06.12.20. Спецкурс. д/з от лекции 12.

(1.) а) $cov(\mu(\theta_k); X_{pe}) = a \cdot \delta_{kp}$.

$$cov(\mu(\theta_k); X_{pe}) = E(\mu(\theta_k) - m)(X_{pe} - m) = E\left[E\left((\mu(\theta_k)-m)(X_{pe}-m) \mid \theta_k\right)\right] =$$

$\mu(\theta_k) = E(X_{ki} \mid \theta_k)$

$m = EX_{ki}$

$$= E\left[(\mu(\theta_k)-m)\, E(X_{pe} - m \mid \theta_k)\right] = E\left[(\mu(\theta_k)-m) \cdot \delta_{kp} \cdot E(\underbrace{(X_{ke} \mid \theta_k)}_{=\mu(\theta_k)} - m)\right] =$$

$$= E\, \delta_{kp} \cdot (\mu(\theta_k) - m)^2 = \delta_{kp} \cdot \underbrace{D(\mu(\theta_k))}_{=a} = \delta_{kp} \cdot a.$$

$E(\mu(\theta_k)) = m$

б) $cov(X_{ki}; X_{pe}) = (a + s^2 \cdot \delta_{ie})\delta_{kp}$.

$$cov(X_{ki}; X_{pe}) = E(X_{ki} - EX_{ki})(X_{pe} - EX_{pe}) = E(X_{ki} \pm E(X_{ki} \mid \theta_k) - EX_{ki})(X_{pe} \pm E(X_{pe} \mid \theta_k) - EX_{pe}) =$$

$$= E(X_{ki} - E(X_{ki} \mid \theta_k))(X_{pe} - E(X_{pe} \mid \theta_k)) +$$

$$+ E(X_{ki} - E(X_{ki} \mid \theta_k)) \cdot (E(X_{pe} \mid \theta_k) - EX_{pe}) +$$

$$+ E(E(X_{ki} \mid \theta_k) - EX_{ki})(X_{pe} - E(X_{pe} \mid \theta_k)) +$$

$$+ E(E(X_{ki} \mid \theta_k) - EX_{ki})(E(X_{pe} \mid \theta_k) - EX_{pe}) =$$

$$= E\left[E\left((X_{ki} - E(X_{ki} \mid \theta_k))(X_{pe} - E(X_{pe} \mid \theta_k)) \mid \theta_k\right)\right] +$$

$$+ E\left[E\left((X_{ki} - E(X_{ki} \mid \theta_k))(E(X_{pe} \mid \theta_k) - EX_{pe}) \mid \theta_k\right)\right] +$$

$$+ E\left[E\left((E(X_{ki} \mid \theta_k) - EX_{ki})(X_{pe} - E(X_{pe} \mid \theta_k)) \mid \theta_k\right)\right] +$$

$$+ cov(E(X_{ki} \mid \theta_k); E(X_{pe} \mid \theta_k)) =$$

$$= E\, cov(X_{ki}; X_{ke} \mid \theta_k) \cdot \delta_{kp}$$

$$+ E\left[E(X_{pe} \mid \theta_k) - EX_{pe})\, \underbrace{E(X_{ki} - E(X_{ki} \mid \theta_k)) \mid \theta_k)}_{=0}\right] +$$

$$+ E\left[E(X_{ki} \mid \theta_k - EX_{ki}) \cdot \underbrace{E(X_{pe} - E(X_{pe} \mid \theta_k)) \mid \theta_k)}_{=0}\right] +$$

$$+ \underbrace{cov(E(X_{ki} \mid \theta_k); E(X_{pe} \mid \theta_k))}_{=\delta_{kp} \cdot D(\underbrace{E(X_{ki} \mid \theta_k)}_{=\mu(\theta_k)})} = \delta_{kp} \cdot \delta_{ie} \cdot \underbrace{E\sigma^2(\theta_k)}_{=s^2} + \delta_{kp} \cdot \underbrace{D(\mu(\theta_k))}_{=a} = \delta_{kp} \cdot (a + \delta_{ie} \cdot s^2).$$

в) $cov(\bar{X}_{k\cdot}(t); \bar{X}_{p\cdot}(t)) = (a + \frac{s^2}{t})\delta_{kp}$.

$$cov(\bar{X}_{k\cdot}(t); \bar{X}_{p\cdot}(t)) = cov\left(\frac{1}{t}\sum_{i=1}^{t} X_{ki}; \frac{1}{t}\sum_{j=1}^{t} X_{pj}\right) = \frac{1}{t^2}\sum_{i=1}^{t}\sum_{j=1}^{t} cov(X_{ki}; X_{pj}) =$$

$$= \frac{1}{t^2}\sum_{i=1}^{t}\sum_{j=1}^{t}(a + s^2\delta_{ij})\delta_{kp} = \frac{\delta_{kp}}{t^2}(at^2 + s^2 \cdot t) = \delta_{kp} \cdot (a + \frac{s^2}{t}).$$

