Applied Mathematical Finance I

Lecture 1: Introduction and Mathematical Prerequisites

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What is this course about?

- In this course, we will focus on theory and practice of interest rate and credit derivatives.
- To give some flavour, here are some questions that we are going to answer during the course
    - What are common types of interest rate and credit derivatives?
    - How are they valued and hedged?
    - What are popular interest rate models and how are they calibrated in practice?
    - What changed with the global financial crisis of 2007-08?
    - In view of LIBOR Transition, what is the new generation of products and models?

Motivation

- Consider a derivative contract with final payoff depending on a future realization of the Key Rate of the Central Bank of Russia. How it can be hedged?
- In case of equity derivative, a trader would hedge market risk by taking an offsetting position in underlying stock (delta-hedging).
- Interest rate, however, does not represent a price of a traded asset so one cannot directly buy or sell interest rate for hedging purposes.
- Our goal is to develop arbitrage-free pricing theory for fixed-income markets and today we briefly discuss some fundamentals of derivatives pricing.

General Setup

- We consider an economy with continuous and frictionless trading activity over a finite time interval $[0, T]$.
- Uncertainty is modelled via a probability space $(\Omega, \mathcal{F}, \mathbb{P})$.
- Information flow is represented by a filtration $\mathbb{F} = \{\mathcal{F}_t,\ 0 \leq t \leq T\}$, a family of sub-$\sigma$-algebras such that $\mathcal{F}_s \subseteq \mathcal{F}_t$ for $s \leq t$ and $\mathcal{F} = \mathcal{F}_T$. For technical reasons, we require that the filtration satisfies the usual conditions*.
- The state of economy at time t is given by a vector of prices of n dividend-free assets $S_t = \left(S_t^1, \ldots S_t^n\right)^\top$.
- Process S is assumed to be adapted to $\mathbb{F}$ meaning that S_t is fully observable at t.

*That is, $\mathcal{F}_t$ is right-continuous for $t \in [0, T]$ and $\mathcal{F}_0$ contains all subsets of sets of zero probability.

$X(t,\omega)$ $\quad \mathcal{B}[0,t] \otimes \mathcal{F}_t$

Trading Strategies

$\forall\, t \in [0,T] \quad X : [0,t] \times \Omega \to \mathbb{R}$

càdlàg
RCLL

- A trading strategy is a progressively measurable process $\phi = (\phi^1, \ldots, \phi^n)^\top$. Interpretation: ϕ^i_t denotes the number of units of the i-th asset held in the portfolio at time t.

$\langle \varphi, S \rangle$

- The value V_t of the trading strategy ϕ at time t is defined as $V_t = \boxed{\phi_t^\top \cdot S_t}$.
- Trading gains (or losses) over a small time interval $[t, t+dt]$ are $\boxed{\phi_t^\top \cdot (S_{t+dt} - S_t)}$.
- When continuously rebalancing the portfolio over time interval $[t, T]$, the total profit and loss (P&L) become

$$\boxed{\text{P\&L}(t,T) = \int_t^T \phi_u^\top \, dS_u.}$$

- To proceed, let us briefly recall some relevant mathematical concepts.

Driving Process

- We will consider a special case of the abstract setup above where information is generated by standard k-dimensional Wiener process $W_t = \left(W_t^1, \ldots W_t^k\right)^\top$
  - W^i is independent of W^j for $i \neq j$.
  - Independent Gaussian increments $W_t^i - W_s^i \sim \mathcal{N}(0, t-s)$ for $s \leq t$.
  - Each component W^i is a continuous process starting at 0.
- Normally, filtration $\mathbb{F}$ is the one generated by W (possibly augmented)

$$\mathcal{F}_t = \mathcal{F}_t^W = \sigma\{W_s,\ 0 \leq s \leq t\}.$$

$\mathbb{R}_+$ $\quad \tau \to \mathbb{R}_+ \cup \{+\infty\}$

Martingales and Local Martingales

- Let X be a real-valued adapted process with $\mathbb{E}^{\mathbb{P}}|X_t| < \infty$ for all $t \in [0, T]$, where $\mathbb{E}^{\mathbb{P}}$ denotes expectation under measure $\mathbb{P}$. X is called a $\mathbb{P}$-martingale on $[0, T]$ if

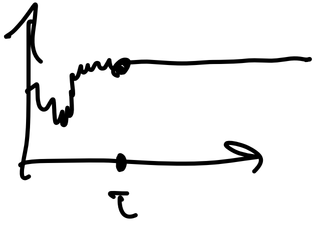
$$\mathbb{E}^{\mathbb{P}}_s[X_t] = \mathbb{E}^{\mathbb{P}}[X_t | \mathcal{F}_s] = X_s \quad \text{a. s.}$$

$\tau : \Omega \to [0,T]$

for all $0 \le s \le t \le T$.

