Applied Mathematical Finance I

Lecture 6: Short-Rate Models

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Introduction

- So far we mainly concentrated on pricing simple products without considering models for evolution of zero-coupon bond price processes $\{p(\cdot, T)\}_{T>0}$.
- It seems quite natural to model the behaviour of the instantaneous short rate r and consider T-bonds as derivative contracts on r.
- This was in fact the earliest approach to interest rate modelling.
- Before we discuss derivatives pricing under short-rate models, let us recall the main points of the classical approach to valuation in the case of a stock market.

Classical Approach to Stock Option Pricing

- In the classical Black-Scholes-Merton framework, we start by modelling risky asset price dynamics under the objective probability measure $\mathbb{P}$ as

$$dS_t = S_t \left[\mu \, dt + \sigma \, dW_t^{\mathbb{P}}\right].$$

- Value $V(t, S)$ of a derivative contract is then defined as a cost of its replication i.e. as initial endowment into a (permissible) self-financing hedging strategy.
- Standard replication argument then leads to a PDE which does not depend on μ.
- Using Feynman-Kac formula, the solution of the PDE (i.e. derivative's price) can be expressed as the expectation of the payoff under the unique measure $\mathbb{Q} \sim \mathbb{P}$ such that dynamics of S under $\mathbb{Q}$ become

$$dS_t = S_t \left[r \, dt + \sigma \, dW_t^{\mathbb{Q}}\right].$$

Short-Rate Models: First Look

- We now consider a market consisting of the only exogenously given asset

$$dB_t = r_t B_t \, dt.$$

- Our goal is to find prices of zero-coupon bonds which are considered as derivatives of the underlying short rate r.
- We assume that dynamics of the short rate under the objective measure $\mathbb{P}$ are given by

$$dr_t = \mu(t, r_t) \, dt + \sigma(t, r_t) \, dW_t^{\mathbb{P}}. \tag{1}$$

  Can we apply arguments from the previous slide to derive T-bond price?
- Actually no. The reason is that the only hedging instrument is the bank account B so we have no possibility to form a self-financing portfolio replicating T-bond. Modelling framework is not complete.

Short-Rate Models: First Look (continued)

- It should be now clear that in our framework arbitrage-free price of a T-bond is not uniquely determined.
- Indeed, as T-bond cannot be replicated, its price can in principle take any positive value.
- However, this does not mean that zero-bond prices for different maturities can take any form. As we will see shorty, bond prices have to satisfy consistency conditions in order to exclude arbitrage opportunities on the bond market.
- Moreover, considering particular T-bond as a benchmark security which price is given, bond prices for other maturities will be uniquely determined in terms of the price of the benchmark bond and dynamics (1).

Deriving the Term-Structure Equation (continued)

- To derive consistency condition for zero-coupon bonds of different maturities, we assume that, for every $T > 0$, the price of a T-bond is given by

$$p(t,T) = F(t, r_t, T), \quad t \le T,$$

where F is a sufficiently smooth deterministic function such that $F(T, r, T) = 1$.

- Given (1), we apply Itô formula to $F_T(t, r_t) = F(t, r_t, T)$ and get

$$dF_T(t,r_t) = F_T(t,r_t)\left[\alpha_T(t,r_t)dt + \beta_T(t,r_t)\, dW_t^{\mathbb{P}}\right],$$

where

$$\alpha_T(t,r_t) = \frac{1}{F_T(t,r_t)}\left[\frac{\partial F_T(t,r_t)}{\partial t} + \mu(t,r_t)\frac{\partial F_T(t,r_t)}{\partial r} + \frac{1}{2}\sigma^2(t,r_t)\frac{\partial^2 F_T(t,r_t)}{\partial r^2}\right],$$

$$\beta_T(t,r_t) = \frac{\sigma(t,r_t)}{F_T(t,r_t)}\frac{\partial F_T(t,r_t)}{\partial r}.$$

Deriving the Term-Structure Equation (continued)

- We now form a self-financing portfolio consisting of two zero-coupon bonds of different maturities S and T.

