Multi-Period Asset Pricing
Part 1

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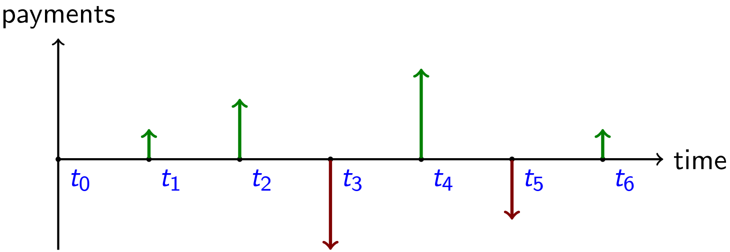
November – December, 2022

Introduction to arbitrage-free pricing

Financial security

$$\boxed{\textbf{Financial Security} = \textbf{Cash Flow}}$$

Example (Interest Rate Swap)



Pricing problem: compute a “fair” value of the security at t_0.

Classification of financial securities

We divide all financial assets into 2 groups:

1. **Traded securities**: prices are *fixed* by the market.
2. **Non-traded securities**: prices have to be *computed*.

Remark

This "black-and-white" classification is quite idealistic. The same security may be considered as traded or non-traded at different times. For example, a call option is usually liquid *at-the-money* and illiquid deeply *out-of-the-money*.

Goal of the course: the **arbitrage-free pricing** (**AFP**) of non-traded securities (for finite financial models).

Financial market

Financial market (FM): all traded securities.

Trading strategy: $X_0 \xrightarrow[(\Delta_n)]{} X_N$, where

$X_n = X_n(\omega)$: the total wealth at $t = n$ (for outcome ω).

$\Delta_n = \Delta_n(\omega)$: the number of the stocks at $t = n$.

Arbitrage strategy: a trading strategy, where we

1. start with $X_0 = 0$ (*nothing*),
2. end with $X_N(\omega) \geq 0$ for all ω and $X_N(\omega') > 0$ for some ω' (*something*).

Assumption (NA)

The financial market is arbitrage free.

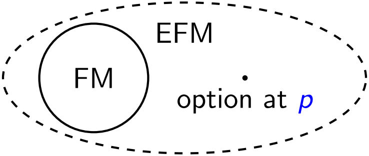
Arbitrage-free price

In addition to FM (with NA), we consider a non-traded option.

Definition (AFP)

The amount p is an **arbitrage-free price (AFP)** if given an opportunity to trade the option at p we still have NA:

$$\text{Extended FM} \triangleq \{\text{FM} + \text{option traded at } p\} \text{ has NA.}$$

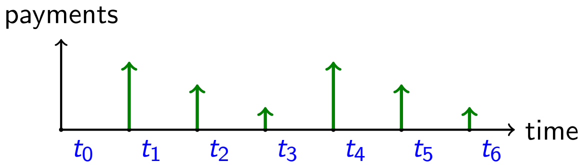


Basic questions:

1. How to compute AFP?
2. When is AFP unique?

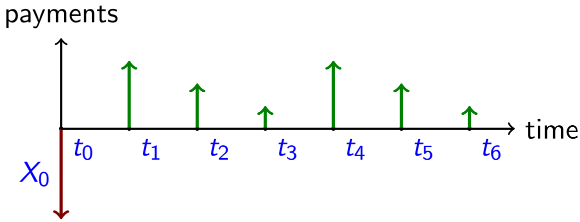
Replication

Cash flow of non-traded option:



Replicating strategy:

1. starts with some initial capital X_0,
2. generates *exactly* the same cash flow in the future:



Methodology of AFP

Theorem

An AFP p is unique if and only if there is a replicating strategy. In this case,

$$p = X_0,$$

where X_0 is the initial capital of a replicating strategy.

Main Principle:

$$\boxed{\textbf{Unique Arbitrage-Free Pricing (!AFP)} = \textbf{Replication}}$$

Remark

A replicating strategy may not be unique. However, its initial capital is unique. Otherwise, we get a contradiction with NA.

Methodology of AFP

Proof.

$\Longleftarrow$: We assume that a replicating strategy exists and that the option is traded at price p.

