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Stochastic models of prediction games

Lecture 1

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Plan of the lectures

Lecture 1

- Games for conditional probabilities
- Games for conditional expectations

Lecture 2

- Games for point processes

Games for conditional probabilities

The model

- A probability space $(\Omega, \mathscr{F}, \mathrm{P})$ with a discrete-time filtration $\mathbb{F} = (\mathscr{F}_t)_{t=0}^{\infty}$.

- An adapted sequence $X = (X_t)_{t=1}^{\infty}$ of random vectors in $\mathbb{R}^N$, which have values in the set $X_t \in \{e_1, \dots, e_N\}$ (indicators of disjoint events).

- M players whose strategies are predictable sequences $h^m = (h_t^m)_{t=1}^{\infty}$, where $h_t^m = (h_t^{m1}, \dots, h_t^{mN}) \in \mathbb{R}_+^N$.

- The wealth of a player is described by an adapted sequence $W^m = (W_t^m)_{t=0}^{\infty}$.

- The strategies satisfy the constraint $\sum\limits_{n=1}^{N} h_t^{mn} \leqslant W_{t-1}^m$.

Wealth dynamics

The wealth sequences W^m are defined by the equation

$$W_t^m = H_t \sum_{n=1}^{N} \rho_t^{mn} X_t^n + h_t^{m0},$$

where

$$H_t = \sum_{k=1}^{M} \sum_{i=1}^{N} h_t^{ki},$$

$$\rho_t^{mn} = \frac{h_t^{mn}}{\sum\limits_{k=1}^{M} h_t^{kn}},$$

$$h_t^{m0} = W_{t-1}^m - \sum_{n=1}^{N} h_t^{mn}.$$

Simplifying assumptions

- There is a sequence $Y = (Y_t)_{t=1}^{\infty}$ with values in a finite set $\{y_1, \ldots, y_L\}$, which represents "available information".

- All the pairs (X_t, Y_t) are i.i.d. and $\mathscr{F}_t = \sigma(X_s, Y_{s+1}; s \leqslant t)$.

- The strategies are defined through proportion functions

$$h_t^{mn} = W_{t-1}^{m} \lambda^{mn}(Y_t),$$

  where $\lambda^{mn}(y) \geqslant 0$ and $\sum\limits_{n=1}^{N} \lambda^{mn}(y) \leqslant 1$.

- Without loss of generality, assume that for all y and n

$$\mathrm{P}(Y_t = y) > 0, \qquad \mathrm{P}(X_t^n = 1 \,|\, Y_t = y) > 0.$$

Asymptotically optimal strategies

Denote

$$p^n(y) = \mathrm{P}(X_t^n = 1 \,|\, Y_t = y).$$

Theorem. Suppose player 1 uses the strategy $\lambda(y) = p(y)$ and player m uses a strategy $\widetilde{\lambda}(y)$ such that for some n, y it holds that $\widetilde{\lambda}^n(y) > 0$ and

$$\frac{\widetilde{\lambda}^n(y)}{\sum\limits_{i=1}^{N} \widetilde{\lambda}^i(y)} \neq p^n(y).$$

Then with probability 1 we have

$$\lim_{t\to\infty} \frac{W_t^m}{W_t^1} = 0.$$

Proof of the theorem

We give a proof in the case when $\sum_{n=1}^{N} \lambda^{mn}(y) = 1$ for all y. The general case will be considered later.