- Let us denote by ϕ_t the number of T-bonds held in the portfolio at time t while the remaining part is invested in S-bonds. Value V_t of the portfolio then satisfies

$$
\begin{aligned}
dV_t &= \phi_t \, dF_t(t, r_t) + \frac{V_t - \phi_t F_T(t, r_t)}{F_S(t, r_t)} \, dF_S(t, r_t) \\
&= V_t \left[\omega_t \frac{dF_T(t, r_t)}{F_T(t, r_t)} + (1 - \omega_t) \frac{dF_S(t, r_t)}{F_S(t, r_t)} \right] \\
&= V_t \left[(\omega_t \, \alpha_T + (1 - \omega_t) \alpha_S) dt + (\omega_t \, \beta_T + (1 - \omega_t) \beta_S) \, dW_t^{\mathbb{P}} \right],
\end{aligned}
$$

with $\omega_t = \dfrac{\phi_t F_T(t, r_t)}{V_t}$ being a proportion of funds invested in T-bonds.

Deriving the Term-Structure Equation (continued)

- Choosing $\omega_t = \dfrac{\beta_S(t, r_t)}{\beta_S(t, r_t) - \beta_T(t, r_t)}$ we get a locally risk-free portfolio

$$dV_t = V_t \frac{\beta_S \alpha_T - \beta_T \alpha_S}{\beta_S - \beta_T} \, dt.$$

- We now need the following result.
  If there exists a self-financing portfolio with value process satisfying

$$dV_t = V_t \, k_t \, dt, \quad V_0 > 0,$$

  for some progressively measurable process k_t, then absence of arbitrage implies

$$k_t = r_t, \quad d\mathbb{P} \otimes dt\text{-a.e.}$$

Consequence from Absence of Arbitrage: Proof

- Consider a self-financing portfolio U_t such that

$$dU_t = \varphi_t \, dV_t + \psi_t \, dB_t, \quad U(0) = 0.$$

- Dynamics of the discounted process is given by

$$d\frac{U_t}{B_t} = \varphi_t \, \frac{V_t}{B_t}(k_t - r_t) \, dt.$$

- Let us choose $\varphi_t = \mathbb{1}_{\{k_t > r_t\}}$. We then have

$$\frac{U_t}{B_t} = \int_0^t \frac{V_s}{B_s} \, (k_s - r_s)_+ \, ds \geq 0 \quad \mathbb{P}\text{-a.s.}$$

Consequence from Absence of Arbitrage: Proof

- Absence of arbitrage then gives

$$\mathbb{E}^{\mathbb{P}}\left[\frac{U_t}{B_t}\right] = \mathbb{E}^{\mathbb{P}}\left[\int_0^t \frac{V_s}{B_s}(k_s - r_s)_+ \, ds\right] = 0$$

which implies that $k_t \leq r_t, \quad d\mathbb{P} \otimes dt\text{-a.e.}$

- On the other hand, we can choose $\varphi_t = -\mathbb{1}_{\{k_t < r_t\}}$. In that case we get

$$\mathbb{E}^{\mathbb{P}}\left[\frac{U_t}{B_t}\right] = \mathbb{E}^{\mathbb{P}}\left[\int_0^t \frac{V_s}{B_s}(r_s - k_s)_+ \, ds\right] = 0$$

which implies that $k_t \geq r_t, \quad d\mathbb{P} \otimes dt\text{-a.e.}$

Deriving the Term-Structure Equation (continued)

- Let us get back to our problem. We have just showed that

$$\frac{\beta_S(t,r_t)\alpha_T(t,r_t) - \beta_T(t,r_t)\alpha_S(t,r_t)}{\beta_S(t,r_t) - \beta_T(t,r_t)} = r_t, \quad d\mathbb{P}\otimes dt\text{-a.e.}$$

- This can be rewritten as

$$\frac{\alpha_T - r_t}{\beta_T} = \frac{\alpha_S - r_t}{\beta_S}, \quad d\mathbb{P}\otimes dt\text{-a.e.}$$

- Now note that the left-hand side of the above relation does not depend on S, whereas the right-hand side does not depend on T.

