Multi-Period Asset Pricing
Part 5

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State processes

State processes in binomial model

The "naive" implementation of the algorithm of the backward induction for AFP is **not practical**.

Example

If we want to price an option with maturity one year and take

$$\text{one period} = \text{one working day},$$

then we obtain $N = 256$ period model. Already for the boundary conditions we need to write

$$2^N = 2^{256} = \text{ practically } \infty$$

equalities.

Idea: adapt the algorithm of backward induction to the *type* of an option.

Standard options in binomial model

We consider a *standard* European option with payoff

$$V_N(\omega) = f_N(S_N(\omega)), \quad \omega \in \Omega,$$

for some (deterministic) function $f_N = f_N(x)$. From the "naive" backward induction we deduce that the AFPs have the form:

$$V_n = f_n(S_n), \quad n = 0, \dots, N,$$

where the functions $f_n = f_n(x)$ are defined recursively:

$$f_n(x) = \frac{1}{1+r}\left(\widetilde{p} f_{n+1}(ux) + \widetilde{q} f_{n+1}(dx)\right),$$
$$x \in \mathrm{range}(S_n) = \left\{S_0 d^n, S_0 d^{n-1} u, \dots, S_0 d u^{n-1}, S_0 u^n\right\}.$$

Note that at $t = n$ we have to perform $n + 1 \ll 2^n$ computations.

[Problem on "power" option] [...] = skipped

Problem

Compute AFP V_0 for the "power" option with the payoff

$$V_N(\omega) = (S_N(\omega))^{\alpha}, \quad \omega \in \Omega,$$

for some exponent $\alpha \in \mathbb{R}$.

Solution

The AFP V_n at $t = n$ has the form: $V_n = f_n(S_n)$, where the functions (f_n) solve the FDE (finite-difference equation):

$$\begin{aligned}
f_N(x) &= x^{\alpha}, \quad x \in \mathrm{range}(S_N), \\
f_n(x) &= \frac{1}{1+r}\left(\widetilde{p} f_{n+1}(ux) + \widetilde{q} f_{n+1}(dx)\right), \\
&\qquad x \in \mathrm{range}(S_n),\ n = 0, \dots, N-1.
\end{aligned}$$

[Problem on "power" option]

We look for the solution in the form

$$f_n(x) = a_n x^\alpha,$$

for some constants (a_n). After the substitution, we deduce that (a_n) solve another FDE:

$$\begin{aligned} a_N &= 1, \\ a_n &= \theta a_{n+1}, \quad n = 0, 1, \ldots, N-1, \end{aligned}$$

where

$$\theta = \frac{1}{1+r}\left(\widetilde{p}u^\alpha + \widetilde{q}d^\alpha\right).$$

It follows that $a_n = \theta^{N-n}$ and

$$V_0 = f_0(S_0) = a_0(S_0)^\alpha = \theta^N(S_0)^\alpha. \qquad \square$$

[Standard options in binomial model]

We can also deduce an "explicit" formula for any standard option. If

$$V_N(\omega) = f_N(S_N(\omega)), \quad \omega \in \Omega,$$

then by the "naive" explicit formula,

$$\begin{aligned} V_0 &= \frac{1}{(1+r)^N} \sum_{\omega \in \Omega} V_N(\omega)\, \widetilde{p}^{\#H(\omega)}\, \widetilde{q}^{\#T(\omega)} \\ &= \frac{1}{(1+r)^N} \sum_{n=0}^{N} f_N(S_0 u^n d^{N-n})\, \widetilde{p}^{n}\, \widetilde{q}^{N-n} C_N^n, \end{aligned}$$

where the *binomial coefficient*

$$C_N^n = \frac{N!}{n!(N-n)!} = |\omega : \ \#H(\omega) = n, \ \#T(\omega) = N-n|$$

is the number of words with n letters H and $N-n$ letters T.

State processes in binomial model

The stock price $S = (S_n)_{0 \le n \le N}$ is an example of a *state process* in binomial model.

