① X_t – кр. Бесселя, $\delta > 1$, $X_0 > 0$, $Y_t = X_t^2$: $dY_t = G_t dt + H_t dB_t$

$$dX_t = \frac{\delta - 1}{2X_t}dt + dB_t$$

$$dX_t^2 = 2X_t dX_t + dt = (\delta - 1)dt + 2X_t dB_t + dt = \delta dt + 2X_t dB_t$$

$$G_t = \delta, \quad H_t = 2X_t$$

② $V(t,x)$ — ?

$$\begin{cases} V'_t(t,x) + \frac{1}{2}V''_{xx}(t,x) - \mu x V'_x(t,x) = 0, & t \in [0,1],\ x \in \mathbb{R} \\ V(1,x) = x^2 \end{cases}$$

$dX_t = -\mu X_t dt + 0 dB_t$ $\quad T = 1$

$V(t,x) = \mathbb{E}_T[X_T^2 \mid X_t = x]$

$dX_t^2 = 2X_t dX_t + dt = (1 - 2\mu X_t^2)dt + 2X_t dB_t$

$X_t^2 = x^2 + \int_0^t (1 - 2\mu X_\tau^2)d\tau + \int_0^t 2X_\tau dB_\tau$

$\mathbb{E}X$

$X_\tau = x e^{-\mu(\tau - t)} + e^{-\mu(\tau - t)}\int_t^\tau e^{\mu(s-t)}dB_s$

$\mathbb{E}_1 X_\tau = x e^{-\mu(1-t)} = x e^{\mu(t-1)}$

$\mathrm{var}_1[X_\tau] = e^{-2\mu(1-t)}\int_t^1 e^{2\mu(s-t)}ds = \frac{1}{2\mu}e^{-2\mu(1-t)} \cdot e^{2\mu(s-t)}\Big|_{s=t}^{s=1} =$

$= \frac{1}{2\mu}e^{2\mu(t-1)} \cdot \left(e^{2\mu(1-t)} - 1\right) = \frac{1}{2\mu}\left(1 - e^{2\mu(t-1)}\right)$

$\mathbb{E}X_1^2 = \frac{1}{2\mu}\left(1 - e^{2\mu(t-1)}\right) + x^2 e^{2\mu(t-1)}$

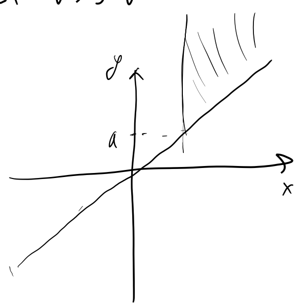
③ ∃ ли? решение ур-ия $dX_t = X_t^2 dt + dB_t$, $X_0 = 1$ $\forall t \in \mathbb{R}_+$

$\rho(x) = \exp\left\{-\int_a^x \frac{2b(y)}{\sigma^2(y)}dy\right\}$, $\quad s(x) = -\int_x^{+\infty}\rho(y)dy$, где некот $a > 0$

$b(y) = y^2$, $\quad \sigma^2(y) = 1$

$\int_a^x 2y^2 dy = \frac{2}{3}(x^3 - a^3) \Rightarrow \rho(x) = \exp\left\{\frac{2}{3}(a^3 - x^3)\right\} \Rightarrow |s(x)| = \int_x^{+\infty}\exp\left\{\frac{2}{3}(a^3 - y^3)\right\}dy$

$\int_a^{+\infty}\rho(x)dx = ?$ — инт сходится ⇒ Тип B либо Тип C



$\int_a^{+\infty}\frac{|s(x)|}{\rho(x)\sigma^2(x)}dx = \int_a^{+\infty}\int_x^{+\infty}\exp\left\{\frac{2}{3}(x^3 - y^3)\right\}dy\,dx = \left\{\begin{array}{l} z = y^{3/2} \\ dz = \frac{3}{2}y^{1/2}dy \end{array}\right\}$

