5-Ангем

$a_{11}x^2 + 2a_{12}xy + a_{22}y^2 + 2a_1x + 2a_2y + a_0 = 0$

$F^{II}(x,y)$

$a, b > 0$

1) $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ (эллипс)

2) $\frac{x^2}{a^2} + \frac{y^2}{b^2} = -1$ (мнимый эллипс)

3) $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 0$ (пара мним. пересек. прям.)

4) $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ (гипербола)

5) $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 0$ (пара пересек. прям.)

6) $y^2 = 2px$, $p>0$ (парабола)

7) $y^2 - a^2 = 0$ (пара парал. прям.)

8) $y^2 + a^2 = 0$ (пара мним. парал. прям.)

9) $y^2 = 0$ (пара совп. прям.)

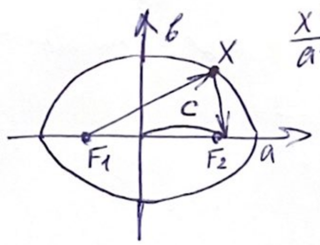
Касательная: $\left.\frac{\partial F}{\partial x}\right|_{(x_0,y_0)}(x - x_0) + \left.\frac{\partial F}{\partial y}\right|_{(x_0,y_0)}(y - y_0) = 0$

Сопряж. диаметр (α, β) неасимпт.: $\alpha(a_{11}x + a_{12}y + a_1) + \beta(a_{12}x + a_{22}y + a_2) = 0$

Ур-ние центра: $\begin{cases} a_{11}x_0 + a_{12}y_0 + a_1 = 0 \\ a_{12}x_0 + a_{22}y_0 + a_2 = 0 \end{cases}$

$\exists!$ центр $\Rightarrow$ (α, β) и (α^*, β^*) сопряжены друг другу, если $\alpha(a_{11}\alpha^* + a_{12}\beta^*) + \beta(a_{12}\alpha^* + a_{22}\beta^*) = 0$

Асимпт. напр. (α, β): $a_{11}\alpha^2 + 2a_{12}\alpha\beta + a_{22}\beta^2 = 0$



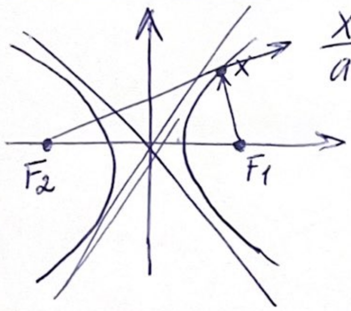
$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

$$F_{1,2}: \left(\pm\sqrt{a^2-b^2}, 0\right)$$

$$|F_1X| + |F_2X| = 2a$$

$$e = \frac{\sqrt{a^2-b^2}}{a} < 1$$

$$d: x = \pm\frac{a}{e} = \pm\frac{a^2}{\sqrt{a^2-b^2}}$$



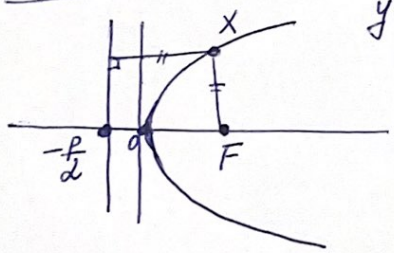
$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$$

$$F_{1,2}: \left(\pm\sqrt{a^2+b^2}, 0\right)$$

$$\big||F_1X| - |F_2X|\big| = 2a$$

$$e = \frac{\sqrt{a^2+b^2}}{a} > 1$$

$$d: x = \pm\frac{a}{e} = \pm\frac{a^2}{\sqrt{a^2+b^2}}$$



$$y^2 = 2px$$

$$F: \left(\frac{p}{2}, 0\right)$$

$$e = 1$$

$$d: x = -\frac{p}{2}$$

$e = \frac{c}{a} = \frac{|F_2X|}{|AX|}$ F_2 – фокус X – т. на кривой

A – т. на дир-се

$\boxed{\text{АНГ-1}}$ Найти все инвариантные прямые аффинного преобр.: $\begin{cases} \tilde{x} = 3x - 2y + 5 \\ \tilde{y} = 2x - y + 5 \end{cases}$

Пусть прообраз имеет вид: $\begin{cases} Ax + By + C = 0 \\ A^2 + B^2 > 0 \end{cases}$