Definition

An adapted sequence $Y = (Y_n)_{0 \le n \le N}$ is called a **state process** if for any option with expiration time $m \le N$ and the payoff

$$V_m = f_m(Y_m) \text{ for some function } f_m = f_m(x),$$

its AFP at any time $n \le m$ has a similar form:

$$V_n = f_n(Y_n) \text{ for some function } f_n = f_n(x).$$

Pricing in practice

General method of AFP: given an option with the payoff

$$V_N = V_N(\omega_1, \dots, \omega_N)$$

find a *state process* Y such that

$$V_N = f_N(Y_N) \text{ for some } f_N = f_N(x).$$

Then (automatically!) for every $0 \leq n \leq N$ the AFP has the form:

$$V_n = f_n(Y_n) \text{ for some } f_n = f_n(x).$$

State processes in binomial model

We observe that

$$\boxed{\#\text{ of computations at } n} = \boxed{\#\text{ of values of } Y_n}$$

"Art" of financial computations: choose state process to minimize the amount of storage.

Remark

State process $(Y_n)_{0 \leq n \leq N}$ can be multi-dimensional:

$$Y_n = (Y_n^1, \ldots, Y_n^d).$$

We should look for a state process with the *minimal* dimension.

“Backward” condition

Lemma

The following conditions are equivalent:

1. Y *is a state process.*
2. *For every* $t = n$ *and every function* $f_{n+1} = f_{n+1}(y)$, $y \in \text{range}(Y_{n+1})$, *there is a function* $f_n = f_n(y)$, $y \in \text{range}(Y_n)$, *such that*

$$f_n(Y_n(\omega_1, \ldots, \omega_n)) = \frac{1}{1+r}\left(\widetilde{p} f_{n+1}(Y_{n+1}(\omega_1, \ldots, \omega_n, H)) \right. \\ \left. + \widetilde{q} f_{n+1}(Y_{n+1}(\omega_1, \ldots, \omega_n, T))\right),$$

for every outcome $(\omega_1, \ldots, \omega_n)$.

Proof.

Follows from the “naive” algorithm of backward induction.

“Forward” condition

The following (“forward”) description is more intuitive.

Lemma (Regression)

Suppose that that for every $n = 0, \dots, N-1$ *the value of* Y_{n+1} *is determined by* Y_n *and* $\omega_{n+1} \in \{H, T\}$, *that is,*

$$Y_{n+1} = g_{n+1}(Y_n, \omega_{n+1}),$$

for some function

$$g_{n+1} = g_{n+1}(y, \omega_1), \quad y \in \mathrm{range}(Y_n),\ \omega_1 \in \{T, H\}.$$

Then (Y_n) *is a state process.*

[Criteria for state processes]

Proof.
Indeed, if $f_{n+1} = f_{n+1}(y)$, $y \in \text{range}(Y_{n+1})$, then

$$f_n(Y_n(\omega_1, \ldots, \omega_n)) = \frac{1}{1+r}\left(\widetilde{p} f_{n+1}(Y_{n+1}(\omega_1, \ldots, \omega_n, H))\right.$$
$$\left. + \widetilde{q} f_{n+1}(Y_{n+1}(\omega_1, \ldots, \omega_n, T))\right),$$

for the function $f_n = f_n(y)$, $y \in \text{range}(Y_n)$, given by

$$f_n(y) = \frac{1}{1+r}\left(\widetilde{p} f_{n+1}(g_{n+1}(y, H)) + \widetilde{q} f_{n+1}(g_{n+1}(y, T))\right).$$

Hence, Y is a state process. □

Problem on up-and-in call

Problem

In N-period binomial model, where $ud = 1$, compute the AFP V_0 of the up-and-in call option with strike K and upper barrier U:

$$V_N = \max(S_N - K, 0)1_{\{M_N \geq U\}},$$

where (M_n) is the historical maximum:

$$M_n = \max_{0 \leq k \leq n} S_k.$$

We present two choices of state processes.

Solution (1)

We observe that $(Y_n = (S_n, M_n))$ is a state process.

Problem on up-and-in call

Indeed, Y_{n+1} is determined by Y_n and ω_{n+1}:

$$Y_{n+1}(\omega_{n+1} = H) = (uS_n, \max(uS_n, M_n)),$$
$$Y_{n+1}(\omega_{n+1} = T) = (dS_n, M_n),$$

Thus, the AFPs (V_n) have the form: $V_n = f_n(S_n, M_n)$, where the functions (f_n) solve the FDE:

$$f_N(s, m) = \max(s - K, 0)1_{\{m \geq U\}}, \quad (s, m) \in \text{range}(S_N, M_N),$$
$$f_n(s, m) = \frac{1}{1+r}\left(\widetilde{p} f_{n+1}(us, \max(us, m)) + \widetilde{q} f_{n+1}(ds, m)\right),$$
$$(s, m) \in \text{range}(S_n, M_n), \; n = 0, 1, \ldots, N-1.$$

