Applied Mathematical Finance I

Lecture 9: CSA Discounting

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Credit Support Annex

- OTC trading between banks is governed by legal document called ISDA Master Agreement.
- Part of it, a Credit Support Annex (CSA), specifies credit risk mitigation in a form of collateral posting. All derivative trades between banks now must be collateralized.
- If bank A owes money to bank B (net present value of derivatives position between A and B is negative to A), A has to post cash collateral to B. Bank B can just keep the collateral should bank A default.
- CSA also specifies
  - Rate r_C paid on collateral (party holding a collateral pays a certain rate to the collateral owner).
  - List of eligible currencies (there may be an option to choose currency).
  - Posting frequency (usually daily), minimum transfer amount, threshold, etc.

Post-Crisis Derivatives Trading

- Let us now describe the mechanics of collateralized trading. Consider an example where at time t bank A sells a derivative to bank B
- Selling derivative at t: A sells the derivative contract to B and receives a premium of $V(t)$. B now has a risk that A will fail to fulfil its obligation on the contract sold.
- Collateral at t: To mitigate that credit risk, A is obliged to post a collateral of amount $C(t)$ to B. Most often, $C(t) = V(t)$, meaning that the trade is fully collateralized. Note that in this case no cash exchange actually happens at t.
- Return of collateral at $t + dt$: At $t + dt$, B returns collateral amount $C(t)$ back and also pays interest on the collateral $C(t)\, r_C(t)\, dt$ to A.
- New collateral at $t + dt$: Derivative price is now $V(t + dt)$. Bank A posts $C(t + dt)$ as collateral to B and the entire procedure is repeated.

Repurchase Agreement (Repo)

- Consider a trader at a bank who needs to buy stocks to hedge option position. Where does he/she get money from?
- The trader could simply borrow funds from bank's treasury desk at some (unsecured) funding rate.
- However, to attract a lower borrowing rate, the trader use stocks just bought as a collateral for the borrow.
- The next day, the trader will get the collateral back, return the loan and pay the overnight repo rate on the money borrowed.
- This is called a repo transaction. Using stocks as collateral, we can attract a lower borrowing rate due to mitigation of credit risk.

The World of Multiple Rates

- Collateralized OTC trading between banks is similar to how derivatives are being traded on exchanges (recall marking to market of a futures contract). Let us now understand how this concept impacts derivatives pricing.

- We will consider a collateralized derivative on a particular asset S with a contractually specified rate $r_C(t)$ paid on cash collateral. We assume a single currecy setup and that collateral posting happens continuously in time.

- We denote the unsecured borrowing/lending funding rate by $r_F(t)$.

- We assume that we can secure our funding by using underlying asset S as a collateral. We denote the corresponding repo rate by $r_R(t)$.

- Also, we assume that $r_C(t) \leq r_R(t) \leq r_F(t)$ and, for simplicity, consider only deterministic rates.

Black-Scholes With Collateral

- We assume that the underlying asset continuously pays dividends at rate $r_D(t)$ and its $\mathbb{P}$-dynamics are given by

$$dS(t) = S(t)\left[\mu(t)\,dt + \sigma(t)\,dW^{\mathbb{P}}(t)\right]$$

- Let $V(t,S)$ denote the time t price of a European derivative security with maturity $T \geq t$ given that $S(t) = S$.
- By Itô's lemma we have for $V(t) = V(t,S(t))$

$$dV = \left[\frac{\partial V}{\partial t} + \frac{1}{2}\frac{\partial^2 V}{\partial S^2}S^2(t)\sigma^2(t)\right]dt + \frac{\partial V}{\partial S}\,dS(t).$$

- Let us also denote by $C(t)$ the cash collateral held against the derivative at time t. We assume that either $C = 0$ (unsecured trading) or $C = V$ (fully collateralized trade).

