Applied Mathematical Finance I

Lecture 7: Heath-Jarrow-Morton Framework and LIBOR Market Model

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Short-Rate Models Revisited

- Last time we saw that in the Markovian setup where short-rate dynamics under risk-neutral measure $\mathbb{Q}$ are given by

$$dr_t = \mu(t, r_t)\, dt + \sigma(t, r_t)\, dW_t^{\mathbb{Q}},$$

  T-bond price $p(t, T) = P(t, r_t, T)$ solves the term structure equation

$$\frac{\partial P}{\partial t} + \mu \frac{\partial P}{\partial r} + \frac{1}{2}\sigma^2 \frac{\partial^2 P}{\partial r^2} - rP = 0, \quad P(T, r, T) = 1.$$

- Choosing μ and σ^2 to be linear functions of r

$$\mu(t, r) = a(t)r + b(t), \quad \sigma^2(t, r) = c(t)r + d(t)$$

  yields affine term structure models where T-bond price is given by

$$p(t, T) = \mathbb{E}_t^{\mathbb{Q}}\left[\mathrm{e}^{-\int_t^T r_s\, ds}\right] = \mathrm{e}^{A(t,T) - B(t,T) r_t} \tag{1}$$

  for some deterministic functions A and B.

Pros and cons of short-rate models

- The main advantage of short-rate models is that they are simple and tractable. Affine term structure models such as Vasiček and CIR give closed-form formulae for prices of zero-coupon bonds and option on bonds which is a good starting point for pricing more complicated derivatives.
- However, it may seem unreasonable to assume that the entire family of zero-coupon bond prices, i.e. the zero curve, is governed by the only stochastic factor given by one-dimensional Wiener process $W^{\mathbb{Q}}$.
- Also, we generally cannot properly calibrate such models to observable market prices due to a small number of model parameters.
- Last but not least: the instantaneous short-rate r is actually unobservable.

Instantaneous Forward Rates

- Recall that instantaneous forward rate $f(t,T)$ is defined as

$$f(t,T) = \lim_{\tau \to 0} F(t,T,T+\tau) = -\frac{\partial \ln p(t,T)}{\partial T}, \tag{2}$$

  meaning that $f(t,T)$ is a forward rate for infinitesimal future period $[T, T+dT]$ as seen at t. Note that $r_t = f(t,t)$.
- Given that $p(t,t) = 1$, formula (2) implies that

$$p(t,T) = \mathrm{e}^{-\int_t^T f(t,u)\,du}. \tag{3}$$

- For short-rate models yielding affine term structure (1), we get

$$f(t,T) = \frac{\partial B(t,T)}{\partial T} r_t - \frac{\partial A(t,T)}{\partial T}.$$

Hull-White Extension of Vasiček Model

- One way to remedy the low parametric nature of Vasiček and CIR models is to make coefficients time-dependent.
- Let us denote by $\{p^\star(0,T)\}_{T\geq 0}$ the term structure of zero-coupon bond prices observed on the market at $t=0$. The corresponding instantaneous forward curve is $f^\star(0,T), T\geq 0$.
- We first consider an auxiliary Vasiček model with $\theta = 0$

$$d\tilde{r}_t = -\kappa \tilde{r}_t \, dt + \sigma \, dW_t^{\mathbb{Q}}, \quad \tilde{r}_0 = f^\star(0,0). \tag{4}$$

- We then introduce a deterministic function $g(t)$ and build time-inhomogeneous short-rate model for $r_t = \tilde{r}_t + g(t)$ so that it fits the forward curve $f^\star(0,T), T\geq 0$.

Hull-White Extension of Vasiček Model (continued)

- Note that bond prices are given by

$$p(t,T) = \mathbb{E}_t^{\mathbb{Q}}\left[\mathrm{e}^{-\int_t^T (\tilde{r}_s + g(s))\,ds}\right] = \mathrm{e}^{-\int_t^T g(s)\,ds} \cdot \mathrm{e}^{A(t,T)-B(t,T)\tilde{r}_t},$$

  where A and B are coefficients in affine term structure representation of T-bond prices in auxiliary model (4).

