21.12.20. ВарОПУ. КР2. Токаева Александра. Вар 1,3,3.

(1.1.) $\int_0^1 (\dot{x}^2 - t^2 x)\,dt \to extr;\ x(0)=x(1)=0$ — простейшая задача

Решение: $L = \dot{x}^2 - t^2 x$

$$L_{\dot{x}} = 2\dot{x}$$
$$L_x = -t^2 \Rightarrow -\frac{d}{dt}(2\dot{x}) - t^2 = 0.$$
$$\Rightarrow \frac{d}{dt}(2\dot{x}) = -t^2$$
$$\Rightarrow 2\dot{x} = -\frac{1}{3}t^3 + \tilde{C}_1$$
$$\Rightarrow \dot{x} = -\frac{1}{6}t^3 + C_1$$
$$\Rightarrow \frac{dx}{dt} = -\frac{1}{6}t^3 + C_1$$
$$\Rightarrow dx = \left(-\frac{1}{6}t^3 + C_1\right)dt$$
$$\Rightarrow x = -\frac{1}{24}t^4 + C_1 t + C_2.$$

$x(0)=0 \Rightarrow C_2 = 0.$

$x(1)=0 \Rightarrow -\frac{1}{24} + C_1 + \cancel{C_2}^{0} = 0. \Rightarrow C_1 = \frac{1}{24}$

$\Rightarrow$ $\hat{x} = -\frac{1}{24}t^4 + \frac{1}{24}t$ — кандидат на экстремум.

Проверим, доставляет ли она экстремум.

Берём h такое, чтобы $\hat{x}+h$ было допустимым, т.е. $h(0)=h(1)=0$.

$$\Rightarrow J(\hat{x}+h) - J(\hat{x}) = \int_0^1 (\dot{\hat{x}} + \dot{h})^2 - t^2(\hat{x}+h)\,dt - \left(\int_0^1 (\dot{\hat{x}})^2 - t^2\hat{x}\,dt\right) =$$

$$= \int_0^1 \left(2\dot{\hat{x}}\dot{h} + (\dot{h})^2 - t^2 h\right)dt = \underbrace{2\dot{\hat{x}}h\Big|_0^1}_{0} - \int_0^1 2\ddot{\hat{x}}h\,dt + \int_0^1 \left((\dot{h})^2 - t^2 h\right)dt =$$

$\dot{\hat{x}} = -\frac{1}{6}t^3 + \frac{1}{24}$

$\ddot{\hat{x}} = -\frac{1}{2}t^2$

$$= -\int_0^1 2\cdot\left(-\frac{1}{2}t^2\right)h\,dt + \int_0^1 (\dot{h})^2 dt - \int_0^1 h t^2 dt = \cancel{\int_0^1 t^2 h\,dt} + \int_0^1 (\dot{h})^2 dt - \cancel{\int_0^1 t^2 h\,dt} = \int_0^1 (\dot{h})^2 dt \geq 0.$$

$\Rightarrow \hat{x} = -\frac{1}{24}t^4 + \frac{1}{24}t \in abs\min$

$$J(\hat{x}) = \int_0^1 \left(-\frac{1}{6}t^3 + \frac{1}{24}\right)^2 - t^2\left(-\frac{1}{24}t^4 + \frac{1}{24}t\right)dt = \int_0^1 \left(\frac{1}{36}t^6 - \frac{1}{3\cdot 24}t^3 + \frac{1}{24\cdot 24} + \frac{1}{24}t^6 - \frac{1}{24}t^3\right)dt =$$

$$= \int_0^1 \left(\frac{5}{12\cdot 6}t^6 - \frac{1}{18}t^3 + \frac{1}{24\cdot 24}\right)dt = \frac{5}{12\cdot 6}\cdot\frac{1}{7} - \frac{1}{18}\cdot\frac{1}{4}^{|24} + \frac{1}{24\cdot 24}^{|3} = \frac{5}{12\cdot 6\cdot 7} - \frac{21}{3\cdot 6\cdot 4\cdot 6\cdot 4} =$$

$$= \frac{5}{3\cdot(6\cdot 4)\cdot 7}^{|24} - \frac{7}{(6\cdot 4)\cdot(6\cdot 4)}^{|21} = \frac{120-147}{21(6\cdot 4)^2} = \frac{-27}{3\cdot 7\cdot 6\cdot 4\cdot 24} = -\frac{3}{28\cdot 2\cdot 24} = -\frac{3}{56\cdot 24} = \frac{-1}{56\cdot 8} = \frac{-1}{448}$$

