Multi-Period Asset Pricing
Part 4

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Arbitrage-free pricing in multi-period binomial model

Why multi-period models?
Backward induction

Why multi-period models?

Single period binomial model: complete, but unrealistic.

Single period trinomial model: more realistic, but incomplete.

Indeed, let d, h, and u be the relative changes in the stock price and r be the interest rate such that

$$d < 1 + r < u, \quad d < h < u.$$

For risk-neutral probabilities $\widetilde{p}_i$, $i = 1, 2, 3$, we get the system:

$$\widetilde{p}_i > 0, \quad i = 1, 2, 3, \tag{a}$$

$$\widetilde{p}_1 + \widetilde{p}_2 + \widetilde{p}_3 = 1, \tag{b}$$

$$\widetilde{p}_1 d + \widetilde{p}_2 h + \widetilde{p}_3 u = 1 + r \quad \left(\widetilde{\mathbb{E}}\,(S_1) = S_0(1 + r)\right) \tag{c}$$

Why multi-period models?

From $(b) + (c)$ we deduce that

$$\widetilde{p}_2 = \frac{u - (1+r) - (u-d)\widetilde{p}_1}{u-h},$$
$$\widetilde{p}_3 = \frac{1 + r - h + (h-d)\widetilde{p}_1}{u-h}.$$

Accounting for (a) we obtain that

$$\max\left(0, \frac{h-(1+r)}{h-d}\right) < \widetilde{p}_1 < \frac{u-(1+r)}{u-d}.$$

Thus, there are ∞-many solutions and

$$\text{FTAPs} \implies \text{NA} + \text{incomplete.}$$

Why multi-period models?

There are 2 ways to get more realistic and still complete model:

1. Add extra traded securities; for instance, liquid options.
2. Trade dynamically, that is, use multi-period models.

Remark (Time scales of financial models)

Single-period: easy to understand and use, but not realistic.

Multi-period: elementary to learn, *very* realistic, but not convenient for computations.

Continuous-time: quite realistic, *very* convenient for computations, but difficult to master (Ito's Calculus!).

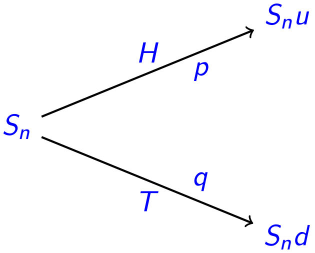
Multi-period binomial model

There are $N+1$ times: $0, 1, \cdots, N$.

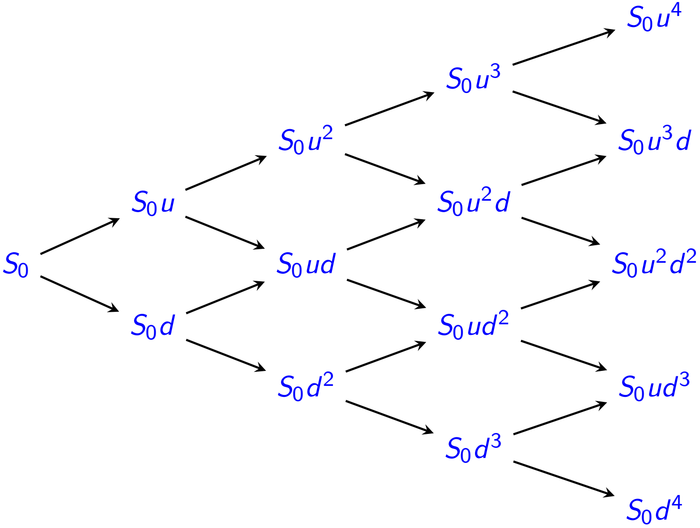
Bank account with interest rate $r > -1$:

$$X_n \text{ at } t = n \quad \longrightarrow \quad X_n(1+r) \text{ at } t = n+1.$$

Stock with parameters u ("up") and d ("down") such that $u > d > 0$:



Multi-period binomial model



Probability space for binomial model

Probability space: $(\Omega, \mathbb{P})$, where elementary event $\omega \in \Omega$ is a *trajectory*:

$$\omega = (\omega_1, \ldots, \omega_N), \quad \omega_i \in \{T, H\}.$$

The number of elementary events:

$$|\Omega| = 2^N. \qquad \text{(Grows very fast!)}$$

As always, we assume that $\mathbb{P}$ is strictly positive:

$$\mathbb{P}(\omega) > 0, \quad \omega \in \Omega.$$

Of course, we also have that

$$\sum_{\omega \in \Omega} \mathbb{P}(\omega) = 1.$$

Adapted sequences in binomial model

Definition

A sequence of random variables $(X_n)_{0 \leq n \leq N}$ is **adapted** if for every $t = n$,

$$X_n = X_n(\omega_1, \dots, \omega_n),$$

that is, X_n depends only of the first n letters $(\omega_1, \dots, \omega_n)$ in $\omega = (\omega_1, \dots, \omega_N)$.

