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Evolutionary finance and dynamic games

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Abstract The paper examines a game-theoretic evolutionary model of an asset market with endogenous equilibrium asset prices. Assets pay dividends that are partially consumed and partially reinvested. The investors use general, adaptive strategies (portfolio rules), distributing their wealth between assets, depending on the exogenous states of the world and the observed history of the game. The main objective of the work is to identify strategies, allowing an investor to "survive", i.e. to possess a positive, bounded away from zero, share of market wealth over the whole infinite time horizon. This work brings together recent studies on evolutionary finance with the classical topic of non-cooperative market games.

Keywords Evolutionary finance · Dynamic games · Stochastic games · Survival strategies

JEL Classification C73 · D52 · G11

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1 Introduction

1.1 The focus of the study

In this paper we examine a game-theoretic dynamic model of an asset market with endogenous equilibrium asset prices. The evolution of the market is determined by the dynamic interaction of the strategies of investors. The investors use general, adaptive strategies (portfolio rules), distributing their wealth between assets in given proportions that depend on the observed exogenous random factors and the history of the market. Randomness is modeled in terms of a stochastic sequence of "states of the world" with a given probability distribution. Assets pay dividends depending on the realization of this process to date. The dividends together with capital gains form traders' budgets, which are partially consumed and partially reinvested. A strategy profile of investors determines the market dynamics with equilibrium asset prices derived from a short run equilibrium of supply and demand. This random dynamical system generates a path of the unfolding simultaneous-move N-player stochastic game, which results in a sequence of market shares (fractions of total wealth) of each of the traders.

The main goal of the study is to identify strategies allowing an investor to "survive" in the market selection process, i.e. to maintain a positive, bounded away from zero, share of total wealth over the infinite time horizon, whatever be the portfolio rules used by the other traders. We construct a strategy, generalizing Kelly's [23] well-known portfolio rule of "betting your beliefs", which possesses this remarkable property of unconditional survival. Moreover, we show that the strategy possessing this property is essentially unique: any other strategy of this kind (belonging to a certain class) is asymptotically similar to the Kelly rule.

1.2 Evolutionary finance

The general idea of this direction of work is to apply evolutionary dynamics—mutation and selection—to the analysis of the long-run performance of investment strategies (portfolio rules). A stock market is understood as a heterogeneous population of frequently interacting portfolio rules in competition for market capital. The ultimate goal is to build a "Darwinian theory" of portfolio selection.

An important role in the formation of this field has been played by the work of Arthur et al. [6], Farmer and Lo [19], LeBaron et al. [25], Blume and Easley [7] , Brock et al. [9] and Lux [27]. Although the above studies provided a starting point for our work, our approach to evolutionary finance is different in the modeling frameworks, focus and objectives. Our principal objective is to use the evolutionary approach for developing new models that would constitute a plausible alternative to conventional general equilibrium, making the theory closer to applications.

A well-known drawback of the conventional framework is the necessity of agents' "perfect foresight" to establish an equilibrium.[1] In particular, the market participants have to agree on the future prices for each of the possible future realizations of the states of the world (without knowing which particular state will be realized). This approach differs radically from the one based on the present evolutionary model. Here, only historical observations and the current

[1] For discussions of this circle of questions see Radner [34], Magill and Quinzii [28] and Dubey et al. [14].

state of the world influence current behavior; no agreement about the future market structure is required and no coordinated actions of the agents are assumed.

Another important distinction between our approach and the conventional general equilibrium paradigm lies in the primitives of the model. We avoid using unobservable agents' characteristics such as individual utilities or subjective beliefs. This makes the theory closer to reality, where typically quantitative information about individuals' utilities is lacking. In addition, investors' strategic behavior often is not fully expressible in terms of the maximization of one quantitative criterion or another. Rather, it may involve satisficing, simple rules of thumb based on experience, and other behavioral notions (e.g., winning in competition, dominating a market segment, etc.).

1.3 Evolutionary finance and game theory

Game theory has become one of the main general tools in mathematics-based research in Economics and Finance. The model we consider in this paper fits into the broad paradigm of the applications of noncooperative game theory to the study of market behavior and can be linked to several distinct strands of literature related to this paradigm. We discuss these links below, by stressing both the common aspects and the key differences between the present model and each of these areas.

We refer, first of all, to the strand of literature on strategic market games initiated by Shapley and Shubik [41–43] and then developed in various directions by several authors. In particular, Sahi and Yao [35] and Amir et al. [2] study models that do not rely on a commodity money, and are thus closer to the present study than others. Identifying our assets with standard commodities, one round of play in the present model shares some important features with static market games of the Shapley–Shubik type. In particular, given players' strategies, the endogenous price formation rule depends only on aggregates and involves setting the value of supply equal to the value of demand for each asset, as the market-clearing condition. All flows across traders are mediated via the market clearing prices. In the way of differences, our model is more easily amenable to dynamic analysis than standard market games for a number of reasons. In particular, wealth is the only critical variable to keep track for each agent, there is no need for a commodity money in its dual role as medium of trade and utility-bearing commodity, and one naturally avoids the usual modeling dilemma of how to deal with fiat money, given that it might introduce end-of-period effects by being worthless in any last period. For other related classes of games see [26,29,39,40].

The central concept in contemporary game theory is that of Nash equilibrium. One fundamental divergence of the present work with the conventional game-theoretic ideology lies in our very solution concept—a survival strategy, which does not involve a Nash equilibrium of any kind. This concept is somewhat reminiscent of various notions of evolutionary stable strategies in evolutionary game theory (Maynard Smith [31] and others, see, e.g., the monograph by Samuelson [36]). However, the latter are typically based on a given static game and random matching in a population of players, in terms of which a selection process is defined. Our notion of survival is defined in the original terms of the dynamic game describing wealth accumulation of investors, which makes it possible to address directly those questions that are of interest in the modeling of asset market dynamics.

The structure of the paper is as follows. Section 2 describes the model. Section 3 states the main results (Theorems 1 and 2). Section 4 outlines a general plan of the proof of the results. The Appendix contains technical details of the proofs.

2 The model

2.1 Asset market

We consider a market where $K \geq 2$ *assets* are traded. The market is influenced by random factors modeled in terms of an exogenous stochastic process $s_1, s_2, \ldots$, where s_t is a random element of a measurable space S_t. At each date $t = 1, 2, \ldots$ assets $k = 1, 2, \ldots, K$ pay dividends $D_{t,k}\left(s^t\right) \geq 0$ depending on the history

$$s^t := (s_1, \ldots, s_t)$$

of states of the world up to date t. The functions $D_{t,k}\left(s^t\right)$ are measurable with respect to the product σ-algebra in $S_1 \times \cdots \times S_t$ and satisfy

$$\sum_{k=1}^{K} D_{t,k}\left(s^t\right) > 0 \text{ for all } t, s^t. \tag{1}$$

This condition means that at each date in each random situation at least one asset yields a strictly positive dividend. The total volume (the number of units) of asset k available in the market at date t is $V_{t,k}\left(s^t\right) > 0$. For $t = 0$, $V_{t,k}$ is a constant number, and for $t \geq 1$, $V_{t,k}\left(s^t\right)$ is a measurable function of s^t.

We denote by $p_t \in \mathbb{R}_+^K$ the vector of market prices of the assets. For each $k = 1, \ldots, K$, the coordinate $p_{t,k}$ of $p_t = (p_{t,1}, \ldots, p_{t,K})$ stands for the price of one unit of asset k at date t. There are $N \geq 2$ *investors* (*traders*) acting in the market. A *portfolio* of investor i at date $t = 0, 1, \ldots$ is specified by a vector $x_t^i = (x_{t,1}^i, \ldots, x_{t,K}^i) \in \mathbb{R}_+^K$ where $x_{t,k}^i$ is the amount (the number of units) of asset k in the portfolio x_t^i. The scalar product $\langle p_t, x_t^i \rangle = \sum_{k=1}^K p_{t,k} x_{t,k}^i$ expresses the value of the investor i's portfolio x_t^i at date t in terms of the prices $p_{t,k}$. The *state of the market* at each date t is characterized by the set of vectors $(p_t, x_t^1, \ldots, x_t^N)$, where p_t is the price vector and $x_t^1, \ldots, x_t^N$ are the traders' portfolios.

At date $t = 0$ the investors have *initial endowments* $w_0^i > 0$ $(i = 1, 2, \ldots, N)$—their budgets at date 0. Investor i's budget at date $t \geq 1$ is

$$\left\langle D_t\left(s^t\right) + p_t, x_{t-1}^i \right\rangle,$$

where

$$D_t\left(s^t\right) := \left(D_{t,1}\left(s^t\right), \ldots, D_{t,K}\left(s^t\right)\right).$$

It consists of two components: the dividends $\left\langle D_t\left(s^t\right), x_{t-1}^i \right\rangle$ paid by the portfolio x_{t-1}^i and the market value $\left\langle p_t, x_{t-1}^i \right\rangle$ of x_{t-1}^i expressed in terms of the today's prices p_t (no exogenous income is accounted for). A fraction $\alpha_t = \alpha_t\left(s^t\right)$ of the budget is invested into assets. We will assume that the *investment rate* $0 < \alpha_t\left(s^t\right) < 1$ is the same for all the traders, although it may depend on time and random factors. The number $1 - \alpha_t$ can represent the *tax rate* or the *consumption rate*. The assumption that $1 - \alpha_t$ is the same for all the investors is quite natural in the former case. In the latter case it is indispensable since we focus in this work on the analysis of the comparative performance of trading strategies (portfolio rules) in the long run. Without this assumption, an analysis of this kind does not make sense: a seemingly worse performance of a portfolio rule in the long run might be simply due to a higher consumption rate of the investor.

