INSTITUTE AND FACULTY OF ACTUARIES

EXAMINERS' REPORT

September 2015

Subject CT1 – Financial Mathematics
Core Technical

Introduction

The Examiners' Report is written by the Principal Examiner with the aim of helping candidates, both those who are sitting the examination for the first time and using past papers as a revision aid and also those who have previously failed the subject.

The Examiners are charged by Council with examining the published syllabus. The Examiners have access to the Core Reading, which is designed to interpret the syllabus, and will generally base questions around it but are not required to examine the content of Core Reading specifically or exclusively.

For numerical questions the Examiners' preferred approach to the solution is reproduced in this report; other valid approaches are given appropriate credit. For essay-style questions, particularly the open-ended questions in the later subjects, the report may contain more points than the Examiners will expect from a solution that scores full marks.

The report is written based on the legislative and regulatory context pertaining to the date that the examination was set. Candidates should take into account the possibility that circumstances may have changed if using these reports for revision.

F Layton
Chairman of the Board of Examiners
December 2015

A. General comments on the *aims of this subject and how it is marked*

1. CT1 provides a grounding in financial mathematics and its simple applications. It introduces compound interest, the time value of money and discounted cashflow techniques which are fundamental building blocks for most actuarial work.

2. Please note that different answers may be obtained to those shown in these solutions depending on whether figures obtained from tables or from calculators are used in the calculations but candidates are not penalised for this. However, candidates may be penalised where excessive rounding has been used or where insufficient working is shown.

B. General comments on *student performance in this diet of the examination*

1. The comments that follow the questions concentrate on areas where candidates could have improved their performance. Where no comment is made the question was generally answered well by most candidates.

2. Student performance was poorer than in previous years. The performance across all questions was of a lower standard which indicated that the lower performance was not as the result of some particularly difficult individual questions.

3. There were some ambiguities in the wording and mark scheme for Q8 but the marking process was adjusted to ensure that candidates were not disadvantaged.

4. There were also elements of non-numerical explanation and analysis required in several questions and, as in previous papers, students performed relatively badly on these sections.

5. Finally, it appeared that many students left themselves short of time for the last question, Q9, which was worth 26 marks.

C. Comparative pass rates for the past 3 years for this diet of examination

Year	*%*
September 2015	44
April 2015	55
September 2014	57
April 2014	60
September 2013	57
April 2013	60

Reasons for any significant change in pass rates in current diet to those in the past:

Historically, the papers have been set by experienced examiners and this has led to very stable pass marks and relatively stable pass rates. This paper was set by the same examining team.

As highlighted in Section B, the performance by candidates on this paper has been significantly poorer than for past exams in this subject. The examiners do not believe that the paper was significantly different this session and any potential ambiguities in the paper were fully allowed for within the marking process.

We have no definite cause for the lower marks. An analysis on a question by question basis shows that candidates on average were only scoring above 60% on average on two questions out of nine, one of which was the 3-mark Q1. This all suggests a relatively weak cohort.

Solutions

Q1 $P = 100(1.03)^{-91/365} = 0.99266$ or £99.266

Let annual simple rate of discount = d

$$\frac{99.266}{100} = 1 - \frac{91}{365}d$$

Therefore $\frac{91}{365}d = 0.00734$; $d = 0.02945$

Q2 (i) (a) $$\delta = \ln(1+i) = \ln\left[\left(1-\frac{d^{(12)}}{12}\right)^{-12}\right] = \ln\left[\left(1-\frac{0.055}{12}\right)^{-12}\right] = 0.055126$$

$= 5.513\%$

(b) $$1+i = \left(1-\frac{d^{(12)}}{12}\right)^{-12} \Rightarrow i = \left(1-\frac{0.055}{12}\right)^{-12} - 1 = 0.056674 = 5.667\%$$

(c) $$1+\frac{i^{(12)}}{12} = \left(1-\frac{d^{(12)}}{12}\right)^{-1} \Rightarrow i^{(12)} = 12\left[\left(1-\frac{0.055}{12}\right)^{-1} - 1\right] = 0.055253$$

$= 5.525\%$

(ii) When interest is paid monthly, the interest that is paid in earlier months itself earns interest. This means that to achieve the same effective rate over the year, the nominal rate must be lower.