Problem on up-and-in call

At $t = n$ we need to perform as many computations as the size of range(S_n, M_n), which is $\approx n^2/2$ as $ud = 1$. $\square$

Solution (2)

Observe that $Y_n = (S_n, Z_n)$, $n = 0, 1, \ldots, N$, is a state process, where

$$Z_n = 1_{\{M_n \geq U\}}$$

is the indicator of the barrier event. Indeed,

$$Y_{n+1}(\omega_{n+1} = H) = (uS_n, Z_n + (1 - Z_n)1_{\{uS_n \geq U\}}),$$
$$Y_{n+1}(\omega_{n+1} = T) = (dS_n, Z_n),$$

and therefore, Y_{n+1} is determined by Y_n and ω_{n+1}.

Problem on up-and-in call

The AFPs (V_n) admit the representation:

$$V_n = f_n(Y_n) = f_n(S_n, Z_n),$$

where the functions (f_n) solve the FDE:

$$\begin{aligned}
f_N(s,z) &= \max(s-K,0)z, \quad s \in \operatorname{range}(S_N),\ z \in \{0,1\}\,, \\
f_n(s,0) &= \frac{1}{1+r}\left(\widetilde{p} f_{n+1}(us, 1_{\{us \geq U\}}) + \widetilde{q} f_{n+1}(ds,0)\right), \\
f_n(s,1) &= \frac{1}{1+r}\left(\widetilde{p} f_{n+1}(us,1) + \widetilde{q} f_{n+1}(ds,1)\right), \\
& s \in \operatorname{range}(S_n),\ n = 0,1,\ldots,N-1.
\end{aligned}$$

At $t = n$ we need to perform $2(n+1)$ computations. $\square$

[Problem on Asian option]

Problem

In N-period binomial model, compute the AFP V_0 of Asian option with payoff

$$V_N = \max(A_N - S_N, 0),$$

where A_n is the historical average:

$$A_n = \frac{1}{n+1} \sum_{k=0}^{n} S_k.$$

We present two choices of state processes.

Solution (1)

Observe that $Y_n = (S_n, A_n)$, $n = 0, 1, \ldots, N$, is a state process.

[Problem on Asian option]

Indeed, Y_{n+1} is determined by Y_n and ω_{n+1}:

$$Y_{n+1}(\omega_{n+1} = H) = \left(uS_n, \frac{1}{n+2} \left((n+1)A_n + uS_n \right) \right),$$

$$Y_{n+1}(\omega_{n+1} = T) = \left(dS_n, \frac{1}{n+2} \left((n+1)A_n + dS_n \right) \right).$$

The AFPs (V_n) admit the representation:

$$V_n = f_n(Y_n) = f_n(S_n, A_n),$$

[Problem on Asian option]

where the functions (f_n) solve the FDE:

$$
\begin{aligned}
f_N(s,a) &= \max(a-s,0), \quad (s,a) \in \mathrm{range}(S_N, A_N), \\
f_n(s,a) &= \frac{1}{1+r}\left[\widetilde{p} f_{n+1}\left(us, \frac{1}{n+2}\left((n+1)a+us\right)\right)\right. \\
&\qquad \left. + \widetilde{q} f_{n+1}\left(ds, \frac{1}{n+2}\left((n+1)a+ds\right)\right)\right], \\
&\qquad (s,a) \in \mathrm{range}(S_n, A_n), \quad n = 0,1,\dots,N-1.
\end{aligned}
$$

It is not easy to estimate the size of $\mathrm{range}(S_n, A_n)$. In continuous time, we get a 2-dimensional PDE. □

[Problem on Asian option]

Solution (2)

Observe that $Z_n = A_n/S_n$, $n = 0, 1, \dots, N$, is the state process. Indeed,

$$
\begin{aligned}
Z_{n+1}(\omega_{n+1} = H) &= \frac{1}{uS_n}\left(\frac{1}{n+2}((n+1)A_n + uS_n)\right) \\
&= \frac{1}{u(n+2)}\left((n+1)Z_n + u\right), \\
Z_{n+1}(\omega_{n+1} = T) &= \frac{1}{dS_n}\left(\frac{1}{n+2}((n+1)A_n + dS_n)\right) \\
&= \frac{1}{d(n+2)}\left((n+1)Z_n + d\right),
\end{aligned}
$$

and therefore, Z_{n+1} is determined by Z_n and ω_{n+1}.