We shall suppose that the function $\alpha_t\left(s^t\right)$ is measurable (for $t = 0$ it is constant) and satisfies the following condition:

$$\alpha_t\left(s^t\right) < V_{t,k}\left(s^t\right) / V_{t-1,k}\left(s^{t-1}\right). \tag{2}$$

This condition holds, in particular, when the total mass $V_{t,k}\left(s^t\right)$ of each asset k does not decrease, i.e. when the right-hand side of (2) is not less than one. But (2) does not exclude the situation when $V_{t,k}$ decreases at some rate, not faster than α_t.

2.2 Investment strategies

For each $t \geq 0$, every trader $i = 1, 2, \ldots, N$ selects a vector of *investment proportions* $\lambda_t^i = \left(\lambda_{t,1}^i, \ldots, \lambda_{t,K}^i\right)$ according to which i plans to distribute the available budget between assets. Vectors λ_t^i belong to the unit simplex

$$\Delta^K := \{(a_1, \ldots, a_K) \geq 0 : a_1 + \cdots + a_K = 1\}.$$

In terms of the game we deal with, the vectors λ_t^i represent the players' (investors') *actions* or *control variables*. The investment proportions at each date $t \geq 0$ are selected by the N traders simultaneously and independently, so that we deal here with a simultaneous-move N-person dynamic game. For $t \geq 1$, players' actions might depend, generally, on the history $s^t = (s_1, \ldots, s_t)$ of the realized states of the world and the *history of the game* $\left(p^{t-1}, x^{t-1}, \lambda^{t-1}\right)$, where $p^{t-1} = (p_0, \ldots, p_{t-1})$ is the sequence of asset price vectors up to time $t-1$, and

$$x^{t-1} := (x_0, x_1, \ldots, x_{t-1}), \quad x_l = \left(x_l^1, \ldots, x_l^N\right),$$
$$\lambda^{t-1} = (\lambda_0, \lambda_1, \ldots, \lambda_{t-1}), \quad \lambda_l = \left(\lambda_l^1, \ldots, \lambda_l^N\right),$$

are the sets of vectors describing the portfolios and the investment proportions of all the traders at all the dates up to $t-1$. The history of the game contains information about the *market history*—the sequence $(p_0, x_0), \ldots, (p_{t-1}, x_{t-1})$ of the states of the market—and about the actions λ_l^i of all the players (investors) $i = 1, \ldots, N$ at all the dates $l = 0, \ldots, t-1$. A vector $\Lambda_0^i \in \Delta^K$ and a sequence of measurable functions with values in Δ^K

$$\Lambda_t^i\left(s^t, p^{t-1}, x^{t-1}, \lambda^{t-1}\right), \quad t = 1, 2, \ldots$$

form an *investment (trading) strategy* Λ^i of trader i, specifying a *portfolio rule* according to which trader i selects investment proportions at each date $t \geq 0$. This is a general game-theoretic definition of a strategy, assuming full information about the history of the game, including the players' previous actions, and the knowledge of all the past and present states of the world.

Among general portfolio rules, we will distinguish those for which Λ_t^i depends only on s^t, and not on the market history $\left(p^{t-1}, x^{t-1}, \lambda^{t-1}\right)$. We will call such portfolio rules *basic*. They play an important role in the present work: the survival strategy we construct belongs to this class.

We would like to emphasize the following. It is not assumed that each investor *should* observe and remember the entire history to date (including the other investors' holdings). The general definition of a strategy *does not exclude* the presence of such extremely well-informed players. The essence of the main result (Theorem 1) lies in the fact that it indicates a relatively simple basic strategy requiring a very limited volume of information and guaranteeing survival in competition with any other strategies which might use all theoretically possible information.

2.3 Dynamic equilibrium

Suppose that at date 0 each investor i has selected some investment proportions $\lambda_0^i = \left(\lambda_{0,1}^i, \ldots, \lambda_{0,K}^i\right) \in \Delta^K$. Then the amount invested in asset k by trader i is $\alpha_0 \lambda_{0,k}^i w_0^i$ and the total amount invested in asset k is $\alpha_0 \sum_{i=1}^N \lambda_{0,k}^i w_0^i$. It is assumed that the market is always in equilibrium (asset supply is equal to asset demand), which makes it possible to determine the equilibrium price $p_{0,k}$ of each asset k from the equations

$$p_{0,k} V_{0,k} = \alpha_0 \sum_{i=1}^N \lambda_{0,k}^i w_0^i, \quad k = 1, 2, \ldots, K. \tag{3}$$

On the left-hand side of (3) we have the total value $p_{0,k} V_{0,k}$ of all the assets of the type k in the market (recall that the amount of each asset k at date 0 is $V_{0,k}$). The right-hand side represents the total wealth invested in asset k by all the investors. Equilibrium implies the equality in (3). The investment proportions $\lambda_0^i = \left(\lambda_{0,1}^i, \ldots, \lambda_{0,K}^i\right)$ chosen by the traders at date 0 determine their portfolios $x_0^i = \left(x_{0,1}^i, \ldots, x_{0,K}^i\right)$ at date 0 by the formula

$$x_{0,k}^i = \frac{\alpha_0 \lambda_{0,k}^i w_0^i}{p_{0,k}}, \quad k = 1, 2, \ldots, K,\ i = 1, \ldots, N. \tag{4}$$

This formula states that the current market value $p_{0,k} x_{0,k}^i$ of the kth position of the portfolio x_0^i of investor i is equal to the fraction $\lambda_{0,k}^i$ of the i's investment budget $\alpha_0 w_0^i$.

Assume now that all the investors have chosen their investment proportion vectors $\lambda_t^i = \left(\lambda_{t,1}^i, \ldots, \lambda_{t,K}^i\right)$ at date $t \geq 1$. Then the equilibrium of asset supply and demand determines the market clearing prices

$$p_{t,k} V_{t,k} = \alpha_t \sum_{i=1}^N \lambda_{t,k}^i \langle D_t\left(s^t\right) + p_t, x_{t-1}^i \rangle, \quad k = 1, \ldots, K. \tag{5}$$

The investment budgets $\alpha_t \langle D_t\left(s^t\right) + p_t, x_{t-1}^i \rangle$ of the traders $i = 1, 2, \ldots, N$ are distributed between assets in the proportions $\lambda_{t,k}^i$, so that the kth position of the trader i's portfolio $x_t^i = \left(x_{t,1}^i, \ldots, x_{t,K}^i\right)$ is

$$x_{t,k}^i = \frac{\alpha_t \lambda_{t,k}^i \left\langle D_t\left(s^t\right) + p_t, x_{t-1}^i \right\rangle}{p_{t,k}}, \quad k = 1, \ldots, K,\ i = 1, \ldots, N. \tag{6}$$

Note that the price vector p_t is determined implicitly as the solution to the system of equations (5). It can be shown that under assumption (2) a non-negative vector p_t satisfying these equations exists and is unique (for any s^t and any feasible x_{t-1}^i and λ_t^i)—see Proposition 1 in Sect. 4.

Given a strategy profile $\left(\Lambda^1, \ldots, \Lambda^N\right)$ of investors and their initial endowments $w_0^1, \ldots, w_0^N$, we can generate a path of the market game by setting

$$\lambda_0^i = \Lambda_0^i, \quad i = 1, \ldots, N, \tag{7}$$

$$\lambda_t^i = \Lambda_t^i\left(s^t, p^{t-1}, x^{t-1}, \lambda^{t-1}\right), \quad t = 1, 2, \ldots, \; i = 1, \ldots, N, \tag{8}$$

and by defining p_t and x_t^i recursively according Eqs. 3–6. The random dynamical system described defines step by step the vectors of investment proportions $\lambda_t^i\left(s^t\right)$, the equilibrium prices $p_t\left(s^t\right)$ and the investors' portfolios $x_t^i\left(s^t\right)$ as measurable vector functions of s^t for each moment of time $t \geq 0$ (for $t = 0$ these vectors are constant). Thus we obtain a random path of the game

$$\left(p_t\left(s^t\right); x_t^1\left(s^t\right), \ldots, x_t^N\left(s^t\right); \lambda_t^1\left(s^t\right), \ldots, \lambda_t^N\left(s^t\right)\right), \tag{9}$$

as a vector stochastic process in $\mathbb{R}_+^K \times \mathbb{R}_+^{KN} \times \mathbb{R}_+^{KN}$.

The above description of asset market dynamics requires clarification. Equations 4 and 6 make sense only if $p_{t,k} > 0$, or equivalently, if the aggregate demand for each asset (under the equilibrium prices) is strictly positive. Those strategy profiles which guarantee that the recursive procedure described above leads at each step to strictly positive equilibrium prices will be called *admissible*. In what follows, we will deal only with such strategy profiles. The hypothesis of admissibility guarantees that the random dynamical system under consideration is well-defined. Under this hypothesis, we obtain by induction that on the equilibrium path all the portfolios $x_t^i = \left(x_{t,1}^i, \ldots, x_{t,K}^i\right)$ are non-zero and the wealth

$$w_t^i := \left\langle D_t + p_t, x_{t-1}^i\right\rangle \tag{10}$$

of each investor is strictly positive. Further, by summing up Eq. 6 over $i = 1, \ldots, N$, we find that

$$\sum_{i=1}^N x_{t,k}^i = \frac{\sum_{i=1}^N \alpha_t \lambda_{t,k}^i \langle D_t + p_t, x_{t-1}^i\rangle}{p_{t,k}} = \frac{p_{t,k} V_{t,k}}{p_{t,k}} = V_{t,k} \tag{11}$$

(the market clears) for every asset k and each date $t \geq 1$. The analogous relations for $t = 0$ can be obtained by summing up Eq. 4. Thus for every equilibrium state of the market $\left(p_t, x_t^1, \ldots, x_t^N\right)$, we have $p_t > 0$, $x_t^i \neq 0$ and (11).