$\forall t \quad \{\tau < t\} \in \mathcal{F}_t$

- Interpretation: "best" prediction of a future state X_t is its current value X_s.
- The notion of a martingale can be generalized in the following way. X is said to be a local martingale (under $\mathbb{P}$) if there exists a sequence of stopping times $\{\tau_n\}_{n=1}^{\infty}$ with $\tau_n \to T$ a. s. when $n \to \infty$, such that $X_{t\wedge\tau_n} = X_{\min\{t,\tau_n\}}$ is a martingale.



- Note that local martingales are not necessarily integrable.

Quadratic Variation and Covariation

- For a stochastic process $X : \mathbb{R}_+ \times \Omega \to \mathbb{R}$, quadratic variation $\langle X \rangle_t$ over $[0, t]$ is defined as

$$\langle X \rangle_t = \lim_{\|P_n\| \to 0} \sum_{i=1}^{n} (X_{t_i} - X_{t_{i-1}})^2,$$

  where $P_n = \{t_0, t_1, \ldots, t_n\}$ is a partition of the interval $[0, t]$ and $\|P_n\| = \max_{1 \le i \le n}(t_i - t_{i-1})$ is the mesh. It is only required that the limit exists under convergence in probability.

  C^1 $\quad \sum |X_{t_i} - X_{t_{i-1}}|$

- Recall that for standard one-dimensional Brownian motion B we have $\boxed{\langle B \rangle_t = t.}$
- Quadratic covariation $\langle X, Y \rangle_t$ of two processes X and Y can be defined via polarization identity

  $\langle X, Y \rangle = \lim \sum (X_{t_i} - X_{t_{i-1}})(Y_{t_i} - Y_{t_{i-1}})$

$$\langle X, Y \rangle_t = \frac{1}{2} \left(\langle X + Y \rangle_t - \langle X \rangle_t - \langle Y \rangle_t \right).$$

$\langle X \rangle = \langle X, X \rangle$

Stochastic Integral

$\int_0^t \sigma_s \, dB_s$ $\quad X(t, \omega)$

- As we have already seen, P&L of a portfolio over interval $[0, t]$ is given by

$\mathbb{F} \; \{\mathcal{F}^W\}$

$$\text{P\&L} = \int_0^t \phi_u^\top \, dS_u$$

and hence stochastic integration plays a central role in quantitative finance.

- How do we define stochastic integral $\int_0^t \sigma_s \, dB_s$, where B is standard one-dimensional Brownian motion?

- First thing to note, is that it cannot be defined pathwise (for fixed $\omega \in \Omega$) as a Riemann–Stieltjes integral because of infinite variation of B on $[0, t]$

$$\sum_{i=1}^{n} |B_{t_i} - B_{t_{i-1}}| \geq \frac{1}{\max_i |B_{t_i} - B_{t_{i-1}}|} \sum_{i=1}^{n} (B_{t_i} - B_{t_{i-1}})^2 \to \infty \text{ a. s. for } \|P_n\| \to 0.$$

$\cancel{t}$ $\quad \to t$

Stochastic Integral (continued)

$[t, t+\delta t)$

- While stochastic integral cannot be defined as the almost sure limit, we can resort to other notions of convergence of random variables.
- Now consider for the moment $\int_0^t B_s dB_s$. Define

$$L_n = \sum_{i=1}^{n} B_{t_{i-1}} (B_{t_i} - B_{t_{i-1}}), \quad R_n = \sum_{i=1}^{n} B_{t_i} (B_{t_i} - B_{t_{i-1}}).$$

$\frac{B_{t_{i-1}} + B_{t_i}}{2}$

$\int \delta \circ dW$

I_{to}

- Note that

$$L^2 - \lim(R_n - L_n) = L^2 - \lim \sum_{i=1}^{n} (B_{t_k} - B_{t_{k-1}})^2 = \langle B \rangle_t = t \neq 0.$$

- Therefore, we expect stochastic integral to depend on the choice of points at which the integrand is evaluated. What choice should we make?

