I. V. Evstigneev declares that he has no conflict of interest.

A. A. Tokaeva declares that she has no conflict of interest.

M. J. Vanaei declares that he has no conflict of interest.

M. V. Zhitlukhin declares that he has no conflict of interest.

Survival strategies in an evolutionary finance model with endogenous asset payoffs

I. V. Evstigneev* A. A. Tokaeva†‡ M. J. Vanaei§ M. V. Zhitlukhin¶

Abstract

Evolutionary Finance explores financial markets as evolving biological systems. Investors pursuing diverse investment strategies compete for the market capital. Some "survive" and some "become extinct". A central goal is to identify strategies guaranteeing survival in the market selection process. The problem is analyzed in frameworks combining stochastic dynamic games and evolutionary game theory. Most of the models currently considered in the field assume that asset payoffs are exogenous and depend only on the underlying stochastic process of states of the world. The present work examines a model where the payoffs are endogenous: they depend on the share of total market wealth invested in the asset.

Keywords: evolutionary finance, behavioral finance, stochastic dynamic games, DSGE, capital growth, survival portfolio rules, martingales.

1. Introduction

Evolutionary Finance (EF) is a rapidly developing research area at the interface of Financial Economics and Mathematical Finance applying the evolutionary approach to the modeling of stochastic dynamics of financial markets. The classical theory (Radner, 1972, 1982) relies upon the hypothesis of full rationality of market players, who are assumed to maximize their utilities subject to budget constraints, i.e. solve well-defined and precisely stated constrained optimization problems. EF models abandon this hypothesis and permit market players to have patterns of investment behavior determined by their individual psychology, not necessarily describable in terms of utility maximization.

*Economics Department, University of Manchester, Oxford Road, Manchester, M13 9PL UK. E-mail: igor.evstigneev@manchester.ac.uk

†Faculty of Mechanics and Mathematics, Lomonosov Moscow State University, 1 Leninskie gory, Moscow, 119991 Russia. Email: galynka@ymail.com

‡Science Development Foundation "Vega Institute of Mathematical Finance and Actuarial Studies", 1 Leninskie gory, Moscow, 119991 Russia.

§Economics Department, University of Manchester, Oxford Road, Manchester, M13 9PL UK. E-mail: mohammadjavad.vanaei@postgrad.manchester.ac.uk

¶Steklov Mathematical Institute of Russian Academy of Sciences, 8 Gubkina st., Moscow, 119991 Russia. Corresponding author. E-mail: mikhailzh@mi-ras.ru

In EF models asset prices are determined endogenously via a short-run equilibrium of supply and demand. Dynamic equilibrium is formed consecutively in each time period in the course of interaction of investment strategies of competing market participants. It is defined directly via the set of strategies of the market players describing their investment behavior. An important feature of EF models is that they employ only objectively observable market data and do not use hidden individual agents' characteristics, such as their utilities and beliefs, which makes them amenable for quantitative practical applications (Schnetzer and Hens, 2022).

The main focus of studies in the field is on questions of survival and extinction of investment strategies in the market selection process. A central goal is to identify those strategies which survive in this process and/or are evolutionary stable (immune to extinction) in one sense or another. Typical results show that under very general assumptions, such strategies exist, are asymptotically unique and in many cases easily computable.

Important contributions to the formation of modern EF as a research area were made in Anderson et al. (1988); Arthur et al. (1997); Blume and Easley (1992); Bottazzi and Dindo (2014); Bottazzi et al. (2018, 2005); Brock et al. (2005); Coury and Sciubba (2012); Farmer (2002); Farmer and Lo (1999); Lo (2004, 2005, 2012, 2017); Lo et al. (2018); Sciubba (2005); Zhang et al. (2014). A recent general survey on EF is provided by Holtfort (2019). The modern state of the art in EF is reflected in papers collected in the special issue of the Proceedings of the National Academy of Sciences of the USA "Evolutionary Models of Financial Markets"[1]. The Editors' introduction to the special issue contains an insightful discussion of conceptual links between EF and Evolutionary Biology. An elementary textbook treatment of EF can be found in Evstigneev et al. Evstigneev et al. (2015), Ch. 20.

