Applied Mathematical Finance I

Lecture 4: Vanilla Interest Rate Derivatives

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Setup

- Consider a market consisting of a money-market account $B_t = \mathrm{e}^{\int_0^t r_s\, ds}$ and a family of zero-coupon bonds $\{p(\cdot, T)\}_{T \geq 0}$. We assume that there exists a risk-neutral measure i.e. a probability measure $\mathbb{Q} \sim \mathbb{P}$ such that, for any $T > 0$,

$$\frac{p(t,T)}{B_t}, \quad t \in [0, T],$$

  is a $\mathbb{Q}-$martingale.

- Arbitrage-free price at t of a derivative contract paying $\mathcal{X}_T$ at $T \geq t$ is given by

$$\mathrm{PV}_t = B_t \cdot \mathbb{E}_t^{\mathbb{Q}}\left[\frac{\mathcal{X}_T}{B_T}\right] = \mathbb{E}_t^{\mathbb{Q}}\left[\mathrm{e}^{-\int_t^T r(s)\, ds} \cdot \mathcal{X}_T\right]. \tag{1}$$

- Choosing T-bond as a numéraire, we can switch to a T-forward measure $\mathbb{Q}^T$

$$\mathrm{PV}_t = p(t,T) \cdot \mathbb{E}_t^T\left[\mathcal{X}_T\right]. \tag{2}$$

Forward Rate Agreement (FRA)

- Forward Rate Agreement (FRA) is an OTC market contract to exchange future spot rate $L(T, T+\tau)$ for a fixed rate K agreed at inception date $t < T$.
- Recall that $L(T, T+\tau)$ is the risk-free rate of return over period $[T, T+\tau]$.
- The market practice for FRA is not to wait until $T+\tau$ but settle the contract at time T when spot rate $L(T, T+\tau)$ is observed. Assuming the unit notional, the actual payment amount (from the viewpoint of a payer of the fixed rate) is then given by theoretical payoff discounted to time T

$$\frac{(L(T, T+\tau) - K)\tau}{1 + \tau L(T, T+\tau)}.$$

- The fixed rate K which sets the contract value to zero at inception is called the FRA par rate.

FRA Par Rate

- By the fundamental pricing equation, we have under risk-neutral measure $\mathbb{Q}$

$$\mathrm{PV}_t^{\mathrm{FRA}} = B_t \cdot \mathbb{E}_t^{\mathbb{Q}}\left[\frac{(L(T,T+\tau)-K)\tau}{1+\tau L(T,T+\tau)} \cdot \frac{1}{B_T}\right].$$

- Given the connection between $L(T,T+\tau)$ and $p(T,T+\tau)$

$$\frac{1}{p(t,T)} = 1+\tau L(t,T),$$

we obtain

$$\begin{aligned}\mathrm{PV}_t^{\mathrm{FRA}} &= B_t \cdot \mathbb{E}_t^{\mathbb{Q}}\left[\frac{1-(1+K\tau)\cdot p(T,T+\tau)}{B_T}\right] \\ &= p(t,T) - (1+K\tau)\cdot p(t,T+\tau).\end{aligned}$$

- Therefore

$$\mathrm{PV}_t^{\mathrm{FRA}} = 0 \iff K = \frac{1}{\tau}\left[\frac{p(t,T)}{p(t,T+\tau)} - 1\right] = F(t,T,T+\tau). \tag{3}$$

More About FRAs

- LIBOR forward rates are break-even rates for FRA contracts.
- Note that static replicating portfolio for FRA contract with strike price K consists of one T-bond bought and $(1 + K\tau)$ units of $(T + \tau)$-bond sold. This result is model independent.
- Could we consider a simpler linear payoff $(L(T, T + \tau) - K)\tau$ at T?
- We actually could, but this turns out to be a much harder case. The thing is that this payoff cannot be statically replicated with zero-coupon bonds.
- On the other hand, it can be replicated by either dynamically trading zero-coupon bonds (this require specifying a model) or with a static portfolio of vanilla options (more on this later).

Futures Contract

- FRAs are forward contracts on LIBOR that allow OTC market participants (institutions) to lock in interest rate for future period.
- In some sense, futures contracts on LIBOR are similar to FRAs but traded on exchanges.
- Futures contracts have the following properties
  - Futures contract costs zero: it can be entered at no cost at any time t.
  - Marking to market: there exists a futures price $f(t, T, T+\tau)$ such that futures contract pays $f(t+dt, T, T+\tau) - f(t, T, T+\tau)$ in the interval $[t, t+dt]$.
  - At maturity, futures price converges to spot rate: $f(T, T, T+\tau) = L(T, T+\tau)$.