- We then have the following result: if the market consisting of B_t and $\{p(\cdot,T)\}_{T>0}$ is free of arbitrage, then there exists a process $\lambda = \lambda(t,r_t)$, called market price of risk, such that

$$\lambda(t,r_t) = \frac{\alpha_T(t,r_t) - r_t}{\beta_T(t,r_t)}, \text{ for every } T > 0. \tag{2}$$

Deriving the Term-Structure Equation (continued)

- Plugging expressions for α_T and β_T into (2), we obtain the fundamental term structure equation

$$\frac{\partial F_T}{\partial t} + (\mu - \lambda\sigma)\frac{\partial F_T}{\partial r} + \frac{1}{2}\sigma^2\frac{\partial^2 F_T}{\partial r^2} - rF_T = 0, \quad F_T(T,r) = 1. \tag{3}$$

- Now define a measure $\mathbb{Q} \sim \mathbb{P}$ such that short rate dynamics become

$$dr_t = [\mu(t,r_t) - \lambda(t,r_t)\sigma(t,r_t)]\, dt + \sigma(t,r_t)\, dW_t^{\mathbb{Q}}.$$

- Consider a discounted price process $X_t = \frac{F_T}{B_t}$. We have under $\mathbb{Q}$

$$\begin{aligned} dX_t &= \frac{1}{B_t}\left[\frac{\partial F_T}{\partial t} + (\mu - \lambda\sigma)\frac{\partial F_T}{\partial r} + \frac{1}{2}\sigma^2\frac{\partial^2 F_T}{\partial r^2} - rF_T\right] dt + \frac{1}{B_t}\frac{\partial F_T}{\partial r}\sigma\, dW^{\mathbb{Q}} \\ &= \frac{1}{B_t}\frac{\partial F_T}{\partial r}\sigma\, dW^{\mathbb{Q}}. \end{aligned}$$

Risk-Neutral Measure and Market Price of Risk

- Assuming that process $\frac{1}{B_t}\frac{\partial F_T}{\partial r}\sigma$ is regular enough, the discounted price of a T-bond is a $\mathbb{Q}$-martingale and hence $\mathbb{Q}$ is a risk-neutral measure and

$$F_T(t, r_t) = B_t\,\mathbb{E}_t^{\mathbb{Q}}\left[\frac{F_T(T, r_T)}{B_T}\right] = \mathbb{E}_t^{\mathbb{Q}}\left[\mathrm{e}^{-\int_t^T r_s\,ds}\right]. \tag{4}$$

- We see that in the bond market there exist infinitely many risk-neutral measures $\mathbb{Q}$ with different measures corresponding to different choices of market price of risk process $\lambda(t, r_t)$.

- The fact that market price of risk λ is not uniquely determined means that there are different possible bond markets all of which are arbitrage-free and consistent with the given $\mathbb{P}$-dynamics of the short rate r. So how do we choose λ?

Risk-Neutral Measure and Market Price of Risk (continued)

- If there is a benchmark T-bond which price process F_T is given a priori, we can infer λ via (2) and then determine S-bond price for any maturity S in terms of F_T and the short rate r by solving the corresponding term structure equation.
- In the real market there already exist traded bonds which essentially means that the market has already chosen some risk-neutral martingale measure $\mathbb{Q}$.
- Given this risk-neutral measure $\mathbb{Q}$, the prices of zero-coupon bonds are uniquely determined by (4).
- Note that we do not need to know μ and λ per se. Collecting observable prices from the market, we can infer the implied risk-neutral drift $\mu - \lambda\sigma$ of the short-rate and this is what we really need due to (3).

Model Calibration

- It should be now clear that it is more convenient to model the short rate dynamics directly under risk-neutral measure $\mathbb{Q}$

$$dr_t = \mu(t, r_t; \gamma)\, dt + \sigma(t, r_t; \gamma)\, dW_t^{\mathbb{Q}}, \tag{5}$$

  where γ is a vector of model parameters and μ now denotes drift under $\mathbb{Q}$.
- Let's assume that at $t = 0$ we observe zero-coupon bond prices for maturities $T_1, \ldots, T_n$. We denote this empirical term structure by $\{p^*(0, T_i)\}_{i=1}^n$.
- Given a particular model (5), we can solve the term structure equation (3) for every parameter $T_1, \ldots, T_n$ to get the theoretical prices $\{F_{T_i}(0, r_0; \gamma)\}_{i=1}^n$.
- Now calibrate parameter vector γ such that theoretical prices $\{F_{T_i}(0, r_0; \gamma)\}_{i=1}^n$ fit empirical prices $\{p^*(0, T_i)\}_{i=1}^n$ as well as possible according to some objective function. This gives the implied model parameter vector γ^*.

