Applied Mathematical Finance I

Lecture 8: Multi-Curve Framework.

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Review the Basics

- Let us assume that we have $p(t,T)$ units of cash at time t. There are two alternative strategies
  - At t, buy a T-bond and enter an agreement to invest a unit of cash for the future period $[T, T+\tau]$ at some rate F.
  - At t, buy $\frac{p(t,T)}{p(t,T+\tau)}$ units of zero coupon bond maturing at $T+\tau$ and just hold to maturity.
- Both strategies yield deterministic cashflow at $T+\tau$ and hence, to avoid arbitrage, we must have

$$1 + F\tau = \frac{p(t,T)}{p(t,T+\tau)} \implies F = F(t,T,T+\tau) = \frac{1}{\tau}\left(\frac{p(t,T)}{p(t,T+\tau)} - 1\right) \quad (1)$$

- We call $F(t,T,T+\tau)$ the implied forward rate for period $[T, T+\tau]$ as seen at t.

Review the Basics (continued)

- Given a tenor structure $t \leq T_0 < T_1 < \cdots < T_n$, consider a swap leg paying floating payments linked to LIBOR rate $L(T_{i-1}, T_i)$ at T_i, $i = 1, \ldots, n$.

- Its present value at t is then given by

$$\mathrm{PV}_t = \sum_{i=1}^{n} p(t, T_i) \cdot \mathbb{E}_t^{T_i} \left[L(T_{i-1}, T_i) \tau_i \right] = \sum_{i=1}^{n} p(t, T_i) F(t, T_{i-1}, T_i) \tau_i. \tag{2}$$

- Using (1), we can rewrite (2) as

$$\mathrm{PV}_t = p(t, T_0) - p(t, T_n). \tag{3}$$

- Present value of a floating leg does not depend on frequency of payments.

Review the Basics (continued)

- Recall that a single-currency basis swap is a contract to exchange floating rates of the same currency with different frequencies, e.g. 3M vs 6M.

- Consider a basis swap where we receive $L(T_{i-1}, T_i) + s$ at times T_i, $i = 1, \ldots, n$, where s is the fixed basis, and pay $L(T_{2j-2}, T_{2j})$ at T_{2j}, $j = 1, \ldots, \frac{n}{2}$.

- In view of (3), the present value of the swap at $t \leq T_0$ is given by

$$\mathrm{PV}_t^{\mathrm{Swap}} = \mathrm{PV}_t^{\mathrm{Leg1}} - \mathrm{PV}_t^{\mathrm{Leg2}} = s \cdot A_t,$$

  where $A_t = \sum_{i=1}^{n} p(t, T_i)\, \tau_i$ is the annuity.

- Therefore, we see that $\mathrm{PV}_t^{\mathrm{Swap}} = 0 \iff s = 0$, i.e. basis spread must be zero.

Does Standard Replication Argument Work?

- Now, consider real market quotes as of 22 Sep 2014
  - USD LIBOR 3M rate: r_{3M} = 0.2356% with the corresponding year fraction in Act360 convention being $\delta_{3M} = 0.25278$.
  - USD LIBOR FRA 3x6 rate: r_{3x6} = 0.25%, $\delta_{3x6} = 0.25$.
- Given these two quotes, we could invest for 6M period by entering 3M LIBOR deposit and then reinvesting in 3M for another 3M period at pre-agreed FRA (forward) rate.
- The 6M deposit LIBOR rate implied from the above quotes is given by

$$r_{6M} = \frac{(1 + r_{3M} \cdot \delta_{3M}) \cdot (1 + r_{3x6} \cdot \delta_{3x6}) - 1}{\delta_{3M} + \delta_{3x6}} = 0.2428\%.$$

- However, the real 6M LIBOR market quote was much higher: 0.3304%.