$\Rightarrow$ её образ: $A\tilde{x} + B\tilde{y} + C = 0 \Leftrightarrow$

$\Leftrightarrow A(3x - 2y + 5) + B(2x - y + 5) + C = 0 \Leftrightarrow$

$\Leftrightarrow \underbrace{(3A + 2B)}_{\tilde{A}}x - \underbrace{(2A + B)}_{\tilde{B}}y + \underbrace{5A + 5B + C}_{\tilde{C}} = 0$

Усл. инвариантности:

$\left\{\begin{aligned} &\tilde{A} = kA = 3A + 2B \Rightarrow B = \frac{k-3}{2}A \\ &\tilde{B} = kB = -2A - B \\ &\tilde{C} = kC = 5A + 5B + C \end{aligned}\right.$ $\Rightarrow \frac{(k+1)(k-3)}{2}A = -2A \Rightarrow$

$\Rightarrow k^2 - 2k + 1 = 0 \Rightarrow \left[\begin{aligned} &k = 1 \\ &A = 0 \end{aligned}\right.$

1сл. $A = 0 \Rightarrow B = 0$ (!!)

2сл. $k = 1 \Rightarrow B = -A$

$\Rightarrow$ $Ax - Ay + C = 0$

$\boxed{\text{АНГ-2}}$

Опред. названи, канонич. вид и канонич. СК

$$F(x,y) = 5x^2 - 6xy + 5y^2 - 6\sqrt{2}\,x - 10\sqrt{2}\,y + 10 = 0$$

$$A = \begin{pmatrix} a_{11} & a_{12} \\ a_{12} & a_{22} \end{pmatrix} = \begin{pmatrix} 5 & -3 \\ -3 & 5 \end{pmatrix}$$

$$\det(A - \lambda E) = 0 \iff \begin{vmatrix} 5-\lambda & -3 \\ -3 & 5-\lambda \end{vmatrix} = 0 \iff$$

$$\iff (5-\lambda)^2 - 9 = 0 \iff \left[\begin{aligned} \lambda &= 2 \\ \lambda &= 8 \end{aligned}\right.$$

(1) $\lambda_1 = 2 \Rightarrow \begin{pmatrix} 3 & -3 \\ -3 & 3 \end{pmatrix} \sim \begin{pmatrix} 1 & -1 \\ 0 & 0 \end{pmatrix} \Rightarrow e_1' = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 1 \end{pmatrix}$

(2) $\lambda_2 = 8 \Rightarrow \begin{pmatrix} -3 & -3 \\ -3 & -3 \end{pmatrix} \sim \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix} \Rightarrow e_2' = \frac{1}{\sqrt{2}}\begin{pmatrix} -1 \\ 1 \end{pmatrix}$

Имеем замену:

$$x'e_1' + y'e_2' = x'\left(\frac{1}{\sqrt{2}}e_1 + \frac{1}{\sqrt{2}}e_2\right) + y'\left(-\frac{1}{\sqrt{2}}e_1 + \frac{1}{\sqrt{2}}e_2\right) =$$

$$= xe_1 + ye_2$$

Приравниваем коэф.:

$$\begin{cases} x = \frac{x'}{\sqrt{2}} - \frac{y'}{\sqrt{2}} \\ y = \frac{x'}{\sqrt{2}} + \frac{y'}{\sqrt{2}} \end{cases} \iff \begin{cases} x' = \frac{x}{\sqrt{2}} + \frac{y}{\sqrt{2}} \\ y' = -\frac{x}{\sqrt{2}} + \frac{y}{\sqrt{2}} \end{cases}$$

$$5x^2 - 6xy + 5y^2 - 6\sqrt{2}\,x - 10\sqrt{2}\,y + 10 =$$

$$= \frac{5}{2}(x'-y')^2 - 3(x'^2 - y'^2) + \frac{5}{2}(x'+y')^2 -$$

$$- 6\sqrt{2}\left(\frac{x'}{\sqrt{2}} - \frac{y'}{\sqrt{2}}\right) - 10\sqrt{2}\left(\frac{x'}{\sqrt{2}} + \frac{y'}{\sqrt{2}}\right) + 10 =$$

$$= 2x'^2 + 8y'^2 - 6x' + 6y' - 10x' - 10y' + 10 =$$

$$= 2x'^2 + 8y'^2 - 16x' - 4y' + 10 = 0 \Rightarrow$$

$\boxed{\text{АНГ-2}}$ 2) 32 $64+\frac{1}{2}-x=10$

$$2x'^2-16x'+\cancel{128}+8y'^2-4y'+\cancel{}\ \frac{1}{2}-\frac{45}{62}=0$$

$$2(x'^2-8x'+16)+2(4y'^2-2y'+\frac{1}{4})=\frac{45}{2}$$

$$\begin{cases} x''=x'-4 \\ y''=y'-\frac{1}{4} \end{cases} \Rightarrow 2(x'')^2+8(y'')^2=\frac{45}{2}$$

