Introduction to modern Game Theory

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Plan

Part 1. **Ideas, methods, applications**. Socio-Economic, political, philosophic links.

Part 2. **Foundations of noncooperative games.** Basic mathematical methods and application to evolutionary biology.

Part 3. **Game-Theoretic approach to financial mathematics.** Pricing derivative securities.

Part 4. **Games of many players in statistical limit.** Mean field games.

? Part 5. **Quantum Games.**

Main books

V. N. Kolokoltsov, O. A. Malafeyev. Introduction to the analysis of many agent systems of competition and cooperation (in Russian). 'Lan', St. Petersburg, 2012.

V. N. Kolokoltsov and O. A. Malafeyev. Understanding Game Theory. World Scientific, 2010, Second Edition 2020.

V. N. Kolokoltsov and O. A. Malafeyev. Many Agent Games in Socio-economic Systems: Corruption, Inspection, Coalition Building, Network Growth, Security. Springer 2019.

Additional books

P. Bernhard, J. Engwerda, B. Roorda, J.M. Schumacher, V. N. Kolokoltsov, P. Saint-Pierre and J.-P. Aubin. The Interval Market Model in Mathematical Finance: Game-Theoretic Methods. Birkhäuser, 2012.

O. A. Malafeyev and V. N. Kolokoltsov. Mathematical and computer analysis of game-theoretic models of project investment with possible corruption taken into account (in Russian). St-Petersburg University, 2019. ISBN 978-5-9651-1243-2.

V. N. Kolokoltsov. Differential equations on measures and functional spaces. Birkhäuser Advanced Texts, Birkhäuser, 2019.

What is it all about?

- What's our life? - A game!
- Who is right or happy here, mate?
- Today it's you, tomorrow me!

The Queen of Spades (opera)

During its life any being attempts to achieve goals which are important to it. However, all beings are in permanent contact with others who are trying to achieve their own goals. Thus any being should always take into consideration the interests and possible actions of other beings: sometimes attempting to outwit an opponent and sometimes forming coalitions with partners that have similar interests. This is precisely what is meant by a game: an attempt to achieve one's goal in an environment where there are other beings that may have opposite or similar, but almost never identical, goals.

Part 1

Ideas, methods, applications

Chapter 1. Around Prisoner's dilemma: static games of two players.

Chapter 2. Auctions and networks: static games of several players.

Chapter 3. Backward induction and repeated games.

Chapter 4. Aggregation of preferences: elections, social agreement, fair distribution.

Chapter 1

Around Prisoner's dilemma. Static games of two players.

Prisoner's dilemma in various forms, social dilemmas, problems of cooperation, dominated strategies, Nash equilibria, Pareto optimality, common knowledge of rationality.

Examples: arms race, preventive war, public good game, common pool resources, battle of the sexes, etc.

An elementary brick of an interaction

Science usually tries to decompose complicated interlaced interactions into simple parts (elementary bricks or links), to analyze their workings separately, and at last to reconstruct the whole chain from these simple parts.
An elementary brick of an interaction with conflict interests: a two-players game.

It is convenient and customary to give the players some names, R and C, say, Ruth and Charlie.

These letters are not randomly chosen: Row and Column players.

Definition

A game of two players, Ruth and Charlie, *in the normal form* is defined by the set S_R of possible strategies of R, the set S_C of possible strategies of C, and by two payoff functions Π_R and Π_C defined on the sets of pairs (s_R, s_C).
The notion of the strategy is capacious.

The payoff functions specify two real numbers $\Pi_R(s_R, s_C)$ and $\Pi_C(s_R, s_C)$ for any pairs of strategies (s_R, s_C), where s_R is from S_R and s_C from S_C. These two numbers describe the payoffs to Ruth and Charlie when they apply their strategies s_R and s_C. The payoffs can be negative, which means, of course, that in this case the player rather loses than wins.
When the number of possible strategies is not large, one can conveniently describe such a game by a table.

Example: Head and Tail (or Matching Pennies)

Ruth and Charlie simultaneously put two coins on the table. If two coins are put in the same way (two heads or two tails), C pays to R one dollar. Otherwise, R pays to C one dollar (the procedure is similar to the game where R announces Tail or Head, and C throws the coin):

C

R		head	tail
	head	1,-1	-1,1
	tail	-1,1	1,-1

Table 1.1

Let us stress here that the first number in a cell shows the winning of R (the Row player), and the second number shows the wining of C (the Column player).

Example: Rock-Paper-Scissors

Children game: R and C simultaneously display their hands in one of three shapes denoting schematically a rock, a paper, or scissors. The rock wins over the scissors as it can shatter them, the scissors win over the paper as they can cut it, and the paper wins over the rock as it can be wrapped around the latter. A winner takes a penny from the opponent. If both players displays the same, then the game is drawn.

		C		
		R	S	P
R	R	0,0	1,-1	-1,1
	S	-1,1	0,0	1,-1
	P	1,-1	-1,1	0,0

Table 1.2

A quite different example of a game is supplied by chess (or go or draughts).

Zero-sum games

Strictly competitive games (or *games with opposite interests*) (or *zero sum games*), "the dinosaurs of the game theory": the gain of one player always equals the lose of another one, i.e. $\Pi_R(s_R, s_C) = -\Pi_C(s_R, s_C)$ for all strategies of R and C. Both examples above are zero sum games. In case of strictly competitive games one can leave only the first number in each cell of the table of the game, as the second number always differs by sign only. For instance, the game of table 1.1 can be specified by the reduced table

		C	
		head	tail
R	head	1	-1
	tail	-1	1

Symmetric games

Symmetric games are the games where each player has the same set of strategies S (i.e. the sets S_R and S_C coincide) and the payoff depends on the pair of strategies only, and not on the name of the player that uses them. In other words, $\Pi_R(s_1, s_2) = \Pi_C(s_2, s_1)$ for any pair of strategies s_1, s_2 from S. For example, the Rock-Paper-Scissor game is symmetric, but matching penny is not.

The table of a symmetric game is a square table, and the second number in each cell coincides with the first number of the cell that is positioned symmetrically with respect to the main diagonal (the diagonal going from the upper left corner to the lower right corner). Hence, symmetric games can also be described by reduced tables with only one (the first) number in each cell.

What to do?

Of course, one is interested in solving the games, which broadly speaking means finding reasonable strategies for players.

But what is meant by “reasonable strategies”, or by “a solution to a game”?

The result depends on the (often unknown) action of your opponent. Say, if you play Head and Tail game described above, choosing Head does not give you any prediction at all about the outcome of the game. So, what is good and what is bad?

There are various approaches to tackle this problem.

Prisoner's dilemma

Two crooks are arrested in connection with a serious crime. However, without a confession the police only have enough evidence to convict the two crooks on a lesser charge. The police offer to both prisoners the same deal (in separate rooms so that no communication between them is possible): if you confess that both of you together committed the serious crime and your partner does not confess, then your sentence will be suspended, and the other will spend 6 years in jail (4 for the crime and 2 for obstructing justice); if both of you confess, then you will both get the 4 year sentence; and if neither of you does so, you both will each spend 2 years in jail for the minor offense.

		C	
		confess	not confess
R	confess	-4,-4	0,-6
	not confess	-6,0	-2,-2

Prisoner's dilemma,II

What should each prisoner do? Imagine yourself ...

As our further examples will show, the Prisoner's Dilemma appears in disguise in various situations.

Solution concept: dominated strategies.
Solution concept: Pareto optimality.

Prisoner's dilemma

Of course, the concrete numbers in table 1.3 are not important. In general, *a prisoner's dilemma* is a game with the table

C

R		defect	cooperate
	defect	p,p	q,r
	cooperate	r,q	s,s

Table 1.4

where $r < p < s < q$.

Exercise. Check that the order $r < p < s < q$ ensures that (defect,defect) is a unique dominant strategy equilibrium, which is however not Pareto optimal, the Pareto optimal solution being (cooperate, cooperate).

Honor among thieves

Suppose a smuggler is going to sell some goods to a criminal boss. Defecting here just means not fulfilling your part of the deal: e.g. the smuggler can bring a fake diamond instead of the real one, and the criminal boss can just shoot the smuggler on the place of exchange.

Of course, defecting gives better payoffs to each of the players, but if both defect they both get a worse outcome than they would by cooperating. You can easily assign some reasonable payoffs for this game.

Another example: Tosca and Cavaradossi.

Arms race and preventive war

Part 1: Two nations develop some deadly new weapon, say H-bomb sticking to the defecting strategy of this conflict, the "Mutually Assured Destruction" (or MAD) strategy.

Part 2: The weapon is developed, you think about using it. In the 50s of the last century the discussion of preventive war was seriously under way in USA. This amoral but seductive doctrine was supported by the arguments of similar kind: if the Soviets bomb us, then we better bomb them as well, but if they don't, we are again better by bombing them first to destroy their power before it is too late.

