11.05.2012. Assignment for Interest Rates and Credit Risk Models

Question 1) Consider forward rates, based on continuous compounding, observed at $t=0$:

$f_0^*(\tau) = 0.1;\ \tau \in [0,6]$, where the time is measured in years.

a) Calculate the bond prices $B_0^*(\tau_i)$ for the times $\tau_i = i;\ i=1\dots6$

$$B_0^*(\tau) = e^{-\int_0^\tau \underbrace{f_0^*(s)}_{0.1}ds} = e^{-0.1\tau}$$

$$\Rightarrow B_0^*(\tau_i) = e^{-0.1\cdot\tau_i} = \boxed{e^{-0.1\cdot i}},\ i=1\dots6.$$

$i=1;\ B_0^*(\tau_1) \approx 0.9048$

$i=2;\ B_0^*(\tau_2) \approx 0.8187$

$i=3;\ B_0^*(\tau_3) \approx 0.7408$

$i=4;\ B_0^*(\tau_4) \approx 0.6703$

$i=5;\ B_0^*(\tau_5) \approx 0.6065$

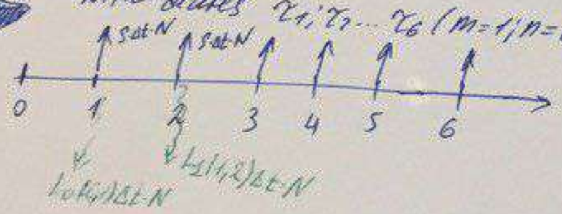
$i=6;\ B_0^*(\tau_6) \approx 0.5488$

b) Calculate the LIBOR rates $L_0(\tau_{i-1};\tau_i);\ i=1\dots6$

$$\begin{cases} L_t(\tau;\tau') = \dfrac{B_t(\tau)-B_t(\tau')}{(\tau'-\tau)\,B_t(\tau')} \\ B_0(\tau) = e^{-0.1\tau} \end{cases}$$

$$\Rightarrow L_0(\tau_{i-1};\tau_i) = \frac{B_0(\tau_{i-1})-B_0(\tau_i)}{\underbrace{(\tau_i-\tau_{i-1})}_{=1}\cdot B_0(\tau_i)} = \frac{B_0(\tau_{i-1})}{B_0(\tau_i)} - 1 = \frac{e^{-0.1\tau_{i-1}}}{e^{-0.1\tau_i}} - 1 = e^{0.1\overbrace{(\tau_i-\tau_{i-1})}^{=1}} - 1 = e^{0.1}-1 \approx \boxed{0.1052}$$

c) Calculate the at-the-money interest rate (swap rate) for an interest rate swap with dates $\tau_1, \tau_2 \dots \tau_6$ ($m=1; n=6$).



$$\Rightarrow \sum_{i=1}^{6} S\cdot\cancel{\Delta t\cdot N}\cdot B_{\tau_i} = \sum_{i=1}^{6} L_{\tau_{i-1}}(\tau_{i-1};\tau_i)\cdot\cancel{\Delta t\cdot N}\cdot B_{\tau_i}$$

$$\Rightarrow S = \frac{\sum_{i=1}^{6} L_{\tau_{i-1}}(\tau_{i-1};\tau_i)\,B_{\tau_i}}{\sum_{i=1}^{6} B_{\tau_i}}$$

но $$L_{\tau_{i-1}}(\tau_{i-1};\tau_i) = \frac{B_{\tau_{i-1}}(\tau_{i-1}) - B_{\tau_{i-1}}(\tau_i)}{\underbrace{(\tau_i-\tau_{i-1})}_{=1}\cdot B_{\tau_{i-1}}(\tau_i)} = \frac{B_{\tau_{i-1}}(\tau_{i-1})}{B_{\tau_{i-1}}(\tau_i)} - 1 = \frac{1}{e^{-0.1\cdot 1}} - 1 = e^{0.1} - 1.$$

$$\Rightarrow S = (e^{0.1}-1)\cdot\frac{\sum_{i=1}^{6} B_{\tau_i}}{\sum_{i=1}^{6} B_{\tau_i}} = e^{0.1} - 1 \approx 0.1052$$

