Applied Mathematical Finance I

Lecture 5: Exotic Derivatives

Vladimir Shangin

Vega Institute Foundation

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Vanilla and Exotic Derivatives

- Last time we discussed vanilla interest rate derivatives. We can split them into the following two groups
  - Model-free products that can be priced based on the prevailing zero curve only (e.g. standard FRAs and swaps).
  - Contracts with non-linear payoffs that depend on a marginal distribution of a single rate (e.g. caplets, floorlets, and swaptions). To price them, one has to specify a model for rate dynamics.
- Today we will consider more complex derivatives that are referred to as exotic. Some of them turn out to be near-vanilla products that can be priced in a model-free way while others require quite sophisticated modelling.

Digital Options: First Look

- Digital caplet with strike price K and expiry T pays at $T+\tau$ an amount given by the indicator function

$$\mathbb{1}_{\{L(T,T+\tau)>K\}}.$$

- By the fundamental pricing theorem

$$\mathrm{PV}_t^{\mathrm{Digital}}(K) = p(t, T+\tau)\mathbb{E}_t^{T+\tau}\left[\mathbb{1}_{\{L(T,T+\tau)>K\}}\right].$$

- Assuming that forward rate $F = F(t, T, T+\tau)$ follows Black model under $\mathbb{Q}^{T+\tau}$, we get

$$\mathrm{PV}_t^{\mathrm{Digital}}(K) = p(t, T+\tau)\Phi(d_-),$$

where $d_- = \frac{\ln(F/K)}{\sigma\sqrt{T-t}} - \frac{\sigma\sqrt{T-t}}{2}$ and Φ stands for standard normal CDF.

Digital Options: Real Case

- As we saw earlier, implied caplet volatility is actually a function of strike, $\sigma = \sigma(K)$, so pricing of a digital caplet becomes more complicated.
- Indeed, note that

$$
\begin{aligned}
\mathrm{PV}_t^{\text{Digital}}(K) &= \lim_{\epsilon \to 0+} \frac{\mathrm{PV}_t^{\text{Caplet}}(K-\epsilon, \sigma(K-\epsilon)) - \mathrm{PV}_t^{\text{Caplet}}(K, \sigma(K))}{\epsilon} \\
&= -\frac{d\mathrm{PV}_t^{\text{Caplet}}(K, \sigma(K))}{dK} \\
&= \underbrace{-\frac{\partial \mathrm{PV}_t^{\text{Caplet}}(K, \sigma(K))}{\partial K}}_{\text{Digital PV in Black model}} - \underbrace{\frac{\partial \mathrm{PV}_t^{\text{Caplet}}(K, \sigma(K))}{\partial \sigma}}_{\text{Caplet vega}} \cdot \underbrace{\frac{\partial \sigma(K)}{\partial K}}_{\text{Vol skew}}.
\end{aligned}
$$

- We see that volatility skew (smile) results into a price adjustment.

Digital Options: Practicalities

- Despite the apparent simplicity, risk management of digital options requires a careful approach.
- Indeed, when close to expiry date T, delta $\frac{\partial \mathrm{PV}^{\mathrm{Caplet}}}{\partial F}$ grows to infinity if F approaches the strike price K.
- When selling a digital option, a trader will actually price, sell and then hedge a super-replicating option spread

$$\frac{\mathrm{PV}_t^{\mathrm{Caplet}}(K-\epsilon) - \mathrm{PV}_t^{\mathrm{Caplet}}(K)}{\epsilon} > \mathbb{1}_{\{L(T,T+\tau)>K\}},$$

  where ϵ is chosen so as to guarantee that delta will not exceed the desired level.
- Obvious advantages: bounded delta plus “free lunch” if $L(T, T+\tau) \in (K-\epsilon, K)$.

