Multi-Period Asset Pricing
Part 3

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Arbitrage-free pricing in single period model

Fundamental theorems

A finite single period model

There are two times: 0 and 1 and $d+1$ traded assets.

Savings account with interest rate $r > -1$ (the same for borrowing and lending):

$$\$1 \text{ at } t = 0 \quad \longrightarrow \quad \$(1+r) \text{ at } t = 1.$$

d stocks with initial prices $S_0 = ({S_0}^i)_{1 \le i \le d}$ and terminal prices $S_1 = ({S_1}^i)_{1 \le i \le d}$.

The terminal prices $S_1 = S_1(\omega)$ are *random variables* on a finite probability space $(\Omega, \mathbb{P})$ such that

$$\mathbb{P}(\omega) > 0, \quad \omega \in \Omega.$$

A finite single period model

Trading strategy: (X_0, Δ_0), where

X_0: the initial wealth,

$\Delta_0 = ({\Delta_0}^i)_{1 \leq i \leq d}$: the initial number of stocks.

Balance equation: $(\langle a, b \rangle \triangleq \sum_{i=1}^{d} a_i b_i)$

$$X_1(\omega) = \underbrace{(X_0 - \langle \Delta_0, S_0 \rangle)(1+r)}_{\text{bank account}} + \underbrace{\langle \Delta_0, S_1(\omega) \rangle}_{\text{stocks}}.$$

Arbitrage strategy: we start with *nothing* and end with *something*,

$$X_0 = 0 \quad \underset{\Delta_0}{\longrightarrow} \quad \begin{array}{l} X_1(\omega) \geq 0, \forall \omega \in \Omega \\ X_1(\omega') > 0, \exists \omega' \in \Omega. \end{array}$$

Question

Is the model arbitrage free?

Arbitrage-free pricing

We assume NA and take a payoff $V_1 = V_1(\omega)$ at $t = 1$.

Main principle:

$$\boxed{\textbf{Pricing} = \textbf{Replication}}$$

Replicating strategy:

$$\underbrace{V_0 = X_0}_{?} \quad \xrightarrow[\Delta_0 - ?]{} \quad \underbrace{X_1(\omega) = V_1(\omega)}_{\text{known}}, \ \omega \in \Omega.$$

We get a linear system:

$$(X_0 - \langle \Delta_0, S_0 \rangle)(1 + r) + \langle \Delta_0, S_1(\omega) \rangle = V_1(\omega), \quad \omega \in \Omega, \quad (*)$$

with $N = |\Omega|$ equations and $1 + d$ unknowns (X_0, Δ_0).

Complete financial models

Question: Is there a solution of $(*)$ for *given* V_1?

$\iff$ Does a replicating strategy exist for *given* option?

Easier question: Is there a solution of $(*)$ for *every* V_1?

$\iff$ Does a replicating strategy exist for *every* option?

$\iff$ Is the model *complete*?

Definition

Financial market (FM) is **complete** if it is arbitrage-free and *every* option is replicable.

Remark

FM is complete $\iff$ AFP exists and is unique for every option.

Arbitrage-free pricing

To find X_0 from $(*)$ we use the same trick as for the single period binomial model. We enumerate the sample space:

$$\Omega = (\omega_n)_{n=1,\dots,N}\,.$$

We choose numbers $(\widetilde{p}_n)$ so that the linear combination of equations $(*)$ with weights $(\widetilde{p}_n)$ has the form:

$$\sum_n \widetilde{p}_n V_1(\omega_n) = X_0(1+r),$$

for every $\Delta_0 = (\Delta_0^i)$. We deduce that

$$\sum_n \widetilde{p}_n = 1 \tag{a}$$

and

$$\sum_n \widetilde{p}_n S_1(\omega_n) = S_0(1+r). \tag{b}$$

Completeness lemma

Lemma

Suppose that the financial market is complete. Then there is a unique solution $(\widetilde{p}_n)$ *of the system* $(a) - (b)$ *and*

$$\widetilde{p}_n > 0, \quad n = 1, \ldots, N.$$

Proof.

As the model is complete, for every payoff $V_1 = V_1(\omega)$ there is a replicating strategy:

$$V_0 = X_0 \quad \underset{\Delta_0}{\longrightarrow} \quad X_1(\omega) = V_1(\omega),\ \omega \in \Omega.$$

Completeness lemma

If the numbers $(\widetilde{p}_n)$ solve $(a) - (b)$, then

$$\sum_n \widetilde{p}_n V_1(\omega_n) = V_0(1 + r).$$

Thus, if we take a "digital" payoff (an Arrow-Debreu security)

$$I_1^n(\omega) = 1_{\{\omega = \omega_n\}}, \quad \omega \in \Omega,$$

and denote by I_0^n its AFP, then we obtain that

$$\widetilde{p}_n = I_0^n(1 + r), \quad n = 1, \ldots, N. \qquad \text{(c)}$$

Observe that NA $\Longrightarrow$ $(\widetilde{p}_n)$ are unique and > 0.

