Seminar 5

① $X_t = \left((B_t^1)^2 + \ldots + (B_t^n)^2\right)^{1/2}, \quad n \geq 2.$

?, $d\tilde{B}_t$?

$\langle \tilde{B}_t \rangle = t \Rightarrow \tilde{B}_t$ - винеровский процесс

$Y_t = X_t^2$

$Y_t = (B_t^1)^2 + \ldots + (B_t^n)^2$

$$\frac{\sqrt{Y_t}}{\sqrt{Y_t}}$$

$$dY_t = 2\sum_{i=1}^n B_t^i dB_t^i + \frac{1}{2}\sum_{i,j} \langle dB_t^i, dB_t^j \rangle dt = n\,dt + 2\sum_{i=1}^n B_t^i dB_t^i = n\,dt + 2\sqrt{Y_t} \underbrace{\sum_{i=1}^n \frac{B_t^i}{\sqrt{(B_t^1)^2 + \ldots + (B_t^n)^2}} dB_t^i}_{} =$$

$$= n\,dt + 2\sqrt{Y_t}\, d\hat{B}_t$$

$X_t = \sqrt{Y_t}$

$$dX_t = \frac{1}{2\sqrt{Y_t}} dY_t - \frac{1}{2\cancel{8}(\cancel{\sqrt{Y_t}})^3\sqrt{Y_t}} \cdot \cancel{4Y_t}\, dt = \frac{1}{2X_t}\left(n\,dt + 2X_t\, d\hat{B}_t\right) - \frac{1}{X_t} dt = \frac{n-1}{2X_t} dt + d\hat{B}_t$$

② $\beta_k(t) = \mathbb{E}[B_t^k], \quad k \geq 2$

$$dB_t^k = kB_t^{k-1} dB_t + \frac{1}{2}k(k-1) B_t^{k-2} dt$$

$$B_t^k = \int_0^t k B_s^{k-1} dB_s + \int_0^t \frac{1}{2}k(k-1) B_s^{k-2} ds \quad \Big| \,\mathbb{E}$$

$$\mathbb{E}[B_t^k] = \beta_k(t) = \frac{1}{2}k(k-1)\int_0^t \mathbb{E}[B_s^{k-2}] ds = \frac{1}{2}k(k-1)\int_0^t \beta_{k-2}(s) ds.$$

$$k=2: \quad \frac{1}{2}\cdot 2 \cdot 1 \cdot \int_0^t 1\, ds = t$$

③
$$\begin{pmatrix} dX_1 \\ dX_2 \end{pmatrix} = \frac{1}{2}\begin{pmatrix} X_1 \\ X_2 \end{pmatrix} dt + \begin{pmatrix} X_2 \\ X_1 \end{pmatrix} dB_t$$

$$\begin{pmatrix} X_1 \\ X_2 \end{pmatrix} = \begin{pmatrix} \mathrm{sh}(B_t) \\ \mathrm{ch}(B_t) \end{pmatrix}.$$

$\alpha\, \mathrm{sh} + \beta\, \mathrm{ch}$ $\quad X_1^0 = \ldots$

$\alpha\, \mathrm{ch} + \beta\, \mathrm{sh}$ $\quad X_2^0 = \ldots$

④ a) $dX_t = X_t dt + dB_t, \quad X_0 = x$

$F_t = e^{-t}$

$$d(F_t X_t) = d(e^{-t}X_t) = -e^{-t}X_t dt + e^{-t}dX_t = \cancel{-e^{-t}X_t dt} + \cancel{e^{-t}X_t dt} + e^{-t}dB_t$$

$$d(e^{-t}X_t) = e^{-t}dB_t$$

$$e^{-t}X_t = \int_0^t e^{-s}dB_s + x$$

$$X_t = xe^t + \int_0^t e^{t-s}dB_s$$

c) $dX_t = r\,dt + \alpha X_t dB_t, \quad X_0 = x$

$$dF_t = \theta_t dt + \gamma_t dB_t$$

$$d(F_t X_t) = F_t dX_t + X_t dF_t + dX_t dF_t = F_t(r\,dt + \alpha X_t dB_t) + X_t(\theta_t dt + \gamma_t dB_t) + \alpha\gamma_t X_t dt =$$

$$= (rF_t + \underbrace{\theta_t X_t + \alpha\gamma_t X_t}_{0})dt + (\underbrace{\alpha X_t F_t + \gamma_t X_t}_{0})dB_t.$$

$$\begin{cases} \theta_t + \alpha\gamma_t = 0 \\ \alpha F_t + \gamma_t = 0 \end{cases} \Rightarrow \quad \begin{matrix} \gamma_t = -\alpha F_t \\ \theta_t = \alpha^2 F_t \end{matrix}$$

$$dF_t = \alpha^2 F_t dt - \alpha F_t dB_t$$

$$F_t = F_0 \exp\left\{-\alpha B_t + \frac{1}{2}\alpha^2 t\right\}, \quad F_0 = 1$$

$$d(F_t X_t) = rF_t dt$$

$$X_t = F_t^{-1}\left(x + \int_0^t rF_s ds\right) = \exp\left\{\alpha B_t - \frac{1}{2}\alpha^2 t\right\}\left(x + r\int_0^t \exp\left\{-\alpha B_s + \frac{1}{2}\alpha^2 s\right\} ds\right) =$$

$$= x\exp\left\{\alpha B_t - \frac{1}{2}\alpha^2 t\right\} + r\int_0^t \exp\left\{\alpha(B_t - B_s) - \frac{1}{2}\alpha^2(t-s)\right\} ds.$$