We give a simple sufficient condition for a strategy profile to be admissible. This condition will hold for all the strategy profiles we shall deal with in the present paper, and in this sense it does not restrict generality. Suppose that some trader, say trader 1, uses a portfolio rule that always prescribes to invest into all the assets in strictly positive proportions $\lambda_{t,k}^1$. Then a strategy profile containing this portfolio rule is admissible. Indeed, for $t = 0$, we get from (3) that $p_{0,k} \geq \alpha_0 V_{0,k}^{-1} \lambda_{0,k}^1 w_0^1 > 0$ and from (4) that $x_0^1 = \left(x_{0,1}^1, \ldots, x_{0,K}^1\right) > 0$ (coordinatewise). Assuming that $x_{t-1}^1 > 0$ and arguing by induction, we obtain

$$\left\langle D_t + p_t, x_{t-1}^1\right\rangle \geq \left\langle D_t, x_{t-1}^1\right\rangle > 0$$

in view of (1), which in turn yields $p_t > 0$ and $x_t^1 > 0$ by virtue of (5) and (6), as long as $\lambda_{t,k}^1 > 0$.

2.4 Comments on the model

The model we consider in this paper is a game-theoretic version of the evolutionary model of a financial market with long-lived dividend-paying assets proposed in Evstigneev et al. [15–17]. In the general methodological perspective, it relies upon the Marshallian [30] principle of temporary equilibrium (not to be confused with a different concept of temporary equilibrium due to Hicks, Lindahl and others; see, e.g. Grandmont [21]). The ideas of Marshall were developed in the framework of mathematical models in economics by Samuelson [37, pp. 321–323]. A key hypothesis needed to study the process of market dynamics by using the Marshallian “moving equilibrium method,” is the co-existence of at least two sets of economic variables changing with different speeds. Then the set of variables changing slower (in our model, the investment proportions) can be temporarily fixed, while the other (in our case, the asset prices) can be assumed to rapidly reach the unique state of short-run equilibrium. For a comprehensive analysis of this approach see Schlicht [38]; in the context of evolutionary finance this modeling principle is discussed in detail in Evstigneev et al. [16,17].

An essential feature of the model is the description of trading strategies in terms of investment proportions. It corresponds to the most commonly used method of quantitative fund management that prescribes to fix investment proportions, specifying the structure of asset allocation, and to maintain these proportions over a certain time period (see, e.g., [12]). In our setting, these proportions are maintained during each of the time periods $(t-1, t]$. In practice, an investor who has decided to keep a certain asset allocation structure rebalances the portfolio on a periodic basis, say monthly, or when a substantial deviation, exceeding some fixed percentage, from the given proportions occurs owing to changes in asset prices [24]. Then the investor sells those assets which are overweighted in the portfolio and buys those which are underweighted, and thus restores the initial balance. This process is assumed to lead in our model to a short-run equilibrium over each time period $(t-1, t]$. The model does not describe how the above process goes within each of the time periods $(t-1, t]$. This question is beyond the scope of the present study. Our framework makes it possible to admit a whole spectrum of mechanisms leading to an equilibrium in the short run, including various auction-type mechanisms.

Active trading strategies based on portfolio rebalancing play an important role in practice because they frequently lead to the growth of the portfolio value. In particular, it is shown in [10,11,13] that in a volatile market, under quite general conditions *any* completely diversified constant-proportions strategy involving periodic portfolio rebalancing is superior to a buy-and-hold strategy. Paradoxically, it may happen that any strategy in the latter class exhibits a negative growth rate, while the growth rate of some strategies in the former class is positive. It should also be noted that in our setting, a buy-and-hold strategy, if implemented, would be driven out of the market by increasing numbers of assets (e.g. when $\gamma_t \geq \gamma > 1$, see (13) below), irrespective of the dynamics of their financial values.

The model at hand, in its present form, focuses on long-lived dividend-paying assets whose prices are determined by a short-run equilibrium of supply and demand. An interesting problem is to extend the present framework so as to include other types of securities (short-lived, with exogenous prices, etc.). Such “hybrid” models would make it possible to substantially enlarge the scope of the applications of the evolutionary approach. Some steps in the development of such models have been made in a recent paper by Evstigneev et al. [18].

3 The main results

3.1 The notion of survival

Let $(\Lambda^1, \ldots, \Lambda^N)$ be an admissible strategy profile of the investors. Consider the path (9) of the random dynamical system generated by this strategy profile and the given initial endowments. Let $w_t^i > 0$ denote the investor i's wealth available at date $t \geq 0$. If $t = 0$, then w_0^i is a constant number, the initial endowment of investor i. If $t \geq 1$, then $w_t^i = w_t^i(s^t)$ is a measurable function of s^t given by formula (10). As we have noted above, $w_t^i(s^t) > 0$.

We are primarily interested in the long-run behavior of the *relative wealth* or the *market shares* $r_t^i := w_t^i / W_t$ of the traders, where $W_t := \sum_{i=1}^N w_t^i$ is the *total market wealth*. We shall say that the portfolio rule Λ^1 (or investor 1 using it) *survives* with probability one if $\inf_{t \geq 0} r_t^1 > 0$ almost surely (a.s.). This means that for almost all realizations of the process of states of the world $s_1, s_2, \ldots$, the market share of the first investor is bounded away from zero by a strictly positive random variable. Alternatively, we can define survival by the requirement that $\liminf_{t \to \infty} r_t^1 > 0$, which is equivalent, as long as the numbers r_t^1 are strictly positive, to the condition that $\inf_{t \geq 0} r_t^1 > 0$.

Let us say that a portfolio rule Λ^1 is a *survival strategy* if investor 1 using it survives with probability one regardless of what portfolio rules are used by the other investors.

We can reformulate the notion of a survival strategy in terms of the wealth processes w_t^i $(i = 1, 2, \ldots, N)$. Survival of a portfolio rule Λ^1 used by investor 1 means that $w_t^1 \geq c \sum_{i=1}^N w_t^i$ (a.s.), where c is a strictly positive random variable. The last inequality holds if and only if

$$w_t^i \leq C w_t^1, \quad i = 1, \ldots, N \text{ (a.s.)}, \tag{12}$$

where C is some random variable. Property (12) expresses the fact that the wealth of any investor i using any strategy Λ^i cannot grow asymptotically faster than the wealth of investor 1 who uses the strategy Λ^1. Thus, we can say that the portfolio rule Λ^1 is *unbeatable*: it cannot be outperformed (in terms of the asymptotic growth rate of wealth) in competition with any set of strategies used by the investor 1's rivals.[2]

3.2 The Kelly rule

Assume that the total mass of each asset grows (or decreases) at the same rate $\gamma_t = \gamma_t(s^t) > 0$:

$$V_{t,k} / V_{t-1,k} = \gamma_t \ (t \geq 1). \tag{13}$$

Thus

$$V_{t,k} = \gamma_t \ldots \gamma_1 V_k, \tag{14}$$

where $V_k > 0$ $(k = 1, 2, \ldots, K)$ are the initial amounts of the assets. In the case of real dividend-paying assets—involving long-term investments in the real economy (e.g., real estate, transportation, media, infrastructure, etc.)—the above assumption means that the economic system under consideration is on a *balanced growth path*.

[2] For a general definition and discussion of the notion of an unbeatable strategy as a game solution concept see Amir et al. [4].

Define the *relative dividends* of the assets $k = 1, \dots, K$ by

$$R_{t,k} = R_{t,k}\left(s^t\right) := \frac{D_{t,k}\left(s^t\right) V_{t-1,k}\left(s^{t-1}\right)}{\sum_{m=1}^{K} D_{t,m}\left(s^t\right) V_{t-1,m}\left(s^{t-1}\right)}, \quad k = 1, \dots, K, \ t \geq 1,$$

and put $R_t\left(s^t\right) = \left(R_{t,1}\left(s^t\right), \dots, R_{t,K}\left(s^t\right)\right)$. By virtue of (14), we have

$$R_{t,k}\left(s^t\right) = \frac{D_{t,k}\left(s^t\right) V_k}{\sum_{m=1}^{K} D_{t,m}\left(s^t\right) V_m}. \tag{15}$$

Further, define

$$\begin{aligned} \rho_t &:= \alpha_t / \gamma_t, \\ \rho_t^l &:= \begin{cases} 1 - \rho_{t+l}, & \text{if } l = 1, \\ \rho_{t+1}\rho_{t+2}\dots\rho_{t+l-1}\left(1 - \rho_{t+l}\right), & \text{if } l > 1, \end{cases} \end{aligned} \tag{16}$$

and assume that

$$\rho_t < 1 - \kappa, \tag{17}$$

where κ is a strictly positive constant. Consider the portfolio rule Λ^* with the vectors of investment proportions $\lambda_t^*\left(s^t\right) = \left(\lambda_{t,1}^*\left(s^t\right), \dots, \lambda_{t,K}^*\left(s^t\right)\right)$ given by

$$\lambda_{t,k}^* = E_t \sum_{l=1}^{\infty} \rho_t^l R_{t+l,k}, \tag{18}$$

where $E_t(\cdot) = E\left(\cdot|s^t\right)$ is the conditional expectation given s^t. If $t = 0$, then $E_t(\cdot) = E_0(\cdot)$ stands for the unconditional expectation $E(\cdot)$. In view of (17), the series of random variables

$$\sum_{l=1}^{\infty} \rho_t^l = \left(1 - \rho_{t+1}\right) + \rho_{t+1}\left(1 - \rho_{t+2}\right) + \rho_{t+1}\rho_{t+2}\left(1 - \rho_{t+3}\right) + \dots$$

converges uniformly, and its sum is equal to one. Therefore the series of random vectors $\sum_{l=1}^{\infty} \rho_t^l R_{t+l,k}$ in (18) converges uniformly to a random vector belonging the unit simplex Δ^K, and so $\lambda_{t,k}^*$ is well-defined.

