Applied Mathematical Finance I

Lectures 1-2: Introduction and Mathematical Prerequisites

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What is this course about?

- In this course, we will focus on theory and practice of interest rate and credit derivatives.
- To give some flavour, here are some questions that we are going to answer during the course
  - What are common types of interest rate and credit derivatives?
  - How are they valued and hedged?
  - What are popular interest rate models and how are they calibrated in practice?
  - What changed with the global financial crisis of 2007-08?
  - In view of LIBOR Transition, what is the new generation of products and models?

Motivation

- Consider a derivative contract with final payoff depending on a future realization of the Key Rate of the Central Bank of Russia. How it can be hedged?
- In case of equity derivative, a trader would hedge market risk by taking an offsetting position in underlying stock (delta-hedging).
- Interest rate, however, does not represent a price of a traded asset so one cannot directly buy or sell interest rate for hedging purposes.
- Our goal is to develop arbitrage-free pricing theory for fixed-income markets and today we briefly discuss some fundamentals of derivatives pricing.

General Setup

- We consider an economy with continuous and frictionless trading activity over a finite time interval $[0, T]$.
- Uncertainty is modelled via a probability space $(\Omega, \mathcal{F}, \mathbb{P})$.
- Information flow is represented by a filtration $\mathbb{F} = \{\mathcal{F}_t,\ 0 \le t \le T\}$, a family of sub-$\sigma$-algebras such that $\mathcal{F}_s \subseteq \mathcal{F}_t$ for $s \le t$ and $\mathcal{F} = \mathcal{F}_T$. For technical reasons, we require that the filtration satisfies the usual conditions*.
- The state of economy at time t is given by a vector of prices of n dividend-free assets $S_t = \left(S_t^1, \ldots S_t^n\right)^\intercal$.
- Process S is assumed to be adapted to $\mathbb{F}$ meaning that S_t is fully observable at t.

*That is, $\mathcal{F}_t$ is right-continuous for $t \in [0, T]$ and $\mathcal{F}_0$ contains all subsets of sets of zero probability.

Trading Strategies

- A trading strategy is a progressively measurable process $\phi = (\phi^1, \ldots \phi^n)^\intercal$. Interpretation: ϕ^i_t denotes the number of units of the i-th asset held in the portfolio at time t.
- The value V_t of the trading strategy ϕ at time t is defined as $V_t = \phi_t^\intercal \cdot S_t$.
- Trading gains (or losses) over a small time interval $[t, t+dt]$ are $\phi_t^\intercal \cdot (S_{t+dt} - S_t)$.
- When continuously rebalancing the portfolio over time interval $[t, T]$, the total profit and loss (P&L) become

$$\text{P\&L}(t, T) = \int_t^T \phi_u^\intercal \, dS_u.$$

- To proceed, let us briefly recall some relevant mathematical concepts.

Driving Process

- We will consider a special case of the abstract setup above where information is generated by standard k-dimensional Wiener process $W_t = \left(W_t^1, \dots W_t^k\right)^\intercal$
  - W^i is independent of W^j for $i \neq j$.
  - Independent Gaussian increments $W_t^i - W_s^i \sim \mathcal{N}(0, t-s)$ for $s \leq t$.
  - Each component W^i is a continuous process starting at 0.
- Normally, filtration $\mathbb{F}$ is the one generated by W (possibly augmented)

$$\mathcal{F}_t = \mathcal{F}_t^W = \sigma\{W_s,\, 0 \leq s \leq t\}.$$

Martingales and Local Martingales

- Let X be a real-valued adapted process with $\mathbb{E}^{\mathbb{P}}|X_t| < \infty$ for all $t \in [0, T]$, where $\mathbb{E}^{\mathbb{P}}$ denotes expectation under measure $\mathbb{P}$. X is called a $\mathbb{P}$-martingale on $[0, T]$ if

$$\mathbb{E}_s^{\mathbb{P}}[X_t] = \mathbb{E}^{\mathbb{P}}[X_t \mid \mathcal{F}_s] = X_s \quad \text{a. s.}$$

  for all $0 \leq s \leq t \leq T$.

- Interpretation: "best" prediction of a future state X_t is its current value X_s.

- The notion of a martingale can be generalized in the following way. X is said to be a local martingale (under $\mathbb{P}$) if there exists a sequence of stopping times $\{\tau_n\}_{n=1}^{\infty}$, with $\tau_n \to T$ a. s. when $n \to \infty$, such that $X_{t \wedge \tau_n} = X_{\min\{t, \tau_n\}}$ is a martingale.

- Note that local martingales are not necessarily integrable.

Quadratic Variation and Covariation

- For a stochastic process $X : \mathbb{R}_+ \times \Omega \to \mathbb{R}$, quadratic variation $\langle X \rangle_t$ over $[0, t]$ is defined as

$$\langle X \rangle_t = \lim_{\|P_n\| \to 0} \sum_{i=1}^{n} (X_{t_i} - X_{t_{i-1}})^2,$$

  where $P_n = \{t_0, t_1, \ldots, t_n\}$ is a partition of the interval $[0, t]$ and $\|P_n\| = \max_{1 \le i \le n}(t_i - t_{i-1})$ is the mesh. It is only required that the limit exists under convergence in probability.

- Recall that for standard one-dimensional Brownian motion B we have $\langle B \rangle_t = t$.

- Quadratic covariation $\langle X, Y \rangle_t$ of two processes X and Y can be defined via polarization identity

$$\langle X, Y \rangle_t = \frac{1}{2} \left(\langle X + Y \rangle_t - \langle X \rangle_t - \langle Y \rangle_t \right).$$

Stochastic Integral

- As we have already seen, P&L of a portfolio over interval $[t, T]$ is given by

$$\text{P\&L} = \int_t^T \phi_u^\intercal \, dS_u$$

  and hence stochastic integration plays a central role in quantitative finance.

- How do we define stochastic integral $\int_0^t \sigma_s \, dB_s$, where B is standard one-dimensional Brownian motion?

