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Asset market games of survival: a synthesis of evolutionary and dynamic games

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Abstract The paper examines a game-theoretic model of a financial market in which asset prices are determined endogenously in terms of a short-run equilibrium. Investors use general, adaptive strategies (portfolio rules) depending on the exogenous states of the world and the observed history of the game. The main goal is to identify portfolio rules, allowing an investor to "survive," i.e., to possess a positive, bounded away from zero, share of market wealth over an infinite time horizon. The model under consideration combines a strategic framework characteristic for stochastic dynamic games with an evolutionary solution concept (survival strategies), thereby linking two fundamental paradigms of game theory.

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1 Introduction

Conventional models of equilibrium and dynamics of asset markets are based on the principles of Walrasian general equilibrium theory.[1] This theory assumes that market participants maximize utilities of consumption subject to budget constraints. The present work develops an alternative equilibrium concept (which can be called *behavioral equilibrium*) admitting that market participants may have different patterns of behavior determined by their individual psychology. Their strategies may involve, for example, mimicking, satisficing, rules of thumb, etc. In the models we consider, the notion of a short-run price equilibrium is defined directly in terms of a strategy profile of the agents, and the process of market dynamics is viewed as a sequence of consecutively related short-run equilibria. This approach, inspired by the ideas of behavioral economics and finance,[2] eliminates a number of drawbacks of the conventional theory. In particular, it does not require the assumption of "perfect foresight" to establish an equilibrium and the knowledge of (unobservable) individual agents' utilities and beliefs to compute it. It opens new possibilities for the modeling of modern financial markets, in particular on the global level, where such objectives as domination, growth or simply survival (especially in crisis environments) play a major role. This approach lies at the basis of models in evolutionary finance,[3] that serve as a general framework for the present line of research.

The model we develop in this paper describes a financial market with endogenous asset prices determined by a short-run equilibrium of supply and demand. Uncertainty on asset payoffs at each period is modeled via an exogenous discrete-time stochastic process governing the evolution of the states of the world. The states of the world are meant to capture various macroeconomic and business cycle variables that may affect investors' behavior. The traders use general, adaptive strategies (portfolio rules), distributing their current wealth between assets at every period, depending on the observed history of the game and the exogenous random factors. The main goal of the study is to identify investment strategies that guarantee the "long-run survival" of any investor using them, in the sense of keeping a strictly positive, bounded away from zero, share of market wealth over the infinite time horizon, irrespective of the investment strategies employed by the other agents in the market. The main result establishes that Kelly (1956) famous portfolio rule of "betting your beliefs" possesses this property of unconditional survival.[4] Moreover, we show that the strategy possessing this property

[1] See Magill and Quinzii (1996) for a textbook treatment of the subject.

[2] Tversky and Kahneman (1991), Shleifer (2000), Shiller (2003), and Thaler (2005).

[3] See a survey in Evstigneev et al. (2009).

[4] The state of the art in the area of research related to the Kelly investment criterion is surveyed in MacLean et al. (2011).

is essentially unique: any other strategy of this kind (belonging to a certain class) is asymptotically similar to the Kelly rule. The result on asymptotic uniqueness we obtain may be regarded as an analogue of turnpike theorems (Dorfman et al. 1958; Nikaido 1968; McKenzie 1986; Arkin and Evstigneev 1987) stating that all optimal or quasi-optimal paths of economic dynamics converge to each other in the long run.

This work brings together the recent line of studies on evolutionary finance with the older literature on stochastic dynamic games pioneered by Shapley (1953).[5] Our modeling framework represents a dynamic non-cooperative market game, in which the mechanisms of short-term price formation and market clearing are reminiscent of those in the models by Shapley and Shubik (1977). On the conceptual level, the approach is related to the *games of survival* first considered by Milnor and Shapley (1957)[6] who studied a constant-sum stochastic game that may be viewed as a natural game-theoretic analogue of the well-known gambler's ruin decision problem (Dubins and Savage 1965). Two players play a zero-sum matrix game at each stage over the infinite time horizon, which determines the process of their wealth dynamics. The outcome of the game is either bankruptcy of one player or a draw. The payoff from the game is either $(1, 0)$ or $(0, 1)$ in the first case and $(q, 1-q)$, with $0 < q < 1$ in the latter.

In a similar vein, Shubik and Whitt (1973) consider a dynamic market game with one unit of a durable good per period, N players and a fixed total wealth distributed across the players in exogenous fixed shares. Each player can bid part or all of his current wealth on the durable good, of which he obtains an amount in proportion to his bid. The total bid is then redistributed to the players according to their fixed shares, and play proceeds to the next period. Each player's objective is to maximize the discounted sum of utilities of consumption, using Markov bidding strategies.

Among the closest to our setting are game-theoretic models considered in capital growth theory; see Bell and Cover (1980, 1988). Results obtained for those models demonstrate that the Kelly portfolio rule is "competitively optimal," the notion of competitive optimality being defined in terms an appropriate zero-sum game. In contrast with the present study, Bell and Cover (1980, 1988) deal with a standard framework of capital growth theory, where asset prices are exogenous. Evolutionary finance models—in particular, the model considered in this paper—extend that theory to the setting with endogenous equilibrium prices.

While the above classes of dynamic investment games are related to the present work in their general focus, there are important differences. A crucial difference is that the fundamental game solution concept we use here is based on the notion of a survival strategy outlined above, rather than on the conventional notion of a Nash equilibrium involving payoff maximization. The notion we deal with is defined in terms of a property holding almost surely, rather than in terms of expectations. No utilities,

[5] For more recent developments, see Neyman and Sorin (2003). In view of this paper's concern with long-run survival only, the class of stochastic games most closely related would be the one with undiscounted rewards (Vieille 2000a,b,c)

[6] For textbook treatments of this class of games, see Luce and Raiffa (1989, Section A8.4) and Maitra and Sudderth (1996, Section 7.16). For more recent research on similar classes of games see Secchi and Sudderth (2001) and references therein. Related questions are discussed in Borch (1966), Shubik and Thompson (1959) and Karni and Schmeidler (1986).

discounted or undiscounted, are assumed to be given in the model, which makes the modeling approach closer to applications, where typically quantitative information about individual investor's preferences is lacking.

The focus on the concept of survival positions this study within the literature developing the evolutionary approach in the social sciences. This approach goes back to Malthus, whose work served as an inspiration for Darwin (on the history of these ideas see Hodgeson 1993). In the twentieth century important contributions in the field were made by Schumpeter (1911), Nelson and Winter (1982), and others. One of the founders of modern evolutionary economics Boulding (1981, p. 108) wrote: "... the perception of potential threats to survival may be much more important in determining behavior than the perceptions of potential profits, so that profit maximization is not really the driving force. It is fear of loss rather than hope of gain that limits our behavior." During many years, these ideas have been primarily theoretical in nature. But the recent crisis changed the situation: what was the subject of abstract theoretical considerations turned into an issue of vital practical importance. The focus on guarantees of survival of financial institutions became a policy of governments and financial regulators.

The emphasis on questions of survival and extinction of investment strategies in a market selection process links our work to evolutionary game theory (Weibull 1995; Vega-Redondo 1996; Samuelson 1997; Hofbauer and Sigmund 1998 and Sandholm 2010). The latter was designed initially with the view to the modeling of biological systems and then received fruitful applications in economics. The notion of a survival portfolio rule, which is stable with respect to the market selection process, is akin to the notions of evolutionary stable strategies (ESS) introduced by Maynard Smith and Price (1973) and Schaffer (1988, 1989). However, the mechanism of market selection in our model is radically distinct from the typical schemes of evolutionary game theory, where repeated random matchings of species or agents in large populations result in their survival or extinction in the long run. Standard frameworks considered in that field deal with models based on a given static game, in terms of which the process of evolutionary dynamics is defined.[7] Players in such models follow relatively simple predefined algorithms, which completely describe their behavior. Our model is quite different in its essence. Although the solution concept we deal with is of an evolutionary nature, the notion of a strategy we use is the one which is characteristic for the conventional setting of dynamic stochastic games. A strategy in this setting is a general rule prescribing what action to take based on the observation of all the previous play and the history of random states of the world. Players are allowed to use any rule of this kind, possess all information needed for this purpose and have a clear goal: guaranteed survival.[8] Thus, the model at hand connects two basic paradigms of game theory—evolutionary and dynamic games.