Arms race and preventive war

John von Neumann, an outstanding mathematician of the 20th century, one of the founders of modern game theory, who also took an active part in the H-bomb project:

"With the Russians it is not a question of whether but when. If you say why not bomb them tomorrow, I say why not today? If you say today at 5 o'clock, I say why not one o'clock".

Conclusion:

Common pool resource or Public goods game

Ruth and Charlie: both are given 20 pounds. Then each of them has a right to either keep this money or anonymously deposit it in a "public account". The money in the public account will be then increased by 50% and shared equally between the players.

		C	
		defect	cooperate
R	defect	20,20	35,15
	cooperate	15,35	30,30

where "cooperate" means here, of course, to deposit your money in a public account.

This game is instructive for a discussion of the Social Contract (think 'taxes').

The usual dilemma for politicians: increase taxes and improve social services (cooperative strategy) or decrease taxes and allow everyone to do the best on his own (defective strategy).

Common knowledge of rationality. Iterated elimination of strictly dominated strategies

The method of the elimination of dominated strategies is based on the assumption that the players are rational. Further: the players are rational, and they all know that the other players are rational, and then further that the players all know that the other players all know that they are rational, and so on ad infinitum. This is the *common knowledge of rationality*. It leads to the method of solving games by the *iterated elimination of strictly dominated strategies*. Example:

C

R		1	2	3
	1	5,0	5,4	0,3
	2	0,4	0,3	5,2

Here R has no dominated strategies, and consequently the game can not be solved by elimination of dominated strategies.

Iterated elimination of strictly dominated strategies

However, for C the last strategy is strictly dominated by the second one and hence can be eliminated as a feasible strategy of C, which reduces the game to the table

		C	
		1	2
R	1	5,0	5,4
	2	0,4	0,3

Of course, R needs to use the Common Knowledge of Rationality (R knows that C is rational) to deduce that C will never play his strictly dominated strategy, and consequently to bring the game to the above reduced form. In this reduced form the second strategy of R is strictly dominated by the first one, so it can be eliminated (second step in the iterated elimination of strictly dominated strategies) and hence the game can be reduced to the table:

Iterated elimination of strictly dominated strategies

R \ C	1	2
1	5,0	5,4

In this game the first strategy of C is strictly dominated by the second one and using "C knows that R is rational and that R knows that C is rational" the player C can now eliminate his first strategy, which leads to a conclusion that the game will be played with R playing the first strategy and C playing the second one.

If applying the iterated elimination of strictly dominated strategies leads to a unique pair of strategies, this pair of strategies is again called a *dominant strategy equilibrium or a solution obtained by the iterated elimination of strictly dominated strategies*.

Weak dominance

A generalization: the method of the elimination of weakly dominated strategies. A strategy s_R^2 of the player R, say, is called *weakly dominated* by a strategy s_R^1 of the same player (or s_R^1 *weakly dominates* s_R^2), if playing s_R^1 is always not worse and sometimes better than playing s_R^2, in other words, if

$$\Pi_R(s_R^1, s_C) \geq \Pi_R(s_R^2, s_C)$$

for all strategies s_C of C and

$$\Pi_R(s_R^1, \tilde{s}) > \Pi_R(s_R^2, \tilde{s})$$

for at least one strategy $\tilde{s}$ of C. Again one can argue that it is reasonable never to use weakly dominated strategies. If a strategy of a player weakly dominates all other strategies, then this strategy is called *the weakly dominant strategy*. If each player has the weakly dominant strategy, then this pair of strategies is called a *(weakly) dominant strategy equilibrium*.

Debtors and creditors

C and R are creditors, and a debtor owes 3000 pounds to each of them, but he has only 5000. The cost of liquidation is 3000. If he defaults on the debt, he loses all his money, but R and C each gets 1000 only. The debtor offers to both C and R 1001 pounds, if both of them agree to cancel his debt. Otherwise, he would declare a default.

		C	
		accept	not accept
R	accept	1001,1001	1000,1000
	not accept	1000,1000	1000,1000

Table 1.5

so that "accept" is dominant.

If R and C adhere to this strategy, then the debtor will retain 2998 for himself.

Is it a reasonable solution?

Nash equilibrium

To grasp the basic idea behind the Nash equilibrium let us look again at the profile (not confess, not confess) of the prisoner's dilemma of table 1.3. Why is such an outcome not feasible, even if we allow for a preliminary agreement of the players to act this way, say, like in the example of Tosca and Scarpia in Section 1.3?

Because an individual deviation from such an agreement would allow any player to do better. In other words, this situation is unstable, if players are allowed to independently explore their most profitable choices. Avoiding such instabilities leads to a general notion of an equilibrium (as a solution of the game) that we are going to define now.

Nash equilibrium

Namely, a pair of strategies $(\tilde{s}_R, \tilde{s}_C)$ of R and C is called a *Nash equilibrium* for a given game if

$$\Pi_R(\tilde{s}_R, \tilde{s}_C) \geq \Pi_R(s_R, \tilde{s}_C)$$

for all strategies s_R of R, and

$$\Pi_C(\tilde{s}_R, \tilde{s}_C) \geq \Pi_C(\tilde{s}_R, s_C)$$

for all strategies s_C of C, i.e., if each of the strategies $\tilde{s}_R$ and $\tilde{s}_C$ is the best reply to another one. In other words, neither player can do better by deviating unilaterally from the profile $(\tilde{s}_R, \tilde{s}_C)$. Saying it in yet another way, if such a pair of actions is agreed between players, then no one would have any reason to break this agreement.
"No regret" outcomes.

Nash equilibrium

It turns out that many games are in fact played according to a Nash equilibrium. What is more surprising, an equilibrium behavior manifests itself also in the biological context, i.e. in the interaction of animals, where one can not expect any rational calculations of the best responses.

The life of John Nash, an outstanding American mathematician who got the 1994 Nobel Prize in Economics, was popularized recently in a spectacular Hollywood film "The Beautiful Mind".

Drawback: multiple equilibria, which leads to the problem Which one to choose?

Nash equilibrium

In tables:

A pair of payoffs in a cell yields a Nash equilibrium, if the first number is a maximum amongst all first numbers from the cells of the same column, and the second number is a maximum amongst all second numbers from the cells of the same row.

Sacrifice game

C

R

	head	tail
head	-1,-1	-1,1
tail	1,-1	-1,-1

Table 1.6

The outcomes (head,tail), (tail, head), (tail,tail) - all Nash.
Elimination of dominant strategies: (tail,tail). Also symmetric.
Is it the best?
But (tail,tail) is not Pareto optimal, and (head,tail), (tail, head) are.

Battle of the sexes

		Wife	
		football	ballet
Husband	football	3,2	1,1
	ballet	0,0	2,3

Table 1.7

The outcomes (3,2) and (2,3) are both Nash equilibria, both efficient (Pareto optimal).
It is not symmetric, but becomes one by changing the order of columns:

		Wife	
		ballet	football
Husband	football	1,1	3,2
	ballet	2,3	0,0

Kant's categorical imperative

In this representation the first strategy means just to be selfish, and consequently a more general representation is

C

R		selfish	non-selfish
	selfish	1,1	3,2
	non-selfish	2,3	0,0

Immanuel Kant's *categorical imperative*: "Act only on such a maxim through which you can at the same time will that it should become a universal law". For the prisoner's dilemma this prescribes "to cooperate" (as you do not want your opponent to defect, you shouldn't do it either) leading to a nice "moral" solution (cooperate, cooperate).

limitation of the famous principle: in battle of sexes not liking people to be selfish yields the worst (for both) outcome (not selfish, not selfish).

Chicken game and the Cuban missile crisis

Two fast cars are driven towards each other by the middle of the road from opposite directions. If one of the drivers swerves before the other, the other, as he passes, shouts "Chicken!" The driver who has swerved becomes an object of contempt.

	not swerve	swerve
not swerve	-1,-1	2,0
swerve	0,2	1,1

Table 1.8

A prime example of such a conflict was the nuclear stalemate between USA and Soviet Union during the famous Cuban Missile Crisis of 1962, where "not to swerve" strategy meant, of course, to continue escalating the conflict with nuclear war at the end of it (the same MAD strategy already mentioned). The analogy is artistically presented by Adreano Celentano in "Innamorato Pazzo".

