JOURNAL OF ECONOMIC THEORY 58, 9–40 (1992)

Evolution and Market Behavior*

LAWRENCE BLUME AND DAVID EASLEY

Department of Economics,
Cornell University, Ithaca, New York 14850

Received February 21, 1992; revised April 15, 1992

In a conventional asset market model we study the evolutionary process generated by wealth flows between investors. Asymptotic behavior of our model is completely determined by the investors' expected growth rates of wealth share. Investment rules are more or less "fit" depending upon the value of this expectation, and more fit rules survive in the market at the expense of the less fit. Using this criterion we examine the long run behavior of asset prices and the common belief that the market selects for rational investors. We find that fit rules need not be rational, and rational rules not be fit. Finally, we investigate how the market selects over various adaptive decision rules *Journal of Economic Literature* Classification Numbers: D90, D80.

1. INTRODUCTION

In this paper we delineate in a conventional asset market model the evolutionary processes which determine the long run behavior of the market. Evolutionary ideas such as natural selection and adaptive behavior have a long history in economic analysis. Two streams of literature have had a significant impact on contemporary economic thought. First are those writings on economic dynamics that hinge on the success or failure of risk-taking entrepreneurs. This view of economic progress is recognizable in both Knight [22] and Schumpeter [27]. A part of this literature are the writings of Alchian [1], of Friedman [17] in his famous essay "On the Methodology of Positive Economics," and of other writers in the early 1950s who attempt to explain why procedural rationality has predictive power. Their answer is an appeal to natural selection which we call the "market selection hypothesis." Market forces favor the survival of economic actors whose decisions are most nearly optimal. According to Friedman [17, p. 21], "... firms behave *as if* they were seeking to maximize their expected returns ..." He reasons [17, p. 22] that "unless the behavior

* Support from NSF Grant SES-8921415 and the Center for Analytic Economics at Cornell University is gratefully acknowledged.

0022-0531/92 $5.00

of businessmen in some way or other approximated behavior consistent with the maximization of returns, it seems unlikely that they would remain in business for long."[1] We want to emphasize the positive content of these arguments. Alchian and Friedman claim that, without assuming much about the behavior of individual economic actors, evolutionary forces drive the economy towards a state which can be described by the conventional neoclassical models with optimizing actors.[2] This claim is significant for such diverse topics as the choice of efficient production techniques and the pricing by arbitrage of financial assets.

A second source of evolutionary ideas is the evolutionary theory of games as developed by Maynard Smith [23] and others. The original motivation for this work was the study of biological, not social, processes. Consequently, tools from mathematical biology such as replicator dynamics were imported to identify solutions. In biologically motivated games the fitness criterion is explicit in the presentation of the game. "Fit" strategies are, by definition, those which perform well against the population. Under replicator dynamics, the proportion of the population adhering to a particular strategy grows according to its relative fitness.

When we turn to markets working through time, however, matters are less straightforward. One might be tempted to apply the biological population processes that have found favor in evolutionary game theory. The implicit hypothesis would be that the population dynamic is the reduced form of a learning process or an adaptive process of strategy revision in a large population of players. (See Crawford [8] for a careful exposition of this view in the game-theoretic context.) We see two problems with applying this approach to markets. First, the choice of fitness criterion is not obvious. Should it be utility, wealth, or something else? Second, there is a natural "population dynamic" in asset markets that emerges from the process of asset accumulation. This process does not depend on individual adaptation. Instead, differential rates of wealth accumulation increase the relative significance of some individuals in determining market outcomes.

Specifically, in a conventional asset market model we describe the "population dynamics" of investment rules when investors do not adapt to the market, and see how it relates to the dynamics of wealth accumulation. We identify the fitness criterion which market dynamics select for, and

[1] Although these early discussions were largely confined to the behavior of firms, they have had an influence on thinking about the rationality of investors, witness Patel, Zeckhauser, and Hendricks [25, p. 232]: "For most economists, it is an article of faith that financial markets reach rational aggregate outcomes, despite the irrational behavior of some participants, since sophisticated players stand ready to capitalize on the mistakes of the naive."

[2] To be fair, Alchian [1, p. 213] is very explicit about the limits of the argument that selection begets perceived rationality: "As in a race, the award goes to the relatively fastest, even if all competitors loaf. Even in a world of stupid men there would still be profits."

examine some of its implications for the long run behavior of asset prices. In the financial asset markets we study, the most fit behavior is that which maximizes the expected growth rate of wealth share accumulation. When one particular investment rule is globally maximal with respect to this criterion, asset prices behave such that expected gains in wealth share converge exponentially to zero. This permits a characterization of the limit behavior of our economies which is not dependent on rationality hypotheses for individual behavior.

With these results we examine the common belief that the market selects for rational investors. We find that the link between rationality and fitness is weak. On the one hand, the criterion of expected growth rate maximization is related to the maximization of expected logarithmic utility for consumption. Thus one can always construct fit rules that are consistent with the Savage axioms. On the other hand, many other investment rules that are rational in this sense behave poorly with respect to the expected growth rate fitness criterion, and can be driven from the market by savings and investment behavior which is inconsistent with the Savage axioms.

Finally we investigate three types of adaptive behavior by individual investors: Bayesian learning, search, and imitation. Our interest is in natural selection over processes of adaptation. When traders have logarithmic expected utility and a common discount factor, we find that the market selects for Bayesian learners and against those for whom the learning process is not "Bayes-like." We also show how various search and imitation rules fare against each other and against Bayesian learning.

The next section of the paper describes the asset market model. Section 3 works out the dynamics of natural selection in the context of some simple rules. A general analysis is carried out in Section 4. Section 5 discusses the market selection hypothesis, and Section 6 treats adaptive behavior. Some concluding remarks are made in Section 7.

2. An Investment Model

Time is discrete and indexed by t. There are S states of the world, indexed by s, one of which will occur at each date. States follow an i.i.d process with distribution $q=(q_1, ..., q_S)$ where $q_s>0$ for all s. Let $\Omega=\prod_0^\infty \{1, ..., S\}$ with representative element $\omega=(\omega_1, \omega_2, ...)$ where ω_t is the state at date t. Define $\Omega^T=\prod_1^T\{1, ..., S\}$ with typical element ω^T for each $T=1, 2, ...$. Let $\mathscr{F}$ denote the product σ-field on Ω and let μ denote the product probability with marginal distribution q. Finally, let $\mathscr{F}_t$ denote the sub-σ-field $\sigma(\omega^t)$ of $\mathscr{F}$.

At each date there is one unit of each of S assets available. If state s occurs at date t then asset s pays off $w_{st}>0$ and all other assets have a zero payoff. So w_{st} will be total wealth in the economy at date t if state s occurs. This wealth will be distributed among the traders proportionately according to the share of asset s each trader owns.[3]

Let $\alpha^i_{st} \geqslant 0$ be the fraction of trader i's wealth at the beginning of t, w^i_{t-1}, that he invests in asset s. We let $\alpha^i_t = (\alpha^i_{1t}, ..., \alpha^i_{St})$ and assume that $\sum_{s=1}^{S} \alpha^i_{st} = 1$. The price of asset s at date t is denoted by ρ_{st}. So trader i owns $\alpha^i_{st} w^i_{t-1}/\rho_{st}$ shares of asset s. Thus, his investment income is $(\alpha^i_{st} w^i_{t-1}/\rho_{st}) w_{st}$ if state s occurs at date t.

After realizing his investment income, trader i consumes fraction $1-\delta^i_{st}$ and saves fraction δ^i_{st} to invest tomorrow. So if state s occurs at date t, trader i's wealth at the beginning of date $t+1$ will be $\delta^i_{st}(\alpha^i_{st} w^i_{t-1}/\rho_{st}) w_{st}$. We refer to $\{\alpha^i_t\}_{t=1}^{\infty}$ as trader i's portfolio rule and the pair $\{\alpha^i_t, \delta^i_t\}_{t=1}^{\infty}$ as trader i's investment rule.

Given the trader's wealths and portfolio rules the price of asset s must satisfy

$$\sum_{i=1}^{I} \frac{\alpha^i_{st} w^i_{t-1}}{\rho_{st}} = 1.$$

Let $p_{st} = \rho_{st}/w_{t-1}$ be a normalized asset price where w_{t-1} is the market wealth at the beginning of date t. Let $r^i_{t-1} = w^i_{t-1}/w_{t-1}$ be trader i's wealth share. Then in equilibrium

$$p_{st} = \sum_{i=1}^{I} \alpha^i_{st} r^i_{t-1}$$

and

$$\sum_{s=1}^{S} p_{st} = 1.$$

Typically α^i_{st} and δ^i_{st} will depend on current market prices and wealth levels, and perhaps on previously observed information as well. It is convenient to suppress the arguments of the α^i_{st} and δ^i_{st}, but it should be kept in mind that these are functions defined on Ω^{t-1}. Moreover, the price equation only determines p_{st} implicitly, and each p_{st} too is an $\mathscr{F}_{t-1}$-measurable random variable. Throughout our analysis, and without further mention, we will assume that the behavior of the α^i_{st} as a function of the p_{st} is such that the equilibrium price equation has a solution. In the literature on temporary equilibrium (of which this is an example of sorts), assumptions are made on the behavior of preferences and beliefs which

[3] This asset structure is derivable from a more general asset structure where assets may pay out in more than one state, but in which a spanning assumption is satisfied as well as a condition on asset demand that eliminates arbitrage possibilities.

guarantee that temporary equilibria exist. In our model, on the other hand, it is appropriate to place assumptions directly on demands since we are not assuming that demands come from maximizing behavior.

We are interested in the limit distribution of wealth shares and market prices. In order to describe the evolution of wealth shares, we need to define the market savings rate,

$$\delta_{st} = \frac{w_t}{w_{st}}.$$

The market savings rate is the share of total asset payout which is saved for investment.

