INSTITUTE AND FACULTY OF ACTUARIES

EXAMINERS' REPORT

September 2017

Subject CT1 – Financial Mathematics
Core Technical

Introduction

The Examiners' Report is written by the Principal Examiner with the aim of helping candidates, both those who are sitting the examination for the first time and using past papers as a revision aid and also those who have previously failed the subject.

The Examiners are charged by Council with examining the published syllabus. The Examiners have access to the Core Reading, which is designed to interpret the syllabus, and will generally base questions around it but are not required to examine the content of Core Reading specifically or exclusively.

For numerical questions the Examiners' preferred approach to the solution is reproduced in this report; other valid approaches are given appropriate credit. For essay-style questions, particularly the open-ended questions in the later subjects, the report may contain more points than the Examiners will expect from a solution that scores full marks.

The report is written based on the legislative and regulatory context pertaining to the date that the examination was set. Candidates should take into account the possibility that circumstances may have changed if using these reports for revision.

Luke Hatter
Chair of the Board of Examiners
December 2017

A. General comments on the *aims of this subject and how it is marked*

1. CT1 provides a grounding in financial mathematics and its simple applications. It introduces compound interest, the time value of money and discounted cashflow techniques which are fundamental building blocks for most actuarial work.

2. Please note that different answers may be obtained to those shown in these solutions depending on whether figures obtained from tables or from calculators are used in the calculations but candidates are not penalised for this. However, candidates may lose marks where excessive rounding has been used or where insufficient working is shown.

B. General comments on *student performance in this diet of the examination*

The comments that follow the questions concentrate on areas where candidates could have improved their performance. Where no comment is made, the question was generally answered well by most candidates.

C. Pass Mark

The Pass Mark for this exam was 60.

Solutions

Q1 (i) (a) $6,000\left(1+\frac{0.03t}{365}\right)=7,600$ [1]

$t=\left(\frac{7,600}{6,000}-1\right)\times\frac{365}{0.03}$ = 3,244.4 days [1]

(b) $6,000(1+0.03)^{t/365}=7,600$ [1]

$t\times\frac{\ln 1.03}{365}=\ln\left(\frac{7,600}{6,000}\right)=\ln 1.26667=0.23639$

$\Rightarrow t=365\times\frac{0.23639}{\ln 1.03}$ = 2,919.0 days [1]

(c) $6,000e^{0.03t/365}=7,600$ [1]

$\Rightarrow t=\frac{365}{0.03}\ln\left(\frac{7,600}{6,000}\right)$ = 2,876.1 days [1]

(ii) Effective interest rate per half year is $\frac{i^{(2)}}{2}$ where

$\left(1+\frac{i^{(2)}}{2}\right)=e^{\delta/2}=e^{0.015}=1.0151131\Rightarrow\frac{i^{(2)}}{2}=1.51131\%$ [1]

[Total 7]

Well answered although some candidates gave $i^{(2)}$ as their final answer to part (ii).

Q2 One party agrees to pay to the other a regular series of fixed amounts… [½]
…for a certain/given term. [½]
In exchange, the second party agrees to pay a series of variable amounts [½]
…based on the level of a short-term interest rate. [½]
[Total 2]

The worst-answered question on the paper even though the above comes directly from the Core Reading.

Q3 Let *d* be the annual simple rate of discount.

The discounted value of 100 in the deposit account would be *X* such that:

$X = 100(1.03)^{-91/365} = 99.26576$ [1]

To provide the same effective rate of return a government bill that pays 100 must have a price of 99.26576 and so $100\left(1-\dfrac{91d}{365}\right) = 99.26576$

$d = \dfrac{365}{91}(1-0.9926576) = 0.029450$ [2]

[Total 3]

There was a potential ambiguity with this question in that the term of the government bill was not separately stated. Most students assumed the term of the bill was also 91 days as the examiners intended but candidates who assumed another term were also given credit.

Q4 Assuming no arbitrage: [1]

Present value of dividends

$= 0.10v_{5\%}^{0.5} + 0.10v_{6\%} = 0.1 \times 0.97590 + 0.1 \times 0.94340 = 0.19193$ [2]

Forward price $= (4 - 0.19193) \times 1.06 = \$4.03655$ [1]

[Total 4]

No comments.

Q5 (i) Let S_{20} = Accumulated value at time 20 of £1 invested at time 0

then $E[S_{20}] = (1+j)^{20}$

$E[100S_{20}] = 100E[S_{20}] = 200 \Rightarrow E[S_{20}] = 2$

$(1+j)^{20} = 2 \Rightarrow j = 0.035265$ [1]

(ii) Let s be the standard deviation of the annual effective rate of return.

