Seminar 6
Stochastic Volatility Models

Vega Institute

Problem 1 🧠

Prove Tanaka's formula

$$|B_t| = \int_0^t \operatorname{sign}(B_s)dB_s + L_t^a,$$

where L_t is adapted non-decreasing process.

Problem 2 💅

Prove the following Bayes formula

$$\mathbb{E}_t^{P_1}(X) = \frac{\mathbb{E}_t^{P_2}\left(\dfrac{dP_1}{dP_2}X\right)}{\mathbb{E}_t^{P_2}\left(\dfrac{dP_1}{dP_2}\right)}$$

where $P_1 \sim P_2$ are equivalent measures, $\mathbb{E}_t$ is conditional expectation wrt $\mathcal{F}_t$ and X is a random variable.

Hint: show that

$$\mathbb{E}^{P_1}\left(\mathbb{E}_t^{P_1}(X)1_G\right) = \mathbb{E}^{P_2}\left(\mathbb{E}_t^{P_1}(X)\mathbb{E}_t^{P_2}\left(\frac{dP_1}{dP_2}\right)1_G\right),$$

for any $G \in \mathcal{F}_t$.

Problem 3 💅

Using Bayes formula, show that $e^{rt}S_t^{-1}$ is a martingale with respect to $\tilde{P} = \dfrac{e^{-rT}S_T}{S_0}P$.

Problem 4 💅

Show that in Heston model characteristic function $\varphi(t,x,u) := \mathbb{E}^P_{t,x,v}e^{iuX_T}$ satisfies the following PDE

$$\begin{cases} \dfrac{\partial \varphi}{\partial t} + \dfrac{v}{2}\dfrac{\partial^2 \varphi}{\partial x^2} + \dfrac{\sigma^2 v}{2}\dfrac{\partial^2 \varphi}{\partial v^2} + \rho\sigma v\dfrac{\partial^2 \varphi}{\partial x \partial v} + (r - \dfrac{1}{2}v)\dfrac{\partial \varphi}{\partial x} + \kappa(\theta - v)\dfrac{\partial \varphi}{\partial v} = 0, \\ \\ \varphi(T,x,v,u) = e^{iux}. \end{cases}$$

Hint: use multi-dimensional Feynman-Kac formula.

Problem 5 💅

Show that $C' = C$ and $D' = D$ (see lecture 8).