$S_{abs\max} = +\infty$:

Берём $x_n = \hat{x} + h$, где $h = n\cdot t(t-1) \Rightarrow \dot{h} = n\cdot(2t-1)$

$$\Rightarrow J(\hat{x}_n + h) = J(\hat{x}) + \int_0^1 (\dot{h})^2 dt = -\frac{1}{448} + n^2\cdot\int_0^1 (2t-1)^2 dt = -\frac{1}{448} + n^2\cdot\frac{11}{6} \to +\infty \text{ при } n \to +\infty.$$

Ответ: $\hat{x} = -\frac{1}{24}t^4 + \frac{1}{24}t \in abs\min$

$S_{abs\min} = -\frac{1}{448}$

$S_{abs\max} = +\infty$

(2.3) $\int_0^1 \dot{x}^2 dt \to extr; \int_0^1 tx\,dt = 1;\ x(0) = x(1) = 0$ — изопериметр. задача.

Решение: $L = \lambda_0 \cdot \dot{x}^2 + \lambda_1 \cdot tx$

$L_{\dot{x}} = 2\lambda_0 \dot{x}$

$L_x = \lambda_1 t \Rightarrow -\frac{d}{dt}(2\lambda_0 \dot{x}) + \lambda_1 t = 0.$

$\Rightarrow \frac{d}{dt}(2\lambda_0 \dot{x}) = \lambda_1 t$

$\Rightarrow 2\lambda_0 \dot{x} = \frac{\lambda_1}{2} \cdot t^2 + C_1$

$\Rightarrow 2\lambda_0 x = \frac{\lambda_1}{6} t^3 + C_1 t + C_2.$

Если $\lambda_0 = 0 \Rightarrow \lambda_1 t = 0 \Rightarrow \lambda_1 = 0$ — так нельзя.

Пусть $\lambda_0 = \frac{1}{2}$ - т.е задача на min

$\Rightarrow x = \frac{\lambda_1}{6} t^3 + C_1 t + C_2.$

$x(0) = 0 \Rightarrow C_2 = 0$

$x(1) = 0 \Rightarrow \frac{\lambda_1}{6} + C_1 = 0 \Rightarrow \lambda_1 = -6C_1$

$\int_0^1 tx\,dt = 1 \Rightarrow \int_0^1 t(-C_1 t^3 + C_1 t)\,dt = -\frac{C_1}{5} + \frac{C_1}{3} = C_1 \cdot \frac{2}{15} = 1 \Rightarrow C_1 = \frac{15}{2}.$

$\Rightarrow \hat{x} = -C_1 t^3 + C_1 t = -\frac{15}{2} t^3 + \frac{15}{2} t$ — кандидат на экстремум.

Проверим, доставляет ли он экстремум.

Берем $x = \hat{x} + h$, где h такое, чтобы $\hat{x} + h$ было допустимым, т.е $\int_0^1 th\,dt = 0;\ h(0) = h(1) = 0.$

$\Rightarrow J(\hat{x} + h) - J(\hat{x}) = \int_0^1 (\dot{\hat{x}} + \dot{h})^2 dt - \int_0^1 (\dot{\hat{x}})^2 dt = \int_0^1 2\dot{\hat{x}}\dot{h}\,dt + \int_0^1 (\dot{h})^2 dt =$

$= \underbrace{2\dot{\hat{x}}h\Big|_0^1}_{=0} - \int_0^1 2\ddot{\hat{x}} h\,dt + \int_0^1 (\dot{h})^2 dt = -\underbrace{\int_0^1 2(-30t)h\,dt}_{=0,\ \text{т.к} \int_0^1 th\,dt = 0} + \int_0^1 (\dot{h})^2 dt = \int_0^1 (\dot{h})^2 dt \geq 0.$

$\dot{\hat{x}} = -\frac{15}{2} \cdot 3t^2 + \frac{15}{2}$

$\ddot{\hat{x}} = -30t$

$\Rightarrow \hat{x} \in absmin$

$S_{absmin} = J(\hat{x}) = \int_0^1 \left(\frac{15}{2}\right)^2 \cdot (-t^3 + t)^2 dt = \frac{225}{4} \cdot \left(\frac{1}{7} - \frac{2}{5} + \frac{1}{3}\right) = \frac{225}{4} \cdot \frac{8}{7 \cdot 5 \cdot 3} = \frac{30}{7}.$