(iii) $100 = 159(1-d)^8 \Rightarrow d = 1-\left(\frac{100}{159}\right)^{1/8} = 0.056319 = 5.632\%$

(iv) $1+i = 1.12^{1/2} \Rightarrow i = 0.058301 = 5.830\%$

> Parts (i) and (iii) were generally calculated well although many candidates chose not to give their final answer to the requested accuracy. Some candidates attempted to use their answers to parts (i)((a) and (i)(b) to find an answer to part (i)(c) – this was not accepted by the examiners . Part (iv) proved more challenging to many candidates. Amongst the marginal candidates, there were very few who gave a clear explanation for part (ii). It is a matter of concern that so many candidates were unable to articulate the relationship between nominal and effective interest rates.

Q3 (i) Duration of the annuity payment is $\frac{(Ia)_{\overline{12}|}}{a_{\overline{12}|}} = \frac{56.6328}{9.3851} =$ 6.0343 years

(ii) Duration of bond is:

$$\frac{5(Ia)_{\overline{8}|} + 800v^8}{5a_{\overline{8}|} + 100v^8}$$

$$= \frac{5\times 28.9133 + 800\times 0.73069}{5\times 6.7327 + 100\times 0.73069}$$

$$= \frac{729.119}{106.733} = 6.8313 \text{ years}$$

(iii) The duration of the assets (the bond) is greater than the duration of the liabilities (pension payments). If there is a rise in interest rates, the present value of the assets will fall by more than the present value of the liabilities and the insurance company will make a loss.

> Parts (i) and (ii) were answered well. Where a term is calculated, it is particularly important to include the units in the final answer. Part (iii) was very poorly answered with many candidates stating that the company must make a loss because the durations were not equal.

Q4 (i) Assuming no arbitrage

The present value of the dividends (in £), I, is:

$I = 0.5(e^{-0.05\times(1/12)} + e^{-0.05\times(7/12)}) = 0.5 \times (0.995842 + 0.971255) = 0.98355$

Hence, forward price $F = (10 - 0.98355)e^{0.05(9/12)} = £9.3610$

(ii) The expected price of the share does not have to be taken into account because, using the no-arbitrage assumption, the purchaser of the forward is simply able to use the current price of the share (and the value of the dividends) given that the forward is simply an alternative way of exposing the investor to the same set of cash flows.

[The expected future price of the share will be taken into account by investors when determining the price they wish to pay for the share and therefore the current share price.]

> Part (i) was often answered well although some candidates miscalculated the timing of the dividends and the statement of the arbitrage assumption was often missed. Part (ii) was poorly answered despite being similar to previous exam questions.

Q5 (i) Present value is

$$\int_3^6 \rho(t)v(t)dt = \int_3^6 5000v(t)dt$$

For $t \geq 3$

$$v(t) = \exp\left(-\int_0^t \delta(t)\,dt\right) = \exp\left(-\int_0^3 0.03 + 0.005t\,dt - \int_3^t 0.005\,dt\right)$$

$$= \exp\left[-0.03t - 0.0025t^2\right]_0^3 \exp\left[-0.005t\right]_3^t$$

$$= \exp\left[-0.1125\right]\exp\left[0.015 - 0.005t\right]$$

$$= \exp(-0.005t - 0.0975)$$

Hence present value is

$$\int_3^6 5,000\exp(-0.005t - 0.0975)dt$$

$$= 5,000e^{-0.0975}\int_3^6 e^{-0.005t}dt$$

$$= \frac{5,000e^{-0.0975}}{-0.005}\left[e^{-0.005t}\right]_3^6 = -1,000,000e^{-0.0975}(e^{-0.03} - e^{-0.015})$$

$$= -880.293.42 + 893,597.35 = £13,303.93$$

(ii) $5,000(\bar{a}_{\overline{6|}} - \bar{a}_{\overline{3|}}) = 13,303.93$

i = 2%: LHS = 13,723
i = 3%: LHS = 13,136

Interpolating

$$i \approx 0.02 + 0.01 \times \frac{13,304 - 13,723}{13,136 - 13,723} = 2.714\% \text{ say } 2.7\%$$

(iii) Accumulation =

$$= 300A(50) = 300\exp(0.005 \times 50 + 0.0975)$$

$$= 300e^{0.3475} = £424.66$$

The discount factor was usually calculated correctly although some candidates just calculated this factor for t = 6 and assumed that the value of a single payment at this time was required. Part (ii) was poorly answered. The important point is that the rate of interest is obtained by equating the amount initially invested as calculated in part (i) with the present value of the annuity.

