Survival strategies in an evolutionary finance model with endogenous asset payoffs

I. V. Evstigneev* A. A. Tokaeva† M. J. Vanaei‡ M. V. Zhitlukhin§

Abstract

We construct survival strategies in a dynamic stochastic market model with short-lived assets and endogenous asset prices. Survival means a possibility of keeping a strictly positive, bounded away from zero share of total market wealth over an infinite time horizon with probability one. We show that the presence of an agent who uses a survival strategy asymptotically determines the market characteristics such as asset prices, wealth distribution and market shares of agents. The main novelty of the paper compared to previous results in the literature, which mostly deal with exogenously defined asset payoffs, is that we assume the asset payoffs may depend on the strategies of market agents in an affine way.

Keywords: evolutionary finance, survival strategies, log-optimal strategies, capital growth, endogenous prices, endogenous payoffs, martingales.

1. Introduction

This paper studies investment strategies in financial markets which are good for the long run. The main goal is to construct strategies which cannot be driven out of the market by other agents in a particular dynamic stochastic market model. We call such strategies *survival*. We show that survival strategies present not only safe investment rules, but also asymptotically determine the market characteristics such as asset prices, wealth distribution and market shares of agents.

In our model a market is viewed from the standpoint of evolutionary behavioral finance (hereinafter EBF), a rapidly developing research area which applies the evolutionary approach to the modeling of financial markets dynamics. A characteristic feature of our model which makes it different, for example, from the theory of growth optimal

*Economics Department, University of Manchester, Oxford Road, Manchester, M13 9PL UK. E-mail: igor.evstigneev@manchester.ac.uk

†Faculty of Mechanics and Mathematics, Lomonosov Moscow State University, 1 Leninskie gory, Moscow, 119991 Russia.

‡Economics Department, University of Manchester, Oxford Road, Manchester, M13 9PL UK. E-mail: mohammadjavad.vanaei@postgrad.manchester.ac.uk

§Steklov Mathematical Institute of Russian Academy of Sciences, 8 Gubkina st., Moscow, 119991 Russia. Corresponding author. E-mail: mikhailzh@mi-ras.ru

investments (Kelly (1956), Breiman (1961), Algoet and Cover (1988), Hakansson and Ziemba (1995) and others), is that asset prices are not specified exogenously, but determined as a result of interaction of the strategies of market participants. At the same time, we do not assume that agents' behavior is fully rational and can be described by well-defined and precisely stated constrained optimization problems as in the classical dynamic equilibrium theory (see Kydland and Prescott (1982); Radner (1972, 1982)). Agents may use arbitrary strategies, in particular their behavior might be determined by their individual psychology. The process of market dynamics is described as a sequence of consecutive short-run equilibria determining equilibrium asset prices over each time period. The notion of a short-run price equilibrium is defined directly via the set of strategies of the market players specifying the patterns of their investment behavior (behavioral equilibrium).

Fundamental contributions to the evolutionary modeling of financial markets were made by, among others, Anderson et al. (1988); Arthur et al. (1997); Blume and Easley (1992); Bottazzi and Dindo (2014); Bottazzi et al. (2005); Brock et al. (2005); Farmer (2002); Farmer and Lo (1999); Hens and Schenk-Hoppé (2005, 2009); Lo (2004, 2005, 2017); Sciubba (2005).

Models integrating evolutionary and behavioral approaches were proposed in Amir et al. (2011, 2013). A survey describing the state of the art in EBF by 2016 and outlining a program for further research was given in Evstigneev et al. (2016). An elementary textbook treatment of the subject can be found in Evstigneev et al. (2015, Ch. 20). For a most recent review of the development of studies related to this area see Holtfort (2019).

The main focus of EBF models is on investment strategies that survive in the market selection process. Survival means a possibility of keeping with probability one a strictly positive, bounded away from zero share of total market wealth over an infinite time horizon irrespectively of the strategies used by the competing market agents. Typical results show that such strategies exist, are asymptotically unique and easily computable. The computations do not require the knowledge of hidden agents' characteristics such as individual utilities and beliefs.

While most of the models in the EBF literature assume *exogenously* specified payoffs, the main novelty of our work is that we consider a model with *endogenous* asset payoffs, which may depend on the strategies of market agents. The first steps in modelling endogenous asset payoffs were made in Amir et al. (2021) (a model with long-lived assets) and Evstigneev et al. (2022) (short-lived assets). However, both of these papers established only a weak property of *local evolutionary stability* of a certain strategy in markets where agents use constant fixed-mix strategies. In our paper, we allow agents to use general strategies and prove a much stronger property – *survival* of a strategy, which can be considered as *global evolutionary stability*.

Our model represents a stochastic discrete-time market consisting of several market agents and several *short-lived* assets which yield payoffs that are distributed between the agents in every time period. Short-lived assets exist for one period and then identically reborn again. Examples of such assets include contracts for production or delivery of a specific amount of some good, service and insurance contracts, etc. Short-lived assets have been used in various models in the economic literature, see, for example, Amir et al. (2013); Araujo et al. (2002); Carbone et al. (2021); Judd et al. (2003). Note that long-lived assets, e.g. common stock, do not fit in this modelling framework. Extension of our results to long-lived assets is an open question.

We assume that the asset payoffs in the model are defined as affine functions (with non-negative slope and intercept coefficients) of the shares of market wealth invested by the agents in each asset. There are two main results obtained in the paper. First, we show that there exists a survival strategy and construct it in a semi-closed form. Then we prove that the aggregate market strategy of all agents converges to this strategy if at least one agent use it, while other agents may use arbitrary strategies. For a model where the payoff functions have i.i.d. coefficients, we prove a stronger result: the survival strategy turns out to be constant and drives out of the market any other constant strategy, thus becoming a single survivor.

The key concept necessary for the construction of a survival strategy is the notion of log-optimality of investment strategies. In our model, by a *log-optimal strategy* we call a strategy which maximizes the expected logarithm of the market share of an agent over each time period. This is similar to log-optimal strategies in the classical capital growth theory (also called growth-optimal strategies, Kelly portfolios), but we operate with market shares of agents, i.e. their *relative wealth*, rather than absolute wealth. Recall that log-optimal strategies in models with exogenous asset prices are those which maximize the expected logarithmic return of a portfolio. It is well-known that such strategies also maximize the asymptotic growth of wealth, so they can be considered as good investment rules for long-term investors; for further details, see, e.g., MacLean et al. (2010, 2011); Ziemba (2015). However, in the classical theory, a log-optimal strategy can be found by solving an optimization problem for a single agent, but in our model we rely on a different approach, since the wealth of an agent depends on the actions of other agents through the endogenous asset prices and payoffs. Nevertheless, in a particular case of our model, which essentially reduces to the standard market model with exogenous asset prices, we show that our notion of log-optimality coincides with the classical one. As a by-product, we obtain a new characterization of classical log-optimal strategies as fixed points of a certain random mapping.

The paper is organized as follows. In Section 2, we describe the model. Section 3 introduces the notions of survival and log-optimal strategies and contains the main results of the paper on their existence and asymptotic behavior. A numerical example is provided in Section 4. Section 5 discusses the relation of our model to other models in the literature. Section 6 contains the proofs of the main results.

2. The model

We consider a discrete-time market where $K \geq 2$ assets are traded among $N \geq 2$ agents. The assets live for one period and are identically reborn at the beginning of each period. The asset prices are determined endogenously through a short-run equilibrium of supply and demand. The supply (the total volume) of each asset is constant and without loss of generality is normalized to 1. The assets yield payoffs which are distributed among the agents at moments of time $t = 1, 2, \ldots$

The market is influenced by random factors modeled in terms of a sequence of random elements $s_1, s_2, \ldots$ with values in a measurable space S. The random element s_t is interpreted as the "state of the world" at time t.

Agent $i = 1, \ldots, N$ in this market is characterized by his/her trading strategy and non-random wealth $w_0^i > 0$ (initial endowment) with which this agent enters the market at time $t = 0$. The wealth w_t^i at time $t \geq 1$ is determined by the dynamics described below.