- First thing to note, is that it cannot be defined pathwise (for fixed $\omega \in \Omega$) as a Riemann–Stieltjes integral because of infinite variation of B on $[0, t]$

$$\sum_{i=1}^{n} |B_{t_i} - B_{t_{i-1}}| \geq \frac{1}{\max_i |B_{t_i} - B_{t_{i-1}}|} \cdot \sum_{i=1}^{n} (B_{t_i} - B_{t_{i-1}})^2 \to \infty \text{ a. s. for } \|P_n\| \to 0.$$

Stochastic Integral (continued)

- While stochastic integral cannot be defined as the almost sure limit, we can resort to other notions of convergence of random variables.
- Now consider for the moment $\int_0^t B_s \, dB_s$. Define

$$L_n = \sum_{i=1}^{n} B_{t_{i-1}} \left(B_{t_i} - B_{t_{i-1}} \right), \quad R_n = \sum_{i=1}^{n} B_{t_i} \left(B_{t_i} - B_{t_{i-1}} \right).$$

- Note that

$$L^2 - \lim(R_n - L_n) = L^2 - \lim \sum_{i=1}^{n} (B_{t_k} - B_{t_{k-1}})^2 = \langle B \rangle_t = t \neq 0.$$

- Therefore, we expect stochastic integral to depend on the choice of points at which the integrand is evaluated. What choice should we make?

Itô Integral

- Recall that P&L of a trading strategy ϕ over $[t, t+dt]$ is $\phi_t^\intercal \cdot (S_{t+dt} - S_t)$.
- This suggests using left endpoint of each subinterval as the evaluation point of the integrand which leads to the notion of the Itô Integral.
- Itô integral $\int_0^t \sigma_s \, dB_s$ can be defined for any real-valued progressively measurable process σ such that $\mathbb{P}\left(\int_0^t |\sigma_s|^2 \, ds < \infty\right) = 1$. Note that in general case construction is rather involved.
- For our needs, it is sufficient to assume that σ is a cádlág* adapted process with $\mathbb{P}\left(\int_0^t |\sigma_s|^2 \, ds < \infty\right) = 1$ in which case Itô integral can be seen as

$$\int_0^t \sigma_s \, dB_s = \lim_{\|P_n\| \to 0} \sum_{i=1}^n \sigma_{t_{i-1}} (B_{t_i} - B_{t_{i-1}}),$$

where the limit is understood in the sense of convergence in probability.

*That is, almost all sample paths are right-continuous with finite left-hand limits.

Multidimensional Itô Integral

- Let $\sigma : \mathbb{R}_+ \times \Omega \to \mathbb{R}^{n\times k}$ be an adapted process such that for all $t \in [0, T]$

$$\int_0^t |\sigma_s|^2 \, ds < \infty,$$

  where $|\sigma_s| = \|\sigma_s\|_F = \sqrt{\mathrm{tr}\left(\sigma_s \cdot \sigma_s^{\mathsf{T}}\right)}$ is the Frobenius norm. In this case we say that σ is in $\mathcal{L}^2_{\mathrm{ad}}$.
- Now define

$$X_t = \int_0^t \sigma_s \, dW_s = \int_0^t \begin{pmatrix} \sigma_s^{11} & \dots & \sigma_s^{1k} \\ \vdots & \ddots & \vdots \\ \sigma_s^{n1} & \dots & \sigma_s^{nk} \end{pmatrix} \begin{pmatrix} dW_s^1 \\ \vdots \\ dW_s^k \end{pmatrix}$$

  as a vector-valued process with $X_t^i = \sum_{j=1}^k \int_0^t \sigma_s^{ij} \, dW_s^j$.

Properties of multidimensional Itô Integral

- Define $I_t = \int_0^t \sigma_s \, dW_s$, where $\sigma : \mathbb{R}_+ \times \Omega \to \mathbb{R}^{n \times k}$ is in $\mathcal{L}^2_{\text{ad}}$. We have
  - I_t is $\mathcal{F}_t$-measurable.
  - I_t is a continuous local martingale.
- Define a space $\mathcal{H}^2_{\text{ad}} \subset \mathcal{L}^2_{\text{ad}}$ as a set of processes σ such $\mathbb{E}^{\mathbb{P}} \left(\int_0^T |\sigma_s|^2 \, ds \right) < \infty$. We then have for $\sigma \in \mathcal{H}^2_{\text{ad}}$
  - I_t is a true martingale and, in particular, $\mathbb{E}^{\mathbb{P}} I_t = 0$ for all $t \in [0, T]$.
  - Itô isometry: $\mathbb{E}^{\mathbb{P}} |I_t|^2 = \mathbb{E}^{\mathbb{P}} \left(\int_0^t |\sigma_s|^2 \, ds \right)$.
  - Auto-covariance: $\mathbb{E}^{\mathbb{P}} \left(I_t \cdot I_s^{\intercal} \right) = \mathbb{E}^{\mathbb{P}} \left(\int_0^{\min\{t,s\}} \sigma_u \cdot \sigma_u^{\intercal} \, du \right)$.

Itô Process

- An adapted continuous n-dimensional process X is called an Itô process if it admits a representation

$$X_t = X_0 + \int_0^t \mu_s \, ds + \int_0^t \sigma_s \, dW_s, \quad \text{a.s.,} \ \forall t \in [0, T], \tag{1}$$

where stochastic processes $\mu : \mathbb{R}_+ \times \Omega \to \mathbb{R}^n$ and $\sigma : \mathbb{R}_+ \times \Omega \to \mathbb{R}^{n \times k}$ are assumed to satisfy suitable integrability conditions.

- It is customary to represent the above formula using the differential notation as

$$dX_t = \mu_t \, dt + \sigma_t \, dW_t.$$

- Integration with respect to Itô process is defined via

$$\int_0^t Y_s \, dX_s = \int_0^t Y_s \, \mu_s \, ds + \int_0^t Y_s \, \sigma_s \, dW_s,$$

where $Y : \mathbb{R}_+ \times \Omega \to \mathbb{R}^{m \times n}$.

