Seminar 3

$\mathbb{E}X = ?$

LLN: $\dfrac{\sum X_n}{n} \xrightarrow[n\to\infty]{} \mathbb{E}X$

$C = D\,\mathbb{E}[g(S_T)]$

$\left[\begin{array}{l} S_T^{(1)}, \dots, S_T^{(n)} \\ D \cdot \frac{1}{n} \sum_i g(S_T^{(i)}) \end{array}\right.$

$\mathbb{P}(|\mathbb{E}X - \bar{X}| > \varepsilon) < \dfrac{Var[\bar{X}]}{\varepsilon^2}$

accuracy: 0,01 , prob. 0.95

$\varepsilon = 0{,}01$

$n - ?$

$var[\bar{x}] = \frac{1}{n} Var[x]$

1.
$$\begin{cases} V_t = G_t B_t + H_t S_t \\ dV_t = G_t\,dB_t + H_t\,dS_t + qH_tS_t\,dt \\ dB_t = rB_t\,dt \\ dS_t = S_t(\mu\,dt + \sigma\,dW_t) \end{cases}$$

$$0 = d(V_t - V(t,S_t)) = G_t\,dB_t + H_t\,dS_t + qH_tS_t\,dt - V_t'(t,S_t)\,dt - V_s'(t,S_t)\,dS_t - \tfrac{1}{2}V_{ss}''(t,S_t)(dS_t)^2 =$$

$$= rB_tG_t\,dt + \underline{(H_t - V_s'(t,S_t))\mu S_t\,dt} + qH_tS_t\,dt - V_t'(t,S_t)\,dt - \tfrac{1}{2}\sigma^2S_t^2V_{ss}''(t,S_t)\,dt + \underline{(H_t - V_s'(t,S_t))\sigma S_t\,dW_t} =$$

$$= \{H_t = V_s'(t,S_t)\} = rB_tG_t\,dt + qV_s'S_t\,dt - V_t'(t,S_t)\,dt - \tfrac{1}{2}\sigma^2S_t^2V_{ss}''(t,S_t)\,dt = \left\{G_t = \frac{V - V_s'(t,S_t)S_t}{B_t}\right\} =$$

$$= rV\,dt + qV_s'S_t\,dt - rV_s'(t,S_t)S_t\,dt - V_t'(t,S_t)\,dt - \tfrac{1}{2}\sigma^2S_t^2V_{ss}''\,dt = 0.$$

$$V_t'(t,s) + (r-q)s\,V_s'(t,s) + \frac{\sigma^2}{2}s^2V_{ss}'' = rV(t,s) \qquad (1)$$

$$\boxed{\begin{array}{l} V(t,s) = e^{at+b}\,\hat{V}(t,s), \quad a - ?,\ b - ? \\ V_t' = a e^{at+b}\,\hat{V}(t,s) + e^{at+b}\,\hat{V}_t'(t,s) \\ V_s' = \hat{V}_s' e^{at+b} \\ V_s'' = \hat{V}_{ss}'' e^{at+b} \end{array}}$$

$$ae^{at+b}\hat{V}(t,s) + e^{at+b}\hat{V}_t'(t,s) + (r-q)s\hat{V}_s'e^{at+b} + \frac{\sigma^2}{2}s^2\hat{V}_{ss}''e^{at+b} = r\hat{V}(t,s)e^{at+b} \quad \Big| : e^{at+b}$$

$$\hat{V}_t'(t,s) + (r-q)s\hat{V}_s' + \frac{\sigma^2}{2}s^2\hat{V}_{ss}'' = (r-a)\hat{V}(t,s) \qquad (2)$$

$a = q$, $b = -qT$

$\exists\ \hat{V}$ – solution of (2) $\Rightarrow V = e^{-q(T-t)}\hat{V}$

$$\hat{V} = S_t\Phi(d_1) - e^{-\hat{r}(T-t)}K\Phi(d_2)$$

$$d_1 = \frac{\ln(S_t/K) + (\hat{r} + \frac{\sigma^2}{2})(T-t)}{\sigma\sqrt{T-t}}, \quad d_2 = d_1 - \sigma\sqrt{T-t}$$

$\hat{r} = r - q$

$$V = e^{-q(T-t)}S_t\Phi(d_1) - e^{-r(T-t)}K\Phi(d_2).$$

$$V(T,s) = (s-K)^+ = e^{aT+b}(s-K)^+ \Rightarrow b = -aT = -qT$$

② $V_t = G_t B_t + H_t S_t$

$dV_t = G_t dB_t + H_t dS_t$

$S_t^* = S_t / B_t$

$V_t^* = V_t / B_t$

$\Rightarrow dV_t^* = H_t \, dS_t^*$

$$dV_t^* = d\left(\frac{V_t}{B_t}\right) = \frac{1}{B_t} dV_t + V_t d\left(\frac{1}{B_t}\right) = \frac{1}{B_t} G_t dB_t + \frac{1}{B_t} H_t dS_t - \frac{1}{B_t^2} V_t dB_t = \underline{\frac{1}{B_t} G_t dB_t} + \frac{1}{B_t} H_t dS_t - \underline{\frac{1}{B_t^2}} (G_t B_t + H_t S_t) dB_t =$$

$$= \frac{1}{B_t} H_t dS_t - \frac{H_t S_t dB_t}{B_t^2} = H_t \left(\frac{dS_t}{B_t} - \frac{S_t dB_t}{B_t^2} \right)$$

$$dS_t^* = d\left(\frac{S_t}{B_t}\right) = \frac{1}{B_t} dS_t + S_t d\left(\frac{1}{B_t}\right) = \frac{1}{B_t} dS_t - \frac{1}{B_t^2} S_t dB_t = \frac{dS_t}{B_t} - \frac{S_t dB_t}{B_t^2}.$$

③ a) $\frac{dP}{dQ} = 1 / \frac{dQ}{dP}$

b) $\mathbb{E}^Q X = \mathbb{E}^P\left[\frac{dQ}{dP} X\right]$

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b) 1) $\exists X = \mathbb{1}_A \Rightarrow \mathbb{E}^Q X = \mathbb{E}^Q \mathbb{1}_A = Q(A) = \int_A \frac{dQ}{dP} dP = \int \frac{dQ}{dP} \mathbb{1}_A dP = \mathbb{E}^P\left[\frac{dQ}{dP} \mathbb{1}_A\right] = \mathbb{E}^P\left[\frac{dQ}{dP} X\right].$

2) $\exists X = \sum a_i \mathbb{1}_{A_i} \Rightarrow$ expectation linearity.

3) Lebesgue th.

a) $\forall A.\ P(A) = \underline{\int_A dP} = \int_A \frac{dP}{dQ} \cdot dQ = \{b)\} = \underline{\int_A \frac{dP}{dQ} \cdot \frac{dQ}{dP} \cdot dP} \quad \forall A.$

$\frac{dP}{dQ} \cdot \frac{dQ}{dP} \overset{a.s.}{=} 1$

Union[float, np.ndarray]

Option = CallStockOption(strike = np.linspace(50, 150, 100), ...)