Home assignments 4 & 5

26.11.2022

This home assignment will not be collected and you do not need to submit it. Use it for your training.

Problem 1

We use N-period binomial model for a non-dividend paying stock with $u = 2$, $d = \frac{1}{2}$, interest rate $r = \frac{1}{2}$, and initial stock price $S_0 > 0$. A derivative security pays amount

$$V_N = \ln S_N$$

at time N. Find a formula for V_0 and Δ_0, the arbitrage-free price and the number of stocks in the replicating strategy at time 0, in terms of S_0 and N.

Remark. Such "log"-option appears in the computations related to the variance swap.

Problem 2

We use N-period binomial model with parameters u, d, and interest rate r such that $d < 1 \le 1+r < u$. The *corridor* option with face value F and barriers L and U, $L < U$, pays at time N the amount

$$V_N = F \frac{1}{N} \sum_{n=1}^{N} 1_{\{L<S_n<U\}},$$

where S_n is the price of the stock at time n.

(i) Show that the arbitrage-free price of the option has the form:

$$V_n = f_n(S_n) + \frac{1}{(1+r)^{N-n}} F \frac{1}{N} \sum_{k=1}^{n} 1_{\{L<S_k<U\}}, \quad n = 0, 1, \ldots, N,$$

and obtain FDE (finite-difference equation) for the functions (f_n).

(ii) Compute $V_0 = f(S_0)$ and Δ_0, the arbitrage-free price and the number of shares in the replicating strategy at time 0, if $N = 2$, $u = 2$, $d = \frac{1}{2}$, $r = \frac{1}{4}$, $S_0 = 4$, and option's parameters $F = 100$, $L = 3$, and $U = 9$.

Problem 3

We use N-period binomial model with parameters S_0, u, d, and interest rate r such that $d < 1 < 1 + r < u$. The *down-and-in rebate* option pays

$$V_N = F 1_{\{\tau \le N\}},$$

at time N, where F is the face value, L is the lower barrier, $L < S_0$, and τ is the exit time of the stock price S through L:

$$\tau = \inf\{n = 1, \ldots, N : \; S_n \le L\}.$$

We set $\tau = N + 1$ if $S_n > L$ for all $n = 0, 1, \ldots, N$. In this case, the option expires worthless.

Remark. Read the conditions carefully. The option expires at time N, not at exit time τ.

(i) Show that the arbitrage-free price of the option *before* τ has the form:

$$V_n = f_n(S_n), \quad n < \tau, \quad n = 0, 1, \ldots, N.$$

Write FDE (finite-difference equation) for functions $f_n = f_n(x)$, $n = 0, 1, \ldots, N$. Do not forget about the boundary conditions and the formulas for the single-period risk-neutral probabilities!

(ii) Compute $V_0 = f_0(S_0)$ and Δ_0, the arbitrage-free price and the number of shares in the replicating strategy at time 0, if $N = 2$, $u = 2$, $d = \frac{1}{2}$, the interest rate $r = \frac{1}{2}$, the initial stock price $S_0 = 4$, and the parameters of the option $F = 27$ and $L = 3$.

Problem 4
We use N-period binomial model for a non-dividend paying stock with parameters u, d and interest rate r. The *BOOST* (Banking On Overall Stability) option with face value F and barriers L and U, $L < S_0 < U$, pays at maturity N the amount

$$V_N = F\frac{\tau - 1}{N},$$

where τ is the first time $n \in \{1, \dots, N\}$ when the stock price S_n exits (L, U):

$$\tau = \inf\{n \in \{1, \dots, N\} : \ S_n \leq L \text{ or } S_n \geq U\}.$$

We set $\tau = N + 1$ if S remains inside of (L, U) for all times $n = 1, \dots, N$. In this case, the option pays F at time N.

(i) Show that the arbitrage-free price of the option *before* τ has the form:

$$V_n = f_n(S_n), \quad n < \tau, \quad n = 0, 1, \dots, N.$$

Write FDE (finite-difference equation) for the functions $f_n = f_n(x)$, $n = 0, 1, \dots, N$. Do not forget about the boundary conditions and the formulas for the single-period risk-neutral probabilities!

