$$\frac{1}{h^2}\sum_{i=1}^{N}(u_i-u_{i-1})^2+\sum_{i=1}^{N-1}p_iu_i^2=\sum_{i=1}^{N-1}f_iu_i$$

К-Б: $u_k^2=\left(\sum_{i=1}^{k}(u_i-u_{i-1})\cdot 1\right)^2\le k\cdot\sum_{i=1}^{k}(u_i-u_{i-1})^2$

$$\sum_{k=1}^{N-1}u_k^2\le\left(\frac{N-1}{2}\right)^2\sum_{i=1}^{N}(u_i-u_{i-1})^2\le$$

$$\le\left(\frac{N-1}{2}\right)^2\sum_{i=1}^{N}(u_i-u_{i-1})+\sum_{i=1}^{N-1}p_iu_i^2=$$

$$=\sum_{i=1}^{N-1}f_iu_i\le\frac{1}{2}\left(\sum_{i=1}^{N-1}f_i^2+\sum_{i=1}^{N-1}u_i^2\right)$$

$$\sum_{i=1}^{N-1}u_i^2\le\sum_{i=1}^{N-1}f_i^2$$

$$\|u_h\|\le\|f_h\|$$

8

$A = A^T \Rightarrow$ Все $\lambda \in R$

$\lambda_{min} \geq \frac{p}{2}$ $\quad \lambda_{max} \leq p + \frac{2}{h^2}$

$\|A^{-1}\| \cancel{\frac{\lambda_{min}}{\lambda_{max}}}$ $\quad \|A^{-1}\| = \frac{1}{\lambda_{min}} \leq \frac{2}{p} \; \forall h$

Значит есть устойчивость.

По теор. Филиппова

1). задача линейна
2). единств. решение
3). аппрокс. на решении
4). устойчивость

$\left.\begin{matrix} 1) \\ 2) \\ 3) \end{matrix}\right\}$ сходимость

N7. $-\frac{u_{i+1} - 2u_i + u_{i-1}}{h^2} + p_i u_i = f_i$

$u_0 = 0 \quad u_N = u_{N-1} \quad (N - \frac{1}{2})h = 1$

$$\sum_{i=1}^{N-1} \frac{(u_{i+1} - u_i)u_i - (u_i - u_{i-1})u_i}{h^2} =$$

$$= \frac{1}{h^2}\sum_{i=2}^{N} (u_i - u_{i-1})u_{i-1} + \frac{1}{h^2}\sum_{i=1}^{N-1} (u_i - u_{i-1})u_i =$$

$$= \frac{1}{h^2}\sum_{i=1}^{N} (u_i - u_{i-1})u_i + \frac{1}{h^2}\sum_{i=1}^{N} (u_i - u_{i-1})u_i =$$

$$= \frac{1}{h^2}\sum_{i=1}^{N} (u_i - u_{i-1})^2$$

7

$$\cancel{\frac{u_N - u_{N-1}}{h} = b + h\,\frac{f_N - p u_N}{2}}$$

$$\underset{N\times N}{A} = \begin{pmatrix} \frac{2}{h^2}+p & -\frac{1}{h^2} & & & \\ -\frac{1}{h^2} & \frac{2}{h^2}+p & -\frac{1}{h^2} & & \\ & & \ddots & & \\ & & -\frac{1}{h^2} & \frac{2}{h^2}+p & -\frac{1}{h^2} \\ & & & -\frac{1}{h^2} & \frac{1}{h^2}+\frac{p}{2} \end{pmatrix}$$

$$\underset{N}{u} = \begin{pmatrix} u_1 \\ \vdots \\ u_N \end{pmatrix} \qquad \underset{N}{f} = \begin{pmatrix} f_1 + \frac{a}{h^2} \\ f_2 \\ \vdots \\ f_{N-1} \\ \frac{b}{h} + \frac{f_N}{2} \end{pmatrix}$$

7 Делаем круги Гершгорина по строкам

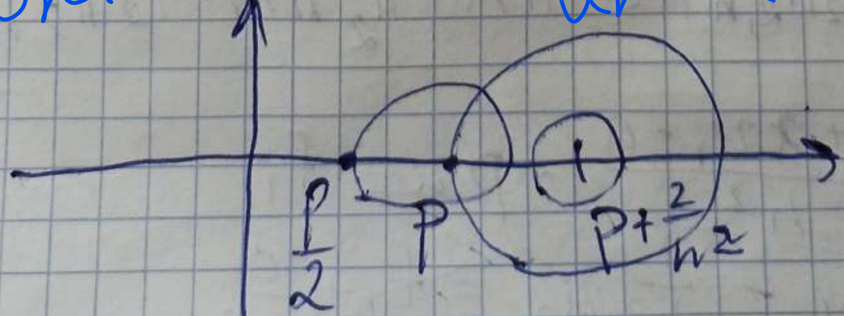
$\cancel{u_{N-1}, z}$ u_1, z: $\quad O = p + \frac{2}{h^2}, \quad r = \frac{1}{h^2}$

u_N: $\quad O = \frac{1}{h^2} + \frac{p}{2}, \quad r = \frac{1}{h^2}$

$u_2, \ldots, u_{N-1}$: $\quad O = p + \frac{2}{h^2}, \quad r = \frac{2}{h^2}$

f ортогональность? иначе — не есть норма.

