Preparation for Vega-Quantathon: set 2

Contents

While implementing the functions below, you need to account for the singularities of the type 0/0.

Cost-of-carry rate curve for the Black model

Input:

$\theta:$ the constant drift term ($\frac{\theta}{\lambda}$ is the mean-reversion level).

$\lambda \geq 0:$ the mean-reversion rate.

$\sigma \geq 0:$ the volatility.

$t_0:$ the initial time given as year fraction.

Output:

$q = (q(t))_{t \geq t_0}:$ the continuously compounded curve of *cost-of-carry* rates.

In the Black model, the log of spot price evolves as

$$\log S(t) = \log S(t_0) + X(t), \quad t \geq t_0,$$

where X is an OU (Ornstein-Uhlenbeck) process driven by Brownian motion B:

$$dX(t) = (\theta - \lambda X(t))dt + \sigma dB(t), \quad X(t_0) = 0.$$

We recall that the forward price curve has the form:

$$F(t) = S(t_0)\exp(q(t)(t - t_0)) = \mathbb{E}\left(S(t)\right), \quad t \geq t_0.$$

The cost-of-carry rate is given by (check the formula):

$$q(t) = \theta\frac{1 - e^{-\lambda(t-t_0)}}{\lambda(t - t_0)} + \frac{\sigma^2}{2}\frac{1 - e^{-2\lambda(t-t_0)}}{2\lambda(t - t_0)}, \quad t \geq t_0.$$

Yield curve for the Vasicek model of interest rates

Input:

$\theta :$ the constant drift term (θ/λ is the mean-reversion level).

$\lambda > 0 :$ the mean-reversion rate.

$\sigma > 0 :$ the volatility.

$r(t_0) :$ the initial short-term interest rate.

$t_0 :$ the initial time given as year fraction.

Output:

$\gamma = (\gamma(t))_{t\geq t_0} :$ the continuously compounded yield curve in the Vasicek model of interest rates.

In the Vasicek model, short-term interest rate $r = (r_t)$ is an OU (Ornstein-Uhlenbeck) process driven by Brownian motion $B = (B_t)$:

$$dr_t = (\theta - \lambda r_t)dt + \sigma dB_t.$$

We recall that the discount curve has the form:

$$D(t) = e^{-\gamma(t)(t-t_0)} = \mathbb{E}\left(e^{-\int_{t_0}^t r_s ds}\right), \quad t \geq t_0.$$

Computations show (please, check) that

$$\gamma(t) = r(t_0)A(t) + \frac{\theta}{\lambda}(1 - A(t)) - \frac{\sigma^2}{2\lambda^2}(1 - 2A(t) + B(t)), \quad t \geq t_0.$$

where

$$A(t) = \frac{1 - e^{-\lambda(t-t_0)}}{\lambda(t - t_0)}, \quad B(t) = \frac{1 - e^{-2\lambda(t-t_0)}}{2\lambda(t - t_0)}.$$

Discount curve for the Vasicek model of interest rates

Input:

θ : the constant drift term (θ/λ is the mean-reversion level).

$\lambda > 0$: the mean-reversion rate.

$\sigma > 0$: the volatility.

$r(t_0)$: the initial short-term interest rate.

t_0 : the initial time given as year fraction.

Output:

$D = (D(t))_{t \geq t_0}$: the discount curve in the Vasicek model of interest rates.

We recall that the discount curve has the form:

$$D(t) = e^{-\gamma(t)(t-t_0)}, \quad t \geq t_0.$$

In the Vasicek model, yield curve $\gamma = \gamma(t)$ is given by

$$\gamma(t) = r(t_0)A(t) + \frac{\theta}{\lambda}(1 - A(t)) - \frac{\sigma^2}{2\lambda^2}(1 - 2A(t) + B(t)), \quad t \geq t_0,$$

where

$$A(t) = \frac{1 - e^{-\lambda(t-t_0)}}{\lambda(t - t_0)}, \quad B(t) = \frac{1 - e^{-2\lambda(t-t_0)}}{2\lambda(t - t_0)}.$$

Forward price curve for an annuity

Input:

q : the coupon rate.

δt : the time interval between coupon payments.

T : the maturity.

