Introduction to modern Game Theory

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Vega, 2021

Part 2

Foundations of noncooperative games. Basic mathematical methods and application to evolutionary biology.

Chapter 1. Nash equilibria for static games with a finite strategy space.

Chapter 2. Evolutionary stable strategies (ESS) and replicator dynamics (RD).

Chapter 3. Dynamic games and dynamic programming.

Chapter 4. Games with a continuous state space.

Chapter 1

Nash equilibria for static games with a finite strategy space

Evolutionary biology, fitness, survival of the fittest, sex ratio game, mixed strategies, the Equality of payoffs lemma, The Nash theorems and the minimax theorem of von Neumann, Kakutani's fixed point theorem.

Hawk and Dove game, Lizards Uta Stasburiana, sex ratio game, tax avoidance and inspection game.

Evolutionary biology

Main ideas. Fitness as the utility function.
Founders and Mainard Smith.
Survival of the fittest. Not only physical traits but behavioral patterns!

Idea of ESS.

Cooperation in nature: zigzags and crocodiles.
Lizards Uta Stasburiana: Rock-Paper-Scissors in Nature.

Sex ratio game: why half of babies males and half females.

Hawk and dove games as social dilemmas

Defect-cooperate dilemma – dilemma between an aggressive and a peaceful behavior, where an aggressor is supposed to win over a peacemaker in a one-shot game.
Names: Hawk and Dove, Lion and Lamb, Wolf and Hare. Any two-player two-strategy symmetric game:

	hawk	dove
hawk	p,p	q,r
dove	r,q	s,s

with $r \leq q$. As shifting does not change the content, we subtract r from all entries leading to the general *Hawk-Dove game*:

	hawk	dove
hawk	H,H	V,0
dove	0,V	D,D

with $V \geq 0$.

Hawk and dove games as social dilemmas

Usual interpretation of this game in the biological context: Fitness, competition over resources.

A dove surrenders the full resource to a hawk without a struggle. Two hawks always fight for the resource, the cost of fighting leading to a decrease in the average value of the resource.

"Hawk-Dove" game is often restricted to the subclass of general symmetric games of table above: $V > 0$ and $H < V$. Often additionally $H < 0$.

Using the reduced form of the table for symmetric games (i.e. omitting the symmetric payoffs for the second player), the game above becomes

	hawk	dove
hawk	H	V
dove	0	D

Hawk and Dove

We shall write further h for hawk and d for dove. Due to the symmetry one writes simply $\Pi(h,h)$, $\Pi(h,d)$, $\Pi(d,h)$, $\Pi(d,d)$ for the payoffs of the player (irrelevant the first or the second) playing respectively h against h, h against d,d against h,d against d. Thus in the table above, $\Pi(h,h) = H$, $\Pi(h,d) = V$, $\Pi(d,h) = 0$, $\Pi(d,d) = D$.

All social dilemmas of Chapter 1 are particular cases of this general Hawk-Dove game with the strategies "defect", "cooperate" being replaced by "hawk" and "dove".

Similar to aggression-peace dilemma, one can analyze other pairs of opposite behavior of animals, say be coy or fast for a female, or be faithful or a philanderer for a male.

Mixed strategies

Playing a *mixed strategy* $\sigma = (p, 1-p)$ (where p is any number between 0 and 1) in a two-action two-player game means to choose the first of the two strategies randomly with probability p. The original strategies clearly correspond to the values $p = 0$ or $p = 1$ and are called *pure strategies* and those with $p \neq 0, 1$ are *strictly mixed*.

The *payoff to a pure strategy s against a mixed strategy* $\sigma = (p, 1-p)$ in the symmetric Hawk-Dove game is defined as the average payoff

$$\Pi(s, \sigma) = p\Pi(s, h) + (1-p)\Pi(s, d).$$

Mixed strategies

Playing a mixed strategy $\eta = (q, 1-q)$ *against a mixed strategy* $\sigma = (p, 1-p)$ yields the payoff

$$\Pi(\eta, \sigma) = q\Pi(h, \sigma) + (1-q)\Pi(d, \sigma),$$

$$= qp\Pi(h,h) + q(1-p)\Pi(h,d) + p(1-q)\Pi(d,h) + (1-q)(1-p)\Pi(d,d).$$

Conclusion: to any two-player and two-action symmetric game there corresponds another game with each player having the infinite number of (mixed) strategies $\sigma = (p, 1-p)$. The strategies with p being different from 0 or 1 are called sometimes pure mixed (or strictly mixed).
Two interpretations: random switching (as above) or "population biology" oriented: mixed strategy $\sigma = (p, 1-p)$ means you are playing a game with a randomly chosen representative of a large population with a fraction p of hawks and the fraction $1-p$ of doves.

General noncooperative games (normal form)

A finite *non-cooperative game* with an arbitrary, finite number m of players $i = 1, 2, ..., m$ in *normal form* is defined by the sets of (pure) strategies $S_1 = \{s_1^1, ..., s_1^{n_1}\}$, $S_2 = \{s_2^1, ..., s_2^{n_2}\}$,..., $S_m = \{s_m^1, ..., s_m^{n_m}\}$, and payoffs $\Pi_i(s_1^{j_1}, s_2^{j_2}, ..., s_m^{j_m})$.

A profile $(s_1^*, \cdots, s_m^*)$ is called a *Nash equilibrium*, if none of the players can win by deviating from this situation: for all i and s_i,

$$\Pi_i(s_1^*, \cdots, s_m^*) \geq \Pi_i(s_1^* \cdots, s_{i-1}^*, s_i, s_{i+1}^*, \cdots, s_m^*).$$

Zero sum games and symmetric games are defined as for two players.

Mixed strategies for a finite number of players

A *mixed strategy* for player i is a probability distribution on the set of his/her pure strategies, i.e. a vector $\sigma = (p_1, ..., p_{n_i})$ of dimension n_i with non-negative coordinates that sum up to 1, i.e. $p_1 + ... + p_{n_i} = 1$. Original pure strategies are, of course, given by vectors with one coordinate being 1 and other being zero.

A collection of mixed strategies is called a *profile or a situation* (in mixed strategies). The payoff for the player i under the profile $\sigma_1 = (p_1^1, ..., p_{n_i}^1)$, $\sigma_2 = (p_1^2, ..., p_{n_i}^2)$,...,$\sigma_m = (p_1^m, ..., p_{n_i}^m)$ is the average payoff (assuming that all players choose their strategies independently):

$$\Pi_i(\sigma_1, \sigma_2, ..., \sigma_m) = \sum_{j_1=1}^{n_1} \sum_{j_2=1}^{n_2} ... \sum_{j_m=1}^{n_m} p_{j_1}^1 p_{j_2}^2 ... p_{j_m}^m \Pi_i(s_1^{j_1}, s_2^{j_2}, ..., s_m^{j_m}),$$

The fundamental theorem of Nash states that any finite game has at least one Nash equilibrium in mixed strategies.

Fixed point existence theorems

Brouwer, Shauder, Tikhonov fixed point theorems: a mapping from a closed convex subset of a Euclidean space, Banach space or locally convex topological space to itself with a compact image has a fixed point.

Kakutani theorem: if S is a nonempty compact convex set in a Euclidean space and $f : S \to 2^S$ such that f has a closed graph and $f(x)$ is nonempty and convex for every x, then there exists a fixed point y: $y \in f(y)$.

Proof of the Nash theorem

To each profile $(\sigma_1, \cdots, \sigma_m)$ let us assign the set of best replies profiles (the product of the sets of best replies), that is a collection of profiles $\tilde{\sigma}_1, \cdots, \tilde{\sigma}_m$ such that

$$\Pi_i(\sigma_1, \cdots, \sigma_{i-1}, \tilde{\sigma}_i, \sigma_{i+1}, \cdots, \sigma_m)$$

$$\geq \Pi_i(\sigma_1, \cdots, \sigma_{i-1}, \sigma_i', \sigma_{i+1}, \cdots, \sigma_m)$$

for all i and σ_i'.
It is straightforward to see that we get a mapping satisfying Kakutani's theorem condition, and hence it has a fixed point.

Dinosaurs: minimax theorem

Minimax theorem. For any finite zero-sum game of two players R and C the *minimax equation* holds:

$$\min_{\sigma_C} \max_{\sigma_R} \Pi_R(\sigma_R, \sigma_C) = \max_{\sigma_R} \min_{\sigma_C} \Pi_R(\sigma_R, \sigma_C).$$

The common value is called the *value* V of the game. Moreover, $\Pi_R(\sigma_R^\star, \sigma_C^\star) = V$ for any Nash equilibrium $(\sigma_R^\star, \sigma_C^\star)$ of this game, and for any two other strategies σ_R, σ_C the *saddle-point condition* holds:

$$\Pi_R(\sigma_R, \sigma_C^\star) \leq V = \Pi_R(\sigma_R^\star, \sigma_C^\star) \leq \Pi_R(\sigma_R^\star, \sigma_C).$$

Dinosaurs: minimax theorem

Additionally

$$\Pi_R(s_R^j, \sigma_C^\star) \leq V = \Pi_R(\sigma_R^\star, \sigma_C^\star) \leq \Pi_R(\sigma_R^\star, s_C^i)$$

holds for all pure strategies s_R^j and s_C^i of the first and second player. Finally

$$\min_{\sigma_C} \max_j \Pi_R(s_R^j, \sigma_C) = V = \Pi_R(\sigma_R^\star, \sigma_C^\star) = \max_{\sigma_R} \min_i \Pi_R(\sigma_R, s_C^i).$$

Proof. Step 1

$$\Pi_R(\sigma_R, \sigma_C) \leq \max_{\sigma_R} \Pi_R(\sigma_R, \sigma_C)$$

for any pair of strategies σ_R, σ_C, and consequently

$$\min_{\sigma_C} \Pi_R(\sigma_R, \sigma_C) \leq \min_{\sigma_C} \max_{\sigma_R} \Pi_R(\sigma_R, \sigma_C)$$

for any strategy σ_R. It implies that the inequality

$$\max_{\sigma_R} \min_{\sigma_C} \Pi_R(\sigma_R, \sigma_C) \leq \min_{\sigma_C} \max_{\sigma_R} \Pi_R(\sigma_R, \sigma_C)$$

holds trivially (without any assumptions on a game).

Proof. Step 2

Suppose now that $(\sigma_R^\star, \sigma_C^\star)$ is a Nash equilibrium. Then

$$\Pi_R(\sigma_R^\star, \sigma_C^\star) \geq \Pi_R(\sigma_R, \sigma_C^\star), \quad \Pi_C(\sigma_R^\star, \sigma_C^\star) \geq \Pi_C(\sigma_R^\star, \sigma_C)$$

for all σ_R, σ_C. By the zero-sum condition $\Pi_R(\sigma_R, \sigma_C) = -\Pi_C(\sigma_R, \sigma_C)$ for all σ_R, σ_C, and the second inequality can be written as

$$\Pi_R(\sigma_R^\star, \sigma_C^\star) \leq \Pi_R(\sigma_R^\star, \sigma_C),$$

which together with the first inequality implies the saddle-point condition.

Proof. Step 3

From the saddle-point condition:

$$\Pi_R(\sigma_R^\star, \sigma_C^\star) = \min_{\sigma_C} \Pi_R(\sigma_R^\star, \sigma_C) \leq \max_{\sigma_R} \min_{\sigma_C} \Pi_R(\sigma_R, \sigma_C),$$

$$\Pi_R(\sigma_R^\star, \sigma_C^\star) = \max_{\sigma_R} \Pi_R(\sigma_R, \sigma_C) \geq \min_{\sigma_C} \max_{\sigma_R} \Pi_R(\sigma_R, \sigma_C).$$

Hence

$$\min_{\sigma_C} \max_{\sigma_R} \Pi_R(\sigma_R, \sigma_C) = \Pi_R(\sigma_R^\star, \sigma_C^\star) = \max_{\sigma_R} \min_{\sigma_C} \Pi_R(\sigma_R, \sigma_C).$$

It follows, in particular, that any Nash equilibrium in a zero-sum game is a saddle-point, and that the payoffs at all Nash equilibria are all the same and coincide with the guaranteed payoff.

Nash equilibria for two-player zero-sum games

Exercise. Q two players two actions zero sum game is given by a matrix $A = (a_{ij})$, $i, j = 1, 2$.
(i) Show that if there do not exist pure strategy Nash equilibria (or saddle points), then either

$$1) \quad \max(a_{11}, a_{22}) \le \min(a_{12}, a_{21}), \quad a_{11} + a_{22} < a_{12} + a_{21},$$

or

$$2) \quad \max(a_{12}, a_{21}) \le \min(a_{11}, a_{22}), \quad a_{11} + a_{22} > a_{12} + a_{21}.$$

(ii) Show that in both cases the value of the game is $\det(A)/t(A)$, where $\det(A)$ is the determinant of A and $t(A) = a_{11} + a_{22} - a_{12} - a_{21}$, and

$$\sigma = \left(\frac{a_{22} - a_{21}}{t}, \frac{a_{11} - a_{12}}{t} \right), \quad \eta = \left(\frac{a_{22} - a_{12}}{t}, \frac{a_{11} - a_{21}}{t} \right)$$

is a pair of minimax strategies.