Social dilemmas

Symmetric two-player, two-action games with some dilemma between cooperation and defection (or selfishness) are sometimes called the simplest social dilemmas.
Stag hunt:

	defect	cooperate
defect	1,1	2,0
cooperate	0,2	3,3

Table 1.10

Utility function

Payoff
Personal utility: measure of happiness
Fitness: reproduction rate

Objectives of game theory

(i) To work out the concepts of reasonable (rational, stable, equilibrium) strategies of behavior and to find when these strategies exist and how to calculate them; to analyze these strategies in situations when a cooperation and a coalition formation is possible;
(ii) To analyze the optimal (equilibrium) strategies dynamically, when a game is developing in time (e.g. chess, pursuit) and one has to correct the behaviour subject to permanently varying circumstances;
(iii) To describe and to assess the methods of the dynamical transition from an arbitrary state to an equilibrium (say, the ways of moving an economy from a crisis to a stable development);
(iv) To clarify the laws of the creation of cooperation from purely individual preferences; these laws can be views as certain social analogs of the laws of the creation of order from chaos.

Some historic remarks

Sun Tsu "The art of war"
Girolamo Cardano "Liber de ludo aleae"
Leo Tolstoy "War and Peace"
Pascal's Wager
Cournot (possibly first formal realization of Nash equilibrium) and Bertrand
Frederick Lanchester: differential equations for military confrontation
Start of modern era: Borel and von Neumann

Chapter 2

Auctions and networks: static games of several players.

Nash equilibria, Braess paradox, Wardrop equilibria.

Volunteer's dilemma, second price auction, increasing bid auction, escalating conflict, Dutch auction.

Volunteer's dilemma

Distinction between *general will* and *the will of all* in one of the most influential books in political and social philosophy, "The Social Contract" of Jean-Jacques Rousseau: "there is often a great difference between the will of all and the general will; the latter regards only the common interest; the former regards private interests, and is merely the sum of particular desires". In the prisoner's dilemma the difference between "common interest" (everyone cooperates) and the "sum of particular desires" (to defect) is quite apparent.

A multi-person version of the chicken and sacrifice games is the "volunteer's dilemma" : disasters, military framework.

Volunteer's dilemma,II

Anecdote: there were four brothers called Anybody, Somebody, Everybody and Nobody who agreed to carry out some job with the following result:
Anybody could have done the job. Everybody thought Somebody would do it, but Nobody did it.

Party game: Ask all participants to submit a bid (written secretly on a piece of paper) asking for either 2 or 5 pounds. If, say, not more than 20% of participants ask for 5, then everyone gets what he/she asked for. Otherwise everyone pays a 1 pound fine.
Exercise. Find all Nash equilibria.

Example

Consider the following simple game with an arbitrary number of participants. All n participants announce simultaneously an integer between 1 and 100, say, k_1, k_2,...,k_n. A winner is a player, whose number turns out to be the nearest (in magnitude) to the half of the average of these numbers, i.e., to the number

$$\frac{k_1 + ... + k_n}{2n}.$$

All losers (those who are not winners) pay a dollar, and the obtained sum is equally divided between the winners. The following exercise shows that this game can be solved by iterated elimination of dominated strategies.

Example

Exercise. (i) First show that the strategy “announce 100” is strictly dominated by the strategy “announce 99”, and hence it can be excluded as a reasonable strategy of the game. Secondly show that when the strategy “announce 100” is eliminated, the strategy “announce 99” is strictly dominated by the strategy “announce 98”, and hence ”announce 99” also can be excluded as a reasonable strategy of the game using the second iteration of the method of elimination of strictly dominated strategies. Going on with this procedure (i.e. eliminating successively 100,99, 98, 97, etc.) show that the only strategy left is the strategy “announce 1”, which is therefore the only dominant strategy in this game.

Example

Exercise. (ii) One may be tempted to think that for arbitrary $k < l$ the strategy "announce l" (i.e the bigger number) is strictly dominated by the strategy "announce k", and hence choosing the smallest number is always optimal, and consequently to choose the dominant strategy "announce 1" one does not need any iterations of the method of elimination of dominated strategies. Show that this is not the case. (Hint: consider three players announcing numbers 2,30,100; half of the average is 22 and the winner is the player that announces 30, and not the smallest number 2.)
(iii) Show, however, that the case of only two players is quite special. In that case, for arbitrary $k < l$ the strategy "announce l" is in fact strictly dominated by the strategy "announce k", and hence the dominant strategy "announce 1" can be obtained by just one step of the method of eliminations.

Second price auction

There are n bidders that submit a single bid, in secret, to the seller of an object, i.e. each player i (with $i = 1, ..., n$) sends a letter to the seller announcing a bid for the object (a positive number), say, v_i. The winner of the object is the player with the highest number among all v_i, but he/she pays not the price he/she announced, but the next highest bid. This auction is sometimes called a Dutch auction, or a Vickrey auction.

Proposition. Truth telling (announcing as his bid his true value t_i, i.e., the price which he is willing to pay for the object) yields the unique Nash equilibrium for all players. (Assume no draw is possible.)

Second price auction

Proof. Let us show that truth telling is the (weakly) dominant strategy (and hence Nash).
To this end we shall compare the strategy t_i of player i with other strategies v_i. We have to consider two cases.
(i) Suppose i wins the object with the bid t_i, and suppose the second highest bid that he actually has to pay is $u < t_i$. Then if his bid were $v_i > u$, he would get the same win for the same price, but if his bid were $v_i < u$, then he would lose the auction thus getting worse payoff than with t_i.
(ii) Suppose i loses the auction (payoff is zero) with the bid t_i, and suppose the highest bid was $h > t_i$. Then if player i would bid $v_i < h$, he would lose anyway, thus changing nothing. If he would bid $v_i > h$, then he would win the auction but had to pay $h > t_i$, i.e. more than he would like to, thus having negative payoff, which is again worse than bidding t_i.

Second price auction

W. Vickrey got the 1996 Nobel Prize in economics, in particular, for his contribution to the theory of the Dutch auction.

One can analyze similarly the *flower auction* in the Netherlands, where one starts with a very high price, which is decreased. The first hand up gets the flowers.
Exercise. Truth telling is again a dominant strategy (thus yielding the unique Nash equilibrium).
Exercise. Compare with the usual auction, where the winner is again the player with the highest bid, but he pays the price he announced. Show that in this case truth telling is not a dominant strategy. What happens if the winner is supposed to pay the third highest price?

Increasing bid auction

There are n bidders in an open auction. The auctioneer begins by bidding at some initial price (stated in dollars, or thousands of dollars, etc.), and raises the price at the rate of one per second. All bidders willing to buy at the price stated put their hands up simultaneously. When only one hand is put up, this highest bidder is the winner and has to pay the price stated.

Exercise. Assume for simplicity that a strategy of any player i is to choose a price v_i and keep a hand up until the auctioneer's price goes higher than v_i. (In fact, it is easy to show that any other strategy is dominated by a strategy of this type.) Show that the optimal strategies are the same as for the second bid auction, i.e., the dominant strategy for each player i is to choose as v_i the actual amount he is willing to pay for the object.

Escalating conflict

Dollar auction game: An auctioneer auctions a prize of value v (say, US dollars) and the players (an arbitrary number of them) take turns in some fixed order. Each player either increases the bid by one or drops out of the game. The game ends when only one player remains. The difference with the usual auction is that both the highest bidder (who gets the prize v) and the second highest bidder (who gets nothing) must pay the auctioneer the amounts of their last bids.

A strategy p_i of player i is, of course, to keep bidding until the level of spent money goes beyond p_i (say, p_i is the amount of money in your pocket) or until everyone else has dropped out.
Exercise. Nash equilibria: one bidder i chooses the level $p_i \geq v - 1$ and all others $j \neq i$ choose $p_i = 0$. Bidder i then gets v.

Escalating conflict, II

The structure of Nash equilibria imply that if two or more players started to bid positive amounts, they would continue to escalate their bids without reaching an equilibrium thus sustaining considerable losses of money.

This is the Macbeth Effect (by Shakespeare): "I am in blood, Stepped in so far that, should I wade no more, Returning were as tedious as to go o'er".

One can observe that the dollar auction game can serve as a model of a variety of real conflicts, not the least important being the arms race that we already mentioned above in connection with prisoner's dilemma.

Escalating conflict, III

Moscow dollar auction: An auctioneer declares to several "players" that together they won a prize v (say, in dollars), but they have to compete for this prize by the following procedure. The players take turns in some fixed order. Each player can either add 10 dollars to the prize, or drop out of the game. The game ends when only one player remains. This player receives the whole prize.

Exercise. (i) Assuming everyone has an unbounded capital, the game above has no Nash equilibria. Hint: in any outcome, those who did not win, could act better than they did (in principle, of course). (ii) What are the equilibria if each player i has a fixed capital p_i?