The model studied in this paper pertains to the family of EF models that has its roots in the papers by Amir et al. (2011, 2013), which initiated a game-theoretic strand in the EF literature. The former paper deals with long-lived dividend-paying assets, while the latter considers short-lived assets. A survey describing the state of the art in this line of research by 2016 and putting forward a program for further studies was given in Evstigneev et al. (2016). For recent progress in the field see Evstigneev et al. (2020); Hens and Naebi (2022); Schnetzer and Hens (2022); Zhitlukhin (2021, 2022, 2023a,b), and references therein.

In nearly all EF models considered in the literature, asset payoffs or dividends are given exogenously and do not depend on the investment strategies of market players. In reality, however, such a dependence is more of a rule than an exception, see e.g. Li et al. (2009); Lintner (1965); Tobin (1969); Tobin and Brainard (1977). At present one of the key general open problems in EF is to develop game-theoretic models of financial

[1] Levin, S. A., Lo, A. W., eds. (2021). Special Issue "Evolutionary Models of Financial Markets", *Proceedings of the National Academy of Sciences of the USA*, 118 (26).

markets that would take into account the endogenous nature of asset payoffs. The first step in this direction was made in Amir et al. (2021), where a game-theoretic framework for the analysis of markets with long-lived dividend-paying securities was developed. The main result was a construction of an evolutionary stable strategy. The notion of evolutionary stability in EF is understood in the sense of classical evolutionary game theory: Maynard Smith and Price (1973) and Schaffer (1988).

The present paper considers a model with short-lived assets and endogenous random payoffs $A_k(\omega, \mu_k)$ of assets $k = 1, \ldots, K$ depending on the share μ_k of total market wealth invested in asset k. First results regarding models of this kind were obtained in Evstigneev and Vanaei (2022) and Evstigneev et al. (2023). In the former paper it was discovered (rather unexpectedly) that in the linear case, when the function $A_k(\omega, \mu_k)$ is linear in μ_k, the problem of constructing a survival strategy reduces to the classical capital growth theory: Algoet and Cover (1988); Breiman (1961); Hakansson and Ziemba (1995); Kelly (1956); MacLean et al. (2010, 2011); Ziemba (2015) and others. It turned out that one can construct a survival strategy by maximizing the expected logarithm of the growth rate – the famous idea of the Kelly portfolio rule (Kelly, 1956). A nonlinear case was considered in Evstigneev et al. (2023), where the existence and uniqueness of an evolutionary stable strategy was established.

The present paper unifies the models considered in Evstigneev and Vanaei (2022) and Evstigneev et al. (2023) in the framework with affine asset payoffs $A_k(\omega, \mu_k) = \alpha_k(\omega) + \beta_k(\omega)\mu_k$. There are three main results in this work. First, we prove that in our model a survival strategy exists and construct it in a semi-closed form as a fixed point of some random operator. In contrast to existing literature on survival strategies, where such strategies can be found explicitly, our construction exhibits a new method which can be applied to models where explicit solutions are not readily available. Our second result shows that in any strategy profile the aggregate market strategy of all agents converges to the survival strategy if at least one agent uses it. As a corollary, this implies that the survival strategy is asymptotically unique in the class of all *basic* strategies (those which depend only of the process of states of the world). The above findings extend previously known facts in this area (see, e.g., Amir et al. (2013)) to models with endogenous asset payoffs. The third result demonstrates that in an i.i.d. case a stronger assertion is true: the survival strategy turns out to be constant and in any strategy profile drives out of the market any other constant strategy, thus becoming a single survivor. This generalizes the early result in Evstigneev et al. (2002) obtained for an i.i.d. model with exogenous payoffs.

The paper is organized as follows. In Section 2, we describe the model. Section 3 states the key results. A numerical example is provided in Section 4. Section 5 discusses the relation of our model to other models in the literature. Section 6 contains the proofs of the main results.

2. The model

We consider a discrete-time market where $K \geq 2$ assets are traded among $N \geq 2$ agents. The assets live for one period and are identically reborn at the beginning of each period. The asset prices are determined endogenously through a short-run equilibrium of supply and demand. The supply (the total volume) of each asset is constant and without loss of generality is normalized to 1. The assets yield payoffs which are distributed among the agents at moments of time $t = 1, 2, \ldots$

The market is influenced by random factors modeled in terms of a sequence of random elements $s_1, s_2, \ldots$ with values in a standard measurable space S. The random element s_t is interpreted as the "state of the world" at time t. Recall that a measurable space is called standard if it is isomorphic to the segment $[0, 1]$ or a discrete finite or countable set with the Borel σ-algebra. This assumption will be needed below to ensure the existence of regular conditional distributions.