Fixed-in-Arrears Instruments

- LIBOR rate $L(T, T+\tau)$ is not restricted to being paid at $T+\tau$.
- Interest rate derivatives paying LIBOR-linked cashflow on the date when the underlying rate fixes are called in-arrears instruments.
- Consider simple LIBOR-in-arrears (LIA) paying $L(T, T+\tau)$ at T. Its value at $t \leq T$ is given by

$$\mathrm{PV}_t^{\mathrm{LIA}} = p(t,T)\mathbb{E}_t^T\left[L(T, T+\tau)\right] = p(t,T)\mathbb{E}_t^T\left[F(T, T, T+\tau)\right].$$

- The difficulty here is that $F(t, T, T+\tau)$ is a martingale under $\mathbb{Q}^{T+\tau}$, not $\mathbb{Q}^T$, and hence the expectation does not equal to the forward rate observed at t

$$\mathbb{E}_t^T\left[F(T, T, T+\tau)\right] \neq F(t, T, T+\tau) = \mathbb{E}_t^{T+\tau}\left[F(T, T, T+\tau)\right].$$

- $\mathbb{E}_t^T\left[F(T, T, T+\tau)\right] - F(t, T, T+\tau)$ is referred to as convexity adjustment.

LIBOR-in-Arrears

- Let us now return to valuing LIA cashflow. Paying $L(T, T+\tau)$ at T is equivalent to paying $\frac{L(T,T+\tau)}{p(T,T+\tau)}$ at $T+\tau$ so we have

$$\mathrm{PV}_t^{\mathrm{LIA}} = p(t, T+\tau)\mathbb{E}_t^{T+\tau}\left[\frac{L(T, T+\tau)}{p(T, T+\tau)}\right].$$

- This can be formally justified as follows. Defining Radon-Nikodym derivative

$$\xi_T = \left.\frac{d\mathbb{Q}^{T+\tau}}{d\mathbb{Q}^T}\right|_{\mathcal{F}_T} = \frac{p(T, T+\tau)}{p(0, T+\tau)} \cdot \frac{p(0, T)}{p(T, T)} = \frac{p(T, T+\tau)}{p(0, T+\tau)} \cdot p(0, T),$$

we get

$$\begin{aligned}\mathrm{PV}_t^{\mathrm{LIA}} &= p(t, T)\mathbb{E}_t^T\left[L(T, T+\tau)\right] = p(t, T)\mathbb{E}_t^T\left[\xi_T\right]\mathbb{E}_t^{T+\tau}\left[\frac{L(T, T+\tau)}{\xi_T}\right] \\ &= p(t, T+\tau)\mathbb{E}_t^{T+\tau}\left[\frac{L(T, T+\tau)}{p(T, T+\tau)}\right]. \qquad (1)\end{aligned}$$

LIBOR-in-Arrears (continued)

- Now note that the factor $\frac{1}{p(T,T+\tau)}$ is a function of rate $L(T, T+\tau)$

  $$\frac{1}{p(T,T+\tau)} = 1 + \tau L(T, T+\tau)$$

  so that (1) can be rewritten as

  $$\begin{aligned} \mathrm{PV}_t^{\mathrm{LIA}} &= p(t, T+\tau)\mathbb{E}_t^{T+\tau}\left[(1+\tau L(T,T+\tau))L(T,T+\tau)\right] \\ &= p(t,T+\tau)F(t,T,T+\tau) + \tau p(t,T+\tau)\mathbb{E}_t^{T+\tau}\left[L(T,T+\tau)^2\right]. \end{aligned}$$

- Therefore, the problem boils down to computing the expectation of $L(T, T+\tau)^2$. Fortunately, density of $L(T, T+\tau)$ under $\mathbb{Q}^{T+\tau}$ can be obtained directly from observable market prices of caplets and floorlets with maturity T.

Carr-Madan Static Replication

- Now, let us invoke the famous variation of the Taylor expansion formula. Consider a function $g(L) \in C^2(\mathbb{R})$ and arbitrary $M \in \mathbb{R}$. Then the following representation holds

$$\begin{aligned} g(L) = {} & g(M) + g'(M) \cdot (L - M) \\ & + \int_{-\infty}^{M} g''(k) \cdot (k - L)_+ \, dk + \int_{M}^{\infty} g''(k) \cdot (L - k)_+ \, dk \end{aligned} \tag{2}$$

- Financial meaning: we can replicate European payoff given by function g with a static position in a continuum of vanilla call and put options.

- Before we proceed, let us briefly recall the notion of Dirac delta function $\delta(x)$.

Dirac Delta Function

- Formally, δ_L can be seen as a probability measure on $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$ such that, for any $B \in \mathcal{B}(\mathbb{R})$, $\delta(B) = 1$ if $L \in B$, and $\delta(B) = 0$ otherwise.