It is worth saying a few words about the stochastic control framework underlying the game under consideration. There are two general modeling approaches in discrete-

[7] The evolutionary process may involve random noise (Foster and Young 1990; Fudenberg and Harris 1992; Cabrales 2000) and the underlying game may be random (Germano 2007).

[8] In Sect. 6 we will show that the goal of survival in the present context is equivalent to the objective of *winning* a certain game associated with the original one ("in order to survive you have to win").

time stochastic control theory—both in its conventional, single-agent version, and in the game-theoretic setting where decisions are made by several agents with different objectives. Models of the first kind are described in terms of transition functions (stochastic kernels) specifying the distribution of the state of the system at time $t+1$ for each given state and control at time t; see, e.g., Shapley (1953), Bertsekas and Shreve (1978) and Dynkin and Yushkevich (1979). In models of the second kind (such as the one in the present work), random factors influencing the system are described in terms of an exogenous random process of states of the world, the distribution of which does not depend on the actions of the agents. This approach is often associated with the term "stochastic programming" (e.g., Dynkin 1972; Rockafellar and Wets 1976; Wallace and Ziemba 2005; Birge and Louveaux 2011). Although theoretically both approaches are in a sense equivalent (see Dynkin and Yushkevich 1979, Section 2.2), in different contexts one is more natural and convenient than the other. Generally, the latter is preferable when the model involves properties of convexity, which is characteristic for economic and financial applications. In a game-theoretic setting, the second approach has been pursued in the work of A. Haurie and co-authors; see, e.g., Haurie et al. (1990). The focus in that line of work is primarily on strategies that depend only on the exogenous states of the world. Haurie et al. (1990) call them $\mathcal{S}$-adapted (adapted to the information structure $\mathcal{S}$); we call them *basic* in this paper. In general, if a solution to the game in the class of such strategies exists, it is rarely subgame perfect. But fortunately this is the case in our model. The Kelly rule, which is a basic strategy, happens to be a subgame-perfect solution in the game at hand, the solution being understood as a strategy which possesses the property of survival in the class of all, not necessarily basic, strategies.

The paper is organized as follows. Section 2 lays out the model description. Section 3 contains the statements of the main results (Theorems 1–3) and their discussion. Section 4 provides their proofs. Section 5 presents a counterexample to a conjecture regarding a possible extension of Theorem 2. In Sect. 6, we discuss links between the notion of survival and other game-theoretic solution concepts. Section 7 concludes.

2 The Model

There are $N \geq 2$ *investors* (*traders*) acting in a market where $K \geq 2$ risky *assets* (*securities*) are traded. A *portfolio* of investor i at date $t = 0, 1, \ldots$ is characterized by a vector $x_t^i := (x_{t,1}^i, \ldots, x_{t,K}^i) \in \mathbb{R}_+^K$ where $x_{t,k}^i$ indicates the amount ("physical units") of asset k in the portfolio x_t^i. The coordinates of x_t^i are non-negative: short sales are ruled out. We denote by $p_t \in \mathbb{R}_+^K$ the vector of market prices of the securities. For each $k = 1, \ldots, K$, the coordinate $p_{t,k}$ of $p_t := (p_{t,1}, \ldots, p_{t,K})$ stands for the price of one unit of asset k at date t. The scalar product $\langle p_t, x_t^i \rangle := \sum_{k=1}^K p_{t,k} x_{t,k}^i$ expresses the market value of investor i's portfolio at date t.

It is supposed that the total amount of each security k available in the market at date 0 is $V_{0,k} > 0$ and in each subsequent time period $t = 1, 2, \ldots$ it is $V_{t,k}(s^t) > 0$, where s_t is the state of the world at date t and $s^t := (s_1, \ldots, s_t)$ is the history of the process (s_t) up to time t. The sequence of states of the world $s_1, s_2, \ldots$ is an exogenous stochastic process with values in a measurable space S. Assets live for one period:

they are traded at the beginning of the period and yield payoffs at the end of it; then the cycle repeats. The payoff $A_{t,k}(s^t) \geq 0$ of asset $k = 1, 2, \ldots, K$ at date $t = 1, 2, \ldots$ depends, generally, on t and s^t. The functions $A_{t,k}(s^t)$ are measurable and satisfy

$$\sum_{k=1}^{K} A_{t,k}(s^t) > 0 \quad \text{for all } t, s^t. \tag{1}$$

The last inequality means that in each random situation at least one asset gives a strictly positive payoff.

At date $t = 0$ investors have initial endowments $w_0^i > 0$ $(i = 1, 2, \ldots, N)$. These initial endowments form the traders' budgets at date 0. Trader i's budget at date $t \geq 1$ is $\langle A_t(s^t), x_{t-1}^i \rangle$, where $A_t(s^t) := (A_{t,1}(s^t), \ldots, A_{t,K}(s^t))$. It is formed by the payoffs of the assets contained in yesterday's portfolio x_{t-1}^i of investor i. This budget is re-invested in the assets available at date t, which will yield payoffs $A_{t+1,k}(s^{t+1})$, $k = 1, \ldots, K$, at date $t + 1$. Dynamics of this kind reflects features of markets where one-period contracts are traded, that are related, e.g., to wholesale trading of energy or natural resources, products of agriculture, real estate, etc.

For each $t \geq 0$, each trader $i = 1, 2, \ldots, N$ selects a vector of *investment proportions* $\lambda_t^i := (\lambda_{t,1}^i, \ldots, \lambda_{t,K}^i)$ according to which he/she plans to distribute the available budget between assets. Vectors λ_t^i belong to the unit simplex

$$\Delta^K := \left\{ (a_1, \ldots, a_K) \in \mathbb{R}_+^K : a_1 + \cdots + a_K = 1 \right\}.$$

The investment proportions at each date $t \geq 0$ are selected by the N traders simultaneously and independently (so that we deal here with a simultaneous-move N-person dynamic game). They represent in the game under consideration the players' *actions* or *decisions*. For $t \geq 1$, these decisions might depend, generally, on the history $s^t := (s_1, \ldots, s_t)$ of the process of the states of the world and the *history of the game*

$$\lambda^{t-1} := (\lambda_l^i), \quad i = 1, \ldots, N, \quad l = 0, \ldots, t-1,$$

containing information about all the previous actions of all the players. A vector $\Lambda_0^i \in \Delta^K$ and a sequence of measurable functions $\Lambda_t^i(s^t, \lambda^{t-1})$, $t = 1, 2, \ldots$, with values in Δ^K form a *portfolio rule*, or an *investment (trading) strategy* Λ^i of trader i, according to which trader i selects investment proportions at each date $t \geq 0$. This is a general game-theoretic definition of a pure strategy, assuming full information about the game history, including the players' previous actions and the knowledge of all the past and present states of the world. In the class of such general portfolio rules, we will distinguish those for which Λ_t^i depends only on s^t, and not on the market history λ^{t-1}. Such portfolio rules will be called *basic*.

Suppose each investor i at date 0 has selected investment proportions $\lambda_0^i = (\lambda_{0,1}^i, \ldots, \lambda_{0,K}^i) \in \Delta^K$. Then the amount invested in asset k by trader i will be $\lambda_{0,k}^i w_0^i$, and the total amount invested in asset k will be equal to $\sum_{i=1}^N \lambda_{0,k}^i w_0^i$. The equilibrium price $p_{0,k}$ of each asset k can be determined from the equations

$$p_{0,k} V_{0,k} = \sum_{i=1}^{N} \lambda_{0,k}^i w_0^i, \quad k = 1, 2, \ldots, K. \tag{2}$$

On the left-hand side of (2), we have the total value, expressed in terms of the prevailing price $p_{0,k}$, of the assets of the kth type purchased by the market participants at date 0 (recall that the amount of each asset k at date 0 is $V_{0,k}$). On the right-hand side, we have the total sum of money invested in asset k by all the investors.

The investors' portfolios $x_0^i = (x_{0,1}^i, \ldots, x_{0,K}^i),\ i = 1, 2, \ldots, N$, at date 0 can be determined from the equations

$$x_{0,k}^i = \frac{\lambda_{0,k}^i w_0^i}{p_{0,k}}, \quad k = 1, 2, \ldots, K, \quad i = 1, \ldots, N, \tag{3}$$

meaning that the current market value $p_{0,k} x_{0,k}^i$ of the kth position of the portfolio x_0^i is equal to the fraction $\lambda_{0,k}^i$ of the trader i's investment budget w_0^i.