Question 2) Consider the time horizon $T=1$ of a bond market and suppose that today's ($t=0$) bond curve is given by
$B_0^*(\tau)=e^{-c\sqrt{1+\tau}}$ for all $\tau\in[0,T]$ with $c\in(0,\infty)$

For a one-factor HJM model with deterministic forward rate volatility
$\sigma_t(\tau)=\sigma^+(\tau-t+1);\ 0\le t\le\tau\le T;\ \sigma^+\in(0;\infty)$

a) Calculate the drift $(\alpha_t(\tau))_{t\in[0,\tau]};\ \tau\in[0,T]$ from the HJM drift condition.

$$\alpha_t(\tau)=\sigma_t(\tau)\int_t^\tau\sigma_t(s)ds=\sigma^+(\tau-t+1)\int_t^\tau\sigma^+(s-t+1)ds=(\sigma^+)^2\cdot(\tau-t+1)\left(\frac{s^2}{2}\Big|_t^\tau-(t-1)(\tau-t)\right)=$$

$$=(\sigma^+)^2(\tau-t+1)\left(\frac{\tau^2-t^2}{2}-(t-1)(\tau-t)\right)=(\sigma^+)^2(\tau-t+1)\left((\tau-t)\left(\frac{\tau+t}{2}-t+1\right)\right)=$$

$$=(\sigma^+)^2\frac{(\tau-t+1)(\tau-t)}{2}(\tau+t-2t+2)=\boxed{\frac{(\sigma^+)^2\cdot(\tau-t+1)(\tau-t)(\tau-t+2)}{2}}$$

b) Fit the initial forward rate $(f_0^*(\tau))_{\tau\in[0,T]}$ to the market data given by the bond curve $(B_0^*(\tau))_{\tau\in[0,T]}$

$$B_0^*(\tau)=e^{-\int_0^\tau f_0^*(s)ds}$$

$$\Rightarrow\ln(B_0^*(\tau))=-\int_0^\tau f_0^*(s)ds$$

$$\Rightarrow[\ln(B_0^*(\tau))]'_\tau=-f_0^*(\tau)$$

$$\Rightarrow f_0^*(\tau)=-(\ln B_0^*(\tau))'_\tau=(c\sqrt{1+\tau})'_\tau=\boxed{\frac{c}{2\sqrt{1+\tau}}}$$

c) Is the factorization $\sigma_t(\tau)=\xi(t)\cdot\psi(\tau)$ satisfied?

$\sigma_t(\tau)=\sigma^+(\tau-t+1)\neq\xi(t)\cdot\psi(\tau)$, where ξ,ψ - deterministic functions

$\Rightarrow$ <u>no</u> Markov property of the short rate

d) Determine the short rate r_T.

HJM model: $df_t(\tau)=\alpha_t(\tau)dt+\sigma_t(\tau)dW_t$

$$\Rightarrow df_t(\tau)=(\sigma^+)^2\cdot\frac{(\tau-t)(\tau-t+1)(\tau-t+2)}{2}dt+\sigma^+(\tau-t+1)dW_t$$

$$\Rightarrow f_t(\tau)=f_0(\tau)+\frac{(\sigma^+)^2}{2}\int_0^t\left[(\tau-s+1)((\tau-s+1)^2-1)\right]ds+\sigma^+\int_0^t(\tau-s+1)dW_s$$

$$\Rightarrow f_t(\tau)=\frac{c}{2\sqrt{1+\tau}}+\frac{(\sigma^+)^2}{2}\cdot\left[\int_0^t(\tau-s+1)^3ds-\int_0^t(\tau-s+1)ds\right]+\sigma^+\int_0^t(\tau-s+1)dW_s$$

$$\Rightarrow f_t(\tau)=\frac{c}{2\sqrt{1+\tau}}+\frac{(\sigma^+)^2}{2}\left[\frac{1}{4}(\tau+1)^4-\frac{1}{4}(\tau-t+1)^4+\frac{1}{2}(\tau-t+1)^2-\frac{1}{2}(\tau+1)^2\right]+\sigma^+\int_0^t(t-s+1)dW_s$$

$$\Rightarrow r_t=f_t(t)=\frac{c}{2\sqrt{1+t}}+\frac{(\sigma^+)^2}{2}\left[\frac{1}{4}(t+1)^4-\frac{1}{4}+\frac{1}{2}-\frac{1}{2}(t+1)^2\right]+\sigma^+\int_0^t(t-s+1)dW_s$$