- The model implied instantaneous forward curve at $t = 0$ is then given by

$$f(0,T) = g(T) - \frac{\partial A(0,T)}{\partial T} + \frac{\partial B(0,T)}{\partial T}\tilde{r}_0, \quad T \geq 0.$$

- By choosing

$$g(T) = f^\star(0,T) + \frac{\partial A(0,T)}{\partial T} - \frac{\partial B(0,T)}{\partial T}\tilde{r}_0, \quad T \geq 0,$$

  we obtain a perfect fit $f(0,T) = f^\star(0,T)$, $T \geq 0$.

Hull-White Extension of Vasiček Model (continued)

- Dynamics of the short rate $r_t = \tilde{r}_t + g(t)$ are then given by

$$
\begin{aligned}
dr_t &= \left(\frac{\partial g}{\partial t} - \kappa \tilde{r}_t\right) dt + \sigma \, dW_t^{\mathbb{Q}} \\
&= \kappa \left(\theta(t) - r_t\right) dt + \sigma \, dW_t^{\mathbb{Q}},
\end{aligned}
$$

  where $\theta(t) = g(t) + \frac{1}{\kappa}\frac{\partial g(t)}{\partial t}$ is time-dependent mean-reversion level.

- The same technique can be applied to CIR and other short-rate models.

- However, the model still suffers from the fact that dynamics of the entire term structure are driven by the only stochastic factor and hence the model cannot produce all possible fluctuations of the zero curve.

The Heath-Jarrow-Morton (HJM) Framework

- Instead of modelling a single state variable r_t we can consider modelling the entire (instantaneous) forward curve as infinite-dimensional state variable.
- Let us now assume that dynamics of $f(t,T)$ under the objective probability measure $\mathbb{P}$ are given by

$$df(t,T) = \mu(t,T)\,dt + \sigma(t,T)\,dW_t^{\mathbb{P}}, \quad f(0,T) = f^{\star}(0,T), \tag{5}$$

where

  - $W^{\mathbb{P}}$ is a standard k-dimensional Wiener process,
  - $\sigma(t,T)$ is a k-dimensional row vector: $\sigma(t,T) = [\sigma_1(t,T),\ldots,\sigma_k(t,T)]$,
  - For any maturity T, $\mu(\cdot,T)$ and $\sigma(\cdot,T)$ are adapted processes such that

$$\int_0^T |\mu(s,T)|\,ds + \int_0^T |\sigma(s,T)|^2\,ds < \infty, \quad \mathbb{P}\text{-a.s.}$$

T-Bond Price Dynamics Under HJM

- We now want to derive dynamics of the T-bond price under $\mathbb{P}$.
- Let us denote $X_t = \ln p(t,T)$. According to (3) and (5), we have that

$$X_t = -\int_t^T f(0,u)\,du - \int_t^T \int_0^t \mu(s,u)\,ds\,du - \int_t^T \int_0^t \sigma(s,u)\,dW_s^{\mathbb{P}}\,du.$$

- Using both standard and stochastic versions of Fubini theorem and splitting $\int_t^T$ into parts, we get

$$\begin{aligned} X_t = &-\int_0^T f(0,u)\,du - \int_0^t \int_s^T \mu(s,u)\,du\,ds - \int_0^t \int_s^T \sigma(s,u)\,du\,dW_s^{\mathbb{P}} \\ &+ \int_0^t f(0,u)\,du + \int_0^t \int_s^t \mu(s,u)\,du\,ds + \int_0^t \int_s^t \sigma(s,u)\,du\,dW_s^{\mathbb{P}}. \end{aligned} \tag{6}$$

T-Bond Price Dynamics Under HJM (continued)

- Note that we have

$$r_u = f(u,u) = f(0,u) + \int_0^u \mu(s,u)\,ds + \int_0^u \sigma(s,u)\,dW_s^{\mathbb{P}}. \tag{7}$$

- Rewriting $\int_0^t \int_s^t (\cdot)\,du\,ds$ as $\int_0^t \int_0^u (\cdot)\,ds\,du$ we then get

$$X_t = X_0 + \int_0^t r_s\,ds + \int_0^t \eta(s,T)\,ds + \int_0^t \nu(s,T)\,dW_s^{\mathbb{P}},$$

  where $\eta(s,T) = -\int_s^T \mu(s,u)\,du$ and $\nu(s,T) = -\int_s^T \sigma(s,u)\,du$.