Itô's Lemma: One-dimensional Case

- Given a function $f : [0, T] \times \mathbb{R} \to \mathbb{R}$ and a scalar Itô process X

$$dX_t = \mu_t \, dt + \sigma_t \, dW_t,$$

  the question is whether the process $Y_t = f(t, X_t)$ is an Itô process.

- Suppose that $f \in C^{1,2}$. Then $Y_t = f(t, X_t)$ is an Itô process such that

$$dY_t = \frac{\partial f}{\partial t}(t, X_t) \, dt + \frac{\partial f}{\partial x}(t, X_t) \, dX_t + \frac{1}{2} \frac{\partial^2 f}{\partial x^2}(t, X_t) \, d\langle X \rangle_t,$$

  where quadratic variation of X is given by

$$\langle X \rangle_t = \int_0^t \sigma_s^2 \, ds.$$

- We usually simply write the above relation as $dX_t^2 = \sigma_t^2 \, dt$.

Itô's Lemma: Multidimensional Case

- Now consider $f : [0,T] \times \mathbb{R}^n \to \mathbb{R}$, $f \in C^{1,2}$, and assume that X is given by formula (1). Then $Y_t = f(t, X_t)$ is an Itô process such that

$$dY_t = \frac{\partial f}{\partial t}(t, X_t)\, dt + \sum_{i=1}^{n} \frac{\partial f}{\partial x_i}(t, X_t)\, dX_t^i + \frac{1}{2} \sum_{i,j=1}^{n} \frac{\partial^2 f}{\partial x_i \partial x_j}(t, X_t)\, d\langle X^i, X^j \rangle_t,$$

  where

$$\langle X^i, X^j \rangle_t = \int_0^t \sum_{l=1}^{k} \sigma_s^{il} \sigma_s^{jl}\, ds.$$

- In particular, taking $f(x_1, x_2) = x_1 x_2$, we get Itô product rule for scalar processes X and Y

$$d(X_t Y_t) = X_t\, dY_t + Y_t\, dX_t + d\langle X, Y \rangle_t.$$

Stochastic Differential Equations

- Usually, we restrict our attention to the case where both coefficients μ and σ in (1) are deterministic measurable functions of time and state.
- In other words, we consider stochastic differential equation (SDE) of the form

$$dX_t = \mu(t, X_t)\, dt + \sigma(t, X_t)\, dW_t, \quad X_0 = x \in \mathbb{R}^n.$$

- A strong solution to the SDE is an Itô process X adapted to $\mathcal{F}_t^W$ and satisfying

$$X_t = x + \int_0^t \mu(s, X_s)\, ds + \int_0^t \sigma(s, X_s)\, dW_s.$$

- As is well known, if $\mu(t, x)$ and $\sigma(t, x)$ satisfy the Lipschitz and linear growth condition in x, then there exists a unique solution to the SDE above. Moreover, it is a Markov process.

Self-financing Strategies

- Now that the basics are covered, let us get back to our setup.
- A trading strategy ϕ is said to be self-financing if, for any $t \in [0, T]$, its value $V_t = \phi_t^\intercal \cdot S_t$ satisfy

$$V_t - V_0 = \int_0^t \phi_u^\intercal \, dS_u,$$

or, in differential notation,

$$dV_t = \phi_t^\intercal \, dS_t.$$

- Interpretation: changes in portfolio value are only due to trading gains or losses, with no funds being added or withdrawn.
- Note that Itô product rule implies

$$S_t^\intercal \, d\phi_t + d\phi_t^\intercal \, dS_t = 0.$$

Arbitrage

- By arbitrage we generally mean a possibility to create "something for nothing".
- Let us denote by Φ the class of all self-financing strategies. Formally, an arbitrage opportunity is a self-financing strategy $\phi \in \Phi$ for which $V_0 = 0$ and, for some $t \in [0, T]$,

$$\mathbb{P}(V_t \geq 0) = 1 \quad \text{and} \quad \mathbb{P}(V_t > 0) > 0$$

  or, equivalently,

$$\mathbb{P}(V_t \geq 0) = 1 \quad \text{and} \quad \mathbb{E}^{\mathbb{P}} V_t > 0.$$

- In economic equilibrium, arbitrage should not exist and requiring the absence of arbitrage strategies is a starting point of asset pricing.

Market Model

- We assume the dynamics of the price vector S are given by an Itô process

$$dS_t = \mu_t \, dt + \sigma_t \, dW_t, \quad S_0 = s.$$

- By market model we mean a pair (S, Ψ), where $\Psi \subseteq \Phi$ is a set of all “allowable” self-financing trading strategies.
- It turns out that arbitrage opportunities are not excluded a priori from the class of self-financing strategies and hence (S, Φ) is generally not free of arbitrage even for “good” choice of price dynamics.
- This means that we have to impose some additional restrictions on the set of allowed trading strategies to get a meaningful model.
- We now turn to the question of characterizing the conditions under which market model is arbitrage-free.

Equivalent Measures and Radon-Nikodym Theorem

- First, we recall that two probability measures $\mathbb{P}$ and $\hat{\mathbb{P}}$ (on the same measurable space) are said to be equivalent ($\mathbb{P} \sim \hat{\mathbb{P}}$) if they have the same null-sets

$$\mathbb{P}(B) = 0 \iff \hat{\mathbb{P}}(B) = 0, \quad \forall B \in \mathcal{F}.$$

- Radon-Nikodym Theorem
  Suppose that $\hat{\mathbb{P}} \sim \mathbb{P}$. There exists a unique (a.s.) random variable ξ, which is called a Radon-Nikodym derivative and is usually denoted as $\frac{d\hat{\mathbb{P}}}{d\mathbb{P}}$, such that

$$\hat{\mathbb{P}}(B) = \int_B \xi \, d\mathbb{P}, \quad \forall B \in \mathcal{F}.$$