$$\Rightarrow r_T=\frac{c}{2\sqrt{1+T}}+\frac{(\sigma^+)^2}{2}\cdot\left[\frac{1}{4}((T+1)^2-1)((T+1)^2+1)-\frac{1}{2}[(T+1)^2-1]\right]+\sigma^+\int_0^T(T-s+1)dW_s$$

$$r_T=\frac{c}{2\sqrt{1+T}}+\frac{(\sigma^+)^2}{2}\cdot\frac{1}{4}T(T+2)(T^2+2T+\cancel{1+1-2})+\sigma^+\int_0^T(T-s+1)dW_s$$

$$\Rightarrow\boxed{r_T=\frac{c}{2\sqrt{1+T}}+\frac{(\sigma^+)^2}{8}\cdot T^2(T+2)^2+\sigma^+\int_0^T(T-s+1)dW_s}$$

Question 3 Consider zero bond dynamics $(B_t(\tau))_{t\in[0,\tau]}$ $\tau \in [0,T]$ defined by the Ho-Lee short rate model $\begin{cases} dr_t = (c+t)dt + \sigma^r dW_t \\ r_0 = r_0^* \end{cases}$,

where the process $(W_t)_{t\in[0,T]}$ follows a Brownian motion on $(\Omega, \mathcal{F}, Q, \mathcal{F}_t)$ with respect to the spot martingale measure Q and $\sigma^r, c, r_0^* \in (0,\infty)$ are fixed

a) Calculate the expectation $E^Q(r_T)$ of the short rate with respect to the spot martingale measure Q in terms of the model parameters $\sigma^r, c, r_0^* \in (0,\infty)$

$\begin{cases} dr_t = (c+t)dt + \sigma^r dW_t \\ r_0 = r_0^* \end{cases}$

$\Rightarrow r_t = r_0^* + \int_0^t (c+s)ds + \sigma^r W_t$

$\Rightarrow r_t = r_0^* + ct + \frac{t^2}{2} + \sigma^r W_t$

$\Rightarrow \boxed{E^Q(r_T) = r_0^* + cT + \frac{T^2}{2}}$

b) Calculate the expectation $E^{Q^T}(r_T)$ of the short rate with respect to the forward martingale measure Q^T.

$E^{Q^T}(r_T) = f_0(T)$

$f_0(T)$ можно найти из того, что $B_0(T) = e^{-\int_0^T f_0(s)ds}$ $\Rightarrow f_0(T) = -(\ln(B_0(T)))'_T$

А $B_0(T)$ мы найдём, зная r_t: $B_0(T) = E^Q\left[e^{-\int_0^T r_s ds}\right]$.

Из $r_s = r_0^* + cs + \frac{s^2}{2} + \sigma^r W_s$

$$\Rightarrow \int_0^T r_s ds = r_0^* T + \frac{cT^2}{2} + \frac{T^3}{6} + \sigma^r \underbrace{\int_0^T W_s ds}_{\substack{= TW_T - \int_0^T s\,dW_s \\ = \int_0^T (T-s)dW_s}}$$

$$\Rightarrow \int_0^T r_s ds \sim N\Big(r_0^* T + \frac{cT^2}{2} + \frac{T^3}{6}; (\sigma^r)^2 \cdot \underbrace{\int_0^T (T-s)^2 ds}_{= \left.\frac{(s-T)^3}{3}\right|_0^T = \frac{T^3}{3}}\Big) = N\Big(r_0^* T + \frac{cT^2}{2} + \frac{T^3}{6}; (\sigma^r)^2 \cdot \frac{T^3}{3}\Big)$$

$$\Rightarrow -\int_0^T r_s ds \sim N\Big(-r_0^* T - \frac{cT^2}{2} - \frac{T^3}{6}; (\sigma^r)^2 \cdot \frac{T^3}{3}\Big) = N(\mu_3; \sigma_3^2)$$

$$\Rightarrow E e^{3} = \int_{-\infty}^{+\infty} \frac{e^{x}}{\sigma_3\sqrt{2\pi}} e^{-\frac{(x-\mu_3)^2}{2\sigma_3^2}} dx = \frac{1}{\sigma_3\sqrt{2\pi}} \int_{-\infty}^{+\infty} e^{-\frac{x^2 + 2x(\mu_3 + \sigma_3^2) - \mu_3^2}{2\sigma_3^2}} dx = e^{\frac{(\mu_3 + \sigma_3^2)^2 - \mu_3^2}{2\sigma_3^2}} = e^{\mu_3 + \frac{\sigma_3^2}{2}}$$