Futures Rate

- Consider trading in a futures market. Let $\phi = (\phi^1, \phi^2)^\intercal$ be a self-financing trading strategy where ϕ^1 corresponds to the money market account B and ϕ^2 is the number of futures contracts with futures price f.
- Since it cost nothing to enter a futures contract, the value of the trading strategy ϕ is simply $V_t = \phi_t^1 B_t$.
- We have under risk neutral measure $\mathbb{Q}$

$$d\frac{V_t}{B_t} = \frac{\phi_t^2}{B_t} df_t$$

  meaning that f is a $\mathbb{Q}$-martingale.
- Futures convexity adjustment is

$$f(t,T,T+\tau) - F(t,T,T+\tau) = -\frac{1}{p(t,T+\tau)} \operatorname{cov}\left[L(T,T+\tau), \mathrm{e}^{-\int_t^{T+\tau} r_s\, ds}\right].$$

Swaps

- An interest rate swap is an agreement between two counterparties to exchange one stream of cashflows for another one based on a specific principal amount (notional). These streams are called the legs of the swap.
- Some common types of swaps are
  - Plain Swaps: One counterparty pays fixed rate and the other one makes payments linked to some floating rate of the same currency.
  - Basis Swaps: Both counterparties make floating rate payments of the same currency but with different frequencies (e.g. 3M vs 6M).
  - Cross-Currency Swaps: One counterparty pays fixed rate and the other one makes payments linked to some floating rate of a different currency. There is usually exchange of notionals at the beginning and the end of a swap.
  - Cross-Currency Basis Swaps: Both counterparties make floating rate payments of different currencies. Again, there is usually exchange of notionals at the beginning and the end of a swap.

Plain Fix-for-Floating Swaps

- Swap schedule is given by a tenor structure i.e. an increasing sequence of times

$$0 \leq T_0 < T_1 < \cdots < T_n, \quad T_i - T_{i-1} = \tau_i, \quad i \in \{1, \ldots, n\}.$$

  For simplicity, we will assume that the schedule is the same for both legs. Here T_0 is called the start of the swap and T_n is the maturity.

- For each of the periods $[T_{i-1}, T_i]$, $i \in \{1, \ldots, n\}$, payer of the fixed rate makes simple interest payments based on the pre-agreed rate K, in return for simple interest payments linked to the LIBOR fixing $L(T_{i-1}, T_i)$ observed at T_{i-1}. The payments are exchanged* at the end of each period i.e. at T_i.

*In practice, the payments are netted meaning that only their difference is actually paid.

Pricing Fix-for-Floating Swap

- By applying (2) to each cash flow individually, the present value of a swap (with unit notional) at time $t \in [0, T_0]$ can be expressed as

$$\begin{aligned} \mathrm{PV}_t^{\mathrm{Swap}} &= \sum_{i=1}^{n} p(t, T_i) \cdot \mathbb{E}_t^{T_i}\left[(L(T_{i-1}, T_i) - K)\tau_i\right] \\ &= \underbrace{\sum_{i=1}^{n} p(t, T_i) F(t, T_{i-1}, T_i)\tau_i}_{\text{Floating leg value}} - \underbrace{\sum_{i=1}^{n} p(t, T_i) K \tau_i}_{\text{Fixed leg value}} . \end{aligned} \tag{4}$$

- By convention, swaps cost zero at inception. Swap par rate, that is, value of the fixed rate K that makes swap cost zero at $t = 0$, is given by the weighted average of forward rates

$$K = \sum_{i=1}^{n} \omega_i F(0, T_{i-1}, T_i), \quad \omega_i = \frac{p(0, T_i)\tau_i}{\sum_{j=1}^{n} p(0, T_j)\tau_j}. \tag{5}$$

Annuity

- Note that formula (4) is model independent: present value only depends on the shape of the zero curve observed at t.
- We can rewrite (4) as

$$\mathrm{PV}_t^{\mathrm{Swap}} = A_t \left(R_t - R_0 \right),$$

where $A_t = \sum_{i=1}^{n} p(t, T_i)\tau_i$ - is the annuity and R_t denotes par rate as seen at t

$$R_t = \sum_{i=1}^{n} \omega_i F(t, T_{i-1}, T_i), \quad \omega_i = \frac{p(t, T_i)\tau_i}{\sum_{j=1}^{n} p(t, T_j)\tau_j}.$$

- Annuity scaled by one basis point (1bp = 0.01%) has a clear financial meaning: it is the change in present value of a swap due to 1bp change in swap par rate.
- A_t is obviously a value of a self-financing portfolio and hence can be used as a valid numéraire.

