№1.1.

$$y'(x) = f(x) \quad (1)$$

$$\frac{y_k - y_{k-2}}{2h} = a_1 f_k + a_0 f_{k-1} + a_{-1} f_{k-2} \quad (3)$$

Аппр. на решении: (3) аппр-т з-чу (1) на решении,

если $\exists\ c, h_0, p$: $\|L_h [y]_h - f_h\| \le c h^p \quad \forall h \le h_0$

и выполнено $\|[f]_h - f_h\| \longrightarrow 0,\ h \to 0$

$$\cancel{[y]_h \begin{pmatrix} y(0) \\ \vdots \\ y(x_k) \\ \vdots \\ y(1) \end{pmatrix}} \qquad [y]_h : \begin{pmatrix} \vdots \\ y(x_k) \\ \vdots \end{pmatrix} \qquad [f]_h \begin{pmatrix} f(x_k) \\ \vdots \end{pmatrix}$$

$$\|L_h [y]_h - f_h\| = \max_{x_k} \left| \frac{y(x_k) - y(x_{k-2})}{2h} - \left(a_1 f(x_k) + a_0 f(x_{k-1}) + a_{-1} f(x_{k-2})\right) \right| \ominus$$

$$y(x_{k-2}) = y(x_k - 2h) = y(x_k) - 2h y'(x_k) + \frac{4h^2}{2} y''(x_k) - \frac{8h^3}{6} y'''(x_k) + \frac{16h^4}{24} y^{IV}(x_k) + o(h^5)$$

$$f(x_{k-1}) = f(x_k - h) = f(x_k) - h f'(x_k) + \frac{h^2}{2} f''(x_k) - \frac{h^3}{6} f'''(x_k) + o(h^4)$$

$$f(x_{k-2}) = f(x_k - 2h) = f(x_k) - 2h f'(x_k) + \frac{4h^2}{2} f''(x_k) - \frac{8h^3}{6} f'''(x_k) + o(h^4)$$

$$\ominus \max_{x_k} \Big| \left(y'(x_k) - h y''(x_k) + \frac{2}{3} h^2 y'''(x_k) - \frac{1}{3} h^3 y^{IV}(x_k)\right) - a_1 f(x_k) - a_0\left(f(x_k) - h f'(x_k) + \frac{h^2}{2} f''(x_k) - \frac{h^3}{6} \ldots\right.$$

$$\left. - a_{-1}\left(f(x_k) - 2h f'(x_k) + \frac{4h^2}{2} f''(x_k) - \frac{8h^3}{6} f'''(x_k)\right)\right) \Big| + o(h^4)$$

Учитывая, что $y'(x) = f(x)$ (т. е. $y'(x_k) = f(x_k)$, $y''(x_k) = f'(x_k)$ и т.д.):

$1:\quad 1 - a_1 - a_0 - a_{-1} = 0$

$h:\quad -1 + a_0 + 2a_{-1} = 0$

$h^2:\quad \frac{2}{3} - \frac{a_0}{2} - 2a_{-1} = 0$

$\Rightarrow \frac{a_0}{2} - \frac{1}{3} = 0 \Rightarrow a_0 = \frac{2}{3}$

$2a_{-1} = 1 - a_0 = \frac{1}{3} \Rightarrow a_{-1} = \frac{1}{6}$

$a_1 = 1 - \frac{2}{3} - \frac{1}{6} = \frac{1}{6}$

Проверим дополнительно h^3:

$$-\frac{1}{3} + \frac{a_0}{6} + \frac{8}{6} a_{-1} = -\frac{1}{3} + \frac{2}{18} + \frac{8}{36} = -\frac{1}{3} + \frac{1}{9} + \frac{2}{9} = 0$$

$$\|[f]_h - f_h\| = \max_{x_k} \Big| \underset{y'(x_k)}{\overset{}{f(x_k)}} - \left(a_1 f(x_k) + a_0 f(x_{k-1}) + a_{-1} f(x_{k-2})\right) \Big| \longrightarrow 0,\ h \to 0$$

уже проверено

+

Ответ: $\frac{y_k - y_{k-2}}{2h} = \frac{1}{6} f_k + \frac{2}{3} f_{k-1} + \frac{1}{6} f_{k-2}$, порядок аппроксимации $= 4$.

