НЕОПР. ИНТЕГРАЛ

№1

$$\int \frac{dx}{(x^2+1)(x^2-3)} = \frac{1}{4}\int \frac{x^2+1+3-x^2}{(x^2+1)(x^2-3)}dx =$$

$$= -\frac{1}{4}\int \frac{dx}{1+x^2} + \frac{1}{4}\int \frac{dx}{x^2-3} = -\frac{1}{4}\operatorname{arctg} x + \frac{1}{8\sqrt{3}}\ln\left|\frac{x-\sqrt{3}}{x+\sqrt{3}}\right|$$

№2

$$\int \frac{\sqrt{1+x^2}+\sqrt{1-x^2}}{\sqrt{1-x^4}}dx = \int \frac{dx}{\sqrt{1-x^2}} + \int \frac{dx}{\sqrt{1+x^2}} =$$

$$= \arcsin x + \ln|x+\sqrt{x^2+1}| + C$$

№3

$$\int (5^x-2^x)^2 dx = \int (25^x - 2\cdot 2^x\cdot 5^x + 4^x)\,dx =$$

$$= \int (e^{x\ln 25} - 2e^{x\ln 10} + e^{x\ln 4})\,dx =$$

$$= \frac{25^x}{\ln 25} - 2\frac{10^x}{\ln 10} + \frac{4^x}{\ln 4} + C$$

№4

$$\int \operatorname{tg}^2 x\,dx = \int (1+\operatorname{tg}^2 x - 1)\,dx = \int \frac{1}{\cos^2 x}dx - \int dx =$$

$$= \operatorname{tg} x - x + C$$

№5

$$\int x\sqrt{1-2x}\,dx = \left\{\begin{matrix} 1-2x=t \\ dx=-\frac{1}{2}dt \end{matrix}\right\} = \int \frac{1-t}{2}\sqrt{t}\cdot\frac{-1}{2}\,dt =$$

$$= \int \left(\frac{1}{4}t^{\frac{3}{2}} - \frac{1}{4}t^{\frac{1}{2}}\right)dt = \frac{1}{10}t^{\frac{5}{2}} - \frac{1}{6}t^{\frac{3}{2}} + C =$$

$$= \frac{(1-2x)^{5/2}}{10} - \frac{(1-2x)^{3/2}}{6} + C$$

№6

$$\int \frac{x-4}{\sqrt{x^2-2}}\,dx = \int \frac{\frac{1}{2}\,dx^2}{\sqrt{x^2-2}} - \int \frac{4\,dx}{\sqrt{x^2-2}} = \sqrt{x^2-2} -$$
$$-4\ln\left|x+\sqrt{x^2-2}\right| + C$$

№7

$$\int \frac{dx}{1+\cos x} = \int \frac{dx}{2\cos^2\frac{x}{2}} = \int \frac{1}{\cos^2\frac{x}{2}}\,d\frac{x}{2} = tg\,\frac{x}{2} + C$$

№8

$$\int \frac{dx}{\cos x} = \int \frac{d\sin x}{1-\sin^2 x} = \frac{1}{2}\ln\left|\frac{1+\sin x}{1-\sin x}\right| + C$$

№9

$$\int x\sin x\,dx = -\int x\,d\cos x = -x\cos x + \int \cos x\,dx =$$
$$= -x\cos x + \sin x + C$$

№10

$$\int x\,ctg^2 x\,dx = \int x\,(ctg^2 x + 1)\,dx - \int x\,dx =$$
$$= \int x\,\frac{1}{\sin^2 x}\,dx - \frac{x^2}{2} = -\frac{x^2}{2} - \int x\,d\,ctg\,x =$$
$$= -\frac{x^2}{2} - x\,ctg\,x + \int ctg\,x\,dx = -\frac{x^2}{2} - x\,ctg\,x +$$
$$+ \ln|\sin x| + C$$

№11

$$\int \ln^2 x\,dx = x\ln^2 x - \int x\,d\ln^2 x = x\ln^2 x -$$
$$- \int x\cdot\frac{2}{x}\ln x\,dx = x\ln^2 x - 2\int \ln x\,dx = x\ln^2 x -$$
$$-2x\ln x + 2\int x\,d\ln x = x\ln^2 x - 2x\ln x + 2x + C$$