[Problem on Asian option]

Since (S_n) is a state process, we deduce that $Y_n = (S_n, Z_n)$ is a state process as well. It follows that the AFPs (V_n) have the form:

$$V_n = f_n(S_n, Z_n), \quad n = 0, 1, \ldots, N,$$

where the functions (f_n) solve the FDE:

$$\begin{aligned}
f_N(s,z) &= \max(sz - s, 0) = s\max(z-1, 0),\\
&\qquad (s,z) \in \mathrm{range}(S_N, Z_N),\\
f_n(s,z) &= \frac{1}{1+r}\left[\widetilde{p} f_{n+1}\left(us, \frac{1}{u(n+2)}((n+1)z+u)\right)\right.\\
&\qquad\left. + \widetilde{q} f_{n+1}\left(ds, \frac{1}{d(n+2)}((n+1)z+d)\right)\right],\\
&\qquad (s,z) \in \mathrm{range}(S_n, Z_n), \quad n = 0, 1, \ldots, N-1.
\end{aligned}$$

[Problem on Asian option]

We look for the solution in the form

$$f_n(s,z) = s g_n(z).$$

After the substitution, we deduce that the functions (g_n) solve the FDE:

$$\begin{aligned} g_N(z) &= \max(z-1,0), \quad z \in \operatorname{range}(Z_N), \\ g_n(z) &= \frac{1}{1+r}\left[\widetilde{p} u g_{n+1}\left(\frac{1}{u(n+2)}((n+1)z+u)\right)\right. \\ &\qquad \left. + \widetilde{q} d\, g_{n+1}\left(\frac{1}{d(n+2)}((n+1)z+d)\right)\right], \\ &\qquad z \in \operatorname{range}(Z_N), \quad n = 0,1,\ldots,N-1. \end{aligned}$$

In continuous time, we get a one-dimensional PDE. □

Summary on state processes

Motivation: "naive" algorithm of backward induction is not practical:

$$\boxed{\#\text{ of computations at } n} = \boxed{\#\text{ of trajectories at } n \ (= 2^n)}$$

Idea: *adapt* the algorithm of backward induction to the *type* of non-traded option.

Method: represent the payoff as a function of a state process Y: $V_N = f_N(Y_N)$. Then (automatically!) $V_n = f_n(Y_n)$ and

$$\boxed{\#\text{ of computations at } n} = \boxed{\#\text{ of values of } Y_n}$$

Goal: *choose* the state process with the *smallest* range (minimal dimension).

Options with random maturity

American options

Options with random maturity

Until now we studied options with *deterministic* maturity. In practice, there are also options whose maturity is *random*.

Example (**Rebate option**)

Rebate option pays $\$1$ as soon as stock price reaches or falls below barrier L; otherwise, it expires worthless. The maturity is given by

$$\tau = \min\{1 \le n \le N : \; S_n \le L\}$$

and is *random*.

Such random maturity is formally defined as a *finite stopping time*.

Finite stopping time

Definition

A random variable $\tau = \tau(\omega)$ is called a **finite stopping time** in the N-period binomial model if

(a) $\tau \in \{0, 1, \dots, N\}$,

(b) for every $t = n$ the event $\{\tau \le n\}$ is determined by the trajectory $(\omega_1, \dots, \omega_n)$ (the event is *"known"* at $t = n$).

Remark

A random variable τ is a finite stopping time if and only if

(a') $\sum_{n=0}^{N} I(\tau = n) = 1$.

(b') The sequence $\left(1_{\{\tau=n\}}\right)_{0 \le n \le N}$ is *adapted*.

Exit through lower barrier

Example

Let $L > 0$ be a lower barrier for (S_n) and set

$$\tau(\omega) = N \quad \text{if} \quad \min_{0 \le n \le N} S_n(\omega) > L$$

and

$$\tau(\omega) = \min\{0 \le n \le N : \ S_n(\omega) \le L\}, \quad \text{otherwise.}$$

Then τ is a finite stopping time. Indeed, clearly, $\tau \in \{0, 1, \ldots, N\}$ and the event

$$\{\omega : \ \tau(\omega) \le n\} = \{\omega : \ \min_{0 \le k \le n} S_k \le L\}$$

depends only on $(\omega_1, \ldots, \omega_n)$. $\square$

AFP of options with random maturity

We consider an option paying the amount

$$G_\tau(\omega) = G_\tau(\omega_1, \ldots, \omega_\tau) = \sum_{n=0}^{N} G_n(\omega_1, \ldots, \omega_n) 1_{\{\tau = n\}}$$

at a finite stopping time τ, where (G_n) is an adapted sequence.