It is reasonably straightforward to show that trader i's wealth share at date T is

$$r^i_T = \prod_{t=1}^{T} \prod_{s=1}^{S} \left(\frac{\delta^i_{st}}{\delta_{st}} \frac{\alpha^i_{st}}{p_{st}} \right)^{1_{st}} r^i_0, \tag{2.1}$$

where 1_{st} is the indicator function which takes the value of 1 if s occurs at date t and 0 otherwise. We analyze the asymptotic behavior of the wealth share processes using suitable laws of large numbers. To prepare for this analysis, we take logs

$$\log r^i_T - \log r^i_0 = \sum_{t=1}^{T} \left(\sum_{s=1}^{S} 1_{st} \log \frac{\delta^i_{st}}{\delta_{st}} + \sum_{s=1}^{S} 1_{st} \log \frac{\alpha^i_{st}}{p_{st}} \right). \tag{2.2}$$

This transformation describes the *growth rate* of wealth share as a sum of random variables. Notice that the random variables being summed are not independent. Even if the rules are independent, the returns are not because today's prices are determined by yesterday's wealth.[4]

Three examples will illustrate the class of investment environments we have in mind. One, concenptually the most familiar, is investing in Arrow securities. The second is the source of much of our intuition—a season of betting at a race track. The third example focuses on the role of risky aggregate wealth.

EXAMPLE 1 (Arrow Securities). There are S securities. A unit of security s pays \$1 if state s occurs, and \$0 otherwise. At date t, investor i uses his wealth w^i_{t-1} to purchase securities whose prices are $p = (p_{1t}, ..., p_{St})$. The number of shares purchased by investor i is

[4] Although at a superficial level there seem to be connections between our problem and the gambler's ruin problem, the two problems are analytically quite distinct.

$\alpha^i_{st} w^i_{t-1}/p_{st}$, so α^i_{st} is in fact just an alternative way to describe investor i's demand for asset s. Investor i's date t budget constraint is

$$\sum_{s=1}^{S} p_{st}\left(\frac{\alpha^i_{st} w^i_{t-1}}{p_{st}}\right) = w^i_{t-1},$$

which reduces to the share identity $\sum_{s=1}^{S} \alpha^i_{st} = 1$. The total wealth disbursed in any state is w_{t-1}, so for each state s we must have

$$\sum_{i=1}^{I} \frac{\alpha^i_{st} w^i_{t-1}}{p_{st}} = w_{t-1},$$

or

$$\sum_{i=1}^{I} \alpha^i_{st} r^i_{t-1} = p_{st}.$$

EXAMPLE 2 (Horse Race). S horses run races against each other. The outcomes are i.i.d., and the win probabilities are given by the probability distribution q. The assets are win bets on the horses. The entire amount of money bet on the horses is divided up among those investors who bet on the winning horse, in proportion to the magnitude of their bets. The track takes no percentage of the amount bet, so there is a portfolio that pays off a certain dollar per dollar invested in it. Thus there is no loss of generality in assuming that no savings is allowed. In this case each "asset" s pays off either w_{t-1} or nothing, depending on whether or not state s is realized. In this model, if the market savings rates are bounded away from 1, then aggregate wealth will converge to 0. But it is wealth share, and not wealth, that determines prices.

EXAMPLE 3 (Exogenous Asset Supply). Each of the S assets has an endogeneously given supply of 1 unit. Asset s pays off amount z_s in state s, and 0 otherwise. Unlike the horse race, asset payouts are not history-dependent. But unlike the first example, there may be social risk: $z_s \neq z_{s'}$ for some distinct states s and s'.

3. SIMPLE RULES—A POSITIVE ANALYSIS

For general investment rules, the analysis of the stochastic process of wealth shares is complex, but it is easy to see what happens for sufficiently simple rules. Suppose that all traders have identical non-zero savings rates and constant (but not necessarily identical) portfolio investment rules. Trader i invests a fixed fraction α^i_s of his wealth in asset s at each date. If trader i sets $\alpha^i_s = 0$ for any asset s, then he will be bankrupt in finite time almost surely. Such traders will have no effect on limit prices, so we restrict attention to portfolios with $\alpha^i_s > 0$ for all s.

In this case everybody saves at the market rate, and so the log-wealth-share evolution equation (2.2) simplifies to

$$\log r^i_T = \sum_{t=1}^{T} \sum_{s=1}^{S} 1_{st}(\log \alpha^i_s - \log p_{st}) + \log r^i_0.$$

As we have observed earlier, the random variables $1_{st}(\log \alpha^i_s - \log p_{st})$ are not independent over time because, although the α^i_s terms are constants, the p_{st} terms are random variables which are not independently distributed. But compare the performance of two investors, i and j:

$$\log\left(\frac{r^i_T}{r^j_T}\right) = \sum_{t=1}^{T} \sum_{s=1}^{S} 1_{st}(\log \alpha^i_s - \log \alpha^j_s) + \log r^i_0 - \log r^j_0.$$

The terms on the right are now i.i.d. random variables, and so, from the SLLN,

$$\lim_{T\to\infty} \frac{1}{T} \log\left(\frac{r^i_T}{r^j_T}\right) = \sum_{s=1}^{S} q_s(\log \alpha^i_s - \log \alpha^j_s)$$

almost surely. The term on the right is a constant. The ratio of i's wealth share to j's converges to 0 or diverges at exponential rate depending on whether this constant is negative or positive.

To interpret the constant in the asymptotic relative share equation, define the *relative entropy of* α *with respect to* q by

$$I_q(\alpha) = \sum_{s=1}^{S} q_s \log\left(\frac{q_s}{\alpha_s}\right).$$

The relative entropy of α with respect to q, although not a metric, does define a measure of distance from α to q. It is always non-negative, and is zero only when $\alpha = q$.

The mean of our i.i.d. random variable, the constant on the right hand side of the asymptotic relative share equation, is

$$I_q(\alpha^j) - I_q(\alpha^i).$$

If this number is negative for some j, then r^i_T converges almost surely to 0. If, on the other hand, it is positive for all $j \neq i$, then r^i_T converges almost surely to 1 (since everyone else's wealth share converges almost surely to 0). In this case it follows from the price equation that p_T converges almost surely to α^i. We summarize these results in the following proposition:

PROPOSITION 3.1. *Suppose all traders have the same savings rate and have constant portfolio rules. If there is a unique investor whose portfolio rule* α^i *has smallest relative entropy with respect to* q, *then* $r^i_T \to 1$ *a.s. and* $r^j_T \to 0$ *a.s. for all* $j \neq i$. *Furthermore,* $p_T \to \alpha^i$ *a.s.*

Our analysis generalizes in a straightforward fashion to the case where investors have distinct, but constant savings rates. In this case the relative share equation becomes

$$\log\left(\frac{r^i_T}{r^j_T}\right) = \sum_{t=1}^{T} \sum_{s=1}^{S} 1_{st}(\log \alpha^i_s - \log \alpha^j_s) + \sum_{t=1}^{T} (\log \delta^i - \log \delta^j) + \log r^i_0 - \log r^j_0.$$

Now, almost surely,

$$\lim_{T\to\infty} \frac{1}{T} \log \frac{r^i_T}{r^j_T} = (I_q(\alpha^j) - \log \delta^j) - (I_q(\alpha^i) - \log \delta^i).$$

PROPOSITION 3.2. *Suppose all traders have constant savings rates and constant portfolio rules. If there is a unique investor* i *such that, for all investors* $j \neq i$,

$$I_q(\alpha^j) - \log \delta^j > I_q(\alpha^i) - \log \delta^i,$$

then $r^i_T \to 1$ *a.s. and* $r^j_T \to 0$ *a.s. for all* $j \neq i$. *Furthermore,* $p_T \to \alpha^i$ *a.s.*

Notice now that the investment rule "nearest" the true probabilities is not necessarily selected for. The increased accumulation rate of nearer investment rules may be offset by lower savings rates.

4. GENERAL RULES—A POSITIVE ANALYSIS

In order to discuss the evolution of market behavior more generally we need to extend the analysis of the preceding section to a larger class of investment and savings rules. We will assume only that rules are measurable and satisfy certain boundedness properties. We require that trader i's investment rule satisfy the following two assumptions:

(1) For all t, α^i_t is $\mathscr{F}_{t-1}$-measurable and δ^i_t is $\mathscr{F}_t$-measurable.

Notice that this assumption has the effect of making the savings rate in period t measurable with respect to the realization of states *through* period t. This is because, with our timing conventions, consumption takes place at the end of the period, after assets pay out. Throughout the paper we shall make the dependence of δ^i_t on the date t outome explicit, writing δ^i_{st} where s indexes the date t realization of ω_t. Thus each δ^i_{st} is also an $\mathscr{F}_{t-1}$-measurable function.

(2) There is a real number $\varepsilon > 0$ such that $\delta^i_{st}, \alpha^i_{st} \geqslant \varepsilon$ for all s, t a.s.

This assumption ensures that traders always have positive wealth.

When all rules in the market are simple rules, a given rule either vanishes in the sense that its wealth share goes to 0, or it comes to dominate the market in the sense that the share of wealth invested with it goes to 1. A third possibility arises when more general classes of rules are considered. Wealth share may fluctuate between 0 and 1.

DEFINITION 4.1. (a) Trader i *vanishes* on the set of sample paths $A \in \mathscr{F}$ if $\limsup r^i_T = 0$ a.s. on A.

(b) Trader i *survives* on the set of sample paths $A \in \mathscr{F}$ if $\limsup r^i_T > 0$ a.s. on A.

(c) Trader i *dominates* on the set of sample paths $A \in \mathscr{F}$ if $\liminf r^i_T > 0$ a.s. on A.

It is possible that aggregate wealth is unbounded from above or converging to zero, so these definitions say nothing about a trader's actual wealth. However, our interest at this point is in equilibrium prices, which are determined by wealth shares.

An example of betting rules that survive but do not dominate comes from the horse race with very intuitive, although non-constant, betting rules. Recall that saving at a zero rate of interest is possible. (There is a bet combination which pays off one certain dollar for every dollar bet.) Suppose that each bettor has a favorite horse, and that every horse is some bettor's favorite. Suppose that each bettor bets according to the following rule: bet some positive fraction, say 1/2, of wealth on your favorite horse, and save the rest. It is easy to show that the lim sup of the total wealth share of those bettors who favor any particular horse is 1 and the lim inf is 0.[5]

To state the main analytical results of this section, it will be convenient to introduce some additional notation. Let

$$X^i_t = \sum_{s=1}^{S} 1_{st} \log \delta^i_{st} \alpha^i_{st}.$$

[5] Furthermore, the time average of market prices converges almost surely to the correct market prices—the inverse of the correct market odds. This result was first established by Diegert and Heath [10]. Surprising as it may be at first glance, the time-average behavior of market prices is very intuitive. Were the result not true, then some bettors would have an average expected return which was strictly positive. Were this to happen, their wealth share would almost surely be infinite, which is not possible.