Examples

The following sequences are adapted: (S_n), $(\max_{k \leq n} S_k)$,

(X_n): the capital of a strategy,

(Δ_n): the number of stocks in a strategy.

Arbitrage in binomial model

Lemma

In the multi-period binomial model,

$$NA \quad \Longleftrightarrow \quad d < 1 + r < u.$$

Proof.

For a multi-period model, an arbitrage exists *globally* if and only if it exists *locally* in one of the branches. The result now follows from the single period case. $\square$

European options in binomial model

A European option with maturity N is determined by its payoff

$$V_N = V_N(\omega) = V_N(\omega_1, \dots, \omega_N).$$

Examples

Lookback option: $V_N(\omega) = M_N - S_N(\omega)$, where

$$M_N = \max_{0 \leq n \leq N} S_n(\omega).$$

Asian call option: $V_N(\omega) = \max(A_N - K, 0)$, where

$$A_N = \frac{1}{N+1} \sum_{n=0}^{N} S_n.$$

AFP in binomial model

$$\boxed{\textbf{!AFP} = \textbf{Replication}}$$

Replicating strategy:

$$X_0 \quad \xrightarrow[(\Delta_n)]{} \quad \underbrace{X_N = V_N}_{\text{known}}$$

or, more precisely,

$$X_0 \xrightarrow[\Delta_0]{} X_1 \xrightarrow[\Delta_1]{} X_2 \xrightarrow[\Delta_2]{} \cdots X_{N-1} \xrightarrow[\Delta_{N-1}]{} \underbrace{X_N = V_N}_{\text{(known)}}$$

To compute it, we move **BACKWARD** in time:

$$\underbrace{X_0 \leftarrow \cdots X_n}_{\text{unknown}} \leftarrow \underbrace{X_{n+1} \leftarrow \cdots \quad \overbrace{X_N = V_N}^{\text{boundary condition}}}_{\text{known}}$$

AFP in binomial model

One-step iteration:

$$\underbrace{X_n}_{?} \xrightarrow[\Delta_n - ?]{} \underbrace{X_{n+1}}_{\text{known}}$$

Given the history $(\omega_1, \ldots, \omega_n)$, we obtain that

$$X_{n+1}(\omega_{n+1} = H) = (X_n - \Delta_n S_n)(1 + r) + \Delta_n S_n u,$$
$$X_{n+1}(\omega_{n+1} = T) = (X_n - \Delta_n S_n)(1 + r) + \Delta_n S_n d,$$

which is the same system as in the single period case.

AFP in binomial model

We deduce that

$$\Delta_n = \frac{X_{n+1}(\omega_{n+1} = H) - X_{n+1}(\omega_{n+1} = T)}{S_{n+1}(\omega_{n+1} = H) - S_{n+1}(\omega_{n+1} = T)},$$

$$X_n = \frac{1}{1+r}\left(\widetilde{p}X_{n+1}(\omega_{n+1} = H) + \widetilde{q}X_{n+1}(\omega_{n+1} = T)\right),$$

where as in the single period case,

$$\widetilde{p} = \frac{1+r-d}{u-d}, \quad \widetilde{q} = 1 - \widetilde{p} = \frac{u-(1+r)}{u-d}.$$

AFP in binomial model

These formulas together with *boundary condition*:

$$X_N(\omega) = V_N(\omega), \quad \omega \in \Omega,$$

form the algorithm of **backward induction** for the AFP in the N-period binomial model.

Remark

As the algorithm shows, we can find a replicating strategy for *every* European option. Hence, the N-period binomial model is *complete*.