(a) If $p > X_0$, then we make an *arbitrage* by buying the replicating strategy and (short) selling the option. We get

$$\text{profit} = p - X_0 > 0.$$

(b) If $p < X_0$, then we make an *arbitrage* by (short) selling the strategy and buying the option. We receive

$$\text{profit} = -p + X_0 > 0.$$

Methodology of AFP

(c) If $p = X_0$, then there is NA. Indeed, a strategy on EFM (original FM + option)

$$Y_0 \xrightarrow[q \text{ options } +(\Delta_n)]{} Y_N(\omega)$$

has the same cash flow as the "twin" strategy on the original FM:

$$Y_0 \xrightarrow[q(\Delta_n^X)+(\Delta_n)]{} Y_N(\omega).$$

Here (Δ_n^X) is the number of stocks in the replicating strategy. To conclude the argument we just recall that FM has NA. $\square$

Problem on forward exchange rates

Problem

There is a financial market with times 0 *and* 1.

Spot FX: $\$S_0 = €1$.

\$ bank: $\$1 \longrightarrow \$(1+r)$.

€ bank: $€1 \longrightarrow €(1+q)$.

Compute the forward FX F_0*:*

$$\$0 \quad \underset{\textit{long position}}{\xrightarrow{\hspace{3cm}}} \quad €1 - \$F_0.$$

Solution

We choose F_0 so that the payoff $€1 - \$F_0$ of the long position in the forward contract can be replicated with $X_0 = \$0$.

Problem on forward exchange rates

Replicating strategy: $(a) + (b)$, where

$$€\,\frac{1}{1+q} \quad \xrightarrow[€\,\text{bank}]{} \quad €\,1, \tag{a}$$

$$-\$\,\frac{F_0}{1+r} \quad \xrightarrow[\$\,\text{bank}]{} \quad -\$F_0. \tag{b}$$

The initial capital in \$ is

$$X_0 = S_0 \frac{1}{1+q} - F_0 \frac{1}{1+r}.$$

In the absence of arbitrage, $X_0 = \$0$. Hence,

$$F_0 = S_0 \frac{1+r}{1+q} \quad (\text{in } \$). \qquad \square$$

Problem on put-call parity

Problem

There is a financial market with times 0 *and* 1.

Discount factor: $D_0 \longrightarrow \$1$.

Call with strike K*:* $C_0 \longrightarrow \max(S_1 - K, 0)$, *where* S_1 *is the stock price at* $t = 1$.

Put with strike K*:* $P_0 \longrightarrow \max(K - S_1, 0)$.

Compute the forward price F_0*:*

$$0 \quad \xrightarrow[\textit{long position}]{} \quad S_1 - F_0.$$

Solution

We choose F_0 so that the payoff $S_1 - F_0$ can be replicated with $X_0 = 0$.

Problem on put-call parity

We write

$$S_1 - F_0 = \max(S_1 - K, 0) - \max(K - S_1, 0) + (K - F_0).$$

Hence, the replicating strategy is

$$\text{call} - \text{put} + (K - F_0) \text{ discount factors}.$$

The initial capital is

$$X_0 = C_0 - P_0 + D_0(K - F_0).$$

In the absence of arbitrage, $X_0 = 0$. It follows that

$$F_0 = \frac{1}{D_0}(C_0 - P_0) + K. \qquad \square$$

Problem on interest rates

Problem

There is a multi-period financial market.

Discount factors: $D_0(n)$ *at* $t = 0 \longrightarrow \$1$ *at* $t = n$.

Bank with stochastic interest rate (r_n)*:*

$$\$1 \text{ at } t = n \longrightarrow \$(1 + r_n) \text{ at } t = n + 1.$$

Compute the AFP of the option paying r_n *at* $t = n + 1$*:*

$$V_0 - ? \longrightarrow V_{n+1} = r_n.$$

Problem on interest rates

Solution

We write

$$V_{n+1} = r_n = 1 + r_n - 1.$$

Replicating strategy: $(a) - (b)$, where

$$D_0(n) \text{ at } t = 0 \xrightarrow[D(n)]{} \$1 \text{ at } t = n \xrightarrow[\text{bank}]{} \$(1 + r_n) \text{ at } t = n + 1, \tag{a}$$

$$D_0(n+1) \text{ at } t = 0 \xrightarrow[D(n+1)]{} \$1 \text{ at } t = n + 1. \tag{b}$$

We obtain that

$$V_0 = D_0(n) - D_0(n+1). \qquad \square$$