$\text{Var}[100S_{20}] = 50^2$

$10{,}000\text{Var}[S_{20}] = 2{,}500 \Rightarrow \text{Var}[S_{20}] = 0.25$ [1]

$$\text{Var}\left[S_{20}\right]=\left((1+j)^2+s^2\right)^{20}-E\left[S_{20}\right]^2$$ [1]

$$0.25=\left(2^{1/10}+s^2\right)^{20}-2^2$$

$$\Rightarrow s^2=\left(0.25+2^2\right)^{1/20}-2^{1/10}=0.00325372$$

$$\Rightarrow s=0.057041$$ [2]

[Total 5]

> *Part (i) was well answered although many candidates struggled with part (ii). The above solution uses the formulae developed in the core reading in the case where the returns in each year are assumed to be independent and identically distributed although these assumptions are not necessary for the calculation of the above answer.*

Q6 Accumulated amount from Fund A

$$=12\times 100\ddot{s}^{(12)}_{\overline{15}|3\%}=1,200\frac{1.03^{15}-1}{12\left(1-1.03^{-1/12}\right)}$$

$$=\$22,679.74$$

[2]

Accumulated amount from Fund B

$$=12\times 100\ddot{s}^{(12)}_{\overline{15}|3.7\%}-12\times 15\ddot{s}^{(12)}_{\overline{1}|3.7\%}\left(1.037\right)^{14}$$

$$=1,200\frac{1.037^{15}-1}{12\left(1-1.037^{-1/12}\right)}-180\frac{1.037-1}{12\left(1-1.037^{-1/12}\right)}\left(1.037\right)^{14}$$

$$=23,967.992-305.313=\$23,662.68$$

[3]

The percentage by which B is greater is found from

$$\frac{23,662.68-22,679.74}{22,679.74}-1=4.33\%$$ [1]

> *A comparatively straightforward question that was poorly done by marginal candidates.*

Q7 (i) Let P be the price per £100 nominal.

$P = 0.8 \times 5 a_{\overline{2}|}^{(2)} + 110 v^2$ with a gross redemption yield of 4% per annum. [1]

$$\Rightarrow P = 0.8 \times 5 \frac{1-1.04^{-2}}{2\left(1.04^{\frac{1}{2}}-1\right)} + 110 \times 1.04^{-2}$$

$\Rightarrow P = 4 \times 1.904771 + 110 \times 0.924556 = £109.320$ [2]

(ii)

Time t	Government bond spot rate $y_t + 1\%$	Present value factor	Payment	Present value of payment
0.5	0.0175	0.99136	2	1.9827
1	0.025	0.97561	2	1.9512
1.5	0.0325	0.95316	2	1.9063
2	0.04	0.92456	112	103.5503

Total present value = £109.391 per £100 nominal [3]

(iii) Forward rate $= \dfrac{(1+y_2)^2}{1+y_1} - 1 = \dfrac{1.03^2}{1.015} - 1 = 0.04522$ [2]

(iv) It may be because interest rates are expected to rise in the future and the yield curve is determined by expectations theory.

And/or because investors might expect inflation to rise leading to expectations of higher interest rates over the longer term.

And/or because investors have a preference for liquidity which puts an upwards bias on the yield curve. A rising curve would be compatible, for example, with constant expectations of interest rates.

And/or because the market segmentation theory holds and short-term bonds might be in demand by investors such as banks.

[1½ each point, maximum 3]
[Total 11]

Marginal candidates struggled with this question with a common error in part (ii) being to assume coupons were annual (which simplified the question considerably).
Part (iv) was poorly answered. Explanations of why the yield curve would be the given shape were required. It was not sufficient just to name the various theories of the yield curve.

Q8 (i) Amount of loan is $50\left(Ia\right)_{\overline{10|}}+50a_{\overline{10|}}$ at 5% per annum effective [1]

$=50\times 39.3738+50\times 7.7217$

= 1968.69 + 386.09 = £2,354.78 [1]

(ii) (a) The outstanding loan after fifth instalment is:

$50\left(Ia\right)_{\overline{5|}}+300a_{\overline{5|}}$ [1]

$=628.32+1,298.85$ = £1,927.17 [1]

The interest component is therefore $0.05\times 1,927.17=£96.36$ [1]

(b) The capital component = 350 - 96.36 = £253.64 [1]

(iii) The capital remaining after the sixth instalment is
1,927.17 – 253.64 = £1,673.53 [1]

Let the new instalment = *X*

$Xa_{\overline{4|}6\%}=1,673.53$

$X=\dfrac{1,673.53}{3.4651}=£482.96$ [2]

[Total 9]

The best answered question on the paper (excluding Q1)

Q9 (i) $A\left(0,10\right)=\exp\int_0^{10}0.09-0.003s\,ds$

$=\exp\left[0.09s-0.0015s^2\right]_0^{10}=\exp\left(0.9-0.15\right)=e^{0.75}$=2.1170 [3]