Impact of the Global Financial Crisis

- So far we have discussed a mathematical framework that worked perfectly till the mid 2007 when replication formula (1) broke down, see Figure 1.
- Moreover, basis spreads could no longer be considered negligible as Figures 2-3 show.
- With defaults of top-rated banks came the realization that LIBORs are not actually risk-free. These rates do bear credit risk and hence must include a corresponding risk premium.
- The main questions we are now going to answer are the following
  - Given that LIBOR is not risk-free, what rates can we consider as risk-free and what rates should we use for discounting?
  - How do we need to adjust our framework to be consistent with the market? Obviously, we should somehow incorporate the credit risk into our modelling.

Market FRA rate vs implied Forward rate

Figure: 3x6 EUR FRA rate vs forward rate implied from quotes of 3M and 6M deposits.

Basis Spread Explosion During the Credit Crunch

Figure: EURIBOR 3M vs 6M basis spread for a 5Y swap.

Basis spreads after the Crisis

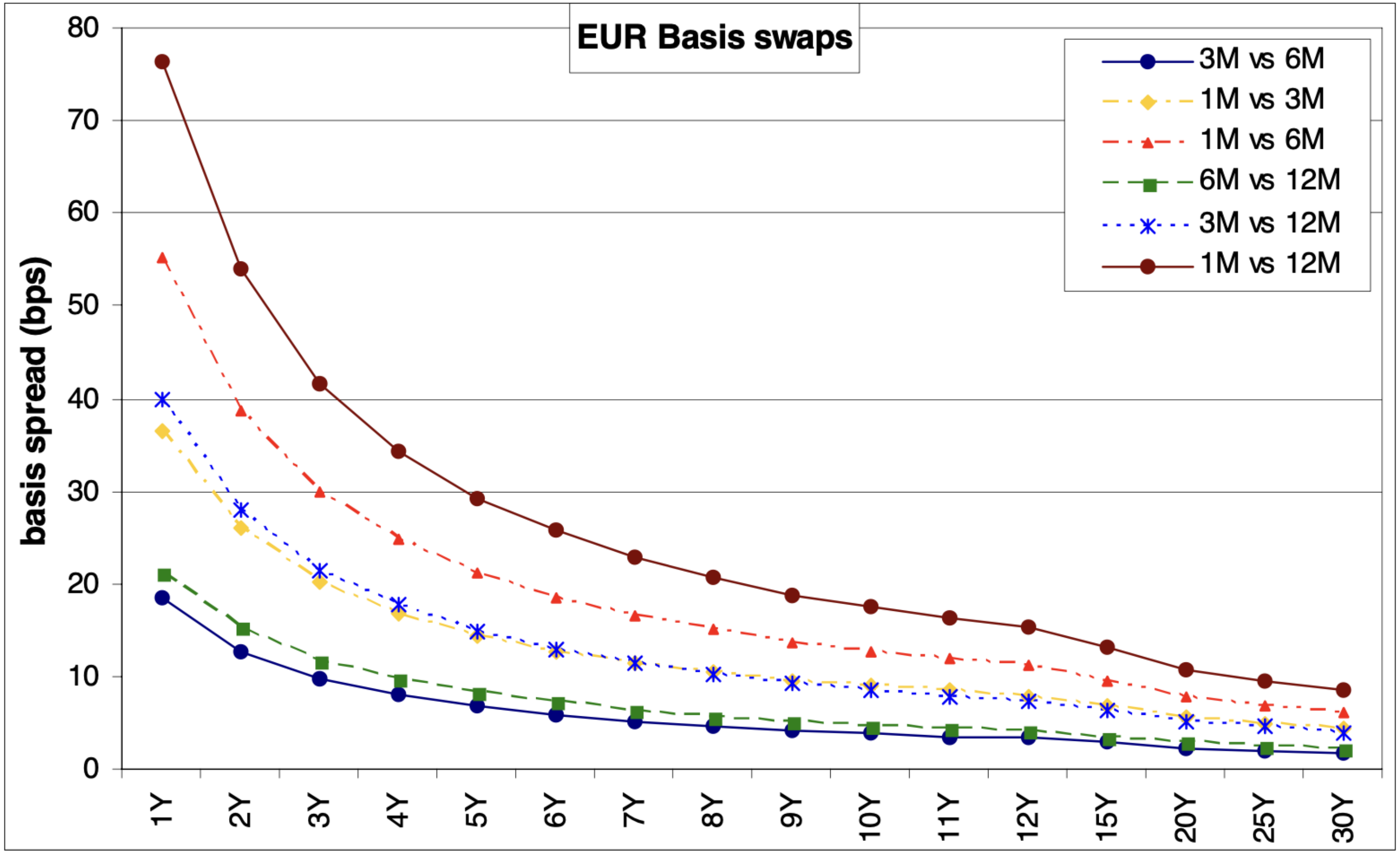

Figure: EUR basis spreads, Feb. 2009.

Drawbacks of LIBOR

- Recall that LIBOR is indicative rate for unsecured borrowing between the top-rated banks.
- Unsecured borrowing means that a lender receives no collateral from a borrower and will at best get back some proportion (recovery rate) of initial investments should the borrower default. Defaults and near-defaults of some top-rated financial firms undermined trust in LIBOR.
- Being the indicative rate, LIBOR also suffers from the following drawbacks
  - Contributor Panel banks can intentionally submit lower rates to give the impression that they have a higher credit quality.
  - Contributor Panel banks can artificially inflate or deflate LIBOR so as to profit from their derivatives positions.
- Such manipulations did take place and, as a result, some major banks were fined a total of about $9 billion.

Overnight rates

- So LIBOR is a bad proxy for risk-free rate. But if not LIBOR, then what?
- USD Secured Overnight Financing Rate (shortly, SOFR)
  - SOFR fixing calculation is based on real transactions in interbank borrowing market. Hence it is not so straightforward to manipulate to;