Vasiček Model

- Vasiček model specifies that $\mathbb{Q}$-dynamics of the short rate is given by
$$dr_t = \kappa(\theta - r_t)\,dt + \sigma\,dW_t^{\mathbb{Q}}, \tag{6}$$
where parameters κ, θ, σ as well as initial short rate r_0 are positive constants.
- It can be easily verified by using Itô formula that the unique strong solution to linear SDE (6) is a Gaussian process
$$r_t = r_0 \mathrm{e}^{-\kappa t} + \theta(1 - \mathrm{e}^{-\kappa t}) + \sigma \int_0^t \mathrm{e}^{-\kappa(t-s)}\,dW_s^{\mathbb{Q}}. \tag{7}$$
- Note that
$$\mathbb{E}^{\mathbb{Q}}[r_t] = r_0 \mathrm{e}^{-\kappa t} + \theta(1 - \mathrm{e}^{-\kappa t}), \quad \mathrm{Var}[r_t] = \frac{\sigma^2}{2\kappa}\left(1 - \mathrm{e}^{-2\kappa t}\right),$$
and hence
$$\lim_{t\to\infty} \mathbb{E}^{\mathbb{Q}}[r_t] = \theta, \quad \lim_{t\to\infty} \mathrm{Var}[r_t] = \frac{\sigma^2}{2\kappa}.$$

Bond Prices Under Vasiček Model

- Our aim now is to compute T-bond price $p(t,T)$ under Vasiček Model. Recall that

$$p(t,T) = \mathbb{E}_t^{\mathbb{Q}}\left[\mathrm{e}^{-\int_t^T r_s\,ds}\right].$$

- Let us denote $\int_t^T r_s\,ds$ by $R(t,T)$. We have

$$R(t,T) = r_0 \int_t^T \mathrm{e}^{-\kappa s}\,ds + \theta \int_t^T (1-\mathrm{e}^{-\kappa s})\,ds + \sigma \int_t^T \int_0^s \mathrm{e}^{-\kappa(s-u)}\,dW_u^{\mathbb{Q}}\,ds.$$

- We then use stochastic Fubini theorem to change the order of integration in the double integral according to

$$\int_t^T \int_0^s (\cdot)\,dW_u^{\mathbb{Q}}\,ds = \int_0^t \int_t^T (\cdot)\,ds\,dW_u^{\mathbb{Q}} + \int_t^T \int_u^T (\cdot)\,ds\,dW_u^{\mathbb{Q}}.$$

Bond Prices Under Vasiček Model (continued)

- We get

$$R(t,T) = r_0 \int_t^T \mathrm{e}^{-\kappa s}\, ds + \theta \int_t^T (1 - \mathrm{e}^{-\kappa s})\, ds$$
$$+ \sigma \int_0^t \int_t^T \mathrm{e}^{-\kappa(s-u)}\, ds\, dW_u^{\mathbb{Q}} + \sigma \int_t^T \int_u^T \mathrm{e}^{-\kappa(s-u)}\, ds\, dW_u^{\mathbb{Q}}$$

- After some manipulations, we obtain

$$\begin{aligned} R(t,T) &= B(T-t) \cdot \left(r_0 \mathrm{e}^{-\kappa t} + \theta(1 - \mathrm{e}^{-\kappa t}) + \sigma \int_0^t \mathrm{e}^{-\kappa(t-s)}\, dW_s^{\mathbb{Q}} \right) \\ &\quad + \kappa\theta \int_t^T B(T-u)\, du + \sigma \int_t^T B(T-u)\, dW_u^{\mathbb{Q}} \\ &= B(T-t) r_t + \kappa\theta \int_t^T B(T-s)\, ds + \sigma \int_t^T B(T-s)\, dW_s^{\mathbb{Q}}, \end{aligned} \tag{8}$$

where $B(t) = \frac{1}{\kappa}(1 - \mathrm{e}^{-\kappa t})$.