Replicating strategy:

$$\underbrace{X_0}_{?} \quad \xrightarrow[(\Delta_n)-?]{} \quad \underbrace{X_\tau = G_\tau}_{\text{known}}$$

Key idea: move *backward* and think *"conditionally"*!

Naive backward induction

We denote by $X_n = X_n(\omega_1, \ldots, \omega_n)$ the capital of replicating strategy at $t = n$ if $\tau \geq n$.

Backward induction:

Time N: on $\{\tau = N\}$ we have that $X_N = G_N$.

$\vdots$

Time n: on $\{\tau \geq n\}$ we have that

1. if $\tau = n$, then $X_n = G_n$,
2. if $\tau > n$, then

$$X_n = \frac{1}{1+r}\left(\widetilde{p}X_{n+1}(\omega_{n+1} = H) + \widetilde{q}X_{n+1}(\omega_{n+1} = T)\right).$$

Down-and-rebate option

Of course, to get a practical algorithm, we need to use state processes.

Problem (Down-and-rebate)

The option pays $G_\tau = 1$ at the first time τ when $S_\tau \leq L$. Here $L > 0$ is the lower barrier. If $S_n > L$ for all times n, then the option expires worthless. Compute the AFP V_0.

Solution

We denote by V_n the AFP at $t = n$ if $\tau \geq n$. We look for V_n in the form

$$V_n = f_n(S_n) \quad \text{on} \quad \{\tau \geq n\}$$

for some function $f_n = f_n(x)$.

Down-and rebate option

In other words,

$f_n(x)$: the AFP at $t = n$ if $S_n = x$ and $\tau \geq n$.

We deduce that the functions (f_n) solve the FDE:

$$
\begin{aligned}
f_N(x) &= 1_{\{x \leq L\}}, \quad x \in \text{range}(S_N), \\
f_n(x) &= 1_{\{x \leq L\}} + 1_{\{x > L\}} \frac{1}{1+r} \big(\widetilde{p} f_{n+1}(ux) + \widetilde{q} f_{n+1}(dx) \big), \\
&\qquad x \in \text{range}(S_n), \quad n = 0, 1, \ldots, N-1.
\end{aligned}
$$

At every time n we have to perform just $n+1$ computations. $\square$

General case with state processes

In general case, when the option pays $G_\tau = G_\tau(\omega_1, \ldots, \omega_\tau)$ at a finite stopping time τ, we look for a state process (Y_n) such that

(a) If $\{\tau \geq n\}$, then

$$\{\tau = n\} = \{Y_n \in A_n\},$$

for some $A_n \subset \mathbb{R}$; with $A_N = \mathbb{R}$ to make τ finite. Equivalently,

$$\tau = \min\{0 \leq n \leq N : \ Y_n \in A_n\}.$$

(b) On the set $\{\tau = n\}$, option's payoff has the form:

$$G_n = g_n(Y_n),$$

for some function $g_n = g_n(y)$.

General case with state processes

Under $(a)+(b)$, the AFP at $t=n$ can be written as

$$V_n = f_n(Y_n) \quad \text{if} \quad \tau \geq n.$$

Here, $f_n(y)$ is the AFP at $t=n$ if $Y_n = y$ and $\tau \geq n$.
If Y has the evolution:

$$Y_{n+1} = h_{n+1}(Y_n, \omega_{n+1}),$$

for some functions (h_n), then the functions (f_n) solve the FDE:

$$\begin{aligned}
f_N(y) &= g_N(y), \quad y \in \text{range}(Y_N),\\
f_n(y) &= 1_{\{y\in A_n\}} g_n(y) + \\
&\quad 1_{\{y\notin A_n\}} \frac{1}{1+r}\left(\widetilde{p} f_{n+1}(h_{n+1}(y,H)) + \widetilde{q} f_{n+1}(h_{n+1}(y,T))\right),\\
&\quad y \in \text{range}(Y_n), \quad n = 0,1,\ldots,N-1.
\end{aligned}$$

American options in binomial model

Example (American put)

An owner of American put with strike K can exercise it at *any time* until maturity. If the option is exercised at $t = n$, then the owner receives the amount:

$$G_n = \max(K - S_n, 0). \qquad \square$$

A general American option is described by its *intrinsic value* adapted sequence

$$G_n = G_n(\omega_1, \ldots, \omega_n), \quad n = 0, \ldots, N.$$

Option's holder chooses a finite stopping time τ and gets the payment $G_\tau = G_\tau(\omega_1, \ldots, \omega_\tau)$.