  - Those transactions are actually overnight which minimizes the probability of the borrower going default during the period;
  - Moreover, the transactions are collateralized with US Treasury Bonds (repo trades).
- The above points make SOFR a good choice for a new USD risk-free rate.
- RUB and EUR overnight rates are respectively RUONIA (Ruble Overnight Index Average) and €STR (Euro short-term rate). Both of them, however, correspond to unsecured overnight borrowing. In USD market, unsecured overnight rate is FEDFUND (federal funds rate).

Overnight Index Swaps

- Overnight Index Swap (OIS) is a fixed-for-floating swap where the floating leg pays at the end of each period the reference overnight rate r_t compounded (more rarely, arithmetically averaged) daily over that period.
- Consider i-th period $[T_{i-1}, T_i]$ of a swap and its partition

$$T_{i-1} = t_{i,0} < t_{i,1} < \cdots < t_{i,n} = T_i, \quad t_{i,j+1} - t_{i,j} = \tau_{i,j+1},$$

  where $t_{i,j}$ is the j-th observation day in the i-th period.
- Daily compounded rate $R(T_{i-1}, T_i)$ for that period is defined as

$$R(T_{i-1}, T_i) = \frac{1}{\tau_i} \cdot \left[\prod_{k=0}^{n-1} (1 + r_{t_{i,k}} \cdot \tau_{i,k+1}) - 1 \right], \; \tau_i = T_i - T_{i-1}, \tag{4}$$

  where $r_{t_{i,k}}$ is the overnight rate fixing observed at $t_{i,k}$.

Overnight forward rate

- We assume that there is a risk-free discounting OIS curve $p_d(t, T), T \geq t$.
- Hereinafter, we mark risk-free discount factors with subscript d (for "discounting") to distinguish them from risky discount factors corresponding to "forwarding" curves considered later. Also, we denote by $\mathbb{E}^{T^d}$ the expectation under the risk-free T-forward measure with numéraire being $p_d(t, T)$.
- For a single risk-free overnight rate observation, we can use the theory developed for LIBORs to define the arbitrage-free overnight forward rate for a period $[T, T + \tau]$ as seen from $t \leq T$ by

$$F(t, T, T + \tau) = \frac{1}{\tau} \left(\frac{p_d(t, T)}{p_d(t, T + \tau)} - 1 \right), \tag{5}$$

where τ is a year fraction corresponding to one business day.