Itô Integral

- Recall that P&L of a trading strategy ϕ over $[t, t+dt]$ is $\phi_t^{\mathsf{T}}(S_{t+dt} - S_t)$.
- This suggests using left endpoint of each subinterval as the evaluation point of the integrand which leads to the notion of the Itô Integral.
- Itô integral $\int_0^t \sigma_s \, dB_s$ can be defined for any real-valued progressively measurable process σ such that $\int_0^t |\sigma_s|^2 \, ds < \infty$ a.s. but in general case construction is rather involved.
- For our needs, it is sufficient to assume that σ is RCLL* adapted process in which case Itô integral can be seen as

$$\int_0^t \sigma_s \, dB_s = \lim_{\|P_n\| \to 0} \sum_{i=1}^{n} \sigma_{t_{i-1}} \left(B_{t_i} - B_{t_{i-1}}\right),$$

$\to L^2$

where the limit is understood in the sense of convergence in probability.

*Trajectories are a.s. right-continuous with left limits.

$(S^1, \ldots, S^n)$ $\int_0^t \varphi_u^T \cdot dS_u$

Multidimensional Itô Integral

- Let $\sigma : \mathbb{R}_+ \times \Omega \to \mathbb{R}^{n\times k}$ be an adapted process such that for all $t \in [0, T]$

$$\int_0^t |\sigma_s|^2 \, ds < \infty,$$

  where $|\sigma_s| = \|\sigma_s\|_F = \sqrt{\operatorname{tr}\left(\sigma_s \cdot \sigma_s^\intercal\right)}$ is the Frobenius norm. In this case we say that σ is in $\mathcal{L}^2_{\text{ad}}$.

- Now define

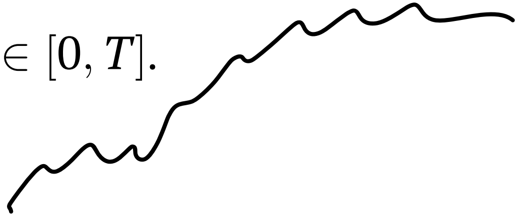
$$X_t = \int_0^t \sigma_s \, dW_s = \int_0^t \begin{pmatrix} \sigma_s^{11} & \cdots & \sigma_s^{1k} \\ \vdots & \ddots & \vdots \\ \sigma_s^{n1} & \cdots & \sigma_s^{nk} \end{pmatrix} \begin{pmatrix} dW_s^1 \\ \vdots \\ dW_s^k \end{pmatrix}$$

$$\begin{pmatrix} X^1 \\ \vdots \\ X^n \end{pmatrix}$$

  as a vector-valued process with $X_t^i = \sum_{j=1}^k \int_0^t \sigma_s^{ij} \, dW_s^j$.

Properties of multidimensional Itô Integral

- Define $I_t = \int_0^t \sigma_s \, dW_s$ where $\sigma : \mathbb{R}_+ \times \Omega \to \mathbb{R}^{n \times k}$ is in $\mathcal{L}^2_{\text{ad}}$. We have
  - I_t is $\mathcal{F}_t$-measurable.
  - I_t is a continuous local martingale.
- Define a space $\mathcal{H}^2_{\text{ad}} \subset \mathcal{L}^2_{\text{ad}}$ as a set of processes σ such $\boxed{\mathbb{E}^{\mathbb{P}} \left(\int_0^T |\sigma_s|^2 \, ds \right) < \infty.}$ We then have for $\sigma \in \mathcal{H}^2_{\text{ad}}$
  - I_t is a true martingale and, in particular, $\mathbb{E}^{\mathbb{P}} I_t = 0$ for all $t \in [0, T]$.
  - Itô isometry: $\mathbb{E}^{\mathbb{P}} |I_t|^2 = \mathbb{E}^{\mathbb{P}} \left(\int_0^t |\sigma_s|^2 \, ds \right)$.
  - Auto-covariance: $\mathbb{E}^{\mathbb{P}} \left(I_t \cdot I_s^{\intercal} \right) = \mathbb{E}^{\mathbb{P}} \left(\int_0^{\min\{t,s\}} \sigma_u \cdot \sigma_u^{\intercal} \, du \right)$.