At every moment of time $t \geq 0$, each agent chooses investment proportions $\lambda_t^i = (\lambda_{t,1}^i, \ldots, \lambda_{t,K}^i)$, according to which he/she allocates the available budget (wealth w_t^i) for purchasing assets at time t, i.e. the budget $\lambda_{t,k}^i w_t^i$ is allocated by agent i for purchasing asset k. The investment proportions are selected by the agents simultaneously and independently. These proportions may depend on the history of states of the world $s^{t-1} := (s_1, \ldots, s_{t-1})$, the game history $\lambda^{t-1} := (\lambda_0, \ldots, \lambda_{t-1})$, where $\lambda_s = (\lambda_s^1, \ldots, \lambda_s^N)$, and the vector of initial endowments $w_0 := (w_0^1, \ldots, w_0^N)$. A *strategy* Λ^i of agent i is defined as a sequence of measurable functions

$$\Lambda_t^i(s^t, w_0, \lambda^{t-1}), \quad t = 0, 1, \ldots,$$

with values in the unit simplex

$$\Delta^K := \{(a_1, \ldots, a_K) \in \mathbb{R}_+^K : a_1 + \ldots + a_K = 1\}.$$

The value of Λ_t^i represents the vector of investment proportions λ_t^i chosen by agent i at time t. These proportions are non-negative, i.e. short sales are not allowed. For $t = 0$, the function $\Lambda_0^i = \Lambda_0^i(w_0)$ does not depend on the history of states of the world and the game history.

Given a vector of initial endowments $w_0 = (w_0^1, \ldots, w_0^N)$ and a strategy profile $\Lambda = (\Lambda^1, \ldots, \Lambda^N)$, the investment proportions chosen by the agents in this market are defined by the recursive relation

$$\lambda_0^i = \Lambda_0^i(w_0), \qquad \lambda_t^i(s^t) = \Lambda_t^i(s^t, w_0, \lambda^{t-1}(s^{t-1})),\ t \geq 1, \tag{1}$$

where $\lambda^t(s^t) = (\lambda_0, \lambda_1(s^1), \ldots, \lambda_t(s^t))$. In what follows, we will omit the argument s^t where it does not lead to ambiguity.

By $p_t = (p_{t,1}, \ldots, p_{t,K})$, we will denote the vector of asset prices. The coordinate $p_{t,k}$ stands for the price of one unit of asset k at time t. We will now define the dynamics of agents' wealth $w_t^i = w_t^i(s^t)$ and asset prices $p_t = p_t(s^t)$ for a fixed strategy profile Λ and a vector of initial endowments w_0.

The prices are formed in equilibrium over each time period as follows. The portfolio of agent i at time $t \geq 0$ is specified by a vector $x_t^i = (x_{t,1}^i, \ldots, x_{t,K}^i)$, where $x_{t,k}^i$ is the amount (the number of units) of asset k in the portfolio. The scalar product $\langle p_t, x_t^i \rangle = \sum_{k=1}^K p_{t,k} x_{t,k}^i$ expresses the value of agent i's portfolio at time t.

At time $t = 0$, the agents' budgets are given by their (non-random) initial endowments w_0^i. Let $A_{t,k} = A_{t,k}(s^t)$, $k = 1, \ldots, K$, denote the asset payoffs at time $t \geq 1$ per one unit of asset. Since we assume the supply of each asset is 1, the quantity $A_{t,k}$ represents the total payoff of asset k. Agent i's budget (wealth) at time $t \geq 1$ is given by

$$w_t^i = \langle A_t, x_{t-1}^i \rangle = \sum_{k=1}^K A_{t,k} x_{t-1,k}^i, \tag{2}$$

i.e. it is constituted of the payoff of the portfolio x_{t-1}^i that was purchased at time $t-1$.

If agent i allocates a fraction $\lambda_{t,k}^i$ of his/her wealth for purchasing asset k at time t, then the number of units of this asset that can be bought is

$$x_{t,k}^i = \frac{\lambda_{t,k}^i w_t^i}{p_{t,k}}. \tag{3}$$

Assume that the market is always in equilibrium: the total asset supply is equal to the total demand (recall that the former is normalized to 1). This implies that for all $t \geq 0$ and $k = 1, \ldots, K$ we have

$$1 = \sum_{i=1}^N x_{t,k}^i = \sum_{i=1}^N \frac{\lambda_{t,k}^i w_t^i}{p_{t,k}},$$

and, consequently, the equilibrium (market clearing) asset prices are given by

$$p_{t,k} = \sum_{i=1}^N \lambda_{t,k}^i w_t^i. \tag{4}$$

If the right-hand side of the above expression is equal to zero on a set of values of s^t, we define $x_{t,k}^i = 0$ in formula (3) on this set.

Thus, given a strategy profile and a vector of initial endowments we can, by using equations (2)–(4), generate recursively the random path of the system specified by the sequences of variables w_t^i (agents' wealth), $p_t = (p_{t,1}, \ldots, p_{t,K})$ (vectors of equilibrium asset prices) and $x_t^i = (x_{t,1}^i, \ldots, x_{t,K}^i)$ (agents' portfolios). In particular, the sequences w_t^i follow the dynamics

$$w_{t+1}^i = \sum_{k=1}^K \frac{\lambda_{t,k}^i w_t^i}{\sum_{j=1}^K \lambda_{t,k}^j w_t^j} A_{t+1,k}, \tag{5}$$

where we treat $0/0 = 0$ under the summation sign.

We assume that the asset payoffs $A_{t,k}$ are *endogenous* in the sense that they may depend on the agents' strategies. In the remaining part of the paper, we will deal with the following particular form of the payoffs, which we call *affine payoffs*.

Let W_t denote the total market wealth at time t and $\mu_{t,k}$ denote the fractions of market wealth allocated by all agents for purchasing asset k at time t:

$$W_t = \sum_{i=1}^{N} w_t^i, \qquad \mu_{t,k} = \frac{1}{W_t} \sum_{i=1}^{N} \lambda_{t,k}^i w_t^i.$$

We will assume that the asset payoffs are affine functions of $\mu_{t,k}$:

$$A_{t+1,k} = \alpha_{t+1,k} + \beta_{t+1,k} \mu_{t,k}, \tag{6}$$

where $\alpha_{t+1,k}$ and $\beta_{t+1,k}$ are random variables of the form

$$\alpha_{t+1,k}(s^{t+1}) = a_{t+1,k}(s^{t+1}, w_0, \lambda^{t-1}(s^{t-1})), \tag{7}$$
$$\beta_{t+1,k}(s^{t+1}) = b_{t+1,k}(s^{t+1}, w_0, \lambda^{t-1}(s^{t-1})) \tag{8}$$

with some measurable non-negative functions $a_{t+1,k}$, $b_{t+1,k}$.

Relations (6)–(8) mean that the asset payoffs $A_{t+1,k}$ at the next moment of time $t+1$ may depend on the current state of the market (which is defined by the history of states s^t, the vector of initial endowments w_0 and the past agents' actions λ^{t-1}), the future random state s_{t+1}, and the investment proportions λ_t chosen by the agents at time t, but the dependence on the latter factor can be expressed only through the fractions $\mu_{t,k}$.

Note that $\mu_{t,k}$ essentially represent the *weighted strategy* of the agents, with the weights being equal to their market shares w_t^i / W_t.

In what follows, to avoid a degenerate model, we will assume that for all $t \geq 1$ and any values of the arguments s^t, w_0, λ^{t-2} it holds that

$$\sum_{k=1}^{K} (a_{t,k}(s^t, w_0, \lambda^{t-2}) + b_{t,k}(s^t, w_0, \lambda^{t-2})) > 0. \tag{9}$$

Remark 1. Although relation (6) states that the asset payoffs depend on the *relative* amounts of market capital $\mu_{t,k}$ invested in the assets, it is not difficult to see that the same model allows to consider payoffs $A_{t,k}$ which are affine functions of the *absolute* amounts $v_{t,k} = \sum_{i=1}^{N} \lambda_{y,k}^i w_t^i$. Indeed, suppose the payoffs are

$$A_{t+1,k} = \alpha_{t+1,k} + \widetilde{\beta}_{t+1,k} v_{t,k}, \tag{10}$$

where $\widetilde{\beta}_{t+1,k}(s^{t+1}) = \widetilde{b}_{t+1,k}(s^{t+1}, w_0, \lambda^{t-1}(s^{t-1}))$ and the functions $\alpha_{t+1,k}$ are as above.