$S_{absmax} = +\infty$:

Берем $x_n = \hat{x} + h$, где $h = n t(t-1)(t + C_0)$, где C_0 таково, чтобы $\int_0^1 th\,dt = 0$, $\int_0^1 t^2(t-1)(t + C_0)\,dt = \frac{1}{60}(-5C_0 - 3) = 0.$ $\Rightarrow C_0 = -\frac{3}{5}.$

$\Rightarrow J(x_n) = J(\hat{x} + h) = J(\hat{x}) + n^2 \cdot \underbrace{\int_0^1 t^2 (t-1)^2 \cdot \left(t - \frac{3}{5}\right)^2 dt}_{= \frac{4}{2625}} \to +\infty$ при $n \to +\infty$.

Ответ: $\hat{x} = -\frac{15}{2} t^3 + \frac{15}{2} t \in absmin$

$S_{absmin} = \frac{30}{7}$

$S_{absmax} = +\infty$

(3.3) $\int_0^{\frac{7\pi}{4}} x\sin t\,dt \to extr;\ x(0)=0;\ |\dot x|\le 1$ – задача оптим. управления.

Решение:

Замена: $\dot x = u$.

$\Rightarrow \int_0^{\frac{7\pi}{4}} x\sin t\,dt \to extr;\ x(0)=0;\ u\in[-1;1]$.

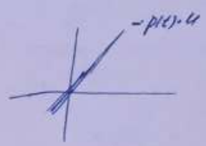
$$\mathcal{L} = \int_0^{\frac{7\pi}{4}} \underbrace{(\lambda_0 x\sin t + p(\dot x - u))}_{=L}\,dt + \underbrace{\lambda_1\cdot x(0)}_{=l}$$

а) $L_{\dot x} = p$

$L_x = \lambda_0 \sin t \Rightarrow -\dot p + \lambda_0 \sin t = 0.$

б) $\begin{cases} p(0) = \lambda_1 \\ p(\frac{7\pi}{4}) = 0 \end{cases}$

в) оптим. по u: $\min\limits_{u\in[-1;1]}\{-p(t)\cdot u\} = -p(t)\cdot \hat u(t)$



$\Rightarrow \hat u(t) = \begin{cases} \text{sign } p(t), \text{ если } p(t)\ne 0 \\ \text{любое из } [-1;1], \text{ если } p(t)=0. \end{cases}$

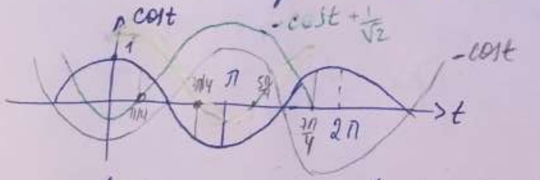
г) неотр: $\lambda_0 \ge 0$ в задаче на min

$\lambda_0 \le 0$ в задаче на max.

• Если $\lambda_0 = 0$, то $\dot p = 0. \Rightarrow p = const \overset{\text{из б)}}{\Rightarrow} p\equiv 0 \Rightarrow (\lambda_0; p) = 0$ – так нельзя.

1) Если $\lambda_0 = 1$, т.е задача на min.

$\Rightarrow \dot p = \sin t \Rightarrow p(t) = -\cos t + C \overset{p(\frac{7\pi}{4})=0}{\Longrightarrow} C = \cos\frac{7\pi}{4} = \frac{1}{\sqrt2} \Rightarrow p(t) = -\cos t + \cos\frac{7\pi}{4} = -\cos t + \frac{1}{\sqrt2} \not\equiv 0.$



$\Rightarrow \underset{=\dot{\hat x}}{\hat u(t)} = \text{sign } p(t) = \begin{cases} -1, & 0\le t\le \frac{\pi}{4} \\ 1, & \frac{\pi}{4}\le t\le \frac{7\pi}{4} \end{cases}$

$\Rightarrow x = \begin{cases} -t + C_1; & 0\le t\le\frac{\pi}{4} \\ t + C_2; & \frac{\pi}{4}\le t\le \frac{7\pi}{4} \end{cases}$

