Introduction to modern Game Theory

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Part 3

Game-Theoretic approach to financial mathematics.
Pricing derivative securities.

Chapter 1. Preliminaries: geometric theory of risk-neutral measures.

Chapter 2. Game-theoretic origins of risk-neutral laws.

Chapter 3. Rainbow options in discrete times.

Chapter 4. Continuous time limit: generalized Black-Scholes equations.

Chapter 5. Pricing credit derivatives.

Part 3

Chapter 6. Comparison with the standard probabilistic approach.

Chapter 7. Dynkin's games and game options.

Exposition of Chapters 1-5 is taken essentially from
V. N. Kolokoltsov, O. A. Malafeyev. Understanding Game Theory. World Scientific, Second Edition 2020,
V. N. Kolokoltsov. Game theoretic analysis of incomplete markets: emergence of probabilities, nonlinear and fractional Black-Scholes equations. Risk and Decision Analysis **4** (2013), 131-161.
P. Bernhard, J. Engwerda, B. Roorda, J.M. Schumacher, V. N. Kolokoltsov, P. Saint-Pierre and J.-P. Aubin. The Interval Market Model in Mathematical Finance: Game-Theoretic Methods. Birkhäuser, 2012.

Chapter 1

Preliminaries: geometric theory of risk-neutral measures.

Barycenter, convexity, risk-neutral measures, extreme points for the sets of measures, positively complete sets, arbitrage opportunities (geometric formulation), fundamental theorem of option pricing.

Notations

For a compact metric space E we denote by $\mathcal{P}(E)$ the set of probability laws on E and by $C(E)$ the Banach space of bounded continuous functions on E. For our purpose here we shall mostly need finite subsets $E = \{\xi_1, \cdots, \xi_k\}$ of $\mathbf{R}^d$ (in which case probability laws are given by the sets of positive numbers $\{p_1, \cdots, p_k\}$ toting up to one).

For $f \in C(E)$, $\mu \in \mathcal{P}(E)$, the standard pairing is given by the integration (probabilistically expectation):

$$(f, \mu) = \int_E f(x)\mu(dx) = \mathbf{E}_\mu f.$$

This pairing also extends to vector valued functions f.

Recall that $\mathcal{P}(E)$ is a compact set in its weak topology.

Definitions

A probability law $\mu \in \mathcal{P}(E)$ on $E \subset \mathbf{R}^d$ is *risk-neutral* (with respect to the origin), if the origin is its barycenter: $\int_E \xi \mu(d\xi) = 0$.

The set of all risk-neutral laws on E will be denoted by $\mathcal{P}_{rn}(E)$.

For a compact subset $E \subset \mathbf{R}^n$ and a continuous mapping $F : E \to \mathbf{R}^d$ let

$$\mathcal{P}(E; F) = \{\mu \in \mathcal{P}(E) : (F, \mu) = \int F(x)\mu(dx) = 0\},$$

so that $\mathcal{P}_{rn}(E) = \mathcal{P}(E; Id)$.

Definitions

A subset $E \subset \mathbf{R}^d$ is called *weakly* (resp. *strongly*) *positively complete*, if there exists no $\omega \in \mathbf{R}^d$ such that $(\omega, \xi) > 0$ (resp. $(\omega, \xi) \geq 0$, $\omega \neq 0$) for all $\xi \in E$.

Geometrically: E does not belong to any open (respectively closed) half-space of $\mathbf{R}^d$.

E is strongly positively complete, if for any $\omega \in \mathbf{R}^d$ there exist vectors $\xi_1, \xi_2 \in E$ such that $(\omega, \xi_1) > 0$ and $(\omega, \xi_2) < 0$.

E is strongly positively complete, if $(\omega, \xi) \geq 0$ for all $\xi \in E$ implies $\omega = 0$.

Examples

A closed E is strongly (weakly) positively complete iff its convex hull so is.

If $E \subset \mathbf{R}^d$ is a compact convex set, then E is weakly positively complete if and only if it contains the origin. It is moreover strongly positively complete whenever the origin is not its boundary point.

Examples. (i) Weakly (not strongly) positively complete:

$$\begin{array}{c} \uparrow \\ \longleftarrow \cdot \longrightarrow \end{array}$$

(ii) Weakly positively complete in $\mathbf{R}^2$ and strongly positively complete in $\mathbf{R}$:

$$\longleftarrow \cdot \longrightarrow$$

Arbitrage opportunities: setting without probability

A set $E \subset \mathbf{R}^d$ is called *arbitrage-free*, if $(\omega, \xi) \geq 0$ for all $\xi \in E$ implies $(\omega, \xi) = 0$ for all $\xi \in E$.

A vector $\omega \in \mathbf{R}^d$ is an *arbitrage opportunity* for a set E if $(\omega, \xi) \geq 0$ for all $\xi \in E$ and $(\omega, \xi) > 0$ for some $\xi \in E$. Thus $E \subset \mathbf{R}^d$ is arbitrage-free iff arbitrage opportunities do not exist.

If E spans the whole $\mathbf{R}^d$, then E is arbitrage-free iff E is strongly positively complete.

Examples above: (i) not arbitrage-free, (ii) arbitrage-free both in $\mathbf{R}^2$ and $\mathbf{R}$.

Arbitrage opportunities: probabilistic setting

Let Y be a random vector in $\mathbf{R}^d$ (defined on some probability space $(\Omega, \mathcal{F}, \mathbf{P})$).
Y is called *arbitrage-free*, if $(\omega, Y) \geq 0$ a.s. implies $(\omega, Y) = 0$ a.s.

A vector $\omega \in \mathbf{R}^d$ is an *arbitrage opportunity* for Y if $(\omega, Y) \geq 0$ a.s. and $\mathbf{P}((\omega, Y) > 0) > 0$.
Thus Y is arbitrage-free iff arbitrage opportunities do not exist.

If the range $E \subset \mathbf{R}^d$ of Y is finite, and all values are taken with positive probabilities, then Y is arbitrage-free iff E is arbitrage-free.

Existence and support

Proposition 1. Let $E \subset \mathbf{R}^n$ be a compact set.

(i) The set $\mathcal{P}(E; F)$ is not empty if and only if $F(E)$ is weakly positively complete in $\mathbf{R}^d$. (This is the *fundamental theorem* of option pricing in geometric form without a priori probability.)

(ii) Let E' be the support of a measure $\mu \in \mathcal{P}(E; F)$. If $F(E')$ does not coincide with the origin, then it is strongly positively complete in the subspace $\mathbf{R}^m \subset \mathbf{R}^d$ generated by $F(E')$ (and thus is arbitrage-free there).

(iii) Let $F(E)$ be weakly positively complete in $\mathbf{R}^d$. Then either the support of any $\mu \in \mathcal{P}(E; F)$ is contained in $F^{-1}(0)$, or there exists a subspace $\mathbf{R}^m \subset \mathbf{R}^d$ such that $F(E) \cap \mathbf{R}^m$ is strongly positively complete in $\mathbf{R}^m$ and any $\mu \in \mathcal{P}(E; F)$ has a support in $F^{-1}(\mathbf{R}^m)$.

Proof (i)

Let $F(E)$ be not weakly positively complete, so $\exists \omega \in \mathbf{R}^d : (\omega, F(x)) > 0$ for all $x \in E$. Then

$$(\omega, \int F(x)\mu(dx)) > 0 \qquad (1)$$

for any $\mu \in \mathcal{P}(E)$, which cannot hold for $\mu \in \mathcal{P}(E; F)$.

Conversely, suppose $\mathcal{P}(E; F) = \emptyset$. Then the image of $\mathcal{P}(E)$ under

$$\mu \to \int F(x)\mu(dx)$$

does not contain the origin and (by the Banach separation theorem) $\exists \omega \in \mathbf{R}^d$:

$$(\omega, \int F(x)\mu(dx)) > 0$$

for all $\mu \in \mathcal{P}(E)$. In particular, $(\omega, F(x)) > 0$ for all $x \in E$.

Proof (ii)

Suppose $F(E') \neq \{0\}$, so that it generates a subspace $\mathbf{R}^m \subset \mathbf{R}^d$ with $m > 0$. Then for any $\omega \in \mathbf{R}^m$, there exists $x \in E'$ such that $(\omega, F(x)) \neq 0$. But since

$$(\omega, \int F(x)\mu(dx)) = 0,$$

there must exist another $y \in E'$ such that the signs of $(\omega, F(y))$ and $(\omega, F(x))$ are different.

Proof (iii)

Suppose $F(E)$ is weakly positively complete. Then either $F(E)$ is strongly positively complete (and we are done), or there exists $\omega \in \mathbf{R}^d$ such that $(\omega, F(x)) \geq 0$ for all $x \in E$ and the set $E_1 = \{x \in E : (\omega, F(x)) = 0\}$ is non-empty.

Let Π be the subspace generated by $F(E_1)$. As it belongs to the orthogonal subspace to ω, its dimension is strictly less than d. Moreover, the support of any $\mu \in \mathcal{P}(E; F)$ is contained in E_1, so that $\mathcal{P}(E; F) = \mathcal{P}(E_1; F)$.

As this set is not empty, $F(E_1)$ is weakly positively complete in $\mathbf{R}^m$.

The proof is now completed by induction in the dimension d.

Fundamental theorem of option pricing (geometric formulation)

Theorem. A random vector Y in $\mathbf{R}^d$ on a probability space $(\Omega, \mathcal{F}, \mathbf{P})$ is arbitrage-free iff there exists a risk neutral measure Q on $(\Omega, \mathcal{F})$ that is equivalent to $\mathbf{P}$ (the latter meaning that $\mathbf{P}(A) = 0$ iff $Q(A) = 0$).

If Y has a finite range $E \subset \mathbf{R}^d$ with E spanning the whole $\mathbf{R}^d$ and with all values having positive probabilities this is equivalent to

Theorem. Let a finite set $E \subset \mathbf{R}^d$ spans the whole $\mathbf{R}^d$. Then E is strongly positively complete (or equivalently arbitrage-free) iff there exists a risk-neutral measure on E with the full support (with the support equal to E). Equivalently: there exists a risk-neutral measure on E with the full support iff the origin is an interior point of the convex hull CE of E.

Fundamental theorem of option pricing: proof

By the previous Proposition claim (iii) the existence of a risk-measure with a full support implies that E is strongly positively complete.

To prove the converse statement assume that E is strongly positively complete. It is sufficient to show that for any $\xi \in E$ there exists a risk-neutral probability on E with a support containing ξ.
Since zero is an interior point of CE, for any $\xi \in E$ there exists $a > 0$ such that $-a\xi$ belongs to the interior point of a face of CE. This face lies in a convex hull of $E \setminus \xi$. Therefore

$$-a\xi = \sum_{\eta \in (E \setminus \xi)} a_\eta \eta.$$

Thus $(a, \{a_\eta\})$ is a risk-neutral probability with a support containing ξ.

Extreme points

Proposition 2. Let $E \subset \mathbf{R}^n$ be compact,
$F = (F^1, \cdots, F^d) : E \to \mathbf{R}^d$, and μ an extreme point of the set $\mathcal{P}(E; F)$.
Then μ is a linear combination of not more than $d+1$ Dirac measures.

Proof. Assuming otherwise, $\exists$ a partition of E in $d+2$ Borel subsets $E = E_1 \cup \cdots \cup E_{d+2}$ that are pairwise disjoint with $\mu(E_i) > 0$ for all i. Then

$$(F^i, \mu) = \sum_{j=1}^{d+2} \int_{E_j} F^i(x)\mu(dx) = 0, \quad i = 1, \cdots, d.$$

Extreme points

Proof (cont.) For any collection of $(d+1)$ numbers $\epsilon_i \in (-1,1)$, let

$$\mu_+ = \sum_{j=1}^{d+2} (1+\epsilon_j)\mu|_{E_j}, \quad \mu_- = \sum_{j=1}^{d+2} (1-\epsilon_j)\mu|_{E_j}.$$

Clearly, $\mu_\pm$ are positive and $\mu = (\mu_+ + \mu_-)/2$. To show that μ is not an extreme point of $\mathcal{P}(E;F)$ it is sufficient to show that there exists a family of ϵ_j such that $\mu_\pm \in \mathcal{P}(E;F)$, that is

$$\sum_{j=1}^{d+2} \epsilon_j \mu(E_j) = 0,$$

$$\sum_{j=1}^{d+2} \epsilon_j \int_{E_j} F^i(x)\mu(dx) = 0, \quad i = 1, \cdots, d,$$

a homogeneous system of $d+1$ linear equations for $d+2$ variables, thus solution space has a positive dimension.

Extreme points

It follows: the set of risk-neutral laws $\mathcal{P}_{rn}(E)$ on a compact subset $E \subset \mathbf{R}^d$ is not empty if and only if E is weakly positively complete, and the support of any risk-neutral law is strongly positively complete in the subspace it generates.

If a finite $E = \{\xi_1, \cdots, \xi_k\} \subset \mathbf{R}^d$ is strongly positively complete, then $k > d$ (by the Banach separation theorem). Minimal families contain precisely $d + 1$ vectors.

Let us say that a finite family of vectors $E = \{\xi_1, \cdots, \xi_k\}$ in $\mathbf{R}^d$ is in *general position* if the vectors of any subset of $\{\xi_1, \cdots, \xi_k\}$ of size d are linearly independent.

Risk neutral measures on $d+1$ points

Theorem 1. Let a finite set $E = \{\xi_1, \cdots, \xi_{d+1}\}$ be strongly positively complete in $\mathbf{R}^d$. Then

(i) the family E is in general position;

(ii) the origin belongs to the interior of the simplex $\Pi[\xi_1, \cdots, \xi_{d+1}]$, defined as the convex hull of the family $\{\xi_1, \cdots, \xi_{d+1}\}$;

(iii) there exists a unique risk-neutral probability law $\{p_1, \cdots, p_{d+1}\}$ on $\{\xi_1, \cdots, \xi_{d+1}\}$, and ...

Risk neutral measures on $d+1$ points

$$p_i = C^{-1}(-1)^{i-1}\det\begin{pmatrix} \xi_1^1 & \cdots & \xi_{i-1}^1 & \xi_{i+1}^1 & \cdots & \xi_{d+1}^1 \\ \xi_1^2 & \cdots & \xi_{i-1}^2 & \xi_{i+1}^2 & \cdots & \xi_{d+1}^2 \\ & & \cdots & & & \\ \xi_1^d & \cdots & \xi_{i-1}^d & \xi_{i+1}^d & \cdots & \xi_{d+1}^d \end{pmatrix}, \tag{2}$$

for $i = 1, \cdots, d$, where

$$C = \det\begin{pmatrix} 1 & 1 & \cdots & 1 \\ \xi_1^1 & \xi_2^1 & \cdots & \xi_{d+1}^1 \\ & & \cdots & \\ \xi_1^d & \xi_2^d & \cdots & \xi_{d+1}^d \end{pmatrix}. \tag{3}$$

Risk neutral measures on $d+1$ points

Proof.

(i) If the vectors $\xi_1, \cdots \xi_d$, say, are dependent, they generate a proper subspace Π of $\mathbf{R}^d$. Then, for any $\omega \in \mathbf{R}^d$ that is orthogonal to Π we shall have either $(\omega, \xi_i) \geq 0$ for all i or $(\omega, \xi_i) \leq 0$ for all i (depending on the position of the vector ξ_{d+1}), contradicting the strong positive completeness.

(ii) Banach separation.

(iii) The existence and uniqueness follows from above.

Risk neutral measures on $d+1$ points

Proof (cont).

Let us consider the vector-valued determinant

$$D = \det \begin{pmatrix} \xi_1 & \xi_2 & \cdots & \xi_{d+1} \\ \xi_1^1 & \xi_2^1 & \cdots & \xi_{d+1}^1 \\ \xi_1^2 & \xi_2^2 & \cdots & \xi_{d+1}^2 \\ & & \cdots & \\ \xi_1^d & \xi_2^d & \cdots & \xi_{d+1}^d \end{pmatrix},$$

defined via its expansion with respect to the first row. Clearly $D = 0$, as each of its coordinates is given by a determinant with two coinciding rows. Hence, expanding it with respect to the first row yields

$$\sum_{i=1}^{d+1} p_i \xi_i = 0,$$

with p_i given by (2).

Risk neutral measures on $d+1$ points

Corollary.
Let a family $E = \{\xi_1, \cdots, \xi_k\}$ be strongly positively complete and in general position. Then the extreme points of the convex set of risk-neutral probabilities on $\{\xi_1, \cdots, \xi_k\}$ are risk-neutral probabilities with supports on strongly positively complete subsets of E of size precisely $d+1$.

General structure of extreme points

Theorem 2. Let a compact set $E \subset \mathbf{R}^d$ be strongly positively complete. Then the extreme points of the set of risk-neutral probabilities on E are the Dirac mass at zero (only when E contains the origin) and the risk-neutral measures with support on families of size $m + 1$, $0 < m \leq d$, that generate a subspace of dimension m and are strongly positively complete in this subspace.