The portfolio rule specified by (18) prescribes to distribute wealth across assets in accordance with the proportions of the expected flow of their discounted future relative dividends. The discount factors ρ_t^l are defined in terms of the investment rate α_t and the growth rate γ_t according to formula (16). It should be emphasized that the investment proportions $\lambda_{t,k}^*\left(s^t\right)$ prescribed by the portfolio rule Λ^* generally depend on time t and the sequence of exogenous states of the world $s^t = (s_1, \dots, s_t)$, but do not depend on the history of the game $\left(p^{t-1}, x^{t-1}, \ \lambda^{t-1}\right)$, so that the strategy Λ^* is basic.

The strategy Λ^* is a generalization of the Kelly portfolio rule of "betting your beliefs" playing an important role in capital growth theory—see Kelly [23], Breiman [8], Algoet and Cover [1], and Hakansson and Ziemba [22]. If $\rho_t = \rho$ is constant, then formula (18) can be written as

$$\lambda_{t,k}^* = E_t \sum_{l=1}^{\infty} \left[(1 - \rho)\rho^{l-1} R_{t+l,k}\right]. \tag{19}$$

Further, if the random elements s_t are independent and identically distributed (i.i.d.) and the relative dividends $R_{t,k}\left(s^t\right) = R_k\left(s_t\right)$ depend only on the current state s_t and do not explicitly depend on t, then $E_t R_k(s_{t+l}) = E R_k\left(s_t\right)$ $(l \geq 1)$, and so

$$\lambda^*_{t,k} = ER_k\left(s_t\right), \tag{20}$$

which means that the strategy Λ^* is formed by the sequence of constant vectors $\left(ER_1\left(s_t\right), \ldots, ER_K\left(s_t\right)\right)$ (independent of t and s^t). Note that in this special case, the formula (20) for Λ^* does not involve the factor ρ. In this case, the "beliefs" at each date t are concerned simply with the expected relative dividends (which do not depend on t), while in the general case, and even when ρ is constant, one has to take into account the expected discounted sum of all the relative dividends at all the future dates after t.

In the classical setting of Kelly's work [23] (a complete market, exogenous asset returns), the portfolio rule of "betting your beliefs" is obtained as a result of the maximization of the expected logarithm of the portfolio return. In our setting, the survival strategy cannot be obtained as a solution to a single-agent optimization problem with a logarithmic, or any other, objective function. However, the function $\ln(\cdot)$ will frequently appear in our arguments, since in a multiplicative context, the functional $E\ln(\cdot)$, rather than $E(\cdot)$ turns out to be a proper characteristic of a random variable.

3.3 The Kelly rule is a survival strategy

Assume that for all k and t,

$$E_t \sum_{l=1}^{\infty} \rho_t^l R_{t+l,k} \ > 0 \text{ (a.s.)}. \tag{21}$$

Clearly (21) holds if and only if $E_t R_{t+l,k} > 0$ (a.s.) for some l. According to this assumption, the conditional expectation involved in the definition of the investment proportions $\lambda^*_{t,k}$ (see (18)) is strictly positive with probability one. Therefore we can select a version $\lambda^*_{t,k}\left(s^t\right)$ of this conditional expectation that is strictly positive for all s^t. This version will be used in the definition of the strategy $\Lambda^* = \left(\lambda^*_t\right)$.

A central result is as follows.

Theorem 1 *The portfolio rule* Λ^* *is a survival strategy.*

This theorem extends to the setting of long-lived dividend-paying assets an analogous result established in the framework of a model with one-period, "short-lived" assets in Amir et al. [4]. That framework, studied also by Blume and Easley [7] and Amir et al. [3], may be regarded as a limiting case as $\rho \to 0$ (with constant $\rho_t = \rho$) of the one considered in the present paper. In the case of short-lived assets, the version of the Kelly rule that turns out to be a survival strategy is defined by the investment proportions $\lambda^*_{t,k} = E_t R_{t+1,k}$. These proportions are limits of those in (19) as $\rho \to 0$.

As we have noted above, if the states of the world s_t are i.i.d. and the functions $R_{t,k}\left(s^t\right) = R_k\left(s_t\right)$ depend only on s_t and do not explicitly depend on t, then the investment proportions $\lambda^*_k = ER_k\left(s_t\right)$ of the strategy Λ^* are constant: they depend neither on time nor on the states of the world (such strategies are called *simple*). A version of the asset market model with long-lived assets in which all the investors use only simple portfolio rules and the states of the world are i.i.d. is considered in Evstigneev et al. [16]. It is shown in that context that the strategy Λ^* not only survives, but outperforms all other simple strategies. Those investors who use Λ^* dominate the market, i.e. gather in the limit total market wealth, while those who use simple strategies distinct from Λ^* vanish: their market shares tend to zero with probability one. This is not so in the model considered in the present paper, where general, not necessarily simple, portfolio rules are allowed. Here, Λ^*-investors survive, i.e. keep market shares bounded away from zero a.s., but they do not necessarily dominate the market.

3.4 Asymptotic uniqueness of the survival strategy

As we have already noted, the portfolio rule Λ^* belongs to the class of basic portfolio rules: the investment proportions $\lambda_t^*\left(s^t\right)$ depend only on the history s^t of the process of states of the world, and do not depend on the market history. The following theorem shows that in this class the survival strategy $\Lambda^* = \left(\lambda_t^*\right)$ is essentially unique: any other basic survival strategy is asymptotically similar to Λ^*.

Theorem 2 *If $\Lambda = (\lambda_t)$ is another basic survival strategy, then*

$$\sum_{t=0}^{\infty} ||\lambda_t^* - \lambda_t||^2 < \infty \ (a.s.). \tag{22}$$

Here, we denote by $||\cdot||$ the Euclidean norm in a finite-dimensional space. Theorem 2 is akin to various *turnpike* results in the theory of economic dynamics, expressing the idea that all optimal or asymptotically optimal paths of an economic system follow in the long run essentially the same route: the turnpike [32,33]. Survival strategies Λ can be characterized by the property that the wealth w_t^j of any investor j cannot grow infinitely faster (with strictly positive probability) than the wealth of investor i using Λ. The class of such investment strategies is similar to the class of "good" paths of economic dynamics, as introduced by Gale [20]—paths that cannot be "infinitely worse" than the turnpike. Theorem 2 is a direct analogue of Gale's turnpike theorem for good paths [20, Theorem 8]; for a stochastic version of this result see Arkin and Evstigneev [5, Chapter 4, Theorem 6].

Note that the class of basic strategies is *sufficient* in the following sense. Any sequence of vectors $r_t = \left(r_t^1, \ldots, r_t^N\right)$ ($r_t = r_t\left(s^t\right)$) of market shares generated by some strategy profile $\left(\Lambda^1, \ldots, \Lambda^N\right)$ can be generated by a strategy profile $\left(\lambda_t^1\left(s^t\right), \ldots, \lambda_t^N\left(s^t\right)\right)$ consisting of basic portfolio rules. The corresponding vector functions $\lambda_t^i\left(s^t\right)$ can be defined recursively by (7) and (8), using (3)–(6). Thus it is sufficient to prove Theorem 1 only for basic portfolio rules; this will imply that the portfolio rule (18) survives in competition with any, not necessarily basic, strategies. Such considerations cannot be automatically applied to the problem of asymptotic characterization of general survival strategies. This problem remains open; it indicates an interesting direction for further research.

4 Mathematical analysis of the model

4.1 Some auxiliary propositions

In this section we set out the program of proving Theorems 1 and 2. The proofs are divided into several steps which are described in several propositions and lemmas below. Based on these auxiliary results, we present at the end of the section the final steps of the proofs of Theorems 1 and 2. Technical details of the derivation of the auxiliary results are relegated to the Appendix.

The first proposition establishes the existence and uniqueness of an equilibrium price vector at each date $t \geq 0$.

Proposition 1 *Let assumption (2) hold. Let $x_{t-1} = \left(x_{t-1}^1, \ldots, x_{t-1}^N\right)$ be a set of vectors $x_{t-1}^i \in \mathbb{R}_+^K$ satisfying (11). Then for any s^t there exists a unique solution $p_t \in \mathbb{R}_+^K$ to Eq. 5. This solution is measurable with respect to all the parameters involved in (5).*

In the next proposition, we derive a system of equations governing the dynamics of the market shares of the investors given their admissible strategy profile $\left(\Lambda^1, \ldots, \Lambda^N\right)$. Consider the path (9) of the random dynamical system generated by $\left(\Lambda^1, \ldots, \Lambda^N\right)$ and the sequence of vectors $r_t = \left(r_t^1, \ldots, r_t^N\right)$, where r_t^i is the investor i's market share at date t.