Q6 (i) $101 = 3a_{\overline{3|}} + 100v^3$

i = 3%: RHS = 100
i = 2.5%: RHS = 101.428

Interpolating

$$i \approx 0.025 + 0.005 \times \frac{101 - 101.428}{100 - 101.428} = 2.65\%$$

(ii) Let i_n = spot rate for term n

One year bond gives

$$101 = 103v_{i_1}$$

$$v_{i_1} = \frac{101}{103} = 0.98058$$

$$\Rightarrow i_1 = \frac{103}{101} - 1 = 1.980\%$$

Two year bond gives

$$101 = 3v_{i_1} + 103v_{i_2}^2$$

$$\Rightarrow v_{i_2}^2 = \frac{101 - 3\frac{101}{103}}{103} = 0.95202$$

$$\Rightarrow i_2 = 2.489\%$$

Three year bond gives

$$101 = 3v_{i_1} + 3v_{i_2}^2 + 103v_{i_3}^3$$

$$\Rightarrow v_{i_3}^3 = \frac{101 - 3\times 0.98058 - 3\times 0.95202}{103} = 0.92429$$

$$\Rightarrow i_3 = 2.659\%$$

(iii) Forward rate is $f_{2,1}$ where

$$1 + f_{2,1} = \frac{(1+i_3)^3}{(1+i_2)^2} = \frac{1.02659^3}{1.02489^2} = 1.03000 \Rightarrow f_{2,1} = 3.000\%$$

(iv) Reasons could include:

Expectations theory suggests that if short-term interest rates are expected to rise then if yields are the same on both long- and short-term bonds, short-term bonds will be more attractive and longer term bonds less attractive and so the yields on short-term bonds will fall relative to those on long-term bonds.

[Expected higher inflation could be a reason for this but could be allowed as a distinct point]

Liquidity preference theory suggests that investors demand higher rates of return for less liquid/longer term-to-maturity investments which are more sensitive to interest rate movements.

Market segmentation with the supply of bonds being restricted at shorter terms or some factor that leads to the demand for bonds of longer terms to be lower

> Common errors on this question included assuming the price of each bond was 100 and that the gross redemption yield of the 3-year bond was equal to the three-year spot rate. In part (iv) many candidates just gave the names of theories of the yield curve without explaining how this applied in this particular scenario. Otherwise, this was the best answered question on the paper apart from Q1.

Q7 (i) Maximum rate of return after 20 years

$$100 = 0.75\left(7a_{\overline{20}|}^{(2)} - 3a_{\overline{10}|}^{(2)}\right) + 130v^{20}$$

Try i = 5%:

$$\text{RHS} = 0.75\left(\frac{7\times(1-1.05^{-20}) - 3\times(1-1.05^{-10})}{2(1.05^{1/2}-1)}\right) + 130\times1.05^{-20}$$

$$= 48.6460 + 48.9956 = 97.6417$$

Try i = 4%:

$$\text{RHS} = 0.75\left(\frac{7\times(1-1.04^{-20}) - 3\times(1-1.04^{-10})}{2(1.04^{1/2}-1)}\right) + 130\times1.04^{-20}$$

$$= 53.6255 + 59.3303 = 112.9558$$

Interpolating

$$i \approx 0.04 + 0.01\times\frac{100-112.9558}{97.6417-112.9558} = 4.846\% = 4.8\%$$

(Exact answer is 4.8338%.)

(ii) Minimum rate of return after 10 years

In this case, the investor invests 100, receives 100 back and receives a net income at a rate of 1.5 per half-year. The rate of return per half-year effective is therefore 1.5 per cent.

The annual effective rate of return is $1.015^2 - 1 = 3.0225\%$.

(iii) There is a 0.5 probability of both early redemption and of late redemption. The expected return is therefore $0.5(4.8338\% + 3.0225\%) \approx 3.928\%$

(iv) If the investor buys the whole loan, the present value of the cash flows from the loan is as follows (per €100 nominal):

$$= 0.75 \times 4a_{\overline{10}|}^{(2)} + 0.5 \times 100v^{10} + 0.5\left(0.75 \times 7a_{\overline{10}|}^{(2)} v^{10} + 130v^{20}\right)$$

At i = 3.928% this is

$$= 3\frac{1 - 1.03928^{-10}}{2(1.03928^{1/2} - 1)} + 50 \times 1.03928^{-10}$$

$$+0.5\left(5.25\frac{1 - 1.03928^{-10}}{2(1.03928^{1/2} - 1)}1.03928^{-10} + 130 \times 1.03928^{-20}\right)$$

$$= 24.6575 + 34.0125 + 0.5(29.3538 + 60.1562)$$

$$= 103.4264$$

This is greater than 100 and so the rate of return will be greater than 3.928% (exact return is 4.212%).