(2.) При выполнении гипотез (BS1) и (BS2):

а) $cov(\mu(\theta_k); X_{pi}) = a \cdot \delta_{kp}$

$$cov(\mu(\theta_k); X_{pi}) = E(\mu(\theta_k) - m)(X_{pi} - m) =$$

$$= E[E((\mu(\theta_k) - m)(X_{pi} - m) \mid \theta_k] = E[(\mu(\theta_k) - m) \cdot E(X_{pi} - m \mid \theta_k)] =$$

$$= E(\delta_{kp} \cdot (\mu(\theta_k) - m) \cdot (E(X_{ki} \mid \theta_k) - m)) = \delta_{kp} \cdot D(\mu(\theta_k)) = \delta_{kp} \cdot a$$

б) $cov(X_{ki}; X_{pj}) = (a + \delta_{ij} \frac{s^2}{w_{ki}}) \delta_{kp}$

$$cov(X_{ki}; X_{pj}) = E((X_{ki} \pm E(X_{ki} \mid \theta_k) - EX_{ki})(X_{pj} \pm E(X_{pj} \mid \theta_k) - EX_{pj}) =$$

раскроем скобки и навесим усл. относ. θ_k

$$= E\,cov(X_{ki}; X_{pj} \mid \theta_k) + cov(E(X_{ki} \mid \theta_k); E(X_{pj} \mid \theta_k)) =$$

$$= \delta_{kp} \cdot \underbrace{E\,cov(X_{ki}; X_{kj} \mid \theta_k)}_{= E \frac{\sigma^2(\theta_k) \cdot \delta_{ij}}{w_{kj}}} + \delta_{kp} \cdot \underbrace{D(\mu(\theta_k))}_{= a} = \delta_{kp} \cdot \left(a + \frac{\delta_{ij}}{w_{kj}} \cdot \underbrace{E\,\sigma^2(\theta_k)}_{= s^2}\right) = \delta_{kp} \cdot \left(a + \delta_{ij} \cdot \frac{s^2}{w_{kj}}\right)$$

в) $cov(X_{ki}; X_{k\cdot}^w) = cov(X_{k\cdot}^w; X_{k\cdot}^w) = a + \frac{s^2}{w_{k\cdot}}$

$w_{k\cdot} = \sum_{j=1}^{t} w_{kj}$

- $cov(X_{ki}; X_{k\cdot}^w) = cov(X_{ki}; \sum_{j=1}^{t} \frac{w_{kj}}{w_{k\cdot}} X_{kj}) = \sum_{j=1}^{t} \frac{w_{kj}}{w_{k\cdot}} cov(X_{ki}, X_{kj}) = \sum_{j=1}^{t} \frac{w_{kj}}{w_{k\cdot}} (a + \delta_{ij} \frac{s^2}{w_{kj}}) =$

$$= a + \sum_{j=1}^{t} \frac{w_{kj}}{w_{k\cdot}} \cdot \delta_{ij} \cdot \frac{s^2}{w_{kj}} = a + \frac{s^2}{w_{k\cdot}}$$

- $cov(X_{k\cdot}^w; X_{k\cdot}^w) = cov(\sum_{i=1}^{t} \frac{w_{ki}}{w_{k\cdot}} X_{ki}; \sum_{j=1}^{t} \frac{w_{kj}}{w_{k\cdot}} X_{kj}) = \frac{1}{(w_{k\cdot})^2} \cdot \sum_{i=1}^{t} \sum_{j=1}^{t} w_{ki} w_{kj} \cdot \underbrace{cov(X_{ki}, X_{kj})}_{= a + \delta_{ij} \cdot \frac{s^2}{w_{ki}}} =$

$$= \frac{(w_{k\cdot})^2 \cdot a}{(w_{k\cdot})^2} + \frac{s^2 \cdot w_{k\cdot}}{(w_{k\cdot})^2} = a + \frac{s^2}{w_{k\cdot}}$$