Proposition 2 *The following equations hold:*

$$r_{t+1}^i = \sum_{k=1}^K \left[\rho_{t+1} \left\langle \lambda_{t+1,k}, r_{t+1} \right\rangle + (1 - \rho_{t+1})\, R_{t+1,k}\right] \frac{\lambda_{t,k}^i r_t^i}{\left\langle \lambda_{t,k}, r_t \right\rangle}, \quad i = 1, \ldots, N, \ t \geq 0. \tag{23}$$

The next proposition shows that it is sufficient to prove Theorem 1 when $N = 2$, i.e., the general model can be reduced to the case of two investors. Define

$$\tilde{\lambda}_{t,k}^2 = \frac{\lambda_{t,k}^2 r_t^2 + \cdots + \lambda_{t,k}^N r_t^N}{1 - r_t^1}. \tag{24}$$

Note that $1 - r_t^1 = r_t^2 + \cdots + r_t^N > 0$, and so $\tilde{\lambda}_{t,k}^2$ is well-defined. Furthermore

$$\sum_{k=1}^K \tilde{\lambda}_{t,k}^2 = \frac{r_t^2 + \cdots + r_t^N}{1 - r_t^1} = 1,$$

which means that the vector $\tilde{\lambda}_t^2 := \left(\tilde{\lambda}_{t,1}^2, \ldots, \tilde{\lambda}_{t,K}^2\right)$ belongs to the unit simplex Δ^K. Thus the sequence of vectors $\tilde{\lambda}_t^2 = \tilde{\lambda}_t^2\left(s^t\right)$ defines a portfolio rule, which will be denoted by $\tilde{\Lambda}$. Define

$$\tilde{r}_t^1 = r_t^1, \ \tilde{r}_t^2 = 1 - r_t^1, \ \tilde{r}_t = \left(\tilde{r}_t^1, \tilde{r}_t^2\right), \ \tilde{\lambda}_{t,k}^1 = \lambda_{t,k}^1, \ \tilde{\lambda}_{t,k} = \left(\tilde{\lambda}_{t,k}^1, \tilde{\lambda}_{t,k}^2\right).$$

Proposition 3 *We have*

$$\tilde{r}_{t+1}^i = \sum_{k=1}^K \left[\rho_{t+1} \left\langle \tilde{\lambda}_{t+1,k}, \tilde{r}_{t+1} \right\rangle + (1 - \rho_{t+1})\, R_{t+1,k}\right] \frac{\tilde{\lambda}_{t,k}^i \tilde{r}_t^i}{\left\langle \tilde{\lambda}_{t,k}, \tilde{r}_t \right\rangle}, \quad i = 1, 2, \ t \geq 0.$$

Thus in the model with two investors $i = 1, 2$ using the strategies Λ and $\tilde{\Lambda}$, respectively, the market share $\tilde{r}_t^1$ of the first investor coincides with r_t^1 (coming from the original model) and the market share $\tilde{r}_t^2$ of the second is equal to $1 - r_t^1$.

Consider the model with two traders ($N = 2$) using strategies $\Lambda^i = \left(\lambda_{t,k}^i\left(s^t\right)\right)$, $i = 1, 2$, and denote by z_t the ratio r_t^1 / r_t^2 of their market shares.

Proposition 4 *The process z_t is governed by the following random dynamical system:*

$$z_{t+1} = z_t \frac{\sum_{k=1}^K \left[\rho_{t+1} \lambda_{t+1,k}^2 + (1 - \rho_{t+1})\, R_{t+1,k}\right] \frac{\lambda_{t,k}^1}{\lambda_{t,k}^1 z_t + \lambda_{t,k}^2}}{\sum_{k=1}^K \left[\rho_{t+1} \lambda_{t+1,k}^1 + (1 - \rho_{t+1})\, R_{t+1,k}\right] \frac{\lambda_{t,k}^2}{\lambda_{t,k}^1 z_t + \lambda_{t,k}^2}}. \tag{25}$$

In the next proposition, we derive an equation which can be used as an equivalent definition of the portfolio rule Λ^*.

Proposition 5 *The portfolio rule $\Lambda^* = \left(\lambda^*_{t,k}\right)$ satisfies*

$$E_t\left[\rho_{t+1}\lambda^*_{t+1,k} + (1-\rho_{t+1})\, R_{t+1,k}\right] = \lambda^*_{t,k} \text{ (a.s.).} \tag{26}$$

It can be shown (by using a contraction principle) that Λ^* is a unique solution to (26), but this fact will not be needed in what follows.

4.2 The plan of the proof of Theorem 1

Proposition 3 shows that we can consider, without loss of generality, the case of two investors. This reduces the dimension of the original random dynamical system from a general N to $N = 2$. Proposition 4 describes a one-dimensional system which governs the evolution of the ratio $z_t = r^1_t/r^2_t$ of the market shares of the two investors, and thus reduces the dimension of the problem to 1. Our goal is to show that the random sequence (z_t) defined recursively by (25) is bounded away from zero almost surely. To this end it turns out to be convenient to take a "step back" and to increase the dimension to K (the number of assets). Assuming that the first trader uses the investment proportions $\lambda^1_{t,k} = \lambda^*_{t,k}\left(s^t\right)$ prescribed by the portfolio rule Λ^* and the second trader employs investment proportions $\lambda^2_{t,k} = \lambda_{t,k}\left(s^t\right)$ specified by some other portfolio rule Λ, we introduce the following change of variables

$$y^k_t = \lambda_{t,k}/z_t, \quad k = 1, \ldots, K, \tag{27}$$

and define $y_t := \left(y^1_t, \ldots, y^K_t\right)$. We examine the dynamics of the random vectors $y_t = y_t\left(s^t\right)$ implied by the system (25). The norm $|y_t| := \sum_k |y^k_t|$ of the vector $y_t \geq 0$ is equal to $\sum_k\left(\lambda_{t,k}/z_t\right) = 1/z_t$, and what we need is to show that $1/|y_t|$ is bounded away from zero a.s. To prove this, we construct a stochastic Lyapunov function—a function of y_t which forms a non-negative supermartingale (ζ_t) along a path (y_t) of the system at hand (see Lemma 3 below). By using the supermartingale convergence theorem, we prove that the stochastic process ζ_t converges a.s., which implies that it is bounded a.s.. We complete the proof by showing that the boundedness of ζ_t implies that $z_t = 1/|y_t|$ is bounded away from zero.

4.3 Some inequalities and supermartingale properties

We begin the realization of the above plan with two lemmas containing inequalities involving the variables y^k_t defined by (27). Define the non-negative random variables

$$Y_t := \ln\left(1 + |y_t|\right) = -\ln r^1_t, \tag{28}$$

$$Z_{t,k} := \ln\left(1 + \frac{y^k_t}{\lambda^*_{t,k}}\right) = \ln\left(1 + \frac{r^2_t\lambda_{t,k}}{r^1_t\lambda^*_{t,k}}\right), \quad Z_t := \sum_{k=1}^{K}\lambda^*_{t,k}Z_{t,k}, \tag{29}$$

and put

$$U_t := Y_t - Z_t. \tag{30}$$

Lemma 1 *The following inequality holds:*

$$\rho_{t+1}Z_{t+1} + (1-\rho_{t+1})\, Y_{t+1} \leq \sum_{k=1}^{K}\left[\rho_{t+1}\lambda^*_{t+1,k} + (1-\rho_{t+1})\, R_{t+1,k}\right] Z_{t,k}. \tag{31}$$

Lemma 2 *We have*

$$U_t = \sum_{k=1}^{K} \lambda^*_{t,k} \ln \frac{\lambda^*_{t,k}}{r^1_t \lambda^*_{t,k} + r^2_t \lambda_{t,k}} \geq 0. \tag{32}$$

From the above results, we derive the following fact.

Lemma 3 *The random sequence*

$$\zeta_t := \rho_t Z_t + (1 - \rho_t) Y_t \tag{33}$$

is a non-negative supermartingale satisfying

$$\zeta_t - E_t \zeta_{t+1} \geq (1 - \rho_t) U_t. \tag{34}$$

The following two lemmas will be used in proof of Theorem 2.

Lemma 4 *Let ζ_t be a supermartingale such that $\inf_t E\zeta_t > -\infty$. Then the series of non-negative random variables $\sum_{t=0}^{\infty} (\zeta_t - E_t \zeta_{t+1})$ converges almost surely.*

Lemma 5 *For any vectors $(a_1, \ldots, a_K) > 0$ and $(b_1, \ldots, b_K) \geq 0$ satisfying $\sum a_k = \sum b_k = 1$, the following inequality holds*

$$\sum_{k=1}^{K} a_k \ln a_k - \sum_{k=1}^{K} a_k \ln b_k \geq \frac{1}{4} \sum_{k=1}^{K} (a_k - b_k)^2 . \tag{35}$$

4.4 Proof of Theorem 1

Since ζ_t is a non-negative supermartingale, the sequence ζ_t converges a.s., and hence it is bounded above a.s. by some random variable C. This implies (see (33) and (17)) that $(1 - \rho_t) Y_t \leq \zeta_t \leq C$ (a.s.), and so

$$-\ln r^1_t = Y_t \leq \zeta_t / (1 - \rho_t) \leq B \text{ (a.s.)},$$

where $B := C/\kappa$. Therefore $r^1_t \geq e^{-B}$ (a.s.). □

4.5 Proof of Theorem 2

Let $\Lambda = (\lambda_t)$ be a basic survival strategy. Suppose that investors $i = 1, 2, \ldots, N-1$ use the strategy $\Lambda^* = (\lambda^*_t)$ and investor N uses Λ. By summing up Eq. 23 with $\lambda^i_t = \lambda^*_t$ over $i = 1, \ldots, N-1$, we obtain

$$\hat{r}^1_{t+1} = \sum_{k=1}^{K} \left[\rho_{t+1} \left(\lambda^*_{t+1,k} \hat{r}^1_{t+1} + \lambda_{t+1,k} \left(1 - \hat{r}^1_{t+1}\right)\right) \right.$$
$$\left. + (1 - \rho_{t+1}) R_{t+1,k}\right] \frac{\lambda^*_{t,k} \hat{r}^1_t}{\lambda^*_{t,k} \hat{r}^1_t + \lambda_{t,k} \left(1 - \hat{r}^1_t\right)}.$$

where $\hat{r}_t^1 := r_t^1 + \cdots + r_t^{N-1}$ is the market share of the group of investors $i = 1, 2, \ldots, N-1$ and $1 - \hat{r}_t = r_t^N$ is the market share of investor N. We used here the fact that

$$\langle \lambda_{t,k}, r_t \rangle = \sum_{i=1}^{N} \lambda_{t,k}^i r_t^i = \sum_{i=1}^{N-1} \lambda_{t,k}^* r_t^i + \lambda_{t,k} r_t^N$$
$$= \lambda_{t,k}^* \sum_{i=1}^{N-1} r_t^i + \lambda_{t,k} r_t^N = \lambda_{t,k}^* \hat{r}_t^1 + \lambda_{t,k} \left(1 - \hat{r}_t^1\right).$$

