INSTITUTE AND FACULTY OF ACTUARIES

EXAMINERS' REPORT

April 2015 examinations

Subject CT1 – Financial Mathematics
Core Technical

Introduction

The Examiners' Report is written by the Principal Examiner with the aim of helping candidates, both those who are sitting the examination for the first time and using past papers as a revision aid and also those who have previously failed the subject.

The Examiners are charged by Council with examining the published syllabus. The Examiners have access to the Core Reading, which is designed to interpret the syllabus, and will generally base questions around it but are not required to examine the content of Core Reading specifically or exclusively.

For numerical questions the Examiners' preferred approach to the solution is reproduced in this report; other valid approaches are given appropriate credit. For essay-style questions, particularly the open-ended questions in the later subjects, the report may contain more points than the Examiners will expect from a solution that scores full marks.

The report is written based on the legislative and regulatory context at the date the examination was set. Candidates should take into account the possibility that circumstances may have changed if using these reports for revision.

F Layton
Chairman of the Board of Examiners

June 2015

General comments on Subject CT1

CT1 provides a grounding in financial mathematics and its simple applications. It introduces compound interest, the time value of money and discounted cashflow techniques which are fundamental building blocks for most actuarial work.

Please note that different answers may be obtained to those shown in these solutions depending on whether figures obtained from tables or from calculators are used in the calculations but candidates are not penalised for this. However, candidates may be penalised where excessive rounding has been used or where insufficient working is shown.

Comments on the April 2015 paper

The comments that follow the questions concentrate on areas where candidates could have improved their performance. Where no comment is made the question was generally answered well by most candidates. In general, the non-numerical questions were answered poorly by marginal candidates.

1 Dividends usually increase annually whereas rents are reviewed less often.
Property is less marketable.
Expenses associated with property investment are higher.
Large, indivisible units of property are less flexible.
On average, dividends will tend to rise more rapidly than rents because dividends benefit from retention and reinvestment of profits in earlier years.

The worst answered question on the paper with over one-third of candidates scoring no marks.

2 (a) Let the answer be t days

$$3{,}000\left(1+0.04\times\frac{t}{365}\right)=3{,}800$$

t = 2,433.33 days

(b) Let the answer be t days:

$$3{,}000\ (1.04)^{\frac{t}{365}}=3{,}800$$

$$\therefore (1.04)^{\frac{t}{365}}=\frac{3{,}800}{3{,}000}$$

$$\frac{t}{365}\ln 1.04=\ln\left(\frac{3{,}800}{3{,}000}\right)$$

$\therefore t$ = 2,199.91 days.

3 (i) $96.5(1.04)^t = 98$

$t \times \ln(1.04) = \ln(98/96.5)$

Therefore, t = 0.3933 years = 143.54 days (144 days)

(ii) The second investor held the bill for 182–144 = 38 days

Therefore $98\left(1+\frac{38}{365}i\right)=100$

$$i=\left(\frac{100}{98}-1\right)\times\frac{365}{38}=0.19603 \text{ or } 19.603\%$$

(iii) The actual rate of interest over 38 days was (100/98) – 1 = 0.020408

Annual effective rate over 1 year would be:

$$(1 + 0.020408)^{365/38} - 1 = 0.21416 \text{ or } 21.416\%$$

4 (i) The "no arbitrage" assumption means that neither of the following applies:

(a) an investor can make a deal that would give her or him an immediate profit, with no risk of future loss;

(b) an investor can make a deal that has zero initial cost, no risk of future loss, and a non-zero probability of a future profit.

(ii) The theoretical price per share of the forward contract is $\pounds 6e^{(0.09-0.035)\times\frac{9}{12}}$ $= \pounds 6.2527$

(iii) In this case the actual forward price is too expensive in relation to the stock.

The investor should borrow $\pounds 6e^{-0.035\times\frac{9}{12}}$ and use this to buy $e^{-0.035\times\frac{9}{12}}$ units of the stock. The investor will also go short in one forward contract. The continuous dividends are reinvested in the stock. (Mark given for general strategy, exact amounts not required).

[After nine months, the investor will have $e^{0.035\times\frac{9}{12}} \times e^{-0.035\times\frac{9}{12}} = 1$ unit of stock that can be sold under the terms of the forward contract for £6.30. The investor will also have to repay cash of $\pounds 6e^{-0.035\times\frac{9}{12}} e^{0.09\times\frac{9}{12}} = \pounds 6.2527$.]