Proof. It is clear that the laws specified in the Proposition are extreme points. Conversely, assume the support E' of an extreme risk-neutral measure contains $m + 1 > 0$ points with $m > 0$. Then $m \leq d$. If E' generates a subspace of dimension less than m, then again it cannot support an extreme risk-neutral measure. Finally, by (ii) of Proposition 1, E' is strongly positively complete in the subspace it generates.

Chapter 2

Game-theoretic origins of risk-neutral laws.

Game with costs linear in one of the strategy parameters, minimax in terms of the risk-neutral laws, minimization of the Legendre transform of convex functions.

Games linear in one control parameter

Objective: calculate

$$\Pi[E](f) = \inf_{\gamma\in\mathbf{R}^d} \sup_{\xi\in E}[f(\xi) - (\xi,\gamma)] \tag{4}$$

and the minimizing γ. More generally, for a compact metric space E and continuous functions $f : E \to \mathbf{R}$, $g : E \to \mathbf{R}^d$, find

$$\Pi[E](f;g) = \inf_{\gamma\in\mathbf{R}^d} \sup_{\xi\in E}[f(\xi) - (g(\xi),\gamma)]. \tag{5}$$

However, this problem can be reduced to the previous, because

$$\Pi[E](f;g) = \inf_{\gamma\in\mathbf{R}^d} \sup_{\eta\in g(E)} [\max_{\xi\in g^{-1}(\eta)} f(\xi) - (\eta,\gamma)]. \tag{6}$$

Similar problems with γ from bounded subset of $\mathbf{R}^d$.

Games linear in one control parameter

We start with a finite set E having precisely $d+1$ points.

A remarkable fact: in this case expression (4) depends linearly on f and the minimizing γ is unique and also depends linearly on f.

Start with a three-point set E in $\mathbf{R}^2$:

$$\Pi[\xi_1, \xi_2, \xi_3](f) = \min_{\gamma \in \mathbf{R}^2} \max_{\xi_1, \xi_2, \xi_3} [f(\xi_i) - (\xi_i, \gamma)], \qquad (7)$$

assuming that the set $E = \{\xi_1, \xi_2, \xi_3\}$ is strongly positively complete in $\mathbf{R}^2$ and thus arbitrage-free there.

Three points problem

Suppose the min in (7) is attained on a vector γ_0 and the corresponding max on a certain ξ_i. Suppose this max is unique, so that

$$f(\xi_i) - (\xi_i, \gamma_0) > f(\xi_j) - (\xi_j, \gamma_0) \tag{8}$$

for all $j \neq i$. As $\xi_i \neq 0$, changing γ_0 by a small amount we can reduce the l.h.s. of (8) by preserving the inequality (8). This possibility contradicts the assumption that γ_0 is a minimal point. Hence, if γ_0 is a minimal point, the corresponding maximum must be attained on at least two vectors.

Three points problem

Suppose it is attained on precisely two vectors, that is

$$f(\xi_i) - (\xi_i, \gamma_0) = f(\xi_j) - (\xi_j, \gamma_0) > f(\xi_m) - (\xi_m, \gamma_0) \qquad (9)$$

for some different i, j, m. Since the angle between ξ_i, ξ_j is strictly less than π (assumed general position), adding a vector

$$\epsilon(\xi_i/|\xi_j| + \xi_j/|\xi_i|)$$

to γ_0 will reduce simultaneously first two expressions from the l.h.s. of (9), but preserve (for small enough ϵ) the inequality on the r.h.s. of (9). This again contradicts the assumption that γ_0 is a minimal point.

Three points problem

Hence, if γ_0 is a minimal point, it must satisfy the equation

$$f(\xi_1) - (\xi_1, \gamma) = f(\xi_2) - (\xi_2, \gamma) = f(\xi_3) - (\xi_3, \gamma), \qquad (10)$$

which is equivalent to the system

$$\begin{cases} (\xi_2 - \xi_1, \gamma_0) = f(\xi_2) - f(\xi_1), \\ (\xi_3 - \xi_1, \gamma_0) = f(\xi_3) - f(\xi_1). \end{cases} \qquad (11)$$

The vectors $\xi_2 - \xi_1, \xi_3 - \xi_1$ are independent. Hence system (11) has a unique solution γ_0.

Three points problem

For a pair of vectors $u, v \in \mathbf{R}^2$, let $D(u, v)$ denote the oriented area of the parallelogram built on u, v and $R(u)$ the result of the rotation of u on 90° anticlockwise. That is, for $u = (u^1, u^2)$, $v = (v^1, v^2)$,

$$D(u, v) = u^1 v^2 - u^2 v^1, \quad R(u) = (u^2, -u^1).$$

Notice that the determinant of system (11) is

$$D(\xi_2 - \xi_1, \xi_3 - \xi_1) = D(\xi_2, \xi_3) + D(\xi_3, \xi_1) + D(\xi_1, \xi_2),$$

Three points problem

By the standard formulas of linear algebra, the unique solution γ_0 is

$$\gamma_0 = \frac{f(\xi_1)R(\xi_2 - \xi_3) + f(\xi_2)R(\xi_3 - \xi_1) + f(\xi_3)R(\xi_1 - \xi_2)}{D(\xi_2, \xi_3) + D(\xi_3, \xi_1) + D(\xi_1, \xi_2)}, \tag{12}$$

and the corresponding optimal value

$$\Pi[\xi_1, \xi_2, \xi_3](f) = \frac{f(\xi_1)D(\xi_2, \xi_3) + f(\xi_2)D(\xi_3, \xi_1) + f(\xi_3)D(\xi_1, \xi_2)}{D(\xi_2, \xi_3) + D(\xi_3, \xi_1) + D(\xi_1, \xi_2)}. \tag{13}$$

Hence we arrive at the following.

Three points problem

Proposition 1. Let $E = \{\xi_1, \xi_2, \xi_3\}$ be arbitrage-free (or strongly positively complete) in $\mathbf{R}^2$, and let $f(\xi_1), f(\xi_2), f(\xi_3)$ be arbitrary numbers. Then expression (7) is given by (13) and the minimum is attained on the single γ_0 given by (12).

Proof. Our discussion above shows that if γ_0 is a minimum point, then it is unique and given by (12). It remains to observe that a minimal point does exist, because

$$\max_{\xi_1,\xi_2,\xi_3} [f(\xi_i) - (\xi_i, \gamma)] \to \infty,$$

as $\gamma \to \infty$.

Three points problem

Corollary. Expression (13) can be written equivalently as

$$\Pi[\xi_1, \xi_2, \xi_3](f) = \mathbf{E} f(\xi),$$

where the expectation is defined with respect to the unique risk neutral probability law $\{p_1, p_2, p_3\}$ on ξ_1, ξ_2, ξ_3:

$$p_i = \frac{D(\xi_j, \xi_m)}{D(\xi_2, \xi_3) + D(\xi_3, \xi_1) + D(\xi_1, \xi_2)}$$

((i, j, k) is either (1,2,3) or (2,3,1) or (3,1,2)).

Problem of $d+1$ points

Aim: to extend to strongly positively complete sets $E = \{\xi_1, \cdots, \xi_{d+1}\}$ of $d+1$ point in $\mathbf{R}^d$:

$$\Pi[\xi_1, \cdots, \xi_{d+1}](f) = \min_{\gamma \in \mathbf{R}^d} \max_{i=1,\cdots,d+1} [f(\xi_i) - (\xi_i, \gamma)]. \tag{14}$$

Comments on orientation

Assume that $\mathbf{R}^d$ is equipped with the standard basis $e_1, \cdots, e_d$ fixing the orientation. Assume the vectors $\{\xi_2, \xi_3, \cdots, \xi_{d+1}\}$ form an oriented basis of $\mathbf{R}^d$.

Let $\{\hat{\xi}_i\}$ denote the ordered subset of $\xi_1, \cdots, \xi_{d+1}$ obtained by removing ξ_i. The basis $\{\hat{\xi}_i\}$ is oriented if and only if i is odd.

For instance, if $d = 3$, the oriented bases form the triples $\{\xi_2, \xi_3, \xi_4\}$, $\{\xi_1, \xi_2, \xi_4\}$, $\{\xi_1, \xi_4, \xi_3\}$ and $\{\xi_1, \xi_3, \xi_2\}$.

Problem of $d+1$ points

The same argument as for $d = 2$ leads us to the conclusion that a minimal point γ_0 must satisfy the equation

$$f(\xi_1) - (\xi_1, \gamma) = \cdots = f(\xi_{d+1}) - (\xi_{d+1}, \gamma), \tag{15}$$

which is equivalent to the system

$$(\xi_i - \xi_1, \gamma_0) = f(\xi_i) - f(\xi_1), \quad i = 2, \cdots, d+1. \tag{16}$$

Problem of $d+1$ points

For a collection of d vectors $u_1, \cdots, u_d \in \mathbf{R}^d$, let $D(u_1, \cdots, u_d)$ denote the oriented volume of the parallelepiped built on $u_1, \cdots, u_d$ and $R(u_1, \cdots, u_{d-1})$ the rotor of the family $(u_1, \cdots, u_{d-1})$:

$$D(u_1, \cdots, u_d) = \det \begin{pmatrix} u_1^1 & \cdots & u_1^d \\ u_2^1 & \cdots & u_2^d \\ & \cdots & \\ u_d^1 & \cdots & u_d^d \end{pmatrix},$$

$$R(u_1, \cdots, u_{d-1}) = \det \begin{pmatrix} e_1 & \cdots & e_d \\ u_1^1 & \cdots & u_1^d \\ & \cdots & \\ u_{d-1}^1 & \cdots & u_{d-1}^d \end{pmatrix}$$

Problem of $d+1$ points

Expanding

$$R(u_1, \cdots, u_{d-1}) = e_1 \det \begin{pmatrix} u_1^2 & \cdots & u_1^d \\ & \cdots & \\ u_{d-1}^2 & \cdots & u_{d-1}^d \end{pmatrix}$$

$$-e_2 \det \begin{pmatrix} u_1^1 & u_1^3 & \cdots & u_1^d \\ & \cdots & & \\ u_{d-1}^1 & u_{d-1}^3 & \cdots & u_{d-1}^d \end{pmatrix} + \cdots .$$

Problem of $d+1$ points

Finally, let us define a multi-linear operator $\tilde{R}$ from an ordered collection $\{u_1, \cdots, u_d\}$ of d vectors in $\mathbf{R}^d$ to $\mathbf{R}^d$:

$$\tilde{R}(u_1, \cdots, u_d) = R(u_2 - u_1, u_3 - u_1, \cdots, u_d - u_1)$$

$$= R(u_2, \cdots, u_d) - R(u_1, u_3, \cdots, u_d) + \cdots + (-1)^{d-1} R(u_1, \cdots, u_{d-1}).$$

Problem of $d+1$ points

The determinant of system (16) equals

$$D = D(\xi_2 - \xi_1, \cdots, \xi_{d+1} - \xi_1)$$

$$= \det \begin{pmatrix} \xi_2^1 - \xi_1^1 & \xi_2^2 - \xi_1^2 & \cdots & \xi_2^d - \xi_1^d \\ & \cdots & & \\ \xi_{d+1}^1 - \xi_1^1 & \xi_{d+1}^2 - \xi_1^2 & \cdots & \xi_{d+1}^d - \xi_1^d \end{pmatrix}.$$

Problem of $d+1$ points

Using the linear dependence of a determinant on columns, this rewrites as

$$D(\xi_2, \cdots, \xi_{d+1}) - \xi_1^1 \det \begin{pmatrix} 1 & \xi_2^2 & \cdots & \xi_2^d \\ & \cdots & & \\ 1 & \xi_{d+1}^2 & \cdots & \xi_{d+1}^d \end{pmatrix}$$

$$-\xi_1^2 \det \begin{pmatrix} \xi_2^1 & 1 & \xi_2^3 & \cdots & \xi_2^d \\ & \cdots & & & \\ \xi_{d+1}^1 & 1 & \xi_{d+1}^3 & \cdots & \xi_{d+1}^d \end{pmatrix} - \cdots .$$

Problem of $d+1$ points

Or

$$D = \sum_{i=1}^{d+1} (-1)^{i-1} D(\{\hat{\xi}_i\}) = \det \begin{pmatrix} 1 & \cdots & 1 \\ \xi_1^1 & \cdots & \xi_{d+1}^1 \\ & \cdots & \\ \xi_1^d & \cdots & \xi_{d+1}^d \end{pmatrix}. \quad (17)$$

Problem of $d + 1$ points

Notice that according to the orientation specified above, $D(\{\hat{\xi}_i\})$ are positive (resp. negative) for odd i (resp. even i), implying that all terms in (17) are positive, so that the collection of numbers

$$p_i = \frac{1}{D}(-1)^{i-1} D(\{\hat{\xi}_i\}) = \frac{(-1)^{i-1} D(\{\hat{\xi}_i\})}{D(\xi_2 - \xi_1, \cdots, \xi_d - \xi_1)}, \qquad (18)$$

with $i = 1, \cdots, d+1$, define a probability law on the set $\xi_1, \cdots, \xi_{d+1}$ with a full support. As one sees directly, these law is precisely the unique risk neutral law on $\{\xi_1, \cdots, \xi_{d+1}\}$ constructed above in Chapter 1.

Problem of $d + 1$ points

By linear algebra, the unique solution γ_0 to system (16) is given by the formulas

$$\gamma_0^1 = \frac{1}{D} \det \begin{pmatrix} f(\xi_2) - f(\xi_1) & \xi_2^2 - \xi_1^2 & \cdots & \xi_2^d - \xi_1^d \\ \cdots & & & \\ f(\xi_{d+1}) - f(\xi_1) & \xi_{d+1}^2 - \xi_1^2 & \cdots & \xi_{d+1}^d - \xi_1^d \end{pmatrix}, \tag{19}$$

$$\gamma_0^2 = \frac{1}{D} \det \begin{pmatrix} \xi_2^1 - \xi_1^1 & f(\xi_2) - f(\xi_1) & \cdots & \xi_2^d - \xi_1^d \\ \cdots & & & \\ \xi_{d+1}^1 - \xi_1^1 & f(\xi_{d+1}) - f(\xi_1) & \cdots & \xi_{d+1}^d - \xi_1^d \end{pmatrix}, \tag{20}$$

and similar for other γ_0^i.

Problem of $d+1$ points

For any i

$$f(\xi_i) - (\gamma_0, \xi_i) = \frac{1}{D}\sum_{i=1}^{d+1}[f(\xi_i)(-1)^{i+1}D(\{\hat{\xi}_i\})], \qquad (21)$$

and

$$\gamma_0 = \frac{1}{D}(f(\xi_2) - f(\xi_1))R(\xi_3 - \xi_1, \cdots, \xi_{d+1} - \xi_1)$$

$$-\frac{1}{D}(f(\xi_3) - f(\xi_1))R(\xi_2 - \xi_1, \xi_4 - \xi_1, \cdots, \xi_{d+1} - \xi_1)$$

$$+ \cdots + \frac{1}{D}(-1)^{d+1}(f(\xi_{d+1}) - f(\xi_1)R(\xi_2 - \xi_1, \cdots, \xi_d - \xi_1),$$

Problem of $d + 1$ points

This rewrites as

$$\gamma_0 = -\frac{1}{D}\big[f(\xi_1)\tilde{R}(\{\hat{\xi}_1\}) - f(\xi_2)\tilde{R}(\{\hat{\xi}_2\})$$

$$+ \cdots + (-1)^d f(\xi_{d+1})\tilde{R}(\{\hat{\xi}_{d+1}\}),\big]$$

or

$$\gamma_0 = -\frac{1}{D}\sum_{j=1}^{d}(-1)^{j-1} f(\xi_j)\tilde{R}(\{\hat{\xi}_j\}) = \sum_{j=1}^{d} p_j f(\xi_j)\frac{\tilde{R}(\{\hat{\xi}_j\})}{D(\{\hat{\xi}_j\})}. \tag{22}$$

Problem of $d + 1$ points

Theorem 1.

Let a family $\{\xi_1, \cdots, \xi_{d+1}\}$ in $\mathbf{R}^d$ be arbitrage-free (or strongly positively complete), and let $f(\xi_1), \cdots, f(\xi_{d+1})$ be arbitrary numbers. Then

$$\Pi[\xi_1, \cdots, \xi_{d+1}](f) = \frac{1}{D} \sum_{i=1}^{d+1} [f(\xi_i)(-1)^{i+1} D(\{\hat{\xi}_i\})] = \mathbf{E} f(\xi), \tag{23}$$

and the minimum in (14) is attained on the single γ_0 given by (22), or equivalently by

$$\gamma_0 = \mathbf{E} \left[f(\xi) \frac{\tilde{R}(\{\hat{\xi}\})}{D(\{\hat{\xi}\})} \right], \tag{24}$$

where the expectation is with respect to the probability law (18), which is the unique risk-neutral probability law on $\{\xi_1, \cdots, \xi_{d+1}\}$.