- Assume $\lambda^{mn}(y) > 0$ for all y (otherwise W_t^m becomes 0 in finite time).
- Let
$$D_t = \ln \frac{W_t^1}{W_t^m} - \ln \frac{W_{t-1}^1}{W_{t-1}^m}.$$
- Rewrite the wealth dynamics equation:
$$\frac{W_t^m}{W_{t-1}^m} = \sum_{n=1}^{N} \frac{\lambda^{mn}(Y_t)}{\sum_{k=1}^{M} h_t^{kn}} X_t^n.$$

- We can change the order of the summation, division and logarithm:

$$D_t = \ln \frac{\displaystyle\sum_{n=1}^{N} \frac{\lambda^n(Y_t)}{\sum_{k=1}^{M} h_t^{kn}} X_t^n}{\displaystyle\sum_{n=1}^{N} \frac{\widetilde{\lambda}^n(Y_t)}{\sum_{k=1}^{M} h_t^{kn}} X_t^n} = \sum_{n=1}^{N} X_t^n \ln \frac{\lambda^n(Y_t)}{\widetilde{\lambda}^n(Y_t)}.$$

- Consequently, D_t are i.i.d. and we have

$$\mathrm{E}\, D_t = \mathrm{E}(\mathrm{E}(D_t \,|\, Y_t)) = \mathrm{E}\left(\sum_{n=1}^{N} p^n(Y_t) \ln \frac{p^n(Y_t)}{\lambda^{mn}(Y_t)} \right) > 0,$$

where the inequality follows from Gibbs' inequality.

- Using the SLLN, we find

$$\lim_{t\to\infty} \frac{1}{t} \ln \frac{W_t^1}{W_t^m} = \lim_{t\to\infty} \frac{1}{t} \left(\frac{W_0^1}{W_0^m} + \sum_{s=1}^{t} D_t \right) = \mathrm{E}\, D_t.$$

- This implies $\ln \dfrac{W_t^1}{W_t^m} \to +\infty$ and $\dfrac{W_t^m}{W_t^1} \to 0$.

Convergence of predictions

Define

$$\pi_t^n = \frac{\sum_{m=1}^{M} h_t^{mn}}{\sum_{m=1}^{M} \sum_{i=1}^{N} h_t^{mi}}.$$

Corollary. If at least one player uses the strategy $\lambda(y) = p(y)$, then for each n we have

$$\lim_{t\to\infty} (\pi_t^n - p^n(Y_t)) = 0 \text{ a.s.}$$

Proof

Again, for simplicity assume $\sum_{n=1}^{N} \lambda^{mn}(y) = 1$ for all m and y.

- Without loss of generality, we can assume that only player 1 uses the strategy $\lambda(y) = p(y)$ and other players use different strategies.

- As follows from the theorem, $W_t^m \to 0$ exponentially for $m \neq 1$. Therefore,

$$\sum_{m=1}^{M} h_t^{mn} \sim h_t^{1n} = p_t^n(Y_t) W_t^1, \qquad \sum_{m=1}^{M} \sum_{n=1}^{N} h_t^{mn} \sim \sum_{n=1}^{N} h_t^{1n} = W_t^1,$$

  where $\xi_t \sim \nu_t$ means $\lim\limits_{t\to\infty} (\xi_t/\nu_t) = 1$.

- Hence $\pi_t^n \sim p^n(Y_t)$. Since π_t^n are bounded, we have $\pi_t^n - p^n(Y_t) \to 0$.

Extension: a case when no player uses the optimal strategy

Theorem. Assume that $\sum_{n=1}^{N} \lambda^{mn}(y) = 1$ for all m, y, and the strategy of player 1 satisfies the condition $\lambda^{1n}(y) > 0$ for all n, y. Suppose for some m it holds that

$$\mathrm{E}\left(\sum_{n=1}^{N} p^n(Y_t) \ln \frac{\lambda^{1n}(Y_t)}{\lambda^{mn}(Y_t)} \right) > 0, \qquad (*)$$

where $\ln(\lambda^{1n}(Y_t)/0) = +\infty$. Then we have

$$\lim_{t\to\infty} \frac{W_t^m}{W_t^1} = 0 \text{ a.s.}$$

Remark. Inequality $(*)$ means that

$$\mathrm{E}(D_{\mathsf{KL}}(p(Y_t) \,\|\, \lambda^1(Y_t))) < \mathrm{E}(D_{\mathsf{KL}}(p(Y_t) \,\|\, \lambda^m(Y_t))),$$

where $D_{\mathsf{KL}}(p \,\|\, q) = \sum_{n=1}^{N} p^n \ln(p^n/q^n)$ if the Kullback–Leibler distance.

