$\int 62$

1) Пусть $\forall v \neq 0$ $\overline{\lim\limits_{t\to s+}} \dfrac{|X_{st}\cdot v|}{|t-s|^{2\alpha}} = +\infty$

$Y_{st} = Y'_s \cdot X_{st} + R_{st}$

$Y_{st} = \tilde{Y}'_s \cdot X_{st} + \tilde{R}_{st}$ $\Rightarrow (Y'_s - \tilde{Y}'_s)\cdot X_{st} = \tilde{R}_{st} - R_{st}$

$$\underbrace{\frac{(Y'_s - \tilde{Y}'_s)\cdot X_{st}}{|t-s|^{2\alpha}}}_{\xrightarrow{t\to s+}\ +\infty} = \underbrace{\frac{\tilde{R}_{st} - R_{st}}{|t-s|^{2\alpha}}}_{\wedge\ C} \Rightarrow Y'_s = \tilde{Y}'_s$$

2) $t = \underset{Y'_0}{0} \cdot \overset{X_{0t}}{t} + \underset{R_{0t}}{t} = \underset{\tilde{Y}'_0}{1} \cdot \overset{X_{0t}}{t} + \underset{\tilde{R}_{0t}}{0} \Rightarrow$ производ. Губинели опр-на неод-и образом.

3) Предп-м, что $\overline{\lim\limits_{t\to s+}} \dfrac{|W_t - W_s|}{|t-s|^{2\alpha}} < \infty$ (положим $V = (1, 0, \dots 0)$)

$$\overline{\lim_{t\to s+}} \frac{|W_t - W_s|}{|t-s|^{2\alpha}} \geq C \cdot \overline{\lim_{t\to s+}} |t-s|^{\frac{1}{2} - 2\alpha} \ln\ln\left(\frac{1}{|t-s|}\right) = +\infty \Rightarrow \text{противоречие}$$

$\alpha > \frac{2}{3}$

$\frac{1}{2} - 2\alpha < \frac{1}{2} - \frac{2}{3} < 0$

$\int 67.$

$$\int_0^T Y_u \circ dX_u = \lim_{\lambda(\Pi)\to 0} \sum_{[u,v]} \frac{Y_u + Y_v}{2} \cdot X_{uv}$$

$$\int_0^T Y_u\, dX_u = \lim_{\lambda(\Pi)\to 0} \sum_{[u,v]} (Y_u \cdot X_{uv} + Y'_u \cdot \mathbb{X}_{uv}) \overset{?}{=} \lim_{\lambda(\Pi)\to 0} \sum_{[u,v]} (Y_w \cdot X_{uv} + Y'_w \cdot \mathbb{X}_{uv}), \quad w \in [u,v]$$

$$\lim_{\lambda(\Pi)\to 0} \sum_{[u,v]} Y_u \cdot X_{uv} = \lim_{\lambda(\Pi)\to 0} \sum_{[u,v]} Y_w \cdot X_{uv} \text{ (св-во инт-ла Юнга)}$$

$$\sum_{[u,v]} \overbrace{(Y'_u - Y'_v)}^{\alpha} \cdot \overset{2\alpha}{\mathbb{X}_{uv}} \leq C \cdot \sum_{[u,v]} |u-v|^{\overset{3\alpha}{\vee}\, 1} \xrightarrow[\lambda(\Pi)\to 0]{} 0$$

$$\int_0^T Y_u\,dX_u = \frac{1}{2}\lim_{\lambda(\pi)\to 0}\sum_{[u,v]}\left(Y_u\cdot X_{uv} + Y'_u\cdot \mathbb{X}_{uv}\right) + \frac{1}{2}\lim_{\lambda(\pi)\to 0}\sum_{[u,v]}\left(Y_v\cdot X_{uv} + Y'_v\cdot \mathbb{X}_{uv}\right) =$$

$$= \int_0^T Y_u\circ dX_u + \lim_{\lambda(\pi)\to 0}\sum_{[u,v]}\frac{Y'_u+Y'_v}{2}\cdot\mathbb{X}_{uv}.$$

утв. из лекции

$$X_t := W_t,\quad \mathbb{X}^{ij}_{st} := \int_s^t W^i_{u\tau}\circ dW^j_\tau \Rightarrow E\Big|\sum_{[u,v]} Y'_u\cdot\mathbb{X}_{uv}\Big|^2 \to 0$$

$$E\Big|\sum_{[u,v]} Y'_v\cdot\mathbb{X}_{uv}\Big|^2 \to 0$$

№70.

