22.09.21. ОМСС. ДЗ от семинара 2.

$$v_{ip}^{\varphi} = v_{ip} \cdot \sqrt{g^{ii} \cdot g^{pp}}$$

(A) Написать v_{ij} через v_i для цилиндрич. с.к. (и потом для $v_{ij}^{\text{физ}}$: $a_{ip} = a_{ip}^{\varphi} \sqrt{g_{ii}} \cdot \sqrt{g_{pp}}$)

Решение:

$$\begin{cases} x_1 = r\cos\varphi \\ x_2 = r\sin\varphi \\ x_3 = z \end{cases}$$

$\bar{\partial}_1 = r_r = (\cos\varphi; \sin\varphi; 0)$

$\bar{\partial}_2 = r_\varphi = (-r\sin\varphi; r\cos\varphi; 0)$

$\bar{\partial}_3 = r_z = (0, 0, 1)$

$$\Rightarrow g_{ij} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & r^2 & 0 \\ 0 & 0 & 1 \end{pmatrix} \Rightarrow g^{ij} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{r^2} & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$\Gamma_{ij}^{k} = \frac{1}{2} g^{ks} \left(\frac{\partial g_{is}}{\partial u^j} + \frac{\partial g_{js}}{\partial u^i} - \frac{\partial g_{ij}}{\partial u^s} \right)$$

$$\boxed{\Gamma_{ij}^{1} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & -r & 0 \\ 0 & 0 & 0 \end{pmatrix}}$$

$$\boxed{\Gamma_{ij}^{2} = \begin{pmatrix} 0 & \frac{1}{r} & 0 \\ \frac{1}{r} & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}} \quad \Gamma_{ij}^{3} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$\Gamma_{11}^{1} = \frac{1}{2} g^{11} \left(\frac{\partial g_{11}}{\partial u^1} \right) = \frac{1}{2} \cdot 1 \cdot 0 = 0$

$\Gamma_{12}^{1} = \frac{1}{2} g^{11} \left(\frac{\partial g_{11}}{\partial u^2} + \frac{\partial g_{21}}{\partial u_1} - \frac{\partial g_{12}}{\partial u^1} \right) = 0.$

$\Gamma_{21}^{1} = \frac{1}{2} g^{11} \left(\frac{\partial g_{21}}{\partial u^1} + \frac{\partial g_{11}}{\partial u^2} - \frac{\partial g_{12}}{\partial u^1} \right) = 0.$

$\Gamma_{22}^{1} = \frac{1}{2} g^{11} \left(\cancel{\frac{\partial g_{12}}{\partial u^1}} + \frac{\partial g_{12}}{\partial u^1} - \frac{\partial g_{22}}{\partial u^1} \right) = \frac{1}{2} \cdot (-2r) = -r.$

$\Gamma_{13}^{1} = \frac{1}{2} g^{11} (\dots) = 0. = \Gamma_{31}^{1}$

$\Gamma_{23}^{1} = 0 = \Gamma_{32}^{1}$

$\Gamma_{33}^{1} = 0.$

$\Gamma_{11}^{2} = \frac{1}{2} g^{22} \left(\frac{\partial g_{12}}{\partial u_1} + \frac{\partial g_{12}}{\partial u_1} - \frac{\partial g_{11}}{\partial u^2} \right) = 0.$

$\Gamma_{12}^{2} = \frac{1}{2} g^{22} \left(\frac{\partial g_{12}}{\partial u^2} + \frac{\partial g_{22}}{\partial u^1} - \frac{\partial g_{12}}{\partial u_2} \right) = \frac{1}{2} \cdot \frac{1}{r^2} \cdot 2r = \frac{1}{r} = \Gamma_{21}^{2}$

$\Gamma_{22}^{2} = \frac{1}{2} g^{22} \left(\frac{\partial g_{22}}{\partial u_2} \right) = 0.$

$\Gamma_{13}^{2} = \Gamma_{31}^{2} = 0.$

$\Gamma_{23}^{2} = \Gamma_{32}^{2} = 0$

$\Gamma_{33}^{2} = 0.$

$\Gamma_{ij}^{3} = 0.$

$$v_{ij} = \frac{1}{2} (\nabla_i v_j + \nabla_j v_i); \quad v_{ij}^{\varphi} = v_{ij} \sqrt{g^{ii} g^{pp}}$$

$$\nabla_i v_j = \frac{\partial v_j}{\partial u_i} - \Gamma_{ij}^{k} v_k$$

$\Rightarrow \nabla_1 v_1 = \frac{\partial v_1}{\partial r} - \Gamma_{11}^{k} \cdot v_k = \frac{\partial v_1}{\partial r}$

$\nabla_1 v_2 = \frac{\partial v_2}{\partial r} - \Gamma_{12}^{k} \cdot v_k = \frac{\partial v_2}{\partial r} - \frac{1}{r} v_2$

$\nabla_1 v_3 = \frac{\partial v_3}{\partial r} - \Gamma_{13}^{k} \cdot v_k = \frac{\partial v_3}{\partial r}$