Itô Process

- An adapted continuous n-dimensional process X is called an Itô process if it admits a representation

$$X_t = X_0 + \int_0^t \mu_s \, ds + \int_0^t \sigma_s \, dW_s, \quad \text{a.s.}, \ \forall t \in [0, T], \tag{1}$$

where stochastic processes $\mu : \mathbb{R}_+ \times \Omega \to \mathbb{R}^n$ and $\sigma : \mathbb{R}_+ \times \Omega \to \mathbb{R}^{n \times k}$ are assumed to satisfy suitable integrability conditions.

- It is customary to represent the above formula using the differential notation as

$$\boxed{dX_t = \mu_t \, dt + \sigma_t \, dW_t.}$$

$$\int_0^t \varphi_u^T \cdot dS_u$$

- Integration with respect to Itô process is defined via

$$\int_0^t Y_s \, dX_s = \int_0^t Y_s \, \mu_s \, ds + \int_0^t Y_s \, \sigma_s \, dW_s,$$

where $Y : \mathbb{R}_+ \times \Omega \to \mathbb{R}^{m \times n}$.

Itô's Lemma: One-dimensional Case

- Given a function $f : [0, T] \times \mathbb{R} \to \mathbb{R}$ and a scalar Itô process X

$$\boxed{dX_t = \mu_t\, dt + \sigma_t\, dW_t,}$$

$f(t, x)$

the question is whether the process $Y_t = f(t, X_t)$ is an Itô process.

- Suppose that $f \in C^{1,2}$. Then $Y_t = f(t, X_t)$ is an Itô process such that

$$dY_t = \boxed{\frac{\partial f}{\partial t}(t, X_t)\, dt + \frac{\partial f}{\partial x}(t, X_t)\, dX_t} + \frac{1}{2}\frac{\partial^2 f}{\partial x^2}(t, X_t)\, d\langle X \rangle_t,$$

where quadratic variation of X is given by

$\langle W \rangle_t = t$

$dX^2 = \sigma^2 dt$

$$\boxed{\langle X \rangle_t = \int_0^t \sigma_s^2\, ds.}$$

$dY = \frac{\partial f}{\partial t} dt + \frac{\partial f}{\partial x} dX + \frac{1}{2}\frac{\partial^2 f}{\partial x^2}(dX)^2$

- We usually simply write the above relation as $dX_t^2 = \sigma_t^2\, dt$.

Itô's Lemma: Multidimensional Case

$X = \begin{pmatrix} X^1 \\ \vdots \\ X^n \end{pmatrix}$

- Now consider $f : [0, T] \times \mathbb{R}^n \to \mathbb{R}$, $f \in C^{1,2}$, and assume that X is given by formula (1). Then $Y_t = f(t, X_t)$ is an Itô process such that

$$dY_t = \frac{\partial f}{\partial t}(t, X_t)\,dt + \sum_{i=1}^{n} \frac{\partial f}{\partial x_i}(t, X_t)\,dX_t^i + \frac{1}{2}\sum_{i,j=1}^{n} \frac{\partial^2 f}{\partial x_i \partial x_j}(t, X_t)\,d\langle X^i, X^j\rangle_t,$$

where

$$\langle X^i, X^j\rangle_t = \int_0^t \sum_{l=1}^{k} \sigma_s^{il}\,\sigma_s^{jl}\,ds.$$

- In particular, taking $f(x_1, x_2) = x_1 x_2$, we get Itô product rule for scalar processes X and Y

$$d(X_t Y_t) = X_t\,dY_t + Y_t\,dX_t + d\langle X, Y\rangle_t.$$

Stochastic Differential Equations

$dX = \operatorname{sgn}(X)\, dW$

$X \quad W$

- Usually, we restrict our attention to the case where both coefficients μ and σ in (1) are deterministic measurable functions of time and state.
- In other words, we consider stochastic differential equation (SDE) of the form

$$dX_t = \mu(t, X_t)\, dt + \sigma(t, X_t)\, dW_t, \quad X_0 = x \in \mathbb{R}^n.$$