$(Y,Y'),\ (Z,Z')$ - контролир. отн. X.

$$\mathcal{U}_t := \int_0^t Z_s\,dX_s\ ;\quad \underbrace{\int_0^T Y_t\,d\mathcal{U}_t}_{№69} \overset{?}{=} \underbrace{\int_0^T Y_tZ_t\,dX_t}_{\text{грубый инт-л}}$$

$$\int_0^T Y_t\,d\mathcal{U}_t = \lim_{\lambda(\pi)\to 0}\sum_{[u,v]} Y_u\cdot\mathcal{U}_{uv} + Y'_u\cdot\overset{Z_u}{\mathcal{U}'_u}\cdot\mathbb{X}_{uv} \ominus$$

$$\mathcal{U}_{uv} = \int_u^v Z_s\,dX_s \approx Z_u X_{uv} + Z'_u\mathbb{X}_{uv}$$ с точностью до 3α-гёльдер. слаг.

$$\ominus \lim_{\lambda(\pi)\to 0}\sum_{[u,v]} Y_u\cdot Z_u\cdot X_{uv} + \underbrace{(Y_u\cdot Z'_u + Y'_u\cdot Z_u)}_{\overset{?}{=}\,(Y\cdot Z)'_u}\mathbb{X}_{uv} = \lim_{\lambda(\pi)\to 0}\sum_{[u,v]} Y_uZ_uX_{uv} + (Y\cdot Z)'_u\cdot\mathbb{X}_{uv} = \int_0^T Y_tZ_t\,dX_t.$$

(?) $$Y_v\cdot Z_v - Y_u\cdot Z_u = (Y_u + Y'_u\cdot X_{uv} + R^Y_{uv})(Z_u + Z'_u\cdot X_{uv} + R^Z_{uv}) - Y_uZ_u =$$

$$= \underbrace{(Y_u\cdot Z'_u + Y'_u\cdot Z_u)}_{(Y\cdot Z)'_u} X_{uv} + R^{YZ}_{uv}$$

№72.

$Y, Z, \tilde Y, \tilde Z \in \mathcal{D}_X^{2\alpha}$; $\|Y\|_{\mathcal D}, \|\tilde Y\|_{\mathcal D}, \|Z\|_{\mathcal D}, \|\tilde Z\|_{\mathcal D} \le M$

$\|Y\|_{\mathcal D} = |Y_0| + |Y'_0| + \|Y'\|_\alpha + \|R\|_{2\alpha}$

$$\int_0^T \tilde Y\, dZ - \int_0^T Y\, dZ = \lim_{\lambda(\Pi)\to 0} \sum_{[u,v]} \left(\tilde Y_u \cdot \tilde Z_{uv} + \tilde Y'_u \cdot \tilde Z'_u \cdot \mathbb X_{uv} - Y_u \cdot Z_{uv} - Y'_u \cdot Z'_u \cdot \mathbb X_{uv}\right)$$

$\Delta Z_{uv} = \tilde Z_{uv} - Z_{uv}$; $\Delta Z_u = \tilde Z_u - Z_u$

$$\tilde Y_u \cdot \tilde Z_{uv} - Y_u \cdot Z_{uv} = Y_u \cdot \Delta Z_{uv} + \Delta Y_u \cdot Z_{uv} + \Delta Y_u \cdot \Delta Z_{uv}$$

$$(\tilde Y'_u \cdot \tilde Z'_u - Y'_u \cdot Z'_u)\cdot \mathbb X_{uv} = (Y'_u \cdot \Delta Z'_u + \Delta Y'_u \cdot Z_u + \Delta Y'_u \cdot \Delta Z'_u)\cdot \mathbb X_{uv}$$

$$A_{uv} := Y_u \cdot Z_{uv} + Y'_u \cdot Z'_u \cdot \mathbb X_{uv}$$

$$A_{st} - A_{su} - A_{ut} = \underline{Y_s \cdot Z_{st}} + \underline{\underline{Y'_s \cdot Z'_s \cdot \mathbb X_{st}}} - \underline{Y_s \cdot Z_{su}} - \underline{\underline{Y'_s \cdot Z'_s \cdot \mathbb X_{su}}} - \underline{Y_u \cdot Z_{ut}} - \underline{\underline{Y'_u \cdot Z'_u \cdot \mathbb X_{ut}}}$$

