INSTITUTE AND FACULTY OF ACTUARIES

EXAMINERS’ REPORT

September 2018

Subject CT1 – Financial Mathematics
Core Technical

Introduction

The Examiners’ Report is written by the Principal Examiner with the aim of helping candidates, both those who are sitting the examination for the first time and using past papers as a revision aid and also those who have previously failed the subject.

The Examiners are charged by Council with examining the published syllabus. The Examiners have access to the Core Reading, which is designed to interpret the syllabus, and will generally base questions around it but are not required to examine the content of Core Reading specifically or exclusively.

For numerical questions the Examiners’ preferred approach to the solution is reproduced in this report; other valid approaches are given appropriate credit. For essay-style questions, particularly the open-ended questions in the later subjects, the report may contain more points than the Examiners will expect from a solution that scores full marks.

The report is written based on the legislative and regulatory context pertaining to the date that the examination was set. Candidates should take into account the possibility that circumstances may have changed if using these reports for revision.

Mike Hammer
Chair of the Board of Examiners
December 2018

A. General comments on the *aims of this subject and how it is marked*

1. CT1 provides a grounding in financial mathematics and its simple applications. It introduces compound interest, the time value of money and discounted cashflow techniques which are fundamental building blocks for most actuarial work.

2. Please note that different answers may be obtained to those shown in these solutions depending on whether figures obtained from tables or from calculators are used in the calculations but candidates are not penalised for this. However, candidates may lose marks where excessive rounding has been used or where insufficient working is shown.

B. General comments on *student performance in this diet of the examination*

1. The comments that follow the questions concentrate on areas where candidates could have improved their performance. Where no comment is made, the question was generally answered well by most candidates. The examiners look most closely at the performance of the candidates close to the pass mark and the comments therefore often relate to those candidates.
2. The number of candidates taking this exam was much lower than in previous diets. This was not surprising given that non-members were not permitted to take this exam due to the fact CT1 without CT5 will not translate to a pass in any subject under the Curriculum 2019 structure.

C. Pass Mark

The Pass Mark for this exam was 60.

Solutions

Q1

(i) Let d be the annual simple rate of discount.

Assume the bank bond also pays out €100.

The present value of the amount invested in the bank bond would be X such that:

$X = 100(1.04)^{-91/365} = 99.0269$ (99.0276 if 365.25 days in a year used) [1]

To provide the same effective rate of return a treasury bill that pays 100 must have a price of 99.0269 and so $100\left(1-\frac{91d}{365}\right)=99.0269$ [1]

$d=\frac{365}{91}(1-0.990269)=0.03903$ (unchanged if 365.25 days in a year used) [1]

(ii) An additional factor could be the risk of the investments [1]

[Total 4]

Part (i) was well answered although some candidates did not explicitly give the price of treasury bill as asked for in the question. In part (ii), answers referring to present value (which is directly related to the rate of return) or term (which was the same for both investments) were not given credit. Credit was given for answers mentioning marketability or liquidity.

Q2

(i) $\delta = \ln(1+i) = -\ln(1-d) = -\ln 0.95 = 0.051293$ [1]

(ii) $\left(1+\frac{i^{(12)}}{12}\right)^{12} = 1+i = \frac{1}{1-d} = 0.95^{-1} \Rightarrow i^{(12)} = 12\left(0.95^{-1/12}-1\right) = 0.051403$ [2]

(iii) $\left(1-\frac{d^{(12)}}{12}\right)^{12} = 1-d = 0.95 \Rightarrow d^{(12)} = 12\left(1-0.95^{1/12}\right) = 0.051184$ [1]

[Total 4]

The best answered question on the paper.