Replication

- Let us consider trading activity from the perspective of a trader of bank A who sold a derivative to another bank B at t and received the premium of $V(t, S)$.
- To replicate the derivative, we form a self-financing portfolio where we hold $\Delta(t) = \frac{\partial V}{\partial S}$ units of stock while the cash position is split among a number of accounts
  - We put amount $C(t)$ in our collateral account at bank B. We will receive rate $r_C(t)$ on the collateral posted.
  - The remaining part $V(t) - C(t)$ of the initial premium received is deposited at treasury at rate $r_F(t)$;
  - To finance the purchase of $\Delta(t)$ units of stock for hedging the derivative, we borrow $\Delta(t)S(t)$ from the treasury and use stocks just bought as a collateral. Effectively, we enter a repo transaction at rate $r_R(t)$.

Replication (continued)

- The cash amount in our portfolio evolves over $[t, t+dt]$ according to

$$\Big[\underbrace{C(t) r_C(t)}_{\text{Interest on collateral}} + \underbrace{(V(t) - C(t)) r_F(t)}_{\text{Interest from unsecured deposit}} - \underbrace{\Delta(t) S(t) r_R(t)}_{\text{Interest paid due to repo}} + \underbrace{\Delta(t) S(t) r_D(t)}_{\text{Dividends received}} \Big] dt$$

- By the self-financing property, the above amount is equal to

$$dV(t) - \Delta(t) dS(t) = \left[\frac{\partial V}{\partial t} + \frac{1}{2} \frac{\partial^2 V}{\partial S^2} S^2(t) \sigma^2(t) \right] dt$$

Pricing PDE

- We then get pricing PDE for derivative value function V

$$\frac{\partial V}{\partial t} + (r_R(t) - r_D(t))S(t)\frac{\partial V}{\partial S} + \frac{1}{2}\frac{\partial^2 V}{\partial S^2}S^2(t)\sigma^2(t) = r_F(t)V(t) - (r_F(t) - r_C(t))C(t).$$

- We can use Feynman-Kac formula to express solution of the above PDE as

$$\begin{aligned} V(t) = & \underbrace{\mathbb{E}_t^{\mathbb{Q}}\left[\mathrm{e}^{-\int_t^T r_F(u)\,du} \cdot V(T)\right]}_{\text{Present value in uncollateralized case}} \\ & + \underbrace{\mathbb{E}_t^{\mathbb{Q}}\left[\int_t^T \mathrm{e}^{-\int_t^u r_F(v)\,dv} \cdot (r_F(u) - r_C(u))C(u)\,du\right]}_{\text{Adjustment due to collateralization}}, \end{aligned} \tag{1}$$

where $\mathbb{Q}$ is a measure under which dynamics of S become

$$dS(t) = S(t)\left[(r_R(t) - r_D(t))\,dt + \sigma(t)\,dW^{\mathbb{Q}}(t)\right].$$

Case of a Fully Collateralized Trade

- Note that (1) can be rewritten as

$$V(t) = \mathbb{E}_t^{\mathbb{Q}}\left[\mathrm{e}^{-\int_t^T r_C(u)\,du} \cdot V(T)\right]$$
$$-\,\mathbb{E}_t^{\mathbb{Q}}\left[\int_t^T \mathrm{e}^{-\int_t^u r_C(v)\,dv} \cdot (r_F(u) - r_C(u))(V(u) - C(u))\,du\right] \tag{2}$$

- Consider a case of a fully collateralized trade where $C(t) = V(t)$. The above formula then reduces to

$$V(t) = \mathbb{E}_t^{\mathbb{Q}}\left[\mathrm{e}^{-\int_t^T r_C(u)\,du} \cdot V(T)\right]$$

- We discount collateralized trades with the rate paid on collateral.

Case of a Non-Nollateralized Trade

- Consider a case where $C(t) \equiv 0$. From (1) we get

$$V(t) = \mathbb{E}_t^{\mathbb{Q}}\left[\mathrm{e}^{-\int_t^T r_F(u)\,du} \cdot V(T)\right]$$

- As mentioned above, derivatives transactions between banks are collateralized. However, a trade between a bank and a corporate firm is not necessarily collateralized and hence in that case we use r_F for discounting cash flows.
- In order to mitigate credit risk in that case, banks additionally charge a credit valuation adjustment (CVA) to a counterparty.
- Discounting rate now depends on a counterparty

Credit Valuation Adjustment (CVA)

- Consider a set of n non-collaterelized derivative trades between bank A and a corporate firm B. We denote by V_t^i the present value of the i-th contract at t.
- From the perspective of A, the positive exposure at t is defined as

$$PE(t) = \left(\sum_{i=1}^{n} V_t^i \right)_+ .$$

- CVA at $t = 0$ is then defined by

$$CVA = \mathbb{E}^{\mathbb{Q}} \left[\frac{PE(\tau)}{B(\tau)} \right], \tag{3}$$

  where B is a money-market account and τ is the time of default of B.
- Generally, quantity (3) is extremely hard to compute.