Two-player two-action symmetric games

Theorem. *Equality of payoffs lemma for symmetric two-player games.*
A profile $(\sigma^\star, \sigma^\star)$ with $\sigma^\star = (p^\star, 1 - p^\star)$ such that $p^\star \neq 0$ and $p^\star \neq 1$ is a symmetric mixed strategy Nash equilibrium in a symmetric Hawk-Dove game if and only if

$$\Pi(h, \sigma^\star) = \Pi(d, \sigma^\star). \tag{1}$$

Proof. The profile $(\sigma^\star, \sigma^\star)$ is a symmetric Nash equilibrium iff

$$\Pi(\sigma^\star, \sigma^\star) \geq \Pi(\sigma, \sigma^\star) \tag{2}$$

for all strategies $\sigma = (p, 1 - p)$. But (1) implies that

$$\Pi(\sigma, \sigma^\star) = \Pi(h, \sigma^\star) = \Pi(d, \sigma^\star) = \Pi(\sigma^\star, \sigma^\star) \tag{3}$$

and consequently (2).

Two-player two-action symmetric games

Vice versa, suppose (1) does not hold, for instance,

$$\Pi(h, \sigma^\star) > \Pi(d, \sigma^\star).$$

Then

$$\Pi(h, \sigma^\star) = (p^\star + 1 - p^\star)\Pi(h, \sigma^\star)$$
$$> p^\star\Pi(h, \sigma^\star) + (1 - p^\star)\Pi(d, \sigma^\star) = \Pi(\sigma^\star, \sigma^\star),$$

which contradicts (2).

Corollary. *If $\sigma^\star = (p^\star, 1 - p^\star)$ is a symmetric strictly mixed (i.e. with $p^\star \neq 0$ and $p^\star \neq 1$) Nash equilibrium, then equality* (3) *holds for all σ. In other words, the inequality* (2) *for all σ implies the corresponding equality* (3) *for all σ.*

Two-player two-action symmetric games

For the Hawk-Dove game (1) reads as

$$p^\star H + (1 - p^\star)V = (1 - p^\star)D,$$

or

$$p^\star = \frac{D - V}{H + D - V}, \tag{4}$$

if $H + D - V \neq 0$. Consequently, subject to the latter condition, there can be at most one symmetric strictly mixed Nash equilibrium. For this equilibrium to exist the number $p^\star$ given by (4) should be between 0 and 1.

Corollary. In the Hawk-Dove game with either $D \neq V$ or $H \neq 0$ there can be at most one symmetric strictly mixed Nash equilibrium. This equilibrium exists and is given by formula (4) only if either (i) $D > V$, $H > 0$, or (ii) $D < V$, $H < 0$.

Two cases of the theorem correspond to the stag hunt dilemma and to the game of chicken respectively.

Two-player two-action games

Theorem. *Equality of payoffs lemma for two-player games.* Let $\sigma_R^\star = (p^\star, 1 - p^\star)$ and $\sigma_C^\star = (q^\star, 1 - q^\star)$ be a profile in a two-player game with actions s_R^1, s_R^2 of Ruth and s_C^1, s_C^2 of Charlie.

(i) If $0 < p^\star < 1$, then $\sigma_R^\star$ is the best response to $\sigma_C^\star$, i.e.

$$\Pi_R(\sigma_R^\star, \sigma_C^\star) \geq \Pi_R(\sigma_R, \sigma_C^\star) \tag{5}$$

for all σ_R, if and only if

$$\Pi_R(s_R^1, \sigma_C^\star) = \Pi_R(s_R^2, \sigma_C^\star) \tag{6}$$

and consequently

$$\Pi_R(\sigma_R^\star, \sigma_C^\star) = \Pi_R(s_R^1, \sigma_C^\star) = \Pi_R(s_R^2, \sigma_C^\star) = \Pi_R(\sigma_R, \sigma_C^\star) \tag{7}$$

for all σ_R.

Two-player two-action games

Theorem (cont.)
(ii) If $0 < q^\star < 1$, then $\sigma_C^\star$ is the best response to $\sigma_R^\star$, i.e.

$$\Pi_R(\sigma_R^\star, \sigma_C^\star) \geq \Pi_R(\sigma_R^\star, \sigma_C) \tag{8}$$

for all σ_C, if and only if

$$\Pi_C(\sigma_R^\star, s_C^1) = \Pi_C(\sigma_R^\star, s_C^2) \tag{9}$$

and consequently

$$\Pi_C(\sigma_R^\star, \sigma_C^\star) = \Pi_C(\sigma_R^\star, s_C^1) = \Pi_C(\sigma_R^\star, s_C^2) = \Pi_C(\sigma_R^\star, \sigma_C) \tag{10}$$

for all σ_C.

Two-player two-action games

Theorem (cont.)
(iii) if $0 < p^\star < 1$ and $0 < q^\star < 1$, then $(\sigma_R^\star, \sigma_C^\star)$ is a Nash equilibrium if and only if both (6) and (9) hold. Thus for strictly mixed strategies inequalities (5) and (8) defining the Nash equilibrium are equivalent to the corresponding equality (7) and (10).

Exercise. Proof the theorem in analogy with the symmetric case above.

Two-player two-action games

Corollary 1. Suppose a two-player game is given by the table

$$
\begin{array}{c c}
 & C \\
R & \begin{array}{|c|c|c|} \hline & 1 & 2 \\ \hline 1 & \text{a,b} & \text{c,d} \\ \hline 2 & \text{e,f} & \text{g,h} \\ \hline \end{array}
\end{array}
\quad \text{or} \quad
\begin{array}{c c}
 & C \\
R & \begin{array}{|c|c|c|} \hline & 1 & 2 \\ \hline 1 & a_{11}, b_{11} & a_{12}, b_{12} \\ \hline 2 & a_{21}, b_{21} & a_{22}, b_{22} \\ \hline \end{array}
\end{array}
$$

with arbitrary numbers a, b, c, d, e, f, g, h, and suppose $(\sigma_R^\star, \sigma_C^\star)$ is a Nash equilibrium with $\sigma_R^\star = (p^\star, 1 - p^\star)$ and $\sigma_C^\star = (q^\star, 1 - q^\star)$. If $0 < p^\star < 1$, then

$$q^\star[(c - g) + (e - a)] = c - g, \tag{11}$$

and if $0 < q^\star < 1$, then

$$p^\star[(h - f) + (b - d)] = h - f. \tag{12}$$

Proof. Equation (11) is equivalent to (6), and equation (12) is equivalent to (9).

Two-player two-action games

Corollary 2. Suppose the "generic conditions" $a \neq e$, $c \neq g$, $b \neq d$, $h \neq f$ hold for a two-player game above.
(i) If at least one of the numbers

$$q^\star = \frac{c-g}{(c-g)+(e-a)}, \quad p^\star = \frac{h-f}{(h-f)+(b-d)} \tag{13}$$

does not belong to the open interval $(0,1)$ (in particular, if it is $\pm\infty$, which may happen if the corresponding denominator vanishes), then there are no mixed strategy Nash equilibria.
(ii) If both these numbers belong to the interval $(0,1)$, then there is a unique mixed strategy Nash equilibrium $((p^\star, 1-p^\star),(q^\star, 1-q^\star))$.
(iii) The number of pure strategy Nash equilibria under "generic conditions" can not exceed two.

Two-player two-action games

Proof.

Let $(\sigma_R = (p, 1-p), \sigma_C = (q, 1-q))$ be a Nash equilibrium with $p \in (0,1)$ (the case with $q \in (0,1)$ is considered similarly). Then $q = q^\star$ by Corollary 1. This would be impossible, if this number would not belong to $[0,1]$. As by "generic condition" it can be neither 0 nor 1, it follows that it belongs to the open interval $(0,1)$. Then again by Corollary 1, it follows that $p = p^\star$, which is again impossible if this number does not belong to $[0,1]$. Hence one concludes again that $p = p^\star \in (0,1)$. And in this case (σ_R, σ_C) is mixed strategy Nash by the equality of payoffs lemma, as required.

Statement (iii) follows from an observation that "generic conditions" exclude the possibility of two neighboring cells in the table to represent a Nash equilibrium.

Two-player two-action games

Example. For the Battle of the Sexes

C

R		1	2
	1	3,2	1,1
	2	0,0	2,3

one finds two pure Nash equilibria $(3, 2)$, $(2, 3)$ (by inspection) and the unique mixed strategy Nash equilibrium $(3/4, 1/4), (1/4, 3/4)$ (by Corollary 2).

Two-player two-action games

Example. The table

		C	
		1	2
R	1	5,0	3,1
	2	5,2	3,0

does not satisfy the generic conditions.

Exercise. Starting with the calculation of the best responses calculate all Nash equilibria (there are infinitely many of them).

Nash equilibria for a finite number of players

The extension of the Equality of Payoffs Lemma to arbitrary finite games is the following.

Theorem. *Equality of payoffs lemma for several players*. Suppose a profile (σ_i, σ) is a Nash equilibrium for a finite game, where $\sigma_i = (p^1, ..., p^m)$ is a mixed strategy of player i that has a collection $\{s^1, ..., s^m\}$ of pure strategies, and where σ denotes the collection of the strategies of all other players in this profile. Suppose $p^j \neq 0$, $p^k \neq 0$ for some $j \neq k$. Then

$$\Pi_i(s^j, \sigma) = \Pi_i(s^k, \sigma).$$

Nash equilibria for a finite number of players

Proof. This is the same as for two players. Namely, assuming $\Pi_i(s^j, \sigma) > \Pi_i(s^k, \sigma)$ one sees that the strategy $\hat{\sigma}_i$ obtained from σ_i by changing p_j to $p_j + p_k$ and p_k to zero, yields a better payoff to i than σ_i.

Practical calculations of Nash equilibria for concrete large games leads to complicated numerical problems of linear algebra and linear programming.

Bimatrix games

Any finite two-players game can be specified by two matrices $A = (a_{ij})$, $B = (b_{ij})$, where a_{ij} and b_{ij} define the payoffs of Ruth and Charlie (first and second player) if they use their strategies i and j respectively. Thus these games are often referred to as *bimatrix games*.

Symmetric games have the property that matrices A and B are square matrices such that $a_{ij} = b_{ji}$ for all i, j, in other words that $B = A^T$ (i.e. B is transpose of A).

Nash equilibria for two-player games

Theorem. *Mixed strategy symmetric Nash equilibria.* A symmetric profile (σ, σ) with $\sigma = (p_1, ..., p_m)$ and all p_j non-vanishing is a symmetric Nash equilibrium for a symmetric game given by $m \times m$-matrix A if and only if

$$A\sigma = \lambda \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \quad \lambda = \Pi(\sigma, \sigma). \tag{14}$$

In particular, if A is non-degenerate, then

$$\sigma = \lambda A^{-1} \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix},$$

and consequently there can be at most one such equilibrium and it exists if and only if all coordinates of the right hand side of the last equation are positive.

Nash equilibria for two-player games

Proof. The first equation of (14) with some constant λ is obvious, since the equality of payoffs lemma means that the coordinates of the vector $A\sigma$ should be equal. The second equation in (14) is obtained by a scalar multiplication of the first one by the vector σ (and using the fact that the sum of the coordinates of σ should be one).

Example

Exercise. Find all symmetric Nash equilibria in the version of the Scissors-Rock-Paper game with fines or credits for a tie, i.e. for the symmetric two-players game specified by the matrix

$$A = \begin{pmatrix} -a & 1 & -1 \\ -1 & -a & 1 \\ 1 & -1 & -a \end{pmatrix} \tag{15}$$

with some $a \neq 0$.
Show that $(1/3, 1/3, 1/3)$ is the unique equilibrium with all probabilities positive.
Show that equilibria with two non-vanishing probabilities exist iff $a < -1$, and there are three of them in this case.
Find out when pure symmetric equilibria exist.

Tax payer against the tax man

Player I – a tax payer, player II – the tax police.
Player I has 2 pure strategies: to hide part of the taxes (H) or to pay them in full (P).
Player II has also 2 strategies: to check player I (C) and to rest (R).
Player I gets the income r if he pays the tax in full. If he chooses the action (H), he gets the additional surplus l. But if he is caught by player II, he has to pay the fine f.

Tax payer against the tax man

In the profile (H,C) player II can discover the unlawful action of player I with the probability p, so that p can be called the efficiency of the police. Choosing (C), player II spends c on the checking procedure. Of course $l, r, f, c > 0$.
Hence we defined a bi-matrix game given by the table (where $\overline{p} = 1 - p$)

Player II (Police)

Player I		Check (C)	Rest (R)
	Hide (H)	$r + \overline{p}l - pf, -c + pf - \overline{p}l$	$r + l, -l$
	Pay (P)	$r, -c$	$r, 0$

Tax payer against the tax man

According to formulas for Nash equilibria, the candidates to the mixed equilibrium are the strategies $(\beta, \overline{\beta})$, $(\alpha, \overline{\alpha})$, where

$$\alpha = \frac{a_{22} - a_{12}}{a_{11} - a_{12} - a_{21} + a_{22}} = \frac{l}{p(l + f)} > 0$$

$$\beta = \frac{b_{22} - b_{21}}{b_{11} - b_{12} - b_{21} + b_{22}} = \frac{c}{p(l + f)} > 0.$$

In order to have these strategies well defined, it is necessary to have $\alpha < 1$ and $\beta < 1$ respectively.