эллипс

$$\text{СК: } \begin{cases} x''=x'-4=\frac{x}{\sqrt{2}}+\frac{y}{\sqrt{2}}-4 \\ y''=y'-\frac{1}{4}=-\frac{x}{\sqrt{2}}+\frac{y}{\sqrt{2}}-\frac{1}{4} \end{cases}$$

Найти канон. вид: $F(x,y)=\underbrace{4x^2+24xy+11y^2}+64x+42y+51=0$

$$A=\begin{pmatrix} a_{11} & a_{12} \\ a_{12} & a_{22} \end{pmatrix}=\begin{pmatrix} 4 & 12 \\ 12 & 11 \end{pmatrix}$$

$$\det(A-\lambda E)=0 \Leftrightarrow \begin{vmatrix} 4-\lambda & 12 \\ 12 & 11-\lambda \end{vmatrix}=0 \Leftrightarrow 44+\lambda^2-15\lambda-144=$$
$$=\lambda^2-15\lambda-100=0$$
$$\lambda^2+5\lambda-20\lambda-100=0$$
$$\lambda(\lambda+5)-20(\lambda+5)=0$$
$$(\lambda-20)(\lambda+5)=0$$

$$\left[\begin{array}{l} \lambda=20 \\ \lambda=-5 \end{array}\right.$$

(1) $\lambda=20$: $\begin{pmatrix} -16 & 12 \\ 12 & -9 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}=0$ $\begin{pmatrix} x \\ y \end{pmatrix}=\begin{pmatrix} 3 \\ 4 \end{pmatrix} \Rightarrow e_1'=\frac{1}{5}\begin{pmatrix} 3 \\ 4 \end{pmatrix}$

(2) $\lambda=-5$: $\begin{pmatrix} 9 & 12 \\ 12 & 16 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}=0$ $\begin{pmatrix} x \\ y \end{pmatrix}=\begin{pmatrix} 4 \\ -3 \end{pmatrix} \Rightarrow e_2'=\frac{1}{5}\begin{pmatrix} 4 \\ -3 \end{pmatrix}$

$$x'e_1'+y'e_2'=xe_1+ye_2$$
$$\| $$
$$x'(\frac{3}{5}e_1+\frac{4}{5}e_2)+y'(\frac{4}{5}e_1-\frac{3}{5}e_2)$$

Приравн. коэф.:

$$\begin{cases} x=\frac{3}{5}x'+\frac{4}{5}y' \\ y=\frac{4}{5}x'-\frac{3}{5}y' \end{cases} \Rightarrow \begin{cases} x'= \\ y'= \end{cases}$$

$\boxed{\text{АНГ-4}}$ $F(2,1)$ $d: x-2y+9=0$

$A=(5;-3)\in$ кривой

1) $\left.\begin{array}{l} \dfrac{\rho(A,F)}{\rho(A,d)}=e \\ \rho(A,F)=\sqrt{(2-5)^2+(1+3)^2}=\sqrt{9+16}=5 \\ \rho(A,d)=\dfrac{|5+6+9|}{\sqrt5}=\dfrac{20}{\sqrt5} \end{array}\right\} \Rightarrow$

$\Rightarrow e=\dfrac{5\cdot\sqrt5}{20}=\dfrac{\sqrt5}{4}<1 \Rightarrow$ эллипс

2) для произв. т. $B=(x,y)\in$ кривой: $\dfrac{\rho(B,F)}{\rho(B,d)}=\dfrac{\sqrt5}{4}$

$$\frac{\sqrt{(x-2)^2+(y-1)^2}}{|x-2y+9|}\cdot\sqrt5=\frac{\sqrt5}{4}$$

$$16(x^2-4x+4+y^2-2y+1)=x^2+4y^2+81-4xy+18x-36y$$

$$16x^2+16y^2-64x-32y+80=x^2+4y^2+81-4xy+18x-36y$$

$$15x^2+4xy-82x+12y^2+4y=1$$

$\dfrac{60}{15}+\dfrac{82}{15}=\dfrac{142}{15}$

$\dfrac{12\cdot15}{15}-\dfrac{4}{15}=\dfrac{4(3\cdot15-1)}{15}=\dfrac{4\cdot44}{15}$

$$\left(\sqrt{15}x+\frac{2}{\sqrt{15}}y-\frac{41}{\sqrt{15}}\right)^2$$

$$15x^2+\frac{4}{15}y^2+\frac{(41)^2}{15}+4xy-82x-\frac{82}{15}y+\frac{4\cdot44}{15}y^2+\frac{142}{15}y+\frac{(41)^2}{15}$$

