Multi-Period Asset Pricing
Part 2

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Arbitrage-free pricing in single period model

Single period binomial model

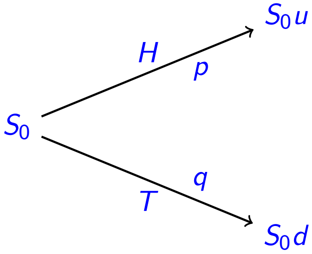
Single period binomial model

There are two times: 0 and 1 and two traded assets:

Bank account with interest rate $r > -1$ (the same rate for borrowing and lending):

$$\$1 \text{ at } t = 0 \longrightarrow \$(1 + r) \text{ at } t = 1.$$

Stock with initial price S_0 and relative changes u ("up") and d ("down") such that $u > d > 0$:



Single period binomial model

Stock's price at $t = 1$ is *random*:

$$S_1 = S_1(\omega), \quad \omega \in \Omega,$$
$$S_1(H) = uS_0, \quad S_1(T) = dS_0,$$

where $\Omega = \{H, T\}$ is the *space of elementary events*: H ("head") and T ("tail").
The probabilities of the elementary events are denoted as

$$p = \mathbb{P}(H), \quad q = \mathbb{P}(T).$$

Of course,

$$p > 0, \quad q > 0, \quad p + q = 1.$$

Arbitrage in single period binomial model

Question

When is the single period binomial model arbitrage-free?

Solution

We always have that

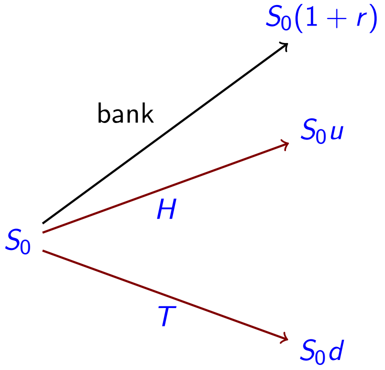
$$0 < d < u, \quad 0 < 1 + r.$$

Thus, there are 3 cases:

1. $1 + r \geq u > d$,
2. $u > 1 + r > d$,
3. $u > d \geq 1 + r$.

Arbitrage in single period binomial model

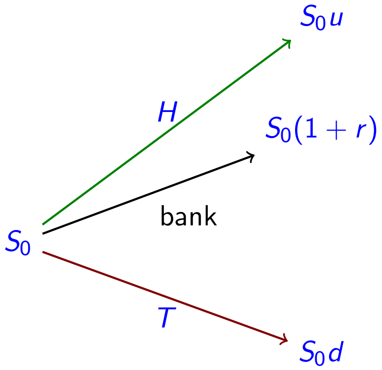
Case 1: $1 + r \geq u > d$.



Arbitrage strategy: sell short the stock and invest into the bank account.

Arbitrage in single period binomial model

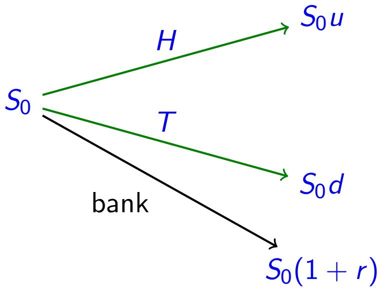
Case 2: $u > 1 + r > d$.



No Arbitrage: we can not gain without a loss.

Arbitrage in single period binomial model

Case 3: $u > d \geq 1 + r$.



Arbitrage strategy: borrow from the bank account and buy the stock.

Lemma

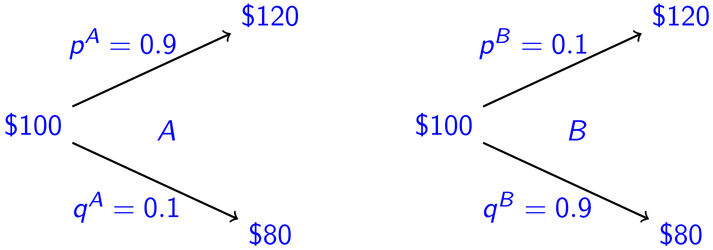
In the single period binomial model,

$$NA \iff d < 1 + r < u.$$

Problem on two calls

Problem

There are two stocks A *and* B *and a bank account. The interest rate* $r = 5\%$ *and the stocks evolve as:*



The call options on A *and* B *have the same strike* $K = \$100$*. Compute the difference* $C^A - C^B$ *of their arbitrage-free prices.*

Problem on two calls

Answer: 0 (no calculations!)

Wrong solution: use the Net Present Value (NPV) formula:

$$C^A = \frac{1}{1+r}\mathbb{E}\left(\max(S_1^A - K, 0)\right) = \frac{0.9 \times 20}{1.05} = \frac{18}{1.05},$$

$$C^B = \frac{1}{1+r}\mathbb{E}\left(\max(S_1^B - K, 0)\right) = \frac{0.1 \times 20}{1.05} = \frac{2}{1.05},$$

$$C^A - C^B = \frac{16}{1.05} \approx 15.2381.$$

(This is wrong!)