Games for conditional expectations

The model

Definition. A prediction game consists of

- $(\Omega, \mathscr{F}, \mathbb{F}, \mathrm{P})$: a filtered probability space,
- $X = (X_t)_{t=1}^{\infty}$: an adapted sequence of random vectors $X_t = (X_t^1, \dots, X_t^N)$ with values in the set $\Delta^N = \{x \in \mathbb{R}_+^N : \sum_{n=1}^N x^n = 1\}$,
- $c = (c_t)_{t=1}^{\infty}$: a non-negative non-random sequence.

Assumption. We will always assume that for all t and n

$$\mathrm{P}(X_t^n > 0 \,|\, \mathscr{F}_{t-1}) > 0.$$

Definition. A strategy of a player is a predictable sequence $\lambda = (\lambda_t)_{t=1}^{\infty}$ of random vectors $\lambda_t = (\lambda_t^1, \dots, \lambda_t^N)$ in $\overline{\Delta}^N = \{x \in \mathbb{R}_+^N : \sum_{n=1}^N x^n \leqslant 1\}$.

For brevity, we will use the notation $\lambda_t^0 = 1 - \sum_{n=1}^N \lambda_t^n$.

Definition. For a given strategy profile $\Lambda = (\lambda^1, \dots, \lambda^M)$ and a vector of initial wealth $W_0 \in \mathbb{R}_+^M$, the wealth of the players at time $t \geqslant 1$ is defined by the relation

$$W_t^m = \left(\sum_{k=1}^M \sum_{n=1}^N \lambda_t^{kn} W_{t-1}^k + c_t \right) \sum_{n=1}^N \frac{\lambda_t^{mn} W_{t-1}^m}{\sum_{k=1}^M \lambda_t^{kn} W_{t-1}^k} X_t^n + \lambda_t^{m0} W_{t-1}^m.$$

Proposition. Suppose there exists a player m with a strategy λ^m such that $\lambda_t^{mn} > 0$ for all t and n, and the initial wealth $W_0^m > 0$.

Then $W_t^m > 0$ and $\sum_{k=1}^{M} \lambda_t^{kn} W_{t-1}^k > 0$.

From now on, we will always assume the condition of this proposition is satisfied.

Further definitions

- Total wealth:

$$\overline{W}_t = \sum_{m=1}^{M} W_t^m.$$

- Relative wealth:

$$R_t^m = \frac{W_t^m}{\overline{W}_t}.$$

- Representative strategy:

$$\bar{\lambda}_t^n = \sum_{m=1}^{M} \lambda_t^{mn} R_t^m.$$

- Vectors of predictions:

$$\pi_t^n = \frac{\bar{\lambda}_t^n}{1 - \bar{\lambda}_t^0}$$

Remark. We have

$$\pi_t^n = \frac{\sum\limits_{m=1}^{M} h_t^{mn}}{\sum\limits_{m=1}^{M} \sum\limits_{i=1}^{N} h_t^{mi}},$$

where $h_t^{mn} = \lambda_t^{mn} W_{t-1}^m$.

Survival strategies

Definition

A strategy λ^m is called survival if in any strategy profile $\Lambda = (\lambda^1, \dots, \lambda^M)$ containing this strategy and for any vector of initial wealth W_0 such that $W_0^m > 0$ it holds that

$$\liminf_{t\to\infty} R_t^m > 0 \text{ a.s.}$$

Main results

Denote by $\mu_t = (\mu_t^1, \dots, \mu_t^N)$ the conditional expectations

$$\mu_t^n = \mathrm{E}(X_t^n \mid \mathscr{F}_{t-1}).$$

Theorem 1. The strategy $\widehat{\lambda}_t = \mu_t$ is survival.

