13.10.21. ОМСС Пунгускова. д/з от семинара 6.

Задача: записать ур-е Эйлера в виде Громеко-Лэмба.

Решение: Дано: $\rho \frac{d\bar{v}}{dt} = -\operatorname{grad} p + \rho \bar{F}$

Хотим: $\frac{\partial \bar{v}}{\partial t} + \frac{1}{2}\operatorname{grad} \bar{v}^2 + [\operatorname{rot}\bar{v}; \bar{v}] = \bar{F} - \frac{1}{\rho}\operatorname{grad} p$

$\Rightarrow$ достаточно док-ть, что:

$$\frac{d\bar{v}}{dt} \overset{?}{=} \frac{\partial \bar{v}}{\partial t} + \frac{1}{2}\operatorname{grad}(\bar{v}^2) + [\operatorname{rot}\bar{v}; \bar{v}].$$

Но $\frac{d\bar{v}}{dt} = \frac{\partial \bar{v}}{\partial t} + \sum_i \frac{\partial \bar{v}}{\partial x_i} \underbrace{\dot{x}_i}_{v_i}$

$\Rightarrow$ $\boxed{\text{Хотим: } \sum_i \frac{\partial \bar{v}}{\partial x_i} v_i \overset{?}{=} \frac{1}{2}\operatorname{grad}(\bar{v}^2) + [\operatorname{rot}\bar{v}; \bar{v}].}$

Имеем: 1) $\sum_i \frac{\partial \bar{v}}{\partial x_i} v_i = \begin{pmatrix} \frac{\partial v_1}{\partial x} v_1 + \frac{\partial v_1}{\partial y} v_2 + \frac{\partial v_1}{\partial z} v_3; \\ \frac{\partial v_2}{\partial x} v_1 + \frac{\partial v_2}{\partial y} v_2 + \frac{\partial v_2}{\partial z} v_3; \\ \frac{\partial v_3}{\partial x} v_1 + \frac{\partial v_3}{\partial y} v_2 + \frac{\partial v_3}{\partial z} v_3 \end{pmatrix}$

2) $\bar{v}^2 = v_1^2 + v_2^2 + v_3^2$

$\Rightarrow \frac{1}{2}\operatorname{grad}(\bar{v}^2) = \begin{pmatrix} v_1 \cdot \frac{\partial v_1}{\partial x} + v_2 \cdot \frac{\partial v_2}{\partial x} + v_3 \cdot \frac{\partial v_3}{\partial x}; \\ v_1 \cdot \frac{\partial v_1}{\partial y} + v_2 \cdot \frac{\partial v_2}{\partial y} + v_3 \cdot \frac{\partial v_3}{\partial y}; \\ v_1 \cdot \frac{\partial v_1}{\partial z} + v_2 \cdot \frac{\partial v_2}{\partial z} + v_3 \cdot \frac{\partial v_3}{\partial z} \end{pmatrix}$

3) $\operatorname{rot}\bar{v} = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ v_1 & v_2 & v_3 \end{vmatrix} = \left(\frac{\partial v_3}{\partial y} - \frac{\partial v_2}{\partial z}; \frac{\partial v_1}{\partial z} - \frac{\partial v_3}{\partial x}; \frac{\partial v_2}{\partial x} - \frac{\partial v_1}{\partial y} \right)$

4) $[\operatorname{rot}\bar{v}; \bar{v}] = \begin{bmatrix} \frac{\partial v_3}{\partial y} - \frac{\partial v_2}{\partial z}; & \frac{\partial v_1}{\partial z} - \frac{\partial v_3}{\partial x}; & \frac{\partial v_2}{\partial x} - \frac{\partial v_1}{\partial y} \\ v_1; & v_2; & v_3 \end{bmatrix} = \begin{pmatrix} v_3\left(\frac{\partial v_1}{\partial z} - \frac{\partial v_3}{\partial x}\right) - v_2\left(\frac{\partial v_2}{\partial x} - \frac{\partial v_1}{\partial y}\right) \\ v_1\left(\frac{\partial v_2}{\partial x} - \frac{\partial v_1}{\partial y}\right) - v_3\left(\frac{\partial v_3}{\partial y} - \frac{\partial v_2}{\partial z}\right) \\ v_2\left(\frac{\partial v_3}{\partial y} - \frac{\partial v_2}{\partial z}\right) - v_1\left(\frac{\partial v_1}{\partial z} - \frac{\partial v_3}{\partial x}\right) \end{pmatrix}$

5) $\Rightarrow \frac{1}{2}\operatorname{grad}(\bar{v}^2) + [\operatorname{rot}\bar{v}; \bar{v}] = \begin{pmatrix} \left(v_1\frac{\partial v_1}{\partial x} + \cancel{v_2\frac{\partial v_2}{\partial x}} + \cancel{v_3\frac{\partial v_3}{\partial x}}\right) + \left(v_3\frac{\partial v_1}{\partial z} - \cancel{v_3\frac{\partial v_3}{\partial x}} - \cancel{v_2\frac{\partial v_2}{\partial x}} + v_2\frac{\partial v_1}{\partial y}\right) \\ \left(\cancel{v_1\frac{\partial v_1}{\partial y}} + v_2\frac{\partial v_2}{\partial y} + \cancel{v_3\frac{\partial v_3}{\partial y}}\right) + \left(v_1\frac{\partial v_2}{\partial x} - \cancel{v_1\frac{\partial v_1}{\partial y}} - \cancel{v_3\frac{\partial v_3}{\partial y}} + v_3\frac{\partial v_2}{\partial z}\right) \\ \left(\cancel{v_1\frac{\partial v_1}{\partial z}} + \cancel{v_2\frac{\partial v_2}{\partial z}} + v_3\frac{\partial v_3}{\partial z}\right) + \left(v_2\frac{\partial v_3}{\partial y} - \cancel{v_2\frac{\partial v_2}{\partial z}} - \cancel{v_1\frac{\partial v_1}{\partial z}} + v_1\frac{\partial v_3}{\partial x}\right) \end{pmatrix} = \begin{pmatrix} v_1\frac{\partial v_1}{\partial x} + v_3\frac{\partial v_1}{\partial z} + v_2\frac{\partial v_1}{\partial y} \\ v_2\frac{\partial v_2}{\partial y} + v_1\frac{\partial v_2}{\partial x} + v_3\frac{\partial v_2}{\partial z} \\ v_3\frac{\partial v_3}{\partial z} + v_2\frac{\partial v_3}{\partial y} + v_1\frac{\partial v_3}{\partial x} \end{pmatrix}$

$\parallel$

$\sum_i \frac{\partial \bar{v}}{\partial x_i} v_i$ чтд.