Whilst it was not required for candidates to give a full mathematical explanation for part (iii), they were expected to recognise that the forward was overpriced and to determine the arbitrage strategy accordingly.

5 We will use the ½-year as the time unit because the interest rate is convertible half yearly. The effective rate of interest is 3% per half year.

Accumulated amount = $\dfrac{120}{\ddot{a}_{\overline{2}|}}\ddot{s}_{\overline{8}|}\times(1.03)^{16}+60\ddot{s}_{\overline{8}|}^{(2)}\times(1.03)^{8}+60\ddot{s}_{\overline{8}|}^{(6)}$ at 3%

We need $d^{(6)}$ from $\left(1-\dfrac{d^{(6)}}{6}\right)^{6}=1-d=\dfrac{1}{1+i}=\dfrac{1}{1.03}$

$$\Rightarrow 1-\frac{d^{(6)}}{6}=\left(\frac{1}{1.03}\right)^{1/6}\Rightarrow \quad d^{(6)}=\left(1-\left(\frac{1}{1.03}\right)^{1/6}\right)\times 6$$

$$= 0\ .029486111$$

Thus accumulated amount =

$$\frac{120}{a_{\overline{2}|}}s_{\overline{8}|}\times(1.03)^{16}+60\frac{i}{d^{(2)}}s_{\overline{8}|}\times(1.03)^{8}+60\frac{i}{d^{(6)}}s_{\overline{8}|} \text{ at } 3\%$$

$$=\frac{120}{1.9135}*8.8923*1.60471+60\times 1.022445*8.8923*1.26677+60*\frac{0.03}{0.029486111}*8.8923$$

= 894.877 + 691.040 + 542.837

= £2,128.75

(above uses factors in formulae and tables book; if book not used then exact answer is £2,128.7**7)**.

Generally well-answered but marginal candidates would have benefited from showing their intermediate working in greater depth and/or with greater clarity.

6 Let:

$i =$ nominal yield
$e =$ inflation rate
$i' =$ real rate

Then

$$1+i=(1+i')(1+e)$$

We first find i and then use above equation to find i':

$$175 = \frac{6}{(1+i)^{1/2}}+\frac{6\times 1.06}{(1+i)^{1\frac{1}{2}}}+\frac{6\times(1.06)^2}{(1+i)^{2\frac{1}{2}}}+\cdots$$

$$=\frac{6}{(1+i)^{1/2}}\left(1+\frac{1.06}{1+i}+\frac{(1.06)^2}{(1+i)^2}+\cdots\right)$$

$$=\frac{6}{(1+i)^{1/2}}\left(\frac{1}{1-\dfrac{1.06}{1+i}}\right)$$

$$=\frac{6(1+i)^{1/2}}{(1+i)\left(1-\dfrac{1.06}{1+i}\right)}$$

$$=\frac{6(1+i)^{1/2}}{i-0.06}$$

Let $i=10\%$, RHS = 157.32
$i=9\%$, RHS = 208.81

Hence

$$i\approx 0.09+0.01\times\left(\frac{208.81-175}{208.81-157.32}\right)$$

$= 0.09657$ (answer to nearest 0.1% is 9.6%)

Then we have

$$1.09657 = (1+i')1.04$$

$\Rightarrow \quad i' = 5.44\%$ p.a. real return (answer to nearest 0.1% is 5.4%)

An alternative method of formulating the equation in real terms to find i' directly was perfectly valid.

7 (i)

Time	*Spot rate of interest*
½	0.01
1½	0.03
2½	0.05
3½	0.07
4½	0.09

Value of liabilities (£m)

$$V = 1\left(v_{1\%}^{0.5} + v_{3\%}^{1.5} + v_{5\%}^{2.5} + v_{7\%}^{3.5}\right) + 2v_{9\%}^{4.5}$$

$_{1\%}v^{½} = 0.99504$
$_{3\%}v^{1.5} = 0.95663$
$_{5\%}v^{2.5} = 0.88517$
$_{7\%}v^{3.5} = 0.78914$
$_{9\%}v^{4.5} = 0.67855$
$V = 4.98308$

(ii) Because expectations of short-term interest rates rise with term and the yield curve is determined by expectations theory.