№1.2.

$$\theta \frac{y_{k+1}-y_k}{h} + (1-\theta)\frac{y_k - y_{k-1}}{h} = f_k \quad , \theta \in [0,1]$$

Исследуем α-устойчивость схемы:

$$\theta\mu^2 - \theta\mu + (1-\theta)\mu - (1-\theta) = 0$$

$$\theta\mu^2 + (1-2\theta)\mu - (1-\theta) = 0$$

- $\theta = 0 \qquad \mu - 1 = 0$

  $\mu = 1 \Rightarrow$ схема α-устойчива

- $\theta > 1$

$$D = (1-2\theta)^2 + 4\theta(1-\theta) = 1 - 4\theta + 4\theta^2 + 4\theta - 4\theta^2 = 1$$

$$\mu_{1,2} = \frac{2\theta - 1 \pm 1}{2\theta} = \left[\begin{array}{l} 1 \\ \frac{\theta-1}{\theta} \end{array}\right.$$

$$-1 \le \frac{\theta-1}{\theta} < 1$$

$$\begin{cases} \theta - 1 \le \theta \\ \theta - 1 \ge -\theta \end{cases} \Rightarrow 1 \ge \theta \ge \frac{1}{2}$$

+

Ответ: схема устойчива при $\theta = 0$, $\theta \in [\frac{1}{2}, 1]$

№1.3.

$$(*)\begin{cases} y' = y \\ y(0) = 1 \end{cases}$$

$$\begin{cases} \dfrac{y_{k+1} - y_k}{h} = \dfrac{y_{k+1} + y_k}{2}, \quad k \geq 0 \\ y_0 = 1 \end{cases}$$

$$y(x_N) - y_N = c_1 h + c_2 h^2 + \ldots, \quad x_N = Nh = 1$$

c_1 - ?

$$y_{k+1}\left(\frac{1}{h} - \frac{1}{2}\right) = y_k\left(\frac{1}{2} + \frac{1}{h}\right)$$

$$y_{k+1} = y_k \frac{\frac{1}{2} + \frac{1}{h}}{\frac{1}{h} - \frac{1}{2}} \Rightarrow y_N = \left(\frac{\frac{1}{h} + \frac{1}{2}}{\frac{1}{h} - \frac{1}{2}}\right)^N y_0 = \left(\frac{\frac{1}{h} + \frac{1}{2}}{\frac{1}{h} - \frac{1}{2}}\right)^N = \left(\frac{\frac{1}{h} + \frac{1}{2}}{\frac{1}{h} - \frac{1}{2}}\right)^{\frac{1}{h}}$$

Решение $(*)$: $y(x) = e^x$

$$y(x_N) - y_N = e^{x_N} - y_N = e - \left(\frac{\frac{1}{h} + \frac{1}{2}}{\frac{1}{h} - \frac{1}{2}}\right)^{\frac{1}{h}} = e - e^{\frac{1}{h}\ln\left(\frac{\frac{1}{h} + \frac{1}{2}}{\frac{1}{h} - \frac{1}{2}}\right)} =$$

$$= e - e^{\frac{1}{h}\ln\left(1 + \underbrace{\frac{1}{\frac{1}{h} - \frac{1}{2}}}_{h \to 0 \,\to\, 0}\right)} = \Big/ \ln(1+x) = x - \frac{x^2}{2} + o(x^3), x \to 0 \Big/ \ominus$$

$$\ominus\ e - e^{\frac{1}{h}\left(\frac{1}{\frac{1}{h} - \frac{1}{2}} - \frac{1}{2}\left(\frac{h}{1 - \frac{h}{2}}\right)^2 + o(h^3)\right)} = \Big/ \frac{h^2}{(1 - \frac{h}{2})^2} = h^2\left(1 - \frac{h}{2}\right)^{-2} = h^2(1 + o(h)) = h^2 + o(h^3) \Big/ =$$

$$= e - e^{\frac{1}{1 - \frac{h}{2}} - \frac{h}{2} + o(h^2)} = e\left(1 - e^{\frac{\frac{h}{2}}{1 - \frac{h}{2}} - \frac{h}{2} + o(h^2)}\right) = \Big/ \frac{h}{2}\left(1 - \frac{h}{2}\right)^{-1} = \frac{h}{2}\left(1 + \cancel{\frac{h}{2}}\right) + o(h^2) \Big/ =$$

$$= e\Big(\underbrace{1 - e^{o(h^2)}}_{\sim o(h^2)}\Big) \sim o(h^2) \Rightarrow c_1 = 0$$

+

Ответ: $c_1 = 0$.

$$\begin{cases} y' + 5y = \sin 2x & = f(x) \\ y(0) = 2 \end{cases}$$

№1.4.