АНГ-5 Гиперболический параболоид:

$$\frac{x^2}{p} - \frac{y^2}{q} = 2z$$

Найти угол между образующими в т. $(2, 1, 0)$

$$\frac{x^2}{4} - \frac{y^2}{1} = 2z$$

$$\left(\frac{x}{\sqrt{p}} - \frac{y}{\sqrt{q}}\right)\left(\frac{x}{\sqrt{p}} + \frac{y}{\sqrt{q}}\right) = 2z$$

2 сем-ва образующих:

$$\begin{cases} \frac{x}{\sqrt{p}} - \frac{y}{\sqrt{q}} = k \\ k\left(\frac{x}{\sqrt{p}} + \frac{y}{\sqrt{q}}\right) = 2z \end{cases} \qquad \begin{cases} \frac{x}{\sqrt{p}} + \frac{y}{\sqrt{p}} = k \\ k\left(\frac{x}{\sqrt{p}} - \frac{y}{\sqrt{q}}\right) = 2z \end{cases}$$

$$\begin{cases} \frac{x}{2} - y = k \\ k\left(\frac{x}{2} + y\right) = 2z \end{cases} \qquad \begin{cases} \frac{x}{2} + y = k \\ k\left(\frac{x}{2} - y\right) = 2z \end{cases}$$

Подставим т. $(2, 1, 0)$:

$1 - 1 = k \Rightarrow k = 0$

$$\begin{cases} z = 0 \\ x = 2y \end{cases}$$

$1 + 1 = k$

$$\begin{cases} \frac{x}{2} - y = z \\ \frac{x}{2} + y = 2 \end{cases} \qquad \begin{cases} x - 2y - 2z = 0 \\ x + 2y - 4 = 0 \end{cases}$$

Имеем напр. вектор:

$$\begin{vmatrix} i & j & k \\ 1 & -2 & 0 \\ 0 & 0 & 1 \end{vmatrix} = -2i - j \qquad \{-2, -1, 0\} = \alpha$$

$$\begin{vmatrix} i & j & k \\ 1 & -2 & -2 \\ 1 & 2 & 0 \end{vmatrix} = 4i - 2j + 4k \qquad \{4, -2, 4\} = \beta$$

$$\cos\angle(\alpha, \beta) = \frac{|(\alpha, \beta)|}{|\alpha| \cdot |\beta|} = \frac{|-8+2|}{\sqrt{5} \cdot \sqrt{36}} = \frac{1}{\sqrt{5}}$$

Отв.: $\arccos \frac{1}{\sqrt{5}}$

$\boxed{\text{АНГ-5}}$ $C(1,0)$ – центр

$A(8,0)$, $B(-2,4)$ – концы сопр. диам.

Ур-ние кривой $2^{\text{го}}$ пор-ка:

$$a_{11}x^2 + 2a_{12}xy + a_{22}y^2 + 2a_1x + 2a_2y + a_0 = 0$$

1) Ур-ние центра:

$$\begin{cases} a_{11}x_0 + a_{12}y_0 + a_1 = 0 \\ a_{12}x_0 + a_{22}y_0 + a_2 = 0 \end{cases} \Rightarrow \begin{cases} a_{11} + a_1 = 0 \\ a_{12} + a_2 = 0 \end{cases} \quad \begin{matrix} a_1 = -a_{11} \\ a_2 = -a_{12} \end{matrix}$$

$(x_0, y_0) = (1, 0)$

2) т. $A, B \in$ кривой:

$$\begin{cases} 64a_{11} + 16a_1 + a_0 = 0 \Leftrightarrow 48a_1 + a_0 = 0 \\ 4a_{11} - 16a_{12} + 16a_{22} - 4a_1 + 8a_2 + a_0 = 0 \Leftrightarrow \end{cases}$$

$$\Leftrightarrow 4a_{11} - 16a_{12} + 16a_{22} + 4a_{11} - 8a_{12} + a_0 = 0$$

$$8a_{11} - 24a_{12} + 16a_{22} + a_0 = 0$$

3) Напр. сопряж. диам.:

$\overline{AC} = (-7, 0) = (\alpha, \beta)$ $\quad \overline{BC} = (3, -4) = (\alpha^*, \beta^*)$

$\exists!$ центр. $\Rightarrow$ (α, β) и (α^*, β^*) – сопряж. $\Leftrightarrow$

$$\Leftrightarrow \alpha(a_{11}\alpha^* + a_{12}\beta^*) + \beta(a_{12}\alpha^* + a_{22}\beta^*) = 0$$

$$-7(3a_{11} - 4a_{12}) = 0 \Rightarrow a_{11} = \frac{4}{3}a_{12}$$

$a_1 = -a_{11}$

$a_2 = -a_{12} = -\frac{3}{4}a_{11}$

$a_{12} = \frac{3}{4}a_{11}$

$a_0 = -48a_1 = 48a_{11}$

$a_{22} = -\frac{1}{16}(8a_{11} - 18a_{11} + 48a_{11}) = -\frac{1}{16} \cdot 38a_{11} = -\frac{19}{8}a_{11}$

$a_{11} = 8 \Rightarrow \ldots$

$\boxed{\text{АНГ-6}}$ Однополостный гиперболоид

$$\frac{x^2}{p}+\frac{y^2}{q}-\frac{z^2}{s^2}=1$$

$$\frac{x^2}{p}-\frac{z^2}{s}=1-\frac{y^2}{q}$$

$$\left(\frac{x}{\sqrt{p}}-\frac{z}{\sqrt{s}}\right)\left(\frac{x}{\sqrt{p}}+\frac{z}{\sqrt{s}}\right)=\left(1-\frac{y}{\sqrt{q}}\right)\left(1+\frac{y}{\sqrt{q}}\right)$$

2 сем-ва прямолинейных образующих:

$$\begin{cases}\lambda\left(\frac{x}{\sqrt{p}}+\frac{z}{\sqrt{s}}\right)=\mu\left(1+\frac{y}{\sqrt{q}}\right)\\ \mu\left(\frac{x}{\sqrt{p}}-\frac{z}{\sqrt{s}}\right)=\lambda\left(1-\frac{y}{\sqrt{q}}\right)\end{cases}$$

$$\begin{cases}\lambda\left(\frac{x}{\sqrt{p}}+\frac{z}{\sqrt{s}}\right)=\mu\left(1-\frac{y}{\sqrt{q}}\right)\\ \mu\left(\frac{x}{\sqrt{p}}-\frac{z}{\sqrt{s}}\right)=\lambda\left(1+\frac{y}{\sqrt{q}}\right)\end{cases}$$

$$x^2+y^2-z^2=1$$

$$\begin{cases}\lambda(x+z)=\mu(1+y)\\ \mu(x-z)=\lambda(1-y)\end{cases} \qquad \begin{cases}\lambda(x+z)=\mu(1-y)\\ \mu(x-z)=\lambda(1+y)\end{cases}$$

Подставим т. $(1,0,0)$:

$\lambda=\mu$

$$\begin{cases}x+z=1+y\\ x-z=1-y\end{cases} \quad \begin{cases}x-y+z-1=0\\ x+y-z-1=0\end{cases}$$

$\lambda=\mu$

$$\begin{cases}x+z=1-y\\ x-z=1+y\end{cases} \quad \begin{cases}x+y+z-1=0\\ x-y-z-1=0\end{cases}$$

Ищем напр. вектор:

$$\begin{vmatrix} i & j & k\\ 1 & -1 & 1\\ 1 & 1 & -1\end{vmatrix}=0\cdot i-2j+2k$$

$\{0,-2,2\}=\bar{\alpha}$

$$\begin{vmatrix} i & j & k\\ 1 & 1 & 1\\ 1 & -1 & -1\end{vmatrix}=0\cdot i+2j-2k$$

$\{0,2,-2\}=\bar{\beta}$

$$\cos\angle(\bar{\alpha},\bar{\beta})=\frac{|(\bar{\alpha},\bar{\beta})|}{|\bar{\alpha}|\cdot|\bar{\beta}|}=\frac{|4-4|}{2\sqrt{2}\cdot 2\sqrt{2}}=0$$

$$\Rightarrow \angle(\alpha,\beta)=\frac{\pi}{2}$$