This was the worst answered question on the paper with many candidates not recognising that the cases where the bond is redeemed after 10 years and after 20 years have to be calculated separately for parts (i) and (ii). If candidates obtained answers for parts (i) and (ii) then part (iii) was usually done well. However, few candidates recognised that substituting the return from part (iii) into the required equation for part (iv) would lead to the required answer.

The question did not state specifically that the coupons in the second 10 years were semi-annual although most students assumed this. Candidates who assumed a different coupon frequency were given full credit.

Q8 (i)

- Generally issued by commercial undertakings and other bodies.
- Shares are held by the owners of a company who receive a share in the company's profits in the form of dividends
- Potential for high returns relative to other asset classes…
- …but high risk particularly risk of capital losses
- Dividends are not fixed or known in advance and…
- …the proportion of profits paid out as dividends will vary from time-to-time
- No fixed redemption date
- Lowest ranking finance issued by companies.
- Return made up of income return and capital gains.

- Initial running yield low but has potential to increase with dividend growth…
- …in line with inflation and real growth in company earnings.
- Marketability depends on the size of the issue.
- Ordinary shareholders receive voting rights in proportion to their holding.

(ii) Let

P = price investor is willing to pay
d = next expected dividend
g = expected annual dividend growth rate
r = annual required return
t = tax rate

Then

$$P = \frac{d(1-t)}{1+r} + \frac{d(1-t)(1+g)}{(1+r)^2} + \frac{d(1-t)(1+g)^2}{(1+r)^3} + \dots$$

$$= \frac{d(1-t)}{1+r}\left[1 + \frac{1+g}{1+r} + \left(\frac{1+g}{1+r}\right)^2 + \left(\frac{1+g}{1+r}\right)^3 + \dots\right]$$

$$= \frac{d(1-t)}{1+r}\frac{1}{1-\frac{1+g}{1+r}} = \frac{d(1-t)}{r-g}$$

(iii) $d = 6p$
$g = 0.01$
$r = 0.06$
$t = 0.2$

$$P = \frac{d(1-t)}{r-g} = \frac{6(1-0.2)}{0.06-0.01} = 96p$$

(iv) If the share were regarded as more risky, then the required return, *r*, would increase. If *r* were to increase, this would reduce the value of the share as *r* is in the denominator (and is positive).

(v) Equation of value would be (working in money terms):

$$960 = 0.8 \times 60v + 1200v - 0.25(1200 - 960)v$$

$$\Rightarrow v = \frac{96}{118.8}$$

Therefore net money rate of return, *i*, is $\frac{118.8}{96} - 1 = 23.75\%$

Net real rate of return is $\frac{1.2375}{126/123} - 1 = 20.80\%$

> There was an error in the paragraph prior to part (v) of this question where the calculation was designed to be based on the purchase/sale of 1,000 shares but the question referred to the sale of "the share". Nearly all candidates based their calculation on the purchase/sale of the same number of shares (whether it be 1 share or 1,000) but candidates who made a different assumption were not penalised.
>
> There was no split of the marks between parts (ii) and part (iii) given on the paper. Candidates who just performed the calculation without the derivation of the formula were give appropriate credit but a formula derivation was required to obtain the full five marks for these parts. The question did not state specifically that the dividends were paid annually although almost all candidates assumed this. Candidates who assumed other payment frequencies were given full credit.

Q9 (i) (a) The payback period is the first time at which the total incoming cash flows are equal or greater in amount than the total outgoing cash flows.

Total incoming cash flows at the beginning of year *t* = 60,000*t*.

Determine *t* for which $60,000t \geq 1,000,000 \Rightarrow t \geq 16.67$

Therefore the payback period is 16 years.

(b) The net total income received in any year from project A is never greater than £60,000. As the costs are incurred at the beginning of the year, there is no point at which the total income from project A is greater than the total income from project B until the very end of the project when the properties are sold. The payback period for B must therefore be less than that for A.

(ii) (a) The discounted payback period occurs where the present value (or accumulated value) of incoming cash flows is equal to or greater than that of outgoing cash flows for the first time.

