Мат-ка

2008

2011 год
3-4 занятие.

-3.1-

1. Основы тенз. алгебры, тенз. анализа

$\bar x = \bar x(\xi)$ — вз. однозначн., дост. гладкая

$\det\left(\frac{\partial x^i}{\partial \xi^j}\right) \neq 0$

(1) $\bar\partial_i = \frac{\partial \bar x}{\partial \xi^i}$

$\bar\partial_i,\ i=1,2,3$ — базис

(2) $g_{ij} = \bar\partial_i \bar\partial_j$, $g=\det(g_{ij}) = \left[\det\left(\frac{\partial x^i}{\partial\xi^j}\right)\right]^2 \neq 0 \Rightarrow$

$\exists\, (g_{ij})^{-1} = (g^{ij})$

(3) $\bar\partial^i = g^{ij}\bar\partial_j$, $\{\bar\partial^i\}$ — взаимн. б-с

(ковектор)

$\bar a = a^i\bar\partial_i = a_i\bar\partial^i$, $a^i = g^{ij}a_j$

Аналогично вводим тензоры. $\bar\partial_i\otimes\bar\partial_j$ ($\bar e_k\otimes\bar e_m$ — ?)

$\hat T = T^{ij}\,\bar\partial_i\otimes\bar\partial_j = T_{ij}\,\bar\partial^i\otimes\bar\partial^j = \dots$

Рассм. 2 с.к. $\xi^i,\ \xi^{i'}$

(4) $T^{i'j'} = T^{ij}\frac{\partial\xi^{i'}}{\partial\xi^i}\frac{\partial\xi^{j'}}{\partial\xi^j}$, и т.д.

$\frac{\partial a^i}{\partial \xi^j}$ — тензор?

$$\frac{\partial}{\partial\xi^j}(a^i\bar\partial_i) = \frac{\partial a^i}{\partial\xi^j}\bar\partial_i + a^i\overbrace{\frac{\partial\bar\partial_i}{\partial\xi^j}}^{\Gamma_{ij}^k\bar\partial_k} = \frac{\partial a^i}{\partial \xi^j}\bar\partial_i + a^i\Gamma_{ij}^k\bar\partial_k =$$

$$= \underbrace{\left(\frac{\partial a^k}{\partial\xi^j} + a^i\Gamma_{ij}^k\right)}_{\nabla_j a^k}\bar\partial_k$$

$\nabla_j a^k$ — ковар-я произв-я

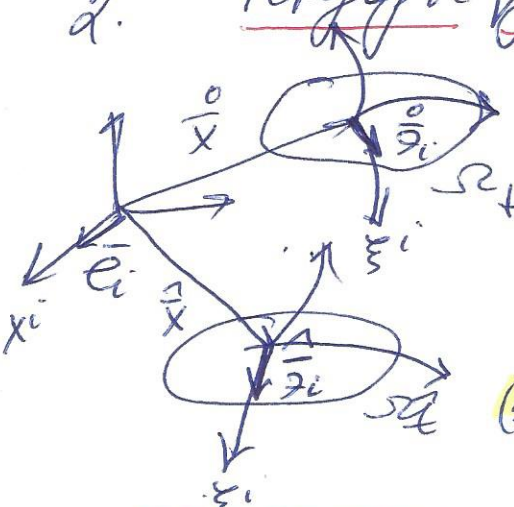
аналогично

$$\nabla_j a_k = \frac{\partial a_k}{\partial\xi^j} - a_i\Gamma_{kj}^i$$

$$\nabla_i T^{jk} = \frac{\partial T^{jk}}{\partial\xi^i} + T^{mk}\Gamma_{mi}^j + T^{jm}\Gamma_{mi}^k$$

Д/з 3.43 $\nabla_k g_{ij} = 0$

2. Тензоры и меры деформации.



$\mathring{\vec\partial}_i = \dfrac{\partial \mathring{\vec x}}{\partial \xi^i}$, $\hat{\vec\partial}_i = \dfrac{\partial \hat{\vec x}}{\partial \xi^i}$

$\mathring g_{ij} = \mathring{\vec\partial}_i \cdot \mathring{\vec\partial}_j$, $\hat g_{ij} = \hat{\vec\partial}_i \cdot \hat{\vec\partial}_j$

(5) $\boxed{2\varepsilon_{ij} := \hat g_{ij} - \mathring g_{ij}}$ — комп. т-ра деформации. В каком б-се?

(6) $\hat{\mathcal{E}} = \varepsilon_{ij}\, \mathring{\vec\partial}^i \otimes \mathring{\vec\partial}^j$ — т-р Грина

$\hat{E} = \varepsilon_{ij}\, \hat{\vec\partial}^i \otimes \hat{\vec\partial}^j$ — т-р Альманзи

$$2\varepsilon_{ij} = \left(\frac{\partial \hat{\vec x}}{\partial \xi^i}\frac{\partial \hat{\vec x}}{\partial \xi^j} - \frac{\partial \mathring{\vec x}}{\partial \xi^i}\frac{\partial \mathring{\vec x}}{\partial \xi^j}\right) = \left(\frac{\partial \hat x^k}{\partial \xi^i}\frac{\partial \hat x^k}{\partial \xi^j} - \frac{\partial \mathring x^k}{\partial \xi^i}\frac{\partial \mathring x^k}{\partial \xi^j}\right)$$

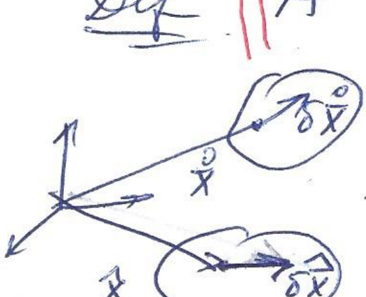
Положим $\xi^i = \mathring x^i \Rightarrow \mathring{\vec\partial}^i = \vec e_i$,