Basis swaps

- In a single-currency basis swap, two parties exchange floating rates of the same currency with different frequencies, e.g. 3M vs 6M.
- We have showed that the initial value of a floating leg is

$$\mathrm{PV}_0^{\text{Floating Leg}} = \sum_{i=1}^{n} p(0, T_i) F(0, T_{i-1}, T_i) \tau_i. \tag{6}$$

- Given that $F(0, T_{i-1}, T_i) = \frac{1}{\tau_i}\left(\frac{p(0,T_{i-1})}{p(0,T_i)} - 1\right)$, we see that (6) reduces to just

$$\mathrm{PV}_0^{\text{Floating Leg}} = p(0, T_0) - p(0, T_n). \tag{7}$$

- Therefore, initial value of a floating leg does not depend on frequency and hence, in our setup, any basis swap* automatically costs zero at inception.

*We assume that both legs have common start and end dates.

Basis Spread Explosion During the Credit Crunch

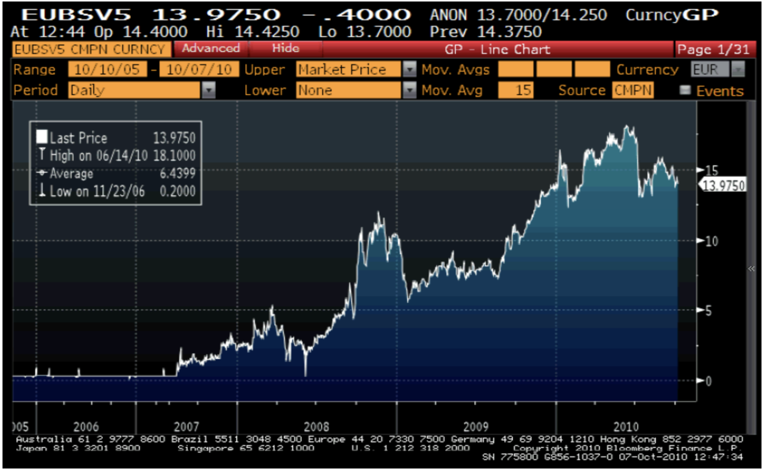

Figure: EURIBOR 3M vs 6M basis spread for a 5Y swap.

Caps & Floors

- A caplet (floorlet) is a call (put) option on a reference interest rate.
- A standard option on a future spot rate $L(T, T+\tau)$ with strike price K pays out at $T+\tau$ (unit notional is assumed)
  - $\tau(L(T, T+\tau) - K)_+$ in case of a caplet.
  - $\tau(K - L(T, T+\tau))_+$ in case of a floorlet.

  Caplets and floorlets are usually simply referred to as optionlets.
- The market, however, directly quotes series of caplets (floorlets) called caps (floors) instead of single period instruments.
- For instance, a standard 2Y cap on 3M LIBOR is a series of 7* caplets, each lasting for 3M. The price of the cap is just the sum of caplet prices.

*The first period is ignored since the corresponding LIBOR rate is already observed.

Optionlet Pricing

- By the pricing equation (2), we obtain for $t \leq T$ under forward measure $\mathbb{Q}^{T+\tau}$

$$\mathrm{PV}_t^{\mathrm{Optionlet}} = p(t, T+\tau) \cdot \mathbb{E}_t^{T+\tau}(\theta\,\tau[L(T, T+\tau) - K])_+, \tag{8}$$

  where $\theta = \pm 1$ depending on the type of the optionlet.

- Given that $L(T, T+\tau) = F(T, T, T+\tau)$ we can rewrite (8) as

$$\mathrm{PV}_t^{\mathrm{Optionlet}} = p(t, T+\tau) \cdot \mathbb{E}_t^{T+\tau}(\theta\,\tau[F(T, T, T+\tau) - K])_+. \tag{9}$$

- Recall that forward rate $F(t, T, T+\tau)$, $t \in [0, T]$, is a $\mathbb{Q}^{T+\tau}$-martingale. Assuming some appropriate dynamics of $F(t, T, T+\tau)$, we can explicitly compute optionlet price.

