Seminar 2
Stochastic Volatility Models

Vega Institute

Problem 1 🧠 ⚰️

Find the solution of SDE and prove its $\exists!$:

a) $dX_t = dt + \sigma X_t dB_t$
b) $dX_t = \dfrac{1}{X_t} dt + dB_t$

Problem 2 🧠

Solve the differential equation using Feynman-Kac formula:

$$\begin{cases} \dfrac{\partial u}{\partial t} + x\dfrac{\partial u}{\partial x} + \dfrac{1}{2}\dfrac{\partial^2 u}{\partial x^2} = u \\ u(T,x) = x^2 \end{cases}$$

Problem 3 🧠

Let $u(t,x)$ be the solution of Black-Scholes PDE:

$$\begin{cases} \dfrac{\partial u}{\partial t} + rx\dfrac{\partial u}{\partial x} + \dfrac{1}{2}\sigma^2 x^2\dfrac{\partial^2 u}{\partial x^2} = ru \\ u(T,x) = (x-K)^+ \end{cases}$$

Prove that $u(t,x)$ can be represented as $u(t,x) = \dfrac{e^{-r(T-t)}}{\sqrt{2\pi(T-t)}}\displaystyle\int_{\mathbb{R}} (xe^{(r-\frac{\sigma^2}{2})(T-t)+\sigma y} - K)^+ e^{-\frac{y^2}{2(T-t)}} dy$ and derive an analytical formula.

Problem 4 🧠

Derive the formula for call and put options in the Bachelier model, where $S_t = S_0 + \mu t + \sigma W_t$ and

a) $B_t = 1$
b) $dB_t = rB_t dt, B_0 = 1$

Problem 5 🧠

Provide an interpretation of $\Phi(d_1)$ and $\Phi(d_2)$ in Black-Scholes formula.

Problem 6 🧠

Prove that in Black-Scholes model for call and put options

$$\frac{\partial C}{\partial \sigma} = \frac{\partial P}{\partial \sigma} = S\varphi(d)\sqrt{T}$$

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Problem 7 💅

Prove put-call parity $C - P = S(0) - Ke^{-rT}$ for a non-dividend stock S and European options using non-arbitrage principle (do not assume any model for asset price dynamics). Using this parity prove the following general properties:

a) $max(0, S(0) - Ke^{-rT}) \leq C < S(0)$
b) $max(0, -S(0) + Ke^{-rT}) \leq P < Ke^{-rT}$
c) C is decreasing in K, P is increasing in K
d) $K' < K'' \Rightarrow C(K') - C(K'') < e^{-rT}(K'' - K')$,
e) $K' < K'' \Rightarrow P(K'') - P(K') < e^{-rT}(K'' - K')$,
f) C and P are convex functions of K
f) C is increasing in S, P is decreasing in S
g) $S' < S'' \Rightarrow C(S'') - P(S') < S'' - S'$
g) $S' < S'' \Rightarrow P(S') - P(S'') < S'' - S'$
h) C and P are convex functions of S

Problem 8 🧠

Provide an example of strategy π_t, which is not admissible and leads to arbitrage (see lecture 4).