$\mathbb{F} = \{\mathcal{F}_t^W\}$

- A strong solution to the SDE is an Itô process X adapted to $\mathcal{F}_t^W$ and satisfying

$X_t \in \mathcal{F}_t$

$$X_t = x + \int_0^t \mu(s, X_s)\, ds + \int_0^t \sigma(s, X_s)\, dW_s.$$

- As is well known, if $\mu(t,x)$ and $\sigma(t,x)$ satisfy the Lipschitz and linear growth condition in x, then there exists a unique solution to the SDE above. Moreover, it is a Markov process.

weak solution

Self-financing Strategies

- Now that the basics are covered, let us get back to our setup.
- A trading strategy ϕ is said to be self-financing if, for any $t \in [0, T]$, its value $V_t = \phi_t^\intercal \cdot S_t$ satisfy

$$\boxed{V_t - V_0 = \int_0^t \phi_u^\intercal \, dS_u,}$$

or, in differential notation,

$$\boxed{dV_t = \phi_t^\intercal \, dS_t.}$$

$S \, d\varphi \quad dS \, d\varphi$

- Interpretation: changes in portfolio value are only due to trading gains or losses, with no funds being added or withdrawn.
- We denote by Φ the class of all self-financing strategies.

Arbitrage

- By arbitrage we generally mean a possibility to create "something for nothing".
- Formally, an arbitrage opportunity is a self-financing strategy $\phi \in \Phi$ for which $V_0 = 0$ and, for some $t \in [0, T]$,

$$\boxed{V_t \geq 0 \text{ a. s., and } \mathbb{P}(V_t > 0) > 0.}$$

- In economic equilibrium, arbitrage should not exist and requiring the absence of arbitrage strategies is a starting point of asset pricing.

Market Model

$dB = rB\,dt$
$dS = \mu S\,dt + \sigma S\,dW_t$

- We assume the dynamics of the price vector S are given by an Itô process

$$dS_t = \mu_t\,dt + \sigma_t\,dW_t, \quad S_0 = s.$$

- By market model we mean a pair (S, Ψ), where $\Psi \subseteq \Phi$ is a set of all "allowable" self-financing trading strategies.

- It turns out that arbitrage opportunities are not excluded a priori from Φ and hence (S, Φ) is generally not free of arbitrage even for simple choice of model dynamics.

- This means that we have to impose some additional restrictions on the set of allowed trading strategies to get a meaningful model.

Equivalent Measures and Radon-Nikodym Theorem

- We now turn to the question of characterizing the conditions under which market model is arbitrage-free.
- First, we recall that two probability measures $\mathbb{P}$ and $\hat{\mathbb{P}}$ (on the same measurable space) are said to be equivalent ($\mathbb{P} \sim \hat{\mathbb{P}}$) if they have the same null-sets

$$\mathbb{P}(B) = 0 \iff \hat{\mathbb{P}}(B) = 0, \quad \forall B \in \mathcal{F}.$$

- Radon-Nikodym Theorem
Suppose that $\mathbb{P} \sim \hat{\mathbb{P}}$. There exists a unique (a.s.) random variable ξ, which is called a Radon-Nikodym derivative and is usually denoted as $\frac{d\hat{\mathbb{P}}}{d\mathbb{P}}$, such that

$$\boxed{\hat{\mathbb{P}}(B) = \int_B \xi \, d\mathbb{P},} \quad \forall B \in \mathcal{F}.$$

$1 = \hat{\mathbb{P}}(\Omega) = \int_{\Omega} \xi \, d\mathbb{P} = \mathbb{E}^{\mathbb{P}}_{\xi}$

- Of course, we must have $\frac{d\hat{\mathbb{P}}}{d\mathbb{P}} \geq 0$ a.s. and $\mathbb{E}^{\mathbb{P}}\left[\frac{d\hat{\mathbb{P}}}{d\mathbb{P}}\right] = 1.$

Numéraires and Equivalent Martingale Measures

N , $\mathbb{Q}^N$ φ $V_t = \varphi_t^T \cdot S_t$

> 0 a.s.