Compounded Forward Rate

- Let us now consider a self-financing strategy where we
  - Sell $\frac{p_d(t,T_{i-1})}{p_d(t,T_i)}$ units of risk-free T_i-bond and time t;
  - Buy one risk-free bond maturing at T_{i-1} and agree to enter into n consecutive (rolling) forward deposits for overnight future periods $[t_{i,j}, t_{i,j+1}]$, $j \in 0, \ldots, n-1$. Note that our initial investment is zero.
- Our net position at T_i is deterministic and given by

$$\underbrace{\prod_{k=0}^{n-1} \left(1 + F(t, t_{i,k}, t_{i,k+1}) \cdot \tau_{i,k}\right)}_{\text{Amount accumulated on the cash account}} - \underbrace{\frac{p_d(t, T_{i-1})}{p_d(t, T_i)}}_{\text{debt on } T_i\text{-bonds}} .$$

- Hence, by no-arbitrage

$$\frac{p_d(t, T_{i-1})}{p_d(t, T_i)} = \prod_{k=0}^{n-1} \left(1 + F(t, t_{i,k}, t_{i,k+1})\, \tau_{i,k+1}\right). \tag{6}$$

Compounded Forward Rate (continued)

- Given (6), we can define compounded forward rate for period $[T_{i-1}, T_i]$ as

$$F(t, T_{i-1}, T_i) = \frac{1}{\tau_i}\left(\frac{p_d(t, T_{i-1})}{p_d(t, T_i)} - 1\right) \tag{7}$$

  and this is essentially equivalent to LIBOR forward rate formula (1).

- Note, however, the fundamental difference between LIBOR rate $L(T_{i-1}, T_i)$ and the compounded rate $R(T_{i-1}, T_i)$ given by (4): LIBOR is observed at the beginning of the period i.e. $L(T_{i-1}, T_i)$ is an $\mathcal{F}_{T_{i-1}}$- measurable random variable, while the compounded rate is not known until the last overnight rate fixing is observed i.e. $R(T_{i-1}, T_i)$ is $\mathcal{F}_{t_{i,n-1}}$- measurable.

- Due to its path-dependent nature, compounded rate $R(T_{i-1}, T_i)$ is often referred to as backward-looking rate since we need to know all past overnight rate fixings to compute it.

Martingale Property of Compounded Forward Rates

- From (7) we see that compounded forward rate $F(t, T_{i-1}, T_i)$, $t \leq T_{i-1}$ is a martingale under T_i^d-forward measure

$$F(t, T_{i-1}, T_i) = \mathbb{E}_t^{T_i^d}\left[R(T_{i-1}, T_i)\right], \quad t \leq T_{i-1}. \tag{8}$$

- We can extend formula (8) to times $t > T_{i-1}$ by introducing “extended” risk-free T-forward bonds.

- Such a bond would correspond to a self-financing strategy that consists of buying the risk-free zero-coupon bond with maturity T, and reinvesting the proceeds received at T at the risk-free rate from time T onwards.

Incorporating Credit Risk

- Our aim now is to incorporate credit risk into LIBOR rates.
- Let us assume that at time t bank A agrees with bank B to enter a forward deposit for future period T to $T + \Delta$ at some fixed rate K.
- We denote by τ_B the random time of default of bank B.
- From the perspective of the bank A, the cash flows are as follows
  - at T: invest a unit of cash given that bank B has not defaulted during $[t, T]$ so the cash flow is -$\mathbb{1}_{\{\tau_B > T\}}$.
  - at $T + \Delta$: get initial investment back and receive the interest given that bank B has not defaulted during $[T, T + \Delta]$ so the cash flow is $(1 + K\Delta)\mathbb{1}_{\{\tau_B > T+\Delta\}}$. Note that zero recovery is assumed.