Further, we have

$$1 - \hat{r}_{t+1}^1 = \sum_{k=1}^{K} \left[\rho_{t+1} \left(\lambda_{t+1,k}^* \hat{r}_{t+1}^1 + \lambda_{t+1,k} \left(1 - \hat{r}_{t+1}^1\right)\right)\right.$$
$$\left. + \left(1 - \rho_{t+1}\right) R_{t+1,k}\right] \frac{\lambda_{t,k} \left(1 - \hat{r}_t^1\right)}{\lambda_{t,k}^* \hat{r}_t^1 + \lambda_{t,k} \left(1 - \hat{r}_t^1\right)}.$$

Thus the dynamics of the market shares $\hat{r}_t^1 = r_t^1 + \cdots + r_t^{N-1}$, $1 - \hat{r}_t^1 = r_t^N$ is exactly the same as the dynamics of the market shares $\hat{r}_t^1, \hat{r}_t^2 = 1 - \hat{r}_t^1$ of two investors $i = 1, 2$ ($N = 2$) using the strategies $\left(\lambda_t^1\right) = \left(\lambda_t^*\right)$ and $\left(\lambda_t^2\right) = (\lambda_t)$, respectively. Since (λ_t) is a survival strategy, the random sequence $r_t^N = 1 - \hat{r}_t^1 = \hat{r}_t^2$ is bounded away from zero almost surely.

Since investor 1 uses the strategy Λ^*, by virtue of Lemma 3 the sequence ζ_t defined by (33) is a non-negative supermartingale, and inequality (34) holds. In view of Lemma 4, the series $\sum_{t=0}^{\infty} (\zeta_t - E_t \zeta_{t+1})$ of non-negative random variables converges a.s. The inequality

$$\zeta_t - E_t \zeta_{t+1} \geq (1 - \rho_t) \sum_{k=1}^{K} \lambda_{t,k}^* \ln \frac{\lambda_{t,k}^*}{\hat{r}_t^1 \lambda_{t,k}^* + \hat{r}_t^2 \lambda_{t,k}}$$

established in Lemmas 2 and 3 and assumption (17) imply that

$$\sum_{t=0}^{\infty} \sum_{k=1}^{K} \lambda_{t,k}^* \ln \frac{\lambda_{t,k}^*}{\hat{r}_t^1 \lambda_{t,k}^* + \hat{r}_t^2 \lambda_{t,k}} < \infty \text{ (a.s.)}. \tag{36}$$

Finally, we observe that

$$\sum_{k=1}^{K} \lambda_{t,k}^* \ln \frac{\lambda_{t,k}^*}{\hat{r}_t^1 \lambda_{t,k}^* + \hat{r}_t^2 \lambda_{t,k}}$$
$$= \sum_{k=1}^{K} \lambda_{t,k}^* \ln \lambda_{t,k}^* - \sum_{k=1}^{K} \lambda_{t,k}^* \ln \left[\hat{r}_t^1 \lambda_{t,k}^* + \hat{r}_t^2 \lambda_{t,k}\right]$$
$$\geq \frac{1}{4} \sum_{k=1}^{K} \left[\lambda_{t,k}^* - \left(1 - \hat{r}_t^2\right) \lambda_{t,k}^* - \hat{r}_t^2 \lambda_{t,k}\right]^2 = \frac{1}{4} \sum_{k=1}^{K} \left(\hat{r}_t^2 \lambda_{t,k}^* - \hat{r}_t^2 \lambda_{t,k}\right)^2$$
$$= \frac{1}{4} \left(\hat{r}_t^2\right)^2 \sum_{k=1}^{K} \left(\lambda_{t,k}^* - \lambda_{t,k}\right)^2 = \frac{1}{4} \left(\hat{r}_t^2\right)^2 ||\lambda_t^* - \lambda_t||^2, \tag{37}$$

(see (35)), where the sequence $\hat{r}_t^2$ is bounded away from zero a.s., as long as (λ_t) is a survival strategy. Therefore

$$\hat{r}_t^2 \geq c > 0 \text{ (a.s.)}, \tag{38}$$

where c is a random variable independent of t. From relations (36)–(38) we conclude that the series $\sum_{t=0}^{\infty} ||\lambda_t^* - \lambda_t||^2$ converges a.s., which completes the proof of Theorem 2. □

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Appendix

Proof of Proposition 1 Fix some t and s^t and consider the operator transforming a vector $p = (p_1, \ldots, p_K) \in \mathbb{R}_+^K$ into the vector $q = (q_1, \ldots, q_K) \in \mathbb{R}_+^K$ with coordinates

$$q_k = \alpha_t V_{t,k}^{-1} \sum_{i=1}^{N} \lambda_{t,k}^i \left\langle D_t + p, x_{t-1}^i \right\rangle.$$

This operator is contracting in the norm $||p||_V := \sum_k |p_k| V_{t-1,k}$. Indeed, by virtue of (2) we have

$$\tilde{\alpha} := \max_{k=1,\ldots,K} \left\{ \alpha_t V_{t-1,k} V_{t,k}^{-1} \right\} < 1,$$

and so

$$\begin{aligned}
||q - q'||_V &= \sum_{k=1}^{K} |q_k - q_k'| V_{t-1,k} \\
&\leq \alpha_t \sum_{k=1}^{K} V_{t-1,k} V_{t,k}^{-1} \sum_{i=1}^{N} \lambda_{t,k}^i |\langle p - p', x_{t-1}^i \rangle| \leq \tilde{\alpha} \sum_{i=1}^{N} \sum_{k=1}^{K} \lambda_{t,k}^i |\langle p - p', x_{t-1}^i \rangle| \\
&= \tilde{\alpha} \sum_{i=1}^{N} |\langle p - p', x_{t-1}^i \rangle| \leq \tilde{\alpha} \sum_{i=1}^{N} \sum_{m=1}^{K} |p_m - p_m'| x_{t-1,m}^i \\
&= \tilde{\alpha} \sum_{m=1}^{K} \sum_{i=1}^{N} |p_m - p_m'| x_{t-1,m}^i = \tilde{\alpha} \sum_{m=1}^{K} |p_m - p_m'| V_{t-1,m} = \tilde{\alpha} ||p - p'||_V,
\end{aligned}$$

where the last but one equality follows from (11). By using the contraction principle, we obtain the existence, uniqueness and measurability of the solution to (5). □

Proof of Proposition 2 From (5) and (6) we get

$$p_{t,k} = V_{t,k}^{-1} \alpha_t \sum_{i=1}^{N} \lambda_{t,k}^i \langle p_t + D_t, x_{t-1}^i \rangle = \alpha_t V_{t,k}^{-1} \sum_{i=1}^{N} \lambda_{t,k}^i w_t^i = \alpha_t V_{t,k}^{-1} \langle \lambda_{t,k}, w_t \rangle, \tag{39}$$

$$x_{t,k}^i = \frac{V_{t,k} \lambda_{t,k}^i w_t^i}{\langle \lambda_{t,k}, w_t \rangle}, \tag{40}$$

where $t \geq 1$, $w_t := \left(w_t^1, \ldots, w_t^N\right)$ and $\lambda_{t,k} := \left(\lambda_{t,k}^1, \ldots, \lambda_{t,k}^N\right)$. The analogous formulas for $t = 0$,

$$p_{0,k} = \alpha_0 V_{0,k}^{-1} \langle \lambda_{0,k}, w_0 \rangle, \ x_{0,k}^i = \frac{V_{0,k} \lambda_{0,k}^i w_0^i}{\langle \lambda_{0,k}, w_0 \rangle}, \tag{41}$$

follow from (3) and (4). Consequently, we have

$$\begin{aligned} w_{t+1}^i &= \sum_{k=1}^K \left(p_{t+1,k} + D_{t+1,k}\right) x_{t,k}^i \\ &= \sum_{k=1}^K \left(\alpha_{t+1} \frac{\langle \lambda_{t+1,k}, w_{t+1} \rangle}{V_{t+1,k}} + D_{t+1,k}\right) \frac{V_{t,k} \lambda_{t,k}^i w_t^i}{\langle \lambda_{t,k}, w_t \rangle} \\ &= \sum_{k=1}^K \left(\alpha_{t+1} \frac{\langle \lambda_{t+1,k}, w_{t+1} \rangle V_{t,k}}{V_{t+1,k}} + D_{t+1,k} V_{t,k}\right) \frac{\lambda_{t,k}^i w_t^i}{\langle \lambda_{t,k}, w_t \rangle}, \quad t \geq 0. \end{aligned} \tag{42}$$

By summing up these equations over $i = 1, \ldots, N$, we obtain

$$\begin{aligned} W_{t+1} &= \sum_{k=1}^K \left(\alpha_{t+1} \frac{\langle \lambda_{t+1,k}, w_{t+1} \rangle V_{t,k}}{V_{t+1,k}} + D_{t+1,k} V_{t,k}\right) \frac{\sum_{i=1}^N \lambda_{t,k}^i w_t^i}{\langle \lambda_{t,k}, w_t \rangle} \\ &= \sum_{k=1}^K \left(\alpha_{t+1} \frac{\langle \lambda_{t+1,k}, w_{t+1} \rangle V_{t,k}}{V_{t+1,k}} + D_{t+1,k} V_{t,k}\right). \end{aligned}$$