- Applying multidimensional Itô's formula to $p(t,T) = \mathrm{e}^{X_t}$, we obtain

$$dp(t,T) = p(t,T)\left[\left(r_t + \eta(t,T) + \frac{1}{2}|\nu(t,T)|^2\right)dt + \nu(t,T)\,dW_t^{\mathbb{P}}\right]. \tag{8}$$

Change of Measure

- Now consider a k-dimensional adapted process λ and define an equivalent probability measure $\mathbb{Q} \sim \mathbb{P}$ via Radon-Nikodym density process Z such that

$$dZ_t = Z_t \lambda_t^\top dW_t^{\mathbb{P}}, \quad Z_0 = 1.$$

- By Girsanov theorem, a process $W^{\mathbb{Q}}$ defined as

$$W_t^{\mathbb{Q}} = W_t^{\mathbb{P}} - \int_0^t \lambda_s \, ds \tag{9}$$

is a $\mathbb{Q}$-Brownian motion.

- Applying Itô's formula to discounted T-bond price process $\frac{p(t,T)}{B_t}$ and using (8) together with (9) we get

$$d\left(\frac{p(t,T)}{B_t}\right) = \frac{p(t,T)}{B_t}\left[\left(\eta(t,T) + \frac{1}{2}|\nu(t,T)|^2 + \nu(t,T)\lambda_t\right) dt + \nu(t,T)\, dW_t^{\mathbb{Q}}\right].$$

HJM Drift Condition

- From the last equation, we see that $\mathbb{Q}$ is a martingale measure if and only if

$$\eta(t,T) + \frac{1}{2}|\nu(t,T)|^2 + \nu(t,T)\lambda_t = 0, \quad \forall T > 0, d\mathbb{P} \otimes dt\text{-a.e.}$$

- Differentiating this equality with respect to maturity time T, we obtain

$$\mu(t,T) = \sigma(t,T) \int_t^T \sigma(t,u)^\intercal \, du - \sigma(t,T)\lambda_t, \quad \forall T > 0, d\mathbb{P} \otimes dt\text{-a.e.} \tag{10}$$

- Inserting (10) in (5) we get the famous HJM drift condition

$$df(t,T) = \sigma(t,T) \left(\int_t^T \sigma(t,u)^\intercal \, du \right) dt + \sigma(t,T) \, dW_t^{\mathbb{Q}}. \tag{11}$$

- The point is that the drift of instantaneous forward rate $f(t,T)$ under martingale measure $\mathbb{Q}$ is fully determined by volatility process $\sigma(t,T)$.

Implied Short-Term Rate Dynamics

- Any particular HJM model implies the dynamics of the short rate by

$$r_t = f(t,t) = f(0,t) + \int_0^t \mu(s,t)\,ds + \int_0^t \sigma(s,t)\,dW_s^{\mathbb{P}}. \tag{12}$$

- Note that short rate r given by (12) is generally not an Itô process. Let us now assume that $\mu(t,T)$, $\sigma(t,T)$, and $f(0,T)$ in (5) are differentiable with respect to T and that the derivatives are bounded. We then have

$$\begin{aligned}\int_0^t \sigma(s,t)\,dW_s^{\mathbb{P}} &= \int_0^t \sigma(s,s)\,dW_s^{\mathbb{P}} + \int_0^t (\sigma(s,t) - \sigma(s,s))\,dW_s^{\mathbb{P}} \\ &= \int_0^t \sigma(s,s)\,dW_s^{\mathbb{P}} + \int_0^t \int_s^t \frac{\partial\sigma(s,u)}{\partial T}\,du\,dW_s^{\mathbb{P}}.\end{aligned}$$

- Applying stochastic Fubini theorem, we get

$$\int_0^t \sigma(s,t)\,dW_s^{\mathbb{P}} = \int_0^t \sigma(s,s)\,dW_s^{\mathbb{P}} + \int_0^t \int_0^u \frac{\partial\sigma(s,u)}{\partial T}\,dW_s^{\mathbb{P}}\,du. \tag{13}$$

Implied Short-Term Rate Dynamics (continued)