Black-76 Formula

- Let us assume that $F(t, T, T+\tau)$, $t \in [0, T]$, follows a geometric Brownian motion with constant volatility σ_{Black} under $\mathbb{Q}^{T+\tau}$

$$\frac{dF(t, T, T+\tau)}{F(t, T, T+\tau)} = \sigma_{\text{Black}}\, dW_t^{T+\tau},$$

  where $W^{T+\tau}$ is a Brownian motion under $\mathbb{Q}^{T+\tau}$.

- We can now easily compute expectation in (8) to get Black's formula

$$\text{PV}_t^{\text{Optionlet}} = p(t, T+\tau)\tau\,\theta\left[F\,\Phi\left(\theta\, d_+\right) - K\,\Phi\left(\theta\, d_-\right)\right], \tag{10}$$

  where $d_+ = \frac{\ln(F/K)}{\sigma_{\text{Black}}\sqrt{T-t}} + \frac{\sigma_{\text{Black}}\sqrt{T-t}}{2}$, $d_- = d_+ - \frac{\sigma_{\text{Black}}\sqrt{T-t}}{2}$, Φ - standard normal CDF.

- We price caps/floors without modelling the evolution of the entire zero curve.

Bachelier Formula

- Under some circumstances rates can become negative so it may be more appropriate to assume that $F(t, T, T+\tau)$, $t \leq T$, follows a normal process

$$dF(t, T, T+\tau) = \sigma_{\text{Bachelier}}\, dW_t^{T+\tau}.$$

- This leads to Bachelier pricing formula

$$\text{PV}_t^{\text{Optionlet}} = p(t, T+\tau)\tau\, \sigma_{\text{Bachelier}} \sqrt{T-t}\left[\theta\, d\, \Phi(\theta\, d) + \phi(d)\right],$$

where $d = \frac{F-K}{\sigma_{\text{Bachelier}}\sqrt{T-t}}$, and $\phi = \Phi'$ - standard normal PDF.

Implied Cap Volatility

- Cap volatility is a single volatility number given to all constituent caplets such that the cap market price is recovered.
- Market quotes imply different volatilities (either Black or Bachelier) for different strikes and maturities, giving the implied cap volatility surface.
- To price single caplets consistently with the market, one needs to bootstrap caplet volatilities (which are not directly observable) from the market cap volatilities.
- For example, a 1Y year cap with quarterly payments will generally have a different volatility (say, σ_1) to a 2Y cap volatility (say, σ_2) for the same strike, thus implying different volatility for the first three caplets. Given σ_1 and σ_2, and picking up an interpolation method, one can construct volatility curve $\sigma(t)$ over [1Y, 2Y] such that the 2Y cap quote is matched.

Cap Market Volatility Surface

Quote: VCAP10

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EUR Caps/Floors - Normal Volatilities

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	STK	ATM	0.00	0.13	0.25	0.38	0.50	1.00	1.50	2.00	2.50	3.00	4.00	5.00	10.0
1Y	0.08	20.20	17.7	21.6	25.2	28.8	32.1	44.7	56.5	67.9	79.2	90.3	112	133	241
18M	0.09	20.19	17.5	21.2	24.6	28.0	31.2	43.1	54.4	65.2	75.8	86.2	106	127	226
2Y	0.11	20.22	17.3	20.8	23.9	27.1	30.1	41.5	52.2	62.5	72.6	82.4	101	120	212
3Y	0.28	21.92	16.6	18.5	21.2	24.7	27.9	39.6	50.2	60.2	69.7	79.0	96.8	113	194
4Y	0.35	28.85	22.3	23.5	26.1	29.6	32.8	44.0	52.9	60.9	68.4	75.6	89.5	103	167
5Y	0.45	36.00	26.8	28.0	30.6	34.2	37.4	49.2	58.4	66.5	74.0	81.2	94.7	107	166
6Y	0.56	42.07	29.7	31.1	33.7	37.3	40.5	52.3	62.6	71.0	78.9	86.2	100	113	172
7Y	0.69	46.93	31.6	33.0	35.7	39.2	42.3	53.7	64.4	72.5	80.0	87.0	99.9	112	167
8Y	0.82	51.26	33.3	34.8	37.4	40.8	43.9	54.9	65.0	72.7	79.7	86.3	98.4	109	161
9Y	0.95	55.01	35.1	36.7	39.2	42.5	45.5	56.0	65.3	72.5	79.0	85.2	96.5	107	156
10Y	1.07	57.96	36.8	38.3	40.8	43.9	46.8	56.8	65.4	72.1	78.2	84.0	94.7	104	152
12Y	1.26	62.03						57.9	65.5	71.5	76.8	82.0	91.9	101	145
15Y	1.47	64.45						58.2	64.8	69.9	74.5	79.3	88.3	97.1	138
20Y	1.68	64.38						57.2	62.7	67.2	70.7	75.0	83.2	91.3	130
25Y	1.76	62.94						55.7	60.7	64.9	68.2	72.2	80.2	88.0	125
30Y	1.80	61.10						54.0	58.7	62.7	65.9	69.8	77.6	85.2	122

