.2)$$

for all n, although as in the previous section (4.2) could be relaxed to a moment condition at the expense of some labor. We shall use the notations

$$\begin{gathered} E[\mathbf{X_{n+1}} - \mathbf{X_n} | \mathbf{X_n} = \mathbf{x}] = \boldsymbol{\mu}(\mathbf{x}) \\ E[(\mathbf{X_{n+1}} - \mathbf{X_n})(\mathbf{X_{n+1}} - \mathbf{X_n})^T | \mathbf{X_n} = \mathbf{x}] = \mathbf{v}(\mathbf{x}) = \{v_{ij}(\mathbf{x})\}. \end{gathered} \qquad (4.3)$$

The idea is to define the process

$$R_n = ||\mathbf{X_n}|| = \left\{ \sum_{i=1}^{s} (X_n^{(i)})^2 \right\}^{1/2} \qquad (4.4)$$

and apply the results of Section III to $\{R_n\}$. The requirement (2.2) that $\limsup R_n = \infty$ a.s. is assumed ad hoc; this is usually easy to verify in particular cases. The process $\{\mathbf{X_n}\}$ is said to be recurrent or transient according as $\{R_n\}$ is.

THEOREM 4.1. *Under the above hypotheses, suppose*

$$\mathbf{v}(\mathbf{x}) = \mathbf{v} + 0(||\mathbf{x}||^{-\delta}) \qquad \text{for some} \qquad \delta > 0, \qquad (4.5)$$

where the matrix $[v_{ij}] = \mathbf{v}$ *is positive definite. Then* $\{\mathbf{X_n}\}$ *is recurrent provided that*

$$\mathbf{x}^T \mathbf{v}^{-1} \boldsymbol{\mu}(\mathbf{x}) \leqslant \frac{(2 - s)}{2} + 0(||\mathbf{x}||^{-\delta}) \qquad (4.6)$$

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[4] Boldface letters denote vectors or matrices throughout this section. Vectors will be considered as column vectors unless written with a superscript T (for transpose).

for all sufficiently large $||\mathbf{x}||$, *while in case for large* $||\mathbf{x}||$

$$\mathbf{x}^T \mathbf{v}^{-1} \boldsymbol{\mu}(\mathbf{x}) \geqslant \frac{2-s}{2} + \varepsilon, \qquad \varepsilon > 0, \tag{4.7}$$

then $\{\mathbf{X_n}\}$ *is transient.*

PROOF. In order to apply Theorem 3.2 to the process $\{R_n\}$, it is necessary to calculate

$$E[R_{n+1} - R_n | R_n = ||\mathbf{x}||]$$
$$= \int_{-B}^{B} \cdots \int_{-B}^{B} \left\{ \left(\sum_{i=1}^{s} (x_i + y_i)^2 \right)^{1/2} - \left(\sum_{i=1}^{s} x_i^2 \right)^{1/2} \right\} dF(y_1, \ldots, y_s; \mathbf{x}).$$

Expanding the integrand and using (4.3) we obtain

$$E[R_{n+1} - R_n | R_n = ||\mathbf{x}||] = \frac{1}{||\mathbf{x}||} \mathbf{x}^T \boldsymbol{\mu}(\mathbf{x}) + \frac{\sum_{i=1}^{s} v_{ii}(\mathbf{x})}{2||\mathbf{x}||} - \frac{\frac{1}{2}\sum_{i=1}^{s} x_i^2 v_{ii}(\mathbf{x}) + \sum_{i<j} x_i x_j v_{ij}(\mathbf{x})}{||\mathbf{x}||^3} + 0(||\mathbf{x}||^{-2}).$$

Now we use (4.5), and consider for the moment the special case when $\mathbf{v}$ is the identity matrix $\mathbf{I}$. This gives

$$E[R_{n+1} - R_n | R_n = ||\mathbf{x}||] = \frac{1}{||\mathbf{x}||} \mathbf{x}^T \boldsymbol{\mu}(\mathbf{x}) + \frac{s-1}{2||\mathbf{x}||} + 0(||\mathbf{x}||^{-\delta-1}). \tag{4.8}$$

A similar calculation, still assuming $\mathbf{v} = \mathbf{I}$, yields

$$E[(R_{n+1} - R_n)^2 | R_n = ||\mathbf{x}||] = 1 + 0(||\mathbf{x}||^{-\delta}). \tag{4.9}$$

Combining (4.6) or (4.7) with (4.8) and comparing the result with (4.9) shows that conditions (3.13) or (3.14), respectively, are satisfied for the $\{R_n\}$ process, and the desired conclusion then follows from Theorem 3.2.