Problem of $d+1$ points: geometric meaning

Each term $(-1)^{i-1}D(\{\hat{\xi}_i\})$ in (17) equals $d!$ times the volume of the pyramid (polyhedron) with vertices $\{0 \cup \{\hat{\xi}_i\}\}$. The determinant D equals $d!$ times the volume of the pyramid $\Pi[\xi_1, \cdots, \xi_{d+1}]$ in the affine space $\mathbf{R}^d$ with vertices being the end points of the vectors ξ_i, $i = 1, \cdots, d+1$.

Consequently, formula (17) expresses the decomposition of the volume of the pyramid $\Pi[\xi_1, \cdots, \xi_{d+1}]$ into $d+1$ parts, the volumes of the pyramids $\Pi[\{0 \cup \{\hat{\xi}_i\}\}]$ obtained by sectioning from the origin, and the weights of the distribution (18) are the ratios of these parts to the whole volume.

Problem of $d+1$ points: geometric meaning

Furthermore, the magnitude of the rotor $R(u_1, \cdots, u_{d-1})$ equals the volume of the parallelepiped built on $u_1, \cdots, u_{d-1}$. Hence $\|\tilde{R}(\{\xi_i\})\|$ equals $(d-1)!$ times the volume (in the affine space $\mathbf{R}^d$) of the $(d-1)$-dimensional face of the pyramid $\Pi[\xi_1, \cdots, \xi_{d+1}]$ with vertices $\{\hat{\xi}_i\}$. Hence the magnitudes of the ratios $\tilde{R}(\{\hat{\xi}_i\})/D(\{\hat{\xi}_i\})$, playing the roles of weights in (24), are the ratios of the $(d-1)!$ times $(d-1)$-dimensional volumes of the bases of the pyramids $\Pi[\{0 \cup \{\hat{\xi}_i\}\}]$ to the $d!$ times their full d-dimensional volumes.

Consequently,

$$\frac{\|\tilde{R}(\{\hat{\xi}_i\})\|}{D(\{\hat{\xi}_i\})} = \frac{1}{h_i}, \tag{25}$$

where h_i is the length of the perpendicular from the origin to the affine hyperspace generated by the end points of the vectors $\{\hat{\xi}_i\}$.

Problem of $d+1$ points: geometric meaning

Hence

$$|\gamma_0| \le \|f\| \max_{i=1,\cdots,d+1} h_i^{-1}. \tag{26}$$

On the other hand,

$$|\gamma_0| \le \|f\| \sum_{i=1}^{d+1} p_i h_i^{-1},$$

and by the above geometric interpretation of $\{p_i\}$

$$|\gamma_0| \le \|f\| \frac{1}{d} \frac{S(\xi_1, \cdots, \xi_{d+1})}{V(\xi_1, \cdots, \xi_{d+1})}, \tag{27}$$

where $S(\xi_1, \cdots, \xi_{d+1})$ is the surface volume of the pyramid $\Pi[\xi_1, \cdots, \xi_{d+1}]$ (the sum of $(d-1)$-dimensional volumes of all its $d+1$ faces) and $V(\xi_1, \cdots, \xi_{d+1})$ is its volume.

General problem

Let us now move to general problem (4) starting with the case of a finite E, that is with the expression

$$\Pi[\xi_1, \cdots, \xi_k](f) = \inf_{\gamma \in \mathbf{R}^d} \max_i [f(\xi_i) - (\xi_i, \gamma)] \tag{28}$$

for a finite set $E = \{\xi_1, \cdots, \xi_k\}$ in $\mathbf{R}^d$.
Theorem 2. Let a family of vectors $\xi_1, \cdots, \xi_k$ in $\mathbf{R}^d$ be strongly positively complete. Then

$$\Pi[\xi_1, \cdots, \xi_k](f) = \max_\mu \mathbf{E}_\mu f(\xi), \tag{29}$$

where max is taken over all extreme points μ of risk-neutral laws on $\{\xi_1, \cdots, \xi_k\}$ and inf in (28) is attained on the corresponding γ given by Theorem 1.

General problem

Proof.

$$\Pi[\xi_1, \cdots, \xi_k](f) = \inf_{\gamma \in \mathbf{R}^d} \max_{p \in \mathcal{P}(E)} \sum_{i=1}^k p_i [f(\xi_i) - (\xi_i, \gamma)]$$

$$\geq \inf_{\gamma \in \mathbf{R}^d} \max_{p \in \mathcal{P}_{rn}(E)} \sum_{i=1}^k p_i [f(\xi_i) - (\xi_i, \gamma)] = \max_{\mu} \mathbf{E}_\mu f(\xi).$$

Moreover, as for the support E' of an extreme risk neutral law

$$\max_{\xi \in E'} [f(\xi) - (\xi, \gamma)] \to \infty,$$

as $\gamma \to \infty$, and hence also

$$\max_{i=1,\dots,k} [f(\xi_i) - (\xi_i, \gamma)] \to \infty,$$

the infinum in (28) is attained on some finite γ.

General problem

Assuming that γ_0 is such a minimum point, let E' denote the subset of all ξ, where the maximum is attained in the expression $f(\xi) - (\xi, \gamma_0)$. We can now conclude, as in the proof of Proposition 1, that E' is weakly positively complete in the subspace it generates. If E' is strongly positively complete in the subspace it generates, we are done.

If otherwise, then there exists ω such that $(\omega, \xi) \geq 0$ for all $\xi \in E'$ and $(\omega, \xi) = 0$ for some proper subset E'' of E'. Then for small enough ϵ, $\gamma_0' = \gamma_0 + \epsilon\omega$ is still a minimum point for (28), but the maximum $f(\xi) - (\xi, \gamma_0')$ is attained only on E''.

Following the same with E'', we can find a subset $\tilde{E}$ of E' and a minimal point $\tilde{\gamma}$ such that $\tilde{E}$ is the set of maximum points of $f(\xi) - (\xi, \tilde{\gamma})$ and either $\tilde{E} = \{0\}$ or it is strongly positively complete in the subspace it generates.

Remark

The max in (29) is attained on a family $\{\xi_i\}_{i\in I}$ if and only if

$$f(\xi_i) - (\gamma_I, \xi_i) \geq f(\xi_r) - (\gamma_I, \xi_r) \tag{30}$$

for any $i \in I$ and any r, where γ_I is the corresponding optimal value.

Consequently, on the convex set of functions f satisfying inequalities (30) for all r, the mapping $\Pi[\xi_1, \cdots, \xi_k](f)$ is linear:

$$\Pi[\xi_1, \cdots, \xi_k](f) = \mathbf{E}_I f(\xi).$$

General problem

It is often important to know the size of a region, where the minimizing γ in expression (28) belongs to. For $d+1$-point sets E such an estimate is given by (26). Let us supply an appropriate estimate for an arbitrary E. Let

$$\varkappa_0 = \varkappa_0(E) = \min_{\omega \in \mathbf{R}^d, |\omega|=1} \max_{\xi \in E}(-\xi, \omega). \tag{31}$$

Clearly $\varkappa_0 > 0$ for any strongly positively complete set E.

Proposition 2. Under the assumptions of Theorem 2, all minimizing γ in expression (28) satisfy the estimate

$$|\gamma| \le \operatorname{osc}(f)/\varkappa_0, \tag{32}$$

where

$$\operatorname{osc}(f) = \max_{\xi \in E} f(\xi) - \min_{\xi \in E} f(\xi).$$

In particular,

$$\Pi[\xi_1, \cdots, \xi_k](f) = \min_{|\gamma| \le osc(f)/\varkappa_0} \max_i [f(\xi_i) - (\xi_i, \gamma)]. \tag{33}$$

General problem

Proof. For any γ

$$\max_i [f(\xi_i) - (\xi_i, \gamma)] \geq \min_i f(\xi_i) + |\gamma| \varkappa_0.$$

On the other hand

$$\Pi[\xi_1, \cdots, \xi_k](f) \leq \max_i f(\xi_i).$$

Therefore, if infinum in (28) is attained on a γ,

$$\min_i f(\xi_i) + |\gamma| \varkappa_0 \leq \max_i f(\xi_i),$$

implying (32).

A mirror result

Theorem 3. Under the assumptions of Theorem 2 the expression

$$\underline{\Pi}[\xi_1, \cdots, \xi_k](f) = \sup_{\gamma \in \mathbf{R}^d} \min_{\xi_1, \cdots, \xi_k} [f(\xi_i) + (\xi_i, \gamma)] \tag{34}$$

can be evaluated by the formula

$$\underline{\Pi}[\xi_1, \cdots, \xi_k](f) = \min_{\mu} \mathbf{E}_\mu f(\xi), \tag{35}$$

where min is taken over all extreme points μ of risk-neutral laws on $\{\xi_1, \cdots, \xi_k\}$. Moreover, sup in (34) is attained on some γ and all maximizing γ satisfy (32).

General problem for infinite sets

Theorem 4. Let a compact set $E \subset \mathbf{R}^d$ be strongly positively complete. Then

$$\Pi[E](f) = \max_{\mu} \mathbf{E}_{\mu} f(\xi), \tag{36}$$

where max is taken over all extreme points μ of risk-neutral laws on E, inf in (28) is attained on some γ, and all minimizing γ satisfy (32).

Proof

As above we show that the l.h.s. of (29) is bounded from below by its r.h.s., which is finite and equals to some $\mathbf{E}_{\mu_0} f(\xi)$. By Prop. 2, the support of μ_0 is a finite set $E' = \{\xi_1, \cdots \xi_k\}$ with $k \leq d + 1$. Let $\{E_n\}$ be an increasing sequence of $1/n$-nets in E containing E' and such that $\varkappa_0(E_n)$ converge to $\varkappa_0(E)$ as $n \to \infty$. By Theorem 2,

$$\Pi[E_n](f) = \max_{\mu} \mathbf{E}_{\mu} f(\xi) = \mathbf{E}_{\mu_0} f(\xi)$$

for all n.

It remains to show that $\Pi[E](f)$ coincides with the limit of $\Pi[E_n](f)$, as $n \to \infty$. But this holds, because inf in the expression for $\Pi[E]$ is attained, and all minimizing γ satisfy (32). In fact, if an increasing sequence of bounded continuous functions on a compact set converges to a continuous function, then their minima converge to the minimum of the limiting function.

Chapter 3

Rainbow options in discrete times.

Colored options, European options and American options, multiple strike options, portfolio options, interval model, robust control approach, Bellman (or Shapley) operator, hedge, risk-neutral probabilities, martingales, submodular payoffs, generalized CRR formula, non-expansive maps, transaction costs.

A market with multiple securities

Recall the notions of European and American options.

Consider a financial market dealing with several securities: the risk-free bonds (or bank account) and J common stocks, $J = 1, 2 \ldots$. In case $J > 1$, the corresponding options are called *colored* or *rainbow options* (J-colors option for a given J). Suppose the prices of the units of these securities, B_m and S_m^i, $i \in \{1, 2, ..., J\}$, change in discrete moments of time $m = 1, 2, \ldots$ according to the recurrent equations $B_{m+1} = \rho B_m$, where the $\rho \geq 1$ is an interest rate which remains unchanged over time, and $S_{m+1}^i = \xi_{m+1}^i S_m^i$, where $\xi_m^i, i \in \{1, 2, ..., J\}$, are unknown sequences taking values in some fixed intervals $M_i = [d_i, u_i] \subset \mathbf{R}$.

Condition: $\rho \in (d_i, u_i)$ for all i.

Interval model

This is the *interval model*, which generalizes the colored version of the classical Cox-Ross-Rubinstein (CRR) model, where a sequence ξ^i_m is confined to take values only among two boundary points d_i, u_i, and it is supposed to be random with some given distribution.

Here any value in the interval $[d_i, u_i]$ is allowed and no probabilistic assumptions are made.

Main examples of coloured options

The type of an option is specified by a given premium function f of J variables. Standard examples:
option delivering the best of J risky assets and cash

$$f(S^1, S^2, ..., S^J) = \max(S^1, S^2, ..., S^J, K), \tag{37}$$

calls on the maximum of J risky assets

$$f(S^1, S^2, ..., S^J) = \max(0, \max(S^1, S^2, ..., S^J) - K), \tag{38}$$

Main examples of coloured options

multiple-strike options

$$f(S^1, S^2, ..., S^J) = \max(0, S^1 - K_1, S^2 - K_2, ..., S^J - K_J), \quad (39)$$

portfolio options

$$f(S^1, S^2, ..., S^J) = \max(0, n_1 S^1 + n_2 S^2 + ... + n_J S^J - K), \quad (40)$$

and *spread* options

$$f(S^1, S^2) = \max(0, (S^2 - S^1) - K). \quad (41)$$

Coloured options as a game against Nature

The investor is supposed to control the growth of his/her capital in the following way. Let X_m denote the capital of the investor at the time $m = 1, 2, \ldots$. At each time $m-1$ the investor determines his portfolio by choosing the numbers γ_m^i of common stocks of each kind to be held so that the structure of the capital is represented by the formula

$$X_{m-1} = \sum_{j=1}^{J} \gamma_m^j S_{m-1}^j + \left[X_{m-1} - \sum_{j=1}^{J} \gamma_m^j S_{m-1}^j\right],$$

The value ξ_m becomes known at the moment m and thus the capital at the moment m becomes

$$X_m = \sum_{j=1}^{J} \gamma_m^j \xi_m^j S_{m-1}^j + \rho\left[X_{m-1} - \sum_{j=1}^{J} \gamma_m^j S_{m-1}^j\right], \qquad (42)$$

if transaction costs are not taken into account.

Coloured options as a game against Nature

If n is the prescribed *maturity date*, then this procedures repeats n times starting from some initial capital $X = X_0$ (selling price of an option) and at the end the investor is obliged to pay the premium f to the buyer. Thus the (final) income of the investor equals

$$G(X_n, S_n^1, S_n^2, ..., S_n^J) = X_n - f(S_n^1, S_n^2, ..., S_n^J). \tag{43}$$

The evolution of the capital can thus be described by the n-step game of the investor with Nature, the behavior of the latter being characterized by unknown parameters ξ_m^j.

Coloured options as a game against Nature

The strategy of the investor is by definition any sequences of vectors $(\gamma_1, \cdots, \gamma_n)$ such that each γ_m could be chosen using the whole previous information: the sequences $X_0, ..., X_{m-1}$ and $S_0^i, ..., S_{m-1}^j$ (for every stock $j = 1, 2, ..., J$). The control parameters γ_m^j can take all real values, i.e. short selling and borrowing are allowed. A position of the game at any time m is characterized by $J+1$ non-negative numbers $X_m, S_m^1, \cdots, S_m^J$ with the final income specified by the function

$$G(X, S^1, ..., S^J) = X - f(S^1, ..., S^J). \tag{44}$$

Coloured options as a game against Nature

The main definition: A strategy $\gamma_1, \cdots, \gamma_n$, of the investor is called a *hedge*, if for any sequence $(\xi_1, \cdots, \xi_n)$ the investor is able to meet his/her obligations, i.e.

$$G(X_n, S_n^1, ..., S_n^J) \geq 0.$$

The minimal value of the capital X_0 for which the hedge exists is called the *hedging price* H of an option.
Looking for the guaranteed payoffs means looking for the worst-case scenario (so called *robust-control approach*), i.e. for the minmax strategies.