  Of course, we must have $\frac{d\hat{\mathbb{P}}}{d\mathbb{P}} > 0$ a.s. and $\mathbb{E}^{\mathbb{P}}\left[\frac{d\hat{\mathbb{P}}}{d\mathbb{P}}\right] = 1.$

Properties of Radon-Nikodym Derivative

- Assume that $\hat{\mathbb{P}} \sim \mathbb{P}$ and let $\mathcal{X}$ be a $\hat{\mathbb{P}}$-integrable random variable. Then

$$\mathbb{E}^{\hat{\mathbb{P}}}\mathcal{X} = \mathbb{E}^{\mathbb{P}}\left[\frac{d\hat{\mathbb{P}}}{d\mathbb{P}}\mathcal{X}\right].$$

- Also, we have $\dfrac{d\mathbb{P}}{d\hat{\mathbb{P}}}\dfrac{d\hat{\mathbb{P}}}{d\mathbb{P}} = 1$ a.s. To see this, we note that for any $B \in \mathcal{F}$

$$\mathbb{P}(B) = \int_B 1\, d\mathbb{P} = \int_B \frac{d\mathbb{P}}{d\hat{\mathbb{P}}}\, d\hat{\mathbb{P}} = \int_B \frac{d\mathbb{P}}{d\hat{\mathbb{P}}}\frac{d\hat{\mathbb{P}}}{d\mathbb{P}}\, d\mathbb{P}.$$

Numéraires and Equivalent Martingale Measures

- It is usually convenient to value assets relatively to some benchmark asset called a numéraire.
- A numéraire N is any self-financing strategy such that its value N_t is strictly positive (a.s.) for all $t \in [0, T]$.
- A measure $\mathbb{Q}^N \sim \mathbb{P}$ is called an equivalent martingale measure induced by N if the normalized asset process $\dfrac{S}{N}$ is a $\mathbb{Q}^N$-martingale. Note that such measure is not necessarily unique.
- If $\mathbb{Q}^N$ exists, $(N, \mathbb{Q}^N)$ is referred to as a numéraire pair.

Numéraires and Self-financing Strategies

- Consider a numéraire pair $(N, \mathbb{Q}^N)$. A normalized price $\dfrac{V}{N}$ of an arbitrary self-financing portfolio ϕ will generally only be a local martingale under $\mathbb{Q}^N$.
- Indeed, applying Itô product rule combined with the definition of a self-financing portfolio, we have

$$\begin{aligned} d\left(\frac{V_t}{N_t}\right) &= \frac{1}{N_t}\,dV_t + V_t\,d\left(\frac{1}{N_t}\right) + dV_t\,d\left(\frac{1}{N_t}\right) \\ &= \frac{1}{N_t}\phi_t^\intercal\,dS_t + \phi_t^\intercal S_t\,d\left(\frac{1}{N_t}\right) + \phi_t^\intercal dS_t\,d\left(\frac{1}{N_t}\right) \\ &= \phi_t^\intercal\,d\left(\frac{S_t}{N_t}\right). \end{aligned}$$

- The fact that $\dfrac{V}{N}$ is a local martingale now follows from Martingale Representation Theorem and properties of Itô integral.

Martingale Representation Theorem

- It follows from properties of Itô integral that all driftless Itô processes of the type

$$dX_t = \sigma_t \, dW_t$$

  are local martingales.

- A converse result is also true: if X_t is a local martingale adapted to filtration generated by W, then there exists a process σ such that the above formula holds.

- Corollary: any integrable random variable $\mathcal{X}$ measurable with respect to $\mathcal{F}_T^W$ can be represented as

$$\mathcal{X} = \mathbb{E}^{\mathbb{P}} \mathcal{X} + \int_0^T \gamma_t \, dW_t$$

  for some process γ.

First Fundamental Theorem of Asset Pricing

- Let N be a numéraire and $\mathbb{Q}^N$ be a corresponding equivalent martingale measure.
- We are now ready to define "allowable" strategies. Self-financing strategy ϕ is called permissible if its normalized price process $\dfrac{V}{N}$ is a true $\mathbb{Q}^N$-martingale. We denote by $\Psi(N)$ the set of all permissible strategies.
- Sufficient Condition for No-Arbitrage
  If there exists a martingale measure $\mathbb{Q}^N$ corresponding to some numéraire N, then market model $(S, \Psi(N))$ is free of arbitrage.

First Fundamental Theorem of Asset Pricing: Proof

- Consider any permissible strategy $\phi \in \Psi(N)$ such that $V_0 = 0$ and $\mathbb{P}\left(V_t \geq 0\right) = 1$ for some time t.
- Given that $\mathbb{Q}^N \sim \mathbb{P}$, we first note that $\mathbb{Q}^N\left(\frac{V_t}{N_t} \geq 0\right) = 1$. Since $\frac{V}{N}$ is a $\mathbb{Q}^N$-martingale, we also have

$$\mathbb{E}^{\mathbb{Q}^N}\left[\frac{V_t}{N_t}\right] = \mathbb{E}^{\mathbb{Q}^N}\left[\frac{V_0}{N_0}\right] = 0$$

and therefore

$$\mathbb{Q}^N\left(\frac{V_t}{N_t} = 0\right) = 1.$$

- The above result implies that $\mathbb{P}\left(V_t = 0\right) = 1$ which means that ϕ is not an arbitrage opportunity and hence $\Psi(N)$ is free of arbitrage.

Derivative Securities

- A derivative security (contingent claim) with maturity T is an $\mathcal{F}_T$-measurable random variable $\mathcal{X}_T$. We will assume that $\mathcal{X}_T$ has a finite variance.
- Interpretation: stochastic payoff at T and no payments before T.
- We say that derivative security $\mathcal{X}_T$ is attainable if there exists a permissible trading strategy ϕ such that

$$V_T = \phi_T^\intercal \cdot S_T = \mathcal{X}_T \quad \text{a.s.}$$

- The trading strategy ϕ in the above formula is said to replicate the derivative security.