Incorporating Credit Risk (continued)

- The fair value of the trade at time t is then given by

$$V_t = -p_d(t,T)\mathbb{E}_t^{T^d}\left[\mathbb{1}_{\{\tau_B > T\}}\right] + p_d(t,T+\Delta)(1+K\Delta)\mathbb{E}_t^{(T+\Delta)^d}\left[\mathbb{1}_{\{\tau_B > T+\Delta\}}\right]. \quad (9)$$

- Let us assume that τ_B is independent of interest rates. We then have

$$p_d(t,T)\,\mathbb{E}_t^{T^d}\left[\mathbb{1}_{\{\tau_B > T\}}\right] = \mathbb{E}_t^{\mathbb{Q}^d}\left[\frac{B_t}{B_T}\mathbb{1}_{\{\tau_B > T\}}\right]$$
$$= \mathbb{E}_t^{\mathbb{Q}^d}\left[\frac{B_t}{B_T}\right]\mathbb{E}_t^{\mathbb{Q}^d}\left[\mathbb{1}_{\{\tau_B > T\}}\right] = p_d(t,T)\,\mathbb{E}_t^{\mathbb{Q}^d}\left[\mathbb{1}_{\{\tau_B > T\}}\right],$$

  where $\mathbb{Q}^d$ is the risk-neutral risk-free measure.

- Formula (9) then simplifies to

$$V_t = -p_d(t,T)\mathbb{E}_t^{\mathbb{Q}^d}\left[\mathbb{1}_{\{\tau_B > T\}}\right] + p_d(t,T+\Delta)(1+K\Delta)\mathbb{E}_t^{\mathbb{Q}^d}\left[\mathbb{1}_{\{\tau_B > T+\Delta\}}\right]. \quad (10)$$

Forward Rate In the Presence of Credit Risk

- Define survival probabilities

$$D(t,T) = \mathbb{E}_t^{\mathbb{Q}^d}\left[\mathbb{1}_{\{\tau_B > T\}}\right] = \mathbb{Q}^d\{\tau_B > T \,|\, \mathcal{F}_t\}.$$

- So the contract value now reads

$$V_t = -p_d(t,T)\,D(t,T) + p_d(t,T+\Delta)\,(1+K\,\Delta)\,D(t,T+\Delta).$$

- Value K which sets $V_t = 0$ is given by

$$K = \tilde{F}(t,T,T+\Delta) = \frac{1}{\Delta}\left[\frac{p_d(t,T)}{p_d(t,T+\Delta)}\,\frac{D(t,T)}{D(t,T+\Delta)} - 1\right].$$

- This can be seen as the new definition of risky forward LIBOR rate.

Risky Discount Factors

- In view of non-zero basis spreads between different tenors, we postulate that there are different adjustment factors $D_\tau(t,T)$ for different tenors τ.
- We can now define risky discount factor from t to T corresponding to tenor τ as

$$p_f^\tau(t,T) = p_d(t,T)\,D_\tau(t,T),$$

  where subscript f stands for "forwarding" as risky discount factors are only used for retrieving risky forward LIBOR rates.
- Risky forward LIBOR rate for tenor τ is then given by

$$\tilde{F}(t,T,T+\tau) = \frac{1}{\tau}\left[\frac{p_f^\tau(t,T)}{p_f^\tau(t,T+\tau)} - 1\right]. \tag{11}$$

- In practice, we usually infer $D_\tau(t,T)$ from market quotes of basis swaps.