(ii) Compute $V_0 = f_0(S_0)$ and Δ_0, the arbitrage-free price and the number of shares in the replicating strategy at time 0, when $N = 3$, $u = 2$, $d = \frac{1}{2}$, $r = \frac{1}{4} = 25\%$, the initial stock price $S_0 = 4$, and the option parameters $F = 150$, $L = 3$, and $U = 9$.

Problem 5
We use N-period binomial model with parameters S_0, u, d, and interest rate r such that $d < 1 < 1 + r < u$. The *up-in-put* option becomes European put with strike K and maturity N as soon as the price of the stock crosses the upper barrier $U > S_0$, that is, at the exit time

$$\tau = \min\{1 \leq n \leq N : \ S_n \geq U\}.$$

We set $\tau = \infty$ if the price of the stock is less than U for all times. In this case, the up-in-put expires worthless.

(i) Write the system of FDEs for the functions $f_n = f_n(x)$ and $g_n = g_n(x)$, $n = 0, 1, \dots, N$, such that

$f_n = f_n(x)$ is the put price at time n if the stock price is x;

$g_n = g_n(x)$ is the up-in-put price at time n if the stock price is x and $n < \tau$.

(ii) Let $N = 3$, $S_0 = 4$, $u = 2$, $d = \frac{1}{2}$, and $r = \frac{1}{4}$. For the up-in put with strike $K = 5$ and upper barrier $U = 7$, determine the price V_0 and the number of stocks Δ_0 in the replicating strategy at time $n = 0$.

Problem 6
We use N-period binomial model with parameters S_0, u, d, and interest rate r such that $d < 1 < 1 + r < u$. The *up-and-out rebate* option with upper barrier U pays the face value F at time N if $S_n < U$ for all times $n = 1, \dots, N$. Otherwise, it expires worthless. Note that the initial time $n = 0$ is not a barrier time.

(i) Show that the arbitrage-free price of the option under the condition that the upper barrier U has not been crossed *before and at* time n, that is, at times $1, \dots, n$, has the form

$$V_n = f_n(S_n), \quad n = 0, 1, \dots, N.$$

Write FDE (finite-difference equation) for the functions $f_n = f_n(x)$, $n = 0, 1, \dots, N$. Do not forget about the boundary conditions and the formulas for the single-period risk-neutral probabilities!

(ii) Compute $V_0 = f_0(S_0)$ and Δ_0, the arbitrage-free price and the number of shares in the replicating strategy at time 0, when $N = 2$, $u = 2$, $d = \frac{1}{2}$, $r = \frac{1}{4}$, the initial stock price $S_0 = 4$, and the option parameters $F = 25$ and $U = 5$.

Problem 7

We use binomial model with $N = 3$ periods, initial stock price $S_0 = 4$, relative factors $u = 2$, $d = \frac{1}{2}$, and interest rate $r = \frac{1}{4}$. Compute the initial prices of the American put, call, and straddle options, whose intrinsic values at time $n = 1, \ldots, N$ are given by

$$G_n^P = \max(K - S_n, 0), \quad G_n^C = \max(S_n - K, 0), \quad G_n^S = |K - S_n|,$$

where $K = 4$ is the strike. Note that the initial time 0 is not an exercise time. Explain why the price of the American straddle is strictly less then the sum of the prices of the American put and call options.

Problem 8

Consider the $N = 6$-period binomial model on a non-dividend paying stock with parameters $S_0 = 4$, $u = 2$, $d = \frac{1}{2}$ and take the interest rate $r = \frac{1}{4}$. Denote by S_n the price of the stock at time n and let $K = 6$ be a fixed amount of cash.

a) The payment process of the American option has the form:

$$G_n = S_n + K, \quad n = 0, 1, \ldots, N.$$

The holder of the option can exercise it at any time $n = 0, 1, \ldots, N$.

b) The payoff of the European option at the maturity N is given by

$$G_N = S_N + K.$$

Compute the difference $V_0^A - V_0^E$ of the arbitrage-free prices of these options.