Braess paradox

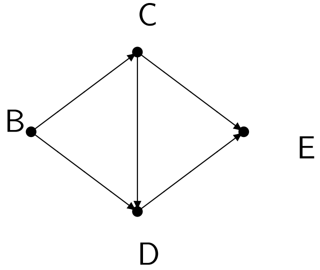


Figure 2.1

In the network of five one-way roads in Figure 2.1 six drivers have to go from B to E. The time needed to traverse each segment depends on the number of cars, n, using this segment and equals (in some units) $10n$ for BC and DE, $50+n$ for BD and CE, and $10+n$ for CD.

Braess paradox

The drivers wish to minimize their driving time by choosing (simultaneously and independently) one of the three possible routes: BCE, BDE, BCDE. This leads to a six-player game, whose outcome can be compactly described by three integers x_1, x_2, x_3: the number of of drivers that have chosen the first, the second and the third route, respectively.

For any such outcome the time of travel is
$10(x_1 + x_3) + 50 + x_1$ for the first route,
$50 + x_2 + 10(x_2 + x_3)$ for the second
$10(x_1 + x_3) + 10 + x_3 + 10(x_2 + x_3)$ for the last one.

Braess paradox

It is not difficult to see that in a Nash equilibrium all these times should be equal (otherwise some of the drivers could do better by switching to another route) leading to the system of equations

$$10(x_1 + x_3) + 50 + x_1 = 50 + x_2 + 10(x_2 + x_3)$$

$$= 10(x_1 + x_3) + 10 + x_3 + 10(x_2 + x_3),$$

$$x_1 + x_2 + x_3 = 6.$$

Exercise 2.7. Show that this system has a unique solution $x_1 = x_2 = x_3 = 2$ yielding the unique Nash equilibrium with total driving time 92.

Braess paradox

Suppose now that segment CD is closed and hence the third route is not available. Similar (and simpler) considerations show that the unique Nash equilibrium becomes $x_1 = x_2 = 3$ leading to a total driving time of 83. Thus reducing the network leads to a better result or, the other way round, increasing capacity yields a worse equilibrium. This effect is called the *Braess paradox*.

Individual choice of shorter (or cheaper) routes, often at the expense of overall performance of the network, is called *selfish routing* in the literature.

Wardrop equilibria

Now: infinitely many drivers. Work in terms of frequencies.
The cost of transportation for an individual: 0 along CD, 1 for BD and CE, the proportion x of the traffic along the segment for BC and DE.
Again there are three paths; x_1, x_2, x_3 the proportions of traffic driving again these routes.
The costs of driving along these routes: $x_1 + x_3 + 1$, $x_2 + x_3 + 1$, $x_1 + x_2 + 2x_3$.
Normalizing condition: $x_1 + x_2 + x_3 = 1$.
The analog of the Nash equilibrium in such networks is called the *Wardrop equilibrium*. As above, in a Wardrop equilibrium, the proportions of traffic going through each possible route are such that the costs of following each route are the same:

$$x_1 + x_3 + 1 = x_2 + x_3 + 1 = x_1 + x_2 + 2x_3, \quad x_1 + x_2 + x_3 = 1.$$

The system has the unique solution $x_1 = x_2 = 0$ and $x_3 = 1$, leading to a cost of transportation of 2 along each route.

Wardrop equilibria

Now, if segment CD is closed and the third route is not available, similar considerations lead to the solution $x_1 = x_2 = 1/2$ with the cost of transportation being $3/2$ (again better than with the road CD).

The ratio of costs under the Nash (or Wardrop) equilibrium (2) and the optimal flow (3/2) is 4/3.

The remarkable and surprising result of the theory: the ratio 4/3 is universal and holds under very general assumptions about the networks (latency of edges is a linear function of their congestions).

Chapter 3.

Backward induction and repeated games.

Subgame perfection, paradoxes of backward induction, extensive and normal form of the game, computer tournaments, infinitely repeated games, tit-for-tat strategy, logical games.

Chain store paradox, Market-entry game, Lion and Man, Ultimatum game, etc.

Dynamic games: Backward induction

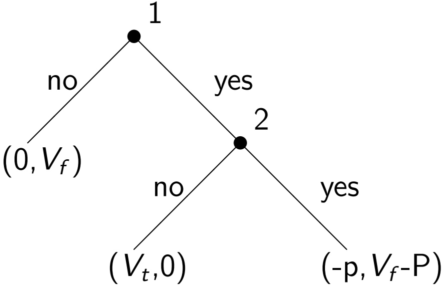
Wise men and their wives.
Imp in the bottle.
Unexpected punishment.
King Solomon's wisdom.

Solomon's wisdom

Better model (Glazer and Ma):
Choose some rough numeric estimates for the values V_t and V_f of the child to the true mother and the false mother respectively. Of course, $V_t > V_f$. Then the women are to play "a game" with all rules and payoffs explained beforehand.
Step 1. The first woman is asked "Is it your child?" If the answer is "No", the child is awarded to the second woman, and the game is over.
Step 2. If the answer is "Yes", then the same question is posed to the second woman: "Is it your child?" If the answer is "No", the child is awarded to the first woman. If the answer is "Yes", the child is awarded to the second woman, but they both get a punishment (money fine, or a prison sentence, or whatever), the punishment of the second woman being of value P and that of the first woman being p, where the values P and p are chosen in such a way that $0 < p < V_f < P < V_t$.

Solomon's wisdom

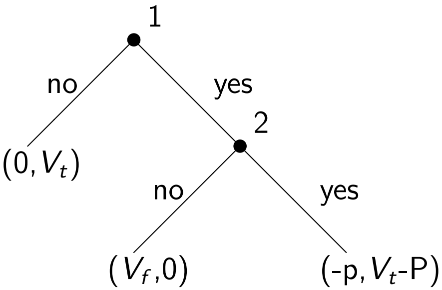
If the first woman is the true mother, the course of the game can be described graphically by the following *game tree*.



where the first number in all brackets always designates the payoff for the first woman, and the second number for the second woman.
Backward induction arguments:

Solomon's wisdom

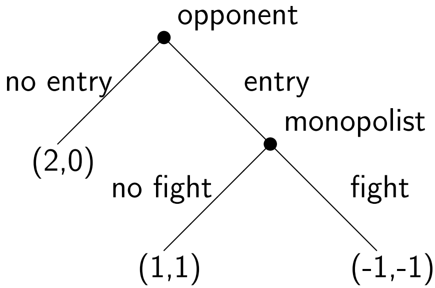
If the second woman is the true mother:



Exercise. Show that the reasonable behavior for the first woman (now the imposter) is to say "No" leaving the child to the second woman, i.e. the child again is awarded to the true mother.

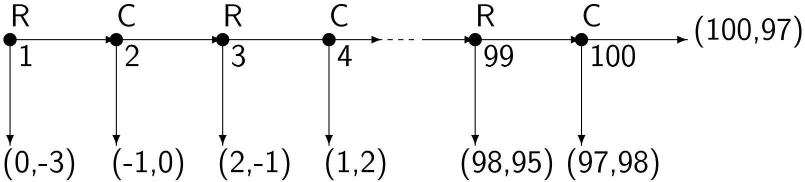
Chain store paradox

Market-entry game:



Chain store paradox. Selten's Nobel Price.

Centipede game



Exercise. Argument of backward induction dictates that the game will be stopped by the first player at the first move (downward).

Curse of backward induction

Solution contradicts common sense. Experiments show: Players choose the right moves (cooperation) long enough, and choose a downwards move (defecting) closer to the end.

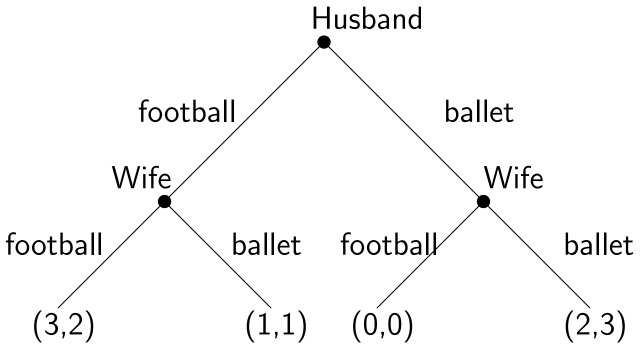
The discrepancies between theoretical predictions and experimental results are common in such situations and are called the *paradoxes of backward induction* or the 'curse of backward induction'. Are the players irrational?

Extensive and normal form of the game

Dynamic games can be described by game trees, also called *extensive forms of games*. In these trees one just denotes by a point each possible position of a game indicating also which player has the right to move in each particular position, and then connects these points by lines if there is a legitimate move of a player which allows a transition between these positions. A description of a dynamic game in a normal form is also possible, but as we mentioned above, the normal form does not catch the dynamics explicitly and hence is not very illustrative.

Example: Dynamic battle of the sexes

Assuming the husband makes the first choice:



Exercise. Backward induction leads to the solution (3.2).
Discussion: biological analogs for the first move advantage.