Let

$$Y_t = \sum_{s=1}^{S} 1_{st} \log \delta_{st} p_{st}.$$

Equation (2.2) becomes

$$\log r^i_T - \log r^i_0 = \sum_{t=1}^{T} X^i_t - \sum_{t=1}^{T} Y_t.$$

Our first result characterizes the ultimate fate of traders in terms of their conditional expected one-period growth rates. Proofs are deferred to the Appendix.

THEOREM 4.1. *Suppose Assumptions* 1 *and* 2 *hold for all traders* i:

(a) *On* $\{\omega \in \Omega : \liminf \sum_{t=1}^{T} E\{X^i_t - Y_t \mid \mathscr{F}_{t-1}\} > -\infty\}$,

(1) *trader* i *dominates, and*

(2) *if all traders have the same savings rate, then* $\lim_T (p_T - \alpha^i_T) = 0$ *a.s.*;

(b) *On* $\{\omega : \limsup \sum_{t=1}^{T} E\{X^i_t - Y_t \mid \mathscr{F}_{t-1}\} > -\infty\}$, *trader* i *survives*;

(c) *On*

$$\left\{\omega : \limsup \frac{\sum_{t=1}^{T} E\{Y_t \mid \mathscr{F}_{t-1}\}}{\sum_{t=1}^{T} E\{X^i_t \mid \mathscr{F}_{t-1}\}} < 1\right\},$$

trader i *vanishes.*

The dominance, survival, and vanishing conditions can all be expressed in terms of the relative entropy measure introduced in the previous section. Computing,

$$\begin{aligned}
&\sum_{t=1}^{T} E\{X^i_t - Y_t \mid \mathscr{F}_{t-1}\} \\
&\quad = \sum_{t=1}^{T} \sum_{s=1}^{S} q_s(\log \alpha^i_{st} - \log p_{st}) + \sum_{t=1}^{T} \sum_{s=1}^{S} q_s \log \frac{\delta^i_{st}}{\delta_{st}} \\
&\quad = \sum_{t=1}^{T} I_q(p_t) - I_q(\alpha^i_t) + \sum_{t=1}^{T} \sum_{s=1}^{S} q_s \log \frac{\delta^i_{st}}{\delta_{st}}.
\end{aligned}$$

Suppose that all traders have identical savings rates. It is easy to see (Lemma A.1 in the Appendix) that $\limsup \sum_{t=1}^{T} I_q(p_t) - I_q(\alpha^i_t) \leqslant -\log r^i_0$. So α^i_t cannot forever be closer to q than is p_t. Otherwise trader i's wealth share would ultimately exceed 1. Trader i dominates when

$\liminf \sum_{t=1}^{T} I_q(p_t) - I_q(\alpha_t^i)$ is finite. This requires that p_t and α_t^i converge together at a rapid rate. As p_t is a wealth-share-weighted average of portfolio rules, this means that asymptotically all survivors invest like investor i. The survival criterion has a similar interpretation in terms of sums over time in the difference of the relative entropy distances of α_t^i and p_t to q. The failure condition (c) implies that $\sum_{t=1}^{T} I_q(p_t) - I_q(\alpha_t^i) = -\infty$. In other words, over time, p_t is closer to q than is α_t^i. In addition, the failure condition studies the rate at which this sum diverges. The term $\sum_{t=1}^{T} E\{X_t^i \mid \mathscr{F}_{t-1}\}$ equals $\sum_{t=1}^{T} \sum_{s=1}^{S} q_s \log \alpha_{st}^i$, which is negative and diverging. If

$$\liminf \frac{1}{\sum_{t=1}^{T} \sum_{s=1}^{S} q_s \log \alpha_{st}^i} \sum_{t=1}^{T} I_q(p_t) - I_q(\alpha_t^i) > 0,$$

then trader i vanishes. By hypothesis, there exist bounds $b, c < 0$ such that $b \geqslant \sum_{s=1}^{S} q_s \log \alpha_{st}^i \geqslant c$. Thus this vanishing requirement amounts to a condition on the averages over time of relative entropy differences: If

$$\liminf \frac{1}{T} \sum_{t=1}^{T} I_q(\alpha_t^i) - I_q(p_t) > 0,$$

then trader i vanishes.

Part (a) of this proposition justifies the use of the word "dominates" in the preceding definition. When savings rates are identical, a trader who dominates actually determines the price asymptotically. His wealth share need not converge to one because there may be other traders who asymptotically have the same portfolio rule, but prices adjust so that his conditional expected gains converge to zero. Theorem 4.1 describes the market selection process. It answers a question raised by Alchian [1] and, more directly, Hirshleifer [20, p. 10]: "According to what criterion does natural selection select when strategies have uncertain outcomes?" Theorem 4.1 identifies the "fitness" criterion the market selects for—the expected growth rate of wealth share.

The condition of part (c) is more commensurate with the remaining parts of the theorem than it looks. The X_t^i's are all bounded away from 0, so $\sum_{t=1}^{T} X_t^i$ diverges, and it follows that the hypotheses of part (c) imply $\limsup \sum_{t=1}^{T} E\{X_t^i - Y_t \mid \mathscr{F}_{t-1}\} = -\infty$. If one were to enumerate all the cases of the asymptotic behavior of $\sum_{t=1}^{T} E\{X_t^i \mid \mathscr{F}_{t-1}\}$ and $\sum_{t=1}^{T} E\{Y_t \mid \mathscr{F}_{t-1}\}$ covered by Theorem 4.1, one case is left out. This is where $\limsup \sum_{t=1}^{T} E\{X_t^i - Y_t \mid \mathscr{F}_{t-1}\} = -\infty$ and

$$\limsup \sum_{t=1}^{T} \frac{E\{X_t^i \mid \mathscr{F}_{t-1}\}}{E\{Y_t \mid \mathscr{F}_{t-1}\}} = 1.$$

To cover this case, more detailed information on the rates of divergence of the various series is required. It is easy to construct examples for which $\limsup \sum_{t=1}^{T} E\{X_t^i - Y_t \mid \mathscr{F}_{t-1}\} = -\infty$ and yet trader i survives. But it is also easy to construct examples where the lim inf of the time averages is 0, and yet the trader vanishes. When the random variables X_t^i and Y_t both have almost-sure limits, a sharper failure criterion is obtainable. In this case, if

$$\lim \frac{1}{\sqrt{T}} \sum_{t=1}^{T} I_q(\alpha_t^i) - I_q(p_t) = +\infty,$$

then trader i vanishes. Corollary 4.1(c), which provides criteria for traders to succeed or fail relative to each other, states and proves a similar result.

DEFINITION 4.2. Trader i *dominates* trader j on the set of sample paths $A \in \mathscr{F}$ if $\liminf r_t^i/r_t^j > 0$ a.s. on A. He *vanishes* relative to trader j on the set of sample paths $A \in \mathscr{F}$ if $\limsup r_t^i/r_t^j = 0$ a.s. on A.

The fitness measure of Theorem 4.1 is hard to deal with analytically because it depends upon market prices which are themselves endogenous. Corollary 4.1 treats relative shares in just the way we did in the horse-race example. We will use this result to construct both examples and general results.

COROLLARY 4.1. *Suppose Assumptions* 1 *and* 2 *hold for all* i:

(a) *On* $\{\omega: \liminf \sum_{t=1}^{T} E\{X_t^i - X_t^j \mid \mathscr{F}_{t-1}\} > -\infty\}$, *trader* i *dominates trader* j.

(b) *On*

$$\left\{\omega: \limsup \frac{\sum_{t=1}^{T} E\{X_t^i \mid \mathscr{F}_{t-1}\}}{\sum_{t=1}^{T} E\{X_t^j \mid \mathscr{F}_{t-1}\}} < 1\right\},$$

trader i *vanishes relative to trader* j.

(c) *If* X_t^i *and* X_t^j *converge almost surely to limit random variables* X^i *and* X^j, *then trader* i *vanishes relative to trader* j *on the set*

$$\left\{\omega: \liminf \frac{1}{\sqrt{T}} \sum_{t=1}^{T} E\{X_t^j - X_t^i \mid \mathscr{F}_{t-1}\} = +\infty\right\}.$$

5. THE MARKET SELECTION HYPOTHESIS

The idea that the market selects for, or at least does not select against, rationality could be formalized in various ways. The conjectures that we investigate are, in increasing order of plausibility: (1) every rational rule survives, (2) every irrational rule vanishes relative to any rational rule, and (3) every irrational rule vanishes relative to some rational rule. It will be obvious that, if we do not control for disparities in savings rates, conjectures (1) and (2) are false. We show that conjectures (1) and (2) are still false when savings rates are all identical. Conjecture (3) is true if we restrict the comparison to rules which save at the same rate, although it is obvious that a rational rule may be defeated by an irrational rule which saves at a higher rate. We conclude that the link between market selection and rationality is weak. Not all rational rules are fit, but some are.

To make these conjectures precise we need a definition of rationality. Clearly the scope of rationality will determine the validity of the three conjectures. We will not rest our claims on a particular version of rationality. Rather, we will describe several hypotheses about what is and is not rational, and trace out their consequences. We suppose that individuals care about investment and savings rules only through their implications for consumption. A consumption plan is a sequence $\{c_t\}_{t=1}^{\infty}$ of non-negative random variables, and each c_t is $\mathscr{F}_t$-measurable. Associated with each investment and savings plan $\{\alpha_t, \delta_t\}_{t=1}^{\infty}$ is its induced consumption plan. A preference relation on consumption plans induces a preference relation on investment and savings plans. A *rational trader* is a trader who uses an investment and savings plan which is maximal with respect to some element of a set P of preference relations which are induced by transitive, complete, and monotonic preference relations on the set of consumption plans. Clearly the precise meaning of rationality will depend upon the set P. But even at this level of generality some constraints are imposed on rules. First, no rule with $\delta_t \equiv 1$ is rational, because any such rule supports only the 0 consumption plan. Second, any two rules which give rise to the same consumption plan are either both rational or both irrational.