Now if $\mathbf{v}$ is any positive definite (and automatically symmetric) matrix, let $\mathbf{P}$ be a non-singular matrix such that $\mathbf{P}\mathbf{v}\mathbf{P}^T = \mathbf{I}$, and consider the process $\{\mathbf{Y_n}\}$, where $\mathbf{Y_n} = \mathbf{P}\mathbf{X_n}$. Let $\boldsymbol{\mu}^*$ and $\mathbf{v}^*$ denote the quantities of (4.3) with the $\{\mathbf{Y_n}\}$ process replacing $\{\mathbf{X_n}\}$. Then we note that if $\mathbf{y} = \mathbf{P}\mathbf{x}$,

$$\mathbf{v}^*(\mathbf{y}) = \mathbf{P}\mathbf{v}(\mathbf{x})\,\mathbf{P}^T = \mathbf{I} + 0(||\mathbf{x}||^{-\delta}) = \mathbf{I} + 0(||\mathbf{y}||^{-\delta}),$$

and also that

$$\mathbf{y}^T \boldsymbol{\mu}^*(\mathbf{y}) = (\mathbf{P}\mathbf{x})^T \mathbf{P}\boldsymbol{\mu}(\mathbf{x}) = \mathbf{x}^T \mathbf{P}^T \mathbf{P}\boldsymbol{\mu}(\mathbf{x}) = \mathbf{x}^T \mathbf{v}^{-1} \boldsymbol{\mu}(\mathbf{x}).$$

The case of the theorem which we have already proved applies to $\{\mathbf{Y_n}\}$, and the general theorem as stated is the result.

EXAMPLE 4. Suppose the process $\{\mathbf{X_n}\}$ consists of sums of independent, identically distributed random vectors with bounded length and a genuinely s-dimensional distribution (ensuring the definiteness of the covariance matrix $\mathbf{v}$). If the means are not 0, the process is easily seen to be transient, so take $\boldsymbol{\mu} = 0$. Then if $s = 1$ or 2 the process is recurrent, while if $s > 2$, it is transient. This is a restricted version of Chung and Fuchs' theorem [1]. Note that if $s > 2$, it is possible to restore recurrence by adding a non-random drift *toward the origin* of the order of $||\mathbf{x}||^{-1}$; if $s = 1$ (Example 2) recurrence may survive the presence of such a drift *away* from the origin, while when $s = 2$ the drift away must be of smaller order of magnitude.

EXAMPLE 5. Here we will consider an s-dimensional *random walk.* By this we mean a Markov chain on the lattice points $\mathbf{x}$ of s-dimensional space whose transition-probability matrix is of the form

$$P_{\mathbf{x},\,\mathbf{x}+\hat{\mathbf{u}}_i} = P_i(\mathbf{x}), \qquad P_{\mathbf{x},\,\mathbf{x}-\hat{\mathbf{u}}_i} = Q_i(\mathbf{x}); \qquad P_{\mathbf{x},\mathbf{y}} = 0 \quad \text{otherwise}, \tag{4.10}$$

where $\hat{\mathbf{u}}_i$ is a unit vector in the direction of the positive i'th coordinate axis. Then clearly

$$\mathbf{v}(\mathbf{x}) = \operatorname{diag}(P_i(\mathbf{x}) + Q_i(\mathbf{x})).$$

This means that the limiting matrix $\mathbf{v}$ is also diagonal (if it exists); call the entries d_i. It is again easy to calculate that

$$\mathbf{x}^T \mathbf{v}^{-1} \boldsymbol{\mu}(\mathbf{x}) = \sum_{i=1}^{s} x_i \frac{P_i(\mathbf{x}) - Q_i(\mathbf{x})}{d_i}.$$

Now if the transition probabilities have a fairly simple form, it is not hard to see whether our results can be applied. For example, suppose that

$$\begin{aligned} P_i(\mathbf{x}) &= P_i + \frac{\alpha_i}{x_i - \xi_i} + 0(||\mathbf{x}||^{-\delta-1}), \qquad P_i > 0, \\ Q_i(\mathbf{x}) &= P_i - \frac{\alpha_i}{x_i - \xi_i} + 0(||\mathbf{x}||^{-\delta-1}). \end{aligned} \tag{4.11}$$