In order to reduce (10) to (6), let

$$b_{t+1,k}(s^{t+1}, w_0, \lambda^{t-1}) = \widetilde{w}_t(s^t, w_0, \lambda^{t-1})\widetilde{b}_{t+1,k}(s^{t+1}, w_0, \lambda^{t-1}),$$

where $\widetilde{w}_t = \sum_{i=1}^N \widetilde{w}_t^i$ with

$$\widetilde{w}_0^i = w_0^i, \qquad \widetilde{w}_t^i(s^t, w_0, \lambda^{t-1}) = \sum_{k=1}^K \frac{\lambda_{t-1,k}^i \widetilde{w}_{t-1}^i}{\sum_{j=1}^K \lambda_{t-1,k}^j \widetilde{w}_{t-1}^j} A_{t,k}, \ t \geq 1.$$

The functions $\widetilde{w}_t^i$ specify the dependence of the agents' wealth on the full information of the game, i.e. knowing the vector of initial endowments, the history of investment proportions λ^{t-1} and the history of random states s^t, it is possible to recover the trajectory of the wealth process by the above relation.

3. Log-optimal and survival strategies

3.1. Definitions

We will be interested in the behavior of the *relative wealth* or the *market shares* of the agents, which are defined by

$$r_t^i := \frac{w_t^i}{W_t}.$$

The following definitions introduce the two main concepts of the paper. Hereinafter, "a.s." means "almost surely", i.e. holding with probability 1.

Definition 1. We call a strategy Λ^i of agent i *survival*, if for any vector of initial endowments w_0 and strategy profile $\Lambda = (\Lambda^1, \dots, \Lambda^N)$ consisting of the given strategy Λ^i and arbitrary strategies Λ^j of agents $j \neq i$, it holds that $w_t^i > 0$ a.s. for all $t \geq 0$ and

$$\inf_{t \geq 0} r_t^i > 0 \text{ a.s.}$$

According to this definition, a survival strategy allows an agent to keep a non-vanishing share of the total market wealth irrespectively of strategies used by the other agents.

A comment is in order. One might think that the focus on survival substantially restricts the scope of the analysis, since one should care about survival only if "things go wrong". It turns out, however, that the class of survival strategies in most of the evolutionary behavioral finance models coincides with the class of unbeatable strategies performing in the long run not worse (or, even, strictly better) in terms of wealth accumulation than any other strategies competing in the market. In other words, *in order to survive one has to win.* Moreover, survival strategies determine the aggregate market structure in the long run. We provide further details on this fact in the context of our model in Section 3.3.

In order to find a survival strategy we will look for a *log-optimal strategy*, the concept of which is introduced in the next definition. To state it, recall that a random sequence ξ_t, which is adapted to the filtration generated by the state process s_t, is called a *submartingale* if $\mathrm{E}|\xi_t| < \infty$ and $\mathrm{E}_t \xi_{t+1} \geq \xi_t$ a.s. for all $t \geq 0$, where $\mathrm{E}_t(\cdot) = \mathrm{E}(\cdot \mid s^t)$ denotes the conditional expectation given $s^t = (s_1, \ldots, s_t)$. For $t = 0$, put $\mathrm{E}_0(\cdot) = \mathrm{E}(\cdot)$.

Definition 2. In the model under consideration, we call a strategy Λ^i *log-optimal*[1], if for any vector of initial endowments w_0 and strategy profile $\Lambda = (\Lambda^1, \ldots, \Lambda^N)$, where Λ^i is the given strategy, it holds that $w_t^i > 0$ a.s. for all $t \geq 0$ and

$$\ln r_t^i \text{ is a submartingale.} \tag{11}$$

This concept is similar to the notion of a log-optimal strategy in the classical capital growth theory for markets with exogenous asset prices. Recall that in this theory a strategy is called log-optimal if no competing strategy can improve the expected logarithmic wealth relative to it. It is well-known that such a strategy maximizes the asymptotic growth rate of an agent's wealth, see, e.g., Algoet and Cover (1988). In Section 5, we will show that in a particular case when our model reduces to a market with exogenous prices, Definition 2 yields the same expected log-wealth maximizing strategy.

However, in the general form of our model, a log-optimal strategy may not maximize the absolute wealth w_t^i of an agent. In fact, a strategy which maximizes w_t^i in one sense or another, irrespectively of the strategies used by the other agents, typically does not exists, since an agent's wealth depends on the whole profile of strategies through the endogenous asset prices and payoffs. Thus, in our model, an agent who uses a log-optimal strategy does not care about his/her absolute wealth, but only wishes to be better (or, at least, not worse) than the market in the sense of (11). It is possible to provide an example of a market where a log-optimal strategy in the sense of Definition 2 results in a "spiteful behavior" and makes the wealth of all the agents vanish asymptotically, with its wealth vanishing slower than the wealth of the other agents. For details, see Drokin and Zhitlukhin (2020).

Proposition 1. *Any log-optimal strategy is a survival strategy.*

Proof. A non-negative submartingale has a finite limit with probability 1 as $t \to \infty$; see, e.g., Shiryaev (2019, Ch. 7.4). Hence, if Λ^i is a log-optimal strategy, then $\lim_{t\to\infty} \ln r_t^i$ is finite, which implies $\inf_{t\geq 0} r_t^i > 0$. □

3.2. Construction of a log-optimal strategy

For $t \geq 1$, define the Δ^K-valued functions $g_t(\lambda^*, s^t, w_0, \lambda^{t-2})$, $\lambda^* \in \Delta^K$, by

$$g_{t,k}(\lambda^*, s^t, w_0, \lambda^{t-2}) = a_{t,k}(s^t, w_0, \lambda^{t-2}) + \lambda_k^* b_{t,k}(s^t, w_0, \lambda^{t-2}).$$

[1] Another term in the context of EBF models is a *relative growth optimal* strategy, see Drokin and Zhitlukhin (2020).

The arguments s^t, w_0, λ^{t-2} have the same meaning as in (7)–(8). For brevity, in what follows we will use the notation $\chi^t = (w_0, \lambda^{t-1})$ for a pair of vector of initial endowments and game history. For example, we will write $g_t = g_t(\lambda^*, s^t, \chi^{t-1})$. For $t = 0$, put $\chi^0 = w_0$.

Let $\mathrm{P}_t(\cdot) = \mathrm{P}(\cdot \mid s^t)$ and $\mathrm{E}_t(\cdot) = \mathrm{E}(\cdot \mid s^t)$ denote the conditional probability and conditional expectation given s^t (where $\mathrm{P}_0(\cdot) = \mathrm{P}(\cdot)$, $\mathrm{E}_0(\cdot) = \mathrm{E}(\cdot)$). Introduce the functions $L_t = L_t(\lambda^*, s^t, \chi^t)$, $t \geq 0$, with values in Δ_K defined by

$$L_{t,k}(\lambda^*, s^t, \chi^t) = \mathrm{E}_t\left(\frac{g_{t+1,k}(\lambda^*, s^{t+1}, \chi^t)}{\sum_{j=1}^K g_{t+1,j}(\lambda^*, s^{t+1}, \chi^t)}\right).$$

We will assume that the conditional probabilities $\mathrm{P}_t(\cdot)$ and expectations $\mathrm{E}_t(\cdot)$ are computed with respect to some fixed variant of the regular conditional distribution of s^{t+1}, which implies that the functions $L_{t,k}$ are jointly measurable with respect to their arguments. For $t = 0$, the function $L_0 = L_0(\lambda^*, \chi^0)$ does not depend on the state of the world.

Proposition 2. *For each $t \geq 0$, there exists a measurable function $\Lambda_t^*(s^t, \chi^t)$ with values in Δ^K which has the following properties:*

(a) for any χ^t, it holds that

$$\mathrm{P}_t\left(\sum_{j=1}^K g_{t+1,j}(\Lambda_t^*(s^t, \chi^t), s^{t+1}, \chi^t) = 0\right) = 0 \ \textit{a.s.}, \tag{12}$$

$$\mathrm{E}_t\left(\frac{b_{t+1,k}(s^{t+1}, \chi^t)}{\sum_{j=1}^K g_{t+1,j}(\Lambda_t^*(s^t, \chi^t), s^{t+1}, \chi^t)}\right) \leq 1 \ \textit{a.s.}, \quad k = 1, \dots, K, \tag{13}$$

(b) Λ_t^* is a fixed point of L_t, i.e. for any χ^t it holds that

$$L_t(\Lambda_t^*(s^t, \chi^t), s^t, \chi^t) = \Lambda_t^*(s^t, \chi^t) \ \textit{a.s.}, \tag{14}$$

where for $t = 0$, we assume that $\Lambda_0^* = \Lambda_0^*(\chi^0)$ depends only on $\chi^0 = w_0$.

The next theorem is the first main result of the paper.