There are two hypotheses on the set P which describe the scope of rationality in a manner convenient for our analysis.

($P1$) P includes all preferences which have a discounted dynamic programming representation with a reward function which is strictly concave and strictly increasing.

($P2$) The set of irrational investment-savings plans is sup-norm dense in the set of all plans.

The first hypothesis says that rationality includes the most popular decision model for choice under uncertainty—expected utility maximization with discounting and a concave reward function. The second hypothesis says that not too many things are rational. It is easy to see that the second hypothesis is met if P is the set of all preferences with concave dynamic programming representations, or even the more general set of recursive preferences discussed by Kreps and Porteus, Epstein, and others. (To see this, recast the problem as buying contingent claims on dollars. Then the solution to maximizing the value of the current action generates Slutsky conditions which impose constraints on investment rules. The set of investment rules not satisfying these conditions is sup-norm dense. Another way to see this is to allow randomized investment plans. These too are sup-norm dense and are never rational when the reward function is strictly concave.)

One simple class of investment rules which are consistent with rational behavior is the class where preferences have a time separable expected utility representation of the form

$$E\left[\sum_{t=1}^{\infty} \gamma^{t-1} \log(c_t)\right],$$

where $E[\cdot]$ is the trader's expectation operator and γ is his discount factor. Given his logarithmic utility, the only aspect of the trader's beliefs that are relevant to his decisions is his probability on exogenous states, say q_t^i for individual i at date t.

THEOREM 5.1. *Suppose trader i's objective function is*

$$E^i\left[\sum_{t=1}^{\infty} (\gamma^i)^{t-1} \log(c_t^i)\right].$$

If beliefs over states at date t are q_t^i and the value above is finite for any investment rule, then the optimal investment rule is the simple rule

$$\delta_t^i = \gamma^i$$
$$\alpha_t^i = q_t^i$$

at each date.

Our first market selection result is that the globally most fit rule is a simple rule. A trader with savings rate at least as large as the market savings rate and the portfolio rule $\alpha_t^i = q$ for all t dominates regardless of others' investment rules. So provided that his beliefs are correct, and his discount factor is large enough, an expected discounted logarithmic utility maximizer will dominate.

THEOREM 5.2. *An investment rule $(\alpha_t^i, \delta_t^i)_{t=1}^{\infty}$ with $\alpha_t^i = q$ and $\delta_t^i \geqslant \delta_t$ a.s. for all t, dominates a.s. regardless of the portfolio rules used by any other trader. If trader i uses this rule, then $p_t \to q$ a.s.*

The most fit rule is the simple rule that maximizes the expected growth rate of wealth share. It has long been known that, when prices are i.i.d., this rule asymptotically gives a higher wealth share than does any other investment rule. This corresponds to a partial equilibrium analysis of investment rules. Theorem 5.2 generalizes this result to a closed equilibrium model, where prices could not possibly be i.i.d.[6] It shows that the market actually selects for this rule, and it describes the consequences of this selection for the asymptotic behavior of equilibrium prices.

Our interest is not in whether simple rules are the *right* investment strategy to follow. As Samuelson [26] has so forcefully shown, it is *right* only for those investors who happen to have logarithmic reward functions. Our interest is only in the positive issues of identifying the rules selected for, and the consequences of that selection for the behavior of market prices.

5.1. *Does Every Rational Rule Survive?*

If P satisfies hypothesis $(P1)$, this conjecture is false. To see this we consider selection between expected discounted logarithmic utility maximizers.

THEOREM 5.3. *Suppose that traders i and j maximize expected discounted logarithmic utility with discount factors* γ^i *and* γ^j *and beliefs* q^i *and* q^j. *Then trader i dominates trader j a.s. and trader j vanishes a.s. if* $I_q(q^j) - \log \gamma^j > I_q(q^i) - \log \gamma^i$.

In any economy where the value of the traders' problems is finite both traders i and j would qualify as rational even according to the restrictive concept of maximizing expected discounted von-Neumann Morgenstern utility. So rational traders may vanish. If we were to further restrict rationality to require that the trader have rational expectations, and thus $q^i = q^j = q$, then survival is determined by discount factors.

In the log utility case, with equal discount factors, the market selects for the most nearly correct beliefs. It is not true, however, that if discount factors, or savings rates, are controlled for that the market generally selects for correct beliefs. The log utility case is special as the traders have the same level of risk aversion. If traders have differing levels of risk aversion and equal savings rates a trader with correct beliefs can be driven out of the market by traders with incorrect beliefs.

THEOREM 5.4. *Consider an economy with two traders and social risk, i.e., for each t there are states s, s' such that* $w_{st} \neq w_{s't}$. *Suppose that each trader saves at the same rate. Suppose that at each date t, trader i's portfolio* α^i_t *is*

[6] See Kelly [21] and Breiman [7]. Other generalizations of this result are known, but they make specific assumptions about the price process which will be hard to verify in an equilibrium model where the price process is endogenously determined.

selected to maximize $E^i[(1-\sigma^i)^{-1}(w^i_{t+1})^{\sigma^i}]$ *for* $0<\sigma^i$ *and beliefs* q^i. *Let* $\sigma^1 \neq 1$ *and* $q^1 = q$. *There is an* $\varepsilon > 0$ *such that for all* $\|q^2 - q\| < \varepsilon$ *and* $|\sigma^2 - 1| < \varepsilon$, *trader* 1 *vanishes and trader* 2 *dominates a.s.*

Controlling for savings rates, the market selects for those investors whose coefficient of relative risk aversion (σ^i) and beliefs (q^i) are most nearly $(1, q)$. Individuals who are either relatively risk averse ($\sigma^i > 1$) or relatively risk neutral ($\sigma^i < 1$) are selected against. Alternatively, controlling for risk aversion and beliefs, the market selects for the most patient investor. This last result is well known for economies without uncertainty.

5.2. *Does Every Irrational Rule Vanish Relative to Any Rational Rule?*

If P satisfies hypotheses ($P1$) and ($P2$) this conjecture is false. A cheap way in which it is false is to imagine investors who act irrationally for a finite number of dates and then act like logarithmic expected utility maximizers with high discount factors and correct beliefs ever after. If their discount factors are as large as any other investors' savings rates, these investors will dominate the market. More interesting is the survival or domination of investors who are not "asymptotically rational." Consider an economy with two traders. One trader is irrational, with an investment-savings plan within ε of the investment plan for a logarithmic expected utility maximizer with correct beliefs and saving rate δ. The second trader is a logarithmic expected utility maximizer with beliefs $q' \neq q$ and savings rate δ. It follows from the continuity of the relative entropy $I_q(\alpha)$ in α that the irrational trader will dominate if ε is sufficiently small or if q' is sufficiently far from q. This does not require that the distance between α_t and q for the irrational trader converge to 0.

5.3. *Does Every Irrational Rule Vanish Relative to Some Rational Rule?*

Suppose now that P satisfies hypothesis ($P1$). The irrational rule $(\alpha_t, \delta_t)_{t=1}^{\infty}$, where $\alpha_t = q$, $\delta_t = 1$ for all t, dominates almost surely. In fact, any rational rule with savings rates bounded away from one will vanish if the above irrational rule is present in the economy. Note, however, that failure of a rational trader who has rational expectations does not mean that the trader makes poor decisions. Since he has rational expectations and his investment rule is optimal for his preferences, he could not do better (from his point of view) by changing his decisions. Dominating or vanishing do not have normative content.

If we control for savings rates, conjecture (3) is essentially correct. In any economy in which the value of the discounted logarithmic utility maximization problem is finite, the simple q rule of Theorem 5.2 is rational and dominates all other rules with the same savings rate. Any trader with the same savings rate as the log-utility trader and using an irrational rule that deviates from the q rule sufficiently often will vanish.

THEOREM 5.5. *Suppose that trader* 1 *maximizes the expectation of the sum of discounted logarithms of consumption, with beliefs* $q_t^1 \equiv q$ *and discount factor* $\gamma^1 < 1$ *such that the value of his decision problem is finite. Then any trader* i *with* $\delta_t^i \leqslant \gamma^1$ *almost surely for large enough* t, *who uses an investment rule* α_t^i *such that* $\liminf (1/T) \sum_{t=1}^{T} I_q(\alpha_t^i) > 0$, *vanishes almost surely.*

Theorem 5.5 applies not only to irrational traders but also to rational traders who do not use the simple q rule. That is, the market selects for a particular investment rule which is consistent with rational expectations and logarithmic utility, and against any trader who does not use this rule. In fact, any rule that does not converge to the q rule sufficiently fast will also vanish. The example following Theorem 6.2 in the section illustrates this phenomenon.

6. ADAPTIVE BEHAVIOR

In this section we consider the fitness of three types of adaptive behavior: Bayesian learning, search, and imitation. Differences in savings rates can overcome differences in fitness of portfolio rules, so we restrict our attention to traders who save at the same rate.

Our general analysis of market dynamics in Sections 3 and 4 has avoided any discussion of the goals which generate investment rules, but this distance is impossible to maintain in analyzing adaptive behavior. In modelling adaptive behavior, we must first identify the stimulus for adaptation. Our results in Section 5 identify a selection pressure towards those investment rules which can be rationalized by subjective expected utility maximization with logarithmic payoff functions. Since, if it is initially available in a non-adapting population, this kind of rule characterizes the limit behavior of the market, our study of adaptation will presume that all individuals evaluate their current decisions from this point of view. That is, individuals will select among simple rules. With logarithmic payoff functions, the optimal rule is to "bet your beliefs." Thus switching rules can be viewed as switching beliefs. Given the logarithmic rationality of simple rules, we can interpret processes of simple rule adaptation as learning processes. The natural learning process to study first is that which itself is derivable from the postulates of expected utility maximization, Bayesian learning. Next, we consider processes in which investors imitate their most successful colleagues. Finally we take up processes which search across beliefs from some pre-selected set.

6.1. *Bayesian Learning Rules*

A *learning rule* for trader i is a sequence of functions $\{\alpha_t^i\}_{t=1}^{\infty}$ where for each date t, $\alpha_t^i: \Omega^{t-1} \to \{\alpha \in R_+^S : \sum_{s=1}^S \alpha_s = 1\}$ is $\mathscr{F}_{t-1}$-measurable. With this observation it is clear that our market selection results for general investment rules apply to selection for learning rules. We need only derive the sum of conditional expected growth rates to decide which learning rules dominate and which ones vanish.