Theorem 2. Suppose some player uses $\widehat{\lambda}$. Then $\lim_{t\to\infty}(\pi_t - \mu_t) = 0$ a.s.

Theorem 3. Suppose that $\inf_{t\geqslant 1}(c_t/\overline{W}_{t-1}) > 0$. Then $\lim_{t\to\infty}(\lambda_t - \mu_t) = 0$ a.s. for any survival strategy λ.

Proof of Theorem 1

- Suppose player m uses the strategy $\widehat{\lambda}$. Let $Z_t = \ln R_t^m$. We are going to show that Z_t is a local submartingale, i.e. $\mathrm{E}(Z_t \mid \mathscr{F}_{t-1}) \geqslant Z_{t-1}$.

- We have

$$
\begin{aligned}
\ln R_t^m - \ln R_{t-1}^m &= \ln \frac{W_t^m / W_{t-1}^m}{\overline{W}_t / \overline{W}_{t-1}} \\
&= \ln \left(\frac{(1 - \bar{\lambda}_t^0)\overline{W}_{t-1} + c_t}{\overline{W}_{t-1} + c_t} \sum_{n=1}^{N} X_t^n \frac{\mu_t^n}{\bar{\lambda}_t^n} \right) \\
&\geqslant \ln \frac{(1 - \bar{\lambda}_t^0)\overline{W}_{t-1} + c_t}{(1 - \bar{\lambda}_t^0)(\overline{W}_{t-1} + c_t)} + \ln \left(\sum_{n=1}^{N} X_t^n \frac{\mu_t^n}{\pi_t^n} \right) \geqslant \sum_{n=1}^{N} X_t^n \ln \frac{\mu_t^n}{\pi_t^n}.
\end{aligned}
$$

- Using Gibbs' inequality, obtain

$$\mathrm{E}(Z_t - Z_{t-1} \,|\, \mathscr{F}_{t-1}) \geqslant \sum_{n=1}^{N} \mu_t^n \ln \frac{\mu_t^n}{\pi_t^n} \geqslant 0.$$

- Therefore, Z_t is a local submartingale, and it is non-positive. Since a non-positive local submartingale has a finite limit, there exists $\lim_{t\to\infty} R_t^m = \exp(\lim_{t\to\infty} Z_t) > 0$.

Proof of Theorem 2

Lemma. Let $x, y \in \Delta^N$ be vectors with strictly positive coordinates. Then

$$\sum_{n=1}^{N} x^n \ln \frac{x^n}{y^n} \geqslant \frac{1}{4} \|x - y\|^2.$$

Using this inequality, we can improve the estimate from the proof of Theorem 1:

$$\mathrm{E}(Z_t - Z_{t-1} \mid \mathscr{F}_{t-1}) \geqslant \frac{1}{4} \|\mu_t - \pi_t\|^2.$$

Consequently, the compensator of Z_t can be bounded from below by

$$A_t \geqslant \frac{1}{4} \sum_{s=1}^{t} \|\mu_t - \pi_t\|^2.$$

Since there exists a finite limit $\lim_{t\to\infty} A_t$, we have $\|\mu_t - \pi_t\| \to 0$.

Proof of Theorem 3

Consider the profile of strategies $\Lambda = (\widehat{\lambda}, \lambda)$, where $\widehat{\lambda}_t = \mu_t$, and λ is another survival strategy.

- As above, we obtain the convergence of the series

$$\sum_{t=1}^{\infty} \ln \frac{(1-\bar{\lambda}_t^0)\overline{W}_{t-1} + c_t}{(1-\bar{\lambda}_t^0)(\overline{W}_{t-1} + c_t)} < \infty.$$

- Since the convergence of $\sum_{t=1}^{\infty} \ln(1+x_t)$, $x_t \geqslant 0$, is equivalent to the convergence of $\sum_{t=1}^{\infty} x_t$, we have

$$\sum_{t=1}^{\infty} \frac{\bar{\lambda}_t^0 c_t}{(1-\bar{\lambda}_t^0)\overline{W}_{t-1} + c_t} < \infty.$$

Because $\overline{W}_t = \overline{W}_{t-1} + c_t$, we find $\sum_{t=0}^{\infty} \bar{\lambda}_t^0 < \infty$.