$$\Rightarrow B_0(T) = E^Q\left[e^{-\int_0^T r_s ds}\right] = e^{\mu_3 + \frac{\sigma_3^2}{2}} = e^{-r_0^* T - \frac{cT^2}{2} - \frac{T^3}{6} + \frac{(\sigma^r)^2 \cdot T^3}{6}}$$

$$\Rightarrow f_0(T) = -(\ln(B_0(T)))'_T = \left(r_0^* T + \frac{cT^2}{2} + \frac{T^3}{6} - \frac{(\sigma^r)^2 T^3}{6}\right)'_T = r_0^* + cT + \frac{T^2}{2} - \frac{(\sigma^r)^2 T^2}{2}$$

$$\Rightarrow E^{Q^T}(r_T) = f_0(T) = \boxed{r_0^* + cT + \frac{T^2}{2} - \frac{(\sigma^r)^2 \cdot T^2}{2}}$$

(Question 4) Consider a one-factor HJM model whose forward rate dynamics follows

$$\begin{cases} df_t(\tau) = \alpha_t(\tau)dt + \sigma_t(\tau)dW_t \\ f_0(\tau) = \lambda \end{cases} ; \quad 0 \le t \le \tau \le T; \ \lambda \in (0,\infty)$$

with a Brownian motion $(W_t)_{t\in[0,T]}$ under the spot martingale measure Q.

Assume that $\sigma_t(\tau) = \sigma^f \cdot \tau \cdot t$; $0 \le t \le \tau \le T$ with a pre-specified parameter $\sigma^f \in (0,\infty)$.

Consider a risky asset following

$$\begin{cases} dS_t = S_t(r_t dt + \sigma^S dW_t) \\ S_0 = S_0^* \in (0,\infty); \ \sigma^S \in (0,\infty) \end{cases}$$

with the short rate $(r_t = f_t(t))_{t\in[0,T]}$.

a) Calculate the volatility $\sigma_t^B(\tau)$ for $0 \le t \le \tau \le T$ defined by

$$dB_t(\tau) = B_t(\tau)(r_t dt + \sigma_t^B(\tau) dW_t).$$

$$\sigma_t^B(\tau) = -\int_t^\tau \sigma_t(s)ds = -\int_t^\tau \sigma^f \cdot s \cdot t \cdot ds = -\sigma^f \cdot t \cdot \int_t^\tau s\,ds = -\sigma^f \cdot t \cdot \frac{s^2}{2}\Big|_t^\tau = -\frac{\sigma^f \cdot t \cdot (\tau^2 - t^2)}{2} = \boxed{\frac{\sigma^f t (t^2 - \tau^2)}{2}}$$

b) With $z_u := \sigma^S - \sigma_u^B(T)$ calculate the quantity $\int_0^T |z_u|^2 du$.

$$|z_u|^2 = |\sigma^S - \sigma_u^B(T)|^2 = \left|\sigma^S - \frac{\sigma^f u}{2}(u^2 - T^2)\right|^2 = (\sigma^S)^2 - \sigma^S \sigma^f u (u^2 - T^2) + \frac{(\sigma^f)^2}{4} u^2 (u^2 - T^2)^2.$$

$$\Rightarrow \int_0^T |z_u|^2 du = \int_0^T \left[(\sigma^S)^2 - \sigma^S \sigma^f u (u^2 - T^2) + \frac{(\sigma^f)^2}{4} u^2 (u^2 - T^2)^2\right] du =$$

$$= \int_0^T \left[(\sigma^S)^2 - \sigma^S\sigma^f u^3 + \sigma^S \sigma^f \cdot u \cdot T^2 + \frac{(\sigma^f)^2}{4} u^6 - \frac{(\sigma^f)^2 \cdot u^4 T^2}{2} + \frac{(\sigma^f)^2}{4} u^2 T^4\right] du =$$