We first analyze Bayesian learning. Suppose a Bayesian considers two models of the i.i.d. process on states, q^1 and q^2. Let $\eta_1 = \Pr\{q = q^1\}$, $0 < \eta_1 < 1$, be the prior probability on model one and $\eta_t = \Pr\{q = q^1 \mid F_{t-1}\}$ the posterior at the beginning of period t. Let $d_t = (d_t^1, ..., d_t^S)$ where for each state s, d_t^s is the number of times that state s has occurred through period t. By Bayes rule, for any $t > 1$, the posterior probability on model 1 is

$$\eta_t = \frac{\eta_1 \prod_{s=1}^S (q_s^1)^{d_{t-1}^s}}{\eta_1 \prod_{s=1}^S (q_s^1)^{d_{t-1}^s} + (1-\eta_1) \prod_{s=1}^S (q_s^2)^{d_{t-1}^s}}.$$

In log odds form this yields

$$\log\left(\frac{\eta_t}{1-\eta_t}\right) = \log\left(\frac{\eta_1}{1-\eta_1}\right) + \sum_{s=1}^S d_{t-1}^s \log q_s^1 - \sum_{s=1}^S d_{t-1}^s \log q_s^2.$$

The process on states is i.i.d. so under the assumption that model 2 is true we have from the SLLN,

$$\lim_{t \to \infty} \frac{1}{t-1} \log \frac{\eta_t}{1-\eta_t} = -I_{q^2}(q^1)$$

almost surely. That is, $\eta_t/(1-\eta_t)$ converges exponentially at rate $I_{q^2}(q^1)$ to zero, almost surely. Thus $\sum_{t=1}^{\infty} \eta_t < \infty$. We summarize these comments in the next lemma.

LEMMA 6.1. *Let trader i be a Bayesian whose prior has finite support containing the true model. Let η_t^k the posterior probability on model k at the beginning of period t. Then almost surely $\sum_{t=1}^{\infty} \eta_t^k < \infty$ for any model k other than the true model.*

The Bayesian trader's portfolio at date t is his predictive probability given η_t: $\alpha_t = \eta_t q^1 + (1-\eta_t) q^2$. We first consider whether a Bayesian dominates or vanishes relative to a trader who has correct beliefs. Suppose investor i is a Bayesian, $\alpha_t^i = \eta_t^i q^1 + (1-\eta_t^i) q^2$, and investor j uses portfolio rule $\alpha_t^j = q^2$ for all t. By Corollary 4.1, investor i will dominate investor j, r_t^i/r_t^j is bounded away from 0 a.s., if $\sum_{t=1}^{\infty} E\{X_t^i - X_t^j \mid \mathscr{F}_{t-1}\}$ is bounded almost surely. Using our definitions this reduces to checking whether $\sum_{t=1}^{\infty} I_{q^2}(\alpha_t^i)$ is bounded from above. Calculation shows that there is a constant c such that

$$\sum_{t=1}^{\infty} I_{q^2}(\alpha_t^i) \leqslant c \sum_{t=1}^{\infty} \eta_t^1 < \infty.$$

Since a Bayesian dominates relative to a trader with correct beliefs, he will dominate relative to any trader and thus dominate.

THEOREM 6.1. *Suppose all traders save at the same rate and bet their beliefs. Any Bayesian whose prior has a finite support which includes the true state distribution dominates almost surely.*

The above analysis shows more than that Bayesians dominate. The only aspect of a learning rule that matters for dominance by investor i is finiteness of $\sum_{t=1}^{\infty} I_q(\alpha_t^i)$.

THEOREM 6.2. *Suppose all traders save at the same rate and bet their beliefs. If trader i's learning rule yields $\sum_{t=1}^{\infty} I_q(\alpha_t^i) < \infty$ a.s., then trader i dominates almost surely.*

Theorem 6.2 demonstrates, at least in the framework of this section, that the market selects for Bayesian or Bayesian-like learning rules. Any Bayesian who puts positive prior probability on the true model dominates regardless of what other (finite) models he considers or how much strictly positive prior weight he puts on the true model.

To see how a slow learner can vanish, consider an economy with only two traders. Both save at the same rate. Trader 1 is a Bayesian and q, a vector of irrational numbers, is in the support of his beliefs. Trader 2 constructs empirical distributions for his beliefs. But he makes observations rarely. He makes his nth observation on the process only when his wealth share falls below $1/n$. It is easy to see that trader 2 will make an infinite number of observations. Suppose he has made some finite number of observations. His belief vector contains only rational numbers, and so is different from q. Ultimately, the Bayesian will have better beliefs, and so the slow learner's wealth share will have to go to 0. Thus at some point he will make another observation. We know from the Glivenko–Cantelli Theorem that his beliefs will ultimately converge to q. But clearly his wealth share goes to 0.

For the case of two consistent learners, part (c) of Corollary 4.1 gives a relative dominance condition. Namely, i fails relative to j if $\liminf (1/\sqrt{T}) \sum_{t=1}^{T} I_q(\alpha_t^i) - I_q(\alpha_t^j) = +\infty$.

The results of this section can be viewed as an extension of the results of Section 5. Given logarithmic prefereces with identical discount factors, the market selects for Bayesian learners.

6.2. *Imitation*

An imitator revises his investment rule by reference to the performance of other traders' rules. Recall that traders evaluate performance based on the growth rate of wealth share. Given our focus on selection across processes of investor adaptation, we consider economies in which some traders used fixed, simple rules and others engage in various forms of imitative behavior. We suppose that only two simple rules are in use in the economy: Traders 1 and 2, respectively, use simple rules q^1 and q^2 with $I_q(q^1) < I_q(q^2)$. We suppose that no rule used by any trader in the market dominates the simple rule q^1. Finally, we consider two types of imitative behavior; one which vanishes and one which dominates.

A trader using imitation rule I adopts at date t the investment rule that yielded the largest actual growth rate of wealth share from date $t-1$ to t. More precisely, if trader i uses imitation rule I,

$$\alpha_t^i \in \underset{\{q^1, q^2\}}{\arg\max} \sum_{s=1}^{S} 1_{st} \log\left(\frac{x}{p_{s,t-1}}\right),$$

for $t > 1$ and α_1^i fixed.

In the economies we consider, such an imitator will switch between investment rules infinitely often. He regularly mimics a rule which is no better than trader two's rule and thus $\lim\inf(1/T)\sum_{t=1}^{\infty} I_{q^1}(\alpha_t^i) > 0$ a.s. So anyone using this rule of "imitating the successful" will vanish almost surely.

The asymptotic performance of rule I is not improved by allowing the trader to observe any finite number of periods and imitate the best performing rule over that period. The trader could, for example, calculate actual T-period growth rates for every trader and then follow the best performing T-period sequence of investment rules. Alternatively, he could simply use the most recent investment rule used by the best performing trader over the past T-periods. In either case the imitator will vanish almost surely.

In order to survive an imitator must settle down to a dominant rule. An imitation rule that has this property is to use at date t the rule used at date $t-1$ by the trader with the largest wealth share. Since the wealth share of investors using rules with relatively low growth rates declines, this rule is asymptotically equivalent to the strategy of imitating that rule which has the largest growth rate of wealth share from date 0 to date $t-1$. Imitation rule II is defined by $\alpha_t^i = \alpha_{t-1}^{j*}$ where $j*$ is such that $r_t^{j*} \geq r_t^j$ for all traders j.

A trader using this rule is imitating the relatively wealthy. Early on he could be imitating those who begin with a large wealth share but follow a poor rule. However, eventually such traders have low wealth share and our imitator imitates only those with dominant rules, i.e., those with $\sum_{t=1}^{\infty} I_{q^i}(\alpha_t) < \infty$. As a result a trader using imitation rule *II* will also dominate almost surely. His limit wealth share may be small, but it will not be zero.

THEOREM 6.3. *Suppose that traders* 1 *and* 2 *invest using simple rules* q^1 *and* q^2, *respectively, and that all other traders use imitation rule I or II. Any trader using imitation rule I vanishes a.s. and any trader using imitation rule II dominates a.s.*

Thus type II imitation increases the selection pressure for correct beliefs. Its asymptotic performance is similar to that of Bayesian learning.

6.3. *Search Rules*

A trader using a search rule has a set of simple portfolio rules that may be chosen at any date. At each date a searcher selects the rule to use in his portfolio choice based on his experience and his search rule. Our interest is in the performance of particular search rules as well as market selection across various adaptive rules. The theorems of Section 4 can be applied to answer these questions given any rule or class of rules under consideration. We consider two prototype search rules.

Trader i considers the finite set B^i of portfolio rules. A trader who uses search rule *I* selects at date t the portfolio rule that would have maximized his date t wealth share given the actual prices and states up to date t. If trader i uses search rule *I* then

$$\alpha_t^i \in \arg\max_{\beta^i \in B^i} \prod_{\tau=1}^{t-1} \prod_{s=1}^{S} \left(\frac{\beta_s^i}{p_{s\tau}}\right)^{1_{s\tau}}.$$

This rule involves relatively sophisticated behavior as the investor must remember the history of the state-price process, compute the objective function, and maximize it. However, this rule perform very well.

Let $B^{i*} = \{\beta^i \in B^i : \beta^i \in \arg\min_{\beta^i \in B^i} I_q(\beta^i)\}$ and let A^i_α be the random set of limit points of α_t^i.

THEOREM 6.4. *If trader i uses search rule I then* $A^i_\alpha \subset B^{i*}$ *a.s. If all traders use search rule I then*

$$\lim_t p_t \in \arg\min_{\beta \in \bigcup_{i=1}^{I} B^i} I_q(\beta).$$

Both a Bayesian whose prior includes q in its support and a trader using search rule *I* with q a feasible rule dominate. The market selects for such traders over anyone whose adaptive rule yields $\lim\inf (1/T) \sum_{t=1}^{\infty} I_q(\alpha_t) > 0$.