$D = (D(t))_{t \geq t_0}$: the discount curve.

t_0 : the initial time.

bClean : the boolean parameter specifying the type of the prices: "clean" or "dirty". The dirty price is the actual amount paid in a transaction. The clean price is the difference between the dirty price and the accrued interest. If t_i is the previous coupon time (or the initial time if no coupons have been paid so far) and t is the settlement time, then the accrued interest is given by

$$A(t) = q(t - t_i).$$

Output:

$F = (F(t))_{t \in [t_0, T]}$: the forward prices for the annuity. Here t is the maturity of the contract and t_0 is the issue time.

The annuity pays coupons $q\delta t$ at times $(t_i)_{i=1,\dots,M}$ such that

$$t_0 < t_1 \leq t_0 + \delta t, \quad t_{i+1} - t_i = \delta t, \quad t_M = T.$$

The buyer pays forward price $F(t)$ at delivery time t and then receives coupons $q\delta t$ at payments times $t_i > t$.

Forward swap rates

Input:

δt : the time interval between payments.

M : the number of payments.

$D = (D(t))_{t \geq t_0}$: the discount curve; t_0 is the initial time.

Output:

> $R^f(t) = R^f(t; t_0, \delta t, M)_{t \geq t_0}$: the forward swap rates computed at t_0 in the contract with period δt and number of payments M.

It costs nothing to enter the forward swap contract with maturity t. At time t, the swap is issued with a notional amount N, the number of payments M, the period between payments δt, and the fixed rate $R^f(t)$.

Forward prices for a stock that pays dividends

Input:

> $S(t_0)$: the spot price.
>
> $(t_i)_{i=1,\ldots,M}$: the dividend times, $t_1 > t_0$.
>
> $(Q_i)_{i=1,\ldots,M}$: the dividend payments.
>
> $D = (D(t))_{t \geq t_0}$: the discount curve.
>
> t_0 : the initial time = the issue time for the forward.

Output:

> $F = (F(t))_{t \in [t_0, t_M]}$: the forward prices for the stock.

The buyer pays forward price $F(t)$ at delivery time t and then receives the stock. If t is a dividend time, then the buyer gets the dividend paid at t. It costs nothing to enter a forward contract at its issue time t_0.

Implied volatility curve for the Hull and White model

Input:

> $\lambda \geq 0$: the mean-reversion rate.
>
> $\sigma > 0$: the short-term volatility.
>
> t_0 : the initial time given as year fraction.

Output: the stationary implied volatility curve for options on discount factors in the Hull and White model of interest rates. It has the form:

$$\Sigma(s,t) = \sigma \frac{1-\exp(-\lambda(t-s))}{\lambda} \sqrt{\frac{1-\exp(-2\lambda(s-t_0))}{2\lambda(s-t_0)}},$$
$$t_0 \le s < t.$$

Here s is the maturity of the option and t is the maturity of the discount factor.

Volatility curve computed from variance curve

Input:

$V = (V(t))_{t \ge t_0}$: the variance curve.

t_0 : the initial time given as year fraction.

Output: continuously compounded volatility curve $\Sigma = (\Sigma(t))_{t \ge t_0}$.

We recall that

$$V(t) = \Sigma^2(t)(t-t_0), \quad t \ge t_0.$$

Discount curve obtained by log linear interpolation

Input:

$(t_i)_{i=1,\dots,M}$: the maturities, $t_i < t_{i+1}$,

$(d_i)_{i=1,\dots,M}$: the discount factors,

t_0 : the initial time, $t_0 < t_1$.

Output: the discount curve

$$d(t) = \exp(l(t)), \quad t \in [t_0, t_M],$$

where function $l = l(t)$ is the linear interpolation of the logs of the market discount factors:

$$l(t) = \mathcal{I}^{\text{linear}}((t_i)_{i=0,1,\dots,M}, (\log d_i)_{i=0,\dots,M}), \quad d_0 = 1.$$

Volatility curve obtained by the linear interpolation of the variance curve

Input:

$(t_i)_{i=1,\dots,M}$: the maturities, $t_1 > t_0$;

$(V(t_i))_{i=1,\dots,M}$: the market volatilities;

t_0 : the initial time.

Output: volatility curve $V = V(t)$ on $[t_0, t_M]$ obtained by the linear interpolation of variance curve

$$D(t) = (t - t_0)V^2(t), \quad t \in [t_0, t_M].$$

In particular, the volatility is constant on $[t_0, t_1]$:

$$V(t) = V(t_1), \quad t \in [t_0, t_1].$$