Additional Aspects

- So far we have assumed that collateral has the same currency as the product.
- This is not always the case in practice. We can consider simple RUB fix-for-floating swap collateralized in USD SOFR.
- How do we price such contract? Obviously, we cannot discount RUB cash flows with USD rate. We need a mechanism to adjust RUB discounting taking into account the fact that the product is collateralized in USD. We will get back to that later.
- Moreover, CSA may specify an additional option for collateral payer to choose the optimal currency of the cash to be posted as collateral.

An Economy Without a Risk-Free Money-Market Account

- Classical derivatives valuation theory starts by assuming the existence of a money market account $B(t)$ which grows at the risk-free rate.
- Now even government bonds cannot be considered credit risk-free so the foundations need to be revisited.
- In modern economy, a credit risk-free asset is an asset fully collateralized on a continuous basis which is fundamentally different from classical money-market account.
- Whereas, with a money-market account one can deposit cash now and withdraw it, credit risk-free, at any future time T, a collateralised asset produces a continuous stream of cash flows from the changes in mark-to-market value.

Collateralization process revisited (continued)

- Consider a fully collateralized transaction where a party A buys at time t an asset worth $V(t)$ from a party B.
- Recall that there is no actual cash exchange at t: A buys the asset worth $V(t)$, and B then must immediately post a cash collateral for an amount of $V(t)$ to A in order to secure the liability just sold.
- At $t + dt$, A returns the collateral amount $V(t)$ to B and also pays the interest $c(t)\,V(t)\,dt$. On the other hand, the new mark-to-market value of the asset is $V(t + dt)$ so B, in turn, posts this amount to A.
- Of course, only net amount is actually to be paid at $t + dt$. Summarizing the above, fully collateralized asset generates continuous stream of payments

$$V(t + dt) - V(t) \cdot (1 + c(t)\,dt) = dV(t) - c(t)\,V(t)\,dt. \tag{4}$$

Two Assets Collateralized With the Same Rate

- Let us consider two risky assets whose dynamics under real-world measure $\mathbb{P}$ are given by

$$dV_i(t) = \mu_i(t)\, V_i(t)\, dt + \sigma_i(t)\, V_i(t)\, dW^{\mathbb{P}}(t), \quad i = 1, 2, \tag{5}$$

  where both price processes are driven by the same Wiener process $W^{\mathbb{P}}(t)$ (case of a stock and an option on that stock).

- Assume that both assets are collateralized at the same rate $c(t)$. According to (4), collateralized assets generate the following cashflows during $[t, t + dt]$

$$dV_i^c(t) = dV_i(t) - c(t)\, V_i(t)\, dt, \quad i = 1, 2. \tag{6}$$

- Note that the process $V_i(t)$ defines the amount of collateral holding while $V_i^c(t)$ can be seen as the cumulative dividend process corresponding to asset i.

- We highlight that we do not assume the existence of a risk-free money market account.

Two Assets Collateralized With the Same Rate (continued)

- Now, consider a portfolio where we buy at time t $\sigma_2(t)\, V_2(t)$ units of the first asset and sell $\sigma_1(t)\, V_1(t)$ units of the second asset.
- The net cashflow at $t + dt$ is then equals to

$$\sigma_2(t)\, V_2(t)\, dV_1^c(t) - \sigma_1(t)\, V_1(t)\, dV_2^c(t).$$

- Given (6), the above expression boils down to

$$\Big[\sigma_2(t)(\mu_1(t) - c(t)) - \sigma_1(t)(\mu_2(t) - c(t))\Big] V_1(t)\, V_2(t)\, dt. \tag{7}$$

