1) $y_0(\tau) = c \cdot \log(1+\tau)$, $\tau \in [0;T]$; $c \in (0;\infty)$

a) Calculate the bond prices $B_0(\tau_i)$; $\tau_i \in \{\tau_1 \dots \tau_5\} = \{1 \dots 5\}$.

$$B_0(\tau_i) = e^{-y_0(\tau_i)\cdot\tau_i} = e^{-c\cdot\log(1+\tau_i)\cdot\tau_i} = \boxed{(1+\tau_i)^{-c\tau_i}}; \ \tau_i \in \{1 \dots 5\}$$

б) Calculate the initial Libor rates $L_0(\tau_i; \tau_{i+1})$; $\tau_i \in \{\tau_1 \dots \tau_4\}$

$$L_t(\tau, \tau') = \frac{B_t(\tau) - B_t(\tau')}{(\tau' - \tau)\cdot B_t(\tau')}$$

$$L_0(\tau_i; \tau_{i+1}) = \frac{B_0(\tau_i) - B_0(\tau_{i+1})}{\underbrace{(\tau_{i+1} - \tau_i)}_{=1}\cdot B_0(\tau_{i+1})} = \frac{B_0(\tau_i)}{B_0(\tau_{i+1})} - 1 = \boxed{\frac{(1+\tau_i)^{-c\tau_i}}{(1+\tau_{i+1})^{-c\tau_{i+1}}} - 1}$$

c) Calculate continuously compounded forward rates $f_0(\tau_i)$; $\tau_i \in \{\tau_1 \dots \tau_5\}$.

$$B_0(\tau) = e^{-\int_0^\tau f_0(s)ds}$$

$$\Rightarrow -\ln B_0(\tau) = \int_0^\tau f_0(s)ds$$

$$\Rightarrow f_0(\tau) = -\frac{\partial}{\partial \tau} \ln B_0(\tau) = -\frac{\partial}{\partial \tau}(-c\log(1+\tau)\cdot\tau) = c\log(1+\tau) + \frac{c\tau}{1+\tau}$$

$$\Rightarrow f_0(\tau_i) = c\left(\log(1+\tau_i) + \frac{\tau_i}{1+\tau_i}\right); \ \tau_i \in \{\tau_1 \dots \tau_5\}$$

d) For a deterministic no-arbitrage time evolution, determine the short rate r_t which explains the initially observed yields.

$$B_0(\tau) = E^Q\left[e^{-\int_0^\tau r_s ds}\right] = e^{-\int_0^\tau r_s ds}$$

$\uparrow$ the deterministic evolution

$$B_0(\tau) = (1+\tau)^{-c\tau}$$

$$\Rightarrow -\int_0^\tau r_s ds = \ln\left((1+\tau)^{-c\tau}\right) = -c\log(1+\tau)\tau$$

$$\Rightarrow \int_0^\tau r_s ds = \log(1+\tau)\cdot\tau$$

$$\Rightarrow r_\tau = (\log(1+\tau)\cdot\tau)' = \boxed{\log(1+\tau) + \frac{\tau}{1+\tau}}$$

2.
$$\begin{cases} dr_t = \theta dt + \sigma^r dW_t \\ r_0 = r_0^* \end{cases} \Rightarrow r_t = r_0^* + \theta t + \sigma^r W_t \sim N(r_0^* + \theta t, (\sigma^r)^2 \cdot t)$$

$$S_T = e^{r_T}$$

a) Calculate the expectation $E^Q(S_T)$ of S_T with respect to the spot measure Q.

$E^Q[S_T] = E^Q[e^{r_T}]$, где $r_T \sim N(\underbrace{r_0^* + \theta T}_{=\mu_\xi}, \underbrace{(\sigma^r)^2 \cdot T}_{=\sigma_\xi^2})$.

$A\ Ee^{\xi} = e^{\mu_\xi + \frac{\sigma_\xi^2}{2}}$

$$\Rightarrow \boxed{E^Q[S_T] = e^{r_0^* + \theta T + \frac{(\sigma^r)^2 \cdot T}{2}}}$$

б) Calculate the expectation $E^{Q^T}(S_T)$ of S_T with respect to the forward measure Q^T.