Agent $i = 1, \ldots, N$ in this market is characterized by his/her trading strategy and non-random wealth $w_0^i > 0$ (initial endowment) with which this agent enters the market at time $t = 0$. The wealth w_t^i at time $t \geq 1$ is determined by the dynamics described below.

At every moment of time $t \geq 0$, each agent chooses investment proportions $\lambda_t^i = (\lambda_{t,1}^i, \ldots, \lambda_{t,K}^i)$, according to which he/she allocates the available budget (wealth w_t^i) for purchasing assets at time t, i.e. the budget $\lambda_{t,k}^i w_t^i$ is allocated by agent i for purchasing asset k. The investment proportions are selected by the agents simultaneously and independently. These proportions may depend on the history of states of the world $s^t := (s_1, \ldots, s_t)$, the game history $\lambda^{t-1} := (\lambda_0, \ldots, \lambda_{t-1})$, where $\lambda_s = (\lambda_s^1, \ldots, \lambda_s^N)$, and the vector of initial endowments $w_0 := (w_0^1, \ldots, w_0^N)$. A *strategy* Λ^i of agent i is defined as a sequence of measurable functions

$$\Lambda_t^i(s^t, w_0, \lambda^{t-1}), \quad t = 0, 1, \ldots,$$

with values in the unit simplex

$$\Delta^K := \{(a_1, \ldots, a_K) \in \mathbb{R}_+^K : a_1 + \ldots + a_K = 1\}.$$

The value of Λ_t^i represents the vector of investment proportions λ_t^i chosen by agent i at time t. These proportions are non-negative, i.e. short sales are not allowed. For $t = 0$, the function $\Lambda_0^i = \Lambda_0^i(w_0)$ does not depend on the history of states of the world and the game history.

A strategy which depends only on the history of states of the worlds but not on the initial wealth or the game history, i.e. $\Lambda_t^i = \Lambda_t^i(s^t)$, will be called *basic*. Basic strategies play a special role in this paper. In particular, the survival strategy which we construct below in Section 3.2 is basic.

Given a vector of initial endowments $w_0 = (w_0^1, \dots, w_0^N)$ and a strategy profile $\Lambda = (\Lambda^1, \dots, \Lambda^N)$, the investment proportions chosen by the agents in this market are defined by the recursive relation

$$\lambda_0^i = \Lambda_0^i(w_0), \qquad \lambda_t^i(s^t) = \Lambda_t^i(s^t, w_0, \lambda^{t-1}(s^{t-1})),\ t \geq 1, \tag{1}$$

where $\lambda^t(s^t) = (\lambda_0, \lambda_1(s^1), \dots, \lambda_t(s^t))$. In what follows, we will omit the argument s^t where it does not lead to ambiguity.

By $p_t = (p_{t,1}, \dots, p_{t,K})$, we will denote the vector of asset prices. The coordinate $p_{t,k}$ stands for the price of one unit of asset k at time t. We will now define the dynamics of agents' wealth $w_t^i = w_t^i(s^t)$ and asset prices $p_t = p_t(s^t)$ for a fixed strategy profile Λ and a vector of initial endowments w_0.

The prices are formed in equilibrium over each time period as follows. The portfolio of agent i at time $t \geq 0$ is specified by a vector $x_t^i = (x_{t,1}^i, \dots, x_{t,K}^i)$, where $x_{t,k}^i$ is the amount (the number of units) of asset k in the portfolio. The scalar product $\langle p_t, x_t^i \rangle = \sum_{k=1}^K p_{t,k} x_{t,k}^i$ expresses the value of agent i's portfolio at time t.

At time $t = 0$, the agents' budgets are given by their (non-random) initial endowments w_0^i. Let $A_{t,k} = A_{t,k}(s^t)$, $k = 1, \dots, K$, denote the asset payoffs at time $t \geq 1$ per one unit of asset. Since we assume the supply of each asset is 1, the quantity $A_{t,k}$ represents the total payoff of asset k. Agent i's budget (wealth) at time $t \geq 1$ is given by

$$w_t^i = \langle A_t, x_{t-1}^i \rangle = \sum_{k=1}^K A_{t,k} x_{t-1,k}^i, \tag{2}$$

i.e. it is constituted of the payoff of the portfolio x_{t-1}^i that was purchased at time $t-1$.