Suppose now that all the investors have chosen their investment proportion vectors $\lambda_t^i = (\lambda_{t,1}^i, \ldots, \lambda_{t,K}^i)$ at date $t \geq 1$. Then the balance between aggregate asset supply and demand implies the formula determining the equilibrium prices

$$p_{t,k} V_{t,k} = \sum_{i=1}^{N} \lambda_{t,k}^i \langle A_t, x_{t-1}^i \rangle, \quad k = 1, \ldots, K, \tag{4}$$

which, in turn, yields the expression for the investors' portfolios $x_t^i = (x_{t,1}^i, \ldots, x_{t,K}^i)$:

$$x_{t,k}^i = \frac{\lambda_{t,k}^i \left\langle A_t, x_{t-1}^i \right\rangle}{p_{t,k}}, \quad k = 1, \ldots, K, \quad i = 1, \ldots, N. \tag{5}$$

Here, in contrast with the case $t = 0$, the traders' budgets at date $t \geq 1$ are not given exogenously as initial endowments, rather they are formed by the payoffs of the previous date's portfolios x_{t-1}^i.

Given a strategy profile $(\Lambda^1, \ldots, \Lambda^N)$ of the investors, we can generate a path of the market game by setting $\lambda_0^i = \Lambda_0^i,\ i = 1, \ldots, N$,

$$\lambda_t^i = \Lambda_t^i(s^t, \lambda^{t-1}), \quad t = 1, 2, \ldots, \quad i = 1, \ldots, N, \tag{6}$$

and by defining p_t and x_t^i recursively according to Eqs. (2)–(5). The random dynamical system described determines the state of the market at each moment of time $t \geq 1$ as a measurable vector function of s^t:

$$(p_t(s^t); x_t^1(s^t), \ldots, x_t^N(s^t); \lambda_t^1(s^t), \ldots, \lambda_t^N(s^t)), \tag{7}$$

where $p_t(s^t)$, $x_t^i(s^t)$ and $\lambda_t^i(s^t)$ are the vectors of equilibrium prices, investors' portfolios and their investment proportions, respectively. (For $t = 0$, these vectors are constant.)

A comment regarding the above description of asset market dynamics is in order. Portfolio positions $x^i_{t,k}$ are well-defined by formulas (3) and (5) only if the prices $p_{t,k}$ are strictly positive, or equivalently, if the aggregate demand $\sum_{i=1}^N \lambda^i_{t,k}\langle A_t, x^i_{t-1}\rangle$ for each asset k is non-zero. Those strategy profiles for which the recursive procedure defined above generates non-zero aggregate demand for each asset will be called *admissible*. In what follows, we will deal only with strategy profiles satisfying this assumption. This assumption guarantees that the random dynamical system under consideration is well-defined and $p_{t,k} > 0$ for all t and k. By summing up equations (5) over $i = 1, \ldots, N$, we find that

$$\sum_{i=1}^N x^i_{t,k} = \frac{\sum_{i=1}^N \lambda^i_{t,k}\left\langle A_t, x^i_{t-1}\right\rangle}{p_{t,k}} = \frac{p_{t,k}V_{t,k}}{p_{t,k}} = V_{t,k}, \tag{8}$$

i.e., the market clears for every asset k and each date $t \geq 0$.

We provide a sufficient condition for a strategy profile to be admissible, that will hold for all the strategy profiles we consider in the present paper (it will follow from hypothesis (12)). Assume that one of the traders, e.g., trader 1, employs a *fully diversified* portfolio rule, prescribing to invest into all the assets in strictly positive proportions $\lambda^1_{t,k}$. Then any strategy profile containing this portfolio rule is admissible. Indeed, for $t = 0$, we get from (2) that $p_{0,k} \geq V^{-1}_{0,k}\lambda^1_{0,k}w^1_0 > 0$ and from (3) that $x^1_0 = (x^1_{0,1}, \ldots, x^i_{0,k}) > 0$ (coordinatewise). Assuming that $x^1_{t-1} > 0$ and arguing by induction, we obtain $\langle A_t, x^1_{t-1}\rangle > 0$ in view of (1), which in turn yields $p_t > 0$ and $x^1_t > 0$ by virtue of (4) and (5), as long as $\lambda^i_t > 0$.

Finally, we note that one could give a seemingly more general definition of a strategy, assuming that the investors can use information not about their own and their rivals' decisions, but also about the previous asset prices $p_0, \ldots, p_{t-1}$ and the portfolios x^i_l, $i = 1, \ldots, N, l = 0, \ldots, t-1$. But all these data are determined by the history s^{t-1} of the states of the world and the investors' decisions (λ^i_l), $i = 1, \ldots, N, l = 0, \ldots, t-1$ up to time $t-1$. Therefore this broader definition does not lead, in fact, to a more general notion of a strategy.

3 The main results

Consider an admissible strategy profile $(\Lambda^1, \ldots, \Lambda^N)$ of the investors and the path (7) of the random dynamical system generated by this strategy profile. Let

$$w^i_t = w^i_t(s^t) := \left\langle A_t(s^t), x^i_{t-1}(s^{t-1})\right\rangle \tag{9}$$

denote investor i's wealth at date $t \geq 1$. The *total market wealth* is equal to

$$W_t := \sum_{i=1}^N w^i_t = \sum_{k=1}^N A_{t,k}(s^t)V_{t-1,k}(s^{t-1}) \ (> 0).$$

We are primarily interested in the long-run behavior of the *relative wealth,* or the *market shares,* $r_t^i := w_t^i / W_t$ of the traders, i.e., in the asymptotic properties of the sequence of vectors $r_t = (r_t^1, \ldots, r_t^N)$ as $t \to \infty$.

Given an admissible strategy profile $(\Lambda^1, \ldots, \Lambda^N)$, we say that the strategy Λ^i (or investor i using it) *survives* with probability one if $\inf_{t \geq 0} r_t^i > 0$ almost surely (a.s.). This means that for almost all realizations of the process of states of the world (s_t), the market share of investor i is bounded away from zero by a strictly positive random constant. A portfolio rule Λ is called a *survival strategy* if investor i using it survives with probability one regardless of what portfolio rules Λ^j, $j \neq i$, are used by the other investors.

To formulate the main result on survival strategies, define the *relative payoffs* by

$$R_{t,k}(s^t) := \frac{A_{t,k}(s^t) V_{t-1,k}(s^{t-1})}{\sum_{m=1}^{K} A_{t,m}(s^t) V_{t-1,m}(s^{t-1})} \tag{10}$$

and put $R_t(s^t) = (R_{t,1}(s^t), \ldots, R_{t,K}(s^t))$. Consider the investment strategy $\Lambda^* = (\lambda_t^*)$ for which

$$\lambda_t^*(s^t) := E_t R_{t+1}(s^{t+1}), \tag{11}$$

where $E_t(\cdot) = E(\cdot|s^t)$ is the conditional expectation given s^t (if $t = 0$, then $E_0(\cdot) = E(\cdot)$). This strategy, depending only on the history s^t of the process (s_t), prescribes to distribute wealth according to the proportions of the conditional expectations of the relative asset payoffs. The portfolio rule (11) is a generalization of the Kelly portfolio rule of "betting your beliefs" well-known in capital growth theory—see Kelly (1956), Breiman (1961), Algoet and Cover (1988), Hakansson and Ziemba (1995), and Thorp (2006).

Assume that for each $k = 1, 2, \ldots, K$,

$$E \ln E_t R_{t+1,k}(s^{t+1}) > -\infty. \tag{12}$$

This assumption implies that the conditional expectation $E_t R_{t+1,k} = E(R_{t+1,k}|s^t)$ is strictly positive a.s., and so we can select a version of this conditional expectation that is strictly positive for all s^t. This version, $\lambda_t^*(s^t)$, will be used in the definition of the portfolio rule (11).

A central result is as follows.

Theorem 1 *The portfolio rule Λ^* is a survival strategy.*

Note that the portfolio rule Λ^* belongs to the class of basic portfolio rules: the investment proportions $\lambda_t^*(s^t)$ depend only on the history s^t of the process of states of the world, and do not depend on the market history. The following theorem shows that in this class the survival strategy $\Lambda^* = (\lambda_t^*)$ is essentially unique: any other basic survival strategy is asymptotically similar to Λ^*.