Completeness lemma

To see that the numbers $(\widetilde{p}_n)$ from (c) indeed solve $(a) - (b)$, we take an arbitrary payoff $V_1 = V_1(\omega)$ and notice that

$$V_1(\omega) = \sum_n V_1(\omega_n) 1_{\{\omega=\omega_n\}} = \sum_n V_1(\omega_n) I_1^n(\omega).$$

It follows that

$$V_0 = \sum_n V_1(\omega_n) I_0^n(\omega) = \frac{1}{1+r} \sum_n \widetilde{p}_n V_1(\omega_n).$$

In particular, choosing $V_1 = 1 + r$ we get (a) and taking $V_1 = S_1^i$, we obtain (b). $\square$

Risk-neutral probability (RNP)

The coefficients $(\widetilde{p}_n)$ define a *risk-neutral probability measure*.

Definition

A probability measure $\widetilde{\mathbb{P}}$ on Ω is called **risk-neutral** (**RNP**) if

1. $\widetilde{\mathbb{P}}(\omega) > 0$ for all $\omega \in \Omega$.
2. For every strategy, the initial and terminal capitals are related as
$$\widetilde{\mathbb{E}}(X_1) = X_0(1 + r). \qquad \square$$

The first item is (a). The second item is equivalent to (b), because of the balance equation:

$$X_1 = (X_0 - \langle \Delta_0, S_0 \rangle)(1 + r) + \langle \Delta_0, S_1 \rangle .$$

Fundamental theorems of asset pricing (FTAPs)

We denote by

$$\widetilde{\mathcal{P}} = \left\{ \widetilde{\mathbb{P}} \right\}.$$

the family of *all* RNPs.

Theorem (1st FTAP)

$$\textit{NA} \quad \Longleftrightarrow \quad \widetilde{\mathcal{P}} \neq \emptyset \quad (\widetilde{\mathcal{P}} \textit{ is nonempty}).$$

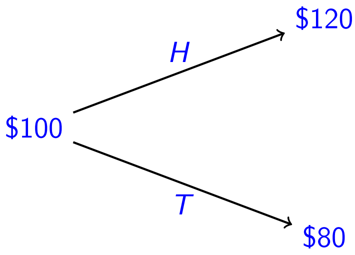
Theorem (2nd FTAP)

$$\textit{Completeness} \quad \Longleftrightarrow \quad |\widetilde{\mathcal{P}}| = 1 \quad (\widetilde{\mathcal{P}} \textit{ contains one element}).$$

Problem on binomial model

Problem

The interest rate $r = 5\%$ and the stock evolves as



Is the model arbitrage-free? Is it complete?

Problem on binomial model

Solution

For $\widetilde{p} = \widetilde{\mathbb{P}}(H)$ and $\widetilde{q} = \widetilde{\mathbb{P}}(T)$ we obtain the system:

$$\begin{gathered} \widetilde{p} > 0, \quad \widetilde{q} > 0, \\ \widetilde{p} + \widetilde{q} = 1, \\ \widetilde{p}120 + \widetilde{q}80 = 105 \quad \left(\widetilde{\mathbb{E}}(S_1) = S_0(1+r)\right). \end{gathered}$$

The system has the unique solution:

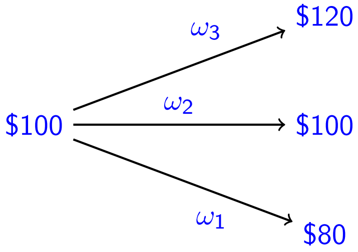
$$\widetilde{p} = 0.625, \quad \widetilde{q} = 0.375.$$

Thus, $|\widetilde{\mathcal{P}}| = 1$ and FTAPs $\implies$ the model is complete (in particular, arbitrage-free). $\square$

Problem on trinomial model

Problem

The interest rate $r = 5\%$ and the stock evolves as



Is the model arbitrage-free? Is it complete?