Надо найти плотность замены меры и поменять винер процесс

Имеем: $r_t = \theta dt + \sigma^r dW_t$

$$\Rightarrow B_t(\tau) = e^{\underbrace{-\theta \int_t^\tau (\tau - u) du}_{= \theta \frac{(\tau-u)^2}{2} \Big|_t^\tau = -\theta \frac{(t-\tau)^2}{2}} + \frac{\sigma^2}{2} \frac{(\tau - t)^3}{3} - (\tau - t) r_t}$$ — было в прошлых, пример.

$$\Rightarrow B_t(\tau) = e^{-\theta \frac{(t-\tau)^2}{2} + \frac{\sigma^2}{2} \frac{(\tau-t)^3}{3} - (\tau - t) r_t}$$

Хотим найти $dB_t(\tau)$:

ну $f(t,x) = e^{-\theta \frac{(t-\tau)^2}{2} + \frac{\sigma^2}{2} \frac{(\tau-t)^3}{3} - (\tau-t)x}$

$$\Rightarrow dB_t(\tau) = \left(-\theta(t-\tau) - \frac{\sigma^2}{2}(\tau-t)^2 + r_t\right) B_t(\tau) dt - (\tau - t) B_t(\tau) \cdot \underbrace{dr_t}_{= \theta dt + \sigma^r dW_t} + \frac{(\sigma^r)^2}{2} \cdot (\tau-t)^2 B_t(\tau) dW_t =$$

$$= B_t(\tau)\left(\cancel{-\theta(t-\tau)} - \frac{\sigma^2}{2}(\tau-t)^2 + r_t \cancel{-(\tau-t)\theta} + \frac{(\sigma^r)^2}{2} \cdot (\tau-t)^2\right) dt - (\tau - t) B_t(\tau) \sigma^r dW_t =$$

$$= B_t(\tau)\left(r_t dt - (\tau - t)\sigma^r dW_t\right)$$

$$\Rightarrow \begin{cases} dB_t(\tau) = B_t(\tau)(r_t dt - (\tau-t)\sigma^r dW_t \\ dB_t = r_t B_t dt \end{cases}$$

$L_t = \frac{B_t(\tau)}{B_t}$; $dL_t = ?$

$f\left(\frac{x}{y}\right) = \frac{x}{y}$; $f_x' = \frac{1}{y}$ $\quad X_t = B_t(\tau)$

$f_y' = \frac{-x}{y^2}$ $\quad Y_t = B_t$

$d\left(\frac{B_t(\tau)}{B_t}\right) = f'_x dx_t + f'_y dy_t + \frac{1}{2} f''_{xx} \underbrace{(dx_t)^2}_{0} + f''_{xy} \underbrace{dx_t \cdot dy_t}_{0} + \frac{1}{2} f''_{yy} \underbrace{(dy_t)^2}_{0} =$

$= \frac{1}{B_t} \cdot B_t(\tau)(r_t dt - (\tau - t)\sigma^r dW_t) - \frac{B_t(\tau)}{B_t^2} \cdot r_t B_t dt =$

$= \frac{B_t(\tau)}{B_t}(\cancel{r_t dt} - (\tau - t)\sigma^r dW_t - \cancel{r_t dt}) =$

$= \frac{B_t(\tau)}{B_t}(-(\tau - t)\sigma^r) dW_t$

$\Rightarrow dL_t = \underbrace{-L_t \cdot (\tau - t)\sigma^r}_{-\mu_t} dW_t$

$\Rightarrow$ по т. Гирсанова: $\tilde{W}_t^{Q^T} = W_t + \int_0^t \mu_s ds$

$\Rightarrow d\tilde{W}_t^{Q^T} = dW_t + \underbrace{\mu_t}_{(\tau-t)\sigma^r} dt$

$\Rightarrow dW_t = d\tilde{W}_t^{Q^T} - (\tau - t)\sigma^r dt$

$\Rightarrow dr_t = \theta dt + \sigma^r dW_t = \theta dt + \sigma^r (d\tilde{W}_t^{Q^T} - (\tau - t)\sigma^r dt) =$