6



$$\|L_h[u] - f_h\| = \|u''(x_{k+1}) + pu(x_k) - f(x_k) + O(h^2)\| = O(h^2).$$

$$\|v\| = \|v\|_{2,h}$$

$$\frac{u(x_N) - u(x_{N-1})}{h} - \delta = O(h^2)$$

$$u'\left(\frac{x_N + x_{N-1}}{2}\right) - \delta = O(h^2)$$

$$u'\left(1 - \frac{h}{2}\right) - \delta = O(h^2)$$

$$u'(1) - \frac{h}{2}u''(1) - \delta = O(h^2)$$

$$6 + \frac{h}{2}(f(1) - pu(1)) - \delta = O(h^2)$$

$$\delta = 6 + \frac{h}{2}(f(1) - pu(1)) = 6 + h\frac{f_N - pu_N}{2}$$

$$\frac{u(x_N) - u(x_{N-1})}{h} = 6 + h\frac{f_N - pu_N}{2}$$

Доказали аппр. на реш. 2-го порядка

Устойчивость: $\|A\|_2$ подчинена $\|\cdot\|_{2,h}$

$$Au_1 = f_1 \quad Au_2 = f_2$$

$$A(u_1 - u_2) = f_1 - f_2$$

$$u_1 - u_2 = A^{-1}(f_1 - f_2)$$

$$\|u_1 - u_2\|_{2,h} \le \|A^{-1}\|_2 \cdot \|f_1 - f_2\|_{2,h}$$

$$\cancel{\frac{-u_2 + 2u_1 - a}{h^2} + pu_1 = f_1}$$

$$\left(\frac{2}{h^2} + p\right)u_1 - \frac{1}{h^2}u_2 = f_1 + \frac{a}{h^2}$$

N1.5. $x_0=0 \quad x_1=h \quad u'(0)=u(0)$

$$u''-2u=\sin x-1$$

$$\frac{u_1-u_0}{h}-u_0=\varphi_h^{\delta}$$

$$\frac{u(h)-u(0)}{h}-u(0)-\varphi_h^{\delta}=O(h^2)$$

$$u'\left(\frac{h}{2}\right)-u(0)-\varphi_h^{\delta}=O(h^2)$$

$$u'\left(\frac{h}{2}\right)-u'(0)-\varphi_h^{\delta}=O(h^2)$$

$$\cancel{u'(0)}+\frac{h}{2}u''(0)\cancel{-u'(0)}-\varphi_h^{\delta}=O(h^2)$$

$$\frac{h}{2}u''(0)-\varphi_h^{\delta}=O(h^2)$$

$$\frac{h}{2}(2u(0)-1)-\varphi_h^{\delta}=O(h^2)$$

$$\delta=\frac{h}{2}(2u(0)-1)=hu_0-\frac{h}{2}$$

$$\boxed{\frac{u_1-u_0}{h}-u_0=hu_0-\frac{h}{2}}=\varphi_h$$

$\varphi_h\to 0,\ h\to 0$

(4)

N1.6. $-u''(x)+pu(x)=f(x)$

$u(0)=a \quad u'(1)=b$

$$-\frac{u_{k+1}+2u_k-u_{k-1}}{h^2}+pu_k=f_k \quad f''(x_k)$$

$u_0=a \quad \cancel{u_N} \quad \frac{u_N-u_{N-1}}{h}=\delta$

$$y(x_N) - y_N = e - \left(\frac{1+\frac{h}{2}}{1-\frac{h}{2}}\right)^N =$$

$$= e - \exp\left(N\left[\ln\left(1+\frac{h}{2}\right) - \ln\left(1-\frac{h}{2}\right)\right]\right) =$$

$$= e - \exp\left(N\left[\frac{h}{2} - \frac{h^2}{4} - \left(-\frac{h}{2} - \frac{h^2}{4}\right) + O(h^3)\right]\right)$$

$$= e - \exp(1 + O(h^2)) = O(h^2)$$

+

$$\boxed{C_1 = 0}$$

№1.4. $y' + 5y = \sin 2x \quad y(0) = 2$

$$\cancel{\frac{y_{k+1} - y_{k-1}}{2h} + 5y_k} \qquad \frac{y_{k+1} - y_k}{h} + \frac{y_{k+1} + y_k}{2} \cdot 5 = f_k$$

$$\cancel{[y(x)] = \left(y\left(\frac{h}{2}\right), \ldots y\left(1+\frac{h}{2}\right)\right)^T}$$