Here $\xi_i \neq 0 \bmod (1)$, and, of course, $P_i(\mathbf{x})$ and $Q_i(\mathbf{x})$ are positive and together sum to one. Changing to the point with coordinates $\{\xi_i\}$ as origin, (4.11) implies that (4.5) holds and that

$$\mathbf{v} = \operatorname{diag}(2P_i), \qquad \mathbf{x}^T \mathbf{v}^{-1} \boldsymbol{\mu}(\mathbf{x}) = \sum_{i=1}^{s} \frac{\alpha_i}{P_i} + 0(||\mathbf{x}||^{-\delta}). \tag{4.12}$$

Applying Theorem (4.1), we obtain the following result:

THEOREM 4.2. *If the transition probabilities of an s-dimensional random walk are positive and of the form* (4.11), *then the walk is recurrent in case*

$$2 \sum_{i=1}^{s} \frac{\alpha_i}{P_i} \leqslant 2 - s,$$

and transient otherwise.

In case $s = 1$, this is essentially the result of Harris again. If for any s we put $P_i = 1/2s$ and $\alpha_i = -\varepsilon/2s$ we have recurrence if and only if $\varepsilon \geqslant (s-2)/2s$. This (if the higher order terms $0(||\mathbf{x}||^{-\delta-1})$ be set equal to zero) is essentially Gillis' result in [9] with his conjecture in the case of equality decided affirmatively.[5]

One other class of random walks is also easily seen to satisfy our hypotheses; suppose that

$$\begin{aligned} P_i(\mathbf{x}) &= P_i\left[1 + \frac{\alpha x_i}{||\mathbf{x}||^2} + 0(||\mathbf{x}||^{-1-\delta})\right], \qquad P_i > 0, \\ Q_i(\mathbf{x}) &= P_i\left[1 - \frac{\alpha x_i}{||\mathbf{x}||^2} + 0(||\mathbf{x}||^{-1-\delta})\right]. \end{aligned} \tag{4.13}$$

Applying Theorem 4.1 as before, we find that *a random walk satisfying* (4.13) *is recurrent if and only if* $2\alpha \leqslant 2 - s$. This example seems to be more deserving than (4.11) of being called a "centrally biased" random walk, at least when each $P_i = 1/2s$.

Appendix: Proof of Theorem 3.2.

Without loss of generality it is possible to operate with conditional probability distributions

$$P[X_{n+1} - X_n \leqslant y | X_n = x, \ X_{n-i} = x_i, \ i = 1, \ldots, n] = F(y; x, x_i). \tag{A.1}$$

[5] This conjecture has also been proved by Flatto [10], by Erdos, and by Gillis himself (both unpublished).

(This matter is fully discussed in [2, Chapter II].) We then have

$$\mu(x; x_i) = E[X_{n+1} - X_n | X_n = x, \quad X_{n-1} = x_i] = \int_{-\infty}^{\infty} y\, dF(y; x, x_i); \quad (A.2)$$

$$v(x; x_i) = E[(X_{n+1} - X_n)^2 | X_n = x, \quad X_{n-i} = x_i] = \int_{-\infty}^{\infty} y^2\, dF(y; x, x_i).$$

It is also convenient to state here a simple fact used below: if $F(y)$ is a distribution function whose absolute $2 + \varepsilon$ moment does not exceed B, then

$$\int_A^{\infty} y^\beta\, dF(y) \leqslant \int_A^{\infty} y^\beta \left(\frac{y}{A}\right)^{2+\varepsilon-\beta} dF(y) \leqslant BA^{\beta-2-\varepsilon} \quad (A.3)$$

for every $A > 0$ and every $0 \leqslant \beta \leqslant 2 + \varepsilon$. Of course, a similar estimate holds on the negative half-line.

The idea of the proof of the theorem is the same as in the case of Theorem 3.1; we show that the process $\{Y_n\}$ defined by

$$Y_n = f(X_n) = \log\log(X_n + e) \quad (A.4)$$

satisfies the hypothesis of Theorem 2.1 provided (3.13) holds. It is necessary to calculate

$$E[Y_{n+1} - Y_n | Y_n = f(x), Y_{n-i}] = \int_{-x}^{\infty} [f(x+y) - f(x)]\, dF(y; x, x_i). \quad (A.5)$$

(The lower limit is $-x$ since $X_{n+1} \geqslant 0$.) The right-hand side of (A.5) can be written as

$$\left\{ \int_{-x}^{-x^\alpha} f(x+y) + \int_{-x^\alpha}^{+x^\alpha} [f(x+y) - f(x)] + \int_{x^\alpha}^{\infty} f(x+y) \right\} dF(y; x, x_i) -$$

$$f(x) \left\{ \int_{-x}^{-x^\alpha} + \int_{x^\alpha}^{\infty} \right\} dF(y; x, x_i), \quad (A.6)$$

and we shall see that this expression is non-positive for all large x.