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Figure: Caps/Floors - Bachelier volatilities.

Swaptions

- A swaption is an option on a swap. The purchaser of a European swaption buys the right to enter into a fix-for-floating interest rate swap with a pre-specified fixed rate on a specific date for a specific period.
- A receiver swaption gives the holder the right to receive a fixed rate while a payer swaption gives the right to pay the fixed rate.
- Swaptions are quoted similar to forward starting swaps. For example, a 1Y10Y swaption is the option to enter into a 10Y swap in 1Y (and hence the maturity of the swaption is 1Y).

Swaption pricing

- Let $\{T_i\}_{i=0}^n$ be an increasing sequence of future times. Consider a swaption with maturity T_0 and strike K that gives the right to enter at T_0 a plain swap with fixed rate K and payments on $\{T_i\}_{i=1}^n$.
- Recall that par swap rate at T_0 is given by

$$R_{T_0} = \frac{\sum_{i=1}^n p(T_0, T_i) \cdot F(T_0, T_{i-1}, T_i) \cdot \tau_i}{\sum_{i=1}^n p(T_0, T_i) \cdot \tau_i}.$$

- The payoff at T_0 of a cash-settled payer swaption can be then written as

$$A_{T_0}(R_{T_0} - K)_+.$$

- Using (1) and (2) the PV of the swaption at $0 \leq t \leq T_0$ is then given by

$$\begin{aligned} \mathrm{PV}_t^{\mathrm{Swaption}} &= B_t \cdot \mathbb{E}_t^{\mathbb{Q}} \left[\frac{A_{T_0}(R_{T_0} - K)_+}{B_{T_0}} \right] \\ &= p(t, T_0) \cdot \mathbb{E}_t^{T_0} \left[A_{T_0}(R_{T_0} - K)_+ \right]. \end{aligned}$$

Swaption pricing and Annuity Measure

- Both of the above expressions for $\mathrm{PV}_t^{\mathrm{Swaption}}$ are hard to compute. Let us invoke a change of numéraire technique again.
- Note that A_t is actually a valid numéraire and we get under the corresponding martingale measure $\mathbb{Q}^A$ (so called annuity measure)

$$\mathrm{PV}_t^{\mathrm{Swaption}} = A_t \cdot \mathbb{E}_t^{\mathbb{Q}^A} (R_{T_0} - K)_+$$

- We see that payer swaption is equivalent to a call option on a swap rate.
- In view of (7), swap rate R_t, considered as a process on $t \in [0, T_0]$, is represented by a value of a self-financing portfolio deflated by numéraire A_t

$$R_t = \frac{p(t, T_0) - p(t, T_n)}{A_t}.$$

- The immediate consequence is that R_t is a martingale under the annuity measure $\mathbb{Q}^A$.

Black and Bachelier Formulae for Swaptions

- Given that R_t, $t \leq T_0$, must be a martingale under $\mathbb{Q}^A$, we again appeal to Black and Bachelier formulae[*]. We can price European swaptions without modelling the dynamics of the entire interest rate curve.
- Market quotes imply different volatilities for swaptions with different specifications. In order to be consistent with a market, we need to put a correct implied volatility (whether it is normal or log-normal) in the corresponding formula.
- An interesting feature of swaption market is that it implies different swaption volatilities for different strikes and different start and end dates of the underlying swap. Therefore, we observe a three-dimensional implied volatility surface which is often referred to as a swaption cube.

[*]Note that in this case the underlying is the swap rate R instead of forward rate F and that the deflator is the annuity A_t instead of zero-coupon bond.