Solution

Recall the main principle:

$$\boxed{\textbf{! AFP} = \textbf{Replication}}$$

Problem on two calls

Replicating strategies:

$$C^A = X_0^A \underset{\Delta_0^A}{\longrightarrow} X_1^A = \max(S_1^A - K, 0),$$
$$C^B = X_0^B \underset{\Delta_0^B}{\longrightarrow} X_1^B = \max(S_1^B - K, 0).$$

Since the prices S^A and S^B have the same "skeletons", we obtain that

$$X_0^A = X_0^B \quad \text{and} \quad \Delta_0^A = \Delta_0^B.$$

It follows that

$$C^A - C^B = X_0^A - X_0^B = 0. \qquad \square$$

Pricing in single period binomial model

We consider a non-traded security, whose payoff at $t = 1$ is

$$V_1 = V_1(\omega), \quad \omega \in \{H, T\}.$$

Examples

$$\begin{aligned} \text{Zero-coupon bond:} \quad & V_1 = F \text{ (deterministic face value)}. \\ \text{Call:} \quad & V_1 = \max(S_1 - K, 0). \\ \text{Put:} \quad & V_1 = \max(K - S_1, 0). \\ \text{Long forward:} \quad & V_1 = S_1 - F. \end{aligned}$$

Problem

Compute the AFP V_0 at $t = 0$.

Pricing in single period binomial model

Solution

We recall the basic principle:

$$\boxed{\textbf{Pricing} \;=\; \textbf{Replication}}$$

Replicating strategy:

$$\underbrace{V_0 = X_0}_{?} \quad \xrightarrow[\Delta_0 - ?]{} \quad \underbrace{X_1 = V_1}_{\text{known}}.$$

1. It starts with some initial capital $X_0 = V_0$.
2. It generates the same payoff as the option:

$$X_1(\omega) = V_1(\omega), \quad \omega \in \{H, T\}.$$

Pricing in single period binomial model

A *portfolio* or a *strategy* is defined by a pair (X_0, Δ_0), where

X_0: the initial capital,

Δ_0: the initial number of shares,

1. $\Delta_0 > 0$ corresponds to a *long* position,
2. $\Delta_0 < 0$ corresponds to a *short* position.

The capital in the bank account at $t = 0$ is

$$X_0 - \Delta_0 S_0.$$

Balance equation: the capital of the portfolio at $t = 1$ is

$$X_1(\omega) = \underbrace{(X_0 - \Delta_0 S_0)(1 + r)}_{\text{bank account}} + \underbrace{\Delta_0 S_1(\omega)}_{\text{stocks}}, \quad \omega \in \Omega.$$

Pricing in single period binomial model

To find a replicating strategy for V_1, we need to solve the system of equations with respect to (X_0, Δ_0):

$$\begin{aligned} (X_1(H) =) \quad & (X_0 - \Delta_0 S_0)(1+r) + \Delta_0 S_0 u = V_1(H), \\ (X_1(T) =) \quad & (X_0 - \Delta_0 S_0)(1+r) + \Delta_0 S_0 d = V_1(T). \end{aligned}$$

Subtracting the second equation from the first equation, we obtain that

$$\Delta_0 S_0 (u - d) = V_1(H) - V_1(T).$$

Thus, the number of stocks

$$\Delta_0 = \frac{V_1(H) - V_1(T)}{S_0(u - d)}.$$

Pricing in single period binomial model

To compute X_0, we take numbers $\widetilde{p}$ and $\widetilde{q}$ such that

$$\widetilde{p} + \widetilde{q} = 1. \tag{a}$$

Multiplying the first equation on $\widetilde{p}$, the second equation on $\widetilde{q}$, and adding the results we obtain that

$$X_0(1+r) + \Delta_0 S_0(\widetilde{p}u + \widetilde{q}d - (1+r)) = \widetilde{p}V_1(H) + \widetilde{q}V_1(T).$$

We now choose $\widetilde{p}$ and $\widetilde{q}$ to get rid of the term containing Δ_0:

$$\widetilde{p}u + \widetilde{q}d - (1+r) = 0. \tag{b}$$

The solution of (a) and (b) is given by

$$\widetilde{p} = \frac{1+r-d}{u-d}, \quad \widetilde{q} = \frac{u-(1+r)}{u-d}.$$

Pricing in single period binomial model

As a result, we get the formula for X_0:

$$(V_0 =) \quad X_0 = \frac{1}{1+r}\left(\widetilde{p}V_1(H) + \widetilde{q}V_1(T)\right). \qquad \square$$

Remark

$$\text{NA} \quad \Longleftrightarrow \quad d < 1 + r < u \quad \Longleftrightarrow \quad \{\widetilde{p} > 0 \;\&\; \widetilde{q} > 0\}.$$

The numbers $\widetilde{p}$ and $\widetilde{q}$ are called the *risk-neutral probabilities*.

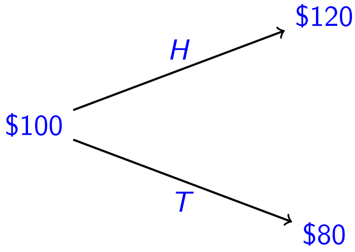
Remark

AFP does not depend on the actual probabilities p and q!

Problem on call option

Problem

The interest rate $r = 5\%$ *and the stock evolves as*



Compute V_0 *and* Δ_0*, the AFP and the number of stocks in the replicating strategy at* $t = 0$*, for the call option with the strike* $K = \$100$.

Problem on call option

Solution

We have that $u = 1.2$ and $d = 0.8$. The risk-neutral probabilities are given by

$$\widetilde{p} = \frac{1 + r - d}{u - d} = 0.625, \quad \widetilde{q} = 1 - \widetilde{p} = 0.375.$$

The arbitrage-free price has the form:

$$V_0 = \frac{1}{1 + r}(\widetilde{p} V_1(H) + \widetilde{q} V_1(T)) = \frac{\$12.50}{1.05} \approx \$11.90.$$

The number of shares in the replicating strategy is given by

$$\Delta_0 = \frac{V_1(H) - V_1(T)}{S_1(H) - S_1(T)} = \frac{20 - 0}{120 - 80} = 0.5. \qquad \square$$