Problem on trinomial model

Solution

For $\widetilde{p}_i = \widetilde{\mathbb{P}}(\omega_i)$, $i = 1, 2, 3$, we obtain the system:

$$\begin{aligned} \widetilde{p}_i > 0, \quad i = 1, 2, 3, \\ \widetilde{p}_1 + \widetilde{p}_2 + \widetilde{p}_3 = 1, \\ \widetilde{p}_1 80 + \widetilde{p}_2 100 + \widetilde{p}_3 120 = 105, \quad \left(\widetilde{\mathbb{E}}(S_1) = S_0(1+r)\right). \end{aligned}$$

It has the infinite number of solutions:

$$0 < \widetilde{p}_1 < 0.375, \quad \widetilde{p}_2 = 0.75 - 2\widetilde{p}_1, \quad \widetilde{p}_3 = \widetilde{p}_1 + 0.25.$$

Hence, $|\widetilde{\mathcal{P}}| = \infty$ and FTAPs $\implies$ the model is arbitrage free and incomplete. $\square$

Risk-neutral valuation

Corollary (Risk-neutral valuation (RNV))

For a complete model with the unique RNP $\widetilde{\mathbb{P}}$ *the AFP of the option paying* $V_1 = V_1(\omega)$ *at* $t = 1$ *is given by*

$$V_0 = \frac{1}{1+r}\widetilde{\mathbb{E}}\left(V_1\right).$$

Proof.

Since the market is complete, there exists a replicating strategy:

$$V_0 = X_0 \quad \underset{\Delta_0}{\longrightarrow} \quad X_1 = V_1.$$

From the definition of $\widetilde{\mathbb{P}}$ we deduce that

$$\widetilde{\mathbb{E}}\left(V_1\right) = \widetilde{\mathbb{E}}\left(X_1\right) = X_0(1+r) = V_0(1+r). \qquad \square$$

Risk-neutral valuation

Remark

The formula

$$V_0 = \frac{1}{1+r}\widetilde{\mathbb{E}}\left(V_1\right),$$

where $\widetilde{\mathbb{P}}$ is an RNP, is called the *Risk-Neutral Valuation (RNV)*.

Arbitrage-free models:

$$\boxed{\textbf{!AFP} = \textbf{Replication}}$$

Complete models:

$$\boxed{\textbf{!AFP} = \textbf{Risk Neutral Valuation}}$$

Pricing in practice

Remark

Practical implementation of arbitrage-free pricing involves the following steps:

1. We start with a class of complete models.
2. We choose a particular complete model (we find $\widetilde{\mathbb{P}}$) by *calibration*.
3. We compute arbitrage-free prices for derivatives by risk-neutral valuation:

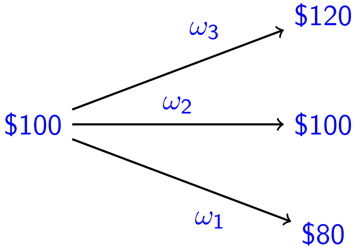
$$V_0 = \frac{1}{1+r}\widetilde{\mathbb{E}}\left(V_1\right).$$

Advantage: there is *no replication*! $\square$

Problem on trinomial model

Problem

A bank account pays the interest rate $r = 5\%$. *A stock evolves as*



Call option has strike $K^C = \$100$ *and is traded at* $C = \$9$.

1. *Is the model arbitrage-free? Is it complete?*
2. *Compute AFPs of the put option with strike* $K^P = \$95$.

Problem on trinomial model

Solution

For the risk-neutral probabilities we get the system:

$$\begin{aligned} \widetilde{p}_i > 0, \quad i = 1, 2, 3, \\ \widetilde{p}_1 + \widetilde{p}_2 + \widetilde{p}_3 = 1, \\ \widetilde{p}_1 80 + \widetilde{p}_2 100 + \widetilde{p}_3 120 = 100(1 + 0.05), \quad &\text{(stock)} \\ \widetilde{p}_3 20 = 9(1 + 0.05). \quad &\text{(call)} \end{aligned}$$

This system has the unique solution:

$$\widetilde{p}_1 = 0.2225, \quad \widetilde{p}_2 = 0.305, \quad \widetilde{p}_3 = 0.4725.$$

Hence, the model is complete. Put's AFP (*no replication!*):

$$P = \frac{1}{1+r} \widetilde{\mathbb{E}} \left(\max(K^P - S_1, 0) \right) = 3.1786. \qquad \square$$

Proof of 1st FTAP (easy part)

Proof.

$\widetilde{\mathcal{P}} \neq \emptyset \implies$ NA: Let $\widetilde{\mathbb{P}}$ be a RNP and take a strategy with initial capital $X_0 = 0$ and nonnegative terminal wealth:

$$X_1(\omega) \geq 0, \quad \omega \in \Omega.$$

Since

$$\sum_{\omega \in \Omega} X_1(\omega) \widetilde{\mathbb{P}}(\omega) = \widetilde{\mathbb{E}}(X_1) = X_0(1+r) = 0$$

and

$$\widetilde{\mathbb{P}}(\omega) > 0, \quad \omega \in \Omega,$$

we deduce that

$$X_1(\omega) = 0, \quad \omega \in \Omega.$$

Hence, the model is arbitrage free. $\square$

Proof of 2nd FTAP

Proof.