Coloured options as a game against Nature

Thus if the final income is specified by a function G, the guaranteed income (discounted to the initial time) of the investor in a one-step game with the initial conditions $X, S^1, ..., S^J$ is given by the *Bellman (or Shapley) operator*

$$\mathbf{B} G(X, S^1, \cdots, S^J)$$

$$= \frac{1}{\rho} \max_{\gamma} \min_{\{\xi^j \in [d_j, u_j]\}} G(\rho X + \sum_{i=1}^{J} \gamma^i \xi^i S^i - \rho \sum_{i=1}^{J} \gamma^i S^i, \xi^1 S^1, \cdots, \xi^J S^J). \tag{45}$$

By backward induction argument, the guaranteed income of the investor in the n-step game with the initial conditions $X_0, S_0^1, ..., S_0^J$ is given by the formula

$$\mathbf{B}^n G(X_0, S_0^1, ..., S_0^J).$$

Coloured options as a game against Nature

In our model G is given by (44). Then $\mathbf{B}G(X, S^1, ..., S^J)$ equals

$$X - \frac{1}{\rho} \min_{\gamma} \max_{\xi} [f(\xi^1 S^1, \xi^2 S^2, \cdots, \xi^J S^J) - \sum_{j=1}^{J} \gamma^j S^j (\xi^j - \rho)],$$

and hence

$$\mathbf{B}^n G(X, S^1, \cdots, S^J) = X - (\mathcal{B}^n f)(S^1, \cdots, S^J),$$

where the *reduced Bellman operator* $(\mathcal{B}f)(z^1, ..., z^J)$ equals

$$\frac{1}{\rho} \min_{\gamma} \max_{\{\xi^j \in [d_j, u_j]\}} [f(\xi^1 z^1, \xi^2 z^2, \cdots, \xi^J z^J) - \sum_{j=1}^{J} \gamma^j z^j (\xi^j - \rho)]. \tag{46}$$

Coloured options as a game against Nature

Hadamard product notation:

$$(x \circ y)_j = x_j y_j.$$

Then

$$(\mathcal{B}f)(z) = \frac{1}{\rho} \min_{\gamma} \max_{\{\xi^j \in [d_j, u_j]\}} [f(\xi \circ z) - (\gamma, \xi \circ z - \rho z)]. \quad (47)$$

Theorem 1. The minimal value of X_0 for which the income of the investor is non-negative (and which by definition is the hedge price H^n in the n-step game) is given by

$$H^n = (\mathcal{B}^n f)(S_0^1, \cdots, S_0^J). \quad (48)$$

Coloured options as a game against Nature

Changing variables $\xi = (\xi^1, \ldots, \xi^J)$ to $\eta = \xi \circ z$ yields

$$(\mathcal{B}f)(z^1, ..., z^J) = \frac{1}{\rho} \min_{\gamma} \max_{\{\eta \in [z^i d_i, z^i u_i]\}} [f(\eta) - \sum_{i=1}^{J} \gamma^i (\eta^i - \rho z^i)], \tag{49}$$

or, by shifting,

$$(\mathcal{B}f)(z^1, ..., z^J) = \frac{1}{\rho} \min_{\gamma} \max_{\{\eta \in [z^i (d_i - \rho), z^i (u_i - \rho)]\}} [f(\eta + \rho z) - (\gamma, \eta)]. \tag{50}$$

Coloured options as a game against Nature

Assuming f is convex, the maxis reduced to the max over the finite set of vectors

$$\eta_I = \xi_I \circ z - \rho z,$$

being the vertices of the rectangular parallelepiped

$$\Pi_{z,\rho} = \times_{i=1}^{J} [z^i(d_i - \rho), z^i(u_i - \rho)],$$

where

$$\xi_I = \{d_i|_{i \in I}, u_j|_{j \notin I}\},$$

are the vertices of

$$\Pi = \times_{i=1}^{J} [d_i, u_i], \tag{51}$$

parametrized by all subsets (including the empty one) $I \subset \{1, \ldots, J\}$.

Coloured options as a game against Nature

Since the origin is an internal point of Π (because $d_i < \rho < u_i$), the family $\{\eta_I\}$ is strongly positively complete. The condition of general position is rough in the sense that it is fulfilled for an open dense subset of pairs (d_i, u_i). Applying Theorem 2 of Chapter 2 and returning back to ξ yields the following formula.

Coloured options as a game against Nature

Theorem 2. If the vertices ξ_I of the parallelepiped Π are in general position in the sense that for any J subsets $I_1, \cdots, I_J$, the vectors $\{\xi_{I_k} - \rho \mathbf{1}\}_{k=1}^J$ are independent in $\mathbf{R}^J$, then

$$(\mathcal{B}f)(z) = \frac{1}{\rho} \max_{\{\Omega\}} \mathbf{E}_\Omega f(\xi \circ z), \quad z = (z^1, \cdots, z^J), \qquad (52)$$

where $\{\Omega\}$ is the collection of all subsets $\Omega = \xi_{I_1}, \cdots, \xi_{I_{J+1}}$ of the set of vertices of Π, of size $J+1$, such that their convex hull contains $\rho\mathbf{1}$ as an interior point ($\mathbf{1}$ is the vector with all coordinates 1), and where $\mathbf{E}_\Omega$ denotes the expectation with respect to the unique probability law $\{p_I\}$, $\xi_I \in \Omega$, on the set of vertices of Π, which is supported on Ω and is risk-neutral with respect to $\rho\mathbf{1}$, that is

$$\sum_{I \subset \{1,\dots,J\}} p_I \xi_I = \rho \mathbf{1}. \qquad (53)$$

Coloured options as a game against Nature

Theorem 2 (cont).
Moreover, if

$$f(\xi\circ z)-(\gamma_{I_1,\cdots,I_{J+1}},(\xi-\rho\mathbf{1})\circ z) \geq f(\zeta\circ z)-(\gamma_{I_1,\cdots,I_{J+1}},(\zeta-\rho\mathbf{1})\circ z)$$

for all vertices ξ, ζ such that $\xi \in \Omega$ and $\zeta \notin \Omega$, where $\gamma_{I_1,\cdots,I_{J+1}}$ is the corresponding optimal value for the polyhedron $\Pi[\xi_{I_1}, \cdots, \xi_{I_{J+1}}]$, then

$$(\mathcal{B}f)(z^1,...,z^J) = \frac{1}{\rho}\mathbf{E}_\Omega f(\xi \circ z). \tag{54}$$

Remarks

(i) Risk-neutrality now corresponds to its usual meaning in finance, i.e. (53) means that all discounted stock prices are martingales: $\mathbf{E}_I \xi^j = \rho$ for all j and

$$\mathbf{E}_I \left(\frac{S^j_{m+1}}{\rho^{j+1}} \,\middle|\, \frac{S^j_m}{\rho^j} \right) = \frac{S^j_m}{\rho^j}.$$

(ii) Calculations are reduced to the iterations $\mathcal{B}^n f$ for a Bellman operator of a controlled Markov chain.
(iii) The number of eligible Ω in (52) is the number of different pyramids (convex polyhedrons with $J+1$ vertices) with vertices taken from the vertices of Π and containing $\rho\mathbf{1}$ as an interior point.

The standard example: CRR (binomial) model

The case $J = 1$ of only one risky security.
There is only one risk- neutral law with

$$p^* = \frac{\rho - d}{u - d}, \quad q^* = 1 - p^* = \frac{u - \rho}{u - d},$$

and the Bellman (Shapley) operator is linear:

$$\mathcal{B}f(z) = \frac{\rho - d}{u - d} f(uz) + \frac{u - \rho}{u - d} f(dz),$$

yielding the classical *CRR formula* for the fair option price:

$$V = \mathcal{B}^n f(S_0) = \rho^{-n} \sum_{k=0}^{n} \binom{n}{k} (p^*)^k (q^*)^{n-k} f(u^k d^{n-k} S_0). \tag{55}$$

Upper and lower prices

Upper (or seller) price concerns the guaranteed (minimal) payoff to an investor selling an option (assuming the Nature is paying with the worst moves).
Lower (or buyer) price concerns the maximal payoff to an investor selling an option (assuming the Nature is paying with the best moves).
Thus the latter is given by the iteration of the *reduced lower Bellman operator*

$$(\mathcal{B}_{low} f)(z) = \frac{1}{\rho} \max_{\gamma} \min_{\{\xi^j \in [d_j, u_j]\}} [f(\xi \circ z) - (\gamma, \xi \circ z - \rho z)]. \quad (56)$$

In this simple interval model and for convex f this expression is trivial, it equals $f(\rho z)$.

Upper and lower prices

If f is concave, or if we allow only boundary points, then

$$(\mathcal{B}_{low} f)(z) = \frac{1}{\rho} \max_{\gamma} \min_{\{\xi^j \in \{d_j, u_j\}\}} [f(\xi \circ z) - (\gamma, \xi \circ z - \rho z)]. \quad (57)$$

Theorem 3 of Chapter 2 applies:

$$(\mathcal{B}_{low} f)(z) = \frac{1}{\rho} \min_{\{\Omega\}} \mathbf{E}_{\Omega} f(\xi \circ z), \quad z = (z^1, \cdots, z^J) \qquad (58)$$

(minimum over the same set of extreme risk-neutral measures). The difference between lower and upper prices can be considered as a measure of intrinsic risk of an incomplete market. If they coincide (only one risk-neutral measure exists), we have a *complete market*.

Nonexpansive property and approximations

The operator $\rho\mathcal{B}$ is *non-expansive*:

$$\|\mathcal{B}(f_1) - \mathcal{B}(f_2)\| \leq \frac{1}{\rho}\|f_1 - f_2\|,$$

and *homogeneous* (both with respect to addition and multiplication):

$$\rho\mathcal{B}(\lambda + f) = \lambda + \rho\mathcal{B}(f), \quad \mathcal{B}(\lambda f) = \lambda\mathcal{B}(f)$$

for any function f and $\lambda \in \mathbf{R}$ (resp. $\lambda > 0$) for the first (resp second) equation.

Nonexpansive property and approximations

Next, if f_p is a power function, that is

$$f_p(z) = (z^1)^{i_1} \cdots (z^J)^{i_J},$$

then $f_p(\xi \circ z) = f_p(\xi) f_p(z)$, implying

$$(\mathcal{B}^n f_p)(z) = ((\mathcal{B} f_p)(\mathbf{1}))^n f_p(z). \tag{59}$$

Consequently, if for a payoff f there exists a power function f_p such that $\|f - f_p\| \le \epsilon$, then

$$\|\mathcal{B}^n f - \lambda^n f_p\| \le \frac{1}{\rho^n} \|f - f_p\| \le \frac{\epsilon}{\rho^n}, \quad \lambda = (\mathcal{B} f_p)(\mathbf{1}), \tag{60}$$

so that an approximate calculation of $\mathcal{B}^n f$ is reduced to the calculation of one number λ.

Sub-modular payoffs

One can get an essential reduction in the combinatorics of Theorem 2 under additional assumptions on the payoff f.
A function $f : \mathbf{R}_+^2 \to \mathbf{R}_+$ is called *sub-modular*, if the inequality

$$f(x_1, y_2) + f(x_2, y_1) \geq f(x_1, y_1) + f(x_2, y_2)$$

holds whenever $x_1 \leq x_2$ and $y_1 \leq y_2$. Similarly, a function $f : \mathbf{R}_+^d \to \mathbf{R}_+$ is called *sub-modular* if

$$f(x \bigvee y) + f(x \bigwedge y) \leq f(x) + f(y),$$

where $\bigvee$ (respectively $\bigwedge$) denotes the Pareto (coordinate-wise) maximum (respectively minimum).
Exercise. If f is twice continuously differentiable, then it is sub-modular if and only if $\frac{\partial^2 f}{\partial z_i \partial z_j} \leq 0$ for all $i \neq j$.
The payoffs of the first three examples of rainbow options, given above are sub-modular.

Sub-modular payoffs: two colors

Let $J = 2$. The polyhedron Π from (51) is then a rectangle. From sub-modularity of f it follows that if Ω from Theorem 2 is either

$$\Omega_{12} = \{(d_1, d_2), (d_1, u_2), (u_1, u_2)\}$$

or

$$\Omega_{21} = \{(d_1, d_2), (u_1, d_2), (u_1, u_2)\},$$

then $(f, \xi) - (\gamma_0, \xi)$ coincide for all vertices ξ of Π. Hence Ω_{12} and Ω_{21} can be discarded in Theorem 2, i.e the maximum is always achieved either on

$$\Omega_d = \{(d_1, d_2), (d_1, u_2), (u_1, d_2)\},$$

or on

$$\Omega_u = \{(d_1, u_2), (u_1, d_2), (u_1, u_2)\}.$$

Sub-modular payoffs: two colors

But the interiors of the triangles formed by Ω_u and Ω_d do not intersect, so that each point of Π (in general position) lies only in one of them (and this position does not depend any more on f). Hence, depending on the position of $\rho\mathbf{1}$ in Π, the expression (52) reduces either to $\mathbf{E}_{\Omega_u}$ or to $\mathbf{E}_{\Omega_d}$.

Theorem 3. Let $J = 2$ and f be convex sub-modular. Denote

$$\kappa = \frac{(u_1 u_2 - d_1 d_2) - \rho(u_1 - d_1 + u_2 - d_2)}{(u_1 - d_1)(u_2 - d_2)}$$

$$= 1 - \frac{\rho - d_1}{u_1 - d_1} - \frac{\rho - d_2}{u_2 - d_2}.$$

Sub-modular payoffs: two colors

Theorem 3 (cont.) If $\kappa \geq 0$, then $\rho(\mathcal{B}f)(z_1, z_2)$ equals

$$\frac{\rho - d_1}{u_1 - d_1} f(u_1 z_1, d_2 z_2) + \frac{\rho - d_2}{u_2 - d_2} f(d_1 z_1, u_2 z_2) + \kappa f(d_1 z_1, d_2 z_2), \tag{61}$$

and the corresponding optimal strategies are

$$\gamma^1 = \frac{f(u_1 z_1, d_2 z_2) - f(d_1 z_1, d_2 z_2)}{z_1(u_1 - d_1)}, \; \gamma^2 = \frac{f(d_1 z_1, u_2 z_2) - f(d_1 z_1, d_2 z_2)}{z_2(u_2 - d_2)}$$

If $\kappa \leq 0$, then $\rho(\mathcal{B}f)(z_1, z_2)$ equals

$$\frac{u_1 - \rho}{u_1 - d_1} f(d_1 z_1, u_2 z_2) + \frac{u_2 - \rho}{u_2 - d_2} f(u_1 z_1, d_2 z_2) + |\kappa| f(u_1 z_1, u_2 z_2), \tag{62}$$

and

$$\gamma^1 = \frac{f(u_1 z_1, u_2 z_2) - f(d_1 z_1, u_2 z_2)}{z_1(u_1 - d_1)}, \; \gamma^2 = \frac{f(u_1 z_1, u_2 z_2) - f(u_1 z_1, d_2 z_2)}{z_2(u_2 - d_2)}$$

Sub-modular payoffs: two colors

The linear operator $\mathcal{B}$ preserves the set of convex sub-modular functions. Hence one can use this formula recursively to obtain all powers of $\mathcal{B}$. For instance for $\kappa = 0$ one obtains for the hedge price the following *two-color extension of the classical CRR formula*:

$$\mathcal{B}^n f(S_0^1, S_0^2)$$
$$= \rho^{-n} \sum_{k=0}^{n} \binom{n}{k} \left(\frac{\rho - d_1}{u_1 - d_1}\right)^k \left(\frac{\rho - d_2}{u_2 - d_2}\right)^{n-k} f(u_1^k d_1^{n-k} S_0^1, d_2^k u_2^{n-k} S_0^2) \tag{63}$$

Sub-modular payoffs: two colors

In terms of the risk-neutral measure

$$\mathcal{B}^n f(S_0^1, S_0^2) = \rho^{-n} \mathbf{E} f(S_n^1, S_n^2),$$

risk neutral evaluation.

Exercise. Write down similar formulas for non-vanishing κ.

General jump-model

Let us turn to *nonlinear jump patterns*. Assume, instead of the stock price evolution model $S_{m+1} = \xi \circ S_m$, that we are given k transformations $g_i : \mathbf{R}^J \to \mathbf{R}^J$, $i = 1, \cdots, k$, which give rise to two models of price dynamics: either
(i) at time $m+1$ the price S_{m+1} belongs to the closure of the convex hull of the set $\{g_i(S_m)\}$, $i = 1, \cdots, k$ (*nonlinear interval model*), or
(ii) S_{m+1} is one of the points $\{g_i(S_m)\}$, $i = 1, \cdots, k$.
Since the first model can be approximated by the second one (by possibly increasing the number of transformations g_i), we shall work with the second model.
For convex payoffs the two models above are fully equivalent.

General jump-model

Assuming for simplicity that possible jump sizes are time independent and the payoff depends only on the end-value of a path, the *reduced Bellman operator* becomes

$$(\mathcal{B}f)(z) = \frac{1}{\rho} \min_{\gamma} \max_{i \in \{1, \cdots, k\}} [f(g_i(z)) - (\gamma, g_i(z) - \rho z)], \quad (64)$$

for $z = (z^1, ..., z^J)$, or equivalently

$$(\mathcal{B}f)(z) = \frac{1}{\rho} \min_{\gamma} \max_{\eta_i = g_i(z) - \rho z, i = 1, \cdots, k} [f(\eta_i + \rho z) - (\gamma, \eta_i)]. \quad (65)$$

General jump-model

The hedge price is still given by iterations and reduced Bellman operator is calculated by Theorem 2 of Chapter 2:

$$(\mathcal{B}f)(z) = \frac{1}{\rho} \max_{\Omega} \mathbf{E}_{\Omega} f(\eta_i + \rho z), \tag{66}$$

where $\mathbf{E}_{\Omega}$ denote expectations with respect to all extreme points of risk-neutral probability laws on vectors
$\eta_i = g_i(z) - \rho z$.
Special case $k = d + 1$.