Because investors have a preference for liquidity which puts an upwards bias on the yield curve (e.g. because long-term bonds are more volatile). A rising curve would be compatible, for example, with constant expectations of interest rates.

Because the market segmentation theory holds and investors short-term bonds might be in demand by investors such as banks (or there is an undersupply of short-term bonds or less demand/more supply for long-term bonds).

(iii) Spot rate to time 4.5 is 9%. Spot rate to time 3.5 is 7%. Therefore:

$$1.09^{4.5}/(1.07)^{3.5} = \text{forward rate from 3.5 to 4.5} = 16.3\%$$

Common errors in part (i) were to assume payments at the end of the year and/or to assume that the payments should be valued with the end of year spot rate (2%, 4%, 6% etc.)

8 (i) $P = 9a_{\overline{10|}7\%} + 100v_{7\%}^{10}$

$v^{10} = 0.50835; \quad a_{\overline{10|}} = 7.02358$

$P = 9 \times 7.02358 + 100 \times 0.50835$

$= 114.047$

(ii) Discounted mean term

$$= \frac{\sum tC_t v^t}{\sum C_t v^t}$$

$$= \frac{9(Ia)_{\overline{10|}} + 10 \times 100v^{10}}{114.047}$$

$$= \frac{9 \times 34.7391 + 10 \times 100 \times 0.50835}{114.047}$$

= 7.199 years

(iii) Duration will be higher because the payments will be more weighted towards the end of the term.

(iv) (a) Effective duration = duration $/(1+i)$

$= 7.199 / 1.07 = 6.728$ years

(b) Effective duration would indicate the extent to which the value of the bond would change if there were a uniform change in interest rates. It is therefore an indication of the risk to which the investor is exposed if interest rates rise and the price of the security falls before it is sold.

Many of the explanations from candidates in part (iii) were very unclear.

9 Present value of outgoings = 4,000,000 + 900,000$v^{½}$

$$@12\% = 4,850,420$$

Present value of income =

$$360,000v\ddot{a}_{\overline{1}|}^{(4)} + 360,000(1+k)v^2\ \ddot{a}_{\overline{1}|}^{(4)}$$

$$+\ldots+360,000v^5(1+k)^4\ \ddot{a}_{\overline{1}|}^{(4)} + 6,800,000\ v^6$$

$$= 360,000\ \ddot{a}_{\overline{1}|}^{(4)}\ v\left(1+v_j+v_j^2+v_j^3+v_j^4\right)+6,800,000\ v^6$$

where $j = \dfrac{1.12}{1+k} - 1$

$$\ddot{a}_{\overline{1}|}^{(4)} = 0.95887 @ 12\%$$

So, present value of income $= 360,000 \times 0.95887 \times \dfrac{1}{1.12} \times \ddot{a}_{\overline{5}|}^{j} + 6,800,000\ v^6$

$$= 308,209\ddot{a}_{\overline{5}|}^{j} + 3,445,092$$

Hence, for IRR = 12%, 4,850,420 = $308,209\ \ddot{a}_{\overline{5}|}^{j} + 3,445,092$

so $\ddot{a}_{\overline{5}|}^{j} = 4.55966$

At 4% $\ddot{a}_{\overline{5}|} = 4.62990$

5% $\ddot{a}_{\overline{5}|} = 4.54595$

$$j \approx 4 + \frac{4.62990 - 4.55966}{4.62990 - 4.54595} = 4.837\%$$

$$j = \frac{1.12}{1+k} - 1$$

$$0.04837 = \frac{1.12}{1+k} - 1$$

$$\therefore 1.04837(1+k) = 1.12$$

$$k = \frac{1.12}{1.04837} - 1 = 0.0683 = 6.83\% \text{ (exact answer is 6.84\%)}$$

Marginal candidates again would have benefited from showing more intermediate working. In project appraisal questions, it is good exam technique to show working and answers for each component of income and outgo separately so that partial marks can be given if any errors are made within a component.