- A numéraire N is any self-financing strategy such that its value N_t is strictly positive for all $t \in [0, T]$.

- A measure $\mathbb{Q}^N \sim \mathbb{P}$ is called an equivalent martingale measure induced by N if the normalized asset process $\frac{S}{N}$ is a $\mathbb{Q}^N$-martingale. $\left(\frac{S^1}{N}, \dots, \frac{S^n}{N}\right)^T$

- Note that a normalized price $\frac{V}{N}$ of an arbitrary self-financing portfolio $\phi \in \Phi$ will generally only be a local martingale under $\mathbb{Q}^N$. $\tilde{\varphi}$ $\frac{\tilde{\varphi}}{N}$ $= \int (\dots) dW^{Q}$

First Fundamental Theorem of Asset Pricing

N $\mathbb{Q}^N$

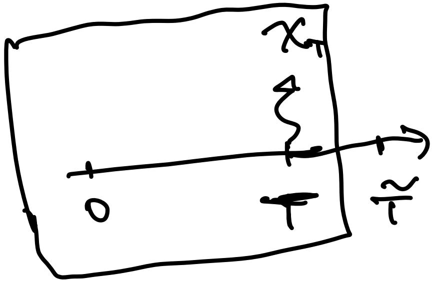
- Let N be a numéraire and $\mathbb{Q}^N$ be a corresponding martingale measure.
- We are now ready to define "allowable" strategies. Self-financing strategy ϕ is called permissible if its normalized price process $\frac{V}{N}$ is a true $\mathbb{Q}^N$-martingale. We denote by $\Psi(N)$ the set of all permissible strategies.

NFLVR

- Sufficient Condition for No-Arbitrage
  If there exists a martingale measure $\mathbb{Q}^N$ corresponding to some numéraire N, then market model $(S, \Psi(N))$ is free of arbitrage.

  $[S, \Psi(N)]$

Derivative Securities



$\mathcal{X}_T \in \mathcal{F}_T$

- A derivative security (contingent claim) with maturity T is an $\mathcal{F}_T$-measurable random variable $\mathcal{X}_T$.

S

- Interpretation: stochastic payoff at T and no payments before T.

- We say that derivative security $\mathcal{X}_T$ is attainable if there exists a permissible trading strategy ϕ such that

$$V_T = \phi_T^{\top} \cdot S_T = \mathcal{X}_T \quad \text{a.s.}$$

- The trading strategy ϕ in the above formula is said to replicate the derivative security.

Derivatives Pricing

$\boxed{\pi_t = V_t}$

Law of one price

- Consider a numéraire N inducing a martingale measure $\mathbb{Q}^N$. Let π_t denote the price of an attainable contingent claim $\mathcal{X}_T$ at time t.
- No-arbitrage principle implies that π_t must be equal to the value of the replicating portfolio V_t. Since the replicating portfolio is not yet know explicitly (we only know it exists), how do we actually compute π_t?
- By the First Fundamental Theorem of Asset Pricing

deflated

$V_T = \mathcal{X}_T$ a.s.

$$\frac{\pi_t}{N_t} = \frac{V_t}{N_t} = \boxed{\mathbb{E}_t^{\mathbb{Q}^N}\left[\frac{V_T}{N_T}\right]} = \mathbb{E}_t^{\mathbb{Q}^N}\left[\frac{\mathcal{X}_T}{N_T}\right]$$

and therefore

PV

$$\boxed{\pi_t = N_t \cdot \mathbb{E}_t^{\mathbb{Q}^N}\left[\frac{\mathcal{X}_T}{N_T}\right].}$$

PDE

x_T

N_T

Complete Markets

$\mathcal{N}$ $\mathbb{Q}^{\mathcal{N}}$

- If all contingent claims $\mathcal{X}_T$ with finite variance can be replicated, the market model is said to be complete.

- Interpretation: in complete markets derivative securities are redundant in a sense that they can be created synthetically by trading the underlying assets $S^1, \ldots, S^n$.

- Second Fundamental Theorem of Asset Pricing
  In the absence of arbitrage, a market model is complete if and only if there exists a numéraire N inducing the unique martingale measure $\mathbb{Q}^N$.