(b) Equation of value for project B is (in £000):

$$60\ddot{a}_{\overline{t}|} \geq 1,000$$

$$\Rightarrow a_{\overline{t}|} \geq \frac{1,000}{60(1+i)} = \frac{1,000}{60.6} = 16.5016$$

Need to solve for t. From inspection of tables, t = 19 and so the discounted payback period is 18 years.

(c) Again, given that the net income from project A is never greater in an individual year, than that from project B, at no rate of interest can the discounted value of the net income from project A be greater than that for the income from project B.

(iii) Internal rate of return from project B is the solution to the following equation of value (all figures in 000s):

$$1,000 = 60\ddot{a}_{\overline{20}|} + 1,000v^{20}$$

This can be solved by general reasoning.

As the investor invests 1,000 and receives an annual income of 60 in advance and receives his capital back at the end, the total rate of return, d, expressed as an effective rate of discount per annum is 6 per cent.

Internal rate of return is $i = \dfrac{d}{1-d} = \dfrac{0.06}{0.94} = 6.383\%$

(iv) If the IRR from project A is higher then it must have a net present value > zero at a rate of interest of 6.383 per cent.

Note that v = 0.94 at i = 6.383%

Present value of costs for project A:

$$= 1,000 + 10(1 + 1.005v + 1.005^2 v^2 + \ldots + 1.005^{19} v^{19})$$

$$= 1,000 + 10\left(\frac{1 - 1.005^{20} v^{20}}{1 - 1.005v}\right)$$

$$= 1,000 + 10\left(\frac{1 - (1.005 \times 0.94)^{20}}{1 - 1.005 \times 0.94}\right)$$

$$=1,000+10\frac{0.67946}{0.0553}=1,122.869$$

Present value of revenue for project A:

$$= 2,000v^{20}+60\ddot{a}_{\overline{4}|}^{(12)}v+60\ddot{a}_{\overline{1}|}^{(12)}v^5(1.005+1.005^2v+\ldots+1.005^{15}v^{14})$$

$$= 2,000v^{20}+60\ddot{a}_{\overline{4}|}^{(12)}v+60\ddot{a}_{\overline{1}|}^{(12)}1.005v^5(1+1.005v+\ldots+1.005^{14}v^{14})$$

$$= 2,000v^{20}+60\ddot{a}_{\overline{4}|}^{(12)}v+60\ddot{a}_{\overline{1}|}^{(12)}1.005v^5\left(\frac{1-1.005^{15}v^{15}}{1-1.005\ \ v}\right)$$

$$= 2,000\times 0.94^{20}+60\times 0.94\left(\frac{1-0.94^4}{12(1-(1-0.06)^{1/12})}\right)$$

$$+60\times 1.005\times 0.94^5\left(\frac{1-0.94}{12(1-(1-0.06)^{1/12})}\right)\left(\frac{1-(1.005\times 0.94)^{15}}{1-1.005\times 0.94}\right)$$

$$=580.212+200.365+446.577=1,227.154$$

NPV of project at IRR from project A is: 1,227.154 – 1,122.869 = 104.285 (= £104,285)

This is clearly positive so project A has a higher IRR.

(v) Project B would be preferred on the basis of both payback period and discounted payback period.

However, both these measures have shortcomings. The first does not take into account interest at all and the second does not take into account cash flows after the discounted payback period [or in the case of project A the occurrence of one large cash flow at the time of the discounted payback period]

Project A would be preferred on the basis of internal rate of return.

The internal rate of return measures the total return on the project and therefore is a better decision criterion than payback period or discounted payback period.

There may be other factors (comparison of NPVs at a particular rate of interest or the risk of the two projects) that should be taken into account.

Other factors could include, for example:

student's need for return of original investment
reliability of estimates of future cashflows

It appeared that many candidates were under time pressure when attempting this question. Nearly all candidates failed to recognise how the payments being at the start of the year for project B would impact on the payback period and the discounted payback period. Many candidates' general reasoning arguments for parts (i)(b) and (ii)(c) were unclear. In part (c), a common mistake was to miss out the return of the original investment in the calculation of the IRR. In part (iv), common errors were to miscount the number of terms (for both costs and revenue). As for similar long questions in previous years, marginal candidates would have benefited from showing their intermediate working in greater depth and/or with greater clarity. There were a wide range of points that could be made to score marks in part (v) but few candidates scored well on this part.

END OF EXAMINERS' REPORT