10 (i) For $0 \le t \le 4$:

$$v(t) = \exp\left(-\int_0^t \delta(s)\,ds\right) = \exp\left(-\int_0^t 0.08ds\right)$$

$$= e^{-0.08t} \text{ and } v(4) = e^{-0.08\times 4} = e^{-0.32}$$

For $4 < t \le 9$:

$$v(t) = \exp\left(-\int_0^4 \delta(s)\,ds - \int_4^t \delta(s)\,ds\right) = e^{-0.32}\exp\left(-\int_4^t 0.12 - 0.01sds\right)$$

$$= e^{-0.32}\exp\left[-0.12s + 0.005s^2\right]_4^t$$

$$= e^{-0.32}\exp\left(0.48 - 0.08 - 0.12t + 0.005t^2\right)$$

$$= e^{0.08-0.12t+0.005t^2}$$

and $v(9) = e^{0.08-0.12\times 9+0.005\times 81} = e^{-0.595}$

For $t > 9$:

$$v(t) = \exp\left(-\int_0^9 \delta(s)\,ds - \int_9^t \delta(s)\,ds\right) = e^{-0.595}\exp\left(-\int_9^t 0.05ds\right)$$

$$= e^{-0.595}\exp\left[-0.05s\right]_9^t = e^{-0.595}\exp\left(0.45 - 0.05t\right) = e^{-0.145-0.05t}$$

(ii) Present value is

$$\int_{10}^{12} 100e^{0.03t}\left(\exp\left(-\int_0^t \delta(s)\,ds\right)\right)dt$$

$$=\int_{10}^{12} 100e^{0.03t}e^{-0.145-0.05t}\,dt$$

$$=\int_{10}^{12} 100e^{-0.145-0.02t}\,dt=\left[\frac{100e^{-0.145-0.02t}}{-0.02}\right]_{10}^{12}$$

$$=\left[-5{,}000e^{-0.145-0.02t}\right]_{10}^{12}=5{,}000\left(e^{-0.345}-e^{-0.385}\right)$$

$$=138.85$$

(iii) Present value $=1{,}000a_{\overline{3}|\delta=8\%}=1{,}000\dfrac{1-e^{-0.08\times 3}}{e^{0.08}-1}=\pounds 2{,}561.89$

11 (i) $(Ia)_{\overline{n}|}=v+2v^2+3v^3+\ldots+nv^n \qquad (1)$

$(1+i)(Ia)_{\overline{n}|}=1+2v+3v^2+\ldots+nv^{n-1} \qquad (2)$

$(2)-(1)\Rightarrow i(Ia)_{\overline{n}|}=1+v+v^2+\ldots+v^{n-1}-nv^n$

$$\Rightarrow (Ia)_{\overline{n}|}=\frac{\left(1+v+v^2+\ldots+v^{n-1}\right)-nv^n}{i}=\frac{\ddot{a}_{\overline{n}|}-nv^n}{i}$$

(ii) Work in months i.e. use a monthly interest rate of 1.25% per month effective:

$$30{,}000=Xv+2Xv^2+\ldots+60Xv^{60}=X\,(Ia)_{\overline{60}|@1.25\%}=X\left(\frac{\ddot{a}_{\overline{60}|}-60v^{60}}{i}\right)$$

$$=X\left(\frac{\dfrac{1-v^{60}}{d}-60v^{60}}{i}\right)=X\left(\frac{\dfrac{1-1.0125^{-60}}{0.0125/1.0125}-60\times 1.0125^{-60}}{0.0125}\right)$$

$$=1126.8774X\Rightarrow X=\pounds 26.62$$

(iii) Equation of value:

$$30,000 = v^{36} 958.32 a_{\overline{60}|}$$

$$\Rightarrow v^{36} a_{\overline{60}|} = 31.3048$$

Try i = 1%: LHS = 31.4202

Try i = 1.1%: LHS = 29.5098

Interpolate: $i = 1\% + 0.1\% \left(\dfrac{31.3048 - 31.4202}{29.5098 - 31.4202} \right) = 1.0060\%$

APR is $(1 + 0.010060)^{12} - 1 = 12.8\%$ to 1 d.p.

The bank is unlikely to be happy to accept the suggestion as it will be earning a lower rate of return compared with the original proposal of 15% per annum convertible monthly (=16.1% per annum effective).

(iv) The student's arrangement will lead to a greater total of payments (60 payments of 36X) when compared to the original arrangement (60 payments of 30.5X on average) but will incur a lower rate of interest. This is because under the student's arrangement no capital or interest will be paid for three years. The extra total of payments will not be sufficient to cover the deferred interest at the bank's preferred rate.