Theorem 1. *A strategy $\Lambda^* = (\Lambda_t^*)_{t=0}^\infty$ which consists of functions satisfying properties (12)–(14) is log-optimal, and, in particular, survival.*

Unfortunately there is no simple method to find a fixed point in (14). Nevertheless, the theorem still presents a great interest because the existence of a log-optimal strategy is a result which is far from evident in the model under consideration. In the examples in the next section, we show that in some cases a log-optimal strategy can be found in a closed form.

Remark 2. (a) The functions Λ_t^* with properties (12)–(14) are, in general, not uniquely defined. Theorem 1 states that any sequence of them constitutes a log-optimal strategy.

A simple example of the non-uniqueness of a log-optimal strategy is as follows. Assume that $a_{t,k} \equiv 0$, $b_{t,k} \equiv 1$ for all t, k, and the total initial market wealth $W_0 = 1$. It is easy to see that equation (5) in this case reads

$$w_{t+1}^i = \frac{w_t^i}{W_t}.$$

Consequently, $W_t = 1$ for all $t \geq 0$ and the agents' wealth w_t^i remain the same no matter what strategies they use.

(b) If a strategy Λ^* satisfies conditions (12) and (14), then a simple sufficient condition for the validity of (13), which we use below, is that for each $t \geq 0$ and χ^t

$$\mathrm{P}_t(a_{t+1,k}(s^{t+1}, \chi^t) > 0) > 0 \text{ a.s.}, \qquad k = 1, \ldots, K.$$

Indeed, in this case (13) is true since $L_{t,k}(\lambda^*, s^t, \chi^t) > 0$ for any $\lambda^* \in \Delta^K$, which implies $\Lambda_{t,k}^*(s^t, \chi^t) > 0$ and therefore

$$\mathrm{E}_t\left(\frac{b_{t+1,k}(s^{t+1}, \chi^t)}{\sum_{j=1}^K g_{t+1,j}(\Lambda_t^*(s^t, \chi^t), s^{t+1}, \chi^t)}\right) \leq \frac{L_{t,k}(\Lambda_t^*(s^t, \chi^t), s^t, \chi^t)}{\Lambda_{t,k}^*(s^t, \chi^t)} = 1.$$

3.3. A log-optimal strategy determines the aggregate market behavior

As was mentioned above, the fractions $\mu_{t,k}$ can be though of as the weighted strategy of the market agents. Our next result shows that, under an additional assumption, if at least one agent uses a log-optimal strategy, then $\mu_{t,k}$ approach this strategy in the limit as $t \to \infty$ with probability 1.

Theorem 2. *Suppose a strategy Λ^* satisfies conditions (12), (14), and the following stronger version of condition (13): there exists $\varepsilon > 0$ such that for any $t \geq 0$ and χ^t it holds that*

$$\mathrm{E}_t\left(\frac{b_{t+1,k}(s^{t+1}, \chi^t)}{\sum_{j=1}^K g_{t+1,j}(\Lambda_t^*(s^t, \chi^t), s^{t+1}, \chi^t)}\right) \leq 1 - \varepsilon \ \textit{a.s.}, \qquad k = 1, \ldots, K. \tag{15}$$

Then, if in a strategy profile $\Lambda = (\Lambda^1, \ldots, \Lambda^N)$ agent i uses the strategy Λ^*, it holds that

$$\sum_{t=1}^{\infty} \|\lambda_t^i - \mu_t\|^2 < \infty \ \textit{a.s.},$$

where $\lambda_t^i = \lambda_t^i(s^t)$ and $\mu_t = \mu_t(s^t)$ denote, respectively, the realization of the strategy of agent i and the realization of the weighted strategy of all agents in this strategy profile (see (1)). In particular, $\|\lambda_t^i - \mu_t\| \to 0$ as $t \to \infty$.

In general, condition (15) is not easy to verify. However, in the case of i.i.d. payoff coefficients $\alpha_{t,k}$, $\beta_{t,k}$ the statement of Theorem 2 becomes more appealing and we can also prove that Λ^* turns out to be a unique survival strategy in the class of all constant strategies (under mild additional assumptions). This is our third main result.

Theorem 3. *Suppose that the sequence of states of the world s_t, $t \geq 1$, consists of i.i.d. random elements and the coefficient $\alpha_{t,k}$, $\beta_{t,k}$ from (6)–(8) depend only on s_t, i.e. $\alpha_{t,k} = a_k(s_t)$, $\beta_{t,k} = b_k(s_t)$. Then the following claims hold true.*

(a) There exists a constant log-optimal strategy $\Lambda_t^* \equiv \Lambda^* \in \Delta^K$.

(b) Assume that, additionally,

$$\mathrm{P}(\alpha_{t,k} > 0) > 0 \text{ for each } k = 1, \dots, K. \tag{16}$$

Then the strategy Λ^* is a unique survival strategy in the class of all constant strategies and $\Lambda_k^* > 0$, $k = 1, \dots, K$. Moreover, Λ^* satisfies (15). In particular, in any strategy profile $\Lambda = (\Lambda^1, \dots, \Lambda^N)$, in which some agent uses the strategy Λ^*, it holds that $\mu_t \to \Lambda^*$ with probability 1 as $t \to \infty$.

(c) Assume that, in addition to (16), it holds that the random variables $\alpha_{t,k}/\Lambda_k^* + \beta_{t,k}$ are linearly independent, i.e. if $\sum_{k=1}^K c_k(\alpha_{t,k}/\Lambda_k^* + \beta_{t,k}) = 0$ a.s. for some constants c_k, then $c_k = 0$ for all $k = 1, \dots, K$.

Then in any strategy profile in which some agent uses the strategy Λ^* and other agents use constant strategies with strictly positive components ($\Lambda_k^i > 0$ for all i, k), it holds that $r_t^i \to 0$ a.s. as $t \to \infty$ for any agent i who uses a strategy $\Lambda^i \neq \Lambda^*$.

4. Examples

Let us illustrate the main results of the paper with simulations and consider the following simple model, in which only two assets are traded. Assume the random states of the world are modeled by a sequence of i.i.d. random vectors $s_t = (s_t^1, s_t^2)$ with values in the set $\{(1,0), (0,1), (1,1)\}$ and symmetric joint distribution

$$\mathrm{P}(s_t = (1,0)) = \mathrm{P}(s_t = (0,1)) = 1 - p, \qquad \mathrm{P}(s_t = (1,1)) = 2p - 1,$$

where $1/2 \leq p < 1/2$ is a parameter. Assume the payoff coefficients $\alpha_{t,k} = a_k(s_t)$, $\beta_{t,k} = b_k(s_t)$ are given by

$$a_k(s_t) = b_k(s_t) = \mathrm{I}(s_t^k = 1), \qquad k = 1, 2.$$

Thus, the payoff of each each is either $1 + \mu_{t,k}$ with probability p or 0 with probability $1 - p$. With probability $2p - 1$, both of the assets yield payoffs simultaneously.

By symmetry, the log-optimal strategy in this model is $\Lambda^* = (1/2, 1/2)$; it is not difficult to check that it indeed satisfies the conditions of Proposition 2. By Theorem 3(b),

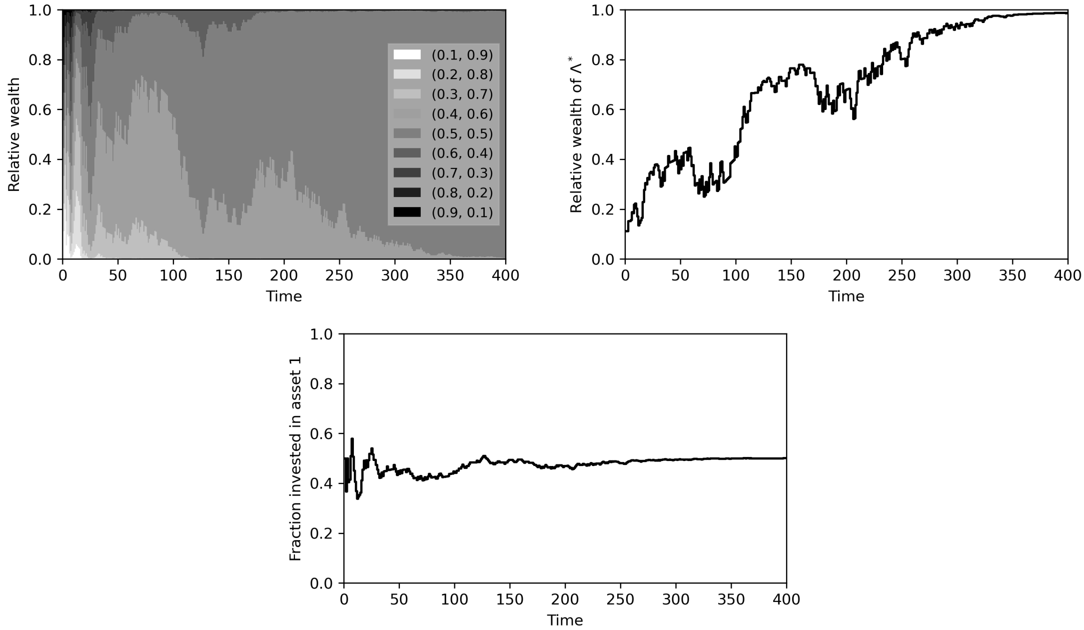

Figure 1: Evolution of agents' wealth in one simulation of the model. Top left: the relative wealth of each strategy $\Lambda^i = (i/10, 1 - i/10)$. Top right: the relative wealth of the log-optimal strategy Λ^*. Bottom: the fraction of the total market wealth invested in the first asset $\mu_{t,1}$.