Pricing Fixed-for-Floating LIBOR Swap with Multicurve

- Fixed-for-floating LIBOR swap pricing formula is

$$\mathrm{PV}_t = \sum_{i=1}^{n} p_d(t, T_i)\, \mathbb{E}_t^{T_i^d} \left[L(T_{i-1}, T_i) - K\right] \tau_i, \quad T_i - T_{i-1} = \tau.$$

- Note that risky forward LIBOR rate $\tilde{F}(t, T_{i-1}, T_i)$ is a martingale under "forwarding" measure corresponding to numéraire $p_f^{\tau}(t, T + \tau)$ so generally

$$\mathbb{E}_t^{T_i^d} L(T_{i-1}, T_i) \neq \tilde{F}(t, T_{i-1}, T_i)$$

- The market consensus is to use approximation

$$\mathrm{PV}_t \approx \sum_{i=1}^{n} p_d(t, T_i) \left[\tilde{F}(t, T_{i-1}, T_i) - K\right] \tau_i. \tag{12}$$

Curve Construction Procedure

- Construct single risk-free discounting curve $p_d(t,T), T \geq t$ from market quotes of OIS instruments.

- Construct multiple (one curve per each tenor τ) forwarding curves $p_f^{\tau}(t,T), T \geq t$ from market quotes of basis swaps (in some cases fixed-for-floating swaps can be used).

- We build the curves one by one. For example for USD market
  - Build the SOFR curve first.
  - Use SOFR curve together with quotes of either 3M LIBOR plain swaps or SOFR vs 3M LIBOR basis swaps to construct 3M LIBOR curve.
  - Given SOFR and 3M LIBOR curves, use 3M vs 6M LIBOR basis swaps to construct the 6M LIBOR curve.

- Given curves built, get discount factors from the discounting curve and retrieve forward rates (11) from the forwarding curve of the respective tenor for pricing derivatives.

SOFR Rates Curves Set

Index	CURVENAME	QUOTE
OIS_USD_1W_SOFR	USD_SOFR	0.0566
OIS_USD_2W_SOFR	USD_SOFR	0.0547
OIS_USD_3W_SOFR	USD_SOFR	0.054
OIS_USD_1M_SOFR	USD_SOFR	0.0536
OIS_USD_2M_SOFR	USD_SOFR	0.0567
OIS_USD_3M_SOFR	USD_SOFR	0.0592
OIS_USD_4M_SOFR	USD_SOFR	0.079
OIS_USD_5M_SOFR	USD_SOFR	0.0984
OIS_USD_6M_SOFR	USD_SOFR	0.1186
OIS_USD_7M_SOFR	USD_SOFR	0.1454
OIS_USD_8M_SOFR	USD_SOFR	0.1734
OIS_USD_9M_SOFR	USD_SOFR	0.1976
OIS_USD_10M_SOFR	USD_SOFR	0.2243
OIS_USD_11M_SOFR	USD_SOFR	0.2523
OIS_USD_1Y_SOFR	USD_SOFR	0.2793

Index	CURVENAME	QUOTE
OIS_USD_18M_SOFR	USD_SOFR	0.4702
OIS_USD_2Y_SOFR	USD_SOFR	0.6604
OIS_USD_3Y_SOFR	USD_SOFR	0.9023
OIS_USD_4Y_SOFR	USD_SOFR	1.02
OIS_USD_5Y_SOFR	USD_SOFR	1.08
OIS_USD_6Y_SOFR	USD_SOFR	1.132
OIS_USD_7Y_SOFR	USD_SOFR	1.174
OIS_USD_8Y_SOFR	USD_SOFR	1.208
OIS_USD_9Y_SOFR	USD_SOFR	1.235
OIS_USD_10Y_SOFR	USD_SOFR	1.262
OIS_USD_12Y_SOFR	USD_SOFR	1.316
OIS_USD_15Y_SOFR	USD_SOFR	1.37
OIS_USD_20Y_SOFR	USD_SOFR	1.417
OIS_USD_25Y_SOFR	USD_SOFR	1.409
OIS_USD_30Y_SOFR	USD_SOFR	1.39

Figure: Market quotes of SOFR instruments.