№12

$$\int \sqrt{2-x^2}\,dx = x\sqrt{2-x^2} - \int x\, d\sqrt{2-x^2} =$$

$$= x\sqrt{2-x^2} + \int \frac{x^2\,dx}{\sqrt{2-x^2}} = x\sqrt{2-x^2} + \int \frac{x^2-2}{\sqrt{2-x^2}}\,dx$$

$$+ \int \frac{2\,dx}{\sqrt{2-x^2}} = x\sqrt{2-x^2} - \int \sqrt{2-x^2}\,dx + 2\arcsin\frac{x}{\sqrt{2}}$$

Обозначим $I = \int \sqrt{2-x^2}\,dx$, получим

$$I = x\sqrt{2-x^2} - I + 2\arcsin\frac{x}{\sqrt{2}}$$

$$I = \frac{x}{2}\sqrt{2-x^2} + \arcsin\frac{x}{\sqrt{2}} + C$$

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INT-14. Найти интеграл

$$\int \frac{dx}{\cos x}$$

Решение.

$$\int \frac{dx}{\cos x} = \int \frac{1}{\cos x} \frac{\frac{1}{\cos x} + \tan(x)}{\frac{1}{\cos x} + \tan(x)} dx = \int \frac{\frac{1}{\cos^2 x} + \frac{1}{\cos x} tg(x)}{\frac{1}{\cos x} + tg(x)} dx \ominus$$

замена перем.: $u = \frac{1}{\cos x} + tg(x)$

$$du = \left(\frac{\sin x}{\cos^2 x} + \frac{1}{\cos^2 x}\right) dx = \left(\frac{\sin(x)+1}{\cos^2 x}\right) =$$

$$= \left(\frac{1}{\cos(x)} \cdot \frac{\sin x}{\cos x} + \frac{1}{\cos^2(x)}\right) dx = \left(\frac{tg(x)}{\cos x} + \frac{1}{\cos^2 x}\right) dx$$

$$\ominus \int \frac{du}{u} = \ln|u| = \boxed{\ln\left|\frac{1}{\cos x} + tg(x)\right| + c}$$

Интегралы

① $\int tg^2x\,dx = \int\frac{\sin^2x}{\cos^2x}dx = \int\frac{1-\cos^2x}{\cos^2x}dx = \int\frac{dx}{\cos^2x} - \int 1\,dx =$

$= tgx - x + C$

② $\int\frac{e^x+e^{2x}}{1-e^x}dx = \int\frac{e^x(1+e^x)}{1-e^x}dx = \int\frac{(de^x)\cdot(1+e^x)}{1-e^x} \overset{e^x=t}{=} \int\frac{(1+t)dt}{(1-t)} =$

$= \int\frac{dt}{1-t} + \int\frac{t\,dt}{1-t} = -\int\frac{d(t-1)}{t-1} - \int dt - \int\frac{d(t-1)}{t-1} = -2\ln|t-1| - t + C =$

$\frac{t}{1-t} \Rightarrow \frac{t}{1-t} = -\frac{t}{t-1} = -\frac{t-1+1}{t-1} = -1-\frac{1}{t-1}$ $\quad = -2\ln|e^x-1| - e^x + C$

③ $\int\frac{dx}{1+\cos x} = \int\frac{1-\cos x}{(1+\cos x)(1-\cos x)}dx = \int\frac{1-\cos x}{\sin^2x}dx = \int\frac{dx}{\sin^2x} - \int\frac{\cos x\,dx}{\sin^2x} =$

$= -ctg\,x - \int\frac{d\sin x}{\sin^2x} = -ctg\,x + \frac{1}{\sin x} + C = \frac{1}{\sin x} - \frac{\cos x}{\sin x} + C = \frac{1-\cos x}{\sin x} + C =$

$= tg\frac{x}{2} + C$

④ $\int arctg\,x\,dx = \int 1\cdot arctg\,dx = x\cdot arctg\,x - \int x\,d(arctg\,x) =$

$(\int u\,dv = uv - \int v\,du)$ $= x\,arctg\,x - \int\frac{x}{1+x^2}dx = x\,arctg\,x - \frac{1}{2}\int\frac{d(x^2+1)}{1+x^2} =$