(7) $$\boxed{\varepsilon_{ij} = \frac{1}{2}\left(\frac{\partial \hat x^k}{\partial \mathring x^i}\frac{\partial \hat x^k}{\partial \mathring x^j} - \delta_{ij}\right)}$$ — комп. т-ра Грина в ДСК

Положим $\xi^i = \hat x^i \Rightarrow \hat{\vec\partial}_i = \vec e_i$

(8) $$\boxed{\varepsilon_{ij} = \frac{1}{2}\left(\delta_{ij} - \frac{\partial \mathring x^k}{\partial \hat x^i}\frac{\partial \mathring x^k}{\partial \hat x^j}\right)}$$ — комп. т-ра А-зи в ДСК

Def $\hat A = \left(\dfrac{\partial \hat x^i}{\partial \mathring x^j}\right)$ — аффинор деф-ции



Пусть $\hat{\vec x} = \hat{\vec x}(\mathring{\vec x}, t)$ — уравнение движения

$$\delta\hat{\vec x} = \hat{\vec x}(\mathring{\vec x} + \delta\mathring{\vec x}, t) - \hat{\vec x}(\mathring{\vec x}, t) = \frac{\partial \hat{\vec x}}{\partial \mathring x^i}\delta \mathring x^i + \vec o(|\delta\mathring{\vec x}|)$$

$\delta\mathring{\vec x} \to 0 \Rightarrow \delta\hat{\vec x} = \dfrac{\partial \hat{\vec x}}{\partial \mathring x^i}\delta\mathring x^i$ или (9) $\boxed{\delta\hat{\vec x} = \hat A\, \delta\mathring{\vec x}}$ осн. ф-ла теории деф-ции

$\Rightarrow$ (9) $\hat{\varepsilon} = \frac{1}{2}\left(\hat{A}^{T}\hat{A} - \hat{I}\right)$ – тр Грина (ДСК)

(10) $\hat{E} = \frac{1}{2}\left(\hat{I} - \hat{A}^{-1T}\hat{A}^{-1}\right)$ – тр А-ди (ДСК)

(11) $\hat{A}^{T}\hat{A} = \hat{C}$ – мера деф. Коши

(12) $\hat{A}^{-1T}\hat{A}^{-1} = \hat{B}$ – мера деф. Коши-Альм-ди

$\Rightarrow \hat{\varepsilon} = \frac{1}{2}(\hat{C} - \hat{I})$, $\hat{E} = \frac{1}{2}(\hat{I} - \hat{B})$.

2012 восн. 2.

Введем (13) $\boxed{\lambda := \frac{|\delta \vec{x}|}{|\delta \mathring{\vec{x}}|}}$ – кратность удлинения волокна.

$|\delta \vec{x}|^2 = (\hat{A}\delta\mathring{\vec{x}})^T \hat{A}\delta\mathring{\vec{x}} = \delta\mathring{\vec{x}}^T \hat{A}^T\hat{A}\,\delta\mathring{\vec{x}} = \delta\mathring{\vec{x}}^T\hat{C}\,\delta\mathring{\vec{x}}$

$\lambda^2 = \mathring{\vec{l}}^T\hat{C}\mathring{\vec{l}}$, $\mathring{\vec{l}} = \frac{\delta\mathring{\vec{x}}}{|\delta\mathring{\vec{x}}|}$

или

$\lambda^2 = \mathring{\vec{l}}^T(2\hat{\varepsilon} + \hat{I})\mathring{\vec{l}}$

Возьмем $\delta\mathring{\vec{x}} = \delta x^{\alpha}\vec{e}_{\alpha} \Rightarrow \mathring{\vec{l}} = \vec{e}_{\alpha} \Rightarrow$

$\lambda_{\alpha}^2 = C_{\alpha\alpha} \Rightarrow \lambda_{\alpha} = \sqrt{C_{\alpha\alpha}} = \sqrt{2\varepsilon_{\alpha\alpha} + 1}$

Если $\|\hat{\varepsilon}\| \ll 1 \Rightarrow \lambda \approx \varepsilon_{\alpha\alpha} + 1$

Относит. удлинение $e := \frac{|\delta\vec{x}| - |\delta\mathring{\vec{x}}|}{|\delta\mathring{\vec{x}}|} = \lambda - 1$

$\Rightarrow e_{\alpha} = \sqrt{C_{\alpha\alpha}} - 1 = \sqrt{2\varepsilon_{\alpha\alpha} + 1} - 1 \approx \varepsilon_{\alpha\alpha}$.

$\rightarrow$ диагон. эл-ты $\hat{\varepsilon}$ – относит. удлинения волокна, до деф-ции лежащего по соотв-й оси ДСК.

!! Д/З 4.1, 4.2, 4.3 (Эглит)

$$a^{i'} = a^{i}\frac{\partial \xi^{i'}}{\partial \xi^{i}}$$

$$\frac{\partial a^{i'}}{\partial \xi^{j'}} = \frac{\partial (a^{i'})}{\partial \xi^{j}}\frac{\partial \xi^{j}}{\partial \xi^{j'}} = \frac{\partial \left(a^{i}\frac{\partial \xi^{i'}}{\partial \xi^{i}}\right)}{\partial \xi^{j}}\frac{\partial \xi^{j}}{\partial \xi^{j'}} =$$

$$= \left(a^{i}\frac{\partial^{2} \xi^{i'}}{\partial \xi^{i}\partial \xi^{j}} + \frac{\partial a^{i}}{\partial \xi^{j}}\frac{\partial \xi^{i'}}{\partial \xi^{j}}\right)\frac{\partial \xi^{j}}{\partial \xi^{j'}}$$