Theorem 2 *If $\Lambda = (\lambda_t)$ is a basic survival strategy, then*

$$\sum_{t=0}^{\infty} ||\lambda_t^* - \lambda_t||^2 < \infty \text{ (a.s.)}. \tag{13}$$

Here, we denote by $||\cdot||$ the Euclidean norm in a finite-dimensional space. Theorem 2 is akin to various *turnpike* results in the theory of economic dynamics, expressing the idea that all optimal or asymptotically optimal paths of an economic system follow in the long run essentially the same route: the turnpike (Nikaido 1968; McKenzie 1986). Survival strategies Λ can be characterized by the property that the wealth w_t^j of any investor j cannot grow asymptotically faster with strictly positive probability than the wealth of investor i using Λ —see Sect. 6. The class of such investment strategies is similar to the class of "good" paths of economic dynamics, as introduced by Gale (1967)—paths that cannot be "infinitely worse" than the turnpike. Theorem 2 is a direct analogue of Gale's turnpike theorem for good paths (Gale 1967, Theorem 8); for a stochastic version of this result see (Arkin and Evstigneev 1987, Chap. 4, Theorem 6).

Note that the class of basic strategies is *sufficient* in the following sense. Any sequence of vectors $r_t = (r_t^1, \ldots, r_t^N)$ $(r_t = r_t(s^t))$ of market shares generated by some strategy profile $(\Lambda^1, \ldots, \Lambda^N)$ can be generated by a strategy profile $(\lambda_t^1(s^t), \ldots, \lambda_t^N(s^t))$ consisting of basic portfolio rules. The corresponding vector functions $\lambda_t^i(s^t)$ can be defined recursively by (6). Thus it is sufficient to prove Theorem 1 only for basic portfolio rules; this will imply that the portfolio rule (11) survives in competition with any, not necessarily basic, strategies. Such considerations cannot be applied to the problem of asymptotic uniqueness of Λ^* in the class of general survival strategies, and Theorem 2 fails to hold in this class—see a counterexample in Sect. 5.

Analogues of Theorems 1 and 2 for another game-theoretic evolutionary finance model are obtained in Amir et al. (2011). That model, developed in Evstigneev et al. (2006,2008), describes a market with long-lived dividend paying assets. Its analysis is based on different (more complex) mathematical techniques.

The next result shows that the portfolio rule Λ^* is a *subgame-perfect* survival strategy, when the notion of a subgame is properly defined. To define this notion we will need the following additional assumption regarding the process $s_1, s_2, \ldots$. We will assume that the given probability measure P on the space of paths $\omega := (s_1, s_2, \ldots)$ can be represented as

$$P(d\omega) = p_1(ds_1)p_2(s_1, ds_2)\cdots p_t(s^{t-1}, ds_t)\cdots, \tag{14}$$

where for each t, $p_t(s^{t-1}, ds_t)$ is a stochastic kernel[9] specifying the conditional distribution of s_t given s^{t-1} (with respect to P). If the stochastic kernels $p_t(s^{t-1}, ds_t)$ are given, formula (14) uniquely defines the probability P by the Ionescu Tulcea theorem (see, e.g., Neveu 1965, Proposition V.1.1). If S is a standard Borel space (e.g.,

[9] A stochastic kernel $p_t(s^t, \Gamma)$ is a function of $s^t \in S^t$ and a measurable set $\Gamma \subseteq S$ such that $p_t(s^t, \Gamma)$ is a probability measure with respect to Γ for each s^t and a measurable function with respect to s^t for each Γ.

Dynkin and Yushkevich 1979, Appendix 1), then *any* probability P can be represented in the form (14) with some $p_t(s^{t-1}, ds_t)$. This follows from the existence theorem for conditional distributions on standard Borel spaces; e.g., Arkin and Evstigneev (1987, Appendix 1).

Fix some moment of time $m > 0$, the history s^m of states of the world, and real numbers $w_m^1 \geq 0$, $w_m^2 \geq 0, \ldots, w_m^N \geq 0$ such that $w_m^1 > 0$ and at least one of the numbers w_m^i, $i \geq 2$, is strictly positive. Put $w_m = (w_m^1, \ldots, w_m^N)$ and define $s_{m+1}^t = (s_{m+1}, \ldots, s_t)$. Consider the game $\mathcal{G}_m(s^m, w_m)$ which is defined like the original game with the play starting at time m (rather than 0), with the players' initial endowments w_m^i, $i = 1, \ldots, N$, with the probability measure

$$P_m^{s^m}(d\omega_{>m}) := p_{m+1}(s^m, ds_{m+1}) \cdots p_{t+1}(s^m, s_{m+1}^t, ds_{t+1}) \cdots ,$$

on the space of sequences $\omega_{>m} := (s_{m+1}, s_{m+2}, \ldots)$ and with the asset payoffs

$$A_t^{s^m}(s_{m+1}^t) := A_t(s^m, s_{m+1}^t), \; t \geq m+1.$$

Assume that the players $i = 1, \ldots, N$ select their actions (investment proportions) λ_t^i at each of the moments of time $t \geq m$ following strategies

$$\Lambda_{m,m}^i, \; \Lambda_{m,t}^i(s_{m+1}^t, \lambda_m^{t-1}), \; t > m.$$

Strategies, market dynamics and survival strategies are defined in a way fully analogous to the case $m = 0$. We defined above the basic strategy Λ^* in terms of the investment proportions $\lambda_t^*(s^t) := E[R_{t+1}(s^{t+1})|s^t]$. Now we shall define $\lambda_t^*(s^t)$ as the version of this conditional expectation representable as

$$\lambda_t^*(s^t) = \int p_{t+1}\left(s^t, ds_{t+1}\right) R_{t+1}\left(s^{t+1}\right),$$

assuming that this function is strictly positive.

The fact that Λ^* is a *subgame-perfect* survival strategy is expressed in the following theorem.

Theorem 3 *For each s^m and w_m the basic portfolio rule defined by the investment proportions*

$$\lambda_t^*(s^m), \; \lambda_t^*\left(s^m, s_{m+1}^t\right), \; t > m,$$

is a survival strategy in the game $\mathcal{G}_m(s^m, w_m)$.

An interesting question which we do not consider in this paper is the analysis of conditions on the strategy profile of investors under which the Kelly rule Λ^* is the *single* survivor in the market selection process. More precisely, suppose investor 1 uses Λ^*, while all the others use some other strategies $\Lambda^2, \ldots, \Lambda^N$ distinct from Λ^*. Under what conditions on $(\Lambda^2, \ldots, \Lambda^N)$ the market share of the first investor tends to one a.s. (so that he/she not only survives, but drives all the others out of the market)? Generally, this is a more delicate question than those we focus on in this work. It was examined by Blume and Easley (1992) in the case of Arrow securities, fixed-mix strategies and independent identically distributed (i.i.d.) states of the world s_t. In this case, when the market is complete, the analysis essentially reduces to the classical capital growth theory with exogenous asset returns (Algoet and Cover 1988), from which the results on the single survival follow. In models of incomplete markets, the analogous question was studied by Evstigneev et al. (2002) (a finite space S and i.i.d. s_t) and Amir et al. (2005) (a finite-state stationary Markov s_t). Only fixed-mix strategies and only basic strategies were considered in the former and in the latter papers, respectively. It should be noted that even under the above restrictive assumptions the results of the two papers do not imply those in the present one and vice versa. We leave the analysis of the single survivor problem in the present, most general setting for further research.

The above-mentioned paper by Blume and Easley (1992) initiated a strand of literature on market selection (e.g., Sandroni 2000; Sciubba 2005) pursuing a different approach in the field. Although this approach is also inspired by the concept of survival, it deals with traditional general equilibrium models in which agents maximize infinite sums of discounted expected utilities. The agents possess their own subjective beliefs based on which they make investment decisions, and the correctness or incorrectness of these beliefs might influence their success or failure in the market selection process. The main questions and goals in that literature are distinct from those in the present paper and the results obtained are rarely (only in some special cases) comparable with ours. For a comprehensive discussion of that direction of research, we refer the reader to the survey by Blume and Easley (2009) (see also Blume and Easley 2006).