↓

инт. сходится ⇒ Тип C ⇒ $\boxed{\text{Нет}}$

④ $\begin{cases} dS_t = S_t(\mu dt + \sigma_t dW_t^1), \ S_0 = 1 \\ \sigma_t = 1 + |W_t^2| \\ W_t^1,\ W_t^2 \text{ – нез} \end{cases}$

многомерная т. Гирсанова

$\mu_s = \begin{pmatrix} \mu/\sigma_s \\ a \end{pmatrix}$, $a \in \mathbb{R}$ — берём $a_1 \neq a_2$ — получаем

2 р.-н. меры

меняем $W_t^1 \longrightarrow$ $\quad dS_t = S_t \sigma_t d\tilde{W}_t^1$ (нез)

$\mathbb{E}dS_t = S_t \cdot \mathbb{E}\sigma_t \cdot \underbrace{\mathbb{E}d\tilde{W}_t^1}_{=0}$

(5) $dS_t^i = S_t^i(\mu^i dt + \sigma^i dW_t^i)$ $S_0^i = s_0^i > 0$, $i=1,2$, W_t^1, W_t^2 нез. $S^i = s^i \exp\left\{\left(\mu^i - \frac{(\sigma^i)^2}{2}\right)t + \sigma^i W_t^i\right\}$

$V_T(s^1, s^2) = (s^2 - s^1)^+$

Numeraire — S^1 $\Rightarrow$ $V_T^{S^1} = \left(\frac{S^2}{S^1} - 1\right)^+$ $\Rightarrow$ $V_T^{\$} = \left(\frac{S^2}{S^1} - 1\right)^+ \cdot S^1$

$V_0^{\$} = S_0^1 \cdot \mathbb{E}_Q\left[\left(\frac{S_T^2}{S_T^1} - 1\right)^+\right]$, Q — т.ч. $\frac{S_t^2}{S_t^1}$ — мартингал

по Гирсанову:

$\left(\mu_2 - \mu_1 - \frac{(\sigma^2)^2}{2} + \frac{(\sigma^1)^2}{2}\right)$

$\tilde{S}_t^1 = 1$

$\tilde{S}_t^2 = \frac{S_t^2}{S_t^1}$ $\tilde{S}_t^2 = \frac{S_t^2}{S_t^1} = \frac{s^2}{s^1}\exp\left\{-\frac{(\sigma^2)^2 - (\sigma^1)^2}{2}t \underbrace{- \sigma^1 W_t^1 + \sigma^2 W_t^2}_{\overset{d}{=}\ \tilde{\sigma}\tilde{W}_t}\right\}$

$\tilde{\sigma} = \sqrt{(\sigma^1)^2 + (\sigma^2)^2}$

$\nu = \mu_1 - \mu_2 - (\sigma^1)^2$ — в Гирсанова

$Z_t = \exp\left\{\int_0^t \nu \, d\tilde{W}_t - \frac{\nu^2}{2}t\right\}$

р-я динамика для $\tilde{S}_t^2$: $d\tilde{S}_t^2 = \tilde{S}_t^2 \tilde{\sigma} \hat{W}_t$

$\mathbb{E}[(\tilde{S}_T^2 - 1)^+]$ — европейский опцион колл, страйк 1.

$C = \tilde{S}_0^2 N(\tilde{d}_1) - N(\tilde{d}_2)$, $\tilde{d}_1 = \frac{\log \tilde{S}_0^2 + \frac{\tilde{\sigma}^2}{2}T}{\tilde{\sigma}\sqrt{T}}$ $\tilde{d}_2 = \tilde{d}_1 - \tilde{\sigma}\sqrt{T}$

$V_0^{\$} = s^1 \cdot \left(\frac{s_2}{s_1} N(d_1) - N(d_2)\right) = s_2 N(d_1) - s_1 N(d_2)$

$d_1 = \frac{\log(s^2/s^1) + \frac{(\sigma^1)^2 + (\sigma^2)^2}{2}T}{\sqrt{((\sigma^1)^2 + (\sigma^2)^2)T}}$, $d_2 = d_1 - \sqrt{(\sigma^1)^2 + (\sigma^2)^2}\, T$