Example: Dynamic battle of the sexes (normal form)

The wife has now four strategies: ff, fb, bf, bb, where the first letter indicates her reaction to her husband's choice f, and the second letter indicates her reaction to her husband's choice b:

		Wife			
		ff	fb	bf	bb
Husband	f	**3,2**	**3,2**	1,1	1,1
	b	0,0	2,3	0,0	**2,3**

There are three Nash equilibria, given in bold. However, only one of them, (f,fb), corresponds to the solution found by backward induction. Hence we conclude that the usual notion of the Nash equilibrium is not sufficient for a successful analysis of dynamic games.

Dynamic games

A *dynamic game* G is given by (i) a number of players; (ii) a set of its possible positions that are divided into two classes: terminal and intermediate positions; (iii) with each intermediate position P there is associated a *local game* G_P usually specified by its normal form with its set of (*local*) strategies and payoffs and a rule that specifies the next position of the game for each profile of local strategies applied at P; (iv) a starting position.

The method of backward induction can be applied for any finite game (that always terminates after a finite number of local games) starting the argument from all terminal positions.

Subgame perfection

But there is another method to distinguish the solution obtained by backward induction, that is more general and can be applied also to infinite games.
A *subgame* of a dynamic game is a game that is conducted by the same rules as the initial game but is started not necessarily from the initial position of the original game, but at any position that can be in principle reached by playing the original game. A Nash equilibrium in a dynamic game is called *subgame perfect* if it remains a Nash equilibrium when reduced to any subgame.

For the battle of the sexes, there are three subgames: initial and the two "small" subgames starting at two positions that can be reached by the husband. Nash equilibria (f,ff) and (b,bb) are not subgame perfect, as they include the actions "answer by f to b" and "answer by b to f" respectively, which are not optimal for the corresponding small subgames.

Games of pursuit and evasion: example

A policeman is placed in the middle of a square and a gangster is in its corner. The policeman can move freely inside the square, and the gangster can move only along its boundary, but the speed of the policeman is two times less than that of the gangster. The objective of the policeman is to achieve a position, where he finds himself on the same side of the square as the gangster.

Exercise. Show that the policeman has a strategy that allows him to achieve his goal. Moreover, the maximal time needed equals the time the policeman moves from a corner of the square to a neighboring one.

Famous example: Lion and Man

It plays on a unit disc on a plain. The Lion starts in the middle O and the Man at any other point A in the interior of the disc. They have equal speeds and cannot touch the boundary of the disc. The Lion wants to catch the Man, that is to turn out in the same point of the disc. The problem of whether the catch was possible has been discussed for some period of time, and it was a surprise for many that the Man has a strategy allowing him to avoid the catch in any finite time.

The main idea is seen when assuming the pursuit is organised by the following simplified discrete setting. The Man moves a distance l_1 from A to a point A_1 and then the Lion moves the same distance l_1 from O to the point O_1, which is the nearest possible to A_1. Then the Man moves a distance l_2 from A_1 to some point A_2 and the Lion moves the same distance l_2 from O_1 to the point O_2, which is the nearest possible to A_2, and so on.

Famous example: Lion and Man

The winning strategy of the Man is as follows. For any j, at a point A_j he chooses A_{j+1} at a distance l_{j+1} in such a way that the segment A_jA_{j+1} is orthogonal to the segment O_jA_j. Moreover, the sequence of distances $\{l_j\}$ is chosen so that $\sum_{j=1}^{\infty} l_j^2 < 1$ (this condition ensures that the Man always remains in the interior of the disc), but $\sum_j l_j = \infty$, which ensures that the pursuit will last for ever.

The situation is quite different if the Lion and the Man are not considered as points!

Ultimatum game

The game is for two persons. The auctioneer gives 10 dollars to the first player that he/she has to divide with the second player by offering him/her any integer amount of dollars (e.g. 1,2,3,...,10). If the second player agrees to take the offer, the first player keeps the rest for himself. However, if the second player rejects the offer, then the whole amount is returned to the auctioneer.

Clearly first player has 10 strategies (to offer 1,2,...,10). The number of the strategies of the second layer is 2^{10} (a strategy of the second player is characterized by a subset of the set of first 10 numbers that he/she is willing to accept).

Ultimatum game

Exercise. There are lots of Nash equilibria in this game (in fact 2^{10}), for instance (offer 6, accept only 2,5, or 6), but there is only one subgame perfect equilibrium that can be also obtained by the arguments of backward induction, namely (offer 1, accept any offer).

The ultimatum game is almost as popular in game theory as the prisoner's dilemma, and a lot of experimental work has been carried out on this game.
Even the comparison between the behaviors of people living in different parts of the globe was investigated.

Infinitely repeated games

Backward induction for repeated games.
The idea of undefined end point.
Repeated games lasting infinitely many times with discounting on each stage by a fixed factor $\delta < 1$.

		C	
		defect	cooperate
R	defect	$\delta^k p,\ \delta^k p$	$\delta^k q,\ \delta^k r$
	cooperate	$\delta^k r,\ \delta^k q$	$\delta^k s,\ \delta^k s$

Practical (bank accounts) and reflects the general philosophy of "Who knows what happens after ten years anyway".

Cooperation by threat and punishment

Denote by c and d the cooperative and defective strategies of the prisoner's dilemma (general form). As our game will be symmetric we shall write $\Pi(d, d)$, $\Pi(c, d)$, $\Pi(d, c)$, $\Pi(c, c)$ for the payoffs of the player (irrelevant if the first or the second) playing respectively d against d, c against d,d against c,c against c. Thus

$$\Pi(d, d) = p, \quad \Pi(d, c) = q, \quad \Pi(c, d) = r, \quad \Pi(c, c) = s.$$

If both players are confined to use always one and the same strategy, i.e. either playing "cooperate" all the time, which strategy we denote by C, or playing d all the time, which strategy we denote by D, then the payoff table of this game becomes the same as for a one stage game, but with all payoffs divided by $(1-\delta)$. In fact, say,

$$\Pi(C, C) = \Pi(c, c)(1 + \delta + \delta^2 + \delta^3 + ...) = \Pi(c, c)\frac{1}{1-\delta}.$$

Cooperation by threat and punishment

Of course, the main interest of an infinite game is the possibility to vary the action from stage to stage. It turns out that unlike a finitely repeated game, threats to punish can now become credible, which can lead to stable cooperation. To see this, let us define the so called *trigger strategy* C_T: start by cooperating (playing c) and continue to cooperate until your opponent defects (plays d), then defect forever after. More precisely, it specifies that if you or your opponent defect at least once, then you defect forever after. One sees directly that if each player plays either C_T or C, the game develops as if both have adopted the cooperating strategy.

Theorem. (C_T, C_T) is a symmetric Nash equilibrium in the above infinitely repeated game (based on the one-shot prisoner's dilemma) whenever

$$\delta \geq \frac{q-s}{q-p}.$$

Cooperation by threat and punishment

Proof. We have to show that C_T is the best response to C_T:

$$\Pi(C_T, C_T) \geq \Pi(S, C_T)$$

for any strategy S. We know already that playing C against C_T yields the same payoff as playing C_T against C_T. So we only need to consider a situation when S includes d on some stage of the game. But whenever the first player plays d, the second player (with the trigger strategy) will play d for ever afterwards, so in order to give a best reply, the first player has also to play d forever afterwards. In other words, the best responses strategies S can be only of the form S_k (with some integer k): play c the first k steps, and then switch to d forever after. Consequently, in order to convince ourselves that C_T is the best response to C_T we only need to check that

$$\Pi(C_T, C_T) \geq \Pi(S_k, C_T).$$

Cooperation by threat and punishment

Proof (cont.)

$$\Pi(C_T, C_T) = \Pi(c,c)(1+\delta+\delta^{k-1}) + \Pi(c,c)(\delta^k+\delta^{k+1}+...),$$

$$\Pi(S_k, C_T) = \Pi(c,c)(1+\delta+\delta^{k-1}) + \Pi(d,c)\delta^k$$
$$+\Pi(d,d)(\delta^{k+1}+\delta^{k+2}+...).$$

Thus $\Pi(C_T, C_T) \geq \Pi(S_k, C_T)$ is equivalent to

$$\Pi(c,c)(\delta^k+\delta^{k+1}+...) \geq \Pi(d,c)\delta^k+\Pi(d,d)(\delta^{k+1}+\delta^{k+2}+...),$$

$$\Longrightarrow \Pi(c,c)\frac{1}{1-\delta} \geq \Pi(d,c) + \Pi(d,d)\frac{\delta}{1-\delta}.$$

Cooperation by threat and punishment

Proof (cont.) Substituting yields

$$\frac{s}{1-\delta} \geq q + \frac{p\delta}{1-\delta},$$

or

$$s \geq q(1-\delta) + p\delta,$$

or $\delta > (q-s)/(q-p)$, as required.
So, a nice solution to the problem of the natural appearance of cooperation is found.
A drawback of the proposed approach lies in the observation that there exists a tremendous number (in fact, infinite) of Nash equilibria in the repeated game considered.