$\nabla_2 v_1 = \frac{\partial v_1}{\partial \varphi} - \frac{1}{r} v_2$

$\nabla_2 v_2 = \frac{\partial v_2}{\partial \varphi} + r \cdot v_1$

$\nabla_2 v_3 = \frac{\partial v_3}{\partial \varphi}$

$\nabla_3 v_1 = \frac{\partial v_1}{\partial z}$

$\nabla_3 v_2 = \frac{\partial v_2}{\partial z}$

$\nabla_3 v_3 = \frac{\partial v_3}{\partial z}$

$\Rightarrow v_{11} = \frac{1}{2}(\nabla_1 v_1 + \nabla_1 v_1) = \frac{\partial v_1}{\partial r}$

$v_{12} = \frac{1}{2}(\nabla_1 v_2 + \nabla_2 v_1) = \frac{1}{2}\left(\frac{\partial v_2}{\partial r} + \frac{\partial v_1}{\partial \varphi}\right) - \frac{1}{r}v_2$

$v_{13} = \frac{1}{2}(\nabla_1 v_3 + \nabla_3 v_1) = \left(\frac{\partial v_3}{\partial r} + \frac{\partial v_1}{\partial z}\right)$

$v_{22} = \nabla_2 v_2 = \frac{\partial v_2}{\partial \varphi} + r v_1$

$v_{23} = \frac{1}{2}(\nabla_2 v_3 + \nabla_3 v_2) = \frac{1}{2}\left(\frac{\partial v_3}{\partial \varphi} + \frac{\partial v_2}{\partial z}\right)$

$v_{33} = \nabla_3 v_3 = \frac{\partial v_3}{\partial z}$

$\Rightarrow v_{ij}^{\varphi} = v_{ij}\sqrt{g^{\alpha\alpha} g^{\beta\beta}}$

$$\boxed{\begin{aligned} v_{11}^{\varphi} &= v_{11} \\ v_{12}^{\varphi} &= v_{21}^{\varphi} = \frac{v_{12}}{r} \\ v_{13}^{\varphi} &= v_{31}^{\varphi} = v_{13} \\ v_{22}^{\varphi} &= \frac{v_{22}}{r^2} \\ v_{23}^{\varphi} &= v_{32}^{\varphi} = \frac{v_{23}}{r} \\ v_{33}^{\varphi} &= v_{33} \end{aligned}}$$

ответ

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(2) $\begin{cases} v_1 = kx_2 \\ v_2 = kx_1 \\ v_3 = 0 \end{cases}$ найти $\bar{\omega}$, v_{ij}, изменение плотности ρ, $\bar{u}(\xi, t)$, $\bar{a}(\xi, t)$, $\bar{a}(\bar{x}, t)$

Решение: 1) $\bar{\omega} = \frac{1}{2}\operatorname{rot}\bar{v} = \frac{1}{2}\begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ ky & kx & 0 \end{vmatrix} = \frac{1}{2}(0; 0; k-k) = (0, 0, 0)$

2) $v_{11} = \frac{\partial v_1}{\partial x_1} = 0$

$v_{12} = v_{21} = \frac{1}{2}\left(\frac{\partial v_1}{\partial x_2} + \frac{\partial v_2}{\partial x_1}\right) = k$

$v_{13} = v_{31} = v_{23} = v_{32} = 0$

$v_{22} = v_{33} = 0$

3) $\frac{\partial \rho}{\partial t} + \operatorname{div}(\rho\bar{v}) = 0$. потому $\operatorname{div}(\rho\bar{v}) = 0$, хотя $\rho = \rho(x_1, x_2, x_3)$?

$\frac{\partial \rho}{\partial t} + k\operatorname{div}(\rho \cdot y; \rho x; 0) = 0$

$\frac{\partial \rho}{\partial t} + \underbrace{k\frac{\partial}{\partial x}(\rho)\cdot y}_{v_1} + \underbrace{k\frac{\partial}{\partial y}(\rho)\cdot x}_{v_2} = 0$

$\Rightarrow \frac{\partial \rho}{\partial t} + \sum_i \frac{\partial \rho}{\partial x_i}\cdot v_i = 0$

$\Rightarrow \frac{d\rho}{dt} = 0 \Rightarrow \rho = const$

4) $\begin{cases} \dot{x}_1 = kx_2 \\ \dot{x}_2 = kx_1 \\ x_3 = \xi_3 \end{cases} \Rightarrow \ddot{x}_1 = k^2 x_1 \Rightarrow \begin{cases} x_1 = \frac{\xi_1+\xi_2}{2}e^{kt} + \frac{\xi_1-\xi_2}{2}e^{-kt} \\ x_2 = \frac{\dot{x}_1}{k} = \frac{\xi_1+\xi_2}{2}\cdot e^{kt} - \frac{\xi_1-\xi_2}{2}\cdot e^{-kt} \\ x_3 = \xi_3 \end{cases}$