SOFR Curve

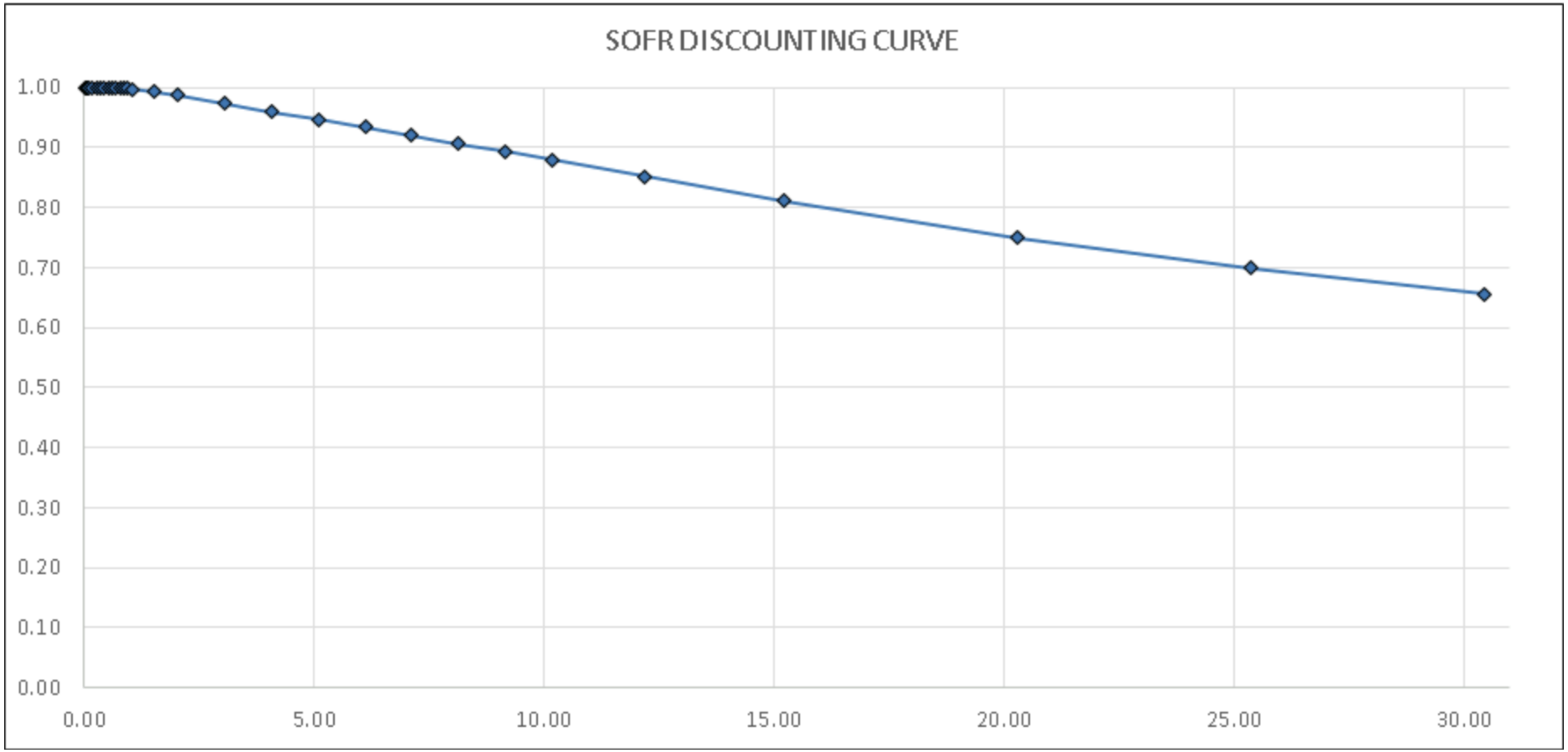

Figure: SOFR discounting curve.