American options in binomial model

Questions

1. *What is an optimal exercise policy $\widehat{\tau}$?*
2. *What is the arbitrage-free price V_0?*

Key idea: move *backward* and think *"conditionally"*!

$V_n = V_n(\omega_1, \ldots, \omega_n)$: the AFP at $t = n$ if $\widehat{\tau} \geq n$.

Boundary condition:

$$V_N = G_N \text{ on } \{\widehat{\tau} = N\}.$$

One-step iteration:

$$\underbrace{V_n \text{ on } \{\widehat{\tau} \geq n\}}_{?} \longleftarrow \underbrace{V_{n+1} \text{ on } \{\widehat{\tau} \geq n+1\}}_{\text{known}}$$

American options in binomial model

Denote by V_n^{stop} and V_n^{cont} the values to *stop* and *continue*. We obtain that

$$\begin{aligned} V_n^{\text{stop}} &= G_n, \\ V_n^{\text{cont}} &= \frac{1}{1+r}(\widetilde{p}V_{n+1}(\omega_{n+1} = H) + \widetilde{q}V_{n+1}(\omega_{n+1} = T)). \end{aligned}$$

Since the owner wants to maximize the value of his position, we have that

$$V_n = \max(V_n^{\text{stop}}, V_n^{\text{cont}}).$$

The first optimal exercise time:

$$\begin{aligned} \widehat{\tau} &= \min\{0 \le n \le N : \; V_n^{\text{stop}} \ge V_n^{\text{cont}}\} \\ &= \min\{0 \le n \le N : \; V_n = G_n\}. \end{aligned}$$

[American options in binomial model]

Remark

The value V_0 we get from the backward induction, is indeed the unique AFP. For example, if $p < V_0$, then we make the profit

$$V_0 - p > 0 \text{ at } t = 0,$$

by using the following strategy:

(a) buy the option at price p and exercise it at time $\widehat{\tau}$;

(b) sell short a replicating strategy (X_n) for the payoff $G_{\widehat{\tau}}$ at $\widehat{\tau}$. Notice, that

$$X_n = V_n = V_n^{\text{cont}} \quad \text{if} \quad \widehat{\tau} > n$$

and in particular, $X_0 = V_0$.

[American options in binomial model]

If, on the other hand, $p > V_0$, then we make the profit

$$p - V_0 > 0 \text{ at } t = 0,$$

and also possible profits

$$V_n - V_n^{\text{cont}} \geq 0 \text{ at } n < \tau,$$
$$V_\tau - G_\tau \geq 0 \text{ at } n = \tau,$$

if we sell the option at $t = 0$, collect the above profits, and invest until the exercise time τ as

$$V_n^{\text{cont}} \xrightarrow[\Delta_n]{} V_{n+1}.$$

Notice that V_n^{cont} is exactly the wealth at $t = n$ of a replicating strategy for V_{n+1}. $\square$

American options and state processes

Again, to get a practical algorithm we have to use state processes. Suppose that the intrinsic value has the form:

$$G_n = g_n(Y_n), \quad n = 0, 1, \ldots, N,$$

for some functions (g_n) and a state process (Y_n). Assume that Y has the evolution:

$$Y_{n+1} = h_{n+1}(Y_n, \omega_{n+1}),$$

for some functions $(h_n = h_n(y, \omega_1))$, where $\omega_1 \in \{H, T\}$.

American options and state processes

In this case, the "conditional" AFPs admit the representation:

$$V_n = f_n(Y_n) \text{ on } \{\widehat{\tau} \geq n\},$$

where the functions (f_n) have the following meaning:

$f_n(y)$: the AFP at $t = n$ if $Y_n = y$ and $\widehat{\tau} \geq n$.

The functions (f_n) solve the FDE:

$$\begin{aligned} f_N(y) &= g_N(y), \quad y \in \mathrm{range}(Y_N), \\ f_n(y) &= \max(g_n(y), f_n^{\mathrm{cont}}(y)), \\ f_n^{\mathrm{cont}}(y) &= \frac{1}{1+r}\left(\widetilde{p} f_{n+1}(h_{n+1}(y, H)) + \widetilde{q} f_{n+1}(h_{n+1}(y, T))\right), \\ & y \in \mathrm{range}(Y_n), \quad n = 0, 1, \ldots, N-1. \end{aligned}$$