it is a unique constant survival strategy. Note that it also satisfies the condition of part (c) of Theorem 3, i.e. $\alpha_{t,k}/\Lambda^*_k + \beta_{t,k}$ are linearly independent random variables.

Let us place Λ^* in a market environment. As an example, consider a market which consists of 9 agents who use the constant strategies $\Lambda^i = (i/10, 1 - i/10)$, where $i = 1, 2, \ldots, 9$. In particular, agent $i = 5$ uses the strategy Λ^*. We do not include the strategies $\Lambda^0 = (0, 1)$ and $\Lambda^{10} = (1, 0)$ since their wealth vanishes in a finite number of time periods.

Figure 1 shows the evolution of the agents' wealth in one simulation of this market over 400 time periods with parameter $p = 2/3$. The first graph displays the relative wealth r^i_t of each agent, with r^i_t corresponding to the width of the corresponding colored area at time t. The second graph shows the relative wealth of the log-optimal strategy. As can be seen from these two graphs, the log-optimal strategy eventually dominates the market and its relative wealth converges to 1, as proved in Theorem 3(c). The third graph shows the fraction of the total market wealth invested in asset 1, i.e. $\mu_{t,1}$ (the fraction invested in the second asset is obviously $\mu_{t,2} = 1 - \mu_{t,1}$), which, as expected, converges to $\Lambda^*_1 = 1/2$.

To show the convergence of the fractions $\mu_{t,k}$ to Λ^*, in Figure 2 we present multiple simulated trajectories of the sequence $\mu_{t,1}$ for the three different parameters $p = 1/2$, $p = 3/4$, $p = 19/20$.

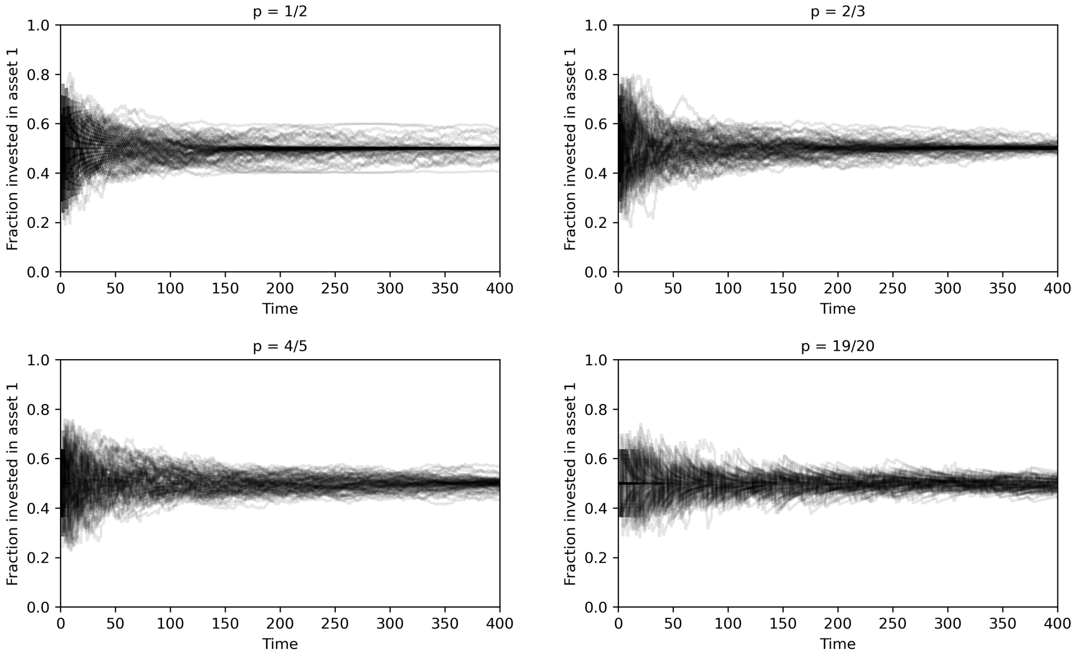

Figure 2: Trajectories of the fraction of the total market wealth invested in the first asset $\mu_{t,1}$ for different value of the probability p. Each graph contains 100 simulated trajectories.

5. Relation to existing models

To demonstrate the generality of our model, let us show how known results on log-optimal and survival strategies can be derived from it.

In the first part of this section, we will consider the evolutionary finance model with short-lived assets of Amir et al. (2013), in which the asset payoffs $A_{t,k}$ are exogenous, i.e. depend only on the random states s_t, but not on the game history. In the second part, we will consider the classical model of an asset market with exogenous asset prices, in which agents' actions do not affect the asset prices and the wealth of other agents.

5.1. The evolutionary finance model with exogenous asset payoffs

Suppose in our model $b_{t,k} = 0$, while $a_{t,k} = a_{t,k}(s^t)$ depend only on the random states but not on the agents' strategies and initial endowments, so that $A_{t,k} = A_{t,k}(s^t) = a_{t,k}(s^t)$. Then equation (5) describing the wealth dynamics reads

$$w_{t+1}^i = \sum_{k=1}^{K} \frac{\lambda_{t,k}^i w_t^i}{\sum_{j=1}^{K} \lambda_{t,k}^j w_t^j} a_{t+1,k}.$$

The unique strategy which satisfies conditions (12)–(14) of Proposition 2 is

$$\Lambda^*_{t,k}(s^t) = \mathrm{E}_t\left(\frac{a_{t+1,k}(s^{t+1})}{\sum_{j=1}^K a_{t+1,j}(s^{t+1})}\right)$$

(with Λ^*_0 being a constant vector). This strategy allocates the current available investment budget among the assets proportionally to the conditional expectations of their relative payoffs. The survival property of this strategy was first established by Amir et al. (2013), but under additional restrictions on the payoffs $A_{t,k}$ it had been known earlier, see Amir et al. (2005); Blume and Easley (1992); Evstigneev et al. (2002).

5.2. The classical model with exogenous asset prices

Let us now consider a market with exogenous asset prices $S_{t,k}(s^t) > 0$, as in the standard textbook model without short sales (see, e.g., Föllmer and Schied (2011, Ch. 5)). The goal of this example is to show that such a market model is a particular case of our model, while the log-optimal strategy constructed in Theorem 1 maximizes the expected logarithm of an agent's wealth or, equivalently, the logarithmic return of portfolio. In particular, this provides a new characterization of an expected log-wealth maximizing strategy as a fixed point of the mapping defined in (14).

Denote by $X_{t+1,k} = S_{t+1,k}/S_{t,k}$ the asset returns. Then the evolution of wealth w_t of an agent who uses a strategy $\Lambda = \Lambda_t(s^t)$ is specified by the relation

$$w_{t+1} = w_t\langle\Lambda_t, X_{t+1}\rangle,$$

where $\langle\cdot,\cdot\rangle$ denotes the scalar product. This model can be obtained from our model if in equation (6) we put

$$\alpha_{t+1,k} = 0, \qquad \beta_{t+1,k} = W_t X_{t+1,k}, \tag{17}$$

where W_t denotes the total market wealth (Remark 1 shows that (17) can be written in terms of the fractions $\mu_{t,k}$).