- In the strategy profile under consideration, we have $\bar{\lambda}_t^0 = R_t^2 \lambda_t^0$, so $\sum_{t=0}^{\infty} \lambda_t^0 < \infty$ because λ survives. Consequently,

$$\lim_{t \to \infty} \lambda_t^0 = 0.$$

- We have

$$\pi_t - \mu_t = \frac{R_t^2(\lambda_t - \mu_t) - \lambda_t^0 \mu_t}{1 - \lambda_t^0}.$$

- From the convergence $\pi_t - \mu_t \to 0$, we see that $\lambda_t - \mu_t \to 0$.

Main results in the extended form (*)

Definition. A strategy λ^m is called survival on a set $A \in \mathscr{F}$, if in any strategy profile containing this strategy and for any vector of initial wealth (with $W_0^m > 0$) it holds that

$$\liminf_{t\to\infty} R_t^m > 0 \text{ a.s. on } A.$$

Theorem 1′. Suppose a strategy λ is strictly positive ($\lambda_t^n > 0$ a.s. for all n, t). Then λ survives on the set

$$\Gamma = \left\{ \omega : \sum_{t=1}^{\infty} \sum_{n=1}^{N} \mu_t^n \ln \frac{\mu_t^n}{\lambda_t^n} < \infty \right\}.$$

If there exists a constant $\varepsilon > 0$ such that $\mu_t^n \geqslant \varepsilon$ for all t, n, then λ survives on the set

$$\Gamma' = \left\{ \omega : \sum_{t=1}^{\infty} \|\mu_t - \lambda_t\|^2 < \infty \right\}.$$

Theorem 2′. Suppose some player uses a strictly positive strategy λ. Then

$$\sum_{t=1}^{\infty} \|\pi_t - \mu_t\|^2 < \infty \text{ a.s. on } \Gamma.$$

In particular, $\lim_{t\to\infty}(\pi_t - \mu_t) = 0$ a.s. on Γ.

Theorem 3′. Assume $\inf_{t\geqslant 1}(c_t/\overline{W}_{t-1}) > 0$ and $\mu_t^n \geqslant \varepsilon > 0$. Then for a strictly positive strategy λ the condition

$$\sum_{t=1}^{\infty} \|\mu_t - \lambda_t\|^2 < \infty \text{ a.s. on a set } A \in \mathscr{F}$$

if necessary and sufficient for survival on A.

Examples

Conditional probabilities

- If at each time $t \geqslant 1$, there are N disjoint events A_t^n such that $\Omega = \bigsqcup_{n=1}^N A_t^n$, we can consider
$$X_t^n = \mathrm{I}(A_t^n).$$
Then $\pi_t^n - \mathrm{P}(A_t^n \mid \mathscr{F}_{t-1}) \to 0$.

- If the events are not disjoint, consider $X_t = (X_t^1, \dots, X_t^N, X_t^{N+1})$, where
$$X_t^n = \frac{1}{N}\mathrm{I}(A_t^n),\ n = 1, \dots, N, \qquad X_t^{N+1} = 1 - \sum_{n=1}^N X_t^n.$$
Then $N\pi_t^n - \mathrm{P}(A_t^n \mid \mathscr{F}_{t-1}) \to 0$.

Conditional moments

Consider a random sequence $Y = (Y_t)_{t=1}^{\infty}$ with values in $[0, 1]$.

Let $X_t = (X_t^{(1)}, \ldots, X_t^{(N)}, X_t^{(N+1)})$, where

$$X_t^{(n)} = \frac{Y_t^n}{N} \text{ for } n = 1, \ldots, N, \qquad X_t^{(N+1)} = 1 - \frac{1}{N}\sum_{n=1}^{M} Y_t^n.$$

Then $N\pi_t^n - \mathrm{E}(Y_t^n \,|\, \mathscr{F}_t) \to 0$ for $n = 1, \ldots, N$.