Q3 (i) Time-weighted rate of return is i where:

$$(1+i)^{2.5} = \frac{70}{60}\frac{300}{70+100} = 2.05882$$ [2]

$$\Rightarrow 1+i = 1.33489 \Rightarrow i = 0.3349$$ [1]

(ii) The money weighted rate of return gives a greater weighting to performance when there is more money in the fund. [½]
The fund was performing better after it had been given the large injection of money on 1 January 2017. [1½]
[Total 5]

Part (i) was answered well. As with similar questions in previous diets, part (ii) was poorly answered. It is important in this type of question to refer to the actual results obtained and the actual data given and the majority of marks in this part were awarded for this.

Q4 (i) (a) Price $= 3a_{\overline{50|}} + 103v^{50}$ at 1.5% working in half-years [1]

$$= 3\times 34.9997 + 103\times 0.47500 = 153.925$$ [1]

(b) Three months later the price will be $= 153.925(1.015)^{\frac{1}{2}} = 155.075$ [1]

(ii) Is there a Capital gain?

$$i^{(2)} = 2\left(1.1^{\frac{1}{2}} - 1\right) = 9.762\%$$ [½]

$$\frac{D}{R}(1-t_1) = \frac{6}{1.03}\times 0.7 = 4.078\%$$ [½]

$$i^{(2)} \le g(1-t) \Rightarrow \text{Capital gain}$$ [½]

Price paid per £100 nominal = P where

$$P = 0.7\times 6a^{(2)}_{\overline{25|}} + 103v^{25} - 0.4(103-P)v^{25} \text{at } 10\%$$

$$P = \frac{0.7 \times 6a_{\overline{25}|}^{(2)} + 0.6 \times 103v^{25}}{1 - 0.4v^{25}}$$
$$= \frac{0.7 \times 6 \times 1.024404 \times 9.0770 + 0.6 \times 103 \times 0.09230}{1 - 0.4 \times 0.09230}$$
$$= \frac{39.05395 + 5.70389}{0.96308} = 46.474$$

Price = £46.474 per £100 nominal [2½]

[Total 7]

The convertible half-yearly interest rate seemed to confuse some candidates but otherwise the questions was generally answered well.

Q5 (i) (a) Options – holder has the right but not the obligation to trade. [1]

Futures – both parties have agreed to the trade and are obliged to do so. [1]

(b) Call Option – right but not the obligation to BUY specified asset in the future at specified price. [1]

Put Option – right but not the obligation to SELL specified asset in the future at specified price. [1]

(ii) Present value of dividends

$= 0.1\left(1.025^{-1/2} + 1.025^{-1} + 1.025^{-3/2}\right) = 0.29270$ [2]

Forward price $= (1.1 - 0.29270) \times 1.025^2 = £0.84817$ [2]

[Total 8]

Well answered although some candidates in part (i) seemed to write down everything that they knew about options whereas the answer required was quite specific. In part (i)(a), partial credit was given for stating the option involved the payment of an initial premium by the holder.

Q6 (i) $(1+i_t) \sim \log N\left(\mu, \sigma^2\right) \Rightarrow S_{10} = \prod_{t=1}^{10}(1+i_t) \sim \log N\left(10\mu, 10\sigma^2\right)$

[1]

$E(1+i_t) = 1 + E(i_t) = 1 + j = 1.08 = e^{\left(\mu + \sigma^2/2\right)}$

$$\mathrm{Var}(1+i_t)=\mathrm{Var}(i_t)=s^2=0.07^2=e^{(2\mu+\sigma^2)}\times\left(e^{\sigma^2}-1\right)$$ [1]

$$\Rightarrow \quad \frac{0.07^2}{(1.08)^2} \quad = \quad e^{\sigma^2}-1$$

$$\Rightarrow \quad \sigma^2=\ln\left[1+\left(\frac{0.07}{1.08}\right)^2\right]= \ 0.0041922$$

[1]

$$1.08=e^{\left(\mu+\frac{0.0041922}{2}\right)}$$

$$\Rightarrow \mu = \ \ln 1.08-\frac{0.0041922}{2}=0.074865$$ [1]

$$\Rightarrow S_{10}\sim \log N\left(0.74865, 0.041922\right)$$ [1]