- Furthermore,

$$\int_0^t \mu(s,t)\,dW_s^{\mathbb{P}} = \int_0^t \mu(s,s)\,dW_s^{\mathbb{P}} + \int_0^t \int_0^u \frac{\partial \mu(s,u)}{\partial T}\,ds\,du, \tag{14}$$

$$f(0,t) = r_0 + \int_0^t \frac{\partial f(0,u)}{\partial T}\,du. \tag{15}$$

Combining formulae (13), (14), and (15), we see that r is an Itô process

$$r_t = r_0 + \int_0^t \alpha_s\,ds + \int_0^t \sigma(s,s)\,dW_s^{\mathbb{P}}, \tag{16}$$

where

$$\alpha_t = \mu(t,t) + \frac{\partial f(0,t)}{\partial T} + \int_0^t \frac{\partial \mu(s,t)}{\partial T}\,ds + \int_0^t \frac{\partial \sigma(s,t)}{\partial T}\,dW_s^{\mathbb{P}}.$$

Example: Hull-White Model

- Let us find out which HJM model, that is a choice of $\sigma(t, T)$, corresponds to Vasiček model with short-rate dynamics given by

$$dr_t = \kappa\,(\theta - r_t)\,dt + \sigma\,dW_t^{\mathbb{Q}}.$$

- Recall that affine term structure of Vasiček model implies that

$$f(t, T) = \frac{\partial B(t, T)}{\partial T} r_t - \frac{\partial A(t, T)}{\partial T}, \quad B(t, T) = \frac{1}{\kappa}\left(1 - \mathrm{e}^{-\kappa(T-t)}\right).$$

- Using Itô's formula, we get

$$df(t, T) = \underbrace{\{\dots\}}_{\text{HJM drift}}\ dt + \sigma \mathrm{e}^{-\kappa(T-t)}\,dW_t^{\mathbb{Q}}.$$

  and hence the corresponding HJM volatility is $\sigma(t, T) = \sigma \mathrm{e}^{-\kappa(T-t)}$.

- Note that since any HJM model exactly fits the initial forward curve $f^\star(0, T)$, the implied short-rate model is actually Hull-White extension of the Vasiček model.

Drawbacks of HJM Framework

- HJM framework is based on the dynamics of instantaneous forward rates which can never be observed in real life.

- Also, note that, in general, model (5) is non-Markov which is undesired in practice. However, if $\sigma(t,T)$ is deterministic (and hence HJM model is Gaussian) and $\sigma(t,T) = g(t)h(T)$ then implied short-rate model will possess Markov property.

- For pricing (exotic) LIBOR-linked derivatives, instead of modelling instantaneous forward rates, we would like to directly model a joint evolution of LIBOR forward rates for a given set of future tenors.

Autocaps

- So far we only considered vanilla and near-vanilla options. Autocap is an example of exotic path-dependent product.
- Autocap is a structured cap comprising a series of contingent caplets which terminates if the number of options expired in the money reaches a pre-agreed threshold number N.
- Consider a tenor structure $T_0 < T_1 < \cdots < T_n$. Mathematically, the payoff at T_i, $i = 1, \ldots, n$, can be expressed as

$$\tau_i \left(L(T_{i-1}, T_i) - K\right)_+ \mathbb{1}_{\{N(T_i) \leq N\}},$$

  where $\tau_i = T_i - T_{i-1}$ and $N(T_i)$ is the total number of Libor rates fixed above K prior to T_i.
- Autocap payoff depends on the joint evolution of LIBOR rates and hence a proper modelling must capture possible movements of the zero curve.

LIBOR Market Model

- Let us now consider log-normal LIBOR Market Model (a.k.a. as Brace-Gatarek-Musiela Model or simply LMM).

- Given that $F_t^i = F(t, T_{i-1}, T_i)$ is a martingale under T_i-forward measure, we postulate that dynamics of each F_t^i under native measure $\mathbb{Q}^{T_i}$ are given by

$$dF_t^i = F_t^i \sigma^i \, dW^{T_i},$$

  where for simplicity we assume that σ^i are positive constants and hence distribution of each F^i is log-normal.

- To price exotic derivatives, such as autocaps, one needs to obtain the dynamic of a whole family of forward rates under a common probability measure.