Bond Prices Under Vasiček Model (continued)

- We note that $R(t,T)$ conditional on $\mathcal{F}_t$ is normally distributed with

$$\mathbb{E}_t^{\mathbb{Q}}\left[R(t,T)\right] = B(T-t)r_t + \kappa\theta \int_0^{T-t} B(s)\,ds, \quad \mathrm{Var}\left[R(t,T)\right] = \sigma^2 \int_0^{T-t} B^2(s)\,ds.$$

- Recall that for a standard normal random variable z we have $\mathbb{E}\left[\mathrm{e}^{a+bz}\right] = \mathrm{e}^{a+\frac{1}{2}b^2}$. We then get

$$p(t,T) = \mathbb{E}_t^{\mathbb{Q}}\left[\mathrm{e}^{-R(t,T)}\right] = \mathrm{e}^{A(T-t)-B(T-t)r_t}, \tag{9}$$

where $A(t) = \int_0^t \left[\frac{1}{2}\sigma^2 B^2(s) - \kappa\theta B(s)\right] ds$.

- The great thing is that T-bond price is given in a closed-form so there is no need to solve the term structure equation (3).

Affine Term Structure

- Formula (9) suggests the following definition. We say that the term structure of zero-coupon prices $\{p(t,T),\ T>0,\ 0\le t\le T\}$ is affine if

$$p(t,T)=\mathrm{e}^{A(t,T)-B(t,T)r_t} \tag{10}$$

  for some deterministic functions A and B. Note that $A(T,T)=B(T,T)=0$.

- Question: for which choices of $\mu(t,r)$ and $\sigma(t,r)$ the $\mathbb{Q}$-dynamics of the sort rate

$$dr_t=\mu(t,r_t)\,dt+\sigma(t,r_t)\,dW_t^{\mathbb{Q}}, \tag{11}$$

  give an affine term structure (ATS)?

- Plugging (10) into the term structure equation (3), we see that model (11) provides an ATS if and only if

$$\frac{\partial A(t,T)}{\partial t}-\left(1+\frac{\partial B(t,T)}{\partial t}\right)r-\mu(t,r)B(t,T)+\frac{1}{2}\sigma^2(t,r)B^2(t,T)=0. \tag{12}$$

Affine Term Structure (continued)

- The form of equation (12) suggests to assume that both μ and σ are linear functions of r

$$\begin{aligned} \mu(t,r) &= a(t)r + b(t), \\ \sigma^2(t,r) &= c(t)r + d(t), \end{aligned}$$

  where we assume that a, b, c, d are continuous functions of t.

- Inserting the above expressions for μ and σ in (12), we get

$$\begin{aligned} &\frac{\partial A(t,T)}{\partial t} - b(t)B(t,T) + \frac{1}{2}d(t)B^2(t,T) \\ &- \left(1 + \frac{\partial B(t,T)}{\partial t} + a(t)B(t,T) - \frac{1}{2}c(t)B^2(t,T)\right) r = 0. \end{aligned} \tag{13}$$

Affine Term Structure (continued)

- Given that equation (13) should hold for all (relevant) t, T, r, we get the following proposition.
  Assume that

$$\mu(t,r) = a(t)r + b(t), \quad \sigma^2(t,r) = c(t)r + d(t). \tag{14}$$

  Then model (11) admits an ATS (10), where A and B solve the system of equations

$$\begin{cases} 1 + \frac{\partial B(t,T)}{\partial t} + a(t)B(t,T) - \frac{1}{2}c(t)B^2(t,T) = 0, \\ B(T,T) = 0. \end{cases} \tag{15}$$

$$\begin{cases} \frac{\partial A(t,T)}{\partial t} - b(t)B(t,T) + \frac{1}{2}d(t)B^2(t,T) = 0, \\ A(T,T) = 0. \end{cases} \tag{16}$$

Cox–Ingersoll–Ross Short-Rate Model

- Cox–Ingersoll–Ross (CIR) model specifies that $\mathbb{Q}$-dynamics of the short rate is given by

$$dr_t = \kappa(\theta - r_t)\,dt + \sigma\sqrt{r_t}\,dW_t^{\mathbb{Q}}, \tag{17}$$

  where parameters κ, θ, σ as well as initial short rate r_0 are positive constants.

- Standard existence theorem cannot be applied due to $\sigma\sqrt{r_t}$ diffusion term. However, it can be shown that the unique strong non-negative solution exists. Moreover, if $2\kappa\theta \geq \sigma^2$, the solution r will be a positive process. Note that there is no closed form for r.

- Given that coefficients in (17) satisfy (14), the model admits an ATS (and hence zero-coupon bond prices are given in closed-form) where A and B solve the corresponding Riccati equations (15)-(16).