$x(0) = 0 \Rightarrow C_1 = 0;$

из условия склейки при $t=\frac{\pi}{4}$: $-\frac{\pi}{4} = \frac{\pi}{4} + C_2 \Rightarrow C_2 = -\frac{\pi}{2}.$

$\Rightarrow \hat x = \begin{cases} -t; & 0\le t\le \frac{\pi}{4} \\ t-\frac{\pi}{2}; & \frac{\pi}{4} < t < \frac{7\pi}{4} \end{cases}$ – кандидат на absmin

Докажем, что $\hat x$ доставляет absmin.

Берём любую допустимую ф-цию x. $h := x - \hat x$.

$\begin{cases} h\in PC[0;\frac{7\pi}{4}];\ |\dot{\hat x} + \dot h|\le 1 \\ h(0)=0 \end{cases} \Rightarrow \begin{cases} |-1+\dot h|\le 1; & t\in[0;\frac{\pi}{4}] \\ |1+\dot h|\le 1, & t\in[\frac{\pi}{4};\frac{7\pi}{4}] \end{cases} \Leftrightarrow \begin{cases} 0\le \dot h\le 2; & t\in[0;\frac{\pi}{4}] \\ -2\le \dot h\le 0; & t\in[\frac{\pi}{4};\frac{7\pi}{4}] \end{cases}$

$$\Rightarrow J(x)-J(\hat x) = J(\hat x+h)-J(\hat x) = \int_0^{\frac{7\pi}{4}}(\hat x+h)\sin t\,dt - \int_0^{\frac{7\pi}{4}}\hat x\sin t\,dt = \int_0^{\frac{7\pi}{4}} h\sin t\,dt =$$

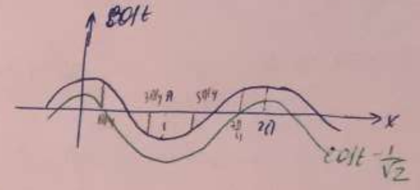
$$= \int_0^{\pi/4} h\sin t\,dt + \int_{\pi/4}^{\frac{7\pi}{4}} h\sin t\,dt \underset{\dot p=\sin t}{=} \int_0^{\pi/4} h\cdot\dot p\,dt + \int_{\pi/4}^{7\pi/4} h\dot p\,dt = \underbrace{h\cdot p\Big|_0^{\pi/4}}_{=0,\ \text{т.к. } h(0)=0,\ p(\pi/4)=0} - \int_0^{\pi/4}\dot h p\,dt + \underbrace{hp\Big|_{\pi/4}^{7\pi/4}}_{=0,\ \text{т.к. } p(\frac{\pi}{4})=0,\ p(\frac{7\pi}{4})=0} - \int_{\pi/4}^{7\pi/4}\dot h p\,dt =$$

$$= -\underbrace{\underbrace{\int_0^{\pi/4}\dot h p\,dt}_{\text{при } 0\le\dot h\le 2,\ p\le 0}}_{\ge 0} - \underbrace{\underbrace{\int_{\pi/4}^{\frac{7\pi}{4}}\dot h p\,dt}_{\text{при } \dot h\le 0,\ p\ge 0}}_{\ge 0} \ge 0 \Rightarrow \hat x \in abs\min$$

$$S_{abs\min} = \int_0^{\pi/4} -t\sin t\,dt + \int_{\pi/4}^{7\pi/4} t\sin t\,dt - \frac{\pi}{2}\cdot\int_{\pi/4}^{7\pi/4}\sin t\,dt = -(\sin t - t\cos t)\Big|_0^{\pi/4} + (\sin t - t\cos t)\Big|_{\pi/4}^{7\pi/4} + \underbrace{\frac{\pi}{2}\cdot\cos t\Big|_{\pi/4}^{\frac{7\pi}{4}}}_{=0} =$$

$$= -\frac{1}{\sqrt2} + \frac{\pi}{4}\cdot\frac{1}{\sqrt2} - \frac{1}{\sqrt2} - \frac{7\pi}{4}\cdot\frac{1}{\sqrt2} - \frac{1}{\sqrt2} + \frac{\pi}{4}\cdot\frac{1}{\sqrt2} = -\frac{3}{\sqrt2} - \frac{5\pi}{4\sqrt2}$$