American options

An *American options* can be exercised at any time. The Bellman operator $\mathbf{B}G(X, S^1, ..., S^J)$ from (45) changes to

$$\mathbf{B}G(X, S^1, \cdots, S^J) = \frac{1}{\rho} \max_{\gamma} \min\Big\{ G(X, S^1, \cdots, S^J),$$

$$\frac{1}{\rho} \min_{\xi} G\Big(\rho X + \sum_{i=1}^{J} \gamma^i \xi^i S^i - \rho \sum_{i=1}^{J} \gamma^i S^i, \xi^1 S^1, \cdots, \xi^J S^J\Big)\Big\}. \quad (67)$$

American options

The corresponding *reduced Bellman operator* takes the form

$$(\mathcal{B}f)(z^1, \dots, z^J) = \frac{1}{\rho} \min_{\gamma} \max\bigl[\rho f(\rho z),$$

$$\max_{\xi}[f(\xi^1 z^1, \xi^2 z^2, \cdots, \xi^J z^J) - \sum_{i=1}^{J} \gamma^i z^i(\xi^i - \rho)]\bigr],$$

$$= \max \Bigg[f(\rho z), \frac{1}{\rho} \min_{\gamma} \max_{\xi}[f(\xi^1 z^1, \xi^2 z^2, \cdots, \xi^J z^J) - \sum_{i=1}^{J} \gamma^i z^i(\xi^i - \rho$$

(68)

American options

Consequently, the main formula via risk-neutral laws becomes

$$(\mathcal{B}f)(z^1, ..., z^J) = \max \left[f(\rho z), \frac{1}{\rho} \max_{\{\Omega\}} \mathbf{E}_\Omega f(\xi \circ z) \right], \qquad (69)$$

The hedge price for the n-step model is given by $\mathcal{B}^n$.

Similar problems arise in the study of *real options*.

Transaction costs (remarks)

To deal with transaction costs, it is convenient to extend the state space of our game, considering the states that are characterized, at time $m-1$, by $2J+1$ numbers

$$X_{m-1}, S^j_{m-1}, v_{m-1} = \gamma^j_{m-1}, \quad j = 1, \cdots, J.$$

When, at time $m-1$, the investor chooses his new control parameters γ_m, the new state at time m becomes

$$X_m, \quad S^j_m = \xi^j_m S^j_{m-1}, \quad v_m = \gamma^j_m, \quad j = 1, \cdots, J,$$

where the value of the portfolio is

$$X_m = \sum_{j=1}^{J} \gamma^j_m \xi^j_m S^j_{m-1} + \rho(X_{m-1} - \sum_{j=1}^{J} \gamma^j_m S^j_{m-1}) - g(\gamma_m - v_{m-1}, S_{m-1}). \tag{70}$$

Transaction costs (remarks)

The corresponding *reduced Bellman operator* takes the form

$$(\mathcal{B}f)(z,v) = \min_{\gamma} \max_{\xi} [f(\xi \circ z, \gamma) - (\gamma, \xi \circ z - \rho z) + g(\gamma - v, z)], \tag{71}$$

where $z, v \in \mathbf{R}^J$, or, changing variables $\xi = (\xi^1, \dots, \xi^J)$ to $\eta = \xi \circ z$ and shifting,

$$(\mathcal{B}f)(z,v)$$

$$= \min_{\gamma} \max_{\{\eta^j \in [z^j(d_j - \rho), z^j(u_j - \rho)]\}} [f(\eta + \rho z, \gamma) - (\gamma, \eta) + g(\gamma - v, z)]. \tag{72}$$

It is possible to extend the theory to cover this nonlinear dependence on γ.

Chapter 4

Continuous time limit: generalized Black-Scholes equations.

Black-Scholes equation, nonlinear Black-Scholes equation, fractional Black-Scholes equation

Scaling

For the general model of nonlinear jumps the reduced Bellman operator is of form (64). Suppose the maturity time is T. Let us decompose the planning time $[0, T]$ into n small intervals of length $\tau = T/n$, and assume

$$g_i(z) = z + \tau^\alpha \phi_i(z), \quad i = 1, \cdots, k, \tag{73}$$

with some functions ϕ_i and a constant $\alpha \in [1/2, 1]$. Thus the jumps during time τ are of the order of magnitude τ^α. As usual, we assume that the risk-free interest rate per time τ equals

$$\rho = 1 + r\tau,$$

with $r > 0$.

Scaling

From (64):

$$\mathcal{B}_\tau f(z) = \frac{1}{1+r\tau} \max_I \sum_{i\in I} p_i^I(z,\tau) f(z+\tau^\alpha \phi_i(z)), \qquad (74)$$

where I are subsets of $\{1,\cdots,k\}$ of size $|I| = J+1$ such that the family of vectors $z+\tau^\alpha\phi_i(z)$, $i \in I$, are in general position and $\{p_i^I(z,\tau)\}$ is the risk-neutral probability law on such a family, with respect to ρz:

$$\sum_{i\in I} p_i^I(z,\tau)(z+\tau^\alpha\phi_i(z)) = (1+r\tau)z. \qquad (75)$$

HJB equation

Let us deduce the HJB equation for the limit, as $\tau \to 0$, of the approximate cost-function $f_t = \mathcal{B}_\tau^{T-t} f_T$, $t \in [0, T]$, with a given final cost f_T, using the standard (heuristic) dynamic programming approach.
By (74):

$$f_{t-\tau}(z) = \frac{1}{1+r\tau} \max_I \sum_{i \in I} p_i^I(z, \tau)$$

$$\times \left[f_t(z) + \tau^\alpha \frac{\partial f_t}{\partial z} \phi_i(z) + \frac{1}{2} \tau^{2\alpha} \left(\frac{\partial^2 f_t}{\partial z^2} \phi_i(z), \phi_i(z) \right) + O(\tau^{3\alpha}) \right].$$

Black-Scholes equation

Since $\{p_i^I\}$ are probabilities and using (75), this rewrites as

$$f_t - \tau \frac{\partial f_t}{\partial t} + O(\tau^2) = \frac{1}{1 + r\tau}\big[f_t(z) + r\tau(z, \frac{\partial f_t}{\partial z})$$

$$+ \frac{1}{2}\tau^{2\alpha} \max_I \sum_{i \in I} p_i^I(z) \left(\frac{\partial^2 f_t}{\partial z^2}\phi_i(z), \phi_i(z) \right) \big] + O(\tau^{3\alpha}),$$

where

$$p_i^I(z) = \lim_{\tau \to 0} p_i^I(z, \tau).$$

This leads to the limiting equations:

$$rf = \frac{\partial f}{\partial t} + r(z, \frac{\partial f}{\partial z}) + \frac{1}{2} \max_I \sum_{i \in I} p_i^I(z) \left(\frac{\partial^2 f}{\partial z^2}\phi_i(z), \phi_i(z) \right) \tag{76}$$

in case $\alpha = 1/2$,

Black-Scholes equation

And to the trivial first order equation

$$rf = \frac{\partial f}{\partial t} + r(z, \frac{\partial f}{\partial z}), \tag{77}$$

with the obvious solution

$$f(t, z) = e^{-r(T-t)} f_T(e^{-r(T-t)} z) \tag{78}$$

in case $\alpha > 1/2$.
Equation (76) is a *nonlinear extension of the classical Black-Scholes equation*.

Upper and lower prices

The hedge price calculated from equation (76) is actually the upper price.
For the lower price, we get similarly the equation

$$rf = \frac{\partial f}{\partial t} + r(z, \frac{\partial f}{\partial z}) + \frac{1}{2} \min_{I} \sum_{i \in I} p_i^I(z) \left(\frac{\partial^2 f}{\partial z^2} \phi_i(z), \phi_i(z) \right). \tag{79}$$

Black-Scholes equation (classical)

In case $J = 1$ and the classical CCR (binomial) setting with

$$\sqrt{\tau}\phi_1 = (u-1)z = \sigma\sqrt{\tau}z, \quad \sqrt{\tau}\phi_2 = (d-1)z = -\sigma\sqrt{\tau}z,$$

(76) becomes the usual Black-Scholes equation

$$rf = \frac{\partial f}{\partial t} + rz\frac{\partial f}{\partial z} + \frac{1}{2}\sigma^2 z^2 \frac{\partial^2 f}{\partial z^2}. \tag{80}$$

Black-Scholes equation (classical)

To solve the Black-Scholes equation one changes the unknown function f to g by

$$f(z) = e^{-r(T-t)} g\left(\frac{1}{\sigma}\log z\right),$$

transforming the equation to the equation with constant coefficient:

$$\frac{\partial g}{\partial t} + \frac{1}{2}(2r-\sigma)\frac{\partial g}{\partial y} + \frac{1}{2}\frac{\partial^2 g}{\partial y^2} = 0, \tag{81}$$

a standard heat equation with a drift. It is solved via the Fourier transform. Namely, the equation for $\tilde{g}(p)$ is

$$\frac{\partial \tilde{g}}{\partial t} = \frac{1}{2}[p^2 - i(2r-\sigma)p]\tilde{g}. \tag{82}$$

Black-Scholes equation (classical)

Hence the inverse Cauchy problem for equation (81) with a given final function g_T equals the convolution of g_T with the inverse Fourier transform of the functions

$$\exp\{-\frac{1}{2}(T-t)[p^2 - i(2r-\sigma)p]\},$$

which equals

$$\frac{1}{2\pi}\int_{\mathbf{R}} \exp\{-\frac{1}{2}(T-t)[p^2 - i(2r-\sigma)p] + ipy\}dp$$

$$= \frac{1}{\sqrt{2\pi(T-t)}} \exp\{-\frac{1}{2}\left(y + \frac{r-\sigma/2}{T-t}\right)^2\}.$$

Exercise. Perform these calculations.

Black-Scholes equation (classical)

Therefore, solution to (81) with a given final function g_T equals

$$g(t,x) = \frac{1}{\sqrt{2\pi(T-t)}} \int \exp\{-\frac{1}{2}\left(x - y + \frac{r-\sigma/2}{T-t}\right)^2\} g_T(y)\, dy.$$

Exercise. (i) Deduce the corresponding formula solving the Black-Scholes equation (80) with a terminal condition $f_T(x)$.
(ii) Applying it to $f_T(x) = (x-K)^+$ derive the celebrated *Black-Scholes formula* for the price of the European call option:

Black-Scholes formula

$$c = S_0\Phi(d_1) - Ke^{-rT}\Phi(d_2), \tag{83}$$

where

$$d_1 = \frac{\ln(S_0/K) + (r + \sigma^2/2)T}{\sigma\sqrt{T}},$$
$$d_2 = \frac{\ln(S_0/K) + (r - \sigma^2/2)T}{\sigma\sqrt{T}}.$$

Black-Scholes equation

More generally, if $k = J + 1$, equation (76) becomes linear:

$$rf = \frac{\partial f}{\partial t} + r(z, \frac{\partial f}{\partial z}) + \frac{1}{2}\sum_{i=1}^{J+1} p_i(z)\left(\frac{\partial^2 f}{\partial z^2}\phi_i(z), \phi_i(z)\right). \quad (84)$$

This is a *generalized Black-Scholes equation* describing a complete market (with randomness coming from J correlated Brownian motions), whenever the diffusion matrix

$$(\sigma^2)_{jk} = \sum_{i=1}^{J+1} p_i(z)\phi_i^j(z), \phi_i^k(z)$$

is non-degenerate.

Two-color example

Let $J = 2$ and the final payoff f_T be submodular. Assume the simplest (and usual) symmetric form for upper and lower jumps (further terms in Taylor expansion are irrelevant for the limiting equation):

$$u_i = 1 + \sigma_i \sqrt{\tau}, \quad d_i = 1 - \sigma_i \sqrt{\tau}, \quad i = 1, 2. \tag{85}$$

Hence

$$\frac{u_i - \rho}{u_i - d_i} = \frac{1}{2} - \frac{r}{2\sigma_i}\sqrt{\tau}, \quad i = 1, 2,$$

and

$$\kappa = -\frac{1}{2} r \sqrt{\tau} \left(\frac{1}{\sigma_1} + \frac{1}{\sigma_2}\right).$$

Two-color example

As $\kappa < 0$, we are in the second case of Theorem 3 of Chapter 3. Hence the only eligible collection of three vectors ϕ is $(d_1, u_2), (u_1, d_2), (u_1, u_2)$, and the (limiting) probability law p_i^l is $(1/2, 1/2, 0)$. Therefore, equation (76) takes the form

$$rf = \frac{\partial f}{\partial t} + r(z, \frac{\partial f}{\partial z}) + \frac{1}{2}\left[\sigma_1^2 z_1^2 \frac{\partial^2 f}{\partial z_1^2} - 2\sigma_1\sigma_2 z_1 z_2 \frac{\partial^2 f}{\partial z_1 \partial z_2} + \sigma_2^2 z_2^2 \frac{\partial^2 f}{\partial z_2^2}\right]. \tag{86}$$

The limiting Black-Scholes type equation is again linear in this example, but with a degenerate second-order part.

Two-color example, lower price

In the analogous stochastic setting, this degeneracy would mean that only one Brownian motion governs the behavior of both underlying stocks. This is not surprising in our approach, where Nature was assumed to be a single player. One could expect uncoupled second derivatives (non-degenerate diffusion) in the limit, if one would choose two independent players for Nature, each playing for each stock.

Thus we are still in the setting of an incomplete market. The lower price is given by the equation (79).

To get a lower price, we shall use approximations of type (58), leading, instead of (76), to the equation

$$rf = \frac{\partial f}{\partial t} + r(z, \frac{\partial f}{\partial z}) + \frac{1}{2} \min_{I} \sum_{i \in I} p_i^I(z) \left(\frac{\partial^2 f}{\partial z^2} \phi_i(z), \phi_i(z) \right) . \tag{87}$$

Two-color example, lower price

If $J = 2$ and the payoff is submodular, the maximum can be taken over the triples (d_1, d_2), (d_1, u_2), (u_1, d_2) or $(d_1, u_2), (u_1, d_2), (u_1, u_2)$ (under (85) only the second triple works). Similarly the minimum can be taken only over the triples $(d_1, d_2), (d_1, u_2), (u_1, u_2)$ or (d_1, d_2), (u_1, d_2), (u_1, u_2). Under (85) both these cases give the same limit as $\tau \to 0$, yielding for the lower price the equation

$$rf = \frac{\partial f}{\partial t} + r(z, \frac{\partial f}{\partial z}) + \frac{1}{2}\left[\sigma_1^2 z_1^2 \frac{\partial^2 f}{\partial z_1^2} + 2\sigma_1\sigma_2 z_1 z_2 \frac{\partial^2 f}{\partial z_1 \partial z_2} + \sigma_2^2 z_2^2 \frac{\partial^2 f}{\partial z_2^2}\right], \tag{88}$$

that differs only by sign at the mixed derivative from the equation for the upper price.

Two-color example, lower price

As f was assumed sub-modular, so that its mixed second derivative is negative, we have

$$\sigma_1\sigma_2 \frac{\partial^2 f}{\partial z_1 \partial z_2} \leq 0 \leq -\sigma_1\sigma_2 \frac{\partial^2 f}{\partial z_1 \partial z_2}.$$

Hence, for the solution f_u of the upper value equation (86), the solution f_l of the lower value equation (88), and the solution f_c of the classical Black-Scholes equation of a complete market based on two independent Brownian motions, i.e. equation (86) or (88) without the term with the mixed derivative (with the same sub-modular initial condition f_T) we have the inequality

$$f_l \leq f_c \leq f_u,$$

as expected.

Two-color example

Equations (86) and (88) can be solved explicitly via the Fourier transform, as with the standard Black-Scholes equation.
Exercise. Show that changing the unknown function f to g by

$$f(z_1, z_2) = e^{-r(T-t)} g\left(\frac{1}{\sigma_1}\log z_1, \frac{1}{\sigma_2}\log z_2\right),$$

so that transforms these equations to the equations with constant coefficients

$$\frac{\partial g}{\partial t} + \frac{1}{2}(2r - \sigma_1)\frac{\partial g}{\partial y_1} + \frac{1}{2}(2r - \sigma_2)\frac{\partial g}{\partial y_2} + \frac{1}{2}\left[\frac{\partial^2 g}{\partial y_1^2} \mp 2\frac{\partial^2 g}{\partial y_1 \partial y_2} + \frac{\partial^2 g}{\partial y_2^2}\right] = 0 \tag{89}$$

(with $\mp$ respectively).

Two-color example

The equation for the Fourier transform $\tilde{g}(p)$ of g is

$$\frac{\partial \tilde{g}}{\partial t} = \frac{1}{2}[(p_1 \mp p_2)^2 - i(2r - \sigma_1)p_1 - i(2r - \sigma_2)p_2]\tilde{g}. \quad (90)$$

Hence the inverse Cauchy problem for equation (89) with a given final function g_T equals the convolution of g_T with the inverse Fourier transform of the functions

$$\exp\{-\frac{1}{2}(T - t)[(p_1 \mp p_2)^2 - i(2r - \sigma_1)p_1 - i(2r - \sigma_2)p_2]\}.$$

Two-color example

Exercise. Calculate these Fourier transforms.