First choose $\alpha < 1$ so that $3 \geqslant d_1 = (2 + \varepsilon)\alpha > 2$, where the positive number ε is that appearing in (3.11). Then inequality (A.3) shows that the last two terms in (A.6) are bounded by

$$2f(x)Bx^{-d_1} = 0(x^{-d_2}) \qquad \text{for} \qquad d_2 < d_1;$$

we may take $d_2 > 2$. The first term in (A.6) is equally simple, for in the range of that integral $0 \leqslant f(x + y) \leqslant f(x)$ so the estimate is the same as for the fourth term. As for the third term, for all large x and $y \geqslant x^\alpha$ we have

$$f(x + y) = \log\log(x + y + e) \leqslant (x + y)^{d_2 - d_1} \leqslant x^{d_2 - d_1} + y^{d_2 - d_1}$$

so that

$$\int_{x^\alpha}^{\infty} f(x + y)\, dF(y; x, x_i) \leqslant \int_{x^\alpha}^{\infty} (x^{d_2 - d_1} + y^{d_2 - d_1})\, dF(y; x, x_i).$$

To this expression (A.3) can be applied, and the result is that the third term too in (A.6) is $0(x^{-d_2})$.

It remains to estimate the second term in (A.6), which can be done with the aid of the expansion

$$f(x + y) - f(x) \tag{A.7}$$

$$= \frac{y}{(x + e)\log(x + e)} - \frac{y^2}{2}\,\frac{\log(x + e) + 1}{(x + e)^2\log^2(x + e)} + \frac{y^3}{3!} f'''(x + \theta y).$$

In the range of integration $(-x^\alpha, x^\alpha)$, $f'''(x + \theta y) = 0(x^{-3})$. But then

$$\int_{-x^\alpha}^{x^\alpha} \frac{y^3}{3!} f'''(x + \theta y)\, dF(y; x, x_i) \leqslant (x^\alpha)^{1-\varepsilon} \int_{-x^\alpha}^{x^\alpha} \frac{|y|^{2+\varepsilon}}{3!} |f'''| dF \leqslant$$
$$0(x^{-3+\alpha(1-\varepsilon)}) = 0(x^{-d_3})$$

where again $d_3 > 2$. We are left with

$$\int_{-x^\alpha}^{x^\alpha} \left\{\frac{y}{(x + e)\log(x + e)} - \frac{y^2}{2}\,\frac{\log(x + e) + 1}{(x + e)^2\log^2(x + e)}\right\} dF(y; x, x_i) + 0(x^{-d_4}) \tag{A.8}$$

as our estimate of (A.5), where $d_4 = \min(d_2, d_3) > 2$. If the integral in (A.8) were over $(-x, \infty)$ we would have

$$\frac{\mu(x, x_i)}{(x + e)\log(x + e)} - \frac{v(x, x_i)}{2}\,\frac{\log(x + e) + 1}{(x + e)^2\log^2(x + e)} + 0(x^{-d_4}). \tag{A.9}$$

In fact, the error involved in the change of the limits can be estimated from (A.3), and is thus seen to be small enough to be incorporated into the $0(x^{-d_4})$. Therefore, (A.9) is a valid expression for the expected value in (A.5).

The proof is now finished: if assumption (3.13) is substituted into (A.9), the result is

$$E[Y_{n+1} - Y_n | Y_n = f(x), Y_{n-i}] \leqslant$$
$$\frac{1}{(x+e)\log(x+e)}\left\{\bar{\mu}(x) - \frac{\underline{v}(x)}{2(x+e)}\left(1 + \frac{1}{\log(x+e)}\right)\right\} + 0(x^{-d_4}) \leqslant$$
$$\frac{-\underline{v}(x)}{2x^2 \log x} + 0(x^{-d_4}), \tag{A.10}$$

which is negative for large x since $d_4 > 2$ and $\underline{v}(x)$ is bounded from 0. Thus by Theorem 2.1, $\{Y_n\}$ and hence $\{X_n\}$ is a recurrent process. Much the same sort of argument (using (3.8) to define the transformed process) proves the converse part of Theorem 3.2, but further unpleasant details will be omitted.

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