For example, for $N = 2$ we have

$$\mathrm{E}(Y_t \,|\, \mathscr{F}_{t-1}) \approx 2\pi_t^1, \qquad \mathrm{Var}(Y_t \,|\, \mathscr{F}_{t-1}) \approx 2\pi_t^2 - 4(\pi_t^1)^2.$$

Appendix: some results from the theory of martingales

Assume given a filtered probability space $(\Omega, \mathscr{F}, \mathbb{F}, \mathrm{P})$.

Definition. An adapted sequence $M = (M_t)_{t=0}^{\infty}$ is called a martingale if:

1. $\mathrm{E}\,|M_t| < \infty$ for all $t \geqslant 0$,
2. $\mathrm{E}(M_t \,|\, \mathscr{F}_{t-1}) = M_{t-1}$ for all $t \geqslant 1$.

If in the second property we replace "$=$" with "$\leqslant$" or "$\geqslant$", then we get the definition of a supermartingale or a submartingale, respectively.

Proposition (Doob decomposition theorem). Any adapted sequence $X = (X_t)_{t=0}^{\infty}$, $\mathrm{E}\,|X_t| < \infty$, can be represented in the form $X_t = X_0 + M_t + A_t$, where M is a martingale, A is a predictable sequence (called the compensator of X), $\mathrm{E}\,|A_t| < \infty$ and $A_0 = M_0 = 0$.

The sequences M and A are a.s.-unique and are given by

$$A_t = \sum_{s=1}^{t} (\mathrm{E}(X_s \,|\, \mathscr{F}_{s-1}) - X_{s-1}), \qquad M_t = \sum_{s=1}^{t} (X_s - \mathrm{E}(X_s \,|\, \mathscr{F}_{s-1})).$$

Corollary. If X is a submartingale, then A is non-decreasing. If X is a supermartingale, then A is non-increasing.

Proposition (Doob convergence theorem). If a submartingale X is bounded from above by an integrable random variable ($X_t \leqslant \xi$ for all $t \geqslant 0$, where $\mathrm{E}\,|\xi| < \infty$), then with probability 1 there exists a finite limit

$$X_\infty = \lim_{t\to\infty} X_t.$$

Definition. A stopping time is a random variable τ with values in $\mathbb{N} \cup \{0\}$ such that $\{\omega : \tau(\omega) = t\} \in \mathscr{F}_t$ for any $t = 0, 1, \dots$

Definition. A local martingale is an adapted sequence $M = (M_t)_{t=0}^{\infty}$ such that there exists a sequence of stopping times τ_k, $k \in \mathbb{N}$, with the following properties:

1. $\tau_{k+1} \geqslant \tau_k$ a.s. for all k,
2. $\lim\limits_{k \to \infty} \tau_k = \infty$ a.s.,
3. $M^{\tau_k} = (M_t^{\tau_k})_{t=0}^{\infty}$, where $M_t^{\tau_k} = M_{\min(t, \tau_k)}$, is a martingale for any k.

In the same way, we define local submartingales and local supermartingales.

Proposition. Let X be an adapted sequence and $\mathrm{E}\,|X_0| < \infty$. Then the following conditions are equivalent.

(a) X is a local submartingale.

(b) Doob decomposition holds true: $X_t = X_0 + M_t + A_t$, where M is a local martingale, A is a non-decreasing predictable sequence, and $M_0 = A_0 = 0$.

(c) $\mathrm{E}(X_t^+ \mid \mathscr{F}_{t-1}) < \infty$ and $\mathrm{E}(X_t \mid \mathscr{F}_{t-1}) \geqslant X_{t-1}$ for all $t \geqslant 1$.

Proposition. Suppose a local submartingale X has a compensator A and is bounded from above by an integrable random variable. Then there exist finite limits X_∞ and A_∞ with probability 1.

Literature

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