$= (\theta - (\tau - t)(\sigma^r)^2) dt + \sigma^r \cdot d\tilde{W}_t^{Q^T}$

$\Rightarrow r_t = r_0^* + \underbrace{\int_0^t (\theta - (\tau - s)(\sigma^r)^2) ds}_{} + \sigma^r \tilde{W}_t^{Q^T}$

$= \theta t + (\sigma^r)^2 \cdot \frac{(s-\tau)^2}{2}\Big|_0^t$

$= \theta t + (\sigma^r)^2 ((t-\tau)^2 - \tau^2)$

$= \theta t + (\sigma^r)^2 t (t - 2\tau)$

$\tau = T$

$\Rightarrow$ в мере Q^T: $r_T \sim N(r_0^* + \theta T + (\sigma^r)^2 T(T - 2\tau); (\sigma^r)^2 \cdot T)$

$\Rightarrow \boxed{E^{Q^T}[S^T] = e^{\mu_S + \frac{\sigma_S^2}{2}} = e^{r_0^* + \theta T - (\sigma^r)^2 T^2 + \frac{(\sigma^r)^2 T}{2}}}$

c) Calculate the forward-futures spread at time $t=0$.

Spread $= E^{Q^T}[S_T] - E^{Q}[S_T] = e^{r_0^* + \theta T + \frac{(\sigma^r)^2 T}{2}} \cdot (e^{-(\sigma^r)^2 T^2} - 1)$

This spread is explained by the non-deterministic nature of r_t and $B_t(\tau)$ and that this model (Ho-Lee) doesn't fit into the time evolution.

(3) $\begin{cases} df_t(\tau) = \alpha_t(\tau)dt + \sigma_t(\tau)dW_t \\ f_0(\tau) = \dfrac{\lambda}{1+\tau} \end{cases}$

$\sigma_t(\tau) = \sigma^t(\tau - t)^3$

a) Determine the expectation $E^Q(r_t)$ of the short rate with respect to Q.

From the HJM drift-condition:

$$\alpha_t(\tau) = \sigma_t(\tau)\int_t^\tau \sigma_t(s)ds = \sigma^t(\tau-t)^3 \int_t^\tau \sigma^t\cdot(s-t)^3 ds = (\sigma^t)^2\cdot(\tau-t)^3\cdot\frac{(s-t)^4}{4}\Big|_t^\tau = (\sigma^t)^2\cdot\frac{(\tau-t)^7}{4}$$

$$\Rightarrow df_t(\tau) = (\sigma^t)^2\cdot\frac{(\tau-t)^7}{4}dt + \sigma^t\cdot(\tau-t)^3 dW_t$$

$$\Rightarrow f_t(\tau) = f_0(\tau) + \frac{(\sigma^t)^2}{4}\int_0^t(\tau-s)^7 ds + \sigma^t\cdot\int_0^t(\tau-s)^4 dW_s$$

$$\Rightarrow f_t(\tau) = \frac{\lambda}{1+\tau} - \frac{(\sigma^t)^2}{32}\cdot(s-\tau)^8\Big|_0^t + \sigma^t\cdot\int_0^t(\tau-s)^4 dW_s$$

$$\Rightarrow f_t(\tau) = \frac{\lambda}{1+\tau} + \frac{(\sigma^t)^2}{32}\cdot\tau^8 - \frac{(\sigma^t)^2}{32}\cdot(t-\tau)^8 + \sigma^t\cdot\int_0^t(\tau-s)^4 dW_s$$

$$\Rightarrow r_t = f_t(t) = \frac{\lambda}{1+t} + \frac{(\sigma^t)^2}{32}\cdot t^8 + \underbrace{\sigma^t\int_0^t(t-s)^4 dW_s}_{\text{Martingale}} \qquad (*)$$

$$\Rightarrow \boxed{E^Q[r_t] = \frac{\lambda}{1+t} + \frac{(\sigma^t)^2}{32}\cdot t^8}$$

б) Determine the Girsanov kernel φ_t from Q to Q^T.