г) $cov(X_{ki}; X_{\cdot\cdot}^w) = \frac{s^2}{w_{\cdot\cdot}} + a \cdot \frac{w_{k\cdot}}{w_{\cdot\cdot}}$

$$cov(X_{ki}; X_{\cdot\cdot}^w) = cov(X_{ki}; \sum_{l=1}^{n} \frac{w_{l\cdot}}{w_{\cdot\cdot}} X_{l\cdot}^w) = \sum_{l=1}^{n} \frac{w_{l\cdot}}{w_{\cdot\cdot}} cov(X_{ki}; X_{l\cdot}^w) = \sum_{l=1}^{n} \frac{w_{l\cdot}}{w_{\cdot\cdot}} cov(X_{ki}; \sum_{j=1}^{t} \frac{w_{lj}}{w_{l\cdot}} X_{lj}) =$$

$$= \sum_{l=1}^{n} \sum_{j=1}^{t} \frac{w_{l\cdot}}{w_{\cdot\cdot}} \cdot \frac{w_{lj}}{w_{l\cdot}} \cdot cov(X_{ki}; X_{lj}) \overset{l=k}{=} \sum_{j=1}^{t} \frac{\cancel{w_{k\cdot}}}{w_{\cdot\cdot}} \cdot \frac{w_{kj}}{\cancel{w_{k\cdot}}} \cdot (a + \delta_{ij} \cdot \frac{s^2}{w_{ki}}) = \frac{w_{k\cdot}}{w_{\cdot\cdot}} a + \frac{s^2}{w_{\cdot\cdot}}$$

д) $cov(X_{k\cdot}^w; X_{\cdot\cdot}^w) = \frac{s^2}{w_{\cdot\cdot}} + \frac{a w_{k\cdot}}{w_{\cdot\cdot}}$

$$cov(X_{k\cdot}^w; X_{\cdot\cdot}^w) = cov(\sum_{j=1}^{t} \frac{w_{kj}}{w_{k\cdot}} X_{kj}; X_{\cdot\cdot}^w) = \sum_{j=1}^{t} \frac{w_{kj}}{w_{k\cdot}} cov(X_{kj}; X_{\cdot\cdot}^w) \overset{\text{см. г)}}{=} \sum_{j=1}^{t} \frac{w_{kj}}{w_{k\cdot}} \left(\frac{w_{k\cdot}}{w_{\cdot\cdot}} a + \frac{s^2}{w_{\cdot\cdot}}\right) = \frac{w_{k\cdot}}{w_{\cdot\cdot}} a + \frac{s^2}{w_{\cdot\cdot}}$$

е) $cov(X_{\cdot\cdot}^w; X_{\cdot\cdot}^w) = \frac{s^2}{w_{\cdot\cdot}} + a \cdot \sum_{k=1}^{n} \left(\frac{w_{k\cdot}}{w_{\cdot\cdot}}\right)^2$

$$cov(X_{\cdot\cdot}^w; X_{\cdot\cdot}^w) = cov(\sum_{l=1}^{n} \frac{w_{l\cdot}}{w_{\cdot\cdot}} X_{l\cdot}^w; X_{\cdot\cdot}^w) = \sum_{l=1}^{n} \frac{w_{l\cdot}}{w_{\cdot\cdot}} cov(X_{l\cdot}^w; X_{\cdot\cdot}^w) \overset{\text{см. д)}}{=} \sum_{l=1}^{n} \frac{w_{l\cdot}}{w_{\cdot\cdot}} \left(\frac{w_{l\cdot}}{w_{\cdot\cdot}} a + \frac{s^2}{w_{\cdot\cdot}}\right) =$$

$$= a \cdot \sum_{l=1}^{n} \left(\frac{w_{l\cdot}}{w_{\cdot\cdot}}\right)^2 + \frac{s^2}{w_{\cdot\cdot}}$$

(2) Пусть $X \leq_{st} Y$.
Можно ли (на том же самом вер. пр-ве) найти $\tilde{Y}$: $\begin{cases} X \leq_1 \tilde{Y} \\ Y \overset{d}{=} \tilde{Y} \end{cases}$

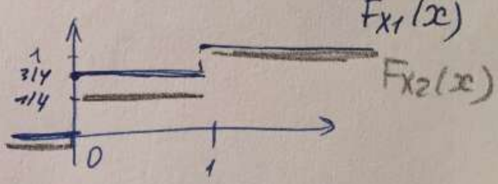
Ответ: нет.

Вот контрпример: $\Omega = \{\omega_1; \omega_2\}$

$P(\omega_1) = 3/4$; $P(\omega_2) = 1/4$

$X_1 = \begin{cases} 0, \text{ на } \omega_1 \\ 1, \text{ на } \omega_2 \end{cases}$, т.е. $X_1 = \begin{cases} 0, \text{ с } p = 3/4 \\ 1, \text{ с } p = 1/4 \end{cases}$

$X_2 = \begin{cases} 0, \text{ на } \omega_2 \\ 1, \text{ на } \omega_1 \end{cases}$, т.е. $X_2 = \begin{cases} 0, \text{ с } p = 1/4 \\ 1, \text{ с } p = 3/4 \end{cases}$



тогда $X_1 \leq_{st} X_2$, т.к. $F_{X_1}(t) \geq F_{X_2}(t)$, $\forall t$

но $P(X_1 > X_2) = P(\omega_2) = 1/4 > 0 \Rightarrow X_1 \not\leq_1 X_2$. чтд.