Swaption Volatility Cube

Strike 0.1%	1y	2y	3y	4y	5y	6y	7y	8y	9y	10y	15y	20y	25y	30y
28-Nov-14	55.27%	72.12%	77.30%	76.20%	77.01%	74.54%	73.61%	72.11%	71.40%	70.86%	68.67%	63.67%	56.71%	54.67%
														55.04%
														61.25%
														80.72%
														55.95%
														74.02%
														87.08%
														97.11%
														91.94%
														79.25%
														73.20%
														70.20%
														74.76%
														74.17%

Strike 0.25%	1y	2y	3y	4y	5y	6y	7y	8y	9y	10y	15y	20y	25y	30y
28-Nov-14	47.71%	56.53%	58.01%	57.63%	59.06%	57.68%	57.40%	56.58%	56.30%	56.10%	55.00%	51.12%	45.40%	43.43%
														43.89%
														49.06%
														64.72%
														44.26%
														58.26%
														68.57%
														76.76%
														72.47%
														61.82%
														56.89%
														54.49%
														57.20%
														56.52%

Strike 0.5%	1y	2y	3y	4y	5y	6y	7y	8y	9y	10y	15y	20y	30y	30y
28-Nov-14	58.14%	60.91%	49.08%	44.78%	45.67%	44.42%	44.36%	43.85%	43.81%	43.82%	43.54%	40.98%	36.93%	36.00%
														35.95%
														39.59%
														51.50%
														35.00%
														46.16%
														54.44%
														61.12%
														57.92%
														49.61%
														45.48%
														43.49%
														45.41%
														44.78%

Strike 1%	1y	2y	3y	4y	5y	6y	7y	8y	9y	10y	15y	20y	25y	30y
28-Nov-14	67.14%	72.50%	56.95%	43.40%	38.57%	34.16%	32.90%	31.87%	31.65%	31.63%	31.77%	30.90%	29.18%	29.87%
28-Jan-15	69.91%	66.17%	52.04%	41.60%	36.33%	33.28%	32.16%	31.73%	31.26%	31.12%	30.57%	30.62%	29.34%	28.99%
28-Apr-15	70.39%	62.28%	49.41%	41.56%	38.15%	35.94%	34.89%	34.35%	33.64%	33.25%	31.86%	32.19%	31.20%	30.37%
28-Oct-15	63.25%	53.06%	46.34%	42.56%	40.76%	39.36%	38.38%	37.49%	36.40%	35.59%	33.30%	33.22%	34.95%	37.56%
28-Oct-16	63.30%	63.68%	59.04%	54.69%	50.60%	48.09%	45.67%	43.33%	40.98%	38.82%	32.61%	23.85%	23.78%	25.49%
30-Oct-17	61.44%	62.75%	58.88%	54.94%	51.27%	48.92%	46.68%	44.69%	42.69%	40.90%	34.37%	29.95%	31.23%	33.86%
29-Oct-18	59.99%	59.08%	56.07%	53.26%	50.57%	48.59%	46.73%	45.00%	43.40%	41.94%	35.69%	33.92%	36.34%	40.05%
28-Oct-19	57.32%	54.94%	52.94%	50.98%	49.11%	47.49%	45.88%	44.60%	43.42%	42.18%	36.55%	36.67%	40.21%	45.08%
28-Oct-21	56.52%	53.87%	51.93%	50.10%	48.24%	46.88%	45.63%	44.34%	43.37%	42.32%	37.23%	37.75%	40.18%	43.22%
28-Oct-24	53.64%	51.73%	49.90%	48.18%	46.25%	45.44%	44.44%	43.35%	42.19%	40.99%	38.86%	37.94%	38.01%	37.79%
29-Oct-29	48.70%	46.13%	43.92%	41.74%	39.67%	39.31%	38.84%	38.29%	37.67%	37.03%	34.23%	33.73%	33.95%	34.41%
30-Oct-34	42.76%	40.46%	38.36%	36.38%	34.50%	34.34%	34.11%	33.82%	33.49%	33.16%	31.66%	31.71%	32.29%	32.77%
28-Oct-39	38.22%	36.25%	35.15%	33.15%	31.83%	32.11%	32.32%	32.46%	32.56%	32.63%	31.36%	32.12%	32.97%	34.17%
28-Oct-44	35.52%	34.71%	32.73%	32.03%	30.96%	31.06%	31.20%	31.36%	31.53%	31.70%	31.87%	32.91%	33.60%	33.63%