Recall that a log-optimal strategy Λ^* in the classical capital growth theory is a strategy which maximizes the expected log-return of a portfolio in each time period, i.e.

$$\Lambda^*_t(s^t) \in \underset{\lambda\in\Delta^K}{\arg\max}\, \mathrm{E}_t \ln\langle\lambda, X_{t+1}(s^{t+1})\rangle. \tag{18}$$

Such a strategy is often referred to as the *Kelly portfolio rule.* Although the above optimization problem may not have a solution when the log-returns are not integrable, if we introduce the *relative returns*

$$R_{t,k} = \frac{X_{t,k}}{\sum_{j=1}^K X_{t,j}}, \quad k = 1,\dots,K,$$

then it is not difficult to show that a log-optimal strategy can be characterized as a solution of the maximization problem for the logarithms of relative returns

$$\Lambda_t^*(s^t) \in \arg\max_{\lambda \in \Delta^K} \mathrm{E}_t \ln\langle \lambda, R_{t+1}(s^{t+1})\rangle. \tag{19}$$

Namely, problem (19) always has a solution and if (18) has at least one solution, then the sets of solutions of (18) and (19) coincide.

Let us investigate the relation between strategies, which satisfy conditions (12)–(14) in our model, and the classical notion of a log-optimal strategy (19). Observe that condition (12) is satisfied by any strategy, since it is equivalent to that

$$\mathrm{P}_t(w_t\langle \Lambda_t^*, X_{t+1}\rangle = 0) = 0,$$

which holds because $X_{t+1,k} > 0$. Conditions (13) and (14) are equivalent to, respectively,

$$\mathrm{E}_t\left(\frac{R_{t+1,k}}{\langle \Lambda_t^*, R_{t+1}\rangle}\right) \le 1, \tag{20}$$

$$\mathrm{E}_t\left(\frac{\Lambda_{t,k}^* R_{t+1,k}}{\langle \Lambda_t^*, R_{t+1}\rangle}\right) = \Lambda_{t,k}^*. \tag{21}$$

It is easy to see that (20) implies (21). Indeed, multiplying the both sides of (20) by $\Lambda_{t,k}^*$, we get the inequality $\mathrm{E}_t(\Lambda_{t,k}^* R_{t+1,k}\langle \Lambda_t^*, R_{t+1}\rangle^{-1}) \le \Lambda_{t,k}^*$, which must actually be an equality with probability 1, since otherwise, by taking the sum of the both sides over $k = 1, \ldots, K$, we would get the contradiction $1 < 1$ with positive probability.

Thus, in the particular case of our model under consideration, conditions (12)–(14) are equivalent to (20).

Proposition 3. *A strategy $\Lambda^* = \Lambda_t^*(s^t)$ satisfies conditions* (12)*–*(14) *of Proposition 2* (*or, equivalently, condition* (20)) *if and only if it is a* (*measurable*) *solution of maximization problem* (19).

Proof. If a strategy Λ^* satisfies condition (20), then for any other strategy $\Lambda = \Lambda_t(s^t)$ we have

$$\begin{aligned}\mathrm{E}_t \ln\langle \Lambda_t, R_{t+1}\rangle - \mathrm{E}_t \ln\langle \Lambda_t^*, R_{t+1}\rangle &= \mathrm{E}_t \ln\left(\frac{\langle \Lambda_t, R_{t+1}\rangle}{\langle \Lambda_t^*, R_{t+1}\rangle}\right) \\ &\le \left\langle \Lambda_t,\ \mathrm{E}_t\left(\frac{R_{t+1}}{\langle \Lambda_t^*, R_{t+1}\rangle}\right)\right\rangle - 1 \le \sum_{k=1}^K \Lambda_{t,k} - 1 = 0,\end{aligned}$$

where the first inequality holds because $\ln x \le x - 1$.

On the other hand, if a strategy Λ_t^* solves problem (19), then it is known (see Algoet and Cover (1988, Th. 1)) that for any other strategy Λ

$$\mathrm{E}_t \frac{\langle \Lambda_t, R_{t+1}\rangle}{\langle \Lambda_t^*, R_{t+1}\rangle} \le 1.$$

Taking $\Lambda_t = (0, \dots, 1, \dots, 0)$ (the unit in the k-th coordinate), we obtain (20). $\square$

6. Proofs of the main results

6.1. Auxiliary results

This section contains several simple lemmas needed to prove the main results.

Lemma 1. *Let $C \subset \mathbb{R}^K$ be a compact set and $(\Omega, \mathcal{F})$ be a measurable space. Suppose a function $L(x, \omega)\colon C \times \Omega \to C$ is continuous in x and measurable in ω. Then L has a measurable fixed point $\xi(\omega)$, i.e. $L(\xi(\omega), \omega) = \xi(\omega)$ for all $\omega \in \Omega$.*

Proof. For a particular $\omega \in \Omega$, a fixed point ξ of $L(x, \omega)$ exists by the Brouwer theorem. Hence the random set $\Gamma(\omega) = \{x \in C : L(x, \omega) = x\}$ is non-empty for each ω. By Filippov's implicit function theorem (see, e.g., Aliprantis and Border (2006, Ch. 18.3)), this random set is weakly measurable[2] and admits a measurable selector $\xi(\omega) \in \Gamma(\omega)$, which gives the sought-for measurable fixed point. $\square$

Lemma 2. *Let $L^n(\omega)$, $n = 1, 2, \dots$, be a sequence of measurable functions on a measurable space $(\Omega, \mathcal{F})$ with values in a compact set $C \subset \mathbb{R}^K$. Then there exists a measurable function $L^*(\omega)$ and a strictly increasing sequence of integer-valued measurable functions $n_i(\omega) \geq 1$, $i = 1, 2, \dots$, such that $\lim_{i \to \infty} L^{n_i(\omega)}(\omega) = L^*(\omega)$ for any ω.*

Proof. This result follows from Lemma 2 in Kabanov and Stricker (2001). $\square$

The next lemma generalizes the Gibbs inequality. It will play a key role in the proof of Theorem 1. By $\|\cdot\|$ we will denote the Euclidean norm.

Lemma 3. *Suppose $x, y \in \mathbb{R}^K_+$ are two vectors such that $\sum_{k=1}^K x_k \leq 1$, $\sum_{k=1}^K y_k \leq 1$, and for each $k = 1, \dots, K$ it holds that if $y_k = 0$, then $x_k = 0$. Then*

$$\sum_{k=1}^K x_k \ln \frac{x_k}{y_k} \geq \frac{\|x - y\|^2}{4} + \sum_{k=1}^K (x_k - y_k), \tag{22}$$

where we define $x_k \ln \frac{x_k}{y_k} = 0$ if $x_k = 0$ or both $x_k = 0$ and $y_k = 0$.

Proof. Using that $\ln a \leq 2(\sqrt{a} - 1)$, we obtain

$$\sum_{k=1}^K x_k \ln \frac{x_k}{y_k} = - \sum_{k : x_k \neq 0} x_k \ln \frac{y_k}{x_k} \geq 2 \sum_{k=1}^K (x_k - \sqrt{x_k y_k})$$
$$= \sum_{k=1}^K (\sqrt{x_k} - \sqrt{y_k})^2 + \sum_{k=1}^K (x_k - y_k).$$

[2] We call a random set $\Gamma\colon \Omega \to 2^{\mathbb{R}^K}$ *weakly measurable* if $\{\omega : \Gamma(\omega) \cap A \neq \emptyset\} \in \mathcal{F}$ for any open set $A \subset \mathbb{R}^K$.

Applying the inequality $(\sqrt{x}_k - \sqrt{y}_k)^2 \geq (x_k - y_k)^2/4$, which is true for any $x_k, y_k \in [0,1]$, we obtain (22). $\square$

The final lemma in this section is a known result from the theory of martingales.