- It follows from the definition that

$$\int_{-\infty}^{\infty} g(x)\, \delta_L(dx) = g(L).$$

- The corresponding CDF is then given by

$$\delta_L((-\infty, x)) = \mathbb{1}_{\{x>L\}} = \begin{cases} 1, \ x > L \\ 0, \ x \leq L, \end{cases}$$

so we have

$$\int_{-\infty}^{\infty} g(x)\, \delta_L(dx) = \int_{-\infty}^{\infty} g(x)\, d\mathbb{1}_{\{x>L\}}.$$

Carr-Madan Formula: Proof

- For simplicity, we consider Carr-Madan formula on $\mathbb{R}_+$. We have

$$\begin{aligned} g(L) = \int_0^\infty g(k)\, d\mathbb{1}_{\{k>L\}} &= \int_0^M g(k)\, d\mathbb{1}_{\{k>L\}} + \int_M^\infty g(k)\, d\mathbb{1}_{\{k>L\}} \\ &= \int_0^M g(k)\, d\mathbb{1}_{\{k>L\}} - \int_M^\infty g(k)\, d\mathbb{1}_{\{k\le L\}}. \end{aligned}$$

- Integrating by parts, we then have

$$\begin{aligned} g(L) &= g(k)\mathbb{1}_{\{k>L\}}\Big|_0^M - \int_0^M g'(k)\mathbb{1}_{\{k>L\}}\, dk \\ &= - g(k)\mathbb{1}_{\{k\le L\}}\Big|_M^\infty + \int_M^\infty g'(k)\mathbb{1}_{\{k\le L\}}\, dk \\ &= g(M) - \int_0^M g'(k)\mathbb{1}_{\{k>L\}}\, dk + \int_M^\infty g'(k)\mathbb{1}_{\{k\le L\}}\, dk. \end{aligned} \tag{3}$$

Carr-Madan Formula: Proof (continued)

- Integrating (3) by parts again, we get

$$
\begin{aligned}
g(L) = g(M) \\
&- g'(k)(k-L)_+\big|_0^M + \int_0^M g''(k)(k-L)_+\,dk \\
&- g'(k)(L-k)_+\big|_M^\infty + \int_M^\infty g''(k)(L-k)_+\,dk \\
&= g(M) + g'(M)(L-M) + \int_0^M g''(k)(k-L)_+\,dk + \int_M^\infty g''(k)(L-k)_+\,dk.
\end{aligned}
$$

LIBOR-in-Arrears (continued)

- Let us get back to the valuation problem. Using Carr-Madan formula with $g(L) = L^2$ and $M = F = F(t, T, T+\tau)$ and switching integration and expectation, we obtain

$$\mathbb{E}_t^{T+\tau}\left[L(T, T+\tau)^2\right] = g(F) + 2\int_0^F \underbrace{\mathbb{E}_t^{T+\tau}\left[(k-L)_+\right]}_{\substack{\text{Undiscounted price of} \\ \text{a floorlet with strike } k}} dk + 2\int_F^\infty \underbrace{\mathbb{E}_t^{T+\tau}\left[(L-k)_+\right]}_{\substack{\text{Undiscounted price of} \\ \text{a caplet with strike } k}} dk. \tag{4}$$

- In practice, the integration is carried out numerically via quadratures.

LIBOR-in-Arrears (continued)

- Formula (4) does not rely on any modelling assumptions and hence is model independent.
- Also, note that (4) heavily relies on the volatility skew/smile at T. Indeed, integration of caplet/floorlet prices over strikes require using the whole slice $\sigma(T, k), k \in [0; +\infty)$, of the implied volatility surface.
- We can apply the same technique to price in-arrears options.

In-Arrears Caplet

- Let us now consider in-arrears caplet paying $(L(T,T+\tau)-K)_+$ at T. Its value is given by

$$\mathrm{PV}_t^{\text{LIA cplt}} = p(t,T)\,\mathbb{E}_t^T\left[(L(T,T+\tau)-K)_+\right]. \tag{5}$$