Exercise. Check (by a direct inspection) the following result.

Tax payer against the tax man

Proposition. 1) If $c \geq p(f+l)$, the pair (H,R) is an equilibrium, and moreover the strategy (R) is dominant for the police (even strictly, if the previous inequality is strict).
2) If $c < p(f+l)$ and $fp \leq \overline{p}l$, the pair (H,C) is an equilibrium and the strategy (H) is dominant (strictly if the previous inequality is strict).
3) If $c < p(f+l)$, $fp > \overline{p}l$, then the unique Nash equilibrium is the profile of mixed strategies $(\beta, \overline{\beta}), (\alpha, \overline{\alpha})$.

Remarks. 1. Consequently, in cases 1) and 2) the actions of the police are not effective. 2. One can check (Exercise!) that the equilibrium in case 3) is stable.

Tax payer against the tax man

It is more interesting to analyze the game obtained by extending the strategy space of player I by allowing him to choose the amount I of tax evasion: $I \in [0, I_M]$, where I_M is the full tax due to player I. For example, we shall assume that the fine is proportional to I, i.e. $f(I) = nI$.

Remark. In the Russian tax legislation (at least 20 years ago) $n = 0.4$.

Under these assumptions:

$$\alpha = \frac{1}{p(n+1)}, \quad \beta = \frac{c}{I}\frac{1}{p(n+1)}.$$

Tax payer against the tax man

Let $H_I(l)$ denote the payoff to player I in the equilibrium when l is chosen. One can distinguish two cases:

1) $p > \frac{1}{n+1} \iff \alpha < 1.$
If

$$l > l_1 = \frac{c}{p(n+1)} \iff \beta < 1,$$

then $(\beta, \overline{\beta}), (\alpha, \overline{\alpha})$ is a stable equilibrium.
If $l < l_1 \iff \beta > 1$, then (H, R) is a stable equilibrium.

Tax payer against the tax man

$$H_I(l < l_1) = r + l < r + l_1,$$

$$H_I(l > l_1) = \beta\alpha(r + \overline{p}l - pf) + \beta\overline{\alpha}(r + l) + \overline{\beta}\alpha r + \overline{\beta}\overline{\alpha}r$$

$$= r + \beta l(\alpha\overline{p} + \overline{\alpha} - \alpha pn) = r.$$

Consequently $H_I(l > l_1) < H_I(l < l_1)$ and therefore player I will avoid tax on the amount $l = l_1$.

Tax payer against the tax man

2) $p < \frac{1}{n+1} \Longleftrightarrow \alpha > 1$.
If $I > I_1 \Longleftrightarrow \beta < 1$, then (H, C) is an equilibrium.
If $I < I_1 \Leftrightarrow \beta > 1$, then (H, R) is an equilibrium. But

$$H_I(I < I_1) = r + I < r + I_1, \quad H_I(I > I_1) = r + (1-p)I - pIn.$$

Hence $H_I(I > I_1) > H_I(I < I_1)$ (the choice $I > I_1$ is reasonable for player I) whenever

$$I(1 - p - pn) \geq I_1 \Longleftrightarrow I \geq \frac{I_1}{1 - p(n+1)}.$$

Tax payer against the tax man

Consequently, if

$$\frac{I_1}{1 - p(n+1)} \leq I_M, \tag{16}$$

the equilibrium strategy for player I is $I = I_M$ and otherwise $I = I_1$.

One can conclude that in both cases it is profitable to avoid tax on the amount I_1, but as the efficiency of tax man increases, it becomes unreasonable to avoid tax on a higher amount.

Tax payer against the tax man

Let us see which condition in the second case would ensure the inequality (16).

Plugging I_1 in (16) yields

$$\frac{c}{p(n+1)(1-p(n+1))} \leq I_M.$$

or

$$x^2 - x + \frac{c}{I_M} \leq 0. \tag{17}$$

with $x = p(n+1) < 1$.

Tax payer against the tax man

The roots of the corresponding equation are

$$x_{1,2} = \frac{1 \pm \sqrt{1 - \frac{4c}{I_M}}}{2}.$$

Hence for $c > I_M/4$ inequality (16) does not hold for any p, and for $c \leq I_M/4$ the solution to (17) is

$$x \in \left[\frac{1 - \sqrt{1 - \frac{4c}{I_M}}}{2}; \frac{1 + \sqrt{1 - \frac{4c}{I_M}}}{2}\right].$$

Thus for

$$c \leq \frac{I_M}{4}, \quad p \in \left[\frac{1 - \sqrt{1 - \frac{4c}{I_M}}}{2(n+1)}; \frac{1 + \sqrt{1 - \frac{4c}{I_M}}}{2(n+1)}\right] \tag{18}$$

it is profitable to avoid tax payment on the amount I_M.

Chapter 2

Evolutionary stable strategies (ESS) and replicator dynamics (RD).

Post-entry population and invasion of mutants, main criterion for ESS, fundamental theorem of natural selection, ESS and the asymptotic stability of the fixed points of RD, RD in genetics, Nash fields.

Evolutionary stable strategies (ESS)

Suppose a symmetric two-players game with a finite set of strategies (the same for each player) $S = \{s_1, s_2, ...\}$ is given. A *population profile* is a probability distribution $\nu = (p_1, p_2, ...)$ on the set S, where each p_j denotes the probability with which the strategy s_i is played. Such a profile can be realized by a variety of ways, for example, by (i) *monomorphic population*: all members of the population use the same mixed strategy $\sigma = \nu$, or by (ii) *polymorphic population*: each individual of the population plays only some pure strategy and p_j denotes the probability to meet an individual playing the pure strategy s_j.

As usual for symmetric games, $\Pi(s_i, s_j)$ will mean the payoff to the player playing the strategy s_i against the strategies s_j.

Evolutionary stable strategies (ESS)

Suppose all individuals are playing a strategy $\sigma^\star = (p_1^\star, p_2^\star, \dots)$, and hence the population is monomorphic with the profile $\nu = \sigma^\star$. Suppose a mutation has occurred that caused a small proportion ϵ of individuals to change their strategy to some mutant strategy σ thus changing the profile ν to the new (so called *post-entry*) profile

$$\nu_\epsilon = \epsilon\sigma + (1-\epsilon)\sigma^\star.$$

Evolutionary stable strategies (ESS)

A mixed strategy $\sigma^\star$ is called *evolutionary stable* (ESS) if there exists $\tilde{\epsilon} > 0$ such that

$$\Pi(\sigma^\star, \nu_\epsilon) > \Pi(\sigma, \nu_\epsilon) \tag{19}$$

for all $\sigma \neq \sigma^\star$ and all ϵ such that $0 < \epsilon < \tilde{\epsilon}$. In other words, if the proportion of mutants is sufficiently small, ESS does strictly better (e.g. produce more offspring) against the post-entry population than the mutant strategy. This implies that any mutants (if appeared in a small proportion) should eventually die out.

Main criterion for ESS

Theorem. The strategy $\sigma^\star$ is ESS if and only if for all strategies $\sigma \neq \sigma^\star$ either
(i) $\Pi(\sigma^\star, \sigma^\star) > \Pi(\sigma, \sigma^\star)$ or
(ii) $\Pi(\sigma^\star, \sigma^\star) = \Pi(\sigma, \sigma^\star)$ and $\Pi(\sigma^\star, \sigma) > \Pi(\sigma, \sigma)$.
Proof. Inequality (19) reads as

$$\Pi(\sigma^\star, \epsilon\sigma + (1-\epsilon)\sigma^\star) > \Pi(\sigma, \epsilon\sigma + (1-\epsilon)\sigma^\star),$$

or (by linearity of the function Π)

$$\epsilon\Pi(\sigma^\star, \sigma) + (1-\epsilon)\Pi(\sigma^\star, \sigma^\star) > \epsilon\Pi(\sigma, \sigma) + (1-\epsilon)\Pi(\sigma, \sigma^\star)$$

or else

$$(\Pi(\sigma^\star, \sigma^\star) - \Pi(\sigma, \sigma^\star))$$
$$+\epsilon[(\Pi(\sigma^\star, \sigma) - \Pi(\sigma, \sigma)) + (\Pi(\sigma^\star, \sigma^\star) - \Pi(\sigma, \sigma^\star)] > 0.$$

A linear function of ϵ is positive in a neighborhood of zero if either it is positive at $\epsilon = 0$ (condition (i) of the Theorem) or it vanishes at $\epsilon = 0$ but has a positive slope (derivative) at $\epsilon = 0$ (condition (ii) of the Theorem).

Hawk and Dove game ESS

Let us use the Theorem for the analysis of the general Hawk-Dove game

	hawk	dove
hawk	H	V
dove	0	D

with $V \geq 0$. Under generic conditions a strictly mixed Nash equilibrium for this game exists if either
(i) $D > V, H > 0$, or (ii) $D < V, H < 0$,
and in both cases this equilibrium $\sigma^\star = (p^\star, 1 - p^\star)$ is unique and is given by formula

$$p^\star = \frac{D - V}{H + D - V}. \tag{20}$$

Hawk and Dove game ESS

Corollary. (i) In the case $D > V, H > 0$ the mixed equilibrium $\sigma^\star$ given by (20) is not ESS; (ii) in the case $D < V, H < 0$ the mixed equilibrium given by (20) is ESS; (iii) if $H > 0$, the pure strategy profile (hawk,hawk) is a Nash equilibrium and the strategy hawk (h) is ESS; (iv) if $D > V$, the pure strategy profile (dove,dove) is a Nash equilibrium and the strategy dove (d) is ESS.

Proof

(i), (ii) From the equality of payoffs lemma $\Pi(\sigma^\star, \sigma^\star) = \Pi(\sigma, \sigma^\star)$ for all σ and hence the ESS condition becomes

$$\Pi(\sigma^\star, \sigma) - \Pi(\sigma, \sigma) > 0$$

for all $\sigma \neq \sigma^\star$. But

$$\Pi(\sigma^\star, \sigma) - \Pi(\sigma, \sigma) = Hp^\star p + Vp^\star(1-p) + D(1-p^\star)(1-p)$$

$$-Hp^2 - Vp(1-p) - D(1-p)^2 = (D - V + p^\star(D + H - V))p$$

$$+(V - H - D)p^2 + (V - D)p^\star = (V - H - D)(p - p^\star)^2.$$

This is positive for $p \neq p^\star$ if $V > D + H$, which holds in case $V > D$, $H < 0$, and hence $\sigma^\star$ is ESS in case (ii). In case (i), the latter expression becomes negative and hence $\sigma^\star$ is not ESS in this case.

Proof

(iii) Finally, $\Pi(h, h) - \Pi(\sigma, h) = (1 - p)H$, which is positive for $H > 0$ and $p \neq 1$, hence strategy h is ESS in this case.

Statement (iv) is proved similarly.

Replicator dynamics

Replicator dynamics (RD) is designed to model a process by which behavioral patterns of populations (created by some mutation, say) are occasionally transferred to an equilibrium (ESS).

Let N denote the size of a population consisting of individuals playing one of m possible strategies $s_1, ..., s_m$ in a two-player symmetric game specified by a payoff function $\Pi(s_i, s_j)$.

As usual in biological context, the payoff expresses the production rate.

Replicator dynamics

Denoting the number of individuals playing s_i, $i = 1, ..., m$, by n_i, and $x_i = n_i/N$. The corresponding population profile is $\nu = (x_1, ..., x_m)$. If the background production rate is c, then

$$\dot{n}_i = (c + \Pi(s_i, \nu))n_i, \quad i = 1, ..., m,$$

where

$$\Pi(s_i, \nu) = \Pi(s_i, s_1)x_1 + ... + \Pi(s_i, s_m)x_m.$$

This system is the *RD in terms of absolute sizes*. It follows:

$$\dot{N} = (c + \Pi(\nu, \nu))N,$$

where

$$\Pi(\nu, \nu) = x_1\Pi(s_1, \nu) + ... + x_m\Pi(s_m, \nu) = \sum_{i,j=1}^{m} x_i x_j \Pi(s_i, s_j)$$

is the average fitness of the population (it represents, of course, the average payoff for playing ν against itself).

Replicator dynamics

Usually one rewrites it in terms of frequencies $x_i = n_i/N$.
From the above dynamics one derives:

$$\dot{x}_i = (\Pi(s_i, \nu) - \Pi(\nu, \nu))x_i, \quad i = 1, ..., m, \tag{21}$$

which is the *RD system of equations in the standard form*.

Exercise. Check is by direct differentiation.