$O(h^2)$

нужно $f(x_k + \frac{h}{2})$ $= \frac{\sin 2(x_k + h) + \sin 2x_k}{2}$

Рассмотрим схему:

$$\begin{cases} \frac{y_{k+1} - y_k}{h} + 5\,\frac{y_{k+1} + y_k}{2} = f_k := \frac{f(x_{k+1}) + f(x_k)}{2}, \quad k = \overline{0, N-1} \\ y_0 = 2 \end{cases}$$ ← ответ

+

Аппрокс. на решении:

$$\|L_h[y]_h - f_h\| = \max_{x_k} \left| \frac{y(x_{k+1}) - y(x_k)}{h} + 5\,\frac{y(x_{k+1}) + y(x_k)}{2} - \frac{f(x_{k+1}) + f(x_k)}{2} \right| =$$

$$= \max_{x_k} \left| y'(x_k) + \frac{h}{2} y''(x_k) + 5y(x_k) + \frac{5}{2} h\, y'(x_k) - \left(f(x_k) + \frac{h}{2} f'(x_k)\right) \right| + o(h^2) =$$

$$= \max_{x_k} \left| f(x_k) + \frac{h}{2} f'(x_k) - \left(f(x_k) + \frac{h}{2} f'(x_k)\right) \right| + o(h^2) = o(h^2)$$

$$\|[f]_h - f_h\| = \max_{x_k} \left| f(x_k) - \frac{f(x_{k+1}) + f(x_k)}{2} \right| = \max_{x_k} \left| f(x_k) - f(x_k) \right| + o(h) \longrightarrow 0, \; h \to 0$$

Итак, порядок аппроксимации равен 2.

$\mu - 1 = 0 \Rightarrow \mu = 1 \Rightarrow$ схема λ-устойчива

№ 1.5.

$$\begin{cases} u'' - 2u = \sin x - 1 \\ u'(0) - u(0) = 0 \end{cases}$$

$x_0 = 0 \quad x_1 = h$

$O(h^2)$

$u(h) = u(0) + hu'(0) + \frac{h^2}{2}u''(0) + O(h^3) \Rightarrow u'(0) = \frac{u(h) - u(0)}{h} - \frac{h}{2}u''(0) + O(h^2)$

$u''(0) = 2u(0) + \sin 0 - 1$

$u'(0) = \frac{u(h) - u(0)}{h} - \frac{h}{2}\left(2u(0) - 1\right) + O(h^2)$

$u'(0) - u(0) = \frac{u(h) - u(0)}{h} - hu(0) + \frac{h}{2} + O(h^2) \qquad u_1 := u(h)\ , u_0 := u(0).$

$\Rightarrow$ искомая аппроксимация: $\frac{u_1 - u_0}{h} - hu_0 + \frac{h}{2} = 0$ ← ответ

+

№1.6.

$$\begin{cases} -u''(x)+pu(x)=f(x), \ p=const>0 & (1) \\ u(0)=a & (2) \\ u'(1)=b & \end{cases}$$

$$\begin{cases} -\dfrac{u_{k+1}-2u_k+u_{k-1}}{h^2}+pu_k=f_k:=f(x_k), \ k=\overline{1,N-1} & (3) \\ u_0=a & \\ \dfrac{u_N-u_{N-1}}{h}=b+\delta & (4) \end{cases}, \ \delta=-\frac{h}{2}f(1)+\frac{ph}{2}u(1)$$

← ответ

Аппроксимация на решении:

$$\|L_h[u]_h-f_h\|=\max_{x_k}\left|-\frac{u(x_{k+1})-2u(x_k)+u(x_{k-1})}{h^2}+pu(x_k)-f(x_k)\right| \ominus$$

$$u(x_{k\pm1})=u(x_k\pm h)=u(x_k)\pm hu'(x_k)+\frac{h^2}{2}u''(x_k)\pm\frac{h^3}{6}u'''(x_k)+o(h^4)$$

$$\ominus \max_{x_k}\left|\underbrace{-u''(x_k)+pu(x_k)}_{f(x_k)}-f(x_k)\right|+o(h^2)=o(h^2)$$

$$\|[f]_h-f_h\|=\max_{x_k}|f(x_k)-\overset{f(x_k)}{f_k}|=0$$

$$\|l_h[u]_h-\varphi_h\|=\max_{x_k}\left|\frac{u(x_N)-u(x_N-h)}{h}-b-\delta\right|=\max_{x_k}\left|u'(1)-\frac{h}{2}u''(1)+o(h^2)-b-\delta\right|=$$

$$=\max\left|-\frac{h}{2}u''(1)-\delta\right|+o(h^2)$$

$$\Rightarrow \delta:=-\frac{h}{2}u''(1)=-\frac{h}{2}f(1)+\frac{ph}{2}u(1)$$

$$-u''(1)=f(1)-pu(1)$$

Теор. Филиппова

> З-чи (1,2) и (3,4) линейные
> ∃! реш-е з-чи (1,2)
> Р.схема аппрокс-т з-чу на реш-и с порядком p.
> Р.схема устойчива
> Тогда реш-е з-чи (3,4) сх-ся к реш-ю з-чи (1,2) с пор-ком не ниже, чем p.