Lemma 4. *Let $\zeta_t = \zeta_t(s^t)$, $t = 0, 1, \ldots$, be a random sequence (ζ_0 is constant) which is uniformly bounded from above (i.e. $\zeta_t \leq c$ a.s. for all t and some constant c) and $\mathrm{E}_{t-1}\zeta_t \geq \zeta_{t-1}$ a.s. for all $t \geq 1$. Then $\mathrm{E}|\zeta_t| < \infty$, so ζ_t is a submartingale.*

Proof. The sequence M_t with $M_0 = 0$ and

$$M_t = \zeta_t - \sum_{s=1}^{t} (\mathrm{E}_{t-1}\zeta_t - \zeta_{t-1}), \quad t \geq 1,$$

is a local martingale because $\mathrm{E}_{t-1} M_t = M_{t-1}$ (Shiryaev, 2019, Ch. 7.1, Th. 1). Since it is bounded from above, it is a true martingale (Shiryaev, 2019, Ch. 7.1, Th. 3), so $\mathrm{E}|M_t| < \infty$. Using that $M_t \leq \zeta_t \leq c$, we obtain $\mathrm{E}|\zeta_t| < \infty$. $\square$

6.2. Proof of Proposition 2

Fix $t \geq 0$. Define Δ^K-valued functions $g_t^n(\lambda^*, s^t, \chi^{t-1})$ and $L_t^n(\lambda^*, s^t, \chi^t)$, $n = 1, 2, \ldots$, by

$$g_{t,k}^n = g_{t,k} + \frac{1}{n},$$
$$L_{t,k}^n(\lambda^*, s^t, \chi^t) = \mathrm{E}_t \left(\frac{g_{t+1,k}^n(\lambda^*, s^{t+1}, \chi^t)}{\sum_{j=1}^K g_{t+1,j}^n(\lambda^*, s^{t+1}, \chi^t)} \right).$$

By treating $\mathrm{E}_t(\cdot)$ as the conditional expectation with respect to a fixed variant of the regular conditional distribution of s^{t+1}, we can assume that the functions L_t^n are continuous in $\lambda^* \in \Delta^K$ and measurable in (s^t, χ^t). Hence by Lemma 1 they have measurable fixed points $\Lambda_t^n = \Lambda_t^n(s^t, \chi^t)$, i.e. for any (s^t, χ^t) it holds that

$$L_t^n(\Lambda_t^n(s^t, \chi^t), s^t, \chi^t) = \Lambda_t^n(s^t, \chi^t). \tag{23}$$

Let

$$\delta_{t,k}^n(s^t, \chi^t) = \mathrm{E}_t \left(\frac{b_{t+1,k}(s^{t+1}, \chi^t)}{\sum_{j=1}^K g_{t+1,j}^n(\Lambda_t^n(s^t, \chi^t), s^{t+1}, \chi^t)} \right).$$

Observe that

$$\delta_{t,k}^n \leq 1, \quad k = 1, \ldots, K, \tag{24}$$

since

$$(1 - \delta_{t,k}^n)\Lambda_{t,k}^n = \mathrm{E}_t \left(\frac{a_{t+1,k} + 1/n}{\sum_{j=1}^K g_{t+1,j}^n(\Lambda_t^n)} \right) > 0.$$

By Lemma 2, it is possible to find an increasing sequence $n_i = n_i(s^t, \chi^t)$, $i = 1, 2, \dots$, such that there exists the limit

$$\Lambda_t^* = \lim_{i\to\infty} \Lambda_t^{n_i}.$$

Now, for a fixed χ^t, by passing to the limit $i \to \infty$ and $n_i \to \infty$ in (24) using Fatou's lemma and taking into account assumption (9), one can see that (12) holds. Then, by the dominated convergence theorem, we obtain (13) from (24), and (14) from (23).

6.3. Proof of Theorem 1

Fix a vector of initial endowments and a strategy profile in which one agent uses the strategy Λ^*. Without loss of generality, assume that Λ^* is used by agent 1.

Define (omitting the argument s^t for brevity)

$$\theta_{t,k} = \frac{\lambda_{t,k}^1}{\mu_{t,k}}.$$

Then equation (5) defining the wealth dynamics can be written as

$$w_{t+1}^1 = r_t^1 \sum_{k=1}^K \theta_{t,k} A_{t+1,k} = r_t^1 \sum_{k=1}^K (\theta_{t,k}\alpha_{t+1,k} + \lambda_{t,k}^1 \beta_{t+1,k}),$$

and the total market wealth satisfies the equation

$$W_{t+1} = \sum_{k=1}^K A_{t+1,k} = \sum_{k=1}^K (\alpha_{t+1,k} + \mu_{t,k}\beta_{t+1,k}).$$

From the above two relations, we find

$$\ln r_{t+1}^1 - \ln r_t^1 = \ln\left(\frac{\sum_{k=1}^K (\theta_{t,k}\alpha_{t+1,k} + \lambda_{t,k}^1 \beta_{t+1,k})}{\sum_{j=1}^K (\alpha_{t+1,j} + \mu_{t,j}\beta_{t+1,j})}\right). \tag{25}$$

Consequently, we can represent

$$\mathrm{E}_t \ln r_{r+1}^1 - \ln r_t^1 = \mathrm{E}_t (F_{t+1} + G_{t+1}),$$

where

$$F_{t+1} = \ln\left(\frac{\sum_{k=1}^K (\theta_{t,k}\alpha_{t+1,k} + \lambda_{t,k}^1 \beta_{t+1,k})}{\sum_{j=1}^K (\alpha_{t+1,j} + \lambda_{t,j}^1 \beta_{t+1,j})}\right), \tag{26}$$

$$G_{t+1} = \ln\left(\frac{\sum_{k=1}^K (\alpha_{t+1,k} + \lambda_{t,k}^1 \beta_{t+1,k})}{\sum_{j=1}^K (\alpha_{t+1,j} + \mu_{t,j}\beta_{t+1,j})}\right). \tag{27}$$

Let us show that $\mathrm{E}_t(F_{t+1} + G_{t+1}) \geq 0$. Consider the argument of the logarithm in

(26) as the convex combination of the values

$$\theta_{t,1},\ \ldots,\ \theta_{t,K}, 1$$

with the coefficients

$$\frac{\alpha_{t+1,k}}{\sum_{j=1}^{K}(\alpha_{t+1,j}+\lambda_{t,j}^{1}\beta_{t+1,j})},\ k=1,\ldots,K, \qquad \frac{\sum_{k=1}^{K}\lambda_{t,k}^{1}\beta_{t+1,k}}{\sum_{j=1}^{K}(\alpha_{t+1,j}+\lambda_{t,j}^{1}\beta_{t+1,j})}.$$

By the concavity of the logarithm, we obtain

$$F_{t+1} \geq \sum_{k=1}^{K} \frac{\alpha_{t+1,k}}{\sum_{j=1}^{K}(\alpha_{t+1,j}+\lambda_{t,j}^{1}\beta_{t+1,j})} \ln \theta_{t,k}. \tag{28}$$

Let

$$\gamma_{t,k} = 1 - \mathrm{E}_t\left(\frac{\beta_{t+1,k}}{\sum_{j=1}^{K}(\alpha_{t+1,j}+\lambda_{t,j}^{1}\beta_{t+1,j})}\right), \qquad k=1,\ldots,K. \tag{29}$$

By (13), we have $\gamma_{t,k} \in [0,1]$, and from (14) it follows that

$$\gamma_{t,k}\lambda_{t,k}^{1} = \mathrm{E}_t\left(\frac{\alpha_{t+1,k}}{\sum_{j=1}^{K}(\alpha_{t+1,j}+\lambda_{t,j}^{1}\beta_{t+1,j})}\right). \tag{30}$$

Taking the expectation in (28), we find

$$\begin{aligned}\mathrm{E}_t F_{t+1} \geq \sum_{k=1}^{K}\gamma_{t,k}\lambda_{t,k}^{1}\ln\theta_{t,k} &= \sum_{k=1}^{K}\gamma_{t,k}\lambda_{t,k}^{1}\ln\frac{\gamma_{t,k}\lambda_{t,k}^{1}}{\gamma_{t,k}\mu_{t,k}} \\ &\geq \frac{1}{4}\sum_{k=1}^{K}(\gamma_{t,k}(\lambda_{t,k}^{1}-\mu_{t,k}))^2 + \sum_{k=1}^{K}\gamma_{t,k}(\lambda_{t,k}^{1}-\mu_{t,k}), \end{aligned} \tag{31}$$

where in the second inequality we applied Lemma 3 to the vectors x, y with coordinates

$$x_k = \gamma_{t,k}\lambda_{t,k}^{1}, \qquad y_k = \gamma_{t,k}\mu_{t,k}.$$

Observe that the conditions of the lemma $\sum_{k=1}^{K} x_k \leq 1$, $\sum_{k=1}^{K} y_k \leq 1$ are met because the vectors λ_t^1 and μ_t have this property and $\gamma_{t,k} \in [0,1]$.