Answer:

$$\frac{1}{2}\frac{1}{\sqrt{2\pi(T-t)}}\delta\left(\frac{y_1+y_2}{2}+(r-\frac{\sigma_1+\sigma_2}{4})(T-t)\right)$$

$$\times\exp\left\{-\frac{1}{8(T-t)}\left(y_1-y_2-\frac{(\sigma_1-\sigma_2)(T-t)}{2}\right)^2\right\}$$

and

$$\frac{1}{2}\frac{1}{\sqrt{2\pi(T-t)}}\delta\left(\frac{y_1-y_2}{2}-\frac{(\sigma_1-\sigma_2)(T-t)}{4}\right)$$

$$\times\exp\left\{-\frac{1}{8(T-t)}\left(y_1+y_2+(2r-\frac{\sigma_1+\sigma_2}{2})(T-t)\right)^2\right\}$$

Two-color example

Exercise. Write down the solutions to the inverse Cauchy problems of equations (86) and (88) as

$$f_{u,l}(t, z_1, z_2) = \int_0^\infty \int_0^\infty G^{\mp}_{T-t}(z_1, z_2; w_1, w_2) f_T(w_1, w_2)\, dw_1 dw_2, \tag{91}$$

with an explicitly given Green functions (or transition probabilities) $G^{\mp}_{T-t}(z_1, z_2; w_1, w_2)$.
Singularity, presented by the δ-functions in formulas above, is due to the degeneracy of the second-order part of the corresponding equations.

Remark: fractional Black-Scholes equation

HJB equation $\partial f/\partial t = L_x f(t,x)$ as the limit of discrete Markov control processes.
If instead use controlled CTRW (continuous time random walks) one gets the fractional extension:

$$D_{a+*}f(t,x) = L_x f(t,x)$$

where the Caputo-Dzherbashyan (CD) derivative of order $\beta \in (0,1)$ is defined as

$$D_{a+*}f(t) = \frac{1}{\Gamma(-\beta)}\int_0^{t-a} \frac{f(t-s)-f(t)}{s^{1+\beta}} + \frac{f(a)-f(t)}{\Gamma(1-\beta)t^{\beta}}.$$

This leads to fractional (nonlinear, etc) Black-Scholes equations.

Chapter 5

Pricing credit derivatives.

Credit default swaps (CDS), instantaneous digital CDS, tranches (e.g. CDOs), mean-field limits.

Instantaneous digital CDS

Consider a market of N securities that can default in discrete time $\delta, 2\delta, \cdots$. Let N_t denote the number of securities defaulted up to time t and set $d_t = N - N_t$ for the number of yet not defaulted ones.

Entering an *instantaneous digital CDS* agreement on jth (not yet defaulted) security at time t means agreeing to pay a (protection) premium $\alpha_j(t)\delta$ and to receive back the compensation of one unit of money in case of j defaulting during the period $(t, t+\delta]$ (and nothing otherwise). The premium $\alpha_j(t)\delta$ is chosen in such a way that there is no charge at inception at time t to enter this contract (alternatively of course, $\alpha_j(t)\delta$ can be considered as a charge for receiving one in case of jth default).

The infinitesimal premium $\alpha_j(t)$ is usually assumed to depend on the whole history of defaults of basic securities until and including time t.

No simultaneous jumps

Standard simplifying assumption that only one default can occur in any given short period of time $(t, t+\delta]$.
That is, for any time t, the possible $(d_t + 1)$ outcomes at time $t+\delta$ are either no default or a default of only one of d_t live (not yet defaulted earlier) securities.

These outcomes can be described symbolically by $d_t + 1$ vectors in $\mathbf{R}^{d_t}$: zero vector e_0 and d_t basis vectors e_i (with ith coordinate 1 and other vanishing), so that any short-term contingent claim starting at t for the period $(t, t+\delta]$ can be described by a function f on $\{e_0, e_1, \cdots, e_{d_t}\}$, $f_i = f(e_i)$.

No simultaneous jumps

Suppose that, in order to replicate any such claim, an investor, with a capital X at time t, is allowed to enter an arbitrary amount γ^i of instantaneous digital CDS agreement on ith security, $i = 1, \cdots, d_t$, (recall that entering such an agreement is costless). Then his capital at time $t + \delta$ in case of event e_i becomes

$$-\delta \sum_{j=1}^{d_t} \gamma^j \alpha_j(t) - f_i + \gamma^i + X = X - f_i + (\gamma, \eta_i),$$

where it is assumed that $\gamma^0 = 0$ and

$$\eta_0 = -\delta\alpha(t) = -\delta(\alpha_1(t), \cdots, \alpha_d(t)) \in \mathbf{R}^d,$$

and $\eta_i = e_i - \delta\alpha(t)$ for $i > 0$.

No simultaneous jumps

Maximising the capital (by choosing appropriate γ against the nature (that chooses $i = 0, \cdots, d_t$) the guaranteed capital equals

$$\max_{\gamma} \min_{i} [X - f_i + (\gamma, \eta_i)] = X - \min_{\gamma} \max_{i} [f_i - (\gamma, \eta_i)]$$

For sufficiently small δ the $d_t + 1$ vectors η_i in $\mathbf{R}^{d_t}$ form a strongly positively complete (arbitrage free) set leading to the simplest complete market setting.

No simultaneous jumps

Hedging price: the minimal value of X needed to be able to fulfill the obligation:

$$C_h = \min_{\gamma \in \mathbf{R}^{d_t}} \max_{i=0,1,\cdots,d_t} [f_i - (\gamma, \eta_i)]. \tag{92}$$

Thus

$$C_h = \mathbf{E}\{f_\cdot\} = \sum_{j=0}^{d_t} p_j(t) f_j,$$

where $\{p_j\}$ are the corresponding unique risk-neutral probabilities on $\{\eta_i\}$:

$$p_j(t) = \frac{\delta p_0 \alpha_j(t)}{1 - \delta\alpha_j}, \quad j = 1, \cdots, d_t, \quad p_0 \left(1 + \sum_j \frac{\delta\alpha_j(t)}{1 - \delta\alpha_j} \right) = 1.$$

Markov models

Exercise. Derive the above formulas for the risk neutral probabilities.

One can look at the law $\{p_{\cdot}(t)\}$ as the transition probabilities from time t to time $t+\delta$ of a certain random process of defaults. It is however not Markovian, as $\alpha(t)$ can depend on the past. To make it Markovian one has to assume additionally that $\alpha_j(t)$ depend only on a current state of defaults. Let us consider three natural Markovian settings.

Markov models

1) The premia $\alpha_j(t) = \alpha(t, d_t)$ with a deterministic function $\alpha(t, d)$. In this case states are numbers from $\{0, 1, \cdots, N\}$ (the number of not defaulted securities) with possible transitions only $m \to m - 1$ with the risk-neutral transition probabilities

$$p_{t,t+\delta}(m, m-1) = \frac{m\alpha(t, m)\delta}{1 - \alpha(t, m)\delta} \left(1 + \frac{m\alpha(t, m)\delta}{1 - \alpha(t, m)\delta}\right)^{-1}$$

$$= \frac{m\alpha(t, m)\delta}{1 + (m-1)\alpha(t, m)\delta}, \quad m \geq 1. \tag{93}$$

Markov models

2) The general case: The state space is the set of all subsets I of $\{1, \cdots, k\}$ and premia at time t depend on the history via the present state $I(t)$, that is, they are given by functions $\alpha_j(t, I)$. Then

$$p_{t,t+\delta}(I, I \setminus j) = \frac{\alpha_j(t,I)\delta}{1 - \alpha_j(t,I)\delta} \left(1 + \sum_{j \in I} \frac{\alpha_j(t,I)\delta}{1 - \alpha_j(t,I)\delta} \right)^{-1}. \tag{94}$$

Markov models

3) The intermediate case. The whole set of securities is decomposed in a finite number k of classes (say, by a rating agency), the infinitesimal premia are the same for securities from the same class and depend on the number of live securities in each class. The states are now vectors $m = (m^1, \cdots, m^k)$ (with non-negative integers as coordinates), with jth coordinate denoting the number of live securities in class j.

Exercise. Write down the corresponding risk-neutral transition probabilities.

Limiting Markov chains in continuous time

As $\delta \to 0$, these models have natural limiting Markov processes in continuous time:

1') A Markov chain on $\{0, 1, \cdots, N\}$ with transition rates

$$q_t(m, m-1) = m\alpha(t, m),$$

2') A Markov chain on subsets of $\{0, 1, \cdots, N\}$ with transition rates

$$q_t(I, I \setminus j) = \alpha_j(t, I);$$

3') A Markov chain on the set of vectors with k nonnegative integer coordinates with transition rates

$$q_t(m, m - e_i) = \alpha_i(t, m).$$

Reference

Our exposition of CDS without simultaneous jumps was close to that of R. Frey and J. Backhaus. Pricing and hedging of portfolio credit derivatives with interacting default intensities. Int. J. Theor. Appl. Finance **11:6** (2008), 611-634.

They described the same Markov models.

We have only stressed their game-theoretic origins.

Simultaneous jumps: completion by tranching

Allowing for simultaneous defaults makes the model incomplete. It can be dealt with by the general techniques.

On the other hand, these models can be completed by allowing trading *instantaneous digital CDS* paying compensation in the event of these *simultaneous defaults*. For definiteness, let us assume that only two simultaneous defaults can occur during each period $(t, t+\delta]$. Assume also that one can enter an *instantaneous digital CDS* on these events, that is, for any pair $i < j$, one can agree to pay a (protection) premium $\alpha_{ij}(t)\delta$ at time t and to receive back the compensation of one unit of money in case of both i and j defaulting during the period $(t, t+\delta]$ (and nothing otherwise). For convenience let us set $\alpha_{ii} = \alpha_i$, where α_i are the premium rates for the usual one-name based CDS.

Simultaneous jumps

We shall denote by M^d the space of upper diagonal $d \times d$ matrices equipped with the scalar product

$$\gamma\eta = \sum_{i \le j} \gamma^{ij}\eta^{ij}.$$

By e_{ij} we denote the matrix with one on the intersection of ith row and jth column and with other elements vanishing. Arguing now as in the previous section we can conclude that for a contingent claim paying f_{ij}, $i < j$, in case of simultaneous default of i and j, paying f_i in case of defaulting i and paying f_0 in case of no default during the period $(t, t+\delta]$ the hedge price at time t becomes

Simultaneous jumps

$$C_h = \min_{\gamma \in M^{d_t}} \max_{\eta \in \{\eta_0, \eta_{ij}\}, 1 \le i \le j} [f(\eta) - (\gamma, \eta)], \qquad (95)$$

where $\eta_0 = -\delta\alpha \in M^{d_t}$ (with α having elements α_{ij}) and $\eta_{ij} = e_{ij} - \delta\alpha$.
We have $1 + d_t(d_t + 1)/2$ vectors η in $d_t(d_t + 1)/2$ dimensional space. This set of vectors is strongly positively complete (arbitrage-free), so that the results of Chapter 1 and 2 yield explicit uniquely defined risk-neutral probabilities that can be looked at as transition probabilities for a Markov chain, just like in the previous section.

Exercise. Write down the corresponding risk-neutral probabilities.

Remark: mean-field limit

Similar limiting continuous-time Markov chains can be identified. These chains are certain death processes.

Later on we shall develop the *dynamic law of large numbers limit (LLN)* with large state spaces for Markov chains arising from the evaluation of CDS.

The mean field limit for the Markov model 3') is a deterministic process on $\mathbf{R}_+^N$ with the evolution described by the ODE

$$\dot{\omega}^j = -\omega_j \alpha_j(t, \omega_1, \cdots, \omega_k). \tag{96}$$

Chapter 6

Comparison with the standard probabilistic approach.

Arbitrage-free markets and arbitrage-free prices, (geometric formulation), superhedge and superreplication, complete markets, second fundamental theorem of option pricing.

Following essentially: H. Föllmer and A. Schied. Stochastic Finance. An introduction in discrete time. First Ed. De Gruyter 2002.

The setting

The market consists of $d+1$ securities with the initial prices

$$\bar{s} = (s_0, s^1, \cdots, s^d) = (s_0, s) \in \mathbf{R}_+^{d+1}, \quad s_i > 0 \, \forall \, i \text{ and } s_0 = 1,$$

and with the next time t prices

$$\bar{S} = (S^0, \cdots, S^d) = (S^0, S) \in \mathbf{R}^+_{d+1},$$

where $S^0 = 1 + r$ with a constant $r \geq 0$, and S an $\mathbf{R}_+^d$-valued r.v. on a probability space $(\Omega, \mathcal{F}, \mathbf{P})$. *Portfolio* is a vector

$$\bar{\gamma} = (\gamma^0, \gamma) \in \mathbf{R}_{d+1},$$

the numbers of shares taken (can be negative).

The setting

If the price of a portfolio is $(\bar{s} \cdot \bar{\gamma})$ at the beginning, it becomes

$$\bar{S} \cdot \bar{\gamma} = \gamma^0(1+r) + S(\omega) \cdot \gamma$$

at time t. And $Y \in \mathbf{R}^d$ with

$$Y^i = \frac{S^i}{1+r} - s_i$$

is the vector of the *discount net gain*.

Def. $\bar{\gamma} \in \mathbf{R}^+_{d+1}$ is an *arbitrage opportunity* if $\bar{s} \cdot \bar{\gamma} \leq 0$, but $\bar{S} \cdot \bar{\gamma} \geq 0$ P-a.s. and $\mathbf{P}(\bar{s} \cdot \bar{\gamma} > 0) > 0$.

Lemma. Arbitrage exists iff $\exists \gamma \in \mathbf{R}^d$: $\gamma \cdot Y \geq 0$ and $\mathbf{P}(\gamma \cdot Y > 0) > 0$.

Proof of the Lemma

(i) Let $\bar{\gamma} = (\gamma^0, \gamma)$ arbitrage. Then γ satisfies the second condition. In fact,

$$\gamma^0 + s \cdot \gamma \leq 0 \quad \text{and } \gamma^0(1+r) + S \cdot \gamma > 0$$

implies

$$(1+r)\gamma \cdot Y = \gamma \cdot S - (1+r)\gamma \cdot s \geq \gamma \cdot S + (1+r)\gamma^0 > 0$$

and the same with positive probability.
(ii) If γ satisfies the second condition, then take $\gamma^0 = -s \cdot \gamma$ and (γ^0, γ) is an arbitrage. In fact,

$$\bar{\gamma} \cdot \bar{S} = \gamma \cdot S - (1+r)\gamma \cdot s = (1+r)\gamma \cdot Y \geq 0.$$

Corollary. Absence of arbitrage holds iff $\gamma \cdot Y \geq 0$ implies

$\gamma \cdot Y = 0$ a.s.

Risk-neutral laws

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Def. P^* is a *risk-neutral (or martingale) probability* if $\bar{s} = \mathbf{E}^*[\bar{S}/(1+r)]$. Of course equivalent to $s = \mathbf{E}^*[S/(1+r)]$ and $\mathbf{E}^* Y = 0$.

Let $\mathcal{P} = \{P^* : P^* \equiv \mathbf{P}$, and it is risk-neutral$\}$.

Theorem 1. (Fundamental theorem of option pricing). Market is arbitrage free iff $\mathcal{P} \neq \emptyset$. In which case $\exists P^* \in \mathcal{P} : dP^*/d\mathbf{P}$ is a bounded function.

Proof. (i) If $P^* \in \mathcal{P}$ and $\bar{\gamma} \in \mathbf{R}^+_{d+1}$ is an *arbitrage opportunity*, then

$$\bar{\gamma} \cdot \bar{s} = \mathbf{E}^* \frac{\bar{\gamma} \cdot \bar{S}}{1+r} > 0,$$

which contradicts to $\bar{\gamma} \cdot \bar{s} \leq 0$.

Proof (cont.)

(ii) We have to show that ($\gamma \cdot Y \geq 0 \Longrightarrow \gamma \cdot Y = 0$ a.s.) implies $\exists P^* : P^* \equiv \mathbf{P}$ and $dP^*/d\mathbf{P}$ bounded and $E^* Y = 0$. First assume $\mathbf{E}|Y| < \infty$. Set $\mathcal{Q}$ the convex set of prob measure such that $Q \equiv \mathbf{P}$ and $dQ/d\mathbf{P}$ bounded, and

$$C = \{\mathbf{E}_Q[Y] : Q \in \mathcal{Q}\}$$

We need to show that C contains the origin. Suppose otherwise. As C convex it follows by the separation theorem that $\exists \gamma \in \mathbf{R}^d$ such that $\mathbf{E}_Q(\gamma \cdot Y) \geq 0$ for all Q and $\mathbf{E}_{Q_0}(\gamma \cdot Y) > 0$ for some Q_0. This implies that $\gamma \cdot Y \geq 0$ a.s. (check why!) and $\mathbf{P}(\gamma \cdot Y > 0) > 0$. Thus an arbitrage opportunity arises.

If $\mathbf{E}|Y| = \infty$ we can define an equivalent measure $\tilde{P}$ such that $\tilde{\mathbf{E}}|Y| < \infty$, for instance choosing the bounded density $d\tilde{P}/d\mathbf{P}$ proportional to $1/(1 + |Y|)$. And we are reduced to the previous case.