Докажем устойчивость р. схемы:

Матрич. вид задачи:

$$A=\begin{pmatrix} \frac{2}{h^2}+p & -\frac{1}{h^2} & 0 & \dots & 0 \\ -\frac{1}{h^2} & \frac{2}{h^2}+p & -\frac{1}{h^2} & & \\ & & \ddots & & \\ & & & -\frac{1}{h^2} & \frac{1}{h^2}+p \end{pmatrix}\begin{pmatrix} u_1 \\ \vdots \\ u_{N-1} \end{pmatrix}=\begin{pmatrix} f_1+\frac{a}{h^2} \\ \vdots \\ f_{N-1}-\frac{b+\delta}{h} \end{pmatrix}$$

$$\|y\|_{2,h}\le\|A^{-1}\|_{2,h}\cdot\|f\|_{2,h}\le const\cdot\|f\|_{2,h} \Rightarrow \text{схема устойчива}$$

$$\|A^{-1}\|_{2,h}=\|A^{-1}\|_2=\frac{1}{|\lambda_{min}(A)|}\le const, \ h\to0$$

+

$$\|u\|_{2,h}=\left(\sum_{i=1}^{N-1}u_i^2\cdot h\right)^{1/2}$$

Ответ

№1.7.

$$\begin{cases} -\dfrac{u_{i+1}-2u_i+u_{i-1}}{h^2}+p_iu_i=f_i, & 1\le i\le N-1,\ p_i\ge 0 \\ u_0=0 & (N-\frac{1}{2})h=1 \\ u_N=u_{N-1} \end{cases}$$

$$-\frac{1}{h^2}\underbrace{\sum_{1}^{N-1}(u_{i+1}-u_i-u_i+u_{i-1})\cdot u_i}+\sum_{1}^{N-1}p_iu_i^2=\sum_{1}^{N-1}f_iu_i$$

$$\underbrace{\sum_{i=1}^{N-1}(u_{i+1}-u_i)u_i}_{\sum_{i=1}^{N}(u_i-u_{i-1})u_{i-1}}-\underbrace{\sum_{i=1}^{N-1}(u_i-u_{i-1})u_i}_{\sum_{i=1}^{N}(u_i-u_{i-1})u_i-(\overbrace{u_N-u_{N-1}}^{=0})u_N}$$

$$+\frac{1}{h^2}\sum_{i=1}^{N}(u_i-u_{i-1})^2+\sum_{i=1}^{N-1}p_iu_i^2=\sum_{i=1}^{N-1}f_iu_i$$

$$u_i=\sum_{n=1}^{i}(u_n-u_{n-1})$$

$$u_i^2=\left(\sum_{n=1}^{i}(u_n-u_{n-1})\cdot 1\right)^2\le\sum_{n=1}^{i}(u_n-u_{n-1})^2\cdot\sum_{n=1}^{i}1\le\sum_{n=1}^{N}(u_n-u_{n-1})^2\cdot N$$

$$\Rightarrow\sum_{i=1}^{N-1}u_i^2\le\ \le(N-1)\cdot N\cdot\sum_{n=1}^{N}(u_n-u_{n-1})^2\le\Big(\underbrace{N-\tfrac{1}{2}}_{}\Big)^2\sum_{n=1}^{N}(u_n-u_{n-1})^2, \quad (N-\tfrac12)^2=\frac{1}{h^2}$$

$$N^2-N\le N^2-N+\frac{1}{4}$$

$$\sum_{i=1}^{N-1}u_i^2\le\frac{1}{h^2}\sum_{n=1}^{N}(u_n-u_{n-1})^2+\sum_{i=1}^{N-1}p_iu_i^2=\sum_{i=1}^{N-1}f_iu_i\le\frac{1}{2}\sum_{i=1}^{N-1}f_i^2+\frac{1}{2}\sum_{i=1}^{N-1}u_i^2$$

$$\Rightarrow\sum_{i=1}^{N-1}u_i^2h\le\sum_{i=1}^{N-1}f_i^2\cdot h$$

+

зачт.

$$\Rightarrow\|u_h\|^2_{L_2,h}\le\|f_h\|^2_{L_2,h}$$

Значит, схема устойчива по опр-ю.