4 Proofs of the main results

Proof of Theorem 1 1st step We begin with the derivation of a system of equations describing the dynamics of the market shares r_t^i. From (2)–(5) and (9), we get

$$p_{t,k}V_{t,k} = \langle \lambda_{t,k}, w_t \rangle, \ \ x_{t,k}^i = \frac{\lambda_{t,k}^i w_t^i V_{t,k}}{\langle \lambda_{t,k}, w_t \rangle}, \tag{15}$$

where $\lambda_{t,k} := (\lambda_{t,k}^1, \ldots, \lambda_{t,k}^N)$ and $w_t := (w_t^1, \ldots, w_t^N)$. Consequently,

$$w_{t+1}^i = \sum_{k=1}^K A_{t+1,k} x_{t,k}^i = \sum_{k=1}^K A_{t+1,k} V_{t,k} \frac{\lambda_{t,k}^i w_t^i}{\langle \lambda_{t,k}, w_t \rangle}. \tag{16}$$

By summing up these equations over $i = 1, \ldots, N$, we obtain

$$W_{t+1} = \sum_{k=1}^K A_{t+1,k} V_{t,k} \frac{\sum_{i=1}^N \lambda_{t,k}^i w_t^i}{\langle \lambda_{t,k}, w_t \rangle} = \sum_{k=1}^K A_{t+1,k} V_{t,k}. \tag{17}$$

Dividing the left-hand side of (16) by W_{t+1}, the right-hand side of (16) by $\sum_{m=1}^{K} A_{t+1,m} V_{t,m}$, and using (17) and (10), we arrive at the system of equations

$$r_{t+1}^i = \sum_{k=1}^{K} R_{t+1,k} \frac{\lambda_{t,k}^i r_t^i}{\langle \lambda_{t,k}, r_t \rangle}, \; i = 1, \ldots, N. \tag{18}$$

2nd step Observe that it is sufficient to prove Theorem 1 in the case of $N = 2$ investors. Consider the random dynamical system (18) and define

$$\tilde{\lambda}_{t,k}^2(s^t) = \begin{cases} (\lambda_{t,k}^2 r_t^2 + \cdots + \lambda_{t,k}^N r_t^N)/(1 - r_t^1) & \text{if } r_t^1 < 1, \\ 1/K & \text{if } r_t^1 = 1. \end{cases} \tag{19}$$

Then we have

$$\lambda_{t,k}^2 r_t^2 + \cdots + \lambda_{t,k}^N r_t^N = \left(1 - r_t^1\right) \tilde{\lambda}_{t,k}^2,$$
$$\langle \lambda_{t,k}, r_t \rangle = r_t^1 \lambda_{t,k}^1 + (1 - r_t^1) \tilde{\lambda}_{t,k}^2,$$

and so

$$r_{t+1}^1 = \sum_{k=1}^{K} R_{t+1,k} \frac{\lambda_{t,k}^1 r_t^1}{r_t^1 \lambda_{t,k}^1 + (1 - r_t^1) \tilde{\lambda}_{t,k}^2}. \tag{20}$$

By summing up equations (18) over $i = 2, \ldots, N$, we obtain

$$1 - r_{t+1}^1 = \sum_{k=1}^{K} R_{t+1,k} \frac{\tilde{\lambda}_{t,k}^2 (1 - r_t^1)}{r_t^1 \lambda_{t,k}^1 + (1 - r_t^1) \tilde{\lambda}_{t,k}^2}. \tag{21}$$

Thus the sequence $(r_t^1(s^t))$ generated by the original N-dimensional system (18) is the same as the analogous sequence generated by the 2-D system (20)–(21) corresponding to the game with two investors $i = 1, 2$ whose investment proportions are $\lambda_{t,k}^1(s^t)$ and $\tilde{\lambda}_{t,k}^2(s^t)$, respectively.

3rd step Assume that $N = 2$ and $\lambda_{t,k}^1 = \lambda_{t,k}^*$. Since $\lambda_{t,k}^* > 0$, our standing hypothesis on the strict positivity of the investment proportions of the first investor is valid. Putting $\kappa_t = \kappa_t(s^t) := r_t^1(s^t)$, we obtain from (18) with $N = 2$:

$$\kappa_{t+1} = \sum_{k=1}^{K} R_{t+1,k} \frac{\lambda_{t,k}^1 \kappa_t}{\lambda_{t,k}^1 \kappa_t + \lambda_{t,k}^2 (1 - \kappa_t)}.$$

Observe that the process $\ln \kappa_t$ is a submartingale. Indeed, we have

$$
\begin{aligned}
E_t \ln \kappa_{t+1} - \ln \kappa_t &= E_t \ln \sum_{k=1}^{K} R_{t+1,k} \frac{\lambda_{t,k}^1}{\lambda_{t,k}^1 \kappa_t + \lambda_{t,k}^2 (1-\kappa_t)} \\
&\geq E_t \sum_{k=1}^{K} R_{t+1,k} \ln \frac{\lambda_{t,k}^1}{\lambda_{t,k}^1 \kappa_t + \lambda_{t,k}^2 (1-\kappa_t)} \\
&= \sum_{k=1}^{K} \lambda_{t,k}^1 \ln \frac{\lambda_{t,k}^1}{\lambda_{t,k}^1 \kappa_t + \lambda_{t,k}^2 (1-\kappa_t)} \\
&= \sum_{k=1}^{K} \lambda_{t,k}^1 \ln \lambda_{t,k}^1 - \sum_{k=1}^{K} \lambda_{t,k}^1 \ln[\lambda_{t,k}^1 \kappa_t + \lambda_{t,k}^2 (1-\kappa_t)] \geq 0 \text{ (a.s.)}.
\end{aligned}
$$

We used here Jensen's inequality for the concave function $\ln x$ and the elementary inequality

$$
\sum_{k=1}^{K} a_k \ln a_k \geq \sum_{k=1}^{K} a_k \ln b_k \quad [\ln 0 := -\infty] \tag{22}
$$

holding for any vectors $(a_1, \ldots, a_K) > 0$ and $(b_1, \ldots, b_K) \geq 0$ with $\sum a_k = \sum b_k = 1$ (see Lemma 2 below).

Further,

$$
\begin{aligned}
\kappa_{t+1} &= \kappa_t \sum_{k=1}^{K} R_{t+1,k} \frac{\lambda_{t,k}^1}{\lambda_{t,k}^1 \kappa_t + \lambda_{t,k}^2 (1-\kappa_t)} \\
&\geq \kappa_t \sum_{k=1}^{K} R_{t+1,k} \left(\min_m \lambda_{t,m}^1 \right) = \kappa_t \left(\min_m \lambda_{t,m}^1 \right).
\end{aligned}
$$

Since $E \min_m \ln \lambda_{t,m}^1 > -\infty$ by virtue of assumption (12) and κ_0 is a strictly positive non-random number, each of the random variables $0 < \kappa_t \leq 1$ satisfies $E|\ln \kappa_t| < \infty$.

The non-positive submartingale $\ln \kappa_t$ has a finite limit a.s., and so $\kappa_t \to \kappa_\infty$ (a.s.), where κ_∞ is a strictly positive random variable. Consequently, the sequence $\kappa_t > 0$ is bounded away from zero with probability one, which means that investor 1 survives almost surely. □

The proof of Theorem 2 is based on two lemmas.

Lemma 1 *Let ξ_t be a submartingale such that $\sup_t E\xi_t < \infty$. Then the series of non-negative random variables $\sum_{t=0}^{\infty}(E_t \xi_{t+1} - \xi_t)$ converges a.s.*

Proof We have $\zeta_t := E_t \xi_{t+1} - \xi_t \geq 0$ by the definition of a submartingale. Further, we have

$$
\sum_{t=0}^{T-1} E\zeta_t = \sum_{t=0}^{T-1} (E\xi_{t+1} - E\xi_t) = E\xi_T - E\xi_0,
$$

and so the sequence $\sum_{t=0}^{T-1} E\zeta_t$ is bounded because $\sup_T E\xi_T < \infty$. Therefore the series of the expectations $\sum_{t=0}^{\infty} E\zeta_t$ of the non-negative random variables ζ_t converges, which implies $\sum_{t=0}^{\infty} \zeta_t < \infty$ a.s. because $E\sum_{t=0}^{\infty} \zeta_t = \sum_{t=0}^{\infty} E\zeta_t$ (see, e.g., Theorem I.12.3 in Saks 1964). □

Lemma 2 *For any vectors* $(a_1, \ldots, a_K) > 0$ *and* $(b_1, \ldots, b_K) \geq 0$ *satisfying* $\sum a_k = \sum b_k = 1$, *the following inequality holds*

$$\sum_{k=1}^{K} a_k \ln a_k - \sum_{k=1}^{K} a_k \ln b_k \geq \frac{1}{4} \sum_{k=1}^{K} (a_k - b_k)^2. \tag{23}$$

Proof We have $\ln x \leq x - 1$, which implies $(\ln x)/2 = \ln \sqrt{x} \leq \sqrt{x} - 1$, and so $-\ln x \geq 2 - 2\sqrt{x}$. By using this inequality, we get

$$\begin{aligned}\sum_{k=1}^{K} a_k(\ln a_k - \ln b_k) &= -\sum_{k=1}^{K} a_k \ln \frac{b_k}{a_k} \geq \sum_{k=1}^{K} a_k(2 - 2\frac{\sqrt{b_k}}{\sqrt{a_k}})\\ &= 2 - 2\sum_{k=1}^{K} \sqrt{a_k b_k} = \sum_{k=1}^{K} (a_k - 2\sqrt{a_k b_k} + b_k)\\ &= \sum_{k=1}^{K} (\sqrt{a_k} - \sqrt{b_k})^2.\end{aligned}$$

This yields (23) because $(\sqrt{a} - \sqrt{b})^2 \geq (a - b)^2/4$ for $0 \leq a, b \leq 1$. □

Remark 1 Lemma 2 can be deduced from an inequality between the *Kullback–Leibler divergence* [(generalizing the expression on the left-hand side of (23)] and the *Hellinger distance* (which reduces in our context to $[\sum(\sqrt{a_k} - \sqrt{b_k})^2]^{1/2}$)—see, e.g., (Borovkov 1998, Section II.31). For the reader's convenience we give a direct and elementary proof of Lemma 2, rather than referring to these general facts.