Linking forward measures together

- Let us find the dynamics of F_t^i under $\mathbb{Q}^{T_j}$, where $j > i$. The change of measure from $\mathbb{Q}^{T_i}$ to $\mathbb{Q}^{T_j}$ in terms of Radon-Nikodym density is given by

$$\left.\frac{d\mathbb{Q}^{T_j}}{d\mathbb{Q}^{T_i}}\right|_{\mathcal{F}_t} = \frac{p(t,T_j)}{p(0,T_j)} \cdot \frac{p(0,T_i)}{p(t,T_i)}.$$

- Noting that

$$\frac{p(t,T_i)}{p(t,T_j)} = \frac{p(t,T_i)}{p(t,T_{i+1})} \cdot \ldots \cdot \frac{p(t,T_{j-1})}{p(t,T_j)} = \prod_{k=i+1}^{j} (1 + \tau_k \cdot F_t^k), \tag{17}$$

we get the following expression for the Radon-Nikodym derivative

$$\left.\frac{d\mathbb{Q}^{T_j}}{d\mathbb{Q}^{T_i}}\right|_{\mathcal{F}_t} = \frac{p(0,T_i)}{p(0,T_j)} \cdot \prod_{k=i+1}^{j} \frac{1}{(1 + \tau_k \cdot F_t^k)}. \tag{18}$$

Another Look At Drift Change

- Consider a process with $\mathbb{P}$-dynamics given by

$$dS_t = S_t \left[\mu\, dt + \sigma\, dW_t^{\mathbb{P}}\right].$$

- Applying Girsanov theorem with kernel λ_t, we can switch to an equivalent measure $\mathbb{Q}$ where dynamic become

$$dS_t = S_t \left[(\mu + \sigma\lambda_t)\, dt + \sigma\, dW_t^{\mathbb{Q}}\right]$$

- Note that the drift correction $\sigma\lambda_t dt$ can actually be represented as $d\langle \ln S, \ln Z\rangle_t$, where

$$Z_t = \exp\left(-\frac{1}{2}\int_0^t \lambda_s^2\, ds + \int_0^t \lambda_s\, dW_s^{\mathbb{P}}\right).$$

Computing the LMM drifts

- Therefore, in order to compute the drift of F_t^i under the T_j-forward measure, we need to calculate $d\langle \ln F_t^i, \ln Z_t\rangle$, with Z_t given by (18). We have

$$
\begin{aligned}
d\langle \ln F_t^i, \ln Z_t\rangle &= \langle d\ln F_t^i, d\ln Z_t\rangle \\
&= \left\langle \frac{dF_t^i}{F_t^i}, -\sum_{k=i+1}^{j} d\ln\left(1+\tau_k\cdot F_t^k\right)\right\rangle \\
&= -\sigma^i\cdot\sum_{k=i+1}^{j}\left\langle dW_t^{T_i}, \frac{\tau_k\cdot F_t^k\cdot\sigma^k\cdot dW_t^{T_k}}{1+\tau_k\cdot F_t^k}\right\rangle \\
&= -\sigma^i\cdot\sum_{k=i+1}^{j}\frac{\tau_k\cdot F_t^k\cdot\sigma^k\cdot\rho_{i,k}}{1+\tau_k\cdot F_t^k}\cdot dt,
\end{aligned}
$$

where $\rho_{i,k}$ is the correlation between W^{T_i} and W^{T_k}.

Pros and cons of LMM

- Summarizing, we get the following dynamics of F_t^i under $\mathbb{Q}^{T_j}$, $j > i$

$$\frac{dF_t^i}{F_t^i} = -\sigma^i \cdot \sum_{k=i+1}^{j} \frac{\tau_k \cdot F_t^k \cdot \sigma^k \cdot \rho_{i,k}}{1 + \tau_k \cdot F_t^k} dt + \sigma^i \, dW^{T_j}.$$

- LMM is almost an ultimate model allowing to properly price highly exotic derivatives such as autocaps, range accruals, etc.

- This is because it models evolution of the entire forward curve and can capture not only changes in curve's level, but also changes in slope and curvature.

- The price we pay for using this model is that we lose simplicity and low-dimensionality. For instance, a 5Y autocap with quarterly payments leads to a 20-dimensional LMM.