$$Y_s \cdot Z_{st} - Y_s \cdot Z_{su} - Y_u \cdot Z_{ut} = Y_s \cdot Z_{ut} - Y_u \cdot Z_{ut} = -Y_{su} \cdot Z_{ut} =$$
$$= -(Y'_s \cdot X_{su} + R^Y_{su})(Z'_u \cdot X_{ut} + R^Z_{ut}) = -Y'_s \cdot X_{su} \cdot Z'_u \cdot X_{ut} +$$
$$- R^Y_{su} \cdot Z'_u \cdot X_{ut} - Y'_s \cdot X_{su} \cdot R^Z_{ut} - R^Y_{su} \cdot R^Z_{ut} = -\cancel{Y'_s \cdot X_{su} \cdot Z'_u \cdot X_{ut}} + S_1$$

$$|R^Y_{su} \cdot Z'_u \cdot X_{ut}| \le \|R^Y\|_{2\alpha} \cdot |t-s|^{2\alpha} \cdot \|X\|_\alpha \cdot |t-s|^\alpha \cdot (|Z'_0| + \|Z'\|_\alpha \cdot T^\alpha)$$
$$\le \|R^Y\|_{2\alpha} \cdot \|X\|_\alpha \cdot (1+T^\alpha) \cdot \|Z\|_{\mathcal D} \cdot |t-s|^\alpha$$

Аналогично оценив-ся оставшиеся слагаемые;

$$Y'_s \cdot Z'_s \cdot \mathbb X_{st} - Y'_s \cdot Z'_s \cdot \mathbb X_{su} - Y'_u \cdot Z'_u \cdot \mathbb X_{uv} = Y'_s \cdot Z'_s (\mathbb X_{ut} + X_{su} X_{ut})$$
$$- (Y'_s + Y'_{su})(Z'_s + Z'_{su}) \cdot \mathbb X_{ut} = \underline{\cancel{Y'_s \cdot Z'_s} \cdot X_{su} \cdot X_{ut}} + S_2$$

$$|S_i| \le C \cdot \|Y\|_{\mathcal D} \cdot \|Z\|_{\mathcal D} \cdot |t-s|^\alpha$$

$\Rightarrow |A_{st} - A_{su} - A_{ut}| \le \tilde{C} \cdot \|Y\|_{\mathcal{D}} \cdot \|Z\|_{\mathcal{D}} \cdot |t-s|^{3\alpha}$

$\Rightarrow \left| \int_s^t Y_u \, dZ_u - Y_s \cdot Z_{st} - Y'_s \cdot Z'_s \cdot \mathbb{X}_{st} \right| \le \tilde{\tilde{C}} \cdot \|Y\|_{\mathcal{D}} \cdot \|Z\|_{\mathcal{D}} \cdot |t-s|^{3\alpha}$

лемма о сшивке $\quad \Rightarrow \left| \int_0^T Y_u \, dZ_u \right| \le C' \|Y\|_{\mathcal{D}} \cdot \|Z\|_{\mathcal{D}}$

$$\int_0^T \tilde{Y} \, d\tilde{Z} - \int_0^T Y \, dZ = \int_0^T Y \, d\Delta Z + \int_0^T \Delta Y \, dZ + \int_0^T \Delta Y \, d\Delta Z$$

$$\left| \int_0^T Y \, d(\Delta Z) \right| \le C \cdot M \cdot \|\Delta Z\|_{\mathcal{D}}$$

$\left| \int_0^T \Delta Y \, dZ \right| \le CM \|\Delta Y\|_{\mathcal{D}} \quad \Rightarrow \left| \int_0^T \tilde{Y} \, d\tilde{Z} - \int_0^T Y \, dZ \right| \le 4CM \left(\|\Delta Y\|_{\mathcal{D}} + \|\Delta Z\|_{\mathcal{D}} \right)$

$$\left| \int_0^T \Delta Y \, d\Delta Z \right| \le 2CM \cdot \|\Delta Z\|_{\mathcal{D}}$$

№75.