Observe that equations RD do not depend on the background rate c.

Fundamental theorem of natural selection

Theorem. If the matrix $\Pi(s_i, s_j)$ is symmetric, the average fitness $\Pi(\nu, \nu)$ of the population does not decrease along any trajectory of RD system (21) and satisfies the equation

$$\frac{d}{dt}\Pi(\nu, \nu) = 2\sum_{i=1}^{m} x_i(\Pi(s_i, \nu) - \Pi(\nu, \nu))^2.$$

Fundamental theorem of natural selection

Proof. Using RD, symmetry of Π and $\sum_i x_i(\Pi(s_i,\nu) - \Pi(\nu,\nu)) = 0$ it follows that

$$\frac{d}{dt}\Pi(\nu,\nu) = 2\sum_{i,j=1}^{m} x_j(\Pi(s_i,s_j)\dot{x}_i$$

$$= 2\sum_{i=1}^{m} \Pi(s_i,\nu)x_i(\Pi(s_i,\nu) - \Pi(\nu,\nu))$$

$$= 2\sum_{i=1}^{m} \Pi(s_i,\nu)x_i(\Pi(s_i,\nu) - \Pi(\nu,\nu))$$

$$-2\sum_{i=1}^{m} \Pi(\nu,\nu)x_i(\Pi(s_i,\nu) - \Pi(\nu,\nu))$$

$$= 2\sum_{i=1}^{m} x_i(\Pi(s_i,\nu) - \Pi(\nu,\nu))^2.$$

Nash and RD

Theorem. If σ is a fixed point of RD equations for a symmetric two-person game G given by a matrix A, then σ defines a symmetric Nash equilibrium either for G or for a symmetric game G' obtained from G by excluding some of its strategies (i.e. given by a matrix that is a major minor of A).

And vice versa, any symmetric Nash equilibrium of either G or a game G' obtained from G by excluding some of its strategies, is a fixed point of RD equations for G.

Proof. Directly from the equality of payoffs lemma.

Stability

A rest point (or a fixed point) y_0 of an ordinary (vector-valued) differential equation $\dot{y} = f(y)$ is called *asymptotically stable (in the Lyapunov sense)*, if there exists an $\epsilon > 0$ such that its solution with any initial point from the ϵ-neighborhood of y_0 tends to y_0 as $t \to \infty$, *neutrally stable (in the Lyapunov sense)*, if for $\epsilon > 0$ there exists a $\delta > 0$ such that any solution starting at a point from the δ-neighborhood of y_0 stays forever in the ϵ-neighborhood of y_0, and *unstable*, if it is not neutrally stable.

The main criterion of stability (*via linear approximation*): let $0 \in \mathbf{R}^d$ be a rest point of the system $\dot{y} = f(y)$ and let $F = \frac{\partial f}{\partial y}(0)$ its Jacobian matrix. Then 0 is asymptotically stable if all eigenvalues of F have negative real parts and unstable if there exists at least one eigenvalue with a positive real part. This theorem is empty for eigenvalues with vanishing real parts.

ESS and stability

Theorem. If σ^* is ESS in a symmetric two-player game, then σ^* is a stable rest point of RD.

Proof. Step 1. ESS means that

$$\epsilon \Pi(\sigma^*, (1-\epsilon)\sigma^* + \epsilon\sigma) > \epsilon \Pi(\sigma, (1-\epsilon)\sigma^* + \epsilon\sigma).$$

Adding $(1-\epsilon)\Pi(\sigma^*, (1-\epsilon)\sigma^* + \epsilon\sigma)$ to both sides yields

$$\Pi(\sigma^*, (1-\epsilon)\sigma^* + \epsilon\sigma) > \Pi((1-\epsilon)\sigma^* + \epsilon\sigma, (1-\epsilon)\sigma^* + \epsilon\sigma).$$

Thus ESS condition implies that

$$\Pi(x^*, x) > \Pi(x, x)$$

for all $x \neq x^*$ in a neighborhood of $x^* = \sigma^*$.

ESS and stability

Proof. Step 2.
Define the *relative entropy function* of two distributions

$$V(x) = V(x, x^*) = -\sum_i x_i^* \ln(x_i/x_i^*).$$

For any two distributions $V(x) \geq 0$. This is equivalent to the famous *Gibbs inequality*:

$$-\sum_i x_i^* \ln x_i^* \leq -\sum_i x_i^* \ln x_i,$$

and it follows from the inequality $\ln x \leq x - 1$ for all $x \geq 0$.
Exercise. Prove all these three inequalities.

ESS and stability

Proof. Step 3.

$$\frac{d}{dt}V(x) = \sum_i \frac{\partial V}{\partial x_i}\dot{x}_i$$

$$= -\frac{x_i^*}{x_i}x_i(\Pi(s_i, x) - \Pi(x, x)) = -(\Pi(x^*, x) - \Pi(x, x)) < 0.$$

Thus V is a strict Lyapunov function in a neighborhood of x^* and thus x^* is asymptotically stable.

ESS and stability

Exercise. For the symmetric two-player game with the table

	A	B	C
A	0	1	1
B	-2	0	3
C	1	1	0

show that $x^* = (1/3, 1/3, 1/3)$ is asymptotically stable in RD, but is not an ESS. Hint: take $\sigma = (1, 1/2, 1/2)$.

RD in genetics and sexual reproduction

Suppose a population is characterized by m types of genes (called *alleles*) $g_1, \ldots, g_m$, and an individual is characterized by a pair (g_i, g_j) of such genes that are placed in a *genetic locus*. An individual with a pair (g_i, g_j) has fitness f_{ij}.

Toy model of sexual reproduction: all genes of all sexually matured individuals are just put in a common pool (all parents are supposed to die), are arbitrary paired, and any pair (g_i, g_j) forms a new individual that will survive to the next stage of sexual reproduction with probability f_{ij}.

RD in genetics and sexual reproduction

Let $n_i = n_i(t)$ (respectively $x_i = x_i(t)$) denote the number (respectively the frequency) of the allele g_i in the population. The fitness of allele i is then $f_i = \sum_{j=1}^m f_{ij} x_j$ and the average fitness of the population is

$$f = \sum_{i=1}^{m} x_i f_i = \sum_{i,j=1}^{m} f_{ij} x_i x_j.$$

The rate of production is proportional to average fitness:

$$\dot{n}_i = n_i \sum_{j=1}^{n} f_{ij} x_j, \quad i = 1, ..., m.$$

But this is RD in terms of absolute sizes with $c = 0$ and $\Pi(s_i, s_j) = f_{ij}$. Hence the frequencies x_i enjoy the corresponding RD. The fundamental theorem of natural selection was first discovered by R.A. Fisher precisely in this context.

RD: two-action two-player games

As $x_1 + x_2 = 1$ for a profile $\nu = (x_1, x_2)$ of a two-actions game, system (21) reduces to just one equation for $x = x_1$ of the form

$$\dot{x} = (\Pi(s_1, \nu) - (x\Pi(s_1, \nu) + (1 - x)\Pi(s_2, \nu))x$$

or better

$$\dot{x} = x(1 - x)(\Pi(s_1, \nu) - \Pi(s_2, \nu)).$$

This RD equation has always two boundary fixed points $x = 0$ and $x = 1$. Mixed strategy fixed points are found from equation $\Pi(s_1, \nu) = \Pi(s_2, \nu)$.

Example: Stability: Hawk and Dove

Let us analyze the Hawk-Dove game in case $H < 0$, $D < V$. This game has only one symmetric Nash equilibrium $\sigma^\star = (p^\star, 1 - p^\star)$ with $p^\star$, and this equilibrium is ESS. Consequently $\sigma^\star$ is a stable fixed point of RD that now takes the form

$$\dot{x} = x(1 - x)[x(H + D - V) - (D - V)].$$

In terms of $y = x - p^\star$:

$$\dot{y} = \left(\frac{D - V}{H + D - V} + y\right)\left(\frac{H}{H + D - V} - y\right) y(H + D - V)$$

$$= y\frac{H(D - V) + O(y)}{H + D - V}.$$

The coefficient at y is negative $\Longrightarrow$ $p^\star$ is stable.

Example: RD for Rock-Scissors-Paper with a tie

$$\begin{cases} \dot{x} = [-ax + y - z + a(x^2 + y^2 + z^2)]x \\ \dot{y} = [-x - ay + z + a(x^2 + y^2 + z^2)]y \\ \dot{z} = [x - y - az + a(x^2 + y^2 + z^2)]z \end{cases}$$

since $\Pi(\nu, \nu) = -a(x^2 + y^2 + z^2)$ for $\nu = (x, y, z)$. Writing this equation around the equilibrium $\sigma^\star = (1/3, 1/3, 1/3)$ in terms of $u = x - 1/3$, $v = y - 1/3$ yields

$$\begin{cases} \dot{u} = \frac{1}{3}[(1 - a)u + 2v] + \dots \\ \dot{v} = \frac{1}{3}[-2u - (1 + a)v] + \dots \end{cases},$$

whereby the dots designate the higher terms (quadratic, etc) in u, v.

Example: RD for Rock-Scissors-Paper with a tie

Thus the linearized equation in a neighborhood of the fixed point is given by the matrix

$$\frac{1}{3}\begin{pmatrix} 1-a & 2 \\ -2 & -(1+a) \end{pmatrix}$$

with eigenvalues $-a \pm i\sqrt{3}$. Consequently $\sigma^\star$ is a stable (hyperbolic) point for $a > 0$, and an unstable hyperbolic point for $a < 0$.

For $a = 0$ (classical Scissors-Rock-Paper) the eigenvalues are purely imaginary, and hence a fixed point of a linearized system is a center with closed orbits around it.

Example: RD for Rock-Scissors-Paper

As the rest point for $a = 0$ is not hyperbolic, one can not deduce from this directly the same property of the full non-linear system.
However, one can check that the function $V(x, y, z) = \ln x + \ln y + \ln z$ is the integral of motion for RD.

Exercise. Check this fact.

Hence $\sigma^\star$ is actually a center for this system, and it has closed orbits around it.
Thus in case $a = 0$, the replicator dynamics generates oscillation, and they are in fact observed by field researchers of Uta Stansburiana lizards, whose interaction is described by suchlike game.

Extension of Rock-Scissors-Paper to n-action

Proposition. Let Γ be a symmetric zero-sum game given by an anti-symmetric $n \times n$-matrix A such that it has a zero vector v, i.e. such that $Av = 0$, with all co-ordinates of v being (strictly) positive. Then the function

$$L(\eta) = x_1^{p_1} x_2^{p_2} \ldots x_n^{p_n}$$

is an integral of the replicator dynamics on the strategies $\eta = (x_1, \ldots, x_n)$, (i.e. it does not change along its trajectories) and the strategy $\sigma = v/(v_1 + \ldots + v_n)$ specifies a neutrally stable symmetric equilibrium.

Exercise. Prove this statement.

Nash fields

A generalized RD for a general non-symmetric game Γ:

$$\dot{x}_i^j = (\Pi_j(s_j^i, \sigma) - \Pi_j(\sigma_j, \sigma))x_i^j, \quad i = 1, ..., n_j, \; j = 1, ..., m, \tag{22}$$

where

$$\sigma_1 = (x_1^1, ..., x_{n_1}^1), \quad \sigma_2 = (x_1^2, ..., x_{n_2}^2), \quad ..., \quad \sigma_m = (x_1^m, ..., x_{n_m}^m)$$

is a profile of (mixed) strategies and σ denotes the collection of the strategies of all players others than j.
This system describes an evolution of the behavior of the players applying a try-and-error method of shifting the strategies in the direction of a better payoff.
It follows from the Equality of payoffs Lemma that a Nash equilibrium is a rest point of this system. Its r.h.s is sometimes called the *Nash vector field* of the game.

Nash fields for bi-matrix games

A Nash equilibrium for a game Γ is called *asymptotically stable, neutrally stable or unstable in the Lyapunov sense (or dynamically)* if it is so for the corresponding dynamics.

Let us write down the RD more explicitly for 2×2 bi-matrix games, i.e. for two-person two actions games with payoffs given by a pair of matrices (a_{ij}), (b_{ij}), $i, j = 1, 2$. The payoffs are thus given by formulae

$$\Pi_1(X, Y) = x_1 a_{11} y_1 + x_1 a_{12} y_2 + x_2 a_{21} y_1 + x_2 a_{22} y_2,$$

$$\Pi_2(X, Y) = x_1 b_{11} y_1 + x_1 b_{12} y_2 + x_2 b_{21} y_1 + x_2 b_{22} y_2,$$

where $X = (x_1, x_2)$, $Y = (y_1, y_2)$ denote the strategies of the two players.