- Re-expressing (5) under $\mathbb{Q}^{T+\tau}$, we get

$$\begin{aligned}\mathrm{PV}_t^{\text{LIA cplt}} &= p(0,T+\tau)\,\mathbb{E}_t^{T+\tau}(L-K)_+ \\ &+ p(0,T+\tau)\,\mathbb{E}_t^{T+\tau}\left[(1+\tau L)(L-K)_+\right] \\ &= p(0,T+\tau)\,\mathbb{E}_t^{T+\tau}\left[(L-K)_+\right] + p(0,T+\tau)\,\tau\,\mathbb{E}_t^{T+\tau}\left[L(L-K)_+\right].\end{aligned}$$

In-Arrears Caplet (continued)

- Applying Carr-Madan formula with $M = K$ to $g(L) = L(L-K)$ and multiplying the result with $\mathbb{1}_{L>K}$, we have

$$\mathbb{E}_t^{T+\tau}\left[L(L-K)_+\right] = K\,\mathbb{E}_t^{T+\tau}\left[(L-K)_+\right] + 2\int_K^{\infty} \mathbb{E}_t^{T+\tau}\left[(L-k)_+\right]\,dk.$$

- Summarizing the above, the price of in-arrears caplet is given by

$$\begin{aligned}\mathrm{PV}_t^{\mathrm{LIA\ cplt}} &= p(0,T+\tau)\,(1+\tau K)\,\mathbb{E}_t^{T+\tau}\left[(L(T,T+\tau)-K)_+\right] \\ &\quad + p(0,T+\tau)\,2\tau\int_K^{\infty} \mathbb{E}^{T+\tau}\left[(L(T,T+\tau)-k)_+\right]\,dk.\end{aligned}$$

LIBOR With Delay

- Let us now move on to a more interesting case of a LIBOR cashflow $L(T, T+\tau)$ with an arbitrary payment date $T_p \geq T$. Note that $T_p = T$ corresponds to LIBOR-in-arrears and $T_p = T + \tau$ corresponds to a regular LIBOR payment. Assuming that $T_p > T$ we generally call this cashflow LIBOR with delay (LD).
- Its value is given by

$$\mathrm{PV}_t^{\mathrm{LD}} = p(t, T_p)\mathbb{E}_t^{T_p}[L(T, T+\tau)].$$

- Switching to $\mathbb{Q}^{T+\tau}$, we get

$$\mathrm{PV}_t^{\mathrm{LD}} = p(t, T+\tau)\mathbb{E}_t^{T+\tau}\left[\frac{p(T, T_p)}{p(T, T+\tau)}L(T, T+\tau)\right]. \tag{6}$$

LIBOR With Delay (continued)

- The problem here is that, generally, we cannot express $p(T, T_p)$ as a function of LIBOR rate $L(T, T+\tau)$ and hence cannot use Carr-Madan formula as is.
- What we can do is to approximate

$$p(T, T_p) \approx \left(\frac{1}{1+\tau L(T, T+\tau)}\right)^{\frac{T_p - T}{\tau}}.$$

Intuition: discount all cash flows after T at the same rate $L(T+\tau)$.

- Under this assumption, formula (6) becomes

$$\mathrm{PV}_t^{\mathrm{LD}} = p(t, T+\tau)\mathbb{E}_t^{T+\tau}\left[L(T, T+\tau)(1+\tau L(T, T+\tau))^{1-\frac{T_p - T}{\tau}}\right].$$

so we can use Carr-Madan formula for pricing.

- Note that this approach is exact for two most common cases $T_p = T$ and $T_p = T + \tau$.

Averaging Swaps

- Averaging swap is a swap where floating rate is defined as a weighted average of observed LIBOR rates. For simplicity of notations, let us consider a single-period instrument.
- Consider a time interval $[S, T]$ and the set of fixing dates $\{t_i\}_{i=1}^n$, where $S = t_1 < t_2 < \cdots < t_n = T$, when the LIBOR rate for tenor τ is observed.
- Floating leg of averaging swap pays at T an amount given by

$$\sum_{i=1}^{n} \omega_i L(t_i, t_i + \tau),$$

where the weights $\omega_i > 0$ and $\sum_{i=1}^n \omega_i = 1$.