Перепишем по формуле Тейлора:

$$y'\left(\frac{x_{k+1} + x_k}{2}\right) + 5y\left(\frac{x_{k+1} + x_k}{2}\right) + O(h^2) =$$

$$= \underbrace{\sin(x_{k+1} + x_k)}_{f_k} + O(h^2)$$

+

аппроксимирует

$\mu = 1$ т.к. α-уст.

$$f_k \to \sin(2x_k), \; h \to 0$$

Также схема α-устойчива, т.к.

$$\mu = \frac{1 - \frac{5}{2}h}{1 + \frac{5}{2}h} \in [0, 1] \text{ при } h \to 0.$$

③

Схема дает 2 пор. аппр. на решении. По теор. Филиппова она имеет 2-й пор. сход.

№2.2. $\Theta \frac{y_{k+1}-y_k}{h} + (1-\Theta)\frac{y_k - y_{k-1}}{h} = f_k$

$\Theta \in [0,1]$

Проверим α-устойчивость:

$\Theta\mu^2 + (1-2\Theta)\mu + \Theta - 1 = 0$

$\mu = \frac{2\Theta - 1 \pm \sqrt{(1-2\Theta)^2 - 4\Theta(\Theta-1)}}{2\Theta}$

$D = (1-2\Theta)^2 - 4\Theta(\Theta-1) = 1$ $\quad \mu = \frac{2\Theta-1\pm 1}{2\Theta}$

$\mu_1 = 1 \quad \mu_2 = 1 - \frac{1}{\Theta}$

$\mu_2 = 1 - \frac{1}{\Theta} \geq -1 \iff \Theta \geq \frac{1}{2}$

При $\Theta = 0 \quad \mu - 1 = 0 \Rightarrow \mu = 1$

$\Theta \in \{0\} \cup [\frac{1}{2}, 1]$ – α-устойчив.

иначе – нет.

+

№2.3. $y^*(0) = 1 \quad y' = y$

$\frac{y_{k+1} - y_k}{h} = \frac{y_{k+1} + y_k}{2} \quad y_0 = 1$

$y(x_N) - y_N = c_1 h + \dots \quad c_1 = ?$ $\quad x_N = Nh = 1$

$y_{k+1}(1 - \frac{h}{2}) = y_k(1 + \frac{h}{2})$

$y_k = \left(\frac{1+\frac{h}{2}}{1-\frac{h}{2}}\right)^k$

2

1.1. $y'(x) = f(x)$ $\qquad \frac{y_k - y_{k-2}}{2h} = a_1 f_k + a_0 f_{k-1} + a_{-1} f_{k-2}$

$$\frac{y_k - y_{k-2}}{2h} = y'(y\, x_{k-1}) + y'''(x_{k-1})\frac{h^2}{3!} + O(h^4)$$

$$a_1 f_k = a_1 (f(x_{k-1}) + f'(x_{k-1})h + f''(x_{k-1})\frac{h^2}{2} + f^{(3)}(x_{k-1})\frac{h^3}{6} + ..$$

$$a_1 f_k + a_0 f_{k-1} + a_{-1} f_{k-2} = (a_1 + a_0 + a_{-1}) f(x_{k-1}) +$$
$$+ (a_1 - a_{-1}) f'(x_{k-1})h + (a_1 + a_{-1}) f''(x_{k-1})\frac{h^2}{2} +$$
$$+ (a_1 - a_{-1}) f'''(x_{k-1})\frac{h^3}{6} + O(h^4)$$

$$\begin{cases} a_1 + a_0 + a_{-1} = 1 \\ a_1 - a_{-1} = 0 \\ a_1 + a_{-1} = \frac{1}{3} \end{cases} \qquad a_0 = \frac{2}{3} \quad a_{-1} = a_1 = \frac{1}{6}$$

Получим 4 порядок аппроксимации.

Выше уже не будет, т.к

$$\frac{y_k - y_{k-2}}{2h} = \ldots + y^{(5)}(x_{k-1})\frac{h^4}{5!} + \cancel{\ldots}$$

$$a_1 f_k + a_0 f_{k-1} + a_{-1} f_{k-2} = \underbrace{(a_1 + a_{-1})}_{1/3} f^{(4)}(x_{k-1})\frac{h^4}{4!}$$

$$\frac{h^4}{5!} \neq \frac{1}{3}\frac{h^4}{4!}$$

+

Ответ: $a_0 = \frac{2}{3}$, $a_{-1} = a_1 = \frac{1}{6}$, $p = 4$.

1