(ii) $\ln S_{10}\sim N\left(0.74865, 0.041922\right)$

and we require X such that $P\left(6,000S_{10}>X\right)=0.975$

$$\Rightarrow P\left(\ln S_{10}>\ln\frac{X}{6,000}\right)=0.975$$ [1]

$$\Rightarrow 1-\Phi\left(\frac{\ln\frac{X}{6,000}-0.74865}{\sqrt{0.041922}}\right)=0.975$$ [1]

$$\Rightarrow \frac{\ln\frac{X}{6,000}-0.74865}{\sqrt{0.041922}}=-1.96$$

$$\Rightarrow X=6,000\exp\left(-1.96\times\sqrt{0.041922}+0.74865\right)=£8,492$$

[1]
[Total 8]

This question proved to be a good differentiator with strong candidates scoring well on both parts but many weaker candidates scoring very little. Full marks could still be scored in part (ii) even if candidates made errors.

Q7 (i) Present value is $v(20) = e^{-\int_0^{20} \delta(s)ds}$ [1]

$$\int_0^{20} \delta(s)\,ds = \int_0^{10} 0.03ds + \int_{10}^{20} 0.003sds$$
$$= \left[0.03s\right]_0^{10} + \left[0.0015s^2\right]_{10}^{20}$$
$$= 0.3 + 0.0015(400-100) = 0.75$$

[2]

$v(20) = e^{-0.75} = 0.47237$ [1]

(ii) Require δ such that $e^{-20\delta} = e^{-0.75} \Rightarrow \delta = 0.0375$ [2]

(iii) Present value $\int_4^8 \rho(t)v(t)\,dt = \int_4^8 e^{-0.06t}e^{-0.03t}dt = \int_4^8 e^{-0.09t}dt$ [2]

$$= \left[\frac{e^{-0.09t}}{-0.09}\right]_4^8$$ [1]

= -5.40836 + 7.75196 = 2.34360 [1]

[Total 10]

Well answered.

Q8 (i) (a) The payback period is the first point at which the total revenues from a project exceed the total cost, with no allowance made for interest. [1½]

(b) The payback period takes no account of interest at all. It is therefore inappropriate for assessing an investment project which should provide the investor with a return or be paid for from borrowings. [1]

The payback period takes no account of what happens after the payback period. In this particular case, it is known that the revenue from the project might be weighted towards the end and the payback period will make no allowance for this. [1½]

(ii) Work in £2017 millions at 6% per annum

PV of initial costs = $15a_{\overline{5|}} = 15 \times 4.2124 = 63.1855$ [1]

PV of running costs = $3\bar{a}_{\overline{30|}} = 3 \times \dfrac{1-1.06^{-30}}{\ln(1.06)} = 3 \times 14.1738 = 42.5213$ [1½]

PV of revenue in first ten years =

$$3.1\bar{a}_{\overline{10}|} = 3.1 \times \frac{1-1.06^{-10}}{\ln(1.06)} = 3.1 \times 7.57875 = 23.4941$$ [1½]

PV of revenue in years 11 to 30 =

$$v^{10}\left(3.2\bar{a}_{\overline{1}|} + 3.2 \times 1.05 v \bar{a}_{\overline{1}|} + \ldots 3.2 \times 1.05^{19} v^{19} \bar{a}_{\overline{1}|}\right)$$ [1½]

$$= 3.2 v^{10} \bar{a}_{\overline{1}|}\left(1 + 1.05v + \ldots (1.05v)^{19}\right)$$

$$= 3.2 v^{10} \bar{a}_{\overline{1}|} \frac{1-(1.05v)^{20}}{1-1.05v}$$

[1]

$$= 3.2 \times 1.06^{-10} \times \frac{1-1.06^{-1}}{\ln 1.06} \frac{1-(1.05/1.06)^{20}}{1-1.05/1.06}$$

$$= 3.2 \times 0.55839 \times 0.97142 \times 18.30506 = 31.7739$$

[1]

Sales proceeds are P such that

$$Pv^{30} = 63.1855 + 42.5213 - 23.4941 - 31.7739 = 50.4388$$

$$P = 50.4388 \times 1.06^{30} = £289.69m$$ [1½]

(iii) Probabilities could be assigned to the cash flows… [1]
… or a higher discount rate could be used to account for risk. [1]
[Total 15]

The calculations in part (ii) were generally done well but parts (i) and (iii) were poorly answered. Part (i) has been asked in previous diets and generally answered better by candidates. Whilst part (iii) has not often been asked, the answer comes directly from the Core Reading.