In order to bound $\mathrm{E}_t G_{t+1}$, by using the inequality $\ln a \geq 1 - a^{-1}$, we find

$$\begin{aligned}\mathrm{E}_t G_{t+1} &\geq \mathrm{E}_t\left(\frac{\sum_{k=1}^{K}(\lambda_{t,k}^{1}-\mu_{t,k})\beta_{t+1,k}}{\sum_{j=1}^{K}(\alpha_{t+1,j}+\lambda_{t,j}^{1}\beta_{t+1,j})}\right) \\ &= \sum_{k=1}^{K}(1-\gamma_{t,k})(\lambda_{t,k}^{1}-\mu_{t,k}) = \sum_{k=1}^{K}\gamma_{t,k}(\mu_{t,k}-\lambda_{t,k}^{1}), \end{aligned} \tag{32}$$

where the last equality holds because $\sum_{k=1}^{K}\lambda_{t,k}^{1} = \sum_{k=1}^{K}\mu_{t,k} = 1$.

From (31) and (32), we obtain

$$\mathrm{E}_t(F_{t+1}+G_{t+1}) \geq \frac{1}{4}\sum_{k=1}^{K}(\gamma_{t,k}(\lambda_{t,k}^1-\mu_{t,k}))^2, \tag{33}$$

so $\mathrm{E}_t(F_{t+1}+G_{t+1}) \geq 0$. By Lemma 4, we conclude that $\ln r_t^1$ is a submartingale.

6.4. Proof of Theorem 2

Assume that the strategy Λ^* is used by agent 1. In the course of proof of Theorem 1, we have shown that $\zeta_t := \ln r_t^1$ is a submartingale. By Doob's decomposition, it can be represented as $\zeta_t = \zeta_0 + M_t + A_t$, where M_t is a martingale, A_t is a predictable non-decreasing sequence (the *compensator* of ζ_t), and $M_0 = A_0 = 0$. Since ζ_t has a finite limit as $t \to \infty$, the compensator converges as well, i.e. $\lim_{t\to\infty} A_t < \infty$ a.s. Inequality (33) implies that

$$A_t := \sum_{u=0}^{t-1}(\mathrm{E}_u\zeta_{u+1}-\zeta_u) \geq \frac{1}{4}\sum_{u=0}^{t-1}\sum_{k=1}^{K}(\gamma_{u,k}(\lambda_{u,k}^1-\mu_{u,k}))^2.$$

From condition (15) and relation (29), it follows that $\gamma_{t,k} \geq \varepsilon > 0$. Then the claim of the theorem follows from the convergence of A_t.

6.5. Proof of Theorem 3

(a) The existence of a constant strategy satisfying conditions (12)–(14), and hence being a log-optimal strategy, easily follows from inspecting the proof of Proposition 2.

(b) Suppose (16) holds. Let

$$\gamma_k = 1 - \mathrm{E}\left(\frac{b_k(s_t)}{\sum_{j=1}^{K}(a_j(s_t)+\Lambda_k^* b_j(s_t))}\right), \qquad k = 1,\dots,K. \tag{34}$$

Then (14) implies (cf. (29)–(30))

$$\gamma_k\Lambda_k^* = \mathrm{E}\left(\frac{a_k(s_t)}{\sum_{j=1}^{K}(a_j(s_t)+\Lambda_j^* b_j(s_t))}\right), \tag{35}$$

hence $\gamma_k\Lambda_k^* > 0$, so $\gamma_k > 0$, which means that condition (15) is satisfied. From Theorem 2, we obtain the convergence $\mu_t \to \Lambda^*$.

If $\widetilde{\Lambda} \in \Delta^K$ is another constant survival strategy, then it must survive in the strategy profile $(\widetilde{\Lambda}, \Lambda^*, \dots, \Lambda^*)$. This means $\inf_{t\geq 0} r_t^1 > 0$ a.s. But then the convergence $\mu_t = r_t^1\widetilde{\Lambda} + (1-r_t^1)\Lambda^* \to \Lambda^*$ takes place only if $\widetilde{\Lambda} = \Lambda^*$. Hence, Λ^* is a unique survival strategy.

(c) Consider a strategy profile in which some agent, say agent 1, uses the strategy Λ^*. Let agent i use a constant strategy $\Lambda^i \neq \Lambda^*$. To prove the theorem, we need to show that $r_t^1/r_t^i \to \infty$ with probability 1 as $t \to \infty$. For this end, we will show that

$$\liminf_{t\to\infty} t^{-1} \ln \frac{r_t^1}{r_t^i} > 0. \tag{36}$$

From equation (25) in the proof of Theorem 1, it follows that

$$D_{t+1} := \ln \frac{r_{t+1}^1}{r_{t+1}^i} - \ln \frac{r_t^1}{r_t^i} = \ln\left(\frac{\sum_{k=1}^K (\theta_{t,k}^1 \alpha_{t+1,k} + \Lambda_k^* \beta_{t+1,k})}{\sum_{k=1}^K (\theta_{t,k}^i \alpha_{t+1,k} + \Lambda_k^i \beta_{t+1,k})}\right),$$

where in the model under consideration $\alpha_{t,k} = a_k(s_t)$, $\beta_{t,k} = b_k(s_t)$, and $\theta_{t,k}^1 = \Lambda_k^*/\mu_{t,k}$, $\theta_{t,k}^i = \Lambda_k^i/\mu_{t,k}$. Then we have

$$t^{-1} \ln \frac{r_t^1}{r_t^i} = t^{-1} \ln \frac{r_0^1}{r_0^i} + t^{-1} \sum_{u=0}^{t-1} \mathrm{E}_u D_{u+1} + t^{-1} \sum_{u=0}^{t-1} (D_{u+1} - \mathrm{E}_u D_{u+1}).$$

It is easy to see that the sequence D_t is uniformly bounded:

$$\frac{1}{c} \le D_t < c,$$

where $c = \max_{i,k} \Lambda_k^i / \min_{i,k} \Lambda_k^i$. Then from the SLLN for martingales, we have $\xi_t := t^{-1} \sum_{u=0}^{t-1} (D_{u+1} - \mathrm{E}_u D_{u+1}) \to 0$ a.s. Thus, to establish (36), it will be sufficient to show that there exists $\varepsilon > 0$ and a random time τ such that for $t \ge \tau$ it holds that

$$\mathrm{E}_t D_{t+1} \ge \varepsilon. \tag{37}$$

By Theorem 1, we have $\mu_{t,k} \to \Lambda_k^*$, so $\theta_{t,k}^1 \to 1$ and $\theta_{t,k}^i \to \Lambda_k^i/\Lambda_k^*$. Consequently, with probability 1 we have

$$\lim_{t\to\infty} \mathrm{E}_t D_{t+1} = \mathrm{E} \ln\left(\frac{\sum_{k=1}^K (\alpha_k + \Lambda_k^* \beta_k)}{\sum_{k=1}^K (\Lambda_k^i \alpha_k / \Lambda_k^* + \Lambda_k^i \beta_k)}\right) =: \mathrm{E} \ln \zeta,$$

where (α, β) is a pair of random vectors in $\mathbb{R}^K$ with the same joint distribution as $(a(s_t), b(s_t))$. Then, in order to prove (37), we need to show that $\mathrm{E} \ln \zeta > 0$, or equivalently $\mathrm{E} \ln \zeta^{-1} < 0$. In view of the strict concavity of the logarithm and Jensen's inequality, it is enough to show that $\mathrm{E}\zeta^{-1} = 1$ and ζ is not constant a.s. To prove the former, use relations (34) and (35), which yield

$$\mathrm{E}\zeta^{-1} = \mathrm{E}\left(\frac{\sum_{k=1}^K (\Lambda_k^i \alpha_k / \Lambda_k^* + \Lambda_k^i \beta_k)}{\sum_{k=1}^K (\alpha_k + \Lambda_k^* \beta_k)}\right) = \sum_{k=1}^K (\gamma_k \Lambda_k^i + (1 - \gamma_k) \Lambda_k^i) = 1.$$

The fact that ζ is not constant follows from the assumption that the random variables

$a_{t,k}/\Lambda_k^* + b_{t,k}$ are linearly independent. Indeed, if $\zeta = c$, then

$$\sum_{k=1}^{K} (c\Lambda_k^* - \Lambda_k^i) \left(\frac{\alpha_k}{\Lambda_k^*} + \beta_k \right) = 0,$$

which implies $c\Lambda_k^* = \Lambda_k^i$ for all k, so $\Lambda^i = \Lambda^*$, which contradicts our assumption.

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