Derivatives Pricing

- Consider a numéraire N inducing a martingale measure $\mathbb{Q}^N$. Let π_t denote the price of an attainable contingent claim $\mathcal{X}_T$ at time t.
- No-arbitrage principle implies that π_t must be equal to the value of the replicating portfolio V_t. Since the replicating portfolio is not yet know explicitly (we only know it exists), how do we actually compute π_t?
- By the First Fundamental Theorem of Asset Pricing

$$\frac{\pi_t}{N_t} = \frac{V_t}{N_t} = \mathbb{E}_t^{\mathbb{Q}^N}\left[\frac{V_T}{N_T}\right] = \mathbb{E}_t^{\mathbb{Q}^N}\left[\frac{\mathcal{X}_T}{N_T}\right]$$

and therefore

$$\pi_t = N_t \cdot \mathbb{E}_t^{\mathbb{Q}^N}\left[\frac{\mathcal{X}_T}{N_T}\right].$$

Complete Markets

- If all contingent claims $\mathcal{X}_T$ can be replicated, the market model is said to be complete.

- Interpretation: in complete markets derivative securities are redundant in a sense that they can be created synthetically by trading the underlying assets $S^1, \ldots, S^n$.

- Second Fundamental Theorem of Asset Pricing
  In the absence of arbitrage, a market model is complete if and only if there exists a numéraire N inducing the unique martingale measure $\mathbb{Q}^N$.

Abstract Bayes' Formula

- Consider two equivalent probability measures $\hat{\mathbb{P}} \sim \mathbb{P}$ on $(\Omega, \mathcal{F}_T)$ and let $\mathcal{X}$ be an $\mathcal{F}_T$-measurable random variable integrable with respect to $\hat{\mathbb{P}}$. Then, for $t \in [0, T]$,

$$\mathbb{E}_t^{\hat{\mathbb{P}}} \mathcal{X} = \frac{\mathbb{E}_t^{\mathbb{P}} \left[\frac{d\hat{\mathbb{P}}}{d\mathbb{P}} \mathcal{X} \right]}{\mathbb{E}_t^{\mathbb{P}} \left[\frac{d\hat{\mathbb{P}}}{d\mathbb{P}} \right]}. \tag{2}$$

- By definition, $\mathbb{E}_t^{\hat{\mathbb{P}}} \mathcal{X}$ is a random variable satisfying the following conditions
  - $\mathbb{E}_t^{\hat{\mathbb{P}}} \mathcal{X}$ is $\mathcal{F}_t$-measurable
  - For any event $A \in \mathcal{F}_t$

$$\int_A \mathbb{E}_t^{\hat{\mathbb{P}}} \mathcal{X} \, d\hat{\mathbb{P}} = \int_A \mathcal{X} \, d\hat{\mathbb{P}}.$$

- Also, recall that for unconditional expectations we have

$$\mathbb{E}^{\hat{\mathbb{P}}} \mathcal{X} = \mathbb{E}^{\mathbb{P}} \left[\frac{d\hat{\mathbb{P}}}{d\mathbb{P}} \mathcal{X} \right].$$

Abstract Bayes' Formula: Proof

- We rewrite formula (2) as

$$\mathbb{E}_t^{\mathbb{P}}\left[\frac{d\hat{\mathbb{P}}}{d\mathbb{P}}\mathcal{X}\right] = \mathbb{E}_t^{\hat{\mathbb{P}}}\mathcal{X}\cdot\mathbb{E}_t^{\mathbb{P}}\left[\frac{d\hat{\mathbb{P}}}{d\mathbb{P}}\right].$$

- The right-hand side of the last formula is $\mathcal{F}_t$-measurable, so it is enough to show

$$\int_A \frac{d\hat{\mathbb{P}}}{d\mathbb{P}}\cdot\mathcal{X}\,d\mathbb{P} = \int_A \mathbb{E}_t^{\hat{\mathbb{P}}}\mathcal{X}\cdot\mathbb{E}_t^{\mathbb{P}}\left[\frac{d\hat{\mathbb{P}}}{d\mathbb{P}}\right]d\mathbb{P}, \quad \forall A\in\mathcal{F}_t.$$

- We have

$$\int_A \frac{d\hat{\mathbb{P}}}{d\mathbb{P}}\cdot\mathcal{X}\,d\mathbb{P} = \int_A \mathcal{X}\,d\hat{\mathbb{P}} = \int_A \mathbb{E}_t^{\hat{\mathbb{P}}}\mathcal{X}\,d\hat{\mathbb{P}} = \int_A \mathbb{E}_t^{\hat{\mathbb{P}}}\mathcal{X}\cdot\frac{d\hat{\mathbb{P}}}{d\mathbb{P}}\,d\mathbb{P}$$
$$= \int_A \mathbb{E}_t^{\mathbb{P}}\left[\mathbb{E}_t^{\hat{\mathbb{P}}}\mathcal{X}\cdot\frac{d\hat{\mathbb{P}}}{d\mathbb{P}}\right]d\mathbb{P} = \int_A \mathbb{E}_t^{\hat{\mathbb{P}}}\mathcal{X}\cdot\mathbb{E}_t^{\mathbb{P}}\left[\frac{d\hat{\mathbb{P}}}{d\mathbb{P}}\right]d\mathbb{P}.$$

Radon-Nikodym Density Process

- For two equivalent probability measures $\hat{\mathbb{P}} \sim \mathbb{P}$ on $(\Omega, \mathcal{F}_T)$, a density process ξ associated with $\hat{\mathbb{P}}$ is defined as

$$\xi_t = \mathbb{E}_t^{\mathbb{P}}\left[\frac{d\hat{\mathbb{P}}}{d\mathbb{P}}\right], \quad t \in [0, T].$$

- Note that ξ is a $\mathbb{P}$-martingale

$$\mathbb{E}_s^{\mathbb{P}}\xi_t = \mathbb{E}_s^{\mathbb{P}}\left[\mathbb{E}_t^{\mathbb{P}}\left[\frac{d\hat{\mathbb{P}}}{d\mathbb{P}}\right]\right] = \mathbb{E}_s^{\mathbb{P}}\left[\frac{d\hat{\mathbb{P}}}{d\mathbb{P}}\right] = \xi_s, \quad s \le t.$$