AFP in binomial model

Remark

The equation for the backward induction can be written in a more compact way:

$$X_n = \mathcal{R}_n \left(X_{n+1} \right),$$

where the operator $\mathcal{R}_n$ acts on $X_{n+1} = X_{n+1}(\omega_1, \ldots, \omega_n, \omega_{n+1})$ as follows:

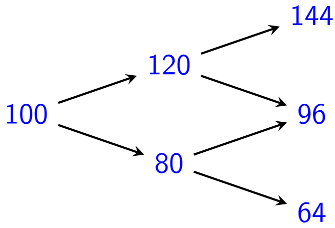
$$\mathcal{R}_n \left(X_{n+1} \right) (\omega_1, \cdots, \omega_n) =$$
$$\frac{1}{1+r} \left(\widetilde{p} X_{n+1}(\omega_1, \cdots, \omega_n, H) + \widetilde{q} X_{n+1}(\omega_1, \cdots, \omega_n, T) \right).$$

The operator $\mathcal{R}_n$ plays a crucial role in software libraries for pricing derivatives. It is called the *rollback operator*.

Pricing of standard call

Problem

The bank pays the interest rate $r = 0.1$. *The stock price follows the two-period binomial model with* $S_0 = 100$, $u = 1.2$, $d = 0.8$:



For the call option with maturity $N = 2$ *and strike* $K = 100$, *compute at* $t = 0$

1. *the arbitrage-free price* V_0,
2. *the number of stocks* Δ_0 *in the replicating strategy.*

Pricing of standard call

Solution

The one-step risk-neutral probabilities are given by

$$\widetilde{p} = \frac{1+r-d}{u-d} = \frac{3}{4}, \quad \widetilde{q} = \frac{u-1-r}{u-d} = \frac{1}{4}.$$

Backward induction:

Time 2: The price of the stock and the value of the option are given in the following table:

ω	stock	call
HH	144	44
TH	96	0
HT	96	0
TT	64	0

Pricing of standard call

Time 1: The value of the call option is given by

$$V_1(H) = \frac{1}{1+r}\left(\widetilde{p}V_2(HH) + \widetilde{q}V_2(HT)\right) = 30,$$
$$V_1(T) = \frac{1}{1+r}\left(\widetilde{p}V_2(TH) + \widetilde{q}V_2(TT)\right) = 0.$$

Time 0: Finally, we obtain that

$$V_0 = \frac{1}{1+r}\left(\widetilde{p}V_1(H) + \widetilde{q}V_1(T)\right) = 20.45,$$
$$\Delta_0 = \frac{V_1(H) - V_1(T)}{S_1(H) - S_1(T)} = \frac{3}{4}.$$

□

Pricing of Asian call

Problem

The bank pays the interest rate $r = 0.1$. *The stock price follows the two-period binomial model with* $S_0 = 100$, $u = 1.2$, $d = 0.8$:

$$\begin{tikzcd}
 & & 144 \\
 & 120 \arrow[ur] \arrow[dr] & \\
100 \arrow[ur] \arrow[dr] & & 96 \\
 & 80 \arrow[ur] \arrow[dr] & \\
 & & 64
\end{tikzcd}$$

Compute the AFP of the Asian call with strike $K = 100$ *and the payoff*

$$V_N = \max\left(\frac{1}{N+1}\sum_{k=0}^{N} S_i - K, 0\right).$$

Pricing of Asian call

Solution

The one-step risk-neutral probabilities are given by

$$\widetilde{p} = \frac{1+r-d}{u-d} = \frac{3}{4}, \quad \widetilde{q} = 1 - \widetilde{p} = \frac{1}{4}.$$

Time 2: The price of the option is given by

$$V_2(HH) = \max\left(\frac{100+120+144}{3} - 100, 0\right) = 21.3333,$$

$$V_2(HT) = \max\left(\frac{100+120+96}{3} - 100, 0\right) = 5.3333,$$

$$V_2(TH) = \max\left(\frac{100+80+96}{3} - 100, 0\right) = 0,$$

$$V_2(TT) = \max\left(\frac{100+80+64}{3} - 100, 0\right) = 0.$$

Pricing of Asian call

Time 1: The value of the Asian call option is given by

$$V_1(H) = \frac{1}{1+r}\left(\widetilde{p}V_2(HH) + \widetilde{q}V_2(HT)\right) = 15.7576,$$
$$V_1(T) = \frac{1}{1+r}\left(\widetilde{p}V_2(TH) + \widetilde{q}V_2(TT)\right) = 0.$$

Time 0: The value of the Asian call option is given by

$$V_0 = \frac{1}{1+r}\left(\widetilde{p}V_1(H) + \widetilde{q}V_1(T)\right) = 10.74. \qquad \square$$