$$dY_t = Y_t \cdot Z_t \cdot dX_t \iff Y_t = y + \underbrace{\int_0^t Y_s \cdot Z_s \, dX_s}_{\overset{\text{№70}}{=} \int_0^t Y_s \, dU_s, \quad U_t = \int_0^t Z_s \, dX_s} = y + \int_0^t Y_s \, dU_s$$

$\Rightarrow$ достаточно решить ур-е $dY_t = Y_t \, dU_t \overset{\text{из лекций}}{\Rightarrow} Y_t = e^{U_t - \frac{1}{2}[U]_t}$

$\Rightarrow Y_t = e^{\int_0^t Z_s \, dX_s - \frac{1}{2} \int_0^t Z_s \otimes Z_s \, d[X]_s}$

№73

№76.

$$dY_t = g(Y_t)\,dt + f(Y_t)\,dX_t$$

$$\hat{X}_t := \begin{pmatrix} t \\ X_t \end{pmatrix}; \quad \hat{\mathbb{X}}_{st} := \begin{pmatrix} \frac{(t-s)^2}{2} & \int_s^t (u-s)\,dX_u \\ \int_s^t X_{su}\,du & \mathbb{X}_{st} \end{pmatrix}$$

Покажем, что $(\hat{X}, \hat{\mathbb{X}})$ – грубая траектория.

1) $X_{st} = X_{su} + X_{ut}$; $\quad t-s = (t-u)+(u-s)$

2) $\frac{(t-s)^2}{2} = \frac{(t-u)^2}{2} + (t-u)(u-s) + \frac{(u-s)^2}{2}$

$$\mathbb{X}^{ij}_{st} = \mathbb{X}^{ij}_{su} + \mathbb{X}^{ij}_{ut} + X^i_{su} \cdot X^j_{ut}$$

$$\int_s^t X_{s\tau}\, d\tau - \int_s^u X_{s\tau}\, d\tau - \int_u^t X_{u\tau}\, d\tau = \int_u^t X_{s\tau}\, d\tau - \int_u^t X_{u\tau}\, d\tau =$$

$$= \int_u^t X_{su}\, d\tau = X_{su} \cdot (t-u)$$

$$\int_s^t (\tau-s)\, dX_\tau - \int_s^u (\tau-s)\, dX_\tau - \int_u^t (\tau-u)\, dX_\tau = (u-s) \cdot X_{ut}$$

=> соотношения Чена вып-ны.

3) $\frac{(t-s)^2}{2} = O(t-s)^{2\alpha}$; $\quad \left|\int_s^t X_{su}\, du\right| \le \|X\|_\alpha \cdot |t-s|^\alpha \cdot |t-s| = \underline{O}(t-s)^{2\alpha}$

$$\left|\int_s^t (u-s)\, dX_u\right| \le ?$$

$$A_{uv} := (u-s) \cdot X_{uv} \Rightarrow |A_{uv} - A_{ur} - A_{rv}| \le \|X\|_\alpha \cdot |u-v|^{1+\alpha}$$

$$\Rightarrow \left|\int_s^t (u-s)\, dX_u\right| \le C \cdot |t-s|^{1+\alpha} = \underline{O}(t-s)^{2\alpha}$$

лемма о швейке => $(\hat{X}, \mathbb{X})$ - грубая траектория.

$\Rightarrow \exists!\, Y_t$ на $[0, \underset{1}{\hat{T}}]$: $\quad dY_t = \hat{f}(Y_t)\, d\hat{X}_t$, $\quad Y_0 = y$, $\quad \hat{f} = \begin{pmatrix} g \\ f \end{pmatrix}$

$$\Rightarrow Y_t = y + \int_0^t \hat{f}(Y_u)\, d\hat{X}_u$$

$$\int_0^t \hat{f}(Y_u)\, d\hat{X}_u = \lim_{\lambda(\Pi) \to 0} \sum_{[u,v]} \left(\hat{f}(Y_u) \cdot \hat{X}_{uv} + (\hat{f}(Y_\cdot))'_u \cdot \mathbb{X}_{uv}\right) =$$

$$= \lim_{\lambda(\pi) \to 0} \sum_{[u,v]} \left(g(Y_u) \cdot (v-u) + f(Y_u) \cdot X_{uv} + (f(Y_\cdot))'_u \cdot \mathbb{X}_{uv} + \underline{O}(|u-v|^{1+\alpha}) \right)$$

$$= \int_0^t g(Y_u)\,du + \int_0^t f(Y_u)\,dX_u .$$