Nash fields for bi-matrix games

The RD becomes

$$\begin{cases} \dot{x}_1 = x_1(a_{11}y_1 + a_{12}y_2 - \Pi_1(X, Y)) \\ \dot{x}_2 = x_2(a_{21}y_1 + a_{22}y_2 - \Pi_1(X, Y)) \\ \dot{y}_1 = y_1(b_{11}x_1 + b_{21}x_2 - \Pi_2(X, Y)) \\ \dot{y}_2 = y_2(a_{12}x_1 + b_{22}x_2 - \Pi_2(X, Y)) \end{cases}$$

Nash fields for bi-matrix games

As $x_1 + x_2 = y_1 + y_2 = 1$, this can be rewritten in terms of just two variables x_1, y_1 as

$$\begin{cases} \dot{x}_1 = x_1(1 - x_1)(Ay_1 - a) \\ \dot{y}_1 = y_1(1 - y_1)(Bx_1 - b) \end{cases}, \tag{23}$$

where

$$A = a_{11} + a_{22} - a_{12} - a_{21}, \quad a = a_{22} - a_{12},$$

$$B = b_{11} + b_{22} - b_{12} - b_{21}, \quad b = b_{22} - b_{21}.$$

Nash fields for bi-matrix games

Under the "general position", i.e. if $A, B, a, b, A - a, B - b$ do not vanish system (23) has always four pure strategy rest points $(0,0), (0,1), (1,0), (1,1)$, and it has the fifth rest point

$$x^\star = \frac{b}{B}, y^\star = \frac{a}{A} \tag{24}$$

whenever

$$0 < a/A < 1, \quad 0 < b/B < 1. \tag{25}$$

Exercise. Show that $(0,0)$ is asymptotically stable for $a, b > 0$, $(0,1)$ is asymptotically stable for $a > 0, b < 0$, $(1,0)$ is asymptotically stable for $a < 0, b > 0$ and $(1,1)$ is asymptotically stable for $a, b < 0$.

Nash fields

Exercise. Show that if $a = b = 0$, then $(0, 0)$ is stable if and only if $A \leq 0$, $B \leq 0$.

For the analysis of the fifth rest point, we rewrite (23) in terms of $\xi = x_1 - x^\star, \eta = y_1 - y^\star$:

$$\left\{ \begin{array}{l} \dot{\xi} = \frac{Ab}{B}(1 - \frac{b}{B})\eta + A(1 - \frac{2b}{B})\xi\eta - A\xi^2\eta, \\ \dot{\eta} = \frac{aB}{A}(1 - \frac{a}{A})\xi + B(1 - \frac{2a}{A})\xi\eta - B\xi\eta^2. \end{array} \right. \tag{26}$$

Nash fields

Exercise. If $AB > 0$ (or $ab > 0$), the equilibrium $(x^\star, y^\star)$ is unstable in the Lyapunov sense.
If $AB < 0$, the eigenvalues of the Jacobian of the r.h.s. of (26) have vanishing real parts so that the stability by linear approximation theorem can not be applied and more subtle methods are required.

Exercise. If $AB < 0$ and (25) hold, the point (24) is neutrally stable, but not asymptotically stable. Hint: proof is based on the observation that the function

$$(B-b)\ln(1-x_1) + b\ln x_1 - (A-a)\ln(1-y_1) - a\ln y_1$$

is a first integral of system (23).

Chapter 3

Dynamic games and dynamic programming.

Dynamic programming, backwards induction, inspection games, Bellman optimality principle and Bellman operator, folk theorem for infinitely repeated games.

Dynamic games, Dynamic programming

Dynamic programming is a mathematical (algorithmic) realization of the idea (and procedure) of backward induction.

Assume X and Y are arbitrary sets, h is a real function on $X \times Y$ and $\mathcal{H} : X \times Y \mapsto \Gamma$ is a mapping with values in the set Γ of zero-sum games of two players.

For any x, y, $\mathcal{H}(x, y)$ is a game $\Gamma_{A,B,H}(x, y)$ with $A = A(x, y)$, $B = B(x, y)$ the sets of strategies of the first and second player, and $H = H(x, y)$ the payoff function to the second player, so that $H(x, y; a, b)$ denotes his payoff in the profile (a, b), $a \in A, b \in B$ (the first player gets $-H(x, y; a, b)$).

Dynamic games, Dynamic programming

Consider now the following two step game $G_{h,\mathcal{H}}$. First the players I and II choose (independently) their strategies $x \in X, y \in Y$. Then the second player obtains $h(x, y)$ and afterwards the game $\mathcal{H}(x, y)$ is played. The strategies of the first (resp. second) player are clearly described by the pairs (x, α) (resp. (y, β)), where $x \in X, y \in Y$ and α, β are functions from $X \times Y$ to A and B respectively.

Dynamic games, Dynamic programming

Theorem. *Dynamic programming for zero-sum games.*
Suppose for any x, y the game $\mathcal{H}(x, y)$ has a value $H(x, y, \alpha_0(x, y), \beta_0(x, y))$ given by certain minimax strategies $\alpha_0(x, y), \beta_0(x, y)$, i.e. $H(x, y, \alpha_0(x, y), \beta_0(x, y))$ is a saddle-point of $H(x, y, a, b)$, i.e.

$$H(x, y, \alpha_0(x, y), b) \leq H(x, y, \alpha_0(x, y), \beta_0(x, y))$$

$$\leq H(x, y, a, \beta_0(x, y))$$

for $a \in A(x, y), b \in B(x, y)$.

Dynamic games, Dynamic programming

Theorem (cont.)
Suppose also that (x_0, y_0) is a saddle-point of the function $h(x, y) + H(x, y, \alpha_0(x, y), \beta_0(x, y))$:

$$h(x_0, y) + H(x_0, y, \alpha_0(x_0, y), \beta_0(x_0, y))$$

$$\leq h(x_0, y_0) + H(x_0, y_0, \alpha_0(x_0, y_0), \beta_0(x_0, y_0))$$
$$\leq h(x, y_0) + H(x, y_0, \alpha_0(x, y_0), \beta_0(x, y_0)), \quad x \in X, y \in Y.$$

Then the game $G_{h,\mathcal{H}}$ has a value that equals

$$h(x_0, y_0) + H(x_0, y_0, \alpha_0(x_0, y_0), \beta_0(x_0, y_0))$$

with minimax strategies being (x_0, α_0) and (y_0, β_0) respectively.

Dynamic games, Dynamic programming

Proof. Straightforward. One has to show that

$$h(x_0, y) + H(x_0, y, \alpha_0(x_0, y), \beta(x_0, y))$$

$$\leq h(x_0, y_0) + H(x_0, y_0, \alpha_0(x_0, y_0), \beta_0(x_0, y_0))$$
$$\leq h(x, y_0) + H(x, y_0, \alpha(x, y_0), \beta_0(x, y_0))$$

for all strategies (x, α) and (y, β), and this follows directly from the inequalities in the condition of the theorem.

The method of dynamic programming consists in using this result for solving k steps games by solving recursively the auxiliary one step games. In particular, it implies the existence of (minimax) solutions in multi-step finite games with alternating moves like chess.

Example. Inspection game

Player I (offender) wants to carry out an unlawful action (a crime). There are N time periods, in which such an action could be carried out.

Player II (inspector), who is interested in preventing the crime, is able to make only one inspection (in any of these periods).

The gain equals 1, if the crime is committed and undiscovered, and equals -1, if the offender is found and prosecuted (this happens if he chooses to commit a crime at the same time as the inspector chooses to carry out the inspection. The gain equals zero, if the offender does not act at all.

Example. Inspection game

In the first period (first step of the game) each player has two alternatives. Player I can act (commit a crime) or not. Player II can carry out an inspection or not doing so.

If player I acts and player II carries out the inspection, the game is over and the gain (of the first player) equals -1. If player I acts and player II does not inspect, the game is over as well and the gain equals 1.

If player I does not act and player II carries out the inspection, then player I can without fear commit the crime in the next period (whenever $N > 1$ of course) and the gain again equals 1 (it is assumed that after each step player I becomes aware of any inspection that has been carried out).

Example. Inspection game

If player I does not act and player II does not inspect, then the game moves to the second step that differs from the previous one only by the remaining number of periods. Consequently the matrix of the (zero-sum game of the) first step can be symbolically expressed as follows:

$$\begin{pmatrix} -1 & 1 \\ 1 & \Gamma_{N-1} \end{pmatrix}$$

Here Γ_{N-1} denotes the necessity to play this game again.

Example. Inspection game

If the values of the games Γ_i are equal respectively v_i, the perspective to play these games is equivalent (in the sense of the expectations) to their values (previous Theorem). Hence the matrix of the game can be written as

$$\begin{pmatrix} -1 & 1 \\ 1 & v_{N-1} \end{pmatrix}$$

This yields the following recursive equation (by $Val A$ we denote the value of a the game with the matrix A):

$$v_N = Val \begin{pmatrix} -1 & 1 \\ 1 & v_{N-1} \end{pmatrix}.$$

Example. Inspection game

Assuming $v_{N-1} \leq 1$ one deduces (see Exercise after the Minimax theorem above)) the following difference equation:

$$v_N = \frac{v_{N-1} + 1}{3 - v_{N-1}}.$$

Exercise. Check this. One may use Exercise after the Minimax theorem above.

Solving this equation combined with the initial condition $v_1 = 0$ yields

$$v_N = \frac{N-1}{N+1}.$$

Example. Inspection game

Thus one gets the value of the game on each step. Then one can calculate also the corresponding equilibrium strategies. Namely, as the matrix takes the form

$$\begin{pmatrix} -1 & 1 \\ 1 & \text{(N-2)/N} \end{pmatrix},$$

the equilibrium strategies for $N \geq 2$ are

$$\left\{ \begin{array}{l} x^N = (\frac{1}{N+1}, \frac{N}{N+1}) \\ y^N = (\frac{1}{N+1}, \frac{N}{N+1}) \end{array} \right\}.$$

Formulation in terms of the Bellman operator

Let X, S_R, S_C be three metric spaces such that S_R and S_C are compact. Let three continuous mappings $F : X \times S_R \times S_C \to X$ (transition rule), $\Pi : X \times S_R \times S_C \to X$ (current payoff) and $f : X \to \mathbf{R}$ (terminal payoff) be given.

Let us consider the following n-step game of two players. Starting from a point $x \in X$ the first player R chooses a move $s_R \in S_R$, then the second player C, being aware of the choice s_R, chooses a move $s_C \in S_C$, and then R pays to C the amount $\Pi(x, s_R, s_C)$ and the position of the game moves to the point $x_1 = F(x, s_R, s_C)$. Then the same procedure repeats from x_1. At the last step n the process reaches certain point x_n and R pays to C an additional terminal payoff $f(x_n)$. The objective of C is to maximise the total payoff received and the objective of R is opposite. We have a well-defined zero-sum game of R and C.

Formulation in terms of the Bellman operator

Let us define the so-called Bellman operator acting on the space of continuous functions $C(X)$ on X by the following rule:

$$\mathcal{B}f(x) = \min_{s_R \in S_R} \max_{s_C \in S_C} [f(x + F(x, s_R, s_C)) + \Pi(x, s_R, s_C)].$$

From the previous theorem (on dynamic programming for zero-sum games) the following statement holds:

The value of the game started at a point x equals $\mathcal{B}^n f(x)$, where $\mathcal{B}^n$ is the nth power (iteration) of the operator $\mathcal{B}$.

Exercise. Convince yourself that this is a correct conclusion.

A consequence that an optimal strategy and an optimal trajectory starting from a middle point x_k remain optimal for the game of the remaining $(n-k)$ steps is referred to as the *Bellman principle of optimality* (for zero-sum games).

Dynamic programming for n-player games

Assume X_i, $i = 1, ..., n$, are arbitrary sets, $h = (h_1, ..., h_n)$ is a collection of real functions on $X = X_1 \times ... \times X_n$ and $\mathcal{H} : X \mapsto \Gamma$ is a mapping with values in the set Γ_n of n person games. More precisely, for any $x = (x^1, ..., x^n)$, $\mathcal{H}(x)$ is a game $\Gamma_{A,H}(x)$, where $A = A_1 \times ... \times A_n$, $A_i = A_i(x)$ are the sets of the strategies of the player i, and $H = (H_1, ..., H_n)(x)$ is the collection of payoff functions, so that $H(x, ; a_1, ..., a_n))$ denotes the payoff to i in the profile $(a_1, ..., a_n)$.

Consider now the following two step game $G_{h,\mathcal{H}}$. First the players choose (independently) their strategies $x_i \in X_i$ forming the first step profile $x = (x^1, ..., x^n)$. Then each player obtains $h_i(x)$ and afterwards the game $\mathcal{H}(x)$ is played.

Dynamic programming for n-player games

The strategies of the player i are clearly described by the pairs (x^i, α^i), where $x^i \in X_i$ and α^i is a functions from X to A_i.

For a profile $\alpha = (\alpha^1, ..., \alpha^n)$ in a game $\Gamma_{A,H}(x)$ we shall denote by $\hat{\alpha}^i$ the collection of the $(n-1)$ strategies of the players $j \neq i$ in the profile α.

Similarly for $x = (x^1, ..., x^n)$ we denote by $\hat{x}^i$ the collection of x^j with $j \neq i$.