$$= (\sigma^S)^2 \cdot T - \frac{\sigma^S \sigma^f T^4}{4} + \frac{\sigma^S \sigma^f \cdot T^4}{2} + \frac{(\sigma^f)^2 \cdot T^7}{28}\,{}^{15} - \frac{(\sigma^f)^2 \cdot T^7}{10}\,{}^{42} + \frac{(\sigma^f)^2 \cdot T^7}{12}\,{}^{35} = \boxed{(\sigma^S)^2 T + \frac{\sigma^S \sigma^f T^4}{4} + \frac{2 (\sigma^f)^2 \cdot T^7}{105}}$$

c) Find $E^Q\left[\frac{1}{B_T} \cdot (S_T - K)^+\right] = ?$

$$E^Q\left[\frac{1}{B_T} \cdot (S_T - K)^+\right] = E^Q\left[\frac{S_T}{B_T} \cdot \mathbb{1}_{(S_T \ge K)}\right] - K \cdot E^Q\left[\frac{1}{B_T} \cdot \mathbb{1}_{(S_T \ge K)}\right] =$$

$$= S_0 \cdot E^{Q^S}\left[\mathbb{1}_{\{S_T \ge K\}}\right] - K \cdot B_0(T) \cdot E^{Q^T}\left[\mathbb{1}_{\{S_T \ge K\}}\right] =$$

$$= \boxed{S_0 \cdot P_{Q^S}(S_T \ge K) - K \cdot B_0(T) \cdot P_{Q^T}(S_T \ge K)}$$

Чтобы посчитать $P_{Q^S}(S_T \ge K)$ и $P_{Q^T}(S_T \ge K)$, нужно узнать распределение S_T в мерах Q^S и Q^T.

А про меры Q^S и Q^T мы знаем только их плотности:

$$L_t^{Q^S} = \frac{S_t \cdot B_0}{B_t \cdot S_0}; \quad L_t^{Q^T} = \frac{B_t(T) \cdot B_0}{B_t \cdot B_0(T)}$$

Тогда из вида $L_t = \frac{B_t(T)}{B_t \cdot B_0(T)}$, применяя ф-лу Ито для $f(x,y) = \frac{x}{y}$ к $X_t = B_t(T)$; $Y_t = B_t$

получаем $\begin{cases} dB_t(T) = B_t(T) \cdot (r_t dt + \sigma_t^B dW_t) \\ dB_t = r_t B_t dt \end{cases} \Rightarrow d\left(\frac{B_t(T)}{B_t}\right) = \frac{B_t(T)}{B_t} \cdot \sigma_t^B(T) \Rightarrow dL_t^{Q^T} = \sigma_t^B L_t dW_t \Rightarrow dW_t^T = dW_t^Q - \sigma_t^B(T)dt$

Аналогично $\begin{cases} dS_t = S_t(r_t dt + \sigma_t^S dW_t) \\ dB_t = r_t B_t dt \end{cases} \Rightarrow d\left(\frac{S_t}{B_t}\right) = \left(\frac{S_t}{B_t}\right) \cdot \sigma_t^S dW_t \Rightarrow dL_t^{Q^S} = \sigma_t^S L_t dW_t \Rightarrow \boxed{dW_t^S = dW_t^Q - \sigma_t^S dt}$

ну ка сделаем по-другому:

докажем, что $d\left(\frac{S_t}{B_t(T)}\right) = \left(\frac{S_t}{B_t(T)}\right)(\sigma^S - \sigma_t^B(T))\,dW_t^T$

Как это док-ть: 1) $\begin{cases} dB_t(T) = B_t(T)(r_t dt + \sigma_t^B dW_t) \\ dB_t = r_t B_t dt \end{cases} \overset{?}{\Rightarrow} d\left(\frac{B_t(T)}{B_t}\right) = \frac{B_t(T)}{B_t}\cdot\sigma_t^B dW_t$

Ну $d\left(\frac{B_t(T)}{B_t}\right) = d\left(\frac{X}{Y}\right) = \frac{1}{Y_t}dX_t - \frac{X_t dY_t}{Y_t^2} + \frac12 \underbrace{f''_{xx}}_{=0}(dX_t)^2 + f''_{xy}\underbrace{dX_t\cdot dY_t}_{=0} + \frac12 f''_{yy}\underbrace{(dY_t)^2}_{=0} =$