Require *i* where $\left(1+i\right)^{10}=2.1170\Rightarrow i=0.077884$ [1]

(ii) $d^{(2)}=2\left(1-\left(1+i\right)^{-1/2}\right)=0.073611$ [1]

(iii) $A(5,10) = \exp \int_5^{10} 0.09 - 0.003s\, ds$

$= \exp\left[0.09s - 0.0015s^2\right]_5^{10} = \exp(0.75 - 0.45 + 0.0375) = e^{0.3375}$

$A(10,15) = e^{5\times 0.06} = e^{0.3}$

$A(5,15) = A(5,10)\,A(10,15) = e^{0.6375} = 1.89175$

Accumulated amount = 1,500 $e^{0.6375}$ =£2,837.62 [3]

(iv) Equivalent annual effective rate of discount is d such that
$(1-d)^{-10} = e^{0.6375} \Rightarrow d = 0.061760$ [1]

(v) For t > 10,

$v(t) = v(10)\exp\left[-\int_{10}^{t} 0.06\, ds\right]$

$= e^{-0.75}\exp\left[-0.06s\right]_{10}^{t}$

$= e^{-0.75}\exp\left[-0.06t + 0.6\right] = e^{-0.06t - 0.15}$

[3]

Present value $= \int_{11}^{15} \rho(t)v(t)\,dt = \int_{11}^{15} 10e^{0.01t}e^{-0.06t-0.15}\,dt = \int_{11}^{15} 10e^{-0.05t-0.15}\,dt$

$= 10\left[\frac{e^{-0.05t-0.15}}{-0.05}\right]_{11}^{15} = -200\left(e^{-0.9} - e^{-0.7}\right)$

$= -81.314 + 99.317 = 18.003$

[3]

Another standard question that was well-answered.

Q10 (i) (a) Work in £ millions

PV of liabilities = $100v^{10} + 200v^{20}$ at 3% per annum

$= 100 \times 0.74409 + 200 \times 0.55368 = 185.145$ [1½]

(b) DMT of liabilities = $\dfrac{1{,}000 \times 0.74409 + 4{,}000 \times 0.55368}{185.145} = \dfrac{2{,}958.797}{185.145}$
=15.981 years [2½]

(ii) PV of assets = $144.054v^{15} + Xa_{\overline{t}|}$ where t is the term of the annuity and X is the annual payment.

So $Xa_{\overline{t}|} = 185.145 - 144.054v^{15} = 185.145 - 144.054 \times 0.64186 = 92.682$ for first condition to be satisfied. [1]

DMT of assets = $\dfrac{144.054 \times 15 \times 0.64186 + X\left(Ia\right)_{\overline{t}|}}{185.145}$ = 15.981 years [2½]

So $X\left(Ia\right)_{\overline{t}|} = 2{,}958.797 - 144.054 \times 15 \times 0.64186 = 1{,}571.859$ for second condition to be satisfied. [1]

Thus $\dfrac{X\left(Ia\right)_{\overline{t}|}}{Xa_{\overline{t}|}} = \dfrac{1{,}571.859}{92.682} \Rightarrow \dfrac{\left(Ia\right)_{\overline{t}|}}{a_{\overline{t}|}} = 16.960$

From inspection of tables, t = 41 years. [1½]

(iii) $Xa_{\overline{41}|} = 92.682 \Rightarrow X = £3.95865m$ [1]

(iv) Redington's third condition requires that the convexity or spread of the terms of the asset proceeds around the discounted mean term is greater than that for the liabilities.
It is likely that this is the case given that the asset proceeds consist in part of an annuity of term 41 years (though not certain). [2]

(v) If the insurance company sells the security and buys one with a shorter term, the discounted mean term of its assets will no longer be equal to that of its liabilities (it will be shorter). This will mean that, if interest rates were to fall, the insurance company would make a loss. [2]

[Total 15]

Part (i) was answered well. In a 'Show that…' question as in part (ii), it is important to show steps clearly. Many marginal candidates did not do this or, more seriously, appeared to claim that incorrect workings led to the required final answer.
Part (v) was answered very poorly with few candidates explaining the precise scenario where a loss would be made.