A trader using search rule *II* also considers a finite set of models B^t. He switches away from his current portfolio rule if and only if his wealth share falls below where it was when the current rule was adopted. When a trader switches he selects a new rule from B^t at random with positive probability on every rule in B^t. Let $\tau(t)$ be the date at which portfolio rule α_t^t was adopted. Search rule *II* is defined by

$$\alpha_t^t = \begin{cases} \alpha_{t-1}^t & \text{if } r_t^t \geq r_{\tau(t)}^t \\ \beta \in B^t & \text{with probability } \varepsilon_\beta^t, \text{ if } r_t^t < r_{\tau(t)}^t, \end{cases}$$

where each $\varepsilon_\beta^t > 0$ and $\sum_{\beta \in B^t} \varepsilon_\beta^t = 1$.

Although the fitness criteria of Corollary 4.2 apply to this case, checking whether search rule *II* is fit is not simple. To illustrate the complexity we consider a three trader, two state example. We assume that the traders begin with equal wealth shares. Let $q = (1/2, 1/2)$ and suppose that trader two uses portfolio rule $\alpha_t^2 = q$ for all t. Trader one uses search rule *II* applied to the set of portfolio rules $B^1 = \{(1/2, 1/2), (1/3, 2/3)\}$. Finally, trader three uses portfolio rule $\alpha_t^3 = (1/3, 2/3)$ for all t. A simple application of Theorem 4.1 shows that trader two dominates, trader three vanishes, and prices converge to q almost surely. Our interest is in what happens to trader one.

The price vector is always in $(1/3, 1/2] \times [1/2, 2/3)$ since it is a wealth share weighted average of the trader's portfolio rules. Suppose trader one adopts q at some date. Then his wealth share rises if state one occurs and falls if state two occurs. The expected growth rate of his wealth share is positive, so with positive probability his wealth share never falls below its initial level and he stays with the correct model forever. But with probability 1/2 he immediately switches away from the correct model. He almost surely does not stay with the wrong model as the expected growth rate of his wealth share is negative when he uses the wrong model. Thus he uses the correct model infinitely often. Every time he adopts the correct model there is positive probability of keeping it, but as prices converge to q the expected growth rate of his wealth share converges to zero when he uses q and remains negative when he uses the wrong model. So although trader one will eventually adopt and keep the correct model his limit wealth share can be arbitrarily close to zero.

Column *I* of Table I provides "limit" wealth share data for trader one when his initial portfolio rule is selected at random from B^1 with probability one-half on each model. The data were constructed by running the economy described above for 1000 periods. The entries in the table are the fraction of 100 observations in which the trader had wealth share in the indicated range after the 1000th period. In this data set no searcher switches after 200 periods and each searcher settles down to the correct

TABLE I

Limit Wealth Shares—Fraction of 100 Observations

Wealth share	Economy I	Economy II		Economy III	
	Searcher	Searcher	Bayesian	Searcher	Bayesian
0–0.1	0.22	0.20	0	0.48	0
0.1–0.2	0.17	0.22	0.16	0.07	0
0.2–0.3	0.14	0.26	0.64	0.17	0
0.3–0.4	0.23	0.16	0.20	0.12	0.04
0.4–0.5	0.24	0.16	0	0.03	0.09
0.5–0.6	0	0	0	0.09	0.03
0.6–0.7	0	0	0	0.04	0.13
0.7–0.8	0	0	0	0	0.17
0.8–0.9	0	0	0	0	0.07
0.9–1.0	0	0	0	0	0.47

model. The observations can be interpreted as limit wealth shares as once the searcher stops switching wealth shares converge exponentially. Although the trader using search rule *II* dominates, the table shows that his limit wealth share may be small.

It is also interesting to compare a trader using search rule *II* with a Bayesian. We consider a three person, two asset economy with $q = (0.5, 0.5)$. Trader one is a Bayesian who places an initial prior probability of one-half on each of two models $(0.5, 0.5)$ and $(0.7, 0.3)$. Trader two uses search rule *II* over the same models. His initial model is selected at random with probability one-half on each model. Trader three has beliefs, and betting rule, q^3.

We know from Theorem 6.1 that the Bayesian dominates and prices converge to q almost surely. How the Bayesian and the searcher do relative to each other depends on the value of q^3. If the third trader is correctly informed then the distribution of the searcher's limit wealth share is more disperse than the distribution of the Bayesian's limit wealth share. The searcher is approximately equally likely to have a larger or a smaller wealth share than the Bayesian. These data are presented in Columns II:Searcher and II:Bayesian of Table I.

Alternatively, if trader three is wrong ($q^3 = (0.6, 0.4)$) then the Bayesian's limit wealth share increases and the searcher is more likely to have an arbitrarily low limit wealth share. See Columns III:Searcher and III:Bayesian of the table. The difference between these two economies (with q^3 correct or wrong) is due to the dependence of the searcher's rule selection on market prices. The Bayesian is learning about an exogenous process and so his beliefs and thus his betting rule are independent of others' behavior. The searcher's model selection is greatly affected, however, by the variance of his wealth share which depends on prices and thus on others' betting rules.

7. Conclusion

In the financial asset markets we have studied, the dynamics of wealth accumulation are such that those investors whose investment and savings rules give rise to the highest conditional expected growth rates come to dominate the market. In other words, natural selection operates to select for those rules most "fit," and the fitness criterion is expected growth rates of wealth share accumulation. This selection criterion has implications for the long run behavior of asset prices. When a particular rule in the market is globally more fit than any other rule, asset prices must behave in such a fashion that gains (in share) from market participation go to 0 at exponential rate. In the case of simple rules, this means that if there is a unique simple rule q' nearest the true state distribution q as measured by relative entropy, then prices converge to q' at exponential rate.

Our ability to identify fit rules permits a characterization of the limit behavior of our economies that is not dependent upon hypotheses concerning investors' motivations, such as expected utility maximization or other rationality hypotheses. Our original motivation for undertaking this research was precisely this point in the context of the "informational efficiency" of markets. This so-called "informational efficiency" of markets has several different interpretations. On the one hand, it is taken to mean that the market efficiently aggregates traders' information or beliefs. On the other hand, it is taken to mean that the market price reflects accurate information. These two views stand in contradiction if some traders have inaccurate beliefs. This contradiction has been resolved by an appeal to the dynamics induced by traders' learning behavior. The claim that markets aggregate information is taken to be a description of short-run or temporary equilibrium. The claim that market price reflects accurate information is taken to be a description of long-run equilibrium. Under some regularity assumptions, if traders' learning can be described by the updating of posterior beliefs according to Bayes rule, then the short-run equilibria will converge to a rational expectations equilibrium, the long-run equilibrium.[7] But this adjustment mechanism is delicate. If traders fail to be perfect Bayesians or if they have too much to learn, then the sequence of temporary equilibria may fail to converge.[8] Furthermore, recent research

[7] This is described in Blume and Easley [5] and Bray and Kreps [6].
[8] Blume and Easley [4] and Feldman [13], respectively.

generalizes to market contexts the insight from bandit problems that it may not be optimal to "completely learn" all the salient features of the market environment.[9]

Our results show that the intuition that the sequence of temporary equilibria converges to a rational expectations equilibria may nonetheless be justified in situations where "belief-adjustment dynamics" fail. There are other market forces at work which may select for traders who are more nearly optimizing and who have better beliefs. Theorem 6.1 demonstrates that it is sufficient to have only some Bayesian traders, so long as they have nearly logarithmic preferences and high discount factors.

Our results also show how the argument that wealth dynamics justify rationality is somewhat wanting. For instance, if all traders have logarithmic preferences, and discount factors are uncorrelated with beliefs, then the view is roughly correct. But if discount factors and beliefs are correlated in the right way, then this view is false. Theorem 5.3 measures precisely how high discount factors can compensate for incorrect beliefs when all traders have logarithmic payoff functions, leading traders with incorrect beliefs to dominate the market, and forcing assets to be asymptotically priced incorrectly. Theorem 5.4 demonstrates the same effect, trading off beliefs against attitudes towards risk.

These results also address issues raised in the literature on the effects of noise traders on asset prices. DeLong *et al.* [9] present a model in which noise trader risk causes a financial asset be the systematically mispriced relative to its fundamental value. In comparing their model with ours, two features stand out. First, there is no opportunity for long-run asset accumulation in their model, so wealth accumulation effects cannot function. Second, given the specification of CARA utilities used in their model, the noise trader's portfolio selection rules may indeed by more fit than the rules used by the sophisticated traders. Our results suggest that systematically misinformed traders may persist, and even come to dominate the market, but this persistence is due to systematic differences in utility functions between correctly informed and misinformed traders. We hope to discuss this issue and the related issue of price-destabilizing speculation in a subsequent paper.

There are other interesting issues to address in our model explicitly concerning evolutionary rules. Our interest in adaptive behavior in this paper is largely with selecting over adaptation processes. To this end we built examples containing several types of adapting agents and watched them compete against each other. We did not study, for instance, how an economy consisting only of imitators behaves. We have not allowed for the

[9] This work includes Easley and Kiefer [11], Feldman and McLennan [14], and McLennan [24].

leakage of new rules into the population which, in an evolutionary context, would correspond to mutation. These questions are hard to ask not because of technical difficulties, but because the results tend to be process-specific, and we have not identified a natural class of individual adaptation processes to study.

APPENDIX

Throughout this section we shall write $\{A\}$ for the event $\{\omega \in \Omega : \omega \in A\}$.

Proof of Theorem 4.1. We first prove two useful lemmas.