In part (i), candidates were expected to show the first and last terms of each series used to derive the result so that the proof is absolutely clear. In part (ii), candidates should show enough steps to demonstrate that they have performed the calculations required to actually prove the answer (e.g. show the numerical values for the factors used) . In part (iv), if interpolating on a monthly interest rate (as in the above solution) the guesses most be close enough together to ensure the estimated annual rate is close enough to the correct answer.

12 (i) Let S_n = Accumulated value at time n of £1 invested at time 0

$$S_n = (1+i_1)(1+i_2)....(1+i_n)$$

$$\Rightarrow E[S_n] = E\left[(1+i_1)(1+i_2)....(1+i_n)\right]$$

$$= E(1+i_1).\,E(1+i_2)....\,.E(1+i_n) \text{ by independence}$$

and $E(1+i_t) = 1 + E(i_t) = 1 + j$

Hence

$$E(S_n) = (1+j)^n$$

Now

$$\text{Var}[S_n] = E\left[S_n^2\right] - \left(E[S_n]\right)^2$$

$$E\left[S_n^2\right] = E\left[(1+i_1)^2(1+i_2)^2 \ldots.(1+i_n)^2\right]$$

$$= E\left[(1+i_1)^2\right].E\left[(1+i_2)^2\right]\ldots. \quad .E\left[(1+i_n)^2\right]$$

by independence

and

$$E\left[(1+i_t)^2\right] = E\left[\left(1+2i_t+i_t^2\right)\right]$$

$$= 1+2\,E(i_t)+E\left(i_t^2\right)$$

and

$$\text{Var}[i_t] = s^2 = E\left(i_t^2\right) - \left[E(i_t)\right]^2$$

$$= E\left(i_t^2\right) - j^2$$

$$\Rightarrow E\left(i_t^2\right) = s^2 + j^2$$

Hence

$$E\left[S_n^2\right] = \left(1+2j+j^2+s^2\right)^n$$

and

$$\text{Var}[S_n] = \left(1+2j+j^2+s^2\right)^n - (1+j)^{2n}$$

(ii) (a) $E(1+i_t) = 1 + E(i_t) = 1 + j = 1.04 = e^{\left(\mu + \sigma^2/2\right)}$

$\text{Var}(1+i_t) = \text{Var}(i_t) = s^2 = 0.12^2 = e^{\left(2\mu+\sigma^2\right)} \times \left(e^{\sigma^2} - 1\right)$

$$\Rightarrow \frac{0.12^2}{(1.04)^2} = e^{\sigma^2} - 1$$

$$\Rightarrow \sigma^2 = Ln\left[1 + \left(\frac{0.12}{1.04}\right)^2\right]$$

$$\Rightarrow \sigma^2 = 0.013226$$

$$1.04 = e^{\left(\mu + \frac{0.013226}{2}\right)}$$

$$\Rightarrow \mu = Ln\,1.04 - \frac{0.013226}{2}$$
$$= 0.032608$$

(b) $Ln(1+i_t) \sim N(0.032608, 0.013226)$

and we require probability $0.06 < i_t < 0.08$

$$= \Pr(1.06 < 1 + i_t < 1.08)$$

$$= \Pr\left(Ln\,1.06 < Ln\,(1+i_t) < Ln\,1.08\right)$$

$$= \Pr\left(\frac{Ln\,1.06 - 0.032608}{\sqrt{0.013226}} < \frac{Ln(1+i_t) - \mu}{\sigma} < \frac{Ln\,1.08 - 0.032608}{\sqrt{0.013226}}\right)$$

$= \Pr(0.22 < Z < 0.39)$ where $Z \sim N(0,1)$

$$= \Phi(0.39) - \Phi(0.22)$$

$$= 0.65173 - 0.58706 = 0.0647$$

i.e. 6% probability (using exact Φ function gives probability of 6.2%)

(iii) The probability in (ii) (b) is small. This is reasonable since the expected return in any year is 4%, and we are being asked to calculate the probability that the return is between 6%, and 8% (i.e. a range which does not include the expected value).

In part (i), it is important to note when the assumption of independence is required for both proofs. Common mistakes in the calculation of μ and σ were to assume that s^2 was 0.12 (rather than s) and 1 + j was 0.04 (rather than 1.04).

END OF EXAMINERS' REPORT