$= \frac{B_t(T)(r_t dt + \sigma_t^B dW_t)}{B_t} - \frac{B_t(T)\cdot r_t dt}{B_t} = \frac{B_t(T)}{B_t}\cdot\sigma_t^B dW_t$

2) $\begin{cases} dS_t = S_t(r_t dt + \sigma_t^S dW_t) \\ dB_t = r_t B_t dt \end{cases} \overset{?}{\Rightarrow} d\left(\frac{S_t}{B_t}\right) = \frac{S_t}{B_t}\cdot\sigma_t^S\cdot dW_t$

(аналогично, применяем ф-лу Ито для $f(x,y)=\frac{x}{y}$)

3) $\begin{cases} d\left(\frac{B_t(T)}{B_t}\right) = \frac{B_t(T)}{B_t}\cdot\sigma_t^B dW_t \quad (Y_t) \\ d\left(\frac{S_t}{B_t}\right) = \frac{S_t}{B_t}\cdot\sigma_t^S\cdot dW_t \quad (X_t) \end{cases} \overset{?}{\Rightarrow} d\left(\frac{S_t}{B_t(T)}\right) = \left(\frac{S_t}{B_t(T)}\right)\cdot(\sigma^S - \sigma_t^B(T))\,dW_t^T$

ну применяем ф-лу Ито для $f(x,y)=\frac{x}{y}$

$\Rightarrow d\left(\frac{S_t}{B_t}\cdot\frac{B_t}{B_t(T)}\right) = \frac{dX_t}{Y_t} - \frac{X_t\, dY_t}{Y_t^2} + \frac12 \underbrace{f''_{xx}}_{=0} + \underbrace{f''_{xy}}_{=-\frac{1}{Y_t^2}}\cdot dX_t\, dY_t + \frac12 \underbrace{f''_{yy}}_{\frac{X_t\cdot 2}{Y_t^3}}(dY_t)^2 =$

$= \frac{X_t\sigma_t^S dW_t}{Y_t} - \frac{X_t Y_t\sigma_t^B dW_t}{Y_t^2} + 0 - \frac{X_t\sigma_t^S\cdot Y_t\sigma_t^B dt}{Y_t^2} + \frac12\cdot\frac{X_t\cdot 2}{Y_t^3}\cdot Y_t^2(\sigma_t^B)^2 dt =$

$= \frac{X_t}{Y_t}\sigma_t^B(\sigma_t^B - \sigma_t^S)dt + \frac{X_t}{Y_t}(\sigma_t^S - \sigma_t^B)dW_t$

$\Rightarrow$ волатильность у $\frac{S_t}{B_t(T)}$ в мере Q равна $\frac{S_t}{B_t(T)}(\sigma_t^S - \sigma_t^B)$. $\Rightarrow$ она останется такой и в мере Q^T, т.к. волатильность при замене меры не меняется.

$\Rightarrow \boxed{d\left(\frac{S_t}{B_t(T)}\right) = \frac{S_t}{B_t(T)}\cdot\underbrace{(\sigma^S - \sigma_t^B(T))}_{=\Sigma_s}\,dW_t^T} \leftarrow$ GBM с $\mu=0$; $\sigma_t = \Sigma_s$.

$\Rightarrow S_T = \overset{=1}{B_T(T)}\cdot\frac{S_0}{B_0(T)}\cdot e^{\int_0^T \Sigma_s dW_s^T - \frac12\int_0^T \Sigma_s^2 ds}$

$B_0(T) = e^{-\int_0^T r(s)ds} = e^{-rT}$

$\Rightarrow P^{Q^T}(S_T \geq K) = P\left(\frac{S_0}{B_0(T)}\cdot e^{\int_0^T \Sigma_s dW_s^T - \frac12\int_0^T \Sigma_s^2 ds} \geq K\right) = P\left(\int_0^T \Sigma_s dW_s^T - \frac12\int_0^T\Sigma_s^2 ds \geq \ln\left(\frac{K\cdot B_0(T)}{S_0}\right)\right) =$