LEMMA A.1. $\limsup \sum_{t=1}^{T} E\{X_t^i - Y_t \mid \mathscr{F}_{t-1}\}\} \leqslant -\log r_0^i$ *a.s.*

Proof. Note that $\mu\{\sum_{t=1}^{T} X_t^i - Y_t \geqslant \sum_{t=1}^{T} E\{X_t^i - Y_t \mid \mathscr{F}_{t-1}\}\} > 0$. So on $\{\sum_{t=1}^{T} E\{X_t^i - Y_t \mid \mathscr{F}_{t-1}\} > -\log r_0^i\}$, $\log r_0^i + \sum_{t=1}^{T} X_t^i - Y_t = \log r_T^i > 0$ with strictly positive probability. This violates the feasibility requirement that no trader's wealth share exceed 1. ∎

LEMMA A.2. *On* $\{\liminf_T \sum_{t=1}^{T} E\{X_t^i - Y_t \mid \mathscr{F}_{t-1}\} > -\infty\}$,

$$\sum_{t=1}^{T} \operatorname{Var}\{X_t^i - Y_t \mid \mathscr{F}_{t-1}\} < \infty \quad a.s.$$

Proof. From Freedman [16, Corollary 4.5(a)] we conclude that $\sup_T \sum_{t=1}^{T} X_t^i - Y_t - E\{X_t^i - Y_t \mid \mathscr{F}_{t-1}\} = \infty$ a.s. on $\{\sum_{t=1}^{\infty} \operatorname{Var}\{X_t^i - Y_t \mid \mathscr{F}_{t-1}\} = \infty\}$. By hypothesis

$$\liminf_T \sum_{t=1}^{T} E\{X_t^i - Y_t \mid \mathscr{F}_{t-1}\} > -\infty$$

and by Lemma A.1,

$$\limsup_T \sum_{t=1}^{T} E\{X_t^i - Y_t \mid \mathscr{F}_{t-1}\} \leqslant -\log r_0^i \qquad \text{a.s.,}$$

so $\sup_T \sum_{t=1}^{T} X_t^i - Y_t = +\infty$ a.s. on $\{\sum_{t=1}^{\infty} \operatorname{Var}\{X_t^i - Y_t \mid \mathscr{F}_{t-1}\} = \infty\}$. But since $\log r_T^i = \sum_{t=1}^{T} X_t^i - Y_t + \log r_0^i$, this contradicts feasibility, thereby proving the lemma. ∎

Now we prove (a)(1). From Lemma A.2 and Freedman [16, Corollary 4.5(b)] it follows that $\sum_{t=1}^{T} X_t^i - Y_t - E\{X_t^i - Y_t \mid \mathscr{F}_{t-1}\}$ has a limit distribution on $\{\liminf_T \sum_{t=1}^{T} E\{X_t^i - Y_t \mid \mathscr{F}_{t-1}\} > -\infty\}$. So on $\{\liminf_T \sum_{t=1}^{T} E\{X_t^i - Y_t \mid \mathscr{F}_{t-1}\} > -\infty\}$, $\liminf_T \sum_{t=1}^{T} X_t^i - Y_t > -\infty$, which is to say that trader i dominates.

Next we prove (a)(2). Consider $\omega \in \Omega$ such that $\lim_t \operatorname{Var}\{X^i_t - Y_t \mid \mathscr{F}_{t-1}\} = 0$. Choose any convergent subsequence $\{p_\tau, \alpha_\tau\}_{\tau=1}^{\infty}$ with limits p^* and α^*, respectively. Of course, $\lim_\tau \operatorname{Var}\{X^i_\tau - Y_\tau \mid \mathscr{F}_{\tau-1}\} = 0$. With identical savings rates, dominance of trader i and zero variance in his growth rate requires $\alpha^{i*}_s = p^*_s$ for all s. Since this is true for any convergent subsequence, $\lim_t (p_t, \alpha^i_t) = (p^*, p^*)$ a.s. on

$$\left\{\liminf_T \sum_{t=1}^{T} E\{X^i_t - Y_t \mid \mathscr{F}_{t-1}\} > -\infty\right\}.$$

Next we prove (b). On $\{\sum_{t=1}^{\infty} \operatorname{Var}\{X^i_t - Y_t \mid \mathscr{F}_{t-1}\} < \infty\}$, $\sum_{t=1}^{\infty} X^i_t - Y_t - E\{X^i_t - Y_t \mid \mathscr{F}_{t-1}\}$ has a limiting distribution. Thus when the hypothesis is satisfied,

$$\limsup_T \sum_{t=1}^{T} X^i_t - Y_t > -\infty \qquad \text{a.s.}$$

On the event $\{\sum_{t=1}^{T} \operatorname{Var}\{X^i_t - Y_t \mid \mathscr{F}_{t-1}\} = \infty\}$ the result follows immediately from Freedman's law [16, Theorem 6.3] of the iterated logarithm for sums of dependent random variables.

Finally we prove (c). From Freedman's version [15] of Levy's strong law and a little calculation we find that

$$\frac{1}{\sum_{t=1}^{T} E\{X^i_t \mid \mathscr{F}_{t-1}\}} \log r^i_T - \left(1 - \frac{\sum_{t=1}^{T} E\{Y_t \mid \mathscr{F}_{t-1}\}}{\sum_{t=1}^{T} E\{X^i_t \mid \mathscr{F}_{t-1}\}}\right) \to 0.$$

Part (c) now follows from the fact that $\sum_{t=1}^{T} E\{X^i_t \mid \mathscr{F}_{t-1}\} \to -\infty$ a.s. ∎

Proof of Corollary 4.1. The proofs of parts (a) and (b) are identical to the corresponding proofs for Theorem 4.1. To prove part (c), write

$$\begin{aligned}\frac{1}{\sqrt{T}} \log \frac{r^j_T}{r^i_T} &= \frac{1}{T} \log \frac{r^j_0}{r^i_0} + \frac{1}{\sqrt{T}} \sum_{t=1}^{T} E\{X^j_t \mid \mathscr{F}_{t-1}\} - E\{X^i_t \mid \mathscr{F}_{t-1}\} \\ &\quad + \frac{1}{\sqrt{T}} \sum_{t=1}^{T} X^i_t - E\{X^i_t \mid \mathscr{F}_{t-1}\} - \frac{1}{\sqrt{T}} \sum_{t=1}^{T} X^i_t - E\{X^i_t \mid \mathscr{F}_{t-1}\}.\end{aligned}$$

Since the random variables X^i_t and X^j_t have almost-sure limits with variances v_i and v_j, respectively, the sum of the conditional variances of each term in the sum $(1/\sqrt{T}) \sum_{t=1}^{T} X^i_t - E\{X^i_t \mid \mathscr{F}_{t-1}\}$ is converging in probability to v_i, and the sum of the conditional variances of each term in the sum $(1/\sqrt{T}) \sum_{t=1}^{T} X^j_t - E\{X^j_t \mid \mathscr{F}_{t-1}\}$ is converging in probability to v_j. Thus it is easy to see that the conditions of a Central Limit Theorem for dependent random variables (Theorem 3.2 of Hall and Heyde [18]) are met. Thus the last two terms in the sum are finite, the first term in the sum converges to 0, and so if the second term diverges, then so does the sum. ▮

Proof of Theorem 5.1. By Hinderer [19, Theorem 17.1] the optimal policy can be derived from Bellman's equation. Using Hinderer's Theorem 14.5 and our assumption on beliefs, we calculate that the date t value function is of the form $V_t(w_t^i, h_t) = (1-\gamma^i)^{-1} \log w_t^i + f_t(h_t)$, where f_t is a measurable function of previous history. Solving Bellman's equation yields

$$\delta_t^i = \gamma^i,$$

$$\alpha_t^i = q_t^i$$

for each t. ▮

Proof of Theorem 5.2. Using $\alpha_t^i = q$ and $\delta_t^i \geqslant \delta_t$ we have that

$$E\{X_t^i - Y_t \mid \mathscr{F}_{t-1}\} \geqslant \sum_{s=1}^{S} q_s \log\left(\frac{q_s}{p_{st}}\right) = I_q(p_t) \geqslant 0,$$

for all t. Thus $\liminf \sum_{t=1}^{T} E\{X_t^i - Y_t \mid \mathscr{F}_{t-1}\} \geqslant 0$ and by Theorem 4.1 trader i dominates. By Lemma A.1, $\sum_{t=1}^{T} E\{X_t^i - Y_t \mid \mathscr{F}_{t-1}\}$ is bounded above almost surely, so $\lim I_q(p_t) = 0$ almost surely. Thus $p_t \to q$ almost surely. ▮

Proof of Theorem 5.3. By Theorem 5.1, $\delta_t^i = \gamma^i$ and $\alpha_t^i = q^i$ a.s., so

$$E\{X_t^i - X_t^j \mid \mathscr{F}_{t-1}\} = \log \gamma^i - \log \gamma^j - I_q(q^i) + I_q(q^j).$$

The result follows from Theorem 4.1. ▮

Proof of Theorem 5.4. First suppose that both traders have correct beliefs. Let $r^i = (\sigma^i)^{-1}$. Calculations show that

$$\alpha_s^i = \frac{q_s^{r^i}(p_s/w_s)^{1-r^i}}{\sum_{v=1}^{S} q_v^{r^i}(p_v/\omega_v)^{1-r^i}} \qquad \text{for} \quad s = 1, ..., S.$$

Furthermore, at $r^i = 1$,

$$\frac{\partial^2 E\{X_t^i - Y_t \mid \mathscr{F}_{t-1}\}}{\partial r^{i^2}} = -\operatorname*{Var}_{\beta} \log \frac{qw}{p},$$

where the variance is computed with respect to the probability β given by

$$\beta_v = \frac{(p_v/w_v)^{1-r^i} q^{r^i}}{\sum_{s=1}^{S} (p_s/w_s)^{1-r^i} q_s^{r^i}} \qquad \text{for} \quad v = 1, ..., S.$$

This variance is strictly positive unless $p_s \equiv q_s w_s$. But in this case, $\alpha_s^i = q_s$ for all i, which implies $p \equiv q$, a contradiction. By continuity we can conclude there is an open neighborhood of qw in which equilibrium prices p_t never land. Thus the curvature of $E\{X_t^i \mid \mathscr{F}_{t-1}\}$ in r^i is bounded away from 0 for all equilibrium prices. So for $|r^2 - 1| < |r^1 - 1|$, $E\{X_t^2 \mid \mathscr{F}_{t-1}\} - E\{X_t^1 \mid \mathscr{F}_{t-1}\} \geqslant \varepsilon > 0$ for all t. By the apparent continuity of portfolios in beliefs, this remains true for q^2 near enough to q. Thus by Theorem 4.1, trader 2 dominates and trader 1 vanishes a.s. ∎

Proof of Theorem 5.5. It follows from Theorem 5.1 that $\alpha_t^1 = q$ and $\delta_t^1 = \gamma^1$ for all t. We consider the case where $\delta_t^i \equiv \gamma^1$ and show that trader i vanishes relative to trader 1. (Obviously the same will then be true if $\delta_{st}^i \leqslant \gamma^1$.) We will apply Corollary 4.1(b). We must show that

$$\limsup \frac{\sum_{t=1}^T E\{X_t^1 \mid \mathscr{F}_{t-1}\}}{\sum_{t=1}^T E\{X_t^i \mid \mathscr{F}_{t-1}\}} < 1.$$

A simple manipulation shows that

$$\frac{\sum_{t=1}^T E\{X_t^1 \mid \mathscr{F}_{t-1}\}}{\sum_{t=1}^T E\{X_t^i \mid \mathscr{F}_{t-1}\}} = \frac{\sum_s q_s \log q_s}{\sum_s q_s \log q_s - (1/T)\sum_{t=1}^T I_q(\alpha_t^i)}.$$

By assumption, $\liminf (1/T)\sum_{t=1}^T I_q(\alpha_t^i) > 0$, so the lim sup of the ratio is less than 1. ∎

Proof of Theorem 6.1. We prove the theorem for a Bayesian whose prior has support $\{q^1, q^2\}$, where $q^2 = q$. The extension to a finite support of arbitrary size is straightforward.