2) Если $\lambda_0 = -1$, т.е задача на max



$$\Rightarrow \dot p = -\sin t \Rightarrow p(t) = \cos t + C \overset{p(\frac{\pi}{4})=0}{\Rightarrow} C = -\frac{1}{\sqrt2} \Rightarrow p(t) = \cos t - \frac{1}{\sqrt2}.$$

$$\Rightarrow \hat u(t) = \operatorname{sign} p(t) = \begin{cases} 1, & t\in[0;\frac{\pi}{4}] \\ -1, & t\in[\frac{\pi}{4};\frac{7\pi}{4}] \end{cases}$$

$$\Rightarrow \hat x = \begin{cases} t+C_1; & t\in[0;\frac{\pi}{4}] \\ -t+C_2; & t\in[\frac{\pi}{4};\frac{7\pi}{4}] \end{cases} \underset{\substack{x(0)=0 \\ \text{склейка при } t=\frac{\pi}{4}}}{\Longrightarrow} \begin{cases} t; & t\in[0;\frac{\pi}{4}] \\ -t+\frac{\pi}{2}; & t\in[\frac{\pi}{4};\frac{7\pi}{4}] \end{cases}$$ – кандидат на max

Докажем, что $\hat x \in abs\max$.

Берем h: $h(0)=0$; $|\dot{\hat x}+\dot h|\le 1$, т.е $\begin{cases} |1+\dot h|\le 1; & t\in[0;\frac{\pi}{4}] \\ |-1+\dot h|\le 1; & t\in[\frac{\pi}{4};\frac{7\pi}{4}] \end{cases} \Leftrightarrow \begin{cases} -2\le\dot h\le 0; & t\in[0;\frac{\pi}{4}] \\ 0\le\dot h\le 2; & t\in[\frac{\pi}{4};\frac{7\pi}{4}] \end{cases}$

$$\Rightarrow J(\hat x+h)-J(\hat x) = \int_0^{\frac{7\pi}{4}}(\hat x+h)\sin t\,dt - \int_0^{\frac{7\pi}{4}}\hat x\sin t\,dt = \int_0^{7\pi/4} h\sin t\,dt =$$

$$= \int_0^{\pi/4} h\sin t\,dt + \int_{\pi/4}^{\frac{7\pi}{4}} h\sin t\,dt \underset{\dot p=-\sin t}{=} -\int_0^{\pi/4} h\dot p\,dt - \int_{\pi/4}^{7\pi/4} h\dot p\,dt = \cancel{-hp\Big|_0^{\pi/4}} + \int_0^{\pi/4}\dot h p\,dt \cancel{-hp\Big|_{\pi/4}^{7\pi/4}} + \int_{\pi/4}^{7\pi/4}\dot h p\,dt =$$

$$= \underbrace{\underbrace{\int_0^{\pi/4}\dot h p\,dt}_{\text{при } \dot h\le 0,\ p\ge 0}}_{\le 0} + \underbrace{\underbrace{\int_{\pi/4}^{7\pi/4}\dot h p\,dt}_{\text{при } \dot h\ge 0,\ p\le 0}}_{\le 0} \le 0 \Rightarrow \hat x \in abs\max$$

$$S_{abs\max} = \int_0^{\pi/4} t\sin t\,dt + \int_{\pi/4}^{7\pi/4}(-t\sin t)\,dt + \frac{\pi}{2}\cdot\int_{\pi/4}^{7\pi/4}\sin t\,dt = \frac{3}{\sqrt2} + \frac{5\pi}{4\sqrt2}.$$

Ответ: $\hat x = \begin{cases} -t; & t\in[0;\frac{\pi}{4}] \\ t-\frac{\pi}{2}; & t\in[\frac{\pi}{4};\frac{7\pi}{4}] \end{cases} \in abs\min$

$S_{abs\min} = -\frac{3}{\sqrt2} - \frac{5\pi}{4\sqrt2}$

$\hat x = \begin{cases} t; & t\in[0;\frac{\pi}{4}] \\ -t+\frac{\pi}{2}; & t\in[\frac{\pi}{4};\frac{7\pi}{4}] \end{cases} \in abs\max$

$S_{abs\max} = \frac{3}{\sqrt2} + \frac{5\pi}{4\sqrt2}$