Q9 (i) The investor pays a purchase price at outset. [½]
The investor receives a series of coupon payments and a capital payment at maturity. [1]
The coupon and capital payments are linked to an index of prices (possibly with a time lag). [½]
[Time lag does not have to be mentioned].

(ii) Let the RPI three months before issue (end 9/2015) = 100
Relevant RPI values are three months before first coupon payment (end 3/2016), three months before second coupon payment (end 9/2016) etc.

Cash payments from the bond are in the following table:

Nominal payment per £100 nominal (*1*)	Base index (*2*)	Index three months before payment (*3*)	(*3*) divided by (*2*) (*4*)	Cash payment (*4*) ×1%×£1m
1	100	$100(1.02)^{0.5}$	1.00995	£10,100
1	100	100(1.02)	1.02	£10,200
1	100	$100(1.02)^{1.5}$	1.0301	£10,301
1	100	$100(1.02)^2$	1.0404	£10,404

There is also the sale value of £1,010,000 [3]

(iii) Equation of value is:

$$1,000,000 = 10,100v^{0.5} + 10,200v + 10,301v^{1.5} + 10,404v^2 + 1,010,000v^2$$ [2½]

Try 2.5%.
RHS of equation of value becomes 1,001,089

Try 3%
RHS of equation of value becomes 991, 538 [1½]

Interpolating:
$$i = 0.025 + (0.03 - 0.025) \times \frac{1,000,000 - 1,001,089}{991,538 - 1,001,089} \approx 0.0256$$ [1]

(iv) The equation of value for the real cash flows is as follows (working in half years):

$$\begin{aligned}1,000,000 = {} & 10,000\frac{RPI(\text{March 2016})}{RPI(\text{September 2015})} \times \frac{RPI(\text{December 2015})}{RPI(\text{June 2016})}v \\ & +10,000\frac{RPI(\text{September 2016})}{RPI(\text{September 2015})} \times \frac{RPI(\text{December 2015})}{RPI(\text{December 2016})}v^2 \\ & +10,000\frac{RPI(\text{March 2017})}{RPI(\text{September 2015})} \times \frac{RPI(\text{December 2015})}{RPI(\text{June 2017})}v^3 \\ & +10,000\frac{RPI(\text{September 2017})}{RPI(\text{September 2015})} \times \frac{RPI(\text{December 2015})}{RPI(\text{December 2017})}v^4 \\ & +1,010,000\frac{RPI(\text{December 2015})}{RPI(\text{December 2017})}v^4\end{aligned}$$

All the RPI factors cancel out except the last two because each is the ratio of RPI at three-month intervals multiplied by the inverse of that ratio. [4]

The equation of value therefore becomes:

$$1,000,000 = 10,000v + 10,000v^2 + 10,000v^3$$
$$+ (10,404 + 1,010,000) \times \frac{RPI\left(\text{December 2015}\right)}{RPI\left(\text{December 2017}\right)} v^4$$

Let rate of inflation per annum in the last three months = x

Equation of value becomes:

$$1,000,000 = 10,000a_{\overline{3}|} + \frac{1,020,404}{1.02^{1.75}(1+x)^{0.25}} v^4 \text{ at } 0.5\%$$

$$\Rightarrow (1+x)^{0.25} = \frac{1,020,404 \times 1.005^{-4}}{\left(1,000,000 - 10,000a_{\overline{3}|}\right) \times 1.02^{1.75}} = 0.995755$$

$$\Rightarrow (1+x) = 0.98313 \Rightarrow x = -0.01687$$ [3]

[Total 17

This proved to be the most difficult question on the paper by some margin. Whilst the examiners expected the calculations in part (iii) and especially part (iv) to be challenging, it was more surprising to see candidates having considerable difficulty with part (ii).