To prove the theorem it is sufficient to show that the Bayesian trader 1 dominates trader 2 whose portfolio rule is $\alpha_t^2 \equiv q^2$ at each date. If this were the case, trader 1 would then dominate any other trader, and thus dominate. Calculation shows that

$$E\{X_t^1 - X_t^2 \mid \mathscr{F}_{t-1}\} = -I_{q^2}(\alpha_t^1).$$

So from Corollary 4.1 it follows that we need only show that $\liminf \sum_{t=1}^T -I_{q^2}(\alpha_t^1) > -\infty$ a.s.

Rewriting the relative entropy we have

$$\begin{aligned} -I_{q^2}(\alpha_t^1) &= \sum_{s=1}^S q_s^2 \log(\alpha_{st}^1/q_s^2) \\ &= \sum_{s=1}^S q_s^1 \log\left(\eta_t^1\left(\frac{q_s^1}{q_s^2} - 1\right) + 1\right). \end{aligned}$$

So it is sufficient to show that

$$\liminf \sum_{t=1}^{T} \log\left(\eta_t^1\left(\frac{q_s^1}{q_s^2}-1\right)+1\right) > -\infty \qquad \text{a.s.}$$

for each s such that $q_s^1/q_s^2 - 1 < 0$.

Let

$$c = \max_{\{(q_s^1/q_s^2-1)<0\}} \left[\frac{\log(\eta_1^1(q_s^1/q_s^2-1)+1)}{\eta_1^1(q_s^1/q_s^2-1)}\right].$$

Then as $\eta_t^1 \to 0$ a.s., the concavity of log implies that on almost all sample paths there is a finite time T such that

$$\log\left(\eta_t^1\left(\frac{q_s^1}{q_s^2}-1\right)+1\right) \geqslant c\left(\frac{q_s^1}{q_s^2}-1\right)\eta_t^1 \qquad \text{for all} \quad t \geqslant T.$$

It follows from Lemma 6.1 that $\sum_{t=1}^{T} \eta_t^1 < \infty$ a.s. So

$$\liminf \sum_{t=1}^{T} \log\left(\eta_t^1\left(\frac{q_s^1}{q_s^2}-1\right)+1\right) \geqslant c\left(\frac{q_s^1}{q_s^2}-1\right)\lim \sum_{t=1}^{T} \eta_t^1 > -\infty. \quad \blacksquare$$

Proof of Theorem 6.2. On the event $\sum_{t=1}^{T} I_q(\alpha_t^i) < \infty$ a.s., $\liminf \sum_{t=1}^{T} E\{X_t^i - Y_t \mid \mathscr{F}_{t-1}\} > -\infty$, and the result follows from Theorem 4.1. $\blacksquare$

Proof of Theorem 6.3. To show that trader i using imitation rule I vanishes, it suffices to show that he vanishes relative to the dominant trader 1. Employing the same calculation and argument as in the proof of Theorem 5.5, it suffices to show that $\liminf(1/T)\sum_{t=1}^{T} I_q(\alpha_t^i) > I_q(q^1)$ a.s. According to imitation rule I, at date t,

$$\alpha_t^i \in \arg\max_{\{q^1, q^2\}} \sum_{s=1}^{S} 1_{st} \log x.$$

So trader i's portfolio choice is an i.i.d. random variable, and q^2 is chosen with positive probability. Since $I_q(q^1) > I_q(q^2)$, $\liminf(1/T)\sum_{t=1}^{T} I_q(\alpha_t^i) > I_q(q^1)$ a.s., and trader i vanishes relative to trader 1.

Now we show that rule II traders dominate. At each date t, all such investors have the same α_t^i, either q^1 or q^2. We know that trader 2 and all rule I traders ultimately vanish. There is a (history dependent) time $T(\omega)$ such that beyond $T(\omega)$, the largest wealth share is held by trader 1 or a rule II trader. On the event where the lim inf of rule II traders' wealth share is 0, rule II traders will infinitely often switch to $\alpha_t^i = q^1$. However, whenever beyond $T(\omega)$ they switch to q^1, the trader with the highest wealth share is and remains trader 1. Thus rule II investors never switch back, which is a contradiction. Thus the event where the lim inf of rule II traders' wealth share is 0 is null, so rule II investors dominate. ▮

Proof of Theorem 6.4. If trader i uses search rule I, then

$$\alpha_t^i \in \arg\max \prod_{\tau=1}^{t-1} \prod_{s=1}^{S} (\beta_s^i)^{1_{\omega}} = \arg\max \sum_{s=1}^{S} f_{t-1}^s \log \beta_s^i,$$

where f_{t-1}^s is the frequency of occurrences of state s in the first $t-1$ periods. It follows from the strong law of large numbers that the maximand converges with t almost surely to

$$\sum_{s=1}^{S} q_s \log \beta_s^i = -I_q(\beta^i) + \sum_{s=1}^{S} q_s \log q_s.$$

Thus the set of limit points A_∞^i of α_t^i is almost surely contained in B^{i*}. The claim about limit prices follows from Proposition 3.2 applied to the set of limit portfolio rules. ▮

References

1. A. Alchian, Uncertainty, evolution and economic theory, *J. Polit. Econ.* **58** (1950), 211–221.
2. W. B. Arthur, Self-reinforcing mechanisms in economics, *in* "The Economy as an Evolving Complex System" (P. W. Anderson, K. Arrow, and D. Pines, Eds.), Addison-Wesley, New York, 1988.
3. L. Blume, M. Bray, and D. Easley, An introduction to the stability of rational expectations equilibrium, *J. Econ. Theory* **26** (1982), 313–317.
4. L. Blume and D. Easley, Learning to be rational, *J. Econ. Theory* **26** (1982), 340–351.
5. L. Blume and D. Easley, Rational expectations equilibrium: An alternative approach, *J. Econ. Theory* **34** (1984), 116–129.
6. M. Bray and D. Kreps, Rational learning and rational expectations, *in* "Essays in Honor of K. J. Arrow" (W. Heller, R. Starr, and D. Starrett, Eds.), Cambridge Univ. Press, Cambridge, 1986.
7. L. Breiman, Optimal gambling systems for favorable games, *in* "Proc. Fourth Berkeley Symp. Math. Statist. Probab., Vol. I" (J. Neyman and E. Scott, Eds.), Univ. of California Press, Berkeley, 1961.
8. V. Crawford, An "evolutionary" interpretation of Van Huyck, Battalio, and Beil's experimental results on coordination, *Games Econ. Behav.* **3** (1991), 25–59.
9. J. B. DeLong, A. Schleifer, L. Summers, and R. Waldman, Noise trader risk in financial markets, *J. Polit. Econ.* **98** (1990), 703–738.
10. C. Diegert and D. Heath, On learning through gambling, unpublished, Department of Operations Research, Cornell University, 1988.
11. D. Easley and N. Kiefer, Controlling a stochastic process with unknown parameters, *Econometrica* **56** (1988), 1045–1064.

12. G. FEIGER, Divergent rational expectations in a dynamic model of a futures market, *J Econ. Theory* **17** (1978), 164–178
13. M. FELDMAN, On the generic nonconvergence of Bayesian beliefs and actions, unpublished, University of Illinois, 1988.
14. M. FELDMAN AND A. MCLENNAN, Learning in a repeated statistical decision problem with normal distributions, unpublished, University of Illinois, 1989.
15. D. FREEDMAN, Another note on the Borel-Cantelli lemma and the strong law, with the Poisson approximation as a by-product, *Ann Probab.* **1** (1973), 910–925.
16. D. FREEDMAN, On tail probabilties for Martingales, *Ann. Probab.* **3** (1975), 100–118.
17 M. FRIEDMAN, "Essays in Positive Economics," Univ. of Chicago Press, Chicago, 1953.
18. P. HALL AND C C. HEYDE, "Martingale Limit Theory and Its Application," Academic Press, New York, 1980.
19. K. HINDERER, "Foundations of Non-Stationary Dynamic Programming with Discrete Time Parameter," Lecture Notes in Operations Research and Mathematical Systems, Springer-Verlag, Berlin, 1970.
20. J. HIRSHLEIFER, Economics from a biological viewpoint, *J. Law Econ.* **20** (1977), 1–52
21. J. L. KELLY, A new interpretation of information rate, *Bell System Tech. J.* **35** (1956), 917–926.
22. F. KNIGHT, "Risk, Uncertainty, and Profit," Houghton Mifflin, Boston, 1921.
23. J MAYNARD SMITH, "Evolution and the Theory of Games," Cambridge Univ. Press, Cambridge, 1982.
24. A. MCLENNAN, Incomplete learning in a repeated statistical decision problem, unpublished, University of Minnesota, 1987.
25. J. PATEL, R. ZECKHAUSER, AND D HENDRICKS, The rationality struggle: Illustrations from financial markets, *Amer. Econ. Rev.* **2** (1991), 232–236.
26. P. SAMUELSON, The "fallacy" of maximizing the geometric mean in long sequences of investing or gambling, *Proc Nat. Acad. Sci. U.S.A.* **68** (1971), 2493–2496.
27. J. A. SCHUMPETER, "The Theory of Economic Development," Harvard Univ. Press, Cambridge, MA, 1934.