$= x\,arctg\,x - \frac{1}{2}\ln(1+x^2) + C$

⑤ $\int\frac{\sqrt{(1+x^2)} + \sqrt{(1-x^2)}}{\sqrt{(1-x^4)}}dx = \int\frac{\sqrt{1+x^2}}{\sqrt{1-x^2}\cdot\sqrt{1+x^2}}dx + \int\frac{\sqrt{1-x^2}}{\sqrt{(1-x^2)}\sqrt{1+x^2}}dx = \int\frac{dx}{\sqrt{1-x^2}} + \int\frac{dx}{\sqrt{1+x^2}} = \arcsin x + \ln|x+\sqrt{x^2+1}| + C$

Проверка: $(\arcsin x)' = \frac{1}{\sqrt{1-x^2}}$

$(\ln|x+\sqrt{x^2+1}|)' = \frac{1}{x+\sqrt{x^2+1}}\cdot\left(1+\frac{1}{2\sqrt{x^2+1}}\cdot 2x\right) = \frac{\sqrt{x^2+1}+x}{\sqrt{x^2+1}\cdot(x+\sqrt{x^2+1})} = \frac{1}{\sqrt{x^2+1}}$

$\frac{1}{\sqrt{1-x^2}} + \frac{1}{\sqrt{x^2+1}} = \frac{\sqrt{x^2+1}+\sqrt{1-x^2}}{\sqrt{1-x^4}}$

⑥ $\int\frac{dx}{\sin^2(x)\cdot(1+tg\,x)} = -\int\frac{d(ctg\,x)}{1+\frac{1}{ctg\,x}} = -\int\frac{ctg\,x\,d(ctg\,x)}{ctg\,x+1} \underset{ctg\,x=t}{=} -\int\frac{t\,dt}{t+1} = -\int 1\,dt + \int\frac{dt}{t+1} =$

$\frac{t}{t+1} = \frac{t+1-1}{t+1} = 1-\frac{1}{t+1}$ $\quad = -t + \ln|t+1| + C = -ctg\,x + \ln|ctg\,x+1| + C$

⑦ $\int\frac{dx}{x^4-1} = \int\frac{dx}{(x^2-1)(x^2+1)} = \frac{1}{2}\int\frac{dx}{x^2-1} - \frac{1}{2}\int\frac{dx}{x^2+1} = -\frac{1}{2}\left(\int\frac{dx}{1-x^2} + \int\frac{dx}{1+x^2}\right) =$

$\frac{1}{(x^2-1)(x^2+1)} = \frac{A}{x^2-1} + \frac{B}{x^2+1} = \frac{1}{2}\cdot\frac{1}{x^2-1} - \frac{1}{2}\cdot\frac{1}{x^2+1}$ $\quad = -\frac{1}{2}\left(\frac{1}{2}\ln\left|\frac{1+x}{1-x}\right| + arctg\,x\right) + C =$

$\begin{cases} A+B=0 \\ A-B=1 \end{cases} \Rightarrow \begin{cases} A=\frac{1}{2} \\ B=-\frac{1}{2} \end{cases}$ $\quad = -\frac{1}{4}\ln\left|\frac{1+x}{1-x}\right| - \frac{1}{2}arctg\,x + C.$

⑧ $\int\sqrt{2-x^2}dx = x\cdot\sqrt{2-x^2} - \int x\,d\sqrt{2-x^2} = x\sqrt{2-x^2} - \int x\frac{1}{2\sqrt{2-x^2}}\cdot(-2x)dx$

$(\int u\,dv = uv - \int v\,du)$ $= x\sqrt{2-x^2} + \int\frac{x^2dx}{\sqrt{2-x^2}} =$

$\frac{x^2}{\sqrt{2-x^2}} = -\frac{2-x^2-2}{\sqrt{2-x^2}} = -\sqrt{2-x^2} + \frac{2}{\sqrt{2-x^2}}$ $\quad = x\sqrt{2-x^2} - \int\sqrt{2-x^2}dx +$

$+ 2\int\frac{dx}{\sqrt{2-x^2}} \Rightarrow$

$\Rightarrow I = x\sqrt{2-x^2} - I + 2\arcsin\frac{x}{\sqrt{2}} + C$

$2I = x\sqrt{2-x^2} + 2\arcsin\frac{x}{\sqrt{2}} + C$

$I = \frac{1}{2}x\sqrt{2-x^2} + \arcsin\frac{x}{\sqrt{2}} + C^*$

⑨ $\int\frac{2x-7}{\sqrt{3x+1}}dx = \frac{2}{3}\int\sqrt{3x+1}\,dx - \frac{23}{3}\int\frac{dx}{\sqrt{3x+1}} =$