Multiple yield curves

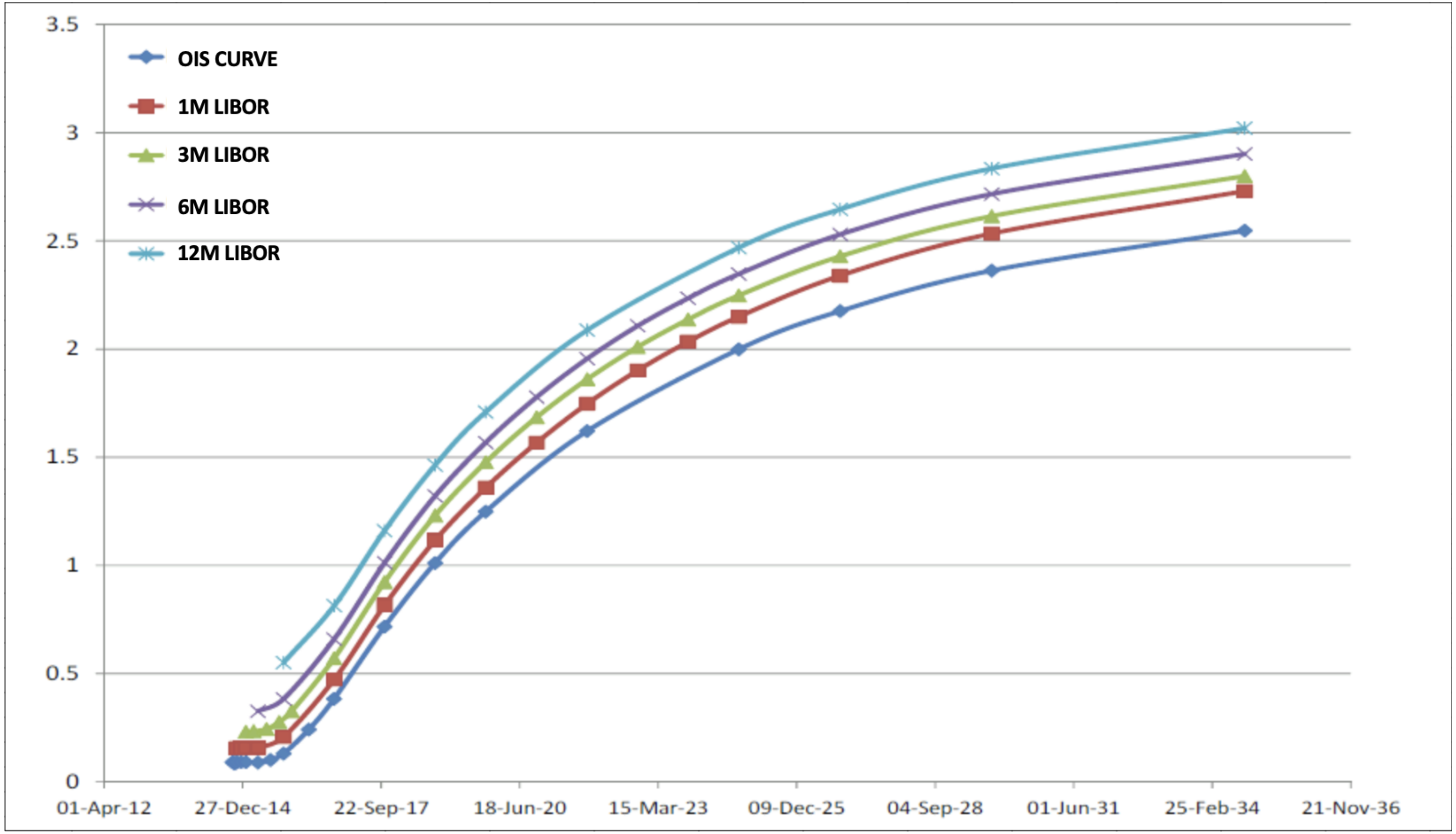

Figure: Multiple yield curves.