- If ζ is the density process for $\mathbb{P}$, i.e., $\zeta_t = \mathbb{E}_t^{\hat{\mathbb{P}}}\left[\frac{d\mathbb{P}}{d\hat{\mathbb{P}}}\right]$, then $\zeta = \frac{1}{\xi}$. Indeed

$$\zeta_t = \mathbb{E}_t^{\hat{\mathbb{P}}}\left[\frac{d\mathbb{P}}{d\hat{\mathbb{P}}}\right] = \frac{\mathbb{E}_t^{\mathbb{P}}\left[\frac{d\mathbb{P}}{d\hat{\mathbb{P}}}\frac{d\hat{\mathbb{P}}}{d\mathbb{P}}\right]}{\mathbb{E}_t^{\mathbb{P}}\left[\frac{d\hat{\mathbb{P}}}{d\mathbb{P}}\right]} = \frac{1}{\mathbb{E}_t^{\mathbb{P}}\left[\frac{d\hat{\mathbb{P}}}{d\mathbb{P}}\right]} = \frac{1}{\xi_t}.$$

Radon-Nikodym Density Process (continued)

- Consider a measurable space $(\Omega, \mathcal{F}_t)$ for $t < T$. Define $\mathbb{P}_t = \mathbb{P}|_{\mathcal{F}_t}$ and $\hat{\mathbb{P}}_t = \hat{\mathbb{P}}|_{\mathcal{F}_t}$.
- There exists $\mathcal{F}_t$-measurable random variable $\dfrac{d\hat{\mathbb{P}}_t}{d\mathbb{P}_t}$ such that

$$\hat{\mathbb{P}}_t(B) = \int_B \frac{d\hat{\mathbb{P}}_t}{d\mathbb{P}_t}\, d\mathbb{P}_t, \quad \forall B \in \mathcal{F}_t.$$

- We note that $\dfrac{d\hat{\mathbb{P}}_t}{d\mathbb{P}_t}$ is given by

$$\frac{d\hat{\mathbb{P}}_t}{d\mathbb{P}_t} = \mathbb{E}_t^{\mathbb{P}}\left[\frac{d\hat{\mathbb{P}}}{d\mathbb{P}}\right].$$

Indeed, for any $B \in \mathcal{F}_t$

$$\int_B \frac{d\hat{\mathbb{P}}_t}{d\mathbb{P}_t}\, d\mathbb{P}_t = \hat{\mathbb{P}}_t(B) = \int_B 1\, d\hat{\mathbb{P}}_t = \int_B \frac{d\hat{\mathbb{P}}}{d\mathbb{P}}\, d\mathbb{P}_t.$$

Change of Numéraire

- Consider a numéraire pair $(N, \mathbb{Q}^N)$. Recall that the arbitrage-free price at time t of an attainable contingent claim $\mathcal{X}_T$ is given by

$$\pi_t = N_t \cdot \mathbb{E}_t^{\mathbb{Q}^N}\left[\frac{\mathcal{X}_T}{N_T}\right]. \tag{3}$$

- Now let us consider another permissible numéraire M. Note that $\frac{M}{N}$ is a $\mathbb{Q}^N$-martingale and, in particular, $\mathbb{E}^{\mathbb{Q}^N}\left[\frac{M_T}{N_T}\right] = \frac{M_0}{N_0}$.
- Define a new equivalent probability measure $\mathbb{Q}^M$ on $(\Omega, \mathcal{F}_T)$ via specifying a Radon-Nikodym derivative ξ_T

$$\xi_T = \frac{d\mathbb{Q}^M}{d\mathbb{Q}^N} = \frac{M_T}{N_T} \cdot \frac{N_0}{M_0}$$

- Note that $\mathbb{Q}^M$ is indeed a probability measure since $\xi_T > 0$ a.s. and $\mathbb{E}^{\mathbb{Q}^N}[\xi_T] = 1$.

Change of Numéraire (continued)

- By applying abstract Bayes' formula to (3), we get

$$\pi_t = N_t \cdot \mathbb{E}_t^{\mathbb{Q}^N}\left[\frac{\mathcal{X}_T}{N_T}\right] = N_t \cdot \mathbb{E}_t^{\mathbb{Q}^N}[\xi_T] \cdot \mathbb{E}_t^{\mathbb{Q}^M}\left[\frac{\mathcal{X}_T}{\xi_T \cdot N_T}\right].$$

- Again, using the fact that $\frac{M}{N}$ is a martingale under $\mathbb{Q}^{\mathbb{N}}$, we have

$$\begin{aligned}\pi_t &= N_t \cdot \frac{M_t}{N_t} \cdot \frac{N_0}{M_0} \cdot \mathbb{E}_t^{\mathbb{Q}^M}\left[\frac{\mathcal{X}_T \cdot M_0 \cdot N_T}{M_T \cdot N_0 \cdot N_T}\right] \\ &= M_t \cdot \mathbb{E}_t^{\mathbb{Q}^M}\left[\frac{\mathcal{X}_T}{M_T}\right].\end{aligned}$$

- This is a very powerful technique since, for a particular payoff $\mathcal{X}_T$, we can try to find a suitable numéraire M so that $\mathbb{E}_t^{\mathbb{Q}^M}\left[\frac{\mathcal{X}_T}{M_T}\right]$ is easy to evaluate.

Doléans-Dade Exponential

- Consider a k-dimensional process θ adapted to filtration generated by Wiener process $W = W^{\mathbb{P}}$. Define a scalar random process Z_t

$$dZ_t = Z_t \theta_t^\intercal dW^{\mathbb{P}}, \quad Z_0 = 1. \tag{4}$$

- By an application of Itô's lemma, it can be easily verified that

$$\begin{aligned} Z_t &= \mathrm{e}^{-\frac{1}{2}\int_0^t \theta_s^\intercal \theta_s \, ds + \int_0^t \theta_s^\intercal \, dW_s^{\mathbb{P}}} \\ &= \mathcal{E}\left(\int_0^t \theta_s^\intercal \, dW_s^{\mathbb{P}}\right), \end{aligned}$$

  where $\mathcal{E}(\cdot)$ is the Doléans-Dade exponential.