Cooperation by threat and punishment

Exercise. Show that under conditions of the above theorem C_T is also subgame perfect. (We make C_T more precise by claiming that C_T cooperates until any of the player defects, then defect forever after).
Hint: subgames that differ from the original one, are specified by the previous round of the game that can be (c, c), (d, c), (c, d), (d, d). From the first of these positions you would play as from the initial one, and from the other three position C_T turns to D, which is again a Nash equilibrium.
Exercise. Suppose each player alternates $c, d, c, ...$ as long as other player does the same, and each would change to d forever whenever the opponents deviates at least once from the adopted pattern $c, d, c, ...$ Show that these pair of strategies also define a subgame perfect symmetric Nash equilibrium for δ being close enough to one.

Tit-for-Tat

The most famous strategy for repeated games is the so called *Tit-for-Tat strategy* T. Tit-for-Tat means "I give to you, what you give to me" or "tooth for tooth and eye for eye" and is defined as a strategy that starts with playing c and then in each stage plays the strategy that the opponent played on the previous stage.

Exercise. Show that (T, T) is a Nash equilibrium, which is however not subgame perfect. (Hint: to show the latter, consider the subgame that follows a round specified by situation (c, d). Then (T, T) specify permanent alternation (d, c), (c, d), (d, c), etc. But alternating $d, c, d, ...$ is not the best reply to the opponent strategy that alternates $c, d, c, ...$)

Cooperation by threat and punishment

The same results can be achieved without any discounting, but if one assumes that after each stage of the game it will stop (never occurs again) with some probability $p > 0$ (which implies, in particular, that almost surely the game will last only a finite number of times).
Exercise. Show that all payoffs (their expectations) are the same as in case of discounting with $\delta = 1 - p$.

In justifying cooperation among humans playing a prisoner's dilemma one can now argue that the human mind treats a large number of repetitions practically as infinity and refer to the above theory.
Another argument claims that it is too much to ask from an individual to carry out large number of rounds of the iterated eliminations of weakly dominated strategies.

Computer tournaments

Axelrod' s experiment:. He invited scientists (game-theorists, sociologists, psychologists, economists) to submit iterated prisoner's dilemma strategies for a computer tournament. Each strategy had to be a computer program that specifies precisely what to do on each step of a game repeated 200 times given the complete history of the interaction. The tournament was organized as a sequence of pairwise contests where each strategy submitted was played 200 times in a repeated game specified by the table

	C		
		defect	cooperate
R	defect	1,1	5,0
	cooperate	0,5	3,3

against all other submitted strategies, against itself, and against a program that chooses cooperation or defection randomly.

The triumph of the strategy Tit-for-tat

To get an overall score, Axelrod averaged each strategy's scores. As the maximum win is 5 in each round, the scores could be any positive number not exceeding 1000.

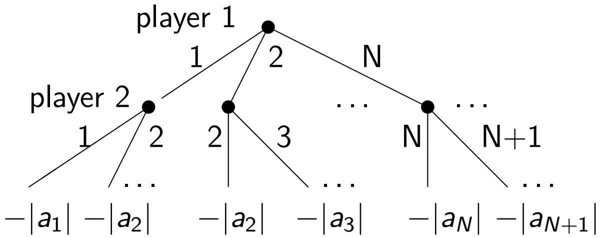
Fourteen strategies were submitted to the first tournament, some of them quite complicated. The highest score of 505 points was achieved by the Tit-For-Tat strategy submitted by A. Rapoport.

Second tournament!
Third tournament with the natural selection!

Logical games

A payoff P is called *attainable (or guaranteed) with arbitrary precision* for a player, if this player can ensure a payoff which is arbitrary close to P: i.e., for an arbitrary $\epsilon > 0$, this player has a strategy that guarantees him a payoff not less than $P - \epsilon$, whatever strategy is used by his opponent.

Now consider the two-player zero-sum game with an infinite number of strategies, given by the tree



Logical games

In other words, the first player can choose any positive integer, say N, and then the second player can choose any integer, say n, that is not less than N. The payoff to the first player becomes $-|a_n|$, which means that he has to pay $|a_n|$ to the second player. The following observation allows for a game theoretic interpretation of the notion of the limit.

Exercise 3.12. Convince yourself that the sequence a_n tends to zero as $n \to \infty$ if and only if the payoff zero is attainable with arbitrary precision for the first player in the game given above.

Banach-Mazur game

A set $A \subset [0,1]$ is fixed. First player, Banach, starts by choosing any subinterval in $[0,1]$, then the second payer, Mazur, chooses a subinterval of the first interval, then Banach chooses a further subinterval of the second interval and so on with additional condition that the lengths of the intervals should eventually tend to zero, so that one final point F will be left at the end. If $F \in A$, the first player wins, otherwise the second wins.

Question: for which A there exists a winning strategy for at least one of two players? Such A are called determinable sets.

Example: If $A = \{a_1, a_2, \cdots,\}$ is countable, the second player has a winning strategy: choose an interval not containing a_1, then an interval not containing a_2, etc.

Axiom of determination: Each set is determinable. It implies: all subsets of $[0,1]$ are Lebesgue measurable, all bounded functions are Lebesque integrable, etc!

Chapter 4.

Aggregation of preferences: elections, social agreement, optimal distribution.

Voting rules, manipulations, minority principle, theorem on dictator, strategic voting, 4 principles of fair distributions, utilitarianism and egalitarianism, simple cooperation games and coalitions, cost-sharing game, core of a game.

Collective choice; problems of voting

Jean de Borda and Jean Condorcet. The tragic fate of marquis de Condorcet.

Several agents (or electors) must make a choice between a given collection of candidates (to choose one of them). Candidates are understood here in a broad sense as people (say, candidates for presidency), variants of a law, etc. Why there are different rule for voting in different countries?

Two candidates – clear: majority. If more : plurality. But:

$$\begin{array}{ll} 4 & a > b > c \\ 3 & b > c > a \\ 2 & c > b > a \end{array}$$

The winner is a who is the worst candidate for the majority (5 out if 9).

Condorcet winner

Condorcet winner is the one (if exists), who beats all others in pairwise contests. This is b in the example above: beats c (7 out of 9) and a (5 out of 9).
But this may not exist. *Voting paradox* (or the *Condorcet paradox*):

$$\begin{array}{ll} 1 & a > b > c \\ 1 & b > c > a \\ 1 & c > a > b \end{array}$$

or

$$\begin{array}{ll} 10 & a > b > c \\ 11 & b > c > a \\ 12 & c > a > b \end{array}$$

Condorcet consistent rule: gives Condorcet winner when exists.

Examples of voting rules

Plurality with runoff (Russia and France presidential election).

Sequential majority comparisons.
(widely used in the congress of USA when voting for a new law and its amendments).

The Borda rule (a scoring voting rule). Each agent declares its full preference on the list of all candidates putting them in order from the best to the worst, say, $a > b > c$ in case of three candidates. Then each candidate gets the corresponding points: the worst gets 0, the next worst gets 1, etc. Then the *Borda score* is calculated for each candidate, which is the total score (sum of points obtained from all agents). The winner is the candidate with the highest total score.

Copeland rule

The Copeland score, say for a: the number of agents that a beats in a pairwise contest minus the number of agents that beat a in a pairwise contest.
Winner: the highest score.
Exercise. For the profile

$$\begin{array}{ll} 1 & a > b > c > d > e \\ 4 & c > d > b > e > a \\ 1 & e > a > d > b > c \\ 3 & e > a > b > d > c \end{array}$$

show that the Borda scores and Copeland scores arrange the candidates in opposite orders.

Condorcet consistency

Plurality with runoff is not Condorcet consistent:

$$\begin{array}{ll} 2 & a > c > b \\ 2 & b > c > a \\ 1 & c > a > b \end{array}$$

Neither the Borda rule is Condorcet consistent, as the profile

$$\begin{array}{ll} 3 & b > c > a \\ 2 & c > a > b \\ 1 & c > b > a \\ 1 & a > b > c \end{array}$$

shows, where the Condorcet winner is b, but the Borda score for candidates a, b, c are 3, 8, 9 respectively, and hence c wins.

Criteria of quality of voting rules

Anonymity. The names of agents are irrelevant.

Neutrality. The name of candidates are irrelevant.

Clearly these properties must hold, if we want all agents and candidates to have equal rights.