Dynamic programming for n-player games

Theorem. *Dynamic programming for n-person games.* Suppose for any x the game $\mathcal{H}(x)$ has a Nash equilibrium given by the profile $\alpha_0(x) = (\alpha_0^1(x), ..., \alpha_0^n(x))$, i.e.

$$H_i(x; \alpha(x)) \geq H_i(x; a_i, \hat{\alpha}_0^i(x))$$

for any i, $a_i \in A_i(x)$. Suppose also that $x_0 = (x_0^1, ..., x_0^n)$ is an equilibrium profile for the n person games with the strategy spaces X_i and the payoffs of i th player given by $h_i(x) + H_i(x; \alpha_0(x))$, i.e.

$$h_i(x_0) + H_i(x_0; \alpha_0(x_0)) \geq h_i(x_i, \hat{x}_0^i) + H_i(x_i, \hat{x}_0^i; \alpha_0(x_i, \hat{x}_0^i))$$

for all i and $x^i \in X_i$.

Dynamic programming for n-player games

Theorem (cont.) Then the profile (x_0, α_0) is a Nash equilibrium in $G_{h,\mathcal{H}}$.

Proof. As above the required inequality

$$h_i(x_0) + H_i(x_0; \alpha_0(x_0))$$

$$\geq h_i(x_i, \hat{x}_0^i) + H_i(x_i, \hat{x}_0^i; \alpha^i(x_i, \hat{x}_0^i), \hat{\alpha}_0^i(x_i, \hat{x}_0^i))$$

follows directly from the conditions of the theorem.

Folk theorem

For a general (not necessarily symmetric) stage game Γ with payoffs $\pi_1(s_1, s_2)$, $\pi_2(s_1, s_2)$, the related infinitely repeated game Γ^δ with discounting has payoffs $\delta^k \pi_j(s_1, s_2)$ on the kth stage and the total payoffs

$$\Gamma_j^\delta = \sum_{k=0}^{\infty} \delta^k \pi_j(s_1^k, s_2^k),$$

where s_j^k are the strategies used in the kth stage.

Playing a profile (σ_1, σ_2) with payoff (ξ_1, ξ_2) in all stages of Γ^δ yields the payoff $(\xi_1/(1-\delta), \xi_2/(1-\delta))$. Thus it is natural to compare the payoffs of a one-stage game (π_1, π_2) with the so-called *average payoffs*

$$(\tilde{\Pi}_1^\delta, \tilde{\Pi}_1^\delta) = (1-\delta)(\Pi_1^\delta, \Pi_2^\delta)$$

of Γ^δ.

Folk theorem

The term *average payoffs* becomes justified if we interpret δ as the probability of the termination of a game. Then the average number of steps of the game is $T = 1/(1-\delta)$ (Exercise: check this!) and thus for a game with payoffs Π_j^δ the numbers $(1-\delta)\Pi_j^\delta$ equal the average payoff per a stage.

Theorem. Let (Π_1^*, Π_2^*) be payoffs for a Nash equilibrium (σ_1^*, σ_2^*) (possibly in mixed states) of a stage game. Let (σ_1, σ_2) be another profile of strategies in a one-stage game yielding individually better payoffs $(\xi_1 = \Pi_1(\sigma_1, \sigma_2), \xi_2 = \Pi_2(\sigma_1, \sigma_2))$, so that $\xi_1 > \Pi_1^*$, $\xi_2 > \Pi_2^*$. Then for large enough δ there exists a subgame perfect Nash equilibrium in the infinitely repeated game with the average payoff (ξ_1, ξ_2).

Proof

Consider the following pair of trigger strategies: begin by using strategies (σ_1, σ_2) and use it until any of the players deviates from it, then switch to (σ_1^*, σ_2^*) for ever.

In order to prove that the pair is Nash, let us assume that the first player starts to deviate from this trigger strategy at a stage k by playing some σ_1' at this stage. Then the total payoff to the first payer starting from the stage k (with the second adhering to the initial trigger) does not exceed

$$\delta^k \left[\Pi_1(\sigma_1', \sigma_2) + \frac{\delta}{\delta - 1} \Pi_1^* \right].$$

This is not better than for the initial strategy if

$$\Pi_1(\sigma_1', \sigma_2) + \frac{\delta}{\delta - 1} \Pi_1^* \leq \frac{\sigma_1}{1 - \delta}.$$

Proof (cont.)

If $\Pi_1(\sigma_1', \sigma_2) \leq \xi_1$, then this holds for all $\delta > 0$. If $\Pi_1(\sigma_1', \sigma_2) > \xi_1$, the inequality rewrites as

$$\delta \geq \delta_1(\sigma_1') = \frac{\Pi_1(\sigma_1', \sigma_2) - \xi_1}{\Pi_1(\sigma_1', \sigma_2) - \Pi_1^*}.$$

Since $\delta_1 \in (0, 1)$ the set of such δ is not empty. Defining similarly δ_2 we conclude that for

$$\delta \geq \max\left(\max_{\sigma_1'} \delta_1, \max_{\sigma_2'} \delta_2\right),$$

our trigger strategy is a Nash equilibrium.

But the average payoffs of the trigger strategies are (ξ_1, ξ_2).

Chapter 4

Games with a continuous state space.

Nash equilibria and RD for continuous strategy spaces, Bellman optimality principle, HJB-Isaacs equation, differential games, controlled Markov processes, generalized solutions.

Cournot's duopoly, Bertrand's duopoly, Stackelberg duopoly, war of attrition.

Cournot's duopoly

Ruth and Charlie are directors of two firms, R and C, that produce and sell a product on the same market. The price of the product is supposed to decrease proportionally to the supply, i.e. if Q_R and Q_C are the quantities of the product produced by R and C, the market price for the unit of the product becomes

$$P(Q) = \begin{cases} P_0(1 - Q/Q_0) & \text{if} \quad Q < Q_0 \\ 0 & \text{if} \quad Q \geq Q_0 \end{cases} \tag{27}$$

where $Q = Q_R + Q_C$ is the aggregate amount produced, and positive constants P_0, Q_0 denote the highest possible price and the highest reasonable production level.

Cournot's duopoly

If the marginal cost of the production is c for both firms, the payoffs clearly are

$$\Pi_R(Q_R, Q_C) = Q_R P(Q) - cQ_R, \quad \Pi_C(Q_R, Q_C) = Q_C P(Q) - cQ_C \tag{28}$$

As for $P_0 \leq c$ the model is meaningless (no profit is available), one always assumes that $P_0 > c$. Thus we have a two-players symmetric game with an infinite set of strategies Q_R, Q_C that are numbers from the interval $[0, Q_0]$ (clearly it makes no sense to produce anything outside this interval).

Cournot's duopoly

Exercise. Analyze this model by the following scheme.
(a) Given a strategy Q_C find the best response of R, i.e. the amount $\hat{Q}_R = \hat{Q}_R(Q_C)$ that maximises the profit for R.
Answer:

$$\hat{Q}_R(Q_C) = \frac{Q_0}{2}\left(1 - \frac{Q_C}{Q_0} - \frac{c}{P_0}\right). \tag{29}$$

(Hint: $\hat{Q}_R$ above is found from the condition

$$\frac{\partial \Pi_R}{\partial Q_R}(\hat{Q}_R) = P_0\left(1 - \frac{\hat{Q}_R + Q_C}{Q_0}\right) - \hat{Q}_R\frac{P_0}{Q_0} - c = 0;$$

then one has to check that the second derivative of Π_R is negative at $\hat{Q}_R$; at last one has to check that $\hat{Q}_R(Q_C) + Q_C \leq Q_0$ for any $Q_C \leq Q_0$. It is still possible that $\hat{Q}_R$ from (29) is negative, so that speaking rigorously, the best response is given by the amount that is the maximum of zero and (29)).

Cournot's duopoly

(b) Similarly the best response of C to a given strategy Q_R is given by

$$\hat{Q}_C(Q_R) = \frac{Q_0}{2}\left(1 - \frac{Q_R}{Q_0} - \frac{c}{P_0}\right). \tag{30}$$

As Nash equilibrium is a pair $Q_R^\star, Q_C^\star$ of strategies, each of which is the best response to another one, this pair have to satisfy the system of equations

$$Q_R^\star = \frac{Q_0}{2}\left(1 - \frac{Q_C^\star}{Q_0} - \frac{c}{P_0}\right), \quad Q_C^\star = \frac{Q_0}{2}\left(1 - \frac{Q_R^\star}{Q_0} - \frac{c}{P_0}\right)$$

Find that the only solution to this system is

$$Q_R^\star = Q_C^\star = Q^\star = \frac{Q_0}{3}\left(1 - \frac{c}{P_0}\right).$$

Cournot's duopoly

(c) Calculate the equilibrium payoff

$$\Pi_R(Q^\star, Q^\star) = \Pi_C(Q^\star, Q^\star) = \frac{Q_0 P_0}{9}\left(1 - \frac{c}{P_0}\right)^2$$

(d) Show that for a monopolist, who acts on the market alone and hence has to maximize the payoff $\Pi_m(Q) = QP(Q) - cQ$ over all $Q \in [0, Q_0]$, the optimal production and the corresponding payoff are

$$Q_m = \frac{Q_0}{2}\left(1 - \frac{c}{P_0}\right), \quad \Pi_m(Q_m) = \frac{Q_0 P_0}{4}\left(1 - \frac{c}{P_0}\right)^2,$$

the latter payoff being of course higher than obtained in duopoly.

Cournot's duopoly

(e) Suppose now that the two firms R and C form a cartel, i.e. they agree to produce an equal amount of product $Q_R = Q_C$. Show that the optimal production would be $Q_R = Q_C = Q_m/2$, which gives better payoff for both of them, than under the Nash equilibrium of the initial game. Note that this is a performance of the prisoner's dilemma: Nash equilibrium yields worse payoffs than what can be achieved by cooperation (cartel), but the cooperation is unstable under unilateral deviation from it.

Analysis of this model was published by Cournot in 1838 and represents historically the first formal realization of the idea of the Nash equilibrium. Therefore some authors use the term Cournot-Nash equilibrium instead of the Nash equilibrium.

Bertrand's duopoly model

Two firms R and C produce and sell a product on the same market as above. But their strategies now are the prices, P_R and P_C, that they assign to the product. The firm "captures the market" if it assigns a lower price. This firm sells the whole product, and the second one sells nothing. In case of the equal prices, the firms share the market equally.

Let c denote the marginal cost of production. The demand function is $Q(P) = Q_0(1 - P/P_0)$ for $P < P_0$, where a constant $P_0 > c$ denotes the highest possible reasonable price for this product (notice that under assumption $c \geq P_0$ the model becomes meaningless: no profit is available).

Bertrand's duopoly model

The payoffs are

$$\Pi_R(P_R, P_C) = \begin{cases} (P_R - c)Q(P_R) & \text{if} \quad P_R < P_C \\ (P_R - c)Q(P_R)/2 & \text{if} \quad P_R = P_C \\ 0 & \text{if} \quad P_R > P_C \end{cases} \tag{31}$$

$$\Pi_C(P_R, P_C) = \begin{cases} (P_C - c)Q(P_C) & \text{if} \quad P_C < P_R \\ (P_C - c)Q(P_R)/2 & \text{if} \quad P_C = P_R \\ 0 & \text{if} \quad P_C > P_R \end{cases} \tag{32}$$

Thus we have a two-players symmetric game with an infinite set of strategies P_R, P_C that are numbers from the interval $[c, P_0]$ (clearly it makes no sense to choose a price outside of this interval).

Let us prove a remarkable feature of this model that the only Nash equilibrium is given by the prices $P_R = P_C = c$, which gives to both players the vanishing payoff:
$\Pi_R(c, c) = \Pi_C(c, c) = 0$.

Bertrand's duopoly model

Let (P_R, P_C) be a Nash equilibrium. Obviously $P_R > P_C$ can not be the best response to P_C whenever $P_C > c$, and similarly $P_C > P_R$ can not be the best response to P_R whenever $P_R > c$. This leads to four remaining possibilities:
(i) $c = P_C < P_R \leq P_0$,
(ii) $c = P_R < P_C \leq P_0$,
(iii) $c < P_C = P_R \leq P_0$,
(iv) $c = P_C = P_R$.
In case (i), P_C is not the best response to P_R, as choosing anything between c and P_R yields a better payoff for C. Thus case (i) does not give a Nash equilibrium. By symmetry the same for (ii).
Case (iii) does not yield Nash equilibrium either, for if P_R is given, a slight decrease in price for C would give him better payoff (he would capture the whole market instead of having half of it).

Bertrand's duopoly model

Exercise. Show that for a monopolist in Bertrand's model, who maximizes the profit $(P - c)Q(p)$, the optimal price would be $P_m = (P_0 + c)/2$ with the profit

$$\Pi_m(P_m) = (P_m - c)Q_0 \left(1 - \frac{P_m}{P_0}\right).$$

The Stackelberg duopoly model

As in Cournot's model above, two firms R and C produce and sell a product on the same market. Prices and payoffs are given by the same formulas (27), (28). But now the decisions of R and C are not simultaneous. The game starts with firm R (called the market leader) making a decision by choosing Q_R. Then C (called the market follower) observes the decision of R and then makes his decision.