Arrow-Debreu security

Problem

Let $\widehat{\omega} = (\widehat{\omega}_1, \ldots, \widehat{\omega}_N)$ *be an outcome. Compute the AFP of the Arrow-Debreu security paying*

$$V_N(\omega) = 1_{\{\omega = \widehat{\omega}\}} \quad \textit{at} \quad t = N.$$

Solution

Let $V_n = V_n(\omega_1, \ldots, \omega_n)$ be the AFP at $t = n$.
If $(\omega_1, \ldots, \omega_n) \neq (\widehat{\omega}_1, \ldots, \widehat{\omega}_n)$, then

$$V_n = 0.$$

If $(\omega_1, \ldots, \omega_n) = (\widehat{\omega}_1, \ldots, \widehat{\omega}_n)$, then

Arrow-Debreu security

$$
\begin{aligned}
V_n &= \frac{1}{1+r}\left(\widetilde{p}V_{n+1}(\omega_{n+1}=H)+\widetilde{q}V_{n+1}(\omega_{n+1}=T)\right)\\
&= \frac{1}{1+r}\left(\widetilde{p}1_{\{\widehat{\omega}_{n+1}=H\}}+\widetilde{q}1_{\{\widehat{\omega}_{n+1}=T\}}\right)\\
&\qquad \times V_{n+1}(\widehat{\omega}_1,\ldots,\widehat{\omega}_n,\widehat{\omega}_{n+1}).
\end{aligned}
$$

We deduce by induction that

$$
\begin{aligned}
V_0 &= \frac{1}{(1+r)^N}\prod_{n=1}^{N}\left(\widetilde{p}1_{\{\widehat{\omega}_{n+1}=H\}}+\widetilde{q}1_{\{\widehat{\omega}_{n+1}=T\}}\right)\\
&= \frac{1}{(1+r)^N}\,\widetilde{p}^{\#H(\widehat{\omega})}\,\widetilde{q}^{\#T(\widehat{\omega})}.
\end{aligned}
$$

Here $\#H(\omega)$ is the number of letters H and $\#T(\omega)$ is the number of letters T in $\omega=(\omega_1,\ldots,\omega_N)$. $\square$

Risk-neutral valuation

The expression for the AFP of an Arrow-Debreu security yields an "explicit" formula for the AFP of any option. Indeed, as

$$V_N(\omega) = \sum_{\widehat{\omega} \in \Omega} V_N(\widehat{\omega}) 1_{\{\omega = \widehat{\omega}\}},$$

we obtain that

$$\begin{aligned} V_0 &= \frac{1}{(1+r)^N} \sum_{\widehat{\omega} \in \Omega} V_N(\widehat{\omega}) \widetilde{p}^{\# H(\widehat{\omega})} \, \widetilde{q}^{\# T(\widehat{\omega})} \\ &= \frac{1}{(1+r)^N} \sum_{\omega \in \Omega} V_N(\omega) \widetilde{p}^{\# H(\omega)} \, \widetilde{q}^{\# T(\omega)}. \end{aligned}$$

Risk-neutral valuation

We observe now that the function

$$\widetilde{\mathbb{P}}(\omega) \triangleq \prod_{n=1}^{N} \left(\widetilde{p}1_{\{\omega_n = H\}} + \widetilde{q}1_{\{\omega_n = T\}}\right) = \widetilde{p}^{\#H(\omega)}\,\widetilde{q}^{\#T(\omega)}, \quad \omega \in \Omega,$$

is a strictly positive probability measure. Indeed,

$$\widetilde{\mathbb{P}}(\omega) > 0, \quad \omega \in \Omega,$$

$$\sum_{\omega \in \Omega} \widetilde{\mathbb{P}}(\omega) = \sum_{i=1,\dots,N} \sum_{\omega_i \in \{H,T\}} \prod_{n=1}^{N} \left(\widetilde{p}1_{\{\omega_n = H\}} + \widetilde{q}1_{\{\omega_n = T\}}\right) = 1.$$

In terms of $\widetilde{\mathbb{P}}$, the expression for V_0 becomes the RNV:

$$V_0 = \frac{1}{(1+r)^N} \sum_{\omega \in \Omega} V_N(\omega)\widetilde{\mathbb{P}}(\omega) = \frac{1}{(1+r)^N} \widetilde{\mathbb{E}}(V_N).$$