- Recall that for a continuous semimartingale X, Doléans-Dade exponential is defined as

$$\mathcal{E}(X)_t = \mathrm{e}^{X_t - X_0 - \frac{1}{2}\langle X \rangle_t}.$$

Girsanov Theorem

- The most important tool when dealing with equivalent measure transformation is Girsanov theorem.
- Suppose that process Z in (4) is a martingale. Then process $W^{\hat{\mathbb{P}}}$ given by

$$W_t^{\hat{\mathbb{P}}} = W_t^{\mathbb{P}} - \int_0^t \theta_s \, ds, \quad t \in [0, T],$$

  is a Wiener process under probability measure $\hat{\mathbb{P}}$ defined via its Radon-Nikodym derivative $\frac{d\hat{\mathbb{P}}}{d\mathbb{P}} = Z_T$.
- Sufficient condition for process Z to be a martingale is the Novikov's condition

$$\mathbb{E}^{\mathbb{P}}\left[\mathrm{e}^{\frac{1}{2}\int_0^T \theta_t^\intercal \theta_t \, dt}\right] < \infty.$$

Girsanov Theorem and Effect on Asset Dynamics

- Recall that asset price process dynamics under $\mathbb{P}$ are given by

$$dS_t = \mu_t \, dt + \sigma_t \, dW_t^{\mathbb{P}}.$$

- We now consider a measure change from $\mathbb{P}$ to $\hat{\mathbb{P}} \sim \mathbb{P}$ given by the exponential martingale kernel Z_T.
- By Girsanov theorem, dynamics of $W^{\mathbb{P}}$ under $\hat{\mathbb{P}}$ become

$$dW_t^{\mathbb{P}} = dW_t^{\hat{\mathbb{P}}} + \theta_t dt.$$

which implies the following $\hat{\mathbb{P}}$-dynamics of S

$$dS_t = (\mu_t + \sigma_t \cdot \theta_t) \, dt + \sigma_t \, dW_t^{\hat{\mathbb{P}}}.$$

Markovian Setup

- In a complete market, any contingent claim can be replicated by a self-financing trading strategy, but it still remains to determine that strategy explicitly.
- Let us now consider a Markovian setup where price process dynamics under $\mathbb{P}$ are given by

$$dS_t = \mu(t, S_t)dt + \sigma(t, S_t)dW_t^{\mathbb{P}}.$$

- Consider an attainable derivative security with payoff of the form $\mathcal{X}_T = F(S_T)$. The Markovian form of the asset dynamics suggests that the derivative price π_t is a function of t and S_t only, $\pi_t = V(t, S_t)$ for some deterministic function $V(t, s)$.

Derivatives Pricing and PDEs

- Assuming that function V above is smooth enough, we apply Itô's lemma to get

$$dV(t,S_t) = \frac{\partial V(t,S_t)}{\partial t}\,dt + \sum_{i=1}^{n} \frac{\partial V(t,S_t)}{\partial s_i}\,\mu^i(t,S_t)_t\,dt + \frac{1}{2}\sum_{i,j=1}^{n} \frac{\partial^2 V(t,S_t)}{\partial s_i \partial s_j}\,\Sigma_{i,j}\,dt$$
$$+ \sum_{i=1}^{n} \frac{\partial V(t,S_t)}{\partial s_i}\,\sigma^i(t,S_t)\,dW_t^{\mathbb{P}},$$

where σ^i is the i-th row of the $n \times k$ matrix σ and $\Sigma_{i,j}$ is the (i,j)-th element of the matrix $\sigma\sigma^{\intercal}$.

- Since contingent claim is attainable, there exists a permissible self-financing replicating portfolio ϕ and hence we must have

$$dV(t,S_t) = \phi_t^{\intercal} dS_t = \sum_{i=1}^{n} \phi_t^i\,\mu^i(t,S_t)dt + \sum_{i=1}^{n} \phi_t^i\,\sigma^i(t,S_t)\,dW_t^{\mathbb{P}}.$$

Derivatives Pricing and PDEs (continued)

- Comparing terms in the two equations above, we see that both equations will hold given that

$$\phi_t^i = \frac{\partial V(t, S_t)}{\partial s_i}, \quad i = 1, \ldots, n,$$

and

$$\frac{\partial V(t,s)}{\partial t} + \frac{1}{2} \sum_{i,j=1}^{n} \frac{\partial^2 V(t,s)}{\partial s_i \partial s_j} \Sigma_{i,j} = 0, \quad V(T,s) = F(s). \tag{5}$$

- What it tells us is that
  - Value function $V(t,s)$ solves the second-order parabolic Cauchy problem (5).
  - To replicate derivative contract paying $F(S_T)$ at time T, one must hold $\frac{\partial V(t,S_t)}{\partial s_i}$ units of asset S^i in portfolio at t.

Black-Scholes-Merton Model

- In the Black-Scholes-Merton model, the economy consists of two assets: the bank (money market) account B and the stock S.
- Dynamics of the bank account are given by

$$dB_t = rB_t dt, \quad B_0 = 1, r \geq 0.$$

Interpretation: cash amount on the account grows at at continuously compounded risk-free interest rate r.

- The stock price dynamics are given by a Geometric Brownian motion

$$dS_t = S_t \left[\mu dt + \sigma dW_t^{\mathbb{P}} \right],$$

where $W^{\mathbb{P}}$ is a one-dimensional Wiener process and μ and $\sigma > 0$ are constants. Interpretation: instantaneous relative returns follow Brownian motion with drift.