If agent i allocates a fraction $\lambda_{t,k}^i$ of his/her wealth for purchasing asset k at time t, then the number of units of this asset that can be bought is

$$x_{t,k}^i = \frac{\lambda_{t,k}^i w_t^i}{p_{t,k}}. \tag{3}$$

Assume that the market is always in equilibrium: the total asset supply is equal to the total demand (recall that the former is normalized to 1). This implies that for all $t \geq 0$ and $k = 1, \dots, K$ we have

$$1 = \sum_{i=1}^N x_{t,k}^i = \sum_{i=1}^N \frac{\lambda_{t,k}^i w_t^i}{p_{t,k}},$$

and, consequently, the equilibrium (market clearing) asset prices are given by

$$p_{t,k} = \sum_{i=1}^N \lambda_{t,k}^i w_t^i. \tag{4}$$

Formulas (2)–(3) make sense only if the asset prices $p_{t,k}$ defined by (4) are non-zero.

In view of that, we shall say that a strategy profile and agents' initial endowments are *admissible*, if $p_{t,k} > 0$ for all t and k. Further results will be obtained only for admissible strategy profiles and vectors of initial endowments.

Given an admissible strategy profile and a vector of initial endowments we can, by using equations (2)–(4), generate recursively the random path of the system specified by the sequences of variables w_t^i (agents' wealth), $p_t = (p_{t,1}, \ldots, p_{t,K})$ (vectors of equilibrium asset prices) and $x_t^i = (x_{t,1}^i, \ldots, x_{t,K}^i)$ (agents' portfolios). In particular, the sequences w_t^i follow the dynamics

$$w_{t+1}^i = \sum_{k=1}^{K} \frac{\lambda_{t,k}^i w_t^i}{\sum_{j=1}^{N} \lambda_{t,k}^j w_t^j} A_{t+1,k}. \tag{5}$$

We assume that the asset payoffs $A_{t,k}$ are *endogenous* in the sense that they may depend on the agents' strategies. In the remaining part of the paper, we will deal with the following particular form of the payoffs, which we call *affine payoffs*.

Let W_t denote the total market wealth at time t, and $w_{t,k}$ denote the total wealth allocated by all agents for purchasing asset k at time t:

$$W_t = \sum_{i=1}^{N} w_t^i, \qquad w_{t,k} = \sum_{i=1}^{N} \lambda_{t,k}^i w_t^i.$$

Denote by $\mu_{t,k}$ the fraction of market wealth allocated for purchasing asset k:

$$\mu_{t,k} = \frac{w_{t,k}}{W_t}.$$

Note that $\mu_{t,k}$ essentially represent the *weighted strategy* of the agents, with the weights being equal to their market shares w_t^i / W_t (cf. the notion of the market portfolio in Markowitz's mean-variance portfolio analysis).

We will assume that the asset payoffs are affine functions of $\mu_{t,k}$ of the form

$$A_{t+1,k} = (\alpha_{t+1,k} + \beta_{t+1,k}\mu_{t,k}) Z_{t+1}, \tag{6}$$

where $\alpha_{t+1,k} = \alpha_{t+1,k}(s^{t+1})$ and $\beta_{t+1,k} = \beta_{t+1,k}(s^{t+1})$ are non-negative random variables, and $Z_{t+1} = z_{t+1}(s^{t+1}, w_{t,1}, \ldots, w_{t,K})$ with some measurable function z_{t+1}.

Relation (6) means that the asset payoffs $A_{t+1,k}$ at the next moment of time $t+1$ may depend on random factors as well as on the investment proportions λ_t (through the fractions $\mu_{t,k}$) chosen by the agents at time t. Multiplication by Z_{t+1} expresses the idea that the payoffs may scale proportionally to the amount of capital allocated to the assets.

Thus, from relations (2)–(6), one can see that the wealth of an agent evolves according to the following dynamics:

$$w_{t+1}^1 = w_t^i Z_{t+1} \sum_{k=1}^{K} \lambda_{t,k}^i \left(\frac{\alpha_{t+1,k}}{