Problem 1. (20 points) Consider the simplest fair lottery: you make a bet V and with probability $\frac{1}{2}$ win (receive $+V$) or lose (receive $-V$). Consider following strategy $\{V_t\}_{t=1}^{\infty}$:

$$V_t = \begin{cases} 2^{t-1}, & \text{if } t = 1 \text{ or you lost all tries before } t. \\ 0, & \text{otherwise.} \end{cases}$$

In other words, you start with $V_1 = 1$ and double your bet until your first win, say at turn τ, and don't play anymore after. Denote your profit (or debt) after t-th turn as π_t with initial condition $\pi_0 = 0$.

1) Prove that $\tau < \infty$ *a.s.*
2) Find $\mathbb{E}[\pi_\tau]$.
3) Find $\mathbb{E}[\pi_t]$.
4) Compare results in 2) and 3). Explain this in terms of Doob's optional sampling theorem.

Problem 2. (20 points) Consider a sample $\{x_i\}_{i=1}^{n}$ from some unknown distribution. Let's order it: $x_{(1)} \leq \ldots \leq x_{(n)}$, so we have $\{x_{(i)}\}_{i=1}^{n}$. Also consider a class of estimators:

$$L(x) = \sum_{i=1}^{n} l_i x_{(i)},$$

i.e. a linear combination of order statistics. This estimator is fully defined by set of parameters $\{l_i\}_{i=1}^{n}$. *Note: it is possible that some* $l_i = 0$ *or can be a function of* i.

For each of following distributions propose *unique* estimator from the above class of each parameter:

(a) $\mathcal{U}[a, b]$,

(b) $\mathcal{N}(\mu, \sigma)$,

(c) $\mathsf{Pois}(\lambda)$.

Problem 3. (60 points) Consider following option strategy. You allow to buy N_1 call options with 90% ATM strike, sell N_2 calls with 110% ATM strike and buy or sell some amount of underlying asset, N_3. All options have the same maturity, T.

1) Use the Garman–Kohlhagen model to price all above options on USD/EUR exchange rate with $T = 1Y$. Explain how you set model parameters and why.
2) For fixed $N = \sum_{i=1}^{3} N_i^*$, find optimal proportions $\left(\frac{N_1^*}{N}, \frac{N_2^*}{N}, \frac{N_3^*}{N}\right)$ of capital allocation which reset to zero delta and gamma of the strategy.
3) Suppose you manage USD100,000 and fully invest this capital in the optimal strategy from 2). Calculate theta of the portfolio and interpret the result. What strategy revenue can you expect?
4) What assumptions must be hold for the model 1) works? What can go wrong in real life? How to fix it?