Q10 (i) A loan repayable by a series of payments at fixed times set in advance. [½]

Typically issued by banks and building societies [½]

Typically long-term … [½]
…e,g. used to fund house purchase
…and secured against the property [½]

Each payment contains an element to pay interest on the loan with the remainder being used to repay capital [½]

In its simplest form, the interest rate will be fixed … [½]
….and the payments will be of fixed equal amounts. [½]

The interest payment portion of the repayments will fall over time… [½]
… and the capital payments will rise over time. [½]

Risk that borrower defaults on loan [½]

Complications might be added such as allowing the loan to be repaid early or allowing the interest rate to vary. [½]
[Max 3]

(ii) (a) Annual repayment is X where

$10,000 = Xa_{\overline{10}|}$ at 4%

$$X = \frac{10,000}{8.1109} = \$1,232.91$$ [2]

(b) DMT of repayments =

$$\frac{\sum_{t=1}^{10} 1,232.91 \times tv^t}{\sum_{t=1}^{10} 1,232.91 \times v^t} = \frac{1,232.91 \times (Ia)_{\overline{10}|}}{10,000}$$ [1½]

$$= \frac{1,232.91 \times 41.9922}{10,000} = 5.1773 \text{ years}$$

[1½ including ½ for units]

(iii) Let the nominal amount invested in the second zero coupon bond = X and let term = n

If the present values are to be equal:

$$10,000 = 5,000v^2 + Xv^n \quad (1)$$ [1]

If the discounted mean terms are to be equal:

$$5.1773 = \frac{2 \times 5,000v^2 + nXv^n}{10,000}$$

$$\Rightarrow 51,773 = 10,000v^2 + nXv^n \quad (2)$$

[1½]

$$\text{Sub } (1) \text{ into } (2) \Rightarrow 51,773 = 10,000v^2 + n\left(10,000 - 5,000v^2\right)$$

$$\Rightarrow n = \frac{51,773 - 10,000v^2}{10,000 - 5,000v^2} = \frac{42,527}{5,377} = 7.909 \text{ years}$$

[2½]

$$\text{Sub back in } (1) \Rightarrow X = \frac{10,000 - 5,000v^2}{v^{7.909}} = \$7,332.81$$ [1]

(iv) (a) Monthly instalment is M where

$$10,000 = 12Ma_{\overline{25}|}^{(12)}$$ [1]

$$10,000 = 12M \frac{\left(1 - 1.04^{-25}\right)}{12\left(1.04^{1/12} - 1\right)}$$

$$\Rightarrow M = \frac{10,000 \times 0.0032737}{0.62488} = \$52.39$$

[1½]

(b) The assets now have a much longer duration than the liabilities. [1]
Therefore, if interest rates rise, the assets will fall in value by more than the liabilities and the bank will make a loss. [1]

(c) After ten years of payments, the capital outstanding is $12 \times 52.39 a^{(12)}_{\overline{15}|}$ [1]

$$= 12 \times 52.39 \frac{1 - 1.04^{-15}}{0.039289} = 7{,}117.15$$ [1]

Interest component of 121st payment $= 7{,}117.15 \times \frac{0.039289}{12} = \$23.30$ [1]

Capital component of 121st payment $= 52.39 - 23.30 = \$29.09$ [½]

[Total 22]

It was pleasing to see that many candidates scored well on this question even if they had struggled with the previous question. Parts (ii), (iii), (iv)(a) and (iv)(c) were also generally answered well. Part (i) was answered less well despite the wide range of mark-scoring points available to be made.

END OF EXAMINERS' REPORT