Proof of Theorem 2 Let $\Lambda = (\lambda_t)$ be a basic survival strategy. Suppose that investors $i = 1, 2, \ldots, N - 1$ use the strategy $\Lambda^* = (\lambda_t^*)$ and investor N uses Λ. By summing up equations (18) with $\lambda_t^i = \lambda_t^*$ over $i = 1, \ldots, N - 1$, we obtain

$$\hat{r}_{t+1}^1 = \sum_{k=1}^{K} R_{t+1,k} \frac{\lambda_{t,k}^* \hat{r}_t^1}{\lambda_{t,k}^* \hat{r}_t^1 + \lambda_{t,k}\left(1 - \hat{r}_t^1\right)},$$

where $\hat{r}_t^1 := r_t^1 + \cdots + r_t^{N-1}$ is the market share of the group of investors $i = 1, 2, \ldots, N - 1$ and $1 - \hat{r}_t^1 = r_t^N$ is the market share of investor N. Further, we have

$$1 - \hat{r}_{t+1}^1 = \sum_{k=1}^{K} R_{t+1,k} \frac{\lambda_{t,k}\left(1 - \hat{r}_t^1\right)}{\lambda_{t,k}^* \hat{r}_t^1 + \lambda_{t,k}\left(1 - \hat{r}_t^1\right)}.$$

Thus the dynamics of the market shares $\hat{r}_t^1 = r_t^1 + \cdots + r_t^{N-1}$, $1 - \hat{r}_t^1 = r_t^N$ is exactly the same as the dynamics of the market shares $\hat{r}_t^1, \hat{r}_t^2 = 1 - \hat{r}_t^1$ of two investors $i = 1, 2$ ($N = 2$) using the strategies $(\lambda_t^1) = (\lambda_t^*)$ and $(\lambda_t^2) = (\lambda_t)$, respectively. Since (λ_t) is a survival strategy, the random sequence $r_t^N = 1 - \hat{r}_t^1 = \hat{r}_t^2$ is bounded away from zero almost surely.

In the course of the proof of Theorem 1 (step 3), we have shown that the sequence $\ln \kappa_t := \ln \hat{r}_t^1$ is a non-positive submartingale satisfying

$$\begin{aligned} &E_t \ln \kappa_{t+1} - \ln \kappa_t \\ &\quad \geq \sum_{k=1}^{K} \lambda_{t,k}^* \ln \lambda_{t,k}^* - \sum_{k=1}^{K} \lambda_{t,k}^* \ln \left[\lambda_{t,k}^* \kappa_t + \lambda_{t,k}(1 - \kappa_t)\right] \text{ (a.s.).} \end{aligned} \tag{24}$$

By virtue of Lemma 1, the series $\sum (E_t \ln \kappa_{t+1} - \ln \kappa_t)$ of non-negative random variables converges a.s., which implies, in view of inequalities (23) and (24), that the sum

$$\sum_{t=0}^{\infty} \sum_{k=1}^{K} \left[\lambda_{t,k}^* - \lambda_{t,k}^* \kappa_t - \lambda_{t,k}(1 - \kappa_t)\right]^2 = \sum_{t=0}^{\infty} (1 - \kappa_t)^2 ||\lambda_t^* - \lambda_t||^2 \tag{25}$$

is finite with probability one. Since $\inf(1 - \kappa_t) = \inf \hat{r}_t^2 > 0$ a.s., the fact that the series in (25) converges a.s. yields (13). □

Proof of Theorem 3 Observe that the game $\mathcal{G}_m(s^m, w_m)$ is isomorphic to the original game in which the role of the measure P is played by $P_m^{s^m}$ and the number of players coincides with the number of those i for which $w_m^i > 0$. By using the definition of the investment proportions λ_t^* and the measure $P_m^{s^m}$, we obtain

$$\begin{aligned} \lambda_t^*(s^m, s_{m+1}^t) &= \int p_{t+1}(s^m, s_{m+1}^t, ds_{t+1}) R_{t+1}\left(s^m, s_{m+1}^{t+1}\right) \\ &= E^{s^m}\left[R_{t+1}(s^m, s_{m+1}^{t+1}) | s_{m+1}^t\right] \; (P_m^{s^m}\text{-almost surely}), \end{aligned} \tag{26}$$

where $E^{s^m}[\cdot | s_{m+1}^t]$ is the conditional expectation with respect to the measure $P_m^{s^m}$. The last equality in (26) follows from the results in Neveu (1965, Proposition V.1.1 and corollaries following it). Thus $\lambda_t^*(s^m, \cdot)$ is the Kelly rule in the game $\mathcal{G}_m(s^m, w_m)$, and consequently, a survival strategy in this game. □

5 A counterexample

In this section we give an example showing that Theorem 2 does not have a natural extension to the class of general, not necessarily basic, portfolio rules. One can conjecture that the following extension of Theorem 2 is valid: if $\Lambda_t^1(\cdot)$ is a general survival strategy of player 1, then for any set of strategies $\Lambda_t^2(\cdot), \ldots, \Lambda_t^N(\cdot)$ of players $2, 3, \ldots, N$, the vectors of investment proportions $\lambda_t^1(s^t)$ of investor 1 generated by the strategy profile $(\Lambda_t^1(\cdot), \Lambda_t^2(\cdot), \ldots, \Lambda_t^N(\cdot))$ converge to $\lambda_t^*(s^t)$ (in one sense or another) as $t \to \infty$. Below we provide a counterexample to this conjecture. The construction of the strategy involved in this counterexample resembles the construction of *trigger strategies* in the "folk theorems" of game theory (see, e.g., Myerson 1991).

Consider the following model. There are two investors ($N = 2$). The state space S consists of two elements: $S = \{1, 2\}$. The random states of the world $s_1, s_2, \ldots$ are independent and identically distributed with probabilities $P\{s_t = 1\} = 1/3$ and $P\{s_t = 2\} = 2/3$. There are two assets $k = 1, 2$ with $V_{t,k}(s^t) = 1$ for all $t, s^t, k = 1, 2$, and $A_{t,k}(s^t) = A_k(s_t)$, where $A(1) = (A_1(1), A_2(1)) = (1, 0)$, $A(2) = (A_1(2), A_2(2)) = (0, 1)$. In this model, $R_{t,k}(s^t) = R_k(s_t)$, where

$$R(1) = (R_1(1), R_2(1)) = (1, 0),\ \ R(2) = (R_1(2), R_2(2)) = (0, 1), \tag{27}$$

and

$$\lambda^* = E_t R(s_{t+1}) = E R(s_{t+1}) = (1/3, 2/3).$$

Consider some vector $0 < b \neq \lambda^*$ in the unit simplex Δ^2 and a set $\Gamma \subseteq \Delta^2$ such that

$$E\frac{\langle a, R(s)\rangle}{\langle b, R(s)\rangle} \leq 1, \quad a \in \Gamma. \tag{28}$$

For example, let $b = (b_1, b_2) = (1/4, 3/4)$ and

$$\Gamma = \{a = (a_1, 1 - a_1) :\ 0 \leq a_1 \leq 1/4\}.$$

Then (28) holds because

$$E\frac{\langle a, R(s)\rangle}{\langle b, R(s)\rangle} = \frac{1}{3}\frac{a_1}{b_1} + \frac{2}{3}\frac{a_2}{b_2} = \frac{4}{3}a_1 + \frac{8}{9}(1 - a_1) = \frac{4a_1 + 8}{9} \leq 1,$$

as long as $a_1 \leq 1/4$.