Let us find the subgame perfect Nash equilibrium.
Method of backward induction leads to the following procedure: for any strategy Q_R find the best response strategy of C (this is given by (30) as we know), and then find the best choice for Q_R assuming that C would use his best response (common knowledge of rationality!).

The Stackelberg duopoly model

Thus we need to find Q_R that maximises the payoff

$$\Pi_R(Q_R, \hat{Q}_C(Q_R)) = Q_R[P(Q_R + \hat{Q}_C(Q_R)) - c],$$

which equals to

$$\Pi_R(Q_R, \hat{Q}_C(Q_R)) = Q_R \frac{P_0}{2}\left(1 - \frac{Q_R}{Q_0} - \frac{c}{P_0}\right),$$

due to (27) and (30). By differentiation one obtains that the maximum is obtained at

$$Q_R^\star = \frac{Q_0}{2}\left(1 - \frac{c}{P_0}\right)$$

leading to the subgame perfect Nash equilibrium

$$Q_R^\star = \frac{Q_0}{2}\left(1 - \frac{c}{P_0}\right), \quad Q_C^\star = \hat{Q}_C(Q_R^\star) = \frac{Q_0}{4}\left(1 - \frac{c}{P_0}\right).$$

The Stackelberg duopoly model

Exercise. Show that under the equilibrium the profits of the players are

$$\Pi_R^\star = \frac{Q_0 P_0}{8}\left(1 - \frac{c}{P_0}\right)^2, \quad \Pi_C^\star = \frac{Q_0 P_0}{16}\left(1 - \frac{c}{P_0}\right)^2,$$

so that $\Pi_R^\star > \Pi_C^\star$ and $Q_R^\star > Q_C^\star$. Show that firm R makes a larger profit in the Stackelberg model, than under the corresponding conditions of the Cournot model, so that the first move leads to an advantage.

War of attrition

Two players compete for a resource of value v. A strategy of the ith player is a persistence time t_i. The player that persists the longest gets all of the resource. If both quit at the same time , neither gets it. Costs (c per time unit) are linked with a waiting time only:

$$\Pi_1(t_1, t_2) = \begin{cases} v - ct_2 & \text{if } t_1 > t_2 \\ -ct_1 & \text{if } t_1 \leq t_2 \end{cases}$$

$$\Pi_2(t_1, t_2) = \begin{cases} v - ct_1 & \text{if } t_2 > t_1 \\ -ct_2 & \text{if } t_2 \leq t_1 \end{cases}$$

War of attrition

Exercise. (a) Check that there are two pure strategy Nash equilibria: $(t_1^* = v/c, t_2^* = 0)$ and $(t_2^* = v/c, t_1^* = 0)$. (b) Compare with the dollar auction game (Part 1).

Let us look at mixed strategies equilibria. It is handy to rewrite in term of the costs: $x = ct_1, y = ct_2$:

$$\Pi_1(x, y) = \begin{cases} v - y & \text{if } x > y \\ -x & \text{if } x \leq y \end{cases}$$

$$\Pi_2(x, y) = \begin{cases} v - x & \text{if } y > x \\ -y & \text{if } y \leq x \end{cases}$$

War of attrition

A mixed strategy for the first and second players are given by probability densities $p(x)$ and $q(x)$ on $[0, \infty)$.

$$\Pi_1(x, q(.)) = \int_0^x (v - y) q(y)\, dy - \int_x^\infty x q(y)\, dy.$$

By a natural extension of the equality of payoff lemma the condition for a Nash equilibrium is that $\Pi_1(x, q(.))$ is a constant (as a function of x): $\partial \Pi_1 / \partial x = 0$, or

$$(v - x) q(x) + x q(x) - \int_x^\infty q(y)\, dy = v q(x) - \int_x^\infty q(y)\, dy = 0.$$

Thus $v q'(x) = -q(x)$ and consequently (the normalising condition arises from $\int q(y) dy = 1$)

$$q(y) = \frac{1}{v} \exp\{-\frac{y}{v}\} \text{ and similarly } p(x) = \frac{1}{v} \exp\{-\frac{x}{v}\}.$$

War of attrition

Exercise. Find out the distribution of the Nash strategies in terms of times. Answer: $p(t) = (c/v)\exp\{-ct/v\}$.

Exercise. Find out the distribution of the duration of the game under the equilibrium. Answer:

$$\mathbf{P}(\text{duration} \le t) = 1 - \exp\{-\frac{2ct}{v}\}.$$

General setting

For a metric compact space Y (for instance, a bounded subset of a Euclidean space $\mathbf{R}^d$) we denote the space of continuous real functions on Y by $C(Y)$ and the set of probability measures on Y by $Y^\star$. The following well known analytic fact is crucial: the set $Y^\star$ is compact in the weak topology, where $\mu_n \to \mu$ as $n \to \infty$ means that $\int f(y)\mu_n(dy) \to \int f(y)\mu(dy)$ for all $f \in C(Y)$.
Consider a non-cooperative n-person game

$$\Gamma_H = \langle I = \{1, ..., n\}, \{X_i\}_1^n, \{H_i\}_1^n \rangle,$$

where I is the set of players, X_i denotes a compact metric space of the strategies of the player i, $i \in I$,
$X = X_1 \times \ldots \times X_n$ is the set of all profiles and $H_i \in C(X)$ is a payoff function of i.

General setting

Let

$$\Gamma_H^* = \langle \{I\}, \{X_i^*\}_1^n, \{H_i^*\}_1^n \rangle$$

denote the mixed strategy extension of Γ_H. Here the set X_i^* is interpreted as the set of mixed strategies of the player i, whose payoff in a profile $P = (p_1, ..., p_n) \in \mathbf{X} = X_1^* \times \ldots \times X_n^*$ equals

$$H_i^*(P) = \int\limits_X H_i(x_1, ..., x_n) dp_1 ... dp_n.$$

Let $H_i^*(P \| x_i)$ denote the integral

$$\int_{X_1 \times ... X_{i-1} \times X_{i+1} \times ... \times X_n} H_i(x_1, ..., x_n) \; dp_1 ... dp_{i-1} dp_{i+1} ... dp_n.$$

Nash equilibria: existence theorem

A situation P is called a *Nash equilibrium*, if

$$H_i^*(P) \geq H_i^*(P\|x_i)$$

for all i and $x_i \in X_i$.

Theorem. (FantSzi – Gliksberg) The game Γ_H^* has an equilibrium.

First proof

Let us choose some probability measures μ_i on X_i, $i = 1, ..., n$, such that any open set in X_i has a (strictly) positive measure. Let

$$C_i(P, y) = \max\{0,\ H_i^*(P\|y) - H_i^*(P)\}, \qquad y \in X_i.$$

The key idea: introduce the mapping $F_{\Gamma_H} = \mathbf{X} \mapsto \mathbf{X}$: If $P = (p_1, ..., p_n) \in \mathbf{X}$, then $F_{\Gamma_H}(P) = \bar{P} = (\bar{p}_1, ..., \bar{p}_n)$ is such that for each Borel set $A_i \subset X_i$

$$\bar{p}_i(A_i) = \frac{p_i(A_i) + \int_{A_i} C_i(p, y)\mu_i(dy)}{1 + \int_{X_i} C_i(p, y)\mu_i(dy)},$$

or equivalently for any $g \in C(X_i)$

$$\int g(y)\bar{p}_i(dy) = \frac{\int g(y)\bar{p}_i(dy) + \int g(y)C_i(p, y)\mu_i(dy)}{1 + \int C_i(p, y)\mu_i(dy)}.$$

First proof (cont.)

From the definition of the weak convergence it is clear that the mapping F_{Γ_H} is continuous in the weak topology. Applying the Schauder-Tikhonov fixed point theorem results in the existence of a fixed point of the mapping F_{Γ_H}.

The final observation is that a profile P is a fixed point of this mapping if and only if it is an equilibrium. In fact, if P is an equilibrium, then all functions $C_i(P, y)$ vanish implying that $F_{\Gamma_H}(P) = P$. Conversely, assume $F_\Gamma(p) = p$. Let us show that all functions $C_i(P, y)$ vanish.

First proof (cont.)

Suppose this does not hold for at least one $i \in I$. Since C_i is continuous, the set $B_i = \{y \in X_i : C_i(P, y) > 0\}$ is open and hence $\mu_i(B_i) > 0$. In particular, $\int C_i(p, y)\mu_i(dy) > 0$, so that for $D_i = X_i \setminus B_i$

$$\bar{p}_i(D_i) = \frac{p_i(D_i)}{1 + \int_{X_i} C_i(p, y)\mu_i(dy)} < p_i(D_i)$$

if $p_i(D_i) > 0$. This holds, because otherwise

$$H_i^\star(P) = \int H_i^\star(P\|y)p_i(dy) = \int_{B_i} H_i^\star(P\|y)p_i(dy)$$
$$> H_i^\star(P)p_i(B_i) = H_i^\star(P),$$

which is a contradiction.

Second proof

Choose an arbitrary number $\varepsilon > 0$ and a finite ε-net

$$X_i^\varepsilon = \{x_{i1}^\varepsilon, ..., x_{iN_i}^\varepsilon\},$$

in X_i, i.e. the subset $X_i^\varepsilon \subset X_i$ enjoys the property that for any point $x \in X_i$ there exists x_{ik}^ε such that $\rho_i(x, x_{ik}^\varepsilon) < \varepsilon$, where ρ_i is the metric of X_i (existence of such a net follows from compactness of X_i). Consider a finite game

$$\Gamma_\varepsilon = \langle I, \{X_i^\varepsilon\}_1^n, \{H_i\}_1^n \rangle,$$

where $H_i : X^\varepsilon = \prod_1^n X_i^\varepsilon \to \mathbf{R}$ is the restriction of H_i on X^ε, and its mixed strategy extension

$$\Gamma_\varepsilon^\star = \langle I, \{(X_i^\varepsilon)^\star\}_1^n, \{H_i^\star\}_1^n \rangle.$$

Second proof (cont.)

Choose an equilibrium $P^\varepsilon = (p_1^\varepsilon, \ldots, p_n^\varepsilon)$ for $\Gamma_\varepsilon^\star$ (which exists according to the Nash theorem on the existence of equilibria for finite games). Here $p_i^\varepsilon = (\xi_i^1, \ldots, \xi_i^{N_i})$ is the probability distribution on the set X_i^ε that can be considered also as a probability measure on X_i with a finite support. The equilibrium condition means that for all $i \in I$, $k_i = 1, ..., N_i$

$$H_i^\star(P^\varepsilon) \geq H_i^\star(P^\varepsilon \| x_{ik_i}^\varepsilon).$$

In the present discrete case

$$H_i^\star(P^\varepsilon \| x_{ik_i}^\varepsilon) = \sum_{j_l=1, l\neq i}^{N_l} H_i(x_{1j_1}^\varepsilon, ..., x_{(i-1)j_{i-1}}^\varepsilon, x_{(i+1)j_{i+1}}^\varepsilon, ..., x_{nj_n}^\varepsilon) \prod_{k\neq i} \xi_k^{j_k}.$$

Second proof (cont.)

Since X_i^ε is a ε-net and due to the continuity (and hence uniform continuity) of H, for arbitrary $y \in X_i$ there exists $x_{ik_i}^\varepsilon \in X_i^\varepsilon$ such that

$$|H_i(y_1, ..., y_{i-1}, y, y_{i+1}, ..., y_n) - H_i(y_1, ..., y_{i-1}, x_{ik_i}^\varepsilon, y_{i+1}, ..., y_n)| < \varepsilon$$

and in particular

$$H_i(y_1, ..., y_{i-1}, x_{ik_i}^\varepsilon, y_{i+1}, ..., y_n) > H_i(y_1, ..., y_{i-1}, y, y_{i+1}, ..., y_n) - \varepsilon$$

for all $y_l \in X_l$, $l \neq i$.

Second proof (cont.)

Integrating this inequality with respect to the probability measures p_l^ε, $l \neq i$, yields

$$H_i^\star(P^\varepsilon \| x_{ik_i}^\varepsilon) \geq H_i^\star(P^\varepsilon \| y) - \varepsilon,$$

and thus

$$H_i^\star(P^\varepsilon) \geq H_i^\star(P^\varepsilon \| y) - \varepsilon \tag{33}$$

for all $y \in X_i$.
Situations satisfying (33) are called ε-*equilibria*.

Second proof (cont.)

Let us take a sequence $\varepsilon_k = 1/k$ and choose a corresponding sequence $\{P^k\}$ of $1/k$-equilibria. Since $\mathbf{X}$ is weakly compact, there exists a subsequence $\{P^{k_m}\}_{m=1}^{\infty}$ weakly converging to a certain $P^0 \in \mathbf{X}$. As

$$H_i^\star(P^{k_m}) \geq H_i^\star(P^{k_m} \| y) - 1/k_m$$

for any $y \in X_i$ it follows by passing to the limit $m \to \infty$ that

$$H_i^\star(P^0) \geq H_i(P^0 \| y),$$

which means that P^0 is an equilibrium in $\Gamma_H^\star$.