$\frac{2x-7}{\sqrt{3x+1}} = \frac{\frac{2}{3}(3x-\frac{21}{2})}{\sqrt{3x+1}} = \frac{\frac{2}{3}(3x+1-\frac{23}{2})}{\sqrt{3x+1}} = \frac{2}{3}\sqrt{3x+1} - \frac{23}{3}\cdot\frac{1}{\sqrt{3x+1}}$

$= \frac{2}{9}\int\sqrt{3x+1}\,(d(3x+1)) - \frac{23}{9}\int(3x+1)^{-1/2}d(3x+1) = \frac{2}{9}\cdot\frac{2}{3}\cdot(3x+1)^{3/2} - \frac{23}{9}\cdot\frac{2}{1}\cdot(3x+1)^{1/2} + C =$

$= \frac{4}{27}(3x+1)^{3/2} - \frac{46}{9}(3x+1)^{1/2} + C = \frac{2}{9}(3x+1)^{1/2}\left(\frac{2}{3}\cdot(3x+1) - 23\right) + C =$

$= \frac{2}{9}(3x+1)^{1/2}\left(2x+\frac{2}{3}-23\right)+C = \frac{2}{27}\sqrt{3x+1}\,(6x-67)+C$

(10). $\int x(x-2)^5 dx = \frac{1}{2}\int (x-2)^5 dx^2 = \frac{1}{2}x^2\cdot(x-2)^5 - \frac{1}{2}\int x^2\cdot 5(x-2)^4 dx =$

$= \frac{1}{2}x^2(x-2)^5 - \frac{5}{6}\int (x-2)^4 dx^3 = \frac{1}{2}x^2(x-2)^5 - \frac{5}{6}x^3(x-2)^4 +$

$+ \frac{5}{6}\int 4(x-2)^3\cdot x^3\, dx = \frac{1}{2}x^2(x-2)^5 - \frac{5}{6}x^3(x-2)^4 +$

$+ \frac{20}{6}\int (x^2-2x)^3 dx = \ldots + \frac{20}{6}\int (x^6 - 3x^4\cdot 2x + 3\cdot x^2\cdot 4x^2 - 8x^3)dx =$

$= \ldots + \frac{20}{6}\int (x^6 + 12x^4 - 6x^5 - 8x^3)dx =$

$= \frac{1}{2}x^2(x-2)^4\left(x-2-\frac{5}{3}x\right) + \frac{20}{6}\left(\frac{x^7}{7} + \frac{12x^5}{5} - \frac{6x^6}{6} - \frac{8x^4}{4}\right) + C =$

$= 1111$

2 способ: $(x-2)^2 = x^2-4x+4$

$(x-2)^4 = (x^2-4x+4)^2 = x^4+16x^2+16-8x^3+8x^2-32x =$

$= x^4-8x^3+24x^2-32x+16$

$(x-2)^5 = (x-2)^4\cdot(x-2) = x^5 - 8x^4 + 24x^3 - 32x^2 + 16x - 2x^4 + 16x^3 - 48x^2 + 64x - 32 = x^5 - 10x^4 + 40x^3 - 80x^2 + 80x - 32$

$\int x(x-2)^5 dx = \int (x^6 - 10x^5 + 40x^4 - 80x^3 + 80x^2 - 32x)dx =$

$= \frac{x^7}{7} - \frac{10x^6}{6} + \frac{40x^5}{5} - \frac{80x^4}{4} + \frac{80x^3}{3} - \frac{32x^2}{2} + C =$

$= \frac{1}{7}x^7 - \frac{5}{3}x^6 + 8x^5 - 20x^4 + \frac{80}{3}x^3 - 16x^2 + C$

(11). $\int \frac{dx}{\cos x} = \int \frac{\cancel{1+t^2}}{1-t^2}\cdot 2\cdot\frac{1}{\cancel{1+t^2}}dt = 2\int\frac{dt}{1-t^2} = 2\cdot\frac{1}{2}\ln\left|\frac{1+t}{1-t}\right| + C =$

