Assignment for Interest Rates and Credit Risk Models
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Submit by 23:59 11 May 2022 by following the instructions below

Question 1 Consider forward rates, based on continuous compounding, observed at $t = 0$

$$f_0^*(\tau) = 0.1, \qquad \tau \in [0, 6]$$

where the time is measured in years.

a) Calculate the bond prices $B_0^*(\tau_i)$ for the times $\tau_i = i$, $i = 1, \dots 6$, (3 marks)

b) Calculate the LIBOR rates $L_0(\tau_{i-1}, \tau_i)$, $i = 1, \dots, 6$. (4 marks)

c) Calculate the at-the-money interest rate (swap rate) for an interest rate swap with dates τ_1, τ_2, $\dots$, τ_6 ($m = 1$, $n = 6$). (3 marks)

Question 2

Consider the time horizon $T = 1$ of a bond market and suppose that today's ($t = 0$) bond curve is given by

$$B_0^*(\tau) = e^{-c\sqrt{1+\tau}} \qquad \text{for all } \tau \in [0, T] \text{ with } c \in]0, \infty[.$$

For a one-factor HJM model with deterministic forward rate volatility

$$\sigma_t(\tau) = \sigma^f \cdot (\tau - t + 1), \qquad 0 \le t \le \tau \le T, \ \ \sigma^f \in]0, \infty[$$

a) Calculate the drift $(\alpha_t(\tau))_{t\in[0,\tau]}$, $\tau \in [0, T]$ from the HJM drift condition. (2 marks)

b) Fit the initial forward rate $(f_0^*(\tau))_{\tau[0,T]}$ to the market data given by the bond curve $(B_0^*(\tau))_{\tau\in[0,T]}$. (3 marks)

c) Is the factorization

$$\sigma_t(\tau) = \xi(t)\psi(\tau) \qquad 0 \le t \le \tau \le T \quad \psi, \xi \text{ deterministic functions}$$

satisfied? (required for Markov property of the short rate) (2 marks)

d) Determine the short rate r_T. (3 marks)

Question 3

Consider zero bond dynamics $(B_t(\tau))_{t\in[0,\tau]}$, $\tau \in [0, T]$ defined by the Ho-Lee short rate model

$$dr_t = (c + t)dt + \sigma^r dW_t, \qquad r_0 = r_0^*$$

where the process $(W_t)_{t\in[0,T]}$ follows a Brownian motion on $(\Omega, \mathcal{F}, \mathbb{Q}, (\mathcal{F}_t)_{t\in[0,T]})$ with respect to the spot martingale measure $\mathbb{Q}$ and $\sigma^r, c, r_0^* \in]0, \infty[$ are fixed.

a) Calculate the expectation $\mathbb{E}^{\mathbb{Q}}(r_T)$ of the short rate with respect to the spot martingale measure $\mathbb{Q}$ in terms of the model parameters $\sigma^r, c, r_0^* \in]0, \infty[$. (5 marks)

b) Calculate the expectation $\mathbb{E}^{\mathbb{Q}^T}(r_T)$ of the short rate with respect to the forward martingale measure $\mathbb{Q}^T$. (5 marks)

Question 4

Consider a one-factor HJM model whose forward rate dynamics follows

$$df_t(\tau) = \alpha_t(\tau)dt + \sigma_t(\tau)dW_t, \qquad f_0(\tau) = \lambda, \quad 0 \le t \le \tau \le T, \;\; \lambda \in]0, \infty[$$

with a Browinan Motion $(W_t)_{t\in[0,T]}$ under the spot martingale measure $\mathbb{Q}$. Assume that

$$\sigma_t(\tau) = \sigma^f \cdot \tau \cdot t \quad 0 \le t \le \tau \le T$$

with a pre-specified parameter $\sigma^f \in]0, \infty[$. Consider a risky asset following

$$dS_t = S_t(r_t dt + \sigma^S dW_t), \qquad S_0 = S_0^* \in]0, \infty[, \;\; \sigma^S \in]0, \infty[$$

with the short rate $(r_t = f_t(t))_{t\in[0,T]}$.

a) Calculate the volatility $\sigma_t^B(\tau)$ for $0 \le t \le \tau \le T$ defined by

$$dB_t(\tau) = B_t(\tau)(r_t dt + \sigma_t^B(\tau)dW_t)$$

(3 marks)

b) With $(\Sigma_u = \sigma^S - \sigma_u^B(T))_{u\in[0,T]}$ calculate the quantity

$$\int_0^T |\Sigma_u|^2 du$$

(3 marks)

c) Calculate the price of the European Call on S_T

$$\mathbb{E}^{\mathbb{Q}}(e^{-\int_0^T r_s ds}(S_T - K)^+) \qquad K \ge 0$$

(4 marks)

Question 5 (10 marks)
Consider two coupon paying bonds (Bond 1 and Bond 2) with face value 10,000 AUD paying coupons monthly at the (annual) coupon rate of 6% (Bond 1) and 9% (Bond 2). Assume that the first coupon just has been paid at $\tau = 0$, the last coupon (in addition to the face value) will be paid at $\tau = 10$ and the bonds are traded now, at $\tau = 0$ at the yields 7% (Bond 1) and 8% (Bond 2). Is it possible to determine the price of a zero bond maturing at $\tau = 10$ using no-arbitrage arguments? If yes, calculate the yield of this bond (continuous compounding).

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