Since the ratio

$$V_{t+1,k} / V_{t,k} = \gamma_{t+1} \tag{43}$$

does not depend on k (see (13)), we have

$$\begin{aligned} W_{t+1} &= \sum_{k=1}^K \left(\alpha_{t+1} \gamma_{t+1}^{-1} \langle \lambda_{t+1,k}, w_{t+1} \rangle + D_{t+1,k} V_{t,k}\right) \\ &= \sum_{k=1}^K \left(\alpha_{t+1} \gamma_{t+1}^{-1} \langle \lambda_{t+1,k}, w_{t+1} \rangle + D_{t+1,k} V_{t,k}\right) = \alpha_{t+1} \gamma_{t+1}^{-1} W_{t+1} + \sum_{k=1}^K D_{t+1,k} V_{t,k}. \end{aligned}$$

This implies the formula

$$W_{t+1} = \frac{1}{1 - \alpha_{t+1} \gamma_{t+1}^{-1}} \sum_{m=1}^K D_{t+1,m} V_{t,m}, \tag{44}$$

where $\alpha_{t+1} \gamma_{t+1}^{-1} = \rho_{t+1}$. From (42) and (43), we find

$$w_{t+1}^i = \sum_{k=1}^K \left(\rho_{t+1} \langle \lambda_{t+1,k}, w_{t+1} \rangle + D_{t+1,k} V_{t,k}\right) \frac{\lambda_{t,k}^i w_t^i}{\langle \lambda_{t,k}, w_t \rangle} \ t \geq 0.$$

Dividing both sides of this equation by W_{t+1} and using (44), we get

$$r_{t+1}^i = \sum_{k=1}^K \left[\rho_{t+1} \langle \lambda_{t+1,k}, r_{t+1} \rangle + (1 - \rho_{t+1}) \frac{D_{t+1,k} V_{t,k}}{\sum_{m=1}^K D_{t+1,m} V_{t,m}}\right] \frac{\lambda_{t,k}^i w_t^i / W_t}{\langle \lambda_{t,k}, w_t \rangle / W_t},$$

which yields (23) by virtue of (13) and (15). □

Proof of Proposition 3 In view of (24) and (23) we have

$$r^1_{t+1} = \sum_{k=1}^{K} \left\{ \rho_{t+1} \left[\lambda^1_{t+1,k} r^1_{t+1} + \left(1 - r^1_{t+1}\right) \tilde{\lambda}^2_{t+1,k} \right] + (1-\rho_{t+1})\, R_{t+1,k} \right\} \frac{\lambda^1_{t,k} r^1_t}{\lambda^1_{t,k} r^1_t + \left(1 - r^1_t\right) \tilde{\lambda}^2_{t,k}}.$$

By summing up Eq. 23 over $i = 2, \ldots, N$, we find

$$\tilde{r}^2_{t+1} = 1 - r^1_{t+1} = \sum_{k=1}^{K} \left[\rho_{t+1} \langle \lambda_{t+1,k}, r_{t+1} \rangle + (1-\rho_{t+1})\, R_{t+1,k} \right] \frac{\left(1 - r^1_t\right) \tilde{\lambda}^2_{t,k}}{\langle \lambda_{t,k}, r_t \rangle}.$$

Thus we obtain

$$\tilde{r}^2_{t+1} = \sum_{k=1}^{K} \left\{ \rho_{t+1} \left[\lambda^1_{t+1,k} r^1_{t+1} + \left(1 - r^1_{t+1}\right) \tilde{\lambda}^2_{t+1,k} \right] + (1-\rho_{t+1})\, R_{t+1,k} \right\} \frac{\left(1 - r^1_t\right) \tilde{\lambda}^2_{t,k}}{\lambda^1_{t,k} r^1_t + \left(1 - r^1_t\right) \tilde{\lambda}^2_{t,k}},$$

which completes the proof. □

Proof of Proposition 4 By using (23) with $N = 2$, we get

$$r^i_{t+1} = \sum_{k=1}^{K} \left[\rho_{t+1} \left(\lambda^i_{t+1,k} r^i_{t+1} + \lambda^j_{t+1,k} \left(1 - r^i_{t+1}\right) \right) + (1-\rho_{t+1})\, R_{t+1,k} \right] \frac{\lambda^i_{t,k} r^i_t}{\lambda^i_{t,k} r^i_t + \lambda^j_{t,k} r^j_t},$$

where $i, j \in \{1, 2\}$ and $i \neq j$. Setting $C^{ij}_{t,k} := \lambda^i_{t,k} r^i_t / \left(\lambda^i_{t,k} r^i_t + \lambda^j_{t,k} r^j_t \right)$, we obtain

$$r^i_{t+1} \left[1 + \rho_{t+1} \sum_{k=1}^{K} \left(\lambda^j_{t+1,k} - \lambda^i_{t+1,k} \right) C^{ij}_{t,k} \right] = \sum_{k=1}^{K} \left[\rho_{t+1} \lambda^j_{t+1,k} + (1-\rho_{t+1})\, R_{t+1,k} \right] C^{ij}_{t,k}.$$

Thus

$$\frac{r^i_{t+1}}{r^j_{t+1}} = \frac{A^{ij}_{t+1} / B^{ij}_{t+1}}{A^{ji}_{t+1} / B^{ji}_{t+1}},$$

where

$$A^{ij}_{t+1} := \sum_{k=1}^{K} \left[\rho_{t+1} \lambda^j_{t+1,k} + (1-\rho_{t+1})\, R_{t+1,k} \right] C^{ij}_{t,k},$$

$$B^{ij}_{t+1} := 1 + \rho_{t+1} \sum_{k=1}^{K} \left(\lambda^j_{t+1,k} - \lambda^i_{t+1,k} \right) C^{ij}_{t,k}.$$

Observe that $B^{ji}_{t+1} = B^{ij}_{t+1}$. Indeed,

$$\begin{aligned} B^{ij}_{t+1} - B^{ji}_{t+1} &= \rho_{t+1} \sum_{k=1}^{K} \left[\left(\lambda^{j}_{t+1,k} - \lambda^{i}_{t+1,k} \right) C^{ij}_{t,k} - \left(\lambda^{i}_{t+1,k} - \lambda^{j}_{t+1,k} \right) C^{ji}_{t,k} \right] \\ &= \rho_{t+1} \sum_{k=1}^{K} \left(\lambda^{j}_{t+1,k} - \lambda^{i}_{t+1,k} \right) = 0 \end{aligned}$$

because $C^{ij}_{t,k} + C^{ji}_{t,k} = 1$. Consequently,

$$\frac{r^1_{t+1}}{r^2_{t+1}} = \frac{A^{12}_{t+1}}{A^{21}_{t+1}} = \frac{r^1_t}{r^2_t} \frac{\sum_{k=1}^{K} \left[\rho_{t+1} \lambda^2_{t+1,k} + (1 - \rho_{t+1})\, R_{t+1,k} \right] \dfrac{\lambda^1_{t,k}}{\lambda^1_{t,k} r^1_t / r^2_t + \lambda^2_{t,k}}}{\sum_{k=1}^{K} \left[\rho_{t+1} \lambda^1_{t+1,k} + (1 - \rho_{t+1})\, R_{t+1,k} \right] \dfrac{\lambda^2_{t,k}}{\lambda^1_{t,k} r^1_t / r^2_t + \lambda^2_{t,k}}},$$

which yields (25). □

Proof of Proposition 5 By virtue of (18), we have

$$\begin{aligned} E_t \left(\rho_{t+1} \lambda^{*}_{t+1,k} \right) &= E_t \left(\rho_{t+1} E_{t+1} \sum_{l=1}^{\infty} \rho^{l}_{t+1} R_{t+1+l,k} \right) \\ &= E_t \left(E_{t+1} \sum_{l=1}^{\infty} \rho_{t+1} \rho^{l}_{t+1} R_{t+1+l,k} \right) = E_t \left(\sum_{l=1}^{\infty} \rho_{t+1} \rho^{l}_{t+1} R_{t+1+l,k} \right), \end{aligned}$$

and so

$$\begin{aligned} E_t \left[\rho_{t+1} \lambda^{*}_{t+1,k} + (1 - \rho_{t+1})\, R_{t+1,k} \right] &= E_t \left[\sum_{l=1}^{\infty} \rho_{t+1} \rho^{l}_{t+1} R_{t+1+l,k} + (1 - \rho_{t+1})\, R_{t+1,k} \right] \\ = E_t \left(\sum_{l=1}^{\infty} \rho^{l+1}_{t} R_{t+l+1,k} + \rho^{1}_{t} R_{t+1,k} \right) &= E_t \sum_{l=1}^{\infty} \rho^{l}_{t} R_{t+l,k} = \lambda^{*}_{t,k} \end{aligned}$$

because $1 - \rho_{t+1} = \rho^1_t$ and

$$\rho^{l+1}_{t} = \rho_{t+1} \rho_{t+2} \ldots \rho_{t+l} \left(1 - \rho_{t+l+1} \right) = \rho_{t+1} \rho^{l}_{t+1}$$

for $l \geq 1$. □

Proof of Lemma 1 From formula (25) with $\lambda^1_{t,k} = \lambda^{*}_{t,k}$ and $\lambda^2_{t,k} = \lambda_{t,k}$, we get

$$\begin{aligned} &\sum_{k=1}^{K} \left[\rho_{t+1} \lambda^{*}_{t+1,k} + (1 - \rho_{t+1})\, R_{t+1,k} \right] \frac{\lambda_{t,k}}{\lambda^{*}_{t,k} z_t + \lambda_{t,k}} \\ &= \sum_{k=1}^{K} \left[\rho_{t+1} \frac{\lambda_{t+1,k}}{z_{t+1}} + (1 - \rho_{t+1}) \frac{R_{t+1,k}}{z_{t+1}} \right] \frac{\lambda^{*}_{t,k} z_t}{\lambda^{*}_{t,k} z_t + \lambda_{t,k}}. \end{aligned}$$