Averaging Swaps: Valuation

- Value of averaging swap at $t \leq S$ is

$$\begin{aligned} \mathrm{PV}_t^{\text{Avg swap}} &= p(t,T)\mathbb{E}_t^T\left[\sum_{i=1}^{n}\omega_i L(t_i, t_i+\tau)\right] \\ &= p(t,T)\sum_{i=1}^{n}\omega_i\,\mathbb{E}_t^T\left[L(t_i,t_i+\tau)\right]. \end{aligned} \tag{7}$$

- Now note that every summand in (7) corresponds to a LIBOR with delay* cashflow and hence can be priced with the approach outlined above.
- Sometimes a rude approximation is used

$$\sum_{i=1}^{n}\omega_i\,\mathbb{E}_t^T\left[L(t_i,t_i+\tau)\right] \approx \sum_{i=1}^{n}\omega_i\,F(t,t_i,t_i+\tau).$$

*The last payment is actually LIBOR-in-arrears.

CMS Rate

- We can go even further and consider swaps whose floating rate is given by the swap rate itself.
- A constant-maturity swap (CMS) rate is defined as a break-even (par) rate of a swap with fixed maturity (tenor), e.g. 10Y.
- Consider a tenor structure $t \leq T_0 < T_1 < \cdots < T_{n+m}$, $\tau_i = T_i - T_{i-1}$. We denote by $R_{k,m}(t)$ a par rate at t of a fix-for-floating swap that starts at T_k, $k \leq n$, and ends at T_{k+m} and by $A_{k,m}(t)$ the corresponding annuity. We know that

$$A_{k,m}(t) = \sum_{i=k+1}^{k+m} p(t, T_i)\tau_i,$$

$$R_{k,m}(t) = \frac{\sum_{i=k+1}^{k+m} p(t, T_i)F(t, T_{i-1}, T_i)\tau_i}{A_{k,m}(t)},$$

and that $R_{k,m}(t)$ is a martingale under $\mathbb{Q}^{A_{k,m}}$.

CMS Swaps

- CMS swap is a swap with floating leg linked to a CMS rate R rather than LIBOR spot rate L. Consider a swap with start date T_0 and end date T_n which references m-period CMS rate. In our notations, its value at $t \leq T_0$ can be written as

$$\mathrm{PV}_t^{\text{CMS swap}} = \sum_{i=0}^{n-1} p(t, T_{i+1})\, \tau_{i+1}\, \mathbb{E}_t^{T_{i+1}} \left[R_{i,m}(T_i) - K \right],$$

where K is the fixed rate.

- While standard fix-for-floating swaps are model-free, meaning that they can be valued solely from using observable term-structure of zero-coupon bond prices, pricing of CMS swaps require making some modelling assumptions.

CMS Swaps: Pricing

- For simplicity we consider a single CMS rate-linked cashflow paid at T_p. Denoting underlying m-period CMS rate observed at $T \leq T_p$ simply as R_T and corresponding annuity as A_T, we have at $t = 0$

$$\mathrm{PV}_0^{\mathrm{CMS}} = p(0, T_p)\mathbb{E}^{T_p}\left[R_T\right] = A_0\mathbb{E}^{\mathbb{Q}^A}\left[\frac{p(T, T_p)}{A_T}R_T\right].$$

- Assuming that $\frac{p(T,T_p)}{A_T}$ is a function of R_T, $\frac{p(T,T_p)}{A_T} = g(R_T)$, and applying Carr-Madan formula, we get

$$\begin{aligned}\mathrm{PV}_0^{\mathrm{CMS}} &= g(0)R(0) \\ &+ \int_0^{R_0} z''(k)p(k)dk + \int_{R_0}^{\infty} z''(k)c(k)dk,\end{aligned}$$

where $z(k) = g(k)k$, $p(k)$ and $c(k)$ are prices of put and call options on S_T with strike k i.e. receiver and payer swaptions respectively.

Autocaps

- Autocap is a structured product comprising a series of contingent caplets which terminates if the number of options expired in the money reaches a pre-agreed threshold N.
- Consider a tenor structure $\{T_i\}_{i=0}^{n}$. Mathematically, the payoff at T_{i+1} can be expressed as

$$\tau_{i+1}\left(L(T_i, T_{i+1}) - K\right)_+ \mathbb{1}_{\{N(T_{i+1})<N\}},$$

  where $\tau_{i+1} = T_{i+1} - T_i$ and $N(T_{i+1})$ is the total number of caplets expired in-the-money before T_{i+1}.
- Autocaps are highly path-dependent products. Their payoff depends on the joint evolution of LIBOR rates and hence a proper modelling must capture possible movements of the zero curve.