Nonredundancy

Let $\mathcal{V} = \{\bar{\gamma} \cdot \bar{S} : \bar{\gamma} \in \mathbf{R}^{d+1}\}$. Though for a $V \in \mathcal{V}$ such $\bar{\gamma}$ may not be unique, say exist $\gamma_1, \gamma_2 : V = \bar{\gamma}_i \cdot \bar{S}$, but $\bar{\gamma}_i \cdot \bar{s}$ will be the same for a no arbitrage case. In fact,

$$\bar{\gamma}_1 \cdot \bar{s} - \bar{\gamma}_2 \cdot \bar{s} = \mathbf{E}^*[\frac{(\bar{\gamma}_1 - \bar{\gamma}_2) \cdot \bar{S}}{1+r}] = 0.$$

Thus one defines the price for a $V \in \mathcal{V}$ as $\pi(V) = \bar{\gamma} \cdot \bar{s}$.

Def. A market is *nonredundant* if $\bar{\gamma} \cdot \bar{S} = 0$ a.s. implies $\bar{\gamma} = 0$. Otherwise there exists $\gamma^i \neq 0$ such that

$$S^i = -\frac{1}{\gamma^i} \sum_{j \neq i} \gamma^j S^j.$$

If there is no arbitrage, then it follows that

$$s^i = -\frac{1}{\gamma^i} \sum_{j \neq i} \gamma^j s^j.$$

Nonredundancy

Proposition. If nonredundant, then the gains are linear independent, that is

$$\gamma \cdot Y = 0 \text{ a.s.} \Longrightarrow \gamma = 0. \tag{97}$$

Proof. $\gamma \cdot Y = 0$ a.s. means $\gamma \cdot S - (1+r)s = 0$ a.s. If $\gamma \neq 0$, there exists $\gamma^i \neq 0$ and then

$$S^i = -\frac{1}{\gamma^i} \sum_{j \neq i, 0} \gamma^j S^j + \frac{1}{\gamma^i}(1+r)s^i.$$

Defining $\bar{\gamma} = (\gamma^0, \gamma)$ with $\gamma^0 = -s^i$ yields linear dependence of $\bar{S}$.

Remark (Exercise). If (97) holds and the market is arbitrage free, then it is nonredundant.

Derivative securities

Def. A *contingent claim* is a non-negative r.v. C. It is a *derivative* of $\bar{S}$ if it is measurable with respect to the σ-algebra generated by $\bar{S}$, or, equivalently, $C = f(\bar{S})$ with some measurable f.

Introducing a claim C with the initial price c is equivalent to introducing a new asset on the market $S^{d+1} = C$ with $s = c$.

Def. A number $c \geq 0$ is an *arbitrage free price* of a contingent claim C iff the market model extended to $S^{d+1} = C$ with $s = c$ is arbitrage-free.

Denote $\Pi(C)$ the set of all such prices. It is convex (why!?) and thus an interval.

Derivative securities

Theorem 2. Let the initial market be arbitrage-free. Then for any C the set $\Pi(C)$ is nonempty and

$$\Pi(C) = \left\{ \mathbf{E}^*[\frac{C}{1+r}] : P^* \in \mathcal{P} \text{ and } \mathbf{E}^*(C) < \infty \right\}.$$

Proof. Only $\Pi(C) \neq \emptyset$ is not obvious. To show this choose $\tilde{P} \equiv \mathbf{P}$ such that $\tilde{E}(C) < \infty$ (always possible, as above). The market is arbitrage free under $\tilde{P}$. Hence, by the fundamental theorem, there exists $P^* \in \mathcal{P}$ with $dP^*/d\tilde{P}$ bounded. Hence

$$\mathbf{E}^*(C) = \tilde{\mathbf{E}}(C\frac{dP^*}{d\tilde{P}}) < \infty.$$

Derivative securities

Def. Arbitrage bounds:

$$\pi_{sup}(C) = \sup \Pi(C), \quad \pi_{inf}(C) = \inf \Pi(C).$$

Remark. Easy to show (each absolutely continuous can be approximated by equivalent)

$$\pi_{sup}(C) = \sup\{\mathbf{E}^*[\frac{C}{1+r}] : P^* \text{ risk n. and abs. cont., } \mathbf{E}^*(C) < \infty\},$$
$$\pi_{inf}(C) = \inf\{\mathbf{E}^*[\frac{C}{1+r}] : P^* \text{ risk n. and abs. cont., } \mathbf{E}^*(C) < \infty\}. \tag{98}$$

Dual characterization

Theorem 3. One has

$$\begin{aligned}\pi_{sup}(C) &= \min\{M\} \\ &= \min\left\{m \in [0,\infty] : \exists\gamma \in \mathbf{R}^d : m + \gamma \cdot Y \geq \frac{C}{1+r} \text{ a.s.}\right\}, \\ \pi_{inf}(C) &= \max\{m \in [0,\infty] : \exists\gamma \in \mathbf{R}^d : m + \gamma \cdot Y \leq \frac{C}{1+r} \text{ a.s.}\}.\end{aligned} \tag{99}$$

Dual characterization: proof (first equation)

For $m \in M$ we have $m \geq \mathbf{E}^*[C/(1+r)]$ for any $P^* \in \mathcal{P}$ and hence $\inf\{M\} \geq \pi_{sup}(C)$.
Let us show that $\inf\{M\} = \pi_{sup}(C)$ by proving that $m > \pi_{sup}(C)$ implies that $m \geq \inf\{M\}$. By definition, if $m > \pi_{sup}(C)$ (and is finite), then there exists an arbitrage opportunity for the extended market with $S^{d+1} = C$, $s^{d+1} = m$, that is a vector $(\gamma, \gamma^{d+1}) \in \mathbf{R}^{d+1}$ such that

$$\gamma \cdot Y + \gamma^{d+1}(\frac{C}{1+r} - m) \geq 0 \text{ a.s.}$$

and is positive with a positive probability. And $\gamma^{d+1} \neq 0$, because the initial market was arbitrage-free.

Dual characterization: proof (first equation)

Moreover, taking expectation we get

$$\gamma^{d+1}\mathbf{E}^*(\frac{C}{1+r} - m) \geq 0,$$

and thus $\gamma^{d+1} < 0$. Hence defining $\zeta = -\gamma/\gamma^{d+1}$ we get $m + \zeta \cdot Y \geq C/(1+r)$ a.s., and thus $m \geq \inf\{M\}$.

Simple limiting argument shows (exercise) that $\inf\{M\} \in M$ completing the proof.

Superreplication and superhedge

Thus for $m \in M$

$$(1+r)(m - s \cdot \gamma) + \gamma \cdot S = \bar{\gamma} \cdot \bar{S} \geq C,$$

and thus $\pi_{sup}(C)$ is the smallest amount of capital which, invested in a risk-free manner, yields a *superhedge* or *superreplication* of C.

Equivalently for a finite range E of Y (with full support):

$$\begin{aligned} \pi_{sup}(C) &= \min_{\gamma} \max_{\xi \in E} \left[\frac{C(\xi)}{1+r} - (\gamma, \xi) \right], \\ \pi_{inf}(C) &= \max_{\gamma} \min_{\xi \in E} \left[\frac{C(\xi)}{1+r} - (\gamma, \xi) \right]. \end{aligned} \tag{100}$$

Superreplication and Replicable claims

Def. Contingent claim C is *attainable* (replicable, redundant), if $C \in \mathcal{V}$, that is $C = \bar{\gamma} \cdot \bar{S}$ and $\bar{\gamma}$ is called a *replicating portfolio* for C.

Theorem 4. Market is arbitrage free and C is a contingent claim. Then (i) If C is attainable, the $\Pi(C)$ is a singleton, it consists of a single element $\bar{\gamma} \cdot \bar{s}$. (ii) If not, then either $\pi_{inf}(C) = \infty$, or $\Pi(C) = (\pi_{inf}, \pi_{sup})$.

Proof

Proof. (i) clear. (ii) One needs to show that $\pi_{inf}, \pi_{sup} \notin \Pi(C)$.

By the dual characterization, $\exists \gamma \in \mathbf{R}^d$ s.t.

$$\pi_{inf}(C) + \gamma \cdot Y \leq \frac{C}{1+r}.$$

This cannot be identity, as C is not attainable. Hence, if $\gamma^0 = s \cdot \gamma - \pi_{inf}(C)$, then $(\gamma^0, -\gamma, 1)$ an arbitrage opportunity for the extended market with $s^{d+1} = \pi_{inf}(C)$, $S^{d+1} = C$. Thus $\pi_{inf}(C)$ is not arbitrage-free.

Exercise. Perform the same argument to show that $\pi_{sup}(C)$ is not arbitrage-free.

Complete market

Def. An arbitrage-free market is called *complete* if every contingent claim is attainable.

Thus $\mathcal{V} = L^0(\Omega, \mathcal{F}, \mathbf{P})$ space of a.s. finite r.v. The dimension of any $L^p(\Omega, \mathcal{F}, \mathbf{P})$ is the max of N such that there exists a partition of Ω in N sets of positive measure.

Theorem 5. Arbitrage-free market is complete iff risk-neutral measure is unique. (This statement is sometimes called the second fundamental theorem of option pricing.) In this case the dimension of $L^0(\Omega, \mathcal{F}, \mathbf{P})$ is not bigger than $d + 1$.

Proof. If complete than the indicators of each A is attainable and, by Theorem 4, $P^*[A] = E^*(\mathbf{1}_A)$ does not depend on P^*, so that P^* is uniquely defined. Conversely, if P^* is unique, then there is a unique arbitrage free price for any C and thus C is attainable. And then $L^0(\Omega, \mathcal{F}, \mathbf{P})$ is contained in $\mathcal{V}$ and thus has dimension not exceeding $d + 1$.

Chapter 7 (uses more advanced probability)

Optimal stopping, Dynkin's games and game options

Dynamic programming for stochastic games, martingales, optimal stopping problem, Snell's envelope.

Literature:

E.B. Dynkin. Game variant of a problem on optimal stopping. Soviet Math -Doklady 10 (1969), 270-274.
J. Neveu. Discrete-Parameter Martingales. North-Holland. Amsterdam, 1975.
Yu. Kifer. Game options. Finance and Stochastics 4:4 (2000), 443-463.

Theorem on *ess* sup

Let F be a family of measurable functions $\Omega \to \bar{\mathbf{R}}$ on a prob. space $(\Omega, \mathcal{F}, \mathbf{P})$. Then

(i) there exists a unique (up to equivalence) measurable function $g : \Omega \to \bar{\mathbf{R}}$ such that (a) $g \geq f$ a.s. $\forall\, f \in F$, (b) if h is measurable and $h \geq f$ a.s. for all $f \in F$, then $h \geq g$ a.s. Such g is denoted *ess* $\sup(F)$.

(ii) There exists a sequence f_n s.t. $g = \sup f_n$ a.s.
If F is a directed set, then f_n can be chosen increasing.

Theorem on *ess* sup

Proof. Turning $\bar{\mathbf{R}}$ to $[0,1]$, say by arctan, reduces the problem to the case of mappings to $[0,1]$.
Let F_c be the class of countable subsets of F. For every $G \in F_c$ define $f_G = \sup_{f \in G} f$. Let $\alpha = \sup_{F_c} \mathbf{E} f_G$. Then sup is attainable on some G (for a defining sequence G_n take $G^* = \cup\{G_n\}$ and $\mathbf{E} f_{G^*} = \alpha$).
Let us prove that $g = f_{G^*}$ is *ess* sup(F).
For (a): Let $f \in F$, the function f_G given by $G = G^* \cup \{f\}$ is $f_G = \max(g, f)$. But $\alpha = \mathbf{E} g \le \mathbf{E} \max(f, g) \le \alpha$ and hence $g = \max(g, f)$ a.s. and thus $g \ge f$. Part (b) is direct (why!?).

Example. $A \in [0,1]$ and $F = \{\mathbf{1}_{\{a\}} : a \in A\}$. Then *ess* sup$(F) = 0$, as all functions equal 0 a.s. But $\sup_{f \in F} f = \mathbf{1}_A$ (can be even nonmeasurable).

Martingales and stopping times (definitions)

Probability basis $(\Omega, \mathcal{F}, \mathcal{F}_n, \mathbf{P})$, where $\mathcal{F}_n$ is a filtration: $\mathcal{F}_n \subset \mathcal{F}_{n+1}$.

A process X_n is adapted if X_n is measurable with respect to $\mathcal{F}_n$. An integrable X_n is a martingale, if $\mathbf{E}(X_{n+1}|\mathcal{F}_n) = X_n$, a supermartingale if $\mathbf{E}(X_{n+1}|\mathcal{F}_n) \leq X_n$, a submartingale if $\mathbf{E}(X_{n+1}|\mathcal{F}_n) \geq X_n$.

An $\mathbf{Z}_+$ r.v. ν on $(\Omega, \mathcal{F}, \mathcal{F}_n, \mathbf{P})$ is a *stopping time* or *Markov time* if $\{\nu = m\} \in \mathcal{F}_m$ for all m.

Snell's problem and Snell's envelope

Theorem. Let Z_n be an adapted integrable process on $(\Omega, \mathcal{F}, \mathcal{F}_n, \mathbf{P})$ with $SZ = \sup_n Z_n^+ \in L^1$ and Λ_n the set of stopping times ν such that $\nu \geq n$ a.s. and $\mathbf{E} Z_\nu^- < \infty$. Then $X_n = \mathit{ess}\ \sup_{\Lambda_n} \mathbf{E}(Z_\nu | \mathcal{F}_n)$ is a supermartinale satisfying

$$X_n = \max(Z_n, \mathbf{E}(X_{n+1} | \mathcal{F}_n)).$$

Sometimes X_n is called *Snell's envelope*.
If Z_n is bounded by below by an integrable r.v., then X_n is the smallest supermartingale dominating Z_n. Finally

$$\mathbf{E} X_n = \sup_{\nu \in \Lambda_n} \mathbf{E} Z_\nu .$$

Proof

X_n are adapted and $Z_n \leq X_n \leq \mathbf{E}(SZ)$, and thus integrable. The key point is that the family $\mathbf{E}(Z_\nu|\mathcal{F}_n), \nu \in \Lambda_n$, is closed under sup. In fact for two $\nu_1, \nu_2 \in \Lambda_n$ we take

$$\nu = \mathbf{1}_{\bar{B}}\nu_1 + \mathbf{1}_B\nu_2, \quad B = \{\mathbf{E}(Z_{\nu_1}|\mathcal{F}_n) < \mathbf{E}(Z_{\nu_2}|\mathcal{F}_n)\}$$

and $\mathbf{E}(Z_\nu|\mathcal{F}_n) = \max(\mathbf{E}(Z_{\nu_1}|\mathcal{F}_n), \mathbf{E}(Z_{\nu_2}|\mathcal{F}_n))$.

Proof (cont.)

By Theorem on *ess* sup, there exist a non-decreasing sequence ν_k in Λ_n s.t. $X_n = \lim \mathbf{E}(Z_{\nu_k}|\mathcal{F}_n)$ a.s. Since this sequence dominates $\mathbf{E}(Z_{\nu_0}|\mathcal{F}_n)$ and is dominated by an integrable r.v., we can apply the dominated (or monotone) convergence to get

$$\mathbf{E}(X_n|\mathcal{F}_{n-1}) = \lim \mathbf{E}(Z_{\nu_k}|\mathcal{F}_{n-1}) \leq X_{n-1} \quad \text{a.s.}$$

Thus X_n is a supermartingale and

$$X_n \geq \max(Z_n, \mathbf{E}(X_{n+1}|\mathcal{F}_n)).$$

Proof (cont.)

To establish the reverse inequality write

$$Z_\nu = Z_n \mathbf{1}_{\nu=n} + Z_{\max(\nu,n+1)} \mathbf{1}_{\nu>n}.$$

From $\mathbf{E}(Z_{\max(\nu,n+1)}|\mathcal{F}_{n+1}) \leq X_{n+1}$ it follows that

$$\mathbf{E}(Z_\nu|\mathcal{F}_n) \leq Z_n \mathbf{1}_{\nu=n} + \mathbf{E}(X_{n+1}|\mathcal{F}_n)\mathbf{1}_{\nu>n} \leq \max(Z_n, \mathbf{E}(X_{n+1}|\mathcal{F}_n))$$

for all $\nu \in \Lambda_n$ and thus

$$X_n \leq \max(Z_n, \mathbf{E}(X_{n+1}|\mathcal{F}_n)).$$

Proof (cont.)

Since the family $\mathbf{E}(Z_\nu|\mathcal{F}_n), \nu \in \Lambda_n$, is directed and increasing, it follows that

$$\mathbf{E}X_n = \mathbf{E}[ess \sup_{\Lambda_n} \mathbf{E}(Z_\nu|\mathcal{F}_n)] = \sup_{\Lambda_n} \mathbf{E}(Z_\nu).$$

In fact, using the sequence ν_k above,

$$\mathbf{E}[ess \sup_{\Lambda_n} \mathbf{E}(Z_\nu|\mathcal{F}_n)] = \lim \mathbf{E}(\mathbf{E}(Z_{\nu_k}|\mathcal{F}_n)) = \lim \mathbf{E}Z_{\nu_k} \leq \sup_{\Lambda_n} \mathbf{E}(Z_\nu).$$

Proof (cont.)