Знаем, что если дано

$$df_t(\tau) = \alpha_t(\tau)dt + \sigma_t(\tau)dW_t,$$

то $dB_t(\tau) = B_t(\tau)\Big(r_t dt - \overbrace{\Big(\int_t^\tau \sigma_t(s)ds\Big)}^{=\sigma^t\cdot\frac{(\tau-t)^4}{4}} dW_t\Big)$

~~А из (*):~~

$$\cancel{dr_t = \Big(-\frac{\lambda}{(1+t)^2} + \frac{(\sigma^t)^2}{4}t^7\Big)dt + \sigma^t(t-s)^4 dW_t + \sigma^t\int_0^t 4(t-s)^3 dW_s}$$

$$\Rightarrow dr_t = \Big(-\frac{\lambda}{(1+t)^2} + \frac{(\sigma^t)^2}{4}t^7\Big)dt$$

А еще нам нужно $dB_t = ?$; $L_t = \dfrac{B_t(\tau)}{B_t}$

$B_t = e^{-\int_0^t r_s ds}$

$dB_t = r_t B_t dt$

$$\Rightarrow \begin{cases} dB_t(\tau) = B_t(\tau)\left(r_t\,dt - \dfrac{\sigma^t(\tau-t)^4}{4}\,dW_t\right) \\ dB_t = r_t B_t\,dt \end{cases}$$

$\Rightarrow$ по ф-ле Ито для $x_t = B_t(\tau)$; $y_t = B_t$

$$d\left(\frac{B_t(\tau)}{B_t}\right) = f'_x\,dx_t + f'_y\,dy_t + \frac{1}{2}f''_{xx}(dx_t)^2 + f''_{xy}\cdot dx_t\cdot dy_t + \frac{1}{2}\cdot f''_{yy}\cdot \underbrace{(dy_t)^2}_{0} =$$

$$= \frac{1}{B_t}B_t(\tau)\left(r_t\,dt - \frac{\sigma^t(\tau-t)^4}{4}\,dW_t\right) - \frac{B_t(\tau)}{B_t^2}\cdot r_t B_t\,dt =$$

$$= \frac{B_t(\tau)}{B_t}\left(\cancel{r_t\,dt - r_t\,dt}\right) - \frac{B_t(\tau)}{B_t}\cdot(\sigma^t)\cdot\frac{(\tau-t)^4}{4}\,dW_t$$

$$\Rightarrow dL_t = -L_t\cdot(\sigma^t)\cdot\frac{(\tau-t)^4}{4}\,dW_t$$

$$\Rightarrow \varphi_s = -\frac{\sigma^t\cdot(\tau-t)^4}{4}\Bigg|_{\tau=T} = \boxed{\frac{-\sigma^t(T-t)^4}{4}}$$

(т.к. Q^T так определено)

4. $B_0^*(\tau) = \frac{1}{1+\tau^2}$

$\sigma_t(\tau) = \sigma^1 \cdot (\tau - t + 1)^{-2}$

a) Calculate the drift $\alpha_t(\tau)$ from the HJM drift condition.

$$\alpha_t(\tau) = \sigma_t(\tau) \int_t^\tau \sigma_t(s)\,ds = \sigma^1 (\tau - t + 1)^{-2} \cdot \int_t^\tau \sigma^1 \cdot \frac{1}{(s-t+1)^2}\,ds = (\sigma^1)^2 \cdot \frac{1}{(\tau-t+1)^2} \cdot \frac{-1}{(s-t+1)}\Bigg|_t^\tau =$$

$$= -(\sigma^1)^2 \cdot \frac{1}{(\tau-t+1)^2}\left(\frac{1}{\tau-t+1} - 1\right) = \boxed{(\sigma^1)^2 \cdot \left((\tau-t+1)^{-2} - (\tau-t+1)^{-3}\right)}$$

b) Fit the initial forward rate $f_0^*(\tau)$ to the market data given by $B_0^*(\tau)$:

$$B_0^*(\tau) = e^{-\int_0^\tau f_0(s)\,ds}$$

$$\Rightarrow f_0^*(\tau) = -\frac{\partial}{\partial \tau} \ln B_0^*(\tau) = -\frac{\partial}{\partial \tau} \ln\left(\frac{1}{1+\tau^2}\right) = \frac{\partial}{\partial \tau} \ln(1+\tau^2) = \boxed{\frac{2\tau}{1+\tau^2}}$$

5. $\begin{cases} df_t(\tau) = \alpha_t(\tau)dt + \sigma_t(\tau)dW_t \\ f_0(\tau) = \lambda \end{cases}$

$$\sigma_t(\tau) = \sigma^f \cdot \frac{\tau}{1+t}$$

$$dS_t = S_t(r_t dt + \sigma^S dW_t), \text{ with } r_t = f_t(t).$$

a) Is the process $r_t = f_t(t)$ Markovian?