Black-Scholes-Merton Model: Probabilistic Approach

- Consider a trading strategy $\phi = (\phi_t^1, \phi_t^2)_{t\in[0,T]}^\intercal \equiv (1,0)^\intercal$ which corresponds to depositing one unit of cash at a bank account and holding it there until time T.
- Obviously, ϕ is self-financing and its value V_t is given by $V_t = 1 \cdot B_t + 0 \cdot S_t = B_t$. Therefore, B itself is a valid numéraire.
- By applying Itô's lemma to normalized stock price $\frac{S}{B}$, we get

$$d\left(\frac{S_t}{B_t}\right) = \frac{S_t}{B_t}\left[(\mu - r)dt + \sigma dW^{\mathbb{P}}\right].$$

- In view of Girsanov theorem, it is clear that, by choosing $\theta_t = \frac{r-\mu}{\sigma}, t \in [0,T]$, we can define an equivalent martingale measure $\mathbb{Q}$ with Radon-Nikodym derivative being

$$\frac{d\mathbb{Q}}{d\mathbb{P}} = \mathrm{e}^{-\frac{1}{2}\frac{(r-\mu)^2}{\sigma^2}T + \frac{r-\mu}{\sigma}W_T^{\mathbb{P}}}. \tag{6}$$

Black-Scholes-Merton Model: Probabilistic Approach

- The probability measure $\mathbb{Q}$ given by (6) is called the risk-neutral measure.
- Under $\mathbb{Q}$, $W_t^{\mathbb{Q}} = W_t^{\mathbb{P}} - \frac{r-\mu}{\sigma} t$ is a Wiener process and

$$d\left(\frac{S_t}{B_t}\right) = \frac{S_t}{B_t}\sigma\, dW^{\mathbb{Q}},$$
$$dS_t = S_t\left[r\, dt + \sigma\, dW^{\mathbb{Q}}\right].$$

- By applying Itô's lemma, it can be easily verified that for stock price S_T we have

$$S_T = S_t e^{r(T-t)}\, e^{-\frac{1}{2}\sigma^2(T-t)+\sigma\left(W_T^{\mathbb{Q}}-W_t^{\mathbb{Q}}\right)}, \quad t \in [0, T].$$

- Note that the risk-neutral measure $\mathbb{Q}$ is unique and hence the market is arbitrage-free and complete.

Call Option Pricing in BSM Setting

- Call option pays out an amount of $\max(S_T - K, 0) = (S_T - K)_+$ at T.
- By arbitrage-free pricing formula, call option price π_t at time $t \le T$ is given by

$$\pi_t = B_t \cdot \mathbb{E}_t^{\mathbb{Q}}\left[\frac{(S_T - K)_+}{B_T}\right] = \mathrm{e}^{-r(T-t)} \cdot \mathbb{E}_t^{\mathbb{Q}}\left[(S_T - K)_+\right].$$

$$= \mathrm{e}^{-r(T-t)} \cdot \int_{-\infty}^{\infty} \left(S_t \mathrm{e}^{r(T-t)}\, \mathrm{e}^{-\frac{1}{2}\sigma^2(T-t)+\sigma z\sqrt{(T-t)}} - K\right)_+ \varphi(z)\, dz,$$

where $\varphi(z) = \Phi'(z), \Phi(z) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{z} \mathrm{e}^{-\frac{y^2}{2}}\, dy$, is the standard Gaussian density.
- In BSM setting, the integral above can be computed explicitly and we get the celebrated Black-Scholes-Merton formula

$$\pi_t = S_t\, \Phi(d_+) - K\mathrm{e}^{-r(T-t)}\, \Phi(d_-),$$

where $d_+ = \frac{\ln \frac{S_t}{K} + (r+\sigma^2)(T-t)}{\sigma\sqrt{T-t}}$, $d_- = d_+ - \sigma\sqrt{T-t}$.

Call Option Pricing via PDEs

- Our derivation of the call option price was so far entirely probabilistic. Let us now show how one can apply the PDE technique to price a call option.
- Writing $\pi_t = V(t, B_t, S_t)$ for some deterministic function $V(t, b, s)$, we see that pricing PDE (5) becomes

$$\frac{\partial V(t,b,s)}{\partial t} + \frac{1}{2}s^2\sigma^2\frac{\partial^2 V(t,b,s)}{\partial s^2} = 0,$$

  subject to the boundary condition $V(T, b, s) = (s-K)_+$.
- Since $B_t = \mathrm{e}^{rt}$ is a deterministic function of t, we can actually introduce function $P(t,s) = V(t, \mathrm{e}^{rt}, s)$. The PDE above, when written in terms of P, becomes the famous Black-Scholes-Merton PDE

$$\frac{\partial P(t,s)}{\partial t} + rS\frac{\partial P(t,s)}{\partial s} + \frac{1}{2}s^2\sigma^2\frac{\partial^2 P(t,s)}{\partial s^2} = rP(t,s).$$

Example

- We would like to price the payoff $(S_T^3 - S_T^2)_+$ in a simple BSM setting with $r = 0$ which implies that $B \equiv 1$ and $dS_t = \sigma S_t dW_t^{\mathbb{Q}}$ under the risk-neutral measure $\mathbb{Q}$.
- The initial price of the contract is then given by

$$\pi_0 = \mathbb{E}^{\mathbb{Q}}\left[(S_T^3 - S_T^2)_+\right] = \mathbb{E}^{\mathbb{Q}}\left[S_T^2\,(S_T - 1)_+\right]$$

and it may look like choosing S^2 as a numéraire allows us to simplify the problem to a pricing of a standard call option under martingale measure $\mathbb{Q}^{S^2}$

$$\pi_0 = S_0^2 \cdot \mathbb{E}^{\mathbb{Q}^{S^2}}\left[(S_T - 1)_+\right].$$

- If $\mathbb{Q}^{S^2}$ exists, the normalized price process $\left(\frac{B}{S}, \frac{S}{S^2}\right) = \left(\frac{1}{S^2}, \frac{1}{S}\right)$ must be a $\mathbb{Q}^{S^2}$-martingale which is impossible. Something went wrong...