If X_n' is another supermartingale dominating Z_n, then by optional sampling

$$X_n' \geq \mathbf{E}(X_\nu' | \mathcal{F}_n) \geq \mathbf{E}(Z_\nu | \mathcal{F}_n)$$

for all $\nu \in \Lambda_n$ and thus $X_n' \geq X_n$ a.s.

Snell's theorem

Let Z_n be an adapted integrable process on $(\Omega, \mathcal{F}, \mathcal{F}_n, \mathbf{P})$ with $SZ = \sup_n Z_n^+ \in L^1$ and Λ_n the set of stopping times ν such that $\nu \geq n$ a.s. and $\mathbf{E} Z_\nu^- < \infty$. The $\sup_{\nu \in \Lambda_0} \mathbf{E} Z_\nu$ is attained iff the stopping time

$$\nu_0 = \inf\{n : X_n = Z_n\}(\text{or } + \infty \, \text{if} \, X_n > Z_n \, \forall \, n)$$

is a.s. finite. If this holds, then $\sup_\nu \mathbf{E} Z_\nu = \mathbf{E} Z_{\nu_0}$ and ν_0 is the smallest finite stoping time satisfying this equation. Moreover, for any $\epsilon > 0$, the formula $\nu_\epsilon = \inf\{n : X_n < Z_n + \epsilon\}$ defines an a.s. finite stopping time such that

$$\sup_\nu \mathbf{E} Z_\nu \leq \mathbf{E} Z_{\nu_\epsilon} + \epsilon.$$

Snell's theorem

Lemma. For every stopping time $\nu \leq \nu_0$ the sequence $X_{\nu\wedge n}$ is an integrable martingale. If ν is a.s. finite, then $\mathbf{E}X_\nu \geq \mathbf{E}X_0$.

Proof. $X_n > Z_n$ on the event $\{\nu_0 > n\}$ and thus on a smaller event $\{\nu > n\}$, and thus $X_n = \mathbf{E}(X_{n+1}|\mathcal{F}_n)$ there. Thus

$$\mathbf{E}(X_{\nu\wedge(n+1)}|\mathcal{F}_n) = X_\nu \mathbf{1}_{\nu\leq n} + \mathbf{E}(X_{n+1}|\mathcal{F}_n)\mathbf{1}_{\nu>n}$$

$$= X_\nu \mathbf{1}_{\nu\leq n} + X_n \mathbf{1}_{\nu>n} = X_{\nu\wedge n}.$$

Thus $X_{\nu\wedge n}$ is an integrable martingale and $\mathbf{E}X_{\nu\wedge n} = \mathbf{E}X_0$. Sequence X_n is dominated by the martingale $SZ_n = \mathbf{E}(SZ|\mathcal{F}_n)$ and hence by Fatou

$$\liminf_n \mathbf{E}(SZ_{\nu\wedge n} - X_{\nu\wedge n}) \geq \mathbf{E}(SZ_\nu - X_\nu),$$

Hence $\mathbf{E}X_\nu \geq \limsup \mathbf{E}X_{\nu\wedge n} = \mathbf{E}X_0$ completing proof of the lemma.

Proof of Snell's theorem

Only the main point:
It follows that if ν_0 is finite, then

$$\mathbf{E}Z_{\nu_0} = \mathbf{E}X_{\nu_0} \geq \mathbf{E}X_0 = \sup_{\Lambda_0} \mathbf{E}Z_{\nu}.$$

Hence ν_0 is optimal.

Snell's theorem: simpler case

Simpler case when sequence is finite: $Z_n : 0 \leq n \leq p$. Then by the backward induction show that the sequence $X_p = Z_p$ and

$$X_k = \max(Z_k, \mathbf{E}(X_{k+1}|\mathcal{F}_k))$$

defines the smallest integrable supermartingale dominating Z_n and

$$\nu_0 = \min\{n : 0 \leq n \leq p, X_n = Z_n\}$$

is optimal, because $X_{\nu_0 \wedge n}$ is a martingale and thus

$$X_0 = \mathbf{E}(X_{\nu_0}|\mathcal{F}_0).$$

Financial application: American options.

Dynamic programming for stochastic games

Assume X and Y are arbitrary finite sets, ξ a (real-valued) random variable on a probability space $(\Omega, \mathcal{F}, \mathbf{P})$, h is a real function on $X \times Y$ and for any $x \in X, y \in Y$, $\omega \in \Omega$, the game $\mathcal{H}(x, y, \xi) = \{H(x, y, \xi(\omega); a, b, \omega)\}$ is defined as the zero-sum game with the sets of strategies of the first and second player given by finite sets $A = A(x, y)$, $B = B(x, y)$, and with the payoffs

$$\mathbf{E}_\xi H(x, y, \xi; a, b, .) = \mathbf{E}(H(x, y, \xi; a, b, .)|\xi), \qquad (101)$$

which are the expectations, given ξ, of the random payoffs $H(x, y, \xi(.); a, b, .)$ to the second player in the profile (a, b), $a \in A, b \in B$ (the first player gets $-H(x, y, \xi(.); a, b, .)$).

Dynamic games, Dynamic programming

Consider now the following two-step zero-sum game $G_{h,\mathcal{H}}$:
First the players I and II choose (independently) their strategies $x \in X, y \in Y$.
Then the second player obtains $h(x,y)$, the values $\xi(\omega)$ are announced and the game $\mathcal{H}(x,y,\xi)$ with the payoffs (101) is played.

The strategies of the first (resp. second) player are described by the pairs (x,α) (resp. (y,β)), where $x \in X, y \in Y$ and $\alpha(x,y,\xi(\omega))$, $\beta(x,y,\xi(\omega))$ are measurable functions from $X \times Y \times \mathbf{R}$ to A and B respectively, and the payoffs are considered as averaged:

$$h(x,y) + \mathbf{E}H(x,y,\xi(.);\alpha(x,y,\xi(.)),\beta(x,y,\xi(.)),.).$$

Dynamic games, Dynamic programming

Theorem. *Dynamic programming for stochastic zero-sum games.*
Suppose for any $x, y, \xi(\omega)$ the game $\mathcal{H}(x, y, \xi(\omega))$ has a value

$$\mathbf{E}_\xi H(x, y, \xi; \alpha^*(x, y, \xi), \beta^*(x, y, \xi), .)$$

$$= \max_b \min_a \mathbf{E}_\xi H(x, y, \xi; a, b, .) = \min_a \max_b \mathbf{E}_\xi H(x, y, \xi; a, b, .)$$

given by certain minimax strategies α^*, β^*, so that, for all $a \in A(x, y)$, $b \in B(x, y)$,

$$\mathbf{E}_\xi H(x, y, \xi; \alpha^*(x, y, \xi), b, .)$$

$$\leq \mathbf{E}_\xi H(x, y, \xi; \alpha^*(x, y, \xi), \beta^*(x, y, \xi), .)$$

$$\leq \mathbf{E}_\xi H(x, y, \xi; a, \beta^*(x, y, \xi), .).$$

Dynamic games, Dynamic programming

Theorem (cont.)
Suppose also that (x_0, y_0) is a saddle-point of the function

$$h(x,y) + \mathbf{E}H(x,y,\xi;\alpha^*(x,y,\xi),\beta^*(x,y,\xi),.) :$$

that is

$$h(x_0,y) + H(x_0,y,\alpha^*(x_0,y,\xi),\beta^*(x_0,y,\xi),.)$$

$$\leq h(x_0,y_0) + \mathbf{E}H(x_0,y_0,\xi;\alpha^*(x_0,y_0,\xi),\beta^*(x_0,y_0,\xi),.)$$

$$\leq h(x,y_0){+}H(x,y_0,\alpha^*(x,y_0,\xi),\beta^*(x,y_0,\xi),.), \quad x \in X, y \in Y.$$

Dynamic games, Dynamic programming

Theorem (cont.)
Then the game $G_{h,\mathcal{H}}$ has a value that equals

$$h(x_0, y_0) + \mathbf{E}H(x_0, y_0, \xi; \alpha^*(x_0, y_0, \xi), \beta^*(x_0, y_0, \xi), .)$$

with minimax strategies being $(x_0, \alpha^*(x_0, y_0, \xi))$ and $(y_0, \beta^*(x_0, y_0, \xi))$, so that

$$h(x_0, y_0) + \mathbf{E}H(x_0, y_0, \xi; \alpha^*(x_0, y_0, \xi), \beta^*(x_0, y_0, \xi), .)$$

$$= \max_{y,\beta(x,y,\xi)} \min_{x,\alpha(x,y,\xi)} [h(x,y) + \mathbf{E}H(x, y, \xi; \alpha(x, y, \xi), \beta(x, y, \xi), .)]$$

$$= \min_{x,\alpha(x,y,\xi)} \max_{y,\beta(x,y,\xi)} [h(x,y) + \mathbf{E}H(x, y, \xi; \alpha(x, y, \xi), \beta(x, y, \xi), .)].$$

Dynamic programming

Proof. One has to show that, for all $y, \beta(x, y, \xi)$ and all $x, \alpha(x, y, \xi)$,

$$h(x_0, y) + \mathbf{E}H(x_0, y, \xi; \alpha^*(x_0, y, \xi), \beta(x_0, y, \xi), .)$$

$$\leq h(x_0, y_0) + \mathbf{E}H(x_0, y_0, \xi; \alpha^*(x_0, y_0, \xi), \beta^*(x_0, y_0, \xi), .)$$

$$\leq h(x, y_0) + \mathbf{E}H(x, y_0, \xi; \alpha(x, y_0, \xi), \beta^*(x, y_0, \xi), .).$$

Dynamic programming

Proof of the right inequality:

$$h(x_0, y_0) + \mathbf{E} H(x_0, y_0, \xi; \alpha^*(x_0, y_0, \xi), \beta^*(x_0, y_0, \xi), .)$$

$$\leq h(x, y_0) + \mathbf{E} H(x, y_0, \xi; \alpha^*(x, y_0, \xi), \beta^*(x, y_0, \xi), .)$$

$$= h(x, y_0) + \mathbf{E}\mathbf{E}_\xi H(x, y_0, \xi; \alpha^*(x, y_0, \xi), \beta^*(x, y_0, \xi), .)$$

$$\leq h(x, y_0) + \mathbf{E}\mathbf{E}_\xi H(x, y_0, \xi; \alpha(x, y_0, \xi), \beta^*(x, y_0, \xi), .).$$

$$= h(x, y_0) + \mathbf{E} H(x, y_0, \xi; \alpha(x, y_0, \xi), \beta^*(x, y_0, \xi), .).$$

Dynkin's game

Assume adapted processes $X_n \geq Z_n \geq Y_n \geq 0$, $n = 1, \cdots, N$, are given on $(\Omega, \mathcal{F}, \mathcal{F}_n, \mathbf{P})$ such that $X_N = Y_N$.

The zero-sum game of two players I and II is performed in N stages by the following rules.
At each moment $n = 1, \cdots, N-1$ each player has two actions: stop the game or continue. If both choose continue, then no payment on this stage occurs and the game moves to the next stage. If I chooses stop and II continue, then I pays X_n to II and the game stops. If II chooses stop and I continue, then I pays Y_n to II and the game stops. If I and II choose stop, then I pays Z_n to II and the game stops.

If the game continues to the final step N, then I pays $X_N = Y_N$ to II and the game stops.

Dynkin's game

For any $n \leq N$ let Σ_n denote the set of all stopping times with values in $[n, N]$.

Then the strategies of player I and player II are the choices of stoping times $\sigma \in \Sigma_1$ and $\tau \in \Sigma_1$, where player I and II decide to stop the game, respectively.

Let us denote by $H(\sigma, \tau)$ the resulting payoff to II.

Dynkin's game

Notice that the condition $X_n \geq Z_n \geq Y_n$ ensures that in the zero-sum two-player two action game

	stop=1	cont=2
stop=1	$\Pi_{11}^n = Z_n$	$\Pi_{12}^n = Y_n$
cont=2	$\Pi_{21}^n = X_n$	$\Pi_{22}^n = \Omega_n$

where player II is the row player that tries to maximise the payoffs given, has a minimax profile (saddle-point) in pure strategies.
The saddle-point is either (cont,stop) if $\Omega_n \geq X_n$, or (cont,cont) if $\Omega_n \in (Y_n, X_n)$, or (stop,cont) if $\Omega_n \leq Y_n$.

The result does not depend on the values Z_n.

Dynkin's game

Suppose the players arrive at the stage $N-1$. Then the remaining game has the value

$$V_{N-1} = \max_{j=1,2} \min_{i=1,2} \Pi_{i,j}^{N-1} = \min_{i=1,2} \max_{j=1,2} \Pi_{i,j}^{N-1}, \tag{102}$$

where $\Pi_{11}^{N-1} = Z_{N-1}, \Pi_{12}^{N-1} = Y_{N-1}$, $\Pi_{21}^{N-1} = X_{N-1}$, $\Pi_{22}^{N-1} = \mathbf{E}_{N-1} X_N$.
Alternatively it can be written as

$$V_{N-1} = \max_{\tau \in \Sigma_{N-1}} \min_{\sigma \in \Sigma_{N-1}} \mathbf{E}_{N-1} H(\sigma, \tau) = \min_{\sigma \in \Sigma_{N-1}} \max_{\tau \in \Sigma_{N-1}} \mathbf{E}_{N-1} H(\sigma, \tau).$$

Dynkin's game

Suppose the player arrive at the stage $N-2$. Then, by dynamic programming, the remaining game has the value

$$V_{N-2} = \max_{j=1,2} \min_{i=1,2} V_{N-1}(ij) = \min_{i=1,2} \max_{j=1,2} V_{N-1}(ij)$$

where $V_{N-1}(11), V_{N-1}(12), V_{N-1}(21)$ are given by the payoffs Z_{N-2}, Y_{N-2}, X_{N-2}, and $V_{N-1}(22)$ is given by (102). Alternatively it can be written as

$$V_{N-2} = \max_{\tau \in \Sigma_{N-2}} \min_{\sigma \in \Sigma_{N-2}} \mathbf{E}_{N-2} H(\sigma, \tau) = \min_{\sigma \in \Sigma_{N-2}} \max_{\tau \in \Sigma_{N-2}} \mathbf{E}_{N-2} H(\sigma, \tau).$$

Looking at the table above one can conclude that

$$V_{N-2} = \begin{cases} X_{N-2}, & \mathbf{E}_{N-2} V_{N-1} \geq X_{N-2}, \\ \mathbf{E}_{N-2} V_{N-1}, & \mathbf{E}_{N-2} V_{N-1} \in (Y_{N-2}, X_{N-2}), \\ Y_{N-2}, & \mathbf{E}_{N-2} V_{N-1} \leq Y_{N-2}. \end{cases} \quad (103)$$

Dynkin's game

By induction we obtain the following.

Theorem. Dynkin's games started from any time $n = 1, \cdots N-1$ have values in pure strategies

$$V_n = \max_{\tau \in \Sigma_n} \min_{\sigma \in \Sigma_n} \mathbf{E}_n H(\sigma, \tau) = \min_{\sigma \in \Sigma_n} \max_{\tau \in \Sigma_n} \mathbf{E}_n H(\sigma, \tau), \tag{104}$$

which are linked by the backward induction recursion:

$$V_n = \begin{cases} X_n, & \mathbf{E}_n V_{n+1} \geq X_n, \\ \mathbf{E}_n V_{n+1}, & \mathbf{E}_n V_{n+1} \in (Y_n, X_n), \\ Y_n, & \mathbf{E}_n V_{n+1} \leq Y_n. \end{cases} \tag{105}$$

Finally the optimal (minimax) strategies for players II and I are, respectively,

$$\tau^* = \min\{n \leq N : V_n \leq Y_n\}, \quad \sigma^* = \max\{n \leq N : V_n \geq X_n\}. \tag{106}$$

Game options and convertible (callable) bonds

Consider the standard Binomial (CRR) market model (or more general discrete market).

Games option is defined by a time period $[0, N]$ and two adapted processes $X_n \geq Y_n$, $n \leq N$ such that $X_N = Y_N$. The seller of the option must pay to the buyer Y_n, whenever she decides to terminate the contract. If the seller decides to termite the contract, he has to pay X_n, the amount $X_n - Y_n$ being the fine for earlier termination.

We are in the setting of the previous result, only the discounting factor should be taken into account.

Game options and convertible (callable) bonds

The fair price (interpretation via hedge from the point of view of the investor or seller, who is now the maximiser) for the option at the start of the contract is given by the value of the game:

$$V_n = \max_{\sigma \in \Sigma_1} \min_{\tau \in \Sigma_1} (1+r)^{-\sigma \wedge \tau} \mathbf{E} H(\sigma, \tau) = \min_{\tau \in \Sigma_1} \max_{\sigma \in \Sigma_1} \mathbf{E} (1+r)^{-\sigma \wedge \tau} H(\sigma, \tau), \tag{107}$$

where $H(\sigma, \tau)$ is the payoff to the investor under stopping times σ, τ.