- Note that we entered the trade at t at zero cost and can terminate it at $t + dt$ at zero additional cost after the amount (7) is paid.
- Thus, in order to exclude arbitrage opportunities, the following relationship must hold

$$\sigma_2(t)(\mu_1(t) - c(t)) = \sigma_1(t)(\mu_2(t) - c(t)). \tag{8}$$

Risk-Neutral Measure

- We rewrite no-arbitrage condition (8) as

$$\frac{\mu_1(t) - c(t)}{\sigma_1(t)} = \frac{\mu_2(t) - c(t)}{\sigma_2(t)}. \tag{9}$$

- Defining new process $W^{\mathbb{Q}}(t)$ via

$$dW^{\mathbb{Q}}(t) = dW^{\mathbb{P}}(t) + \frac{\mu_1(t) - c(t)}{\sigma_1(t)}dt = dW^{\mathbb{P}}(t) + \frac{\mu_2(t) - c(t)}{\sigma_2(t)}dt.$$

  and using Girsanov theorem, we can rewrite (5) as

$$dV_i(t) = c(t)\,V_i(t)\,dt + \sigma_i(t)\,V_i(t)\,dW^{\mathbb{Q}}(t), \quad i = 1, 2.$$

- We see, that there exists a measure $\mathbb{Q} \sim \mathbb{P}$ such that

$$V_i(t) = \mathbb{E}_t^{\mathbb{Q}}\left[\mathrm{e}^{-\int_t^T c(s)ds}\,V_i(T)\right] \quad i = 1, 2.$$

Different Collateral Rates

- Collateral rate $c(t)$ may be specific to a particular counterparty.
- As can be easily seen, the similar result applies to the case where the two assets are collateralized at different rates, say $c_1(t)$ and $c_2(t)$.
- In particular, the same arguments as above lead to the following no-arbitrage condition

$$\frac{\mu_1(t) - c_1(t)}{\sigma_1(t)} = \frac{\mu_2(t) - c_2(t)}{\sigma_2(t)}.$$

- Hence, we can still switch to the risk-neutral measure $\mathbb{Q}$ under which the following relations hold

$$V_i(t) = \mathbb{E}_t^{\mathbb{Q}}\left[\mathrm{e}^{-\int_t^T c_i(s)ds}\, V_i(T)\right] \quad i = 1, 2.$$

Extension to Multiple Dimensions

- We can easily extend the above results to a multi-dimensional case.
- Consider a general framework where we have $N+1$ assets, all collateralized with the same rate $c(t)$, which are driven by a standard N-dimensional Wiener process $W^{\mathbb{P}}$

$$dV_i(t) = \mu_i(t) V_t(t) dt + V_i(t) \cdot \sum_{k=1}^{N} \sigma_{i,k} \, dW_k^{\mathbb{P}}, \quad i = 1 \ldots N+1.$$

- For simplicity of notations, we rewrite that in matrix form as

$$dV = \mu V dt + \Sigma dW^{\mathbb{P}},$$

where μV is a column vector with elements $\mu_i V_i$ and Σ is an $(N+1) \times N$ matrix of a full rank with inputs $\Sigma_{i,j} = V_i \, \sigma_{i,j}$.

- Note that there exists a column vector ω such that

$$\omega^{\mathsf{T}} \cdot \Sigma = 0. \tag{10}$$

Extension to Multiple Dimensions (continued)

- Then the portfolio $\omega^\intercal \cdot V$ has no dW component and hence, by the no-arbitrage arguments, we must have that

$$\omega^\intercal \cdot (\mu V - cV) = 0.$$

- This implies that $\mu V - cV$ belongs to the N-dimensional subspace of vectors orthogonal to ω.
- In view of (10), this sub-space also contains all column vectors of matrix Σ. Note that they actually form a basis of this sub-space due to the full rank assumption.
- Therefore, there exists an N-dimensional vector λ such that

$$\begin{aligned}
\mu V - cV &= \Sigma\lambda. \\
dV &= \mu V dt + \Sigma dW^{\mathbb{P}} = cVdt + \Sigma(dW^{\mathbb{P}} + \lambda dt).
\end{aligned}$$

- By multidimensional Girsanov theorem, we can switch to a measure $\mathbb{Q}$ where $dW^{\mathbb{Q}} = dW^{\mathbb{P}} + \lambda dt$ is a driftless Brownian motion.