Let us define the strategy $\Lambda_t^1(s^t, \lambda^{t-1})$, $t = 0, 1, \ldots$, of player 1 as follows. For $t = 0$ put $\Lambda_0^1 = b$. For $t \geq 1$, define

$$\Lambda_t^1(\lambda_0^2, \ldots, \lambda_{t-1}^2) = \begin{cases} b\,, & \text{if } \lambda_l^2 \in \Gamma, l = 0, \ldots, t-1; \\ \lambda^*\,, & \text{otherwise.} \end{cases} \tag{29}$$

The action prescribed by this strategy does not depend on s^t and the previous actions λ_l^1, $l = 0, \ldots, t-1$ of player 1. It depends only on the previous actions $\lambda_l^2, l = 0, \ldots, t-1$ of player 2, the rival of player 1. According to this strategy, if the rival at least once selects a vector of investment proportions outside the set Γ, then 1 immediately starts playing λ^* and does this forever. Otherwise, 1 always plays b.

Proposition 1 *The portfolio rule $\Lambda_t^1(\lambda_0^2, \ldots, \lambda_{t-1}^2)$ is a survival strategy. There exists a whole class $\mathcal{L}$ of strategies $\Lambda_t^2(\cdot)$ of player 2 such that the vectors $\lambda_t^1(s^t)$ of investment proportions of player 1 generated by the strategy profile $((\Lambda_t^1(\cdot), \Lambda_t^2(\cdot))$ coincide for all t with the constant vector $b \neq \lambda^*$ (and hence do not converge to λ^* in any sense). The class $\mathcal{L}$ includes all the basic strategies $(\lambda_t^2(s^t))$ such that $\lambda_t^2(s^t) \in \Gamma$ for all t, s^t.*

Proof Consider any strategy $\Lambda_t^2(\cdot)$ of player 2. The strategy profile $(\Lambda_t^1(\cdot), \Lambda_t^2(\cdot))$ is admissible because $\Lambda_t^1(\cdot) > 0$. Therefore the path of the game is well-defined and the market shares r_t^1, r_t^2 of players $i = 1, 2$ satisfy

$$r_{t+1}^i = \sum_{k=1}^{2} R_k(s_{t+1}) \frac{\lambda_{t,k}^i r_t^i}{\lambda_{t,k}^1 r_t^1 + \lambda_{t,k}^2 r_t^2}, \ i = 1, 2 \tag{30}$$

(see (18)), where $(\lambda_{t,1}^i, \lambda_{t,2}^i) = (\lambda_{t,1}^i(s^t), \lambda_{t,2}^i(s^t))$ are the sequences of vectors of investment proportions of players $i = 1, 2$ generated by the strategy profile $((\Lambda_t^1(\cdot), \Lambda_t^2(\cdot))$. We have to show that r_t^1 is bounded away from zero (a.s.), or equivalently, that the random sequence $\gamma_t := r_t^2/r_t^1$ is bounded above (a.s.).

By using (27), we obtain

$$\gamma_{t+1} = \frac{\lambda_{t,s_{t+1}}^2 r_t^2/(\lambda_{t,s_{t+1}}^1 r_t^1 + \lambda_{t,s_{t+1}}^2 r_t^2)}{\lambda_{t,s_{t+1}}^1 r_t^1/(\lambda_{t,s_{t+1}}^1 r_t^1 + \lambda_{t,s_{t+1}}^2 r_t^2)} = \frac{\lambda_{t,s_{t+1}}^2 r_t^2}{\lambda_{t,s_{t+1}}^1 r_t^1} = \frac{\langle \lambda_t^2, R(s_{t+1})\rangle}{\langle \lambda_t^1, R(s_{t+1})\rangle}\gamma_t. \tag{31}$$

Denote by $\tau = \tau(s_1, s_2, \ldots)$ the moment of time such that $\lambda_\tau^2 \notin \Gamma$, $\lambda_{\tau-1}^2 \in \Gamma$, ..., $\lambda_0^2 \in \Gamma$ if such a moment of time exists; otherwise, put $\tau = \infty$. Thus, $\tau < \infty$ is the first moment of time when the random sequence λ_t^2 leaves the set Γ, and $\tau = \infty$ if this sequence always stays in Γ.

If $\tau = \infty$, then $\Gamma_{t+1} = \beta_{t+1}\Gamma_0$, where

$$\beta_{t+1} = \frac{\langle \lambda_t^2, R(s_{t+1})\rangle}{\langle b, R(s_{t+1})\rangle} \cdots \frac{\langle \lambda_0^2, R(s_1)\rangle}{\langle b, R(s_1)\rangle}$$

[(see (29) and (31)]. If $\tau < \infty$, then for each $t > \tau$ we have

$$\gamma_{t+1} = \frac{\langle \lambda_t^2, R(s_{t+1})\rangle}{\langle \lambda^*, R(s_{t+1})\rangle} \cdots \frac{\langle \lambda_{\tau+1}^2, R(s_{\tau+2})\rangle}{\langle \lambda^*, R(s_{\tau+2})\rangle} \cdot \frac{\langle \lambda_\tau^2, R(s_{\tau+1})\rangle}{\langle b, R(s_{\tau+1})\rangle} \cdots \frac{\langle \lambda_0^2, R(s_1)\rangle}{\langle b, R(s_1)\rangle}\gamma_0 = \alpha_{t+1}\Delta,$$

where

$$\alpha_{t+1} := \frac{\langle \lambda_t^2, R(s_{t+1})\rangle}{\langle \lambda^*, R(s_{t+1})\rangle} \cdots \frac{\langle \lambda_0^2, R(s_1)\rangle}{\langle \lambda^*, R(s_1)\rangle}, \ \Delta := \frac{\langle \lambda^*, R(s_{\tau+1})\rangle}{\langle b, R(s_{\tau+1})\rangle} \cdots \frac{\langle \lambda^*, R(s_1)\rangle}{\langle b, R(s_1)\rangle}\gamma_0.$$

Fix some element $\bar{a} \in \Gamma$ and define $\bar{\lambda}_t^2 := \bar{a}$ if $\lambda_t^2 \notin \Gamma$ and $\bar{\lambda}_t^2 := \lambda_t^2$ if $\lambda_t^2 \in \Gamma$. Then if $\tau = \infty$, we have $\beta_{t+1} = \bar{\beta}_{t+1}$ $(t = 0, 1, \ldots)$, where

$$\bar{\beta}_{t+1} := \frac{\langle \bar{\lambda}_t^2, R(s_{t+1}) \rangle}{\langle b, R(s_{t+1}) \rangle} \cdots \frac{\langle \bar{\lambda}_0^2, R(s_1) \rangle}{\langle b, R(s_1) \rangle}$$

because in this case $\lambda_t^2 \in \Gamma$, and so $\bar{\lambda}_t^2 = \lambda_t^2$, for all t. Observe that the process $\bar{\beta}_t$ is a non-negative supermartingale (with respect to the system of σ-algebras generated by s^t). Indeed, by virtue of (28),

$$E(\bar{\beta}_{t+1}|s^t) = \bar{\beta}_t E \frac{\langle a, R(s) \rangle}{\langle b, R(s) \rangle} \le \bar{\beta}_t$$

where $a := \bar{\lambda}_t^2(s^t) \in \Gamma$. Thus the process $\bar{\beta}_t$ converges a.s. and hence is bounded a.s.. Consequently, if $\tau = \infty$, the process $\gamma_t = \beta_t \gamma_0 = \bar{\beta}_t \gamma_0$ is bounded a.s..

Further, observe that the sequence α_t is a non-negative martingale. Indeed, if we fix s^t and put $a := (a_1, a_2) = \lambda_t^2(s^t)$, then

$$E(\alpha_{t+1}|s^t) = \alpha_t E \frac{\langle a, R(s) \rangle}{\langle \lambda^*, R(s) \rangle} = \alpha_t$$

because

$$E \frac{\langle a, R(s) \rangle}{\langle \lambda^*, R(s) \rangle} = \frac{1}{3} \frac{\langle a, R(1) \rangle}{\langle \lambda^*, R(1) \rangle} + \frac{2}{3} \frac{\langle a, R(2) \rangle}{\langle \lambda^*, R(2) \rangle} = \frac{1}{3} \frac{a_1}{1/3} + \frac{2}{3} \frac{a_2}{2/3} = 1.$$

Therefore α_t is bounded a.s., and so if $\tau < \infty$, then

$$\sup_{t > \tau} \gamma_{t+1} = \delta \sup_{t > \tau} \alpha_{t+1} < \infty \text{ (a.s.)}.$$

Thus we have obtained that the process γ_t is bounded a.s. both when $\tau = \infty$ and when $\tau < \infty$. This proves that $\Lambda_t^1(\cdot)$ is a survival strategy.