By using the notation $y_t^k = \lambda_{t,k}/z_t$ and the fact that $|y_t| = 1/z_t$, we write

$$\sum_{k=1}^{K} \left[\rho_{t+1}\lambda^*_{t+1,k} + (1-\rho_{t+1})\, R_{t+1,k}\right] \frac{y_t^k}{\lambda^*_{t,k} + y_t^k}$$
$$= \sum_{k=1}^{K} \left[\rho_{t+1} y_{t+1}^k + (1-\rho_{t+1})\, R_{t+1,k}|y_{t+1}|\right] \frac{\lambda^*_{t,k}}{\lambda^*_{t,k} + y_t^k},$$

which implies

$$\rho_{t+1} \sum_{k=1}^{K} \frac{\lambda^*_{t,k} y_{t+1}^k - \lambda^*_{t+1,k} y_t^k}{\lambda^*_{t,k} + y_t^k} + (1-\rho_{t+1}) \sum_{k=1}^{K} R_{t+1,k} \frac{\lambda^*_{t,k}|y_{t+1}| - y_t^k}{\lambda^*_{t,k} + y_t^k} = 0. \tag{45}$$

We have

$$\frac{\lambda^*_{t,k} y_{t+1}^k - \lambda^*_{t+1,k} y_t^k}{\lambda^*_{t,k} + y_t^k} = \lambda^*_{t+1,k} \frac{y_{t+1}^k/\lambda^*_{t+1,k} - y_t^k/\lambda^*_{t,k}}{1 + y_t^k/\lambda^*_{t,k}}$$
$$= \lambda^*_{t+1,k} \left(\frac{1 + y_{t+1}^k/\lambda^*_{t+1,k}}{1 + y_t^k/\lambda^*_{t,k}} - 1 \right) \ge \lambda^*_{t+1,k} \ln \frac{1 + y_{t+1}^k/\lambda^*_{t+1,k}}{1 + y_t^k/\lambda^*_{t,k}}, \tag{46}$$

where the last relation follows from the inequality $a - 1 \ge \ln a$ $(a > 0)$. By using (46) and the same inequality, we find

$$\sum_{k=1}^{K} \frac{\lambda^*_{t,k} y_{t+1}^k - \lambda^*_{t+1,k} y_t^k}{\lambda^*_{t,k} + y_t^k} \ge \sum_{k=1}^{K} \lambda^*_{t+1,k} \left[\ln\left(1 + \frac{y_{t+1}^k}{\lambda^*_{t+1,k}}\right) - \ln\left(1 + \frac{y_t^k}{\lambda^*_{t,k}}\right)\right]$$
$$= \sum_{k=1}^{K} \lambda^*_{t+1,k} \left(Z_{t+1,k} - Z_{t,k}\right) = Z_{t+1} - \sum_{k=1}^{K} \lambda^*_{t+1,k} Z_{t,k}. \tag{47}$$

Further, we have

$$\frac{\lambda^*_{t,k}|y_{t+1}| - y_t^k}{\lambda^*_{t,k} + y_t^k} = \frac{\lambda^*_{t,k}|y_{t+1}| + \lambda^*_{t,k}}{\lambda^*_{t,k} + y_t^k} - 1$$
$$\ge \ln \frac{\lambda^*_{t,k}|y_{t+1}| + \lambda^*_{t,k}}{\lambda^*_{t,k} + y_t^k} = \ln \frac{|y_{t+1}| + 1}{1 + y_t^k/\lambda^*_{t,k}},$$

and so

$$\sum_{k=1}^{K} R_{t+1,k} \frac{\lambda^*_{t,k}|y_{t+1}| - y_t^k}{\lambda^*_{t,k} + y_t^k} \ge \sum_{k=1}^{K} R_{t+1,k} \ln \frac{1 + |y_{t+1}|}{1 + y_t^k/\lambda^*_{t,k}}$$
$$= \ln(1 + |y_{t+1}|) - \sum_{k=1}^{K} R_{t+1,k} \ln\left(1 + y_t^k/\lambda^*_{t,k}\right) = Y_{t+1} - \sum_{k=1}^{K} R_{t+1,k} Z_{t,k}$$

(see (28) and (29)), which yields

$$\sum_{k=1}^{K} R_{t+1,k} \frac{\lambda^*_{t,k}|y_{t+1}| - y_t^k}{\lambda^*_{t,k} + y_t^k} \ge Y_{t+1} - \sum_{k=1}^{K} R_{t+1,k} Z_{t,k}. \tag{48}$$

By combining (45), (47) and (48), we find

$$0 \geq \rho_{t+1}\left(Z_{t+1} - \sum_{k=1}^{K} \lambda^*_{t+1,k} Z_{t,k}\right) + (1-\rho_{t+1})\left(Y_{t+1} - \sum_{k=1}^{K} R_{t+1,k} Z_{t,k}\right)$$
$$= \rho_{t+1} Z_{t+1} + (1-\rho_{t+1})\, Y_{t+1} - \sum_{k=1}^{K} \left[\rho_{t+1}\lambda^*_{t+1,k} + (1-\rho_{t+1})\, R_{t+1,k}\right] Z_{t,k},$$

which proves (31). □

Proof of Lemma 2 To prove the first relation in (32) we proceed as follows:

$$U_t = Y_t - Z_t = -\ln r_t^1 - \sum_{k=1}^{K} \lambda^*_{t,k} \ln\left[1 + \frac{r_t^2 \lambda_{t,k}}{r_t^1 \lambda^*_{t,k}}\right]$$
$$= \sum_{k=1}^{K} \lambda^*_{t,k} \ln \frac{1}{r_t^1} + \sum_{k=1}^{K} \lambda^*_{t,k} \ln \frac{r_t^1 \lambda^*_{t,k}}{r_t^1 \lambda^*_{t,k} + r_t^2 \lambda_{t,k}} = \sum_{k=1}^{K} \lambda^*_{t,k} \ln \frac{\lambda^*_{t,k}}{r_t^1 \lambda^*_{t,k} + r_t^2 \lambda_{t,k}}.$$

The last relation in (32) follows from the elementary inequality $\sum_{k=1}^{K} a_k \ln a_k - \sum_{k=1}^{K} a_k \ln b_k \geq 0$, which is presented in a somewhat refined form in Lemma 5. □

Proof of Lemma 3 It is clear that $\zeta_t \geq 0$. By taking the conditional expectation $E_t(\cdot)$ of both sides of inequality (31) and using (26), we obtain

$$E_t \zeta_{t+1} \leq \sum_{k=1}^{K} Z_{t,k} E_t \left[\rho_{t+1}\lambda^*_{t+1,k} + (1-\rho_{t+1})\, R_{t+1,k}\right] = \sum_{k=1}^{K} Z_{t,k} \lambda^*_{t,k} = Z_t. \tag{49}$$

In view of (30), we get

$$E_t \zeta_{t+1} + (1-\rho_t)\, U_t \leq Z_t + (1-\rho_t)\, Y_t - (1-\rho_t)\, Z_t = \rho_t Z_t + (1-\rho_t)\, Y_t = \zeta_t,$$

which proves (34). Thus $E_t \zeta_{t+1} \leq \zeta_t - (1-\rho_t)\, U_t \leq \zeta_t$ because $U_t \geq 0$ (see Lemma 2). The last inequality implies $E\zeta_t \leq E\zeta_0 = \zeta_0 < +\infty$. Since $\zeta_t \geq 0$, we have $E|\zeta_t| < \infty$, and so ζ_t is a supermartingale. □

Proof of Lemma 4 The random variables $\eta_t := \zeta_t - E_t\zeta_{t+1}$ are non-negative by the definition of a supermartingale. Further, we have

$$\sum_{t=0}^{T-1} E\eta_t = \sum_{t=0}^{T-1} (E\zeta_t - E\zeta_{t+1}) = E\zeta_0 - E\zeta_T,$$

and so the sequence $\sum_{t=0}^{T-1} E\, \eta_t$ is bounded because $\sup_T (-E\zeta_T) = -\inf_T E\, \zeta_T < +\infty$. Therefore the series of the expectations $\sum_{t=0}^{\infty} E\, \eta_t$ of the non-negative random variables η_t converges, which implies $\sum_{t=0}^{\infty} \eta_t < \infty$ a.s. because $E \sum_{t=0}^{\infty} \eta_t = \sum_{t=0}^{\infty} E\, \eta_t$ (the last equality holds for any sequence $\eta_t \geq 0$). □

Proof of Lemma 5 We have $\ln x \le x - 1$, which implies $(\ln x)/2 = \ln \sqrt{x} \le \sqrt{x} - 1$, and so $-\ln x \ge 2 - 2\sqrt{x}$. By using this inequality, we get

$$\sum_{k=1}^{K} a_k (\ln a_k - \ln b_k) = -\sum_{k=1}^{K} a_k \ln \frac{b_k}{a_k} \ge \sum_{k=1}^{K} a_k \left(2 - 2\frac{\sqrt{b_k}}{\sqrt{a_k}}\right)$$
$$= 2 - 2\sum_{k=1}^{K} \sqrt{a_k b_k} = \sum_{k=1}^{K} \left(a_k - 2\sqrt{a_k b_k} + b_k\right) = \sum_{k=1}^{K} \left(\sqrt{a_k} - \sqrt{b_k}\right)^2.$$

This yields (35) because $\left(\sqrt{a} - \sqrt{b}\right)^2 \ge (a-b)^2/4$ for $0 \le a, b \le 1$. □

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