Completeness $\implies |\widetilde{\mathcal{P}}| = 1$: This statement has been proved in the Completeness Lemma.

$|\widetilde{\mathcal{P}}| = 1 \implies$ Completeness: We take a payoff V_1 and denote by $\widehat{X}_1$ its best least square approximation in the set $\mathcal{X}$ of terminal capitals of trading strategies:

$$\widetilde{\mathbb{E}}\left((\widehat{X}_1 - V_1)^2\right) = \min_{X_1 \in \mathcal{X}} \widetilde{\mathbb{E}}\left((X_1 - V_1)^2\right).$$

We shall show that $\widehat{X}_1$ replicates V_1:

$$\widehat{X}_1 = V_1.$$

Proof of 2nd FTAP

We denote

$$G = \widehat{X}_1 - V_1.$$

For any $X_1 \in \mathcal{X}$ the function

$$f(y) = \widetilde{\mathbb{E}}\left((yX_1 + \widehat{X}_1 - V_1)^2\right), \quad y \in \mathbb{R},$$

attains its minimum at $y = 0$. It follows that

$$0 = f'(0) = 2\widetilde{\mathbb{E}}\left((\widehat{X}_1 - V_1)X_1\right) = 2\widetilde{\mathbb{E}}\left(GX_1\right).$$

We have obtained the *first-order condition* for the optimality of $\widehat{X}_1$:

$$\widetilde{\mathbb{E}}\left((\widehat{X}_1 - V_1)X_1\right) = \widetilde{\mathbb{E}}\left(GX_1\right) = 0, \quad X_1 \in \mathcal{X}.$$

Proof of 2nd FTAP

We take a sufficiently small $\epsilon > 0$ so that

$$1 + \epsilon G(\omega) > 0, \quad \omega \in \Omega,$$

and define a strictly positive function

$$\widehat{\mathbb{P}}(\omega) \triangleq (1 + \epsilon G(\omega))\widetilde{\mathbb{P}}(\omega), \quad \omega \in \Omega.$$

We have that $\widehat{\mathbb{P}}(\omega) > 0$, $\omega \in \Omega$, and

$$\sum_{\omega \in \Omega} \widehat{\mathbb{P}}(\omega) = \sum_{\omega \in \Omega} (1 + \epsilon G(\omega))\widetilde{\mathbb{P}}(\omega) = \widetilde{\mathbb{E}}(1 + \epsilon G) = 1.$$

Thus, $\widehat{\mathbb{P}}$ is a strictly positive probability measure.

Proof of 2nd FTAP (difficult part)

If $X_0 \longrightarrow X_1$ is a strategy, then

$$\widehat{\mathbb{E}}\left(X_1\right) = \widetilde{\mathbb{E}}\left((1+\epsilon G)X_1\right) = \widetilde{\mathbb{E}}\left(X_1\right) + \epsilon\widetilde{\mathbb{E}}\left(GX_1\right) = X_0(1+r).$$

Hence, $\widehat{\mathbb{P}}$ is a RNP.
However, by our assumption,

$$|\widetilde{\mathcal{P}}| = 1.$$

It follows that

$$\widehat{\mathbb{P}}\left(\omega\right) = \widetilde{\mathbb{P}}\left(\omega\right), \quad \omega \in \Omega,$$

and therefore,

$$\widehat{X}_1 - V_1 = G = 0.$$

Thus, V_1 is replicable and the model is complete. $\square$

Proof of 1st FTAP (difficult part)

Proof.

NA $\implies \widetilde{\mathcal{P}} \neq \emptyset$: We state (without a proof) the following lemma.

Lemma

Every non-traded option with payoff $V_1 = V_1(\omega)$ *admits (maybe non unique) AFP* V_0. *In other words, the extended financial model, where one can trade*

1. *the original securities,*
2. *the option at the price* V_0

is arbitrage-free.

Proof of 1st FTAP (difficult part)

The lemma yields the existence of AFPs for Arrow-Debreu securities:

$$I_n(\omega) = 1_{\{\omega=\omega_n\}}, \quad \omega \in \Omega,\ n = 1, \dots, N = |\Omega|.$$

After such extension, the model becomes complete, because every payoff V_1 is a linear combination of the Arrow-Debreu securities:

$$V_1(\omega) = \sum_{i=1}^{N} V_1(\omega_i) I_i(\omega).$$

2nd FTAP $\implies$ existence and uniqueness of RNP $\widetilde{\mathbb{P}}$ in the extended model. Clearly, $\widetilde{\mathbb{P}}$ is also a RNP in the original model. $\square$