Effectiveness or Pareto optimality. If a is better than b for all agents, then b can not win the elections.

Let us check the validity of these basic properties (axioms) on our model examples given above.
It is easy to see that the plurality rule, plurality with runoff, the Copeland and the Borda rules all satisfy the axioms of anonymity, neutrality and efficiency.

Criteria of quality of voting rules

On the other hand, the method of sequential majority comparisons (though being anonymous) is neither neutral nor efficient.

Monotonicity (and manipulation). Suppose a candidate *a* wins under a certain profile of preferences, and then this profile is changed in such away that the position of *a* is improved and the relative comparison of any other pair of candidates remains the same for all agents. Then *a* should win under the new profile as well. Shortly, improving the position of a candidate can not lead to the defeat of this candidate.

This property sounds as obvious. In fact, it is easy to check that the plurality rule and the rules of Copland and Borda are monotone.

Criteria of quality of voting rules

But the plurality rule with runoff is not monotone, which the following two profiles show

6	$a > b > c$	6	$a > b > c$
5	$c > a > b$	5	$c > a > b$
4	$b > c > a$	4	$b > c > a$
2	$b > a > c$	2	$a > b > c$

In fact, under the first profile, a and b go to the second round and a wins. The second profile is the same but for two agents who change $b > a > c$ to $a > b > c$ so that a becomes better than before. At the same time, under this profile a and c go to the second round and c wins.

Strategic voting

At last, let us consider the following two axioms.
Reinforcement. If two disjoint groups of agents consider the same set of candidates, and both these groups choose a candidate a (voting independently of each other), then a should win if these groups vote together.

Axiom of participation. Suppose a group of agents chooses a candidate a from a given set of candidates. Suppose another agent joins this group of agents. Then the new group will choose either a or a candidate which is better than a in the opinion of the new agent.

Strategic voting

The reinforcement axiom seems to be quite reasonable when a choosing organ is subdivided on several sections (chambers of a parliament, regional committees of a party, etc). And the axiom of participation ensures that for any agent it can only be useful to take part in voting and to express frankly his/her opinion.

The non-fulfilment of the axioms of monotonicity, reinforcement or participation is a key to the game theory involvement in the analysis of voting, because this indicates that telling truth is not always the best strategy to achieve one's goal. This leads to *strategic voting*.

Minority principle

So far, we discussed the quality of voting rules from the general point of view that a fair rule should reflect the preferences of the majority of voters in the best way. This point of view can not serve as a sole guide for fairness. The majority of, say, 51% of agents ignoring the opinion of the minority can become a real dictator.

"However the minority can not depend completely on the majority: justice which is the negation of force, demands that minority has its guarantees.".

It is surely not fair, when a superpower invades a weak country and rules it (in the full agreement with the principle of majority). This arguments lead to the search of voting rules satisfying *the minority principle*, which states that any coalition, whatever small, can exercise at least some influence on the decision making.

Minority principle

Voting by successive veto. The agents that are ordered in some way, declare their preferences. Then the least preferred candidate of the first agent is eliminated, then from the remaining set the worst candidate of the second agent is eliminated, etc. This continues until only one candidate remains. He is declared the winner.
This rule allows any coalition of k agents to exclude any set of k candidates, whenever k is less than the number of candidates.

Dictators

Let the set of agents $I = \{1, \ldots, N\}$, and the set of alternatives $A = \{x, y, \ldots, z\}$ are given. Each agent has its preference on the set $A = \{x, y, \ldots, z\}$, that is a permutation of A. The set of such permutations is denoted by $L(A)$.
The point $R(N) = (R_1, \cdots, R_N)$ in the product $(L(A))^N = L(A) \times \cdots \times L(A)$ (N times) is called the preference profile. Building the map $f\colon (L(A))^N \longrightarrow L(A)$ is the *preference aggregation problem*. The natural conditions for such *preference aggregation function f* are:
Condition 1 (Pareto, or a unanimity condition). If the preferences of all agents are the same and equal to R, that is $R_i = R$ for all i, then $f(R(N)) = R$.
Condition 2 (Independence of Irrelevant alternatives). Group preference $f(R(N))$ on any pair of alternatives (x, y) depends only on the preferences of agents on the same pair of alternatives. *Condition 3 (Monotonicity)*.

Dictators

The following Arrow's theorem (on dictators) on the aggregation of preferences is valid.

Theorem. Let the number of alternatives be not less than three. Under the conditions 1- 3, $f(R(N)) = R_i$, where i is some fixed agent, that is, the only way to aggregate the preferences of society is to choose a leader representing its interests (which is a well established practice for many centuries, both in the animal world and in any society).

Dictators

Proof. We shall write $x >_i y$, if the agent i prefers x to y. For a coalition K (a subset of the set of agents) we write $x >_K y$ if $x >_i y$ for all $i \in K$.
Let us denote by $W(x, y)$ the set of coalitions K such that $x >_K y$ implies $x > y$. For example, $I \in W(x, y)$ because of the conditions of unanimity. Let $W = \cup_{x,y} W(x, y)$ and C be a minimal coalition of W (that is, no proper subset of C belongs to W). Then C belongs to some $W(x, y)$. We claim that C consists of a single agent. Indeed, let $i \in C$ and z be an alternative other than x and y. Let us consider the following profile $R(N)$:

$$x >_i y >_i z, \quad z >_{C \setminus \{i\}} x >_{C \setminus \{i\}} y, \quad y >_{\bar{C}} z >_{\bar{C}} x,$$

where $\bar{C}$ is the complement of C in $\{1, \cdots, N\}$ and where the preferences on other agents are not indicated, since they are irrelevant by Condition 2. Let us write $x > y$ if x is preferred to y by the collective preference $f(R(N))$.

Dictators

Proof (cont.) As $x >_C y$ and $C \in W(x, y)$, then $x > y$. Since $z >_{C\setminus\{i\}} y$ and otherwise for other agents, and $C \setminus \{i\} \notin W(z, y)$, the preference $z > y$ is not satisfied, so that $y > z$. By transitivity, $x > z$. But $x >_i z$ and otherwise for other agents. Hence $\{i\} \in W(x, z)$ for any $z \neq x$. Thus C is not minimal leading to contradiction.

It remains to check that the agent i is decisive for any pair of alternatives. Take two alternatives y, z other than x, and consider the profile $Q(N)$:

$$y >_i x >_i z, \quad z >_{N\setminus\{i\}} y >_{N\setminus\{i\}}> x.$$

Let us now write $x > y$ if x is preferred to y by the collective preference $f(Q(N))$. We have $x > z$ (since $\{i\}$ belongs to $W(x, z)$) and $y > x$ by unanimity and independence from exterior alternatives). Hence $y > z$ and thus $\{i\} \in W(y, z)$, so that i is the leader.

Principles of fair distribution

There are two related approaches:
(1) the players have to decide a fair solution between themselves. These kind of problems are usually referred to as bargaining problems.
(ii) a solution can be supplied by an external advisor, sometimes called a benevolent dictator.

Philosophical basis for the theory of distribution justice is supplied by Aristotle's thesis: "Equal should be treated equally, and unequals unequally, in proportion to the relevant similarities and differences".
(Recall the "Animal farm": "All animals are equal, but some of them are more equal than others".)

Principles of fair distribution, II

More Aristotle: a tyrant arises from the people against the nobility, ... so that the people does not suffer injustice from them, ... the justice by general agreement is some kind of equality.

Plato: Tyrant is put forward by the people. Punishing the nobles he promises the cancelation of debts and redistribution of land.

Kautilya (Chanakya): if power is not effective, the habit of fishes arises: the strong eat the weak (Arthashastra, the Science of politics).

Principles of fair distribution, III

Four simple ideas govern the practical applications of this principle revealing more clearly what should be meant by "relevant similarities and differences".

These ideas are *exogenous rights, compensation, reward and fitness.*

Flute assignment problem (Plato).

Principles of fair distribution, IV

The fitness argument demands that resources go to whomever makes the best use of them for the benefit of all. Ideologically it is close to the program of classical *utilitarianism* that favors a distribution that maximizes the overall utility of a communion (say, the sum of all individual utilities) ignoring the needs of particular members of a community.

Exogenous rights: classical *egalitarianism*.
Unequal exogenous rights are also common, say shareholders in a publicly traded company, or parties of different size in a parliament should have unequal shares of decision power.

Principles of fair distribution, V

Exercise. Suppose there is a shortage of a medical supply (or care) in a military hospital. Identify the principles, on which the following policies of distribution are based:
favor the most severely wounded,
favor the bravest soldiers,
give priority according to a military rank,
maximize the number of recoveries that allow to return to fighting (in particular, favor several lightly wounded soldiers and sacrifice a badly wounded one).