Symmetric equilibria

A game Γ_H is called *symmetric* if all strategy spaces X_i are the same, i.e. $X_i = Y$ with a certain compact space Y for all i and the payoffs are symmetric, i.e. there exists a function $H(y; x_1, ..., x_{n-1})$ on Y^n, which is symmetric with respect to $x_1, ..., x_{n-1}$ (i.e. is invariant under permutations of all but the first variables) such that

$$H_i(x_1, ..., x_n) = H(x_i; x_1, ..., x_{i-1}, x_{i+1}, ..., x_n).$$

Symmetric equilibria

Theorem. The mixed strategy extension Γ_H^* of a symmetric game Γ_H (with compact Y and continuous H) has a symmetric equilibrium, that is an equilibrium of the form $P = (p, ..., p)$ with $p \in Y^\star$.

Proof. Looking at the first proof of the previous theorem one observes that under the condition of symmetry the set of symmetric profiles $(p, ..., p)$ is invariant under the mapping F_{Γ_H} so that this mapping has a symmetric fixed point.

Approximate equilibria

In many situations one can not find an equilibrium exactly, and hence one is looking only for its approximation, i.e. ε-*equilibria*.

Theorem. Let X_i be metric compact spaces, and H_i are bounded on $X = X_1 \times \ldots \times X_n$ and upper uniformly semi-continuous with respect to x_i, i.e. for each $x_i \in X_i$ there exists its neighborhood $V(x_i)$ such that $x_i' \in V(x_i)$ implies

$$H_i(y_1, ..., y_{i-1}, y, y_{i+1}, ..., y_n) - H_i(y_1, ..., y_{i-1}, x^{\varepsilon}_{ik_i}, y_{i+1}, ..., y_n) < \varepsilon$$

for all $y_1, ..., y_n$. Then the game Γ_H has (mixed) ε-equilibria for any $\varepsilon > 0$.

Proof. Looking at the second proof of the FantSzi – Gliksberg Theorem, observe that the above condition is actually all one needs to establish the existence of ε-equilibrium.

General RD

Consider (the mixed strategy extension of) a symmetric k-person game with strategies of each players belonging to a compact space X and with payoffs given by a continuous function $H(x; y_1, ..., y_{k-1})$ on X^k symmetric with respect to the last variables $y_1, ..., y_{k-1}$.

A natural generalization of the *replicator dynamics* (RD) of two person finite game represents the measure-valued evolution described by the weak equation

$$\frac{d}{dt}\int_X g(x)\nu_t(dx) = \int_X (H^\star(\nu_t \| x) - H^\star(\nu_t))g(x)\nu_t(dx), \quad (34)$$

which has to hold for all $g \in C(X)$.

General RD

If a reference measure M on X is chosen, equation (34) can be written in terms of the densities f_t of ν_t with respect to M as

$$\dot{f}_t(x) = f_t(x)(H^\star(f_t M \| x) - H^\star(f_t M)). \tag{35}$$

RD and Nash

Theorem. (i) If ν defines a symmetric Nash equilibrium for symmetric k-person game (its mixed strategy extension) specified by payoff $H(x; y_1, ..., y_{k-1})$ on X^k (X again a compact space), then ν is a fixed point for the RD. If ν is such that any open set in X has a positive ν measure (pure mixed profile), then the inverse statement holds.

Proof. (i) By definition, ν defines a symmetric Nash equilibrium if and only if

$$H^\star(\nu \| x) \leq H^\star(\nu) \tag{36}$$

for all $x \in X$. But the set $M = \{x : H^\star(\nu\|x) < H^\star(\nu)\}$ should have ν-measure zero (otherwise integrating (36) would lead to a contradiction). Thus, for all g,

$$\int_X (H^\star(\nu\|x) - H^\star(\nu))g(x)\nu_t(dx) = 0. \tag{37}$$

RD and Nash

Proof(cont.) (ii) Conversely assuming (37) holds for all g implies (taking into account here that ν is purely mixed profile)

$$H^{\star}(\nu \| x) = H^{\star}(\nu)$$

on a open dense subset of X and hence everywhere, due to the continuity of H.

RD and Nash

A direct extension of a finite state space result:
Theorem. Consider a mixed strategy extension of a two-person symmetric game with a compact space of pure strategies X of each player and a payoff matrix being an antisymmetric function H on X^2, i.e. $H(x, y) = -H(y, x)$. Assume there exists a positive finite measure M on X such that $\int H(x, y)M(dy) = 0$ for all x. Then M specifies a symmetric Nash equilibrium. Moreover, the function

$$L(f) = \int \ln f_t(x)M(dx)$$

is the first integral (i.e. it is constant on the trajectories) of the system (35) (RD on densities with respect to M).

Differential games

Assume an agent has a position $x \in \mathbf{R}^d$ at time t, which can be moved to a new position $x(t+\tau)$ during a small time τ according to the ODE $\dot{x} = g(x, u)$, where the 'control parameter' u can be chosen by the agent from some given set U. The agent has to pay the charge of $J(x, u)$ per unit of time during the transition.

At the next time $t + \tau$ the agent can choose another u for the next transition during the time interval $[t+\tau, t+2\tau]$ and so on, until the terminal time T, where the agent receives the award $V_T(x(T))$ depending on the final position.

Bellman differential equation

Then for the total optimal payoff $S(t,x)$ of the agent starting at x at time t we have for small τ (*Bellman optimality principle*):

$$S(t,x) = \sup_{u \in U} [S(t+\tau, x + g(x,u)\tau) - J(x,u)\tau].$$

Expanding S by the first order Taylor expansion yields

$$S(t,x) = \sup_{u \in U} [S(t,x) + \frac{\partial S}{\partial t}(t,x)\tau + \frac{\partial S}{\partial x}(t,x) g(x,u)\tau - J(x,u)\tau].$$

Cancelling $S(t,x)$ from both sides yields the so-called *Bellman equation*:

$$\frac{\partial S}{\partial t} + \sup_{u \in U} \left[\left(g(x,u), \frac{\partial S}{\partial x} \right) - J(x,u) \right] = 0. \qquad (38)$$

HJB equation

This equation is nothing else but the Hamilton-Jacobi equation with the specific Hamiltonian

$$H(x, p) = \sup_{u \in U} [(g(x, u), p) - J(x, u)]. \tag{39}$$

Therefore, equations (38) are often referred to as the *Hamilton-Jacobi-Bellman equations* or shortly HJB.

HJB-Isaaks equation for zero-sum games

Two agents have positions $x, y \in \mathbf{R}^d$ at time t, which can be moved to new position $x(t+\tau), y(t+\tau)$ during a small time τ according to the ODEs $\dot{x} = g(x, y, u)$, $\dot{x} = h(x, y, v)$, where $u \in U$ and $v \in V$ can be chosen by the first and the second players respectively. The first agent has to pay to the second one the charge of $J_1(x, u) + J_2(y, v)$ per unit of time during the transition.

At the next time the procedure repeats, etc, until the terminal time T, where the first agent pays to the second one the terminal payoff $V_T(x(T))$. First agent (respectively the second) tries to minimize (respectively maximise) to total payoff.
Let $S(t, x, y)$ denote the total optimal (minimax) payoff to the second agent when the game starts at time$t$ at positions x, y.

HJB-Isaaks equation for zero-sum games

Similar procedure yields the *HJB-Isaaks equation* (main equation of differential games with separated dynamics):

$$\frac{\partial S}{\partial t} + \inf_{u \in U} \left[\left(g(x,u), \frac{\partial S}{\partial x} \right) + J_1(x,u) \right] + \sup_{v \in V} \left[\left(h(y,v), \frac{\partial S}{\partial y} \right) + J_2(y,v) \right] = 0. \tag{40}$$

How to solve? Generalized solutions: viscosity solutions (Krushkov, Crandall-Lions), minimax solutions (Krasovski-Subbotin), tropical mathematics approach.

Bellman equation for controlled jump processes

Equation (38) arises from a control problem, when an agent can control the velocity of its movement. Different equations arises when the motion is subject to jumps. Namely, if jumps are given by some functions $x \mapsto y_1(x), \cdots, y_m(x)$ and occur with intensities $u_j \nu_j(x)$ controlled by an agent, the equation for the optimal payoff gets the form

$$\frac{\partial S}{\partial t} + \sup_{u \in U} \left[\sum_{j=1}^{m} u_j \nu_j(x)(S(t, y(x_j)) - S(t, x)) - J(x, u) \right] = 0, \tag{41}$$

which represents the *Bellman equation* for *controlled jump-processes*.

Bellman equation for controlled jump processes

In fact, saying that jumps $x \mapsto y_1(x), \cdots, y_m(x)$ occur with the rates $u_j\nu_j(x)$ means that the probability of having a jump in a small time τ equals approximately $R = \tau \sum u_j\nu_j(x)$, and the probability to have the jump $x \to y_j(x)$ (when a jump occurs) is $u_j\nu_j(x)/R$. The approximate equation for the optimal payoff becomes

$$S(t,x) = \sup_{u \in U} [\tau \sum_{j=1}^{m} u_j\nu_j(x) S(t+\tau, y_j(x)) - J(x,u)\tau$$

$$+(1 - \tau \sum_{j=1}^{m} u_j\nu_j(x)) S(t+\tau, x)].$$

Expanding in the Taylor series in small τ, yields (41).
Similar a game-theoretic version for processes with jumps can be obtained.

General Markov processes

The *transition operator* (for a time homogeneous Markov process)

$$\Phi_t S(x) = \mathbf{E} S(X_x(t)).$$

The *chain rule, semigroup property, Chapman-Kolmogorov equation*: $\Phi_t \circ \Phi_s = \Phi_{t+s}$.

$$\left.\frac{d}{dt}\right|_{t=0} \Phi_t S(x) = LS(x),$$

with L called the *generator of the Markov process*.

$$\Phi_\tau S(t,x) = \mathbf{E} S(t+\tau, X_x(\tau)),$$

and

$$\left.\frac{d}{dt}\right|_{t=0} \Phi_t S(t,x) = \frac{\partial S}{\partial t}(t,x) + LS(t,x).$$

Controlled Markov processes

$$\Phi^u_\tau S(t,x) = \mathbf{E}^u S(t+\tau, X_x(\tau)),$$

and

$$\left.\frac{d}{dt}\right|_{t=0} \Phi^u_t S(t,x) = \frac{\partial S}{\partial t}(t,x) + L_u S(t,x).$$

Bellman optimality principle:

$$S(t,x) = \sup_{u\in U}[\mathbf{E}^u S(t+\tau, X_x(\tau)) - J(x,u)\tau].$$

Expanding S by the first order Taylor expansion yields

$$S(t,x) = S(t,x) + \sup_{u\in U}[\frac{\partial S}{\partial t}(t,x) + L_u S(t,x) - J(x,u)]\tau,$$

yielding the *HJB equation*

$$\frac{\partial S}{\partial t}(t,x) + \sup_{u\in U}[L_u S(t,x) - J(x,u)] = 0.$$

HJB equations

For instance, for diffusions L is a second order differential operator:

$$LS(x) = \sum_{i,j} a_{ij}(x) \frac{\partial^2 S}{\partial x_i \partial x_j}(x) + \sum_j b_j(x) \frac{\partial S}{\partial x_j}(x).$$

For a Brownian motion $L = \Delta$.
For a drift-controlled Brownian motion the HJB becomes

$$\frac{\partial S}{\partial t}(t,x) + \Delta S + \sup_{u \in U}[\sum_j b_j(x,u) \frac{\partial S}{\partial x_j}(x) - J(x,u)] = 0. \quad (42)$$

More general form:

$$\frac{\partial S}{\partial t}(t,x) + \Delta S + H\left(x, \frac{\partial S}{\partial x}(x)\right) = 0, \quad (43)$$

with $H(x,p)$ called the *Hamiltonian*.

HJB-Isaaks equations

The HJB-Isaaks equation for stochastic zero-sum differential games:

$$\frac{\partial S}{\partial t}(t,x) + \Delta S + \sup_{u \in U} \inf_{v \in V} [\sum_j b_j(u) \frac{\partial S}{\partial x_j}(x) + J(x,u)] = 0.$$

One says that the Isaaks condition is satisfied if $\sup_{u \in U} \inf_{v \in V} = \inf_{v \in V} \sup_{u \in U}$ here for all S.

This holds for separated dynamics:
$H(x,p) = H_1(x,p) + H_2(x,p)$, H_1 controlled by u and H_2 controlled by v.

Generalized solutions of HJB (remarks)

Nonexistence of classical solutions.

Non-uniqueness of almost sure solutions.

Viscosity solutions.

Mild solutions: for equation (43) with the terminal condition S_T:

$$S(t,x) = e^{(T-t)\Delta} S_T + \int_t^T e^{(s-t)\Delta} H(x, \frac{\partial S}{\partial x}(s,x)\, ds. \quad (44)$$