$= P\left(\underbrace{\int_0^T \Sigma_s dW_s^T}_{\sim N(0,\int_0^T \Sigma_s^2 ds)} \geq \ln\left(\frac{K\cdot B_0(T)}{S_0}\right) + \frac12\int_0^T\Sigma_s^2 ds\right) = P\left(\xi\sim N(0,1) \geq \frac{\ln\left(\frac{K\cdot B_0(T)}{S_0}\right) + \frac12\int_0^T\Sigma_s^2 ds}{\sqrt{\int_0^T\Sigma_s^2 ds}}\right) = \Phi\left(\frac{\ln\left(\frac{S_0 e^{rT}}{K}\right) - \frac12\int_0^T\Sigma_s^2 ds}{\sqrt{\int_0^T\Sigma_s^2 ds}}\right) =: \Phi(d_2)$

Аналогично, $P^{Q^S}(S_T \geq K) = \Phi\left(\frac{\ln\left(\frac{S_0 e^{rT}}{K}\right) - \frac12\int_0^T\Sigma_s^2 ds}{\sqrt{\int_0^T\Sigma_s^2 ds}}\right) =: \Phi(d_1)$

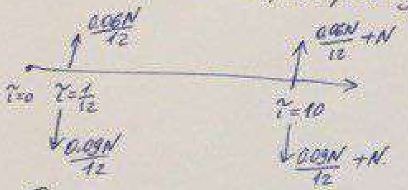
$\Rightarrow E^Q\left[\frac{1}{B_T}(S_T - K)^+\right] = S_0\cdot P_{Q^S}(S_T\geq K) - K\cdot B_0(T)\cdot P_{Q^T}(S_T \geq K) = \boxed{S_0\cdot\Phi(d_1) - K\cdot e^{-rT}\cdot\Phi(d_2)}$

Question 5

Consider two coupon paying bonds (Bond$_1$ and Bond2) with face value 10.000 AUD paying coupons monthly at the (annual) coupon rate of 6% (Bond1) and 9% (Bond2). Assume that the first coupon just has been paid at $\tau=0$, the last coupon (in addition to the face value) will be paid at $\tau=10$ and the bonds are traded now, at $\tau=0$, at the yields 7% (Bond1) and 8% (Bond2). Is it possible to determine the price of a zero bond maturing at $\tau=10$ using no-arbitrage arguments? If yes, calculate the yield of this bond (continuous compounding).

Bond1: YTM = 7%; coupon = 6%; $PV_1 = N\cdot\left(e^{-\frac{0.07}{12}}\right)^{120} + \sum_{i=1}^{120}\left(\frac{0.06}{12}\right)\cdot N\cdot\left(e^{-\frac{0.07}{12}}\right)^{i} \approx 9268.26.$

Bond2: YTM = 8%; coupon = 9%; $PV_2 = N\cdot\left(e^{-\frac{0.08}{12}}\right)^{120} + \sum_{i=1}^{120}\left(\frac{0.09}{12}\right)\cdot N\cdot\left(e^{-\frac{0.08}{12}}\right)^{i} \approx 10667.71.$



Пусть мы купили α шт. bond1 и продали β шт. bond2.
α и β должны удовл. соотношению, чтобы купоны все сократились, т.к у zero bond с номиналом FV и тек. стоимостью PV — нет купонов.

$$\Rightarrow \begin{cases} \frac{0.06N}{12}\cdot\alpha - \frac{0.09N}{12}\cdot\beta = 0 \\ \alpha N - \beta N = FV \\ PV = \alpha\, PV_1 - \beta\cdot PV_2 \end{cases} \Rightarrow \begin{cases} \alpha = \frac{9}{6}\beta = \frac{3}{2}\beta \\ \underbrace{(\alpha-\beta)}_{0.5\beta}\cdot N = FV \\ PV = \frac{3}{2}\beta PV_1 - \beta\cdot PV_2 \end{cases} \Rightarrow FV = 0.5\beta N$$

$$\Rightarrow \frac{PV}{FV} = \frac{\frac{3}{2}\beta PV_1 - \beta PV_2}{0.5\beta\cdot N} = \frac{\frac{3}{2}PV_1 - PV_2}{0.5N} = \frac{3PV_1 - 2PV_2}{N} \approx \frac{6469.365898}{10000} \approx 0.646965\ldots$$

$\Rightarrow PV = 0.6469\cdot FV$

Ищем contin. yield of this bond:

$$e^{-10\cdot r} = 0.6469$$

$$\Rightarrow r = -\frac{\ln(0.6469)}{10}$$

$$\Rightarrow \boxed{r = 4.355\%}$$