Principles of fair distribution, VI

Exercise. *The lifeboat story*. There is a limited number of seats in a boat that can save passengers after a shipwreck (recall Titanic). Describe various policies that can be used to select people to be taken on the boat, and for each policy specify one of four basic principles, which this policy is based on. What would be your policy, if you would have to decide?
Exercise. In his "Republic" Plato proposed to place philosophers at the reigns of government. Which of the four basic principles can be used to support such a proposal?

Would you agree? Practicality? Plato's real experience.

Modern era: the development of artificial intelligence, see the film 'I Robot'.

Utilitarianism and egalitarianism

An acknowledged classic of the philosophy of *utilitarianism* was Jeremy Bentham, who was called by Karl Marx in his 'Capital' "the genius of bourgeois stupidity" not withstanding the fact that Robert Owen, a student of Bentham, became later a founder of the socialism.
Roughly speaking the *utilitarianism* calls for the assessment of collective actions of a society only on the basis of the utility levels of its members (free wills of individuals).

Given n utility functions (e.g. payoffs) $U_1, ..., U_n$ on a set of outcomes X (so that an agent i prefers an outcome x to an outcome y whenever $U_i(x) > U_i(y)$) the problem is to work out a *collective utility function* (CUF) U on X. The maximum of this U would be then considered as the best outcome for the group of n agents.

Utilitarianism and egalitarianism

Assuming that all U_i are expressed in the same units the *classical utilitarianism* suggests to choose the CUF

$$U(x) = U_1(x) + ... + U_n(x),$$

i.e. to think only of the total utility. Another reasonable choice is the *egalitarian CUF*

$$U(x) = \min(U_1(x), ..., U_n(x)),$$

that concentrates on the best outcome for the weakest agent. Applying classical utilitarian CUF could lead to the situation, when the most capable members gets the lowest rewards (thus urging them to hide their talents) and using egalitarian CUF to its extremes could lead to blocking the work of the whole system due to a weakness of a single member.

Distribution of costs and gain; core of a game

Consider a construction by neighboring cities (or houses, or firms), of a supply system (of some product of common use, like water, electricity, gas, etc). For simplicity, we shall discuss the case of three cities only.
Suppose the costs for the construction of the system for the three cities A, B, C are given by $c_1 = 20$, $c_2 = 20$, $c_3 = 50$ (thousands of dollars, or rubles etc) respectively.

However, if the three cities join their efforts they can manage the same construction with the better cost $c_{123} = 60$, than $c_1 + c_2 + c_3$ independently. And at last, any pair of cities can create a coalition and construct a system on their own with the costs $c_{12} = 15$, $c_{13} = 54$, $c_{23} = 54$ for the first and the second, first and third, second and the third cities respectively.

Distribution of costs and gain; core of a game

The question: is it reasonable for the three cities to join their efforts (which seems intuitively clear), and what should be a fair distribution of the cost c_{123} between them?
The first idea is to share the cost equally so that each city pays $c_{123}/3 = 20$, which of course is unreasonable, because the first two players can do much better forming a coalition without the third player. In fact $c_{12} < 40$.
The next more sophisticated idea could be to calculate the global cost saving $c_1 + c_2 + c_3 - c_{123} = 30$ and share it equally between the cities giving the allocation of costs
$20 - 30/3 = 10$ for the first and the second city, and
$50 - 30/3 = 40$ for the third one. But this distribution should again be rejected by the same reason as above, namely the first two cities can do better forming a coalition with the cost $c_{12} = 15$, than with what they should pay $20 = 10 + 10$ according to the distribution proposed above.

Distribution of costs and gain; core of a game

Conclusion: any reasonable distribution of costs should satisfy the *stand-alone principle*: any coalition should not pay more than what it needs to pay to provide service by itself.

A *cost-sharing game* for three players is defined by a set of costs (non-negative numbers) c_1, c_2, c_3 that each player should pay on its own, the set of costs c_{12}, c_{13}, c_{23} that all possible coalitions of two players should pay, and the cost c_{123} required if all players join their efforts. A *cost allocation* is any collection of non-negative numbers x_1, x_2, x_3 such that $x_1 + x_2 + x_3 = c_{123}$.
A cost allocation is said to belong to the *core of the game* if the *stand-alone principle* is satisfied:

$$x_1 + x_2 \le c_{12}, \quad x_2 + x_3 \le c_{23}, \quad x_1 + x_3 \le c_{13},$$
$$x_1 \le c_1, \quad x_2 \le c_2, \quad x_3 \le c_3.$$

Distribution of costs and gain; core of a game

Let us calculate the core of the game in our example above. The above conditions become

$$x_1 + x_2 \leq 15, \quad x_2 + x_3 \leq 54, \quad x_1 + x_3 \leq 54,$$

$$x_1 \leq 20, \quad x_2 \leq 20, \quad x_3 \leq 50.$$

It is convenient to rewrite the problem in terms of the gain-sharing, namely in terms of $y_1 = c_1 - x_1 = 20 - x_1$, $y_2 = c_2 - x_2 = 20 - x_2$, $y_3 = c_3 - x_3 = 50 - x_3$ so that

$$y_1 + y_2 + y_3 = c_1 + c_2 + c_3 - c_{123} = 30,$$

$$y_1 + y_2 \geq c_1 + c_2 - c_{12} = 25,$$

$$y_2 + y_3 \geq c_2 + c_3 - c_{23} = 16,$$

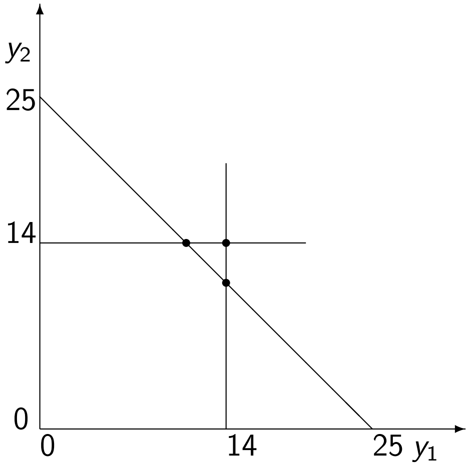
$$y_1 + y_3 \geq c_1 + c_3 - c_{13} = 16,$$

$$0 \leq y_1 \leq 20, \; 0 \leq y_2 \leq 20, \; 0 \leq y_3 \leq 50.$$

Distribution of costs and gain; core of a game

Using $y_3 = 30 - y_1 - y_2$ leads to the conditions:

$$0 \le y_1 \le 14, \quad 0 \le y_2 \le 14, \quad 25 \le y_1 + y_2 \le 30.$$



Numbers near the center of the triangle provide a fair

Distribution of costs and gain; core of a game

Exercise 5.1. Check that if in the above example we change the value c_{12} from 15 to 12 (leaving other values the same), the core of the game would consist of one point only: $y_1 = y_2 = 14$, $y_3 = 2$. If we further make c_{12} smaller than 12, the core would become empty.

Exercise 5.2. Show that the core in a cost-sharing game for three players is non empty if and only if

$$c_{123} \leq c_1 + c_2 + c_3,$$

$$c_{123} \leq c_{12} + c_3, \quad c_{123} \leq c_{23} + c_1, \quad c_{123} \leq c_{13} + c_2,$$

$$c_{123} \leq \frac{1}{2}(c_{12} + c_{13} + c_{23}).$$

Games with transferable utility

The transformation from a cost-sharing problem to a gain-sharing problem that we carried out above, leads to the following general concept.

A *gain-sharing game* (also called a *game with transferable utility*) for three players is specified by a set of payoffs (non-negative numbers) p_1, p_2, p_3 that can be obtained by each player separately, the set of payoffs p_{12}, p_{13}, p_{23} that can be obtained by all possible coalitions of two players, and the payoff p_{123} that can be gained if the three players join their efforts. An allocation is any collection of non-negative numbers y_1, y_2, y_3 such that $y_1 + y_2 + y_3 = p_{123}$. One says that such an allocation belongs to the *core of the game* if the *stand-alone principle* is satisfied, namely if

$$y_1 + y_2 \geq p_{12}, \quad y_2 + y_3 \geq p_{23}, \quad y_1 + y_3 \geq p_{13},$$

$$y_1 \geq p_1, \quad y_2 \geq p_2, \quad y_3 \geq p_3.$$

End of Part 1

Main keywords:
Static games in normal form with finite sets of strategies, Nash equilibria, Pareto optimality, common knowledge of rationality, social dilemmas, utility function,
Auctions, Braess paradox and selfish routing, Wardrop equilibriua,
Dynamic games in extensive form, backward induction, subgame perfection, infinitely repeated games, cooperation via threat and punishment, computer tournaments,
Models of voting and elections, minority principle, strategic voting theorem on dictator, principles of fair distribution, cooperative games, stand-alone principle, the core of the game.