Q11 (i) $PV_A = 10{,}000\ddot{a}_{\overline{1}|}^{(12)} + 10{,}000 \times 1.05v \times \ddot{a}_{\overline{1}|}^{(12)} + 10{,}000 \times 1.05^2 v^2 \times \ddot{a}_{\overline{1}|}^{(12)}$ [2]

$$= 10,000\ddot{a}_{\overline{1}|}^{(12)}\left(1+1.05v+(1.05v)^2\right)$$

$$= 10,000\ddot{a}_{\overline{1}|}^{(12)}\frac{1-(1.05v)^3}{1-1.05v}$$

$$= 10,000\frac{1-v}{d^{(12)}}\frac{1-(1.05v)^3}{1-1.05v}$$

$$= 10,000\times 0.986579\times 3.058629$$

$$= £30,176$$

[2]

[or from 2nd line in 1 above:

$$= 10,000\frac{1-v}{d^{(12)}}\times(1+1.019417+1.039212)$$

$$= 10,000\times 0.986579\times 3.058629$$

$$= £30,176]$$

(ii) $$PV_B(1+i)^6 = 1,300\ddot{a}_{\overline{45}|}^{(4)}+200\left(\ddot{a}_{\overline{45}|}^{(4)}-\ddot{a}_{\overline{15}|}^{(4)}\right)+300\left(\ddot{a}_{\overline{45}|}^{(4)}-\ddot{a}_{\overline{30}|}^{(4)}\right)$$ [2]

$$PV_B = v^6\left[1,800\ddot{a}_{\overline{45}|}^{(4)}-200\ddot{a}_{\overline{15}|}^{(4)}-300\ddot{a}_{\overline{30}|}^{(4)}\right]$$

$$\Rightarrow PV_B = 1.03^{-6}\frac{\left[1,800\left(1-v^{45}\right)-200\left(1-v^{15}\right)-300\left(1-v^{30}\right)\right]}{4\left(1-1.03^{-1/4}\right)}$$

$$\Rightarrow PV_B = 0.837484\times\frac{1,800\times 0.735561-200\times 0.358138-300\times 0.588013}{0.0294499} = £30,598$$

[3]

[or

$$PV_B(1+i)^6 = 1,300\ddot{a}_{\overline{15}|}^{(4)}+1,500v^{15}\ddot{a}_{\overline{15}|}^{(4)}+1,800v^{30}\ddot{a}_{\overline{15}|}^{(4)}$$

$$= \ddot{a}_{\overline{15}|}^{(4)}\left(1,300+1,500v^{15}+1,800v^{30}\right)$$

$$= \frac{\left(1-v^{15}\right)}{4\left(1-1.03^{-1/4}\right)}\times(1,300+1,500\times 0.641862+1,800\times 0.411987)$$

$$= \frac{0.358138}{0.0294499}\times 3,004.3696$$

$$\Rightarrow PV_B = 0.837484\times 36,535.91 = £30,598]$$

(iii) Option A has the lower present value out of A and B. Therefore, the student has to calculate the salary level so that $PV_C = 30{,}176$ [1]

Let the initial salary level in relation to option C be S_C

$$30{,}176 = 0.03S_C v^3\left(v + 1.03v^2 + \ldots + 1.03^9 v^{10}\right) + 0.03S_C 1.03^{10} v^{18}\left(v + 1.01v^2 + \ldots + 1.01^{29} v^{30}\right)$$

[3]

$$= 0.03S_C v^4\left(10 + 1.03^{10} v^{15}\left(1 + 1.01v \; + \ldots + 1.01^{29} v^{29}\right)\right)$$

$$= 0.03S_C v^4\left(10 + v^5\,\frac{1 - 1.01^{30} v^{30}}{1 - 1.01v}\right)$$

$$= 0.03S_C 1.03^{-4}\left(10 + 0.862609 \times 22.90226\right) = 0.79313S_C$$

$$\Rightarrow S_C = £38{,}047$$ [3]

Therefore, the starting salary has to be less than £38,047 for option C to have the lowest net present value. [1]

(iv) The risks to the students of the three options are very different. For example, the payments under option C vary with salaries and probably with general inflation and the time spent out of the labour market, whereas under options A and B payments are fixed. Therefore, it does not seem reasonable to use the same interest rate (and therefore risk premium) to evaluate all three options. [2]

(v) Possible risks could be:

Student defaults on loan payments (for those that choose option B)

Student salaries are less than the university expects (for those that choose option C) – this could include lower than expected general inflation

Salary earning periods being shorter than expected (for those that choose option C) e.g. because of periods of maternity/paternity leave.

Mortality risk: e.g. under options B and C, if mortality were higher than expected, payments received would be lower than expected.

Students select against the university with those expecting low salaries or poor employment prospects choosing C and those expecting high salaries choosing options A or B.

Students choosing option C artificially restrict official salary (e.g. 'cash-in-hand', payment via dividends, working abroad). [1½ each point, maximum 4]

[Total 23]

There was an ambiguity in part (iii) where the examiners intended for the maximum initial level of salary to be given as the answer. All marginal candidates appeared to read this part as the examiners had intended.

Parts (i) and (ii) were answered well but the later parts were answered poorly, possibly as a result of time pressure. Parts (iv) and (v) did not *require reference to the earlier calculations but were still not answered well by marginal candidates.*

END OF EXAMINERS' REPORT