$= \ln\left|\frac{1+\operatorname{tg}\frac{x}{2}}{1-\operatorname{tg}\frac{x}{2}}\right| + C$

$\left(\cos x = \frac{1-\operatorname{tg}^2\frac{x}{2}}{1+\operatorname{tg}^2\frac{x}{2}};\ t = \operatorname{tg}\frac{x}{2}\right.$

$dt = \frac{1}{\cos^2\frac{x}{2}}\cdot\frac{1}{2}dx \Rightarrow dx = 2\cos^2\frac{x}{2}\,dt = 2\cdot\frac{1}{\operatorname{tg}^2\frac{x}{2}+1}dt = \left.\frac{2}{1+t^2}dt\right)$

$1+\operatorname{tg}^2\frac{x}{2} = \frac{1}{\cos^2\frac{x}{2}}$

(12). $\int (5^x - 2^x)^2 dx = \int (5^{2x} - 2\cdot 10^x + 2^{2x})dx = \frac{1}{2}\int 5^{2x} d(2x) - 2\int 10^x dx + \frac{1}{2}\int 2^{2x} d(2x) =$

$= \frac{1}{2}\cdot\frac{5^{2x}}{\ln 5} - 2\cdot\frac{10^x}{\ln 10} + \frac{1}{2}\cdot\frac{2^{2x}}{\ln 2} + C$

(13). $\int x\sqrt{1-2x}\,dx = \int \frac{1-t}{2}\cdot\sqrt{t}\cdot\left(-\frac{1}{2}\right)dt = +\frac{1}{4}\int\sqrt{t}\,(t-1)\,dt = \frac{1}{4}\int (t^{\frac{3}{2}} - t^{\frac{1}{2}})dt =$

$1-2x = t \quad dt = -2dx$

$2x = 1-t \quad dx = -\frac{1}{2}dt$

$x = \frac{1-t}{2}$

$= \frac{1}{4}\cdot\frac{t^{5/2}}{5/2} - \frac{1}{4}\cdot\frac{t^{3/2}}{3/2} + C =$

$= \frac{(1-2x)^{5/2}}{10} - \frac{(1-2x)^{3/2}}{6} + C =$

$= \frac{1}{2}(1-2x)^{3/2}\cdot\left[\frac{1-2x}{5} - \frac{1}{3}\right] + C = \frac{1}{2}(1-2x)^{3/2}\cdot\frac{3-6x-5}{15} + C$

$= \frac{(-3x-1)(1-2x)^{3/2}}{15} + C$

(14). $\int\frac{x-4}{\sqrt{x^2-2}}dx = \int\frac{x\,dx}{\sqrt{x^2-2}} - 4\int\frac{dx}{\sqrt{x^2-2}} = \frac{1}{2}\int\frac{d(x^2-2)}{\sqrt{x^2-2}} - 4\int\frac{dx}{\sqrt{x^2-(\sqrt{2})^2}} =$

$= \sqrt{x^2-2} - 4\ln\left|x + \sqrt{x^2-2}\right| + C$

(15). $\int \ln^2 x\,dx = x\ln^2 x - \int x\,d\ln^2 x = x\ln^2 x - \int x\cdot 2\ln x\cdot\frac{1}{x}dx =$

$= x\ln^2 x - 2\int \ln x\,dx = x\ln^2 x - 2x\ln x + 2\int x\,d\ln x =$

$= x\ln^2 x - 2x\ln x + 2\int x\cdot\frac{1}{x}dx = x\ln^2 x - 2x\ln x + 2x + C$

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INT-14 Найти интеграл

$$\int \frac{dx}{\cos x}$$

$$\int \frac{dx}{\cos x} = \int \frac{\cos x}{\cos^2 x} dx = \int \frac{1}{1-\sin^2 x} d\sin x =$$

Замена: $\sin x = t$

$$= \int \frac{1}{1-t^2} dt = (\text{табличный интеграл}) = \frac{1}{2} \ln\left|\frac{1+t}{1-t}\right| + C =$$

$$= \frac{1}{2} \ln\left|\frac{1+\sin x}{1-\sin x}\right| + C$$

Ответ: $\int \frac{dx}{\cos x} = \frac{1}{2} \ln\left|\frac{1+\sin x}{1-\sin x}\right| + C \quad (C \in \mathbb{R})$