$\Rightarrow \bar{u} = \bar{x} - \bar{\xi} = \begin{pmatrix} \xi_1(\operatorname{ch}kt - 1) + \xi_2\operatorname{sh}kt \\ \xi_1\operatorname{sh}kt + \xi_2(\operatorname{ch}kt - 1) \\ 0 \end{pmatrix}$

5) $\bar{a}(\bar{x}, t) = \frac{d\bar{v}}{dt} = (k\cdot v_2; kv_1; 0) = (k^2 x_1; k^2 x_2; 0)$

6) $\bar{a}(\xi, t) = \frac{\partial \bar{u}(\xi, t)}{\partial t} = k(\xi_1\operatorname{sh}kt + \xi_2\operatorname{ch}kt;\ \xi_1\operatorname{ch}kt + \xi_2\operatorname{sh}kt; 0)$

②
$$\begin{cases} u_1 = \xi_1(e^{k_1 t}-1) \\ u_2 = \xi_2(e^{k_2 t}-1) \\ u_3 = \xi_3(e^{k_3 t}-1) \end{cases} \Rightarrow \bar{x} = \bar{u} + \bar{\xi} = (\xi_1 e^{k_1 t}; \xi_2 e^{k_2 t}; \xi_3 e^{k_3 t})$$

найти $\bar{v}(x,t)$, v_{ij}, линии тока и траектории

Решение: 1) $\bar{v}(\bar{x},t) = \dfrac{\partial \bar{u}(\bar{\xi},t)}{\partial t} = (k_1 \cdot \xi_1 e^{k_1 t}; k_2 \cdot \xi_2 e^{k_2 t}; k_3 \cdot \xi_3 e^{k_3 t}) = (k_1 x_1; k_2 x_2; k_3 x_3)$

2) $v_{11} = \dfrac{\partial v_1}{\partial x_1} = k_1$; $v_{21} = v_{12} = v_{13} = v_{31} = 0$

$v_{22} = \dfrac{\partial v_2}{\partial x_2} = k_2$

$v_{33} = \dfrac{\partial v_3}{\partial x_3} = k_3$

3) Траектории:
$$\begin{cases} \dot{x}_1 = k_1 x_1 \\ \dot{x}_2 = k_2 x_2 \\ \dot{x}_3 = k_3 x_3 \end{cases} \Rightarrow \begin{cases} x_1 = \xi_1 e^{k_1 t} \\ x_2 = \xi_2 e^{k_2 t} \\ x_3 = \xi_3 e^{k_3 t} \end{cases}$$

4) линии тока:
$$\frac{dx_1}{k_1 x_1} = \frac{dx_2}{k_2 x_2} = \frac{dx_3}{k_3 x_3} \Rightarrow k_2 \ln x_1 = k_1 \ln x_2 + C_1$$
$$\Rightarrow \begin{cases} x_1^{k_2} = C_1 \cdot x_2^{k_1} \\ x_1^{k_3} = C_3 \cdot x_3^{k_1} \\ x_2^{k_3} = C_2 \cdot x_3^{k_2} \end{cases}$$

③
$$\begin{cases} v_1 = \dfrac{bc}{a}(x_2 - x_3) \\ v_2 = \dfrac{ca}{b}(x_3 - x_1) \\ v_3 = \dfrac{ab}{c}(x_1 - x_2) \end{cases}$$

$a, b, c = const$

Траектории плоские?

Решение:
$$\begin{cases} \dot{x}_1 = \dfrac{bc}{a}(x_2 - x_3) \\ \dot{x}_2 = \dfrac{ca}{b}(x_3 - x_1) \\ \dot{x}_3 = \dfrac{ab}{c}(x_1 - x_2) \end{cases} \Rightarrow a^2 \dot{x}_1 + b^2 \dot{x}_2 + c^2 \dot{x}_3 = 0$$

$\Rightarrow a^2 x_1 + b^2 x_2 + c^2 x_3 = d$ $\Rightarrow$ Траектории плоские!

const $= a^2\xi_1 + b^2\xi_2 + c^2\xi_3$

④
$$\begin{cases} v_1 = cx_2 - bx_3 \\ v_2 = ax_3 - cx_1 \\ v_3 = bx_1 - ax_2 \end{cases}$$

$a, b, c = const$

Док-ть, что движение сферо-посм-е.

Решение:
$$\begin{cases} \dot{x}_1 = cx_2 - bx_3 \\ \dot{x}_2 = ax_3 - cx_1 \\ \dot{x}_3 = bx_1 - ax_2 \end{cases} \Rightarrow x_1 \dot{x}_1 + x_2 \dot{x}_2 + x_3 \dot{x}_3 = 0$$

$$\Rightarrow \left(\frac{x_1^2}{2}\right)^{\cdot} + \left(\frac{x_2^2}{2}\right)^{\cdot} + \left(\frac{x_3^2}{2}\right)^{\cdot} = 0$$

$$\Rightarrow x_1^2 + x_2^2 + x_3^2 = const \text{ — ч.т.д.}$$