Yes, it is, because $\sigma_t(\tau) = \sigma^f \cdot \tau \cdot \frac{1}{1+t} = \psi(\tau) \cdot \xi(t)$ – представимо в такой форме

b) Calculate the volatility $\sigma_t^B(\tau)$ для

$$dB_t(\tau) = B_t(\tau)(r_t dt + \sigma_t^B(\tau) dW_t)$$

$$\sigma_t^B(\tau) = -\int_t^\tau \sigma_t(s)ds = -\int_t^\tau \sigma^f s \frac{1}{1+t} ds = -\sigma^f \frac{1}{1+t} \cdot \frac{s^2}{2}\Big|_t^\tau = \boxed{\frac{\sigma^f (t-\tau)^2}{2(1+t)}}$$

c) The fraction $Z_t = \frac{S_t}{B_t(T)}$ satisfies the following SDE:

$$dZ_t = Z_t(a(t)dt + m(t)dW_t).$$

Найти $a(t)$ и $m(t)$.

$\begin{cases} dS_t = S_t(r_t dt + \sigma^S dW_t) \\ dB_t(\tau) = B_t(\tau)(r_t dt + \sigma_t^B(\tau) dW_t) \end{cases}$

$\Rightarrow$ по ф-ле Ито для $f(x,y) = \frac{x}{y}$: $f'_x = \frac{1}{y}$, $f'_y = -\frac{x}{y^2}$, $f''_{xx} = 0$, $f''_{xy} = -\frac{1}{y^2}$, $f''_{yy} = \frac{2x}{y^3}$; $X_t = S_t$, $Y_t = B_t(\tau)$

$$\Rightarrow d\left(\frac{S_t}{B_t(\tau)}\right) = f'_x dX_t + f'_y dY_t + \frac{1}{2}\underbrace{f''_{xx}}_{=0}(dX_t)^2 + f''_{xy} \cdot dX_t \cdot dY_t + \frac{1}{2} \cdot f''_{yy} \cdot (dY_t)^2 =$$

$$= \frac{1}{B_t(\tau)} \cdot S_t(r_t dt + \sigma^S dW_t) - \frac{S_t}{(B_t(\tau))^2} \cdot B_t(\tau) \cdot (r_t dt + \sigma_t^B(\tau) dW_t) -$$

$$- \frac{1}{(B_t(\tau))^2} \cdot S_t \cdot \sigma^S \cdot \cancel{B_t(\tau)} \cdot \sigma_t^B(\tau) dt + \frac{S_t}{(B_t(\tau))^3} \cdot \cancel{(B_t(\tau))^2} \cdot (\sigma_t^B(\tau))^2 dt =$$

$$= \frac{S_t}{B_t(\tau)}\left(\cancel{r_t dt} + \sigma^S dW_t - \cancel{r_t dt} - \sigma_t^B(\tau) dW_t - \sigma^S \cdot \sigma_t^B(\tau) dt + (\sigma_t^B)^2 dt\right) =$$

$$= \underbrace{\frac{S_t}{B_t(\tau)}}_{=Z_t}\left(\underbrace{\sigma_t^B(\tau)(\sigma_t^B(\tau) - \sigma^S)}_{=a(t)} dt + \underbrace{(\sigma^S - \sigma_t^B(\tau))}_{=m(t)} dW_t\right)$$

$$\Rightarrow \boxed{\begin{cases} a(t) = \sigma_t^B(T)(\sigma_t^B(T) - \sigma^S) \\ m(t) = \sigma^S - \sigma_t^B(T) \end{cases}}$$