It remains to observe that if player 2 uses a strategy (e.g. a basic one) for which $\lambda_t^2(s^t) \in \Gamma$ for all t, s^t, then we have $\lambda_t^1 = b \neq \lambda^*$ for all t, s^t. □

6 Strategies of survival as unbeatable strategies

Consider an abstract game of N players $i = 1, \ldots, N$ selecting strategies Λ^i from some given sets. Let $w^i = w^i(\Lambda^1, \ldots, \Lambda^N)$ be the outcome of the game for player i corresponding to the strategy profile $(\Lambda^1, \ldots, \Lambda^N)$. Possible outcomes w^i are elements of a set $\mathcal{W}^i$. Suppose that a preference relation

$$w^i \succcurlyeq_{ij} w^j, w^i \in \mathcal{W}^i, \; w^j \in \mathcal{W}^j, \; i \neq j,$$

is given, making it possible to compare relative performance of players i and j. Let us say that a strategy Λ of player i is *unbeatable* if for any admissible strategy profile $(\Lambda^1, \Lambda^2, \ldots, \Lambda^N)$ in which $\Lambda^i = \Lambda$, we have

$$w^i(\Lambda^1, \Lambda^2, \ldots, \Lambda^N) \succcurlyeq_{ij} w^j(\Lambda^1, \Lambda^2, \ldots, \Lambda^N) \text{ for all } j \neq i. \tag{32}$$

Thus, if player i uses the strategy Λ, she cannot be outperformed by any of her rivals $j \neq i$, irrespective of what strategies they use.

Let us return to the game we consider in this paper. Denote by $\mathcal{W}$ the set of sequences of positive random variables and define the following relation $\succcurlyeq$ between two sequences $\alpha = (\alpha_t)$ and $\beta = (\beta_t)$ in $\mathcal{W}$. Let us write $\beta \succcurlyeq \alpha$ (or $\alpha \preccurlyeq \beta$) if there exists a random variable $H > 0$ such that $\alpha_t \leq H\beta_t$ (a.s.). The relation $\beta \succcurlyeq \alpha$ means that almost surely the sequence α does not grow asymptotically faster than β. Consider the asset market game with N players having initial endowments $w_0^1 > 0, \ldots, w_0^N > 0$. Given an admissible strategy profile $(\Lambda^1, \ldots, \Lambda^N)$ of N investors, we obtain for each $i = 1, 2, \ldots, N$ a sequence $w^i := (w_0^i, w_1^i, \ldots)$ of positive random variables, where w_t^i stands for investor i's wealth at date t. As we have seen, the evolution of the vectors $w_t := (w_t^1, \ldots, w_t^N)$ is governed by the random dynamical system (16). Let us regard the sequence $w^i = w^i(\Lambda^1, \ldots, \Lambda^N)$ as an outcome of the game for player i corresponding to the strategy profile $(\Lambda^1, \ldots, \Lambda^N)$. Let us apply the above definition with $\mathcal{W}^i = \mathcal{W}$ and the preference relation $\succcurlyeq$ (independent of i and j) to the specific game at hand. Its meaning in the present context is as follows: a portfolio rule Λ is an unbeatable strategy of investor i if it guarantees the asymptotically fastest growth rate of investor i's wealth, irrespective of what strategies the other investors employ.

It is easily seen that survival strategies are those and only those which are unbeatable.

Proposition 2 *A portfolio rule is a survival strategy if and only if it is unbeatable.*

Proof Since the market shares are expressed as $r_t^i = w_t^i / W_t$, relations (32) hold if and only if

$$r^i \succcurlyeq r^j \quad \text{for all } j \neq i. \tag{33}$$

If the market share r_t^i of investor i satisfies $r_t^i \geq c$ (a.s.), where c is a strictly positive random variable, then $w_t^i \geq cW_t \geq cw_t^j$ (a.s.) for all j. Thus $w_t^j \leq c^{-1}w_t^i$ (a.s.), and so (32) holds. Conversely, if $w_t^j \leq Hw_t^i$ (a.s.) for some random variable $H > 0$, then $W_t \leq [(N-1)H + 1]w_t^i$ (a.s.), which yields $r_t^i \geq [(N-1)H + 1]^{-1}$ (a.s.). □

The idea of an unbeatable (or *winning*) strategy was central in the pre-Nash game theory. At those times, solving a game meant primarily finding a winning strategy. This question was considered in the paper by Bouton (1901–1902), apparently the earliest mathematical paper in the field. Borel (1921) wrote: "One may propose to investigate whether it is possible to determine a method of play better than all others; i.e., one that gives the player who adopts it a superiority over every player who does not adopt it." A large amount of work has been done around the notion of *determinacy* of a game.

This notion is formulated in terms of winning strategies: a game is determined if one of the players has a winning strategy. Problems related to determinacy of chess were considered in the paper by Zermelo (1913); for an English translation and discussion of this paper see Schwalbe and Walker (2001). A deep mathematical analysis of this circle of questions, initiated by Gale and Stewart (1953), has led to important achievements in set theory and topology (Martin 1975; Telgársky 1987).

The problem of finding a winning strategy in two-person zero-sum games essentially reduces to finding an equilibrium (minimax) strategy. Therefore questions of this kind were typically formulated and examined in terms of minimax strategies in zero-sum games, especially after the seminal paper by von von Neumann (1928). In the 1950s, when game theory started developing primarily as a mathematical framework of economic modeling, non-zero sum N-player games came to the fore, and the notion of Nash equilibrium became central to the field.

The concept of an unbeatable strategy as such emerged again in theoretical biology, which served as the starting point for the development of evolutionary game theory. Hamilton (1967) used this notion, and the term "unbeatable strategy"—without a rigorous formalization—in his paper on the analysis of sex ratios in populations of some species. Maynard Smith and Price (1973) formalized Hamilton's idea, but at the same time somewhat changed its content. The notion usually referred to as the Maynard Smith's ESS can be called a *conditionally unbeatable* strategy. It cannot be beaten if the rival is "weak enough." In the context of evolutionary biology, Maynard Smith's ESS is a strategy which cannot be beaten if the fraction of the rivals (mutants) in the population is sufficiently small.[10] This definition requires the population to be infinite, since one has to speak of its arbitrarily small fractions. Versions of Maynard Smith's ESS applicable to finite populations were considered by Schaffer (1988, 1989). Schaffer's notions of ESS, the weaker and the stronger ones, are also conditionally unbeatable strategies. The former requires that the population contains only *one* mutant, the latter assumes that there are several *identical* mutants. Having in mind this, evolutionary, branch of the history of unbeatable strategies, it is not surprising that they have appeared in our evolutionary finance context.

7 Concluding remarks

Research at the interface of mathematical finance and evolutionary game theory seems promising to us as it can potentially serve as the basis for quantitative financial applications. Models developed in this field rely upon behavioral approaches, rather than the traditional schemes of utility maximization. Mathematical finance has a record of outstanding successes when a mathematical theory influenced financial practice. It is a unique field, where theoretical highlights have led to the creation of new markets—a result of the revolutionary Black–Scholes formula. By and large, these achievements were due to the successful exploitation of utility-independent pricing principles based on the considerations of arbitrage and hedging. An important step towards quantitative game-theoretic modeling in finance was made in the pioneering work of Kifer

[10] An unconditional variant of the Maynard Smith's ESS was considered by Kojima (2006).

(2000), who employed Dynkin (1969) stopping games for the design of a new type of derivative securities—game (Israeli) options. Kifer combined the classical setting of zero-sum games with the no-arbitrage pricing principle, a cornerstone of finance. We believe that the continuation of research along these lines will lead to new fruitful applications in investment science.

Finally, it would be of interest to consider the possibility of the application of the present approach in theoretical biology. One can try to employ it in models that do not use the notion of a fitness function (a counterpart of utility) and focus on survival as the fundamental criterion of evolutionary stability. The results of the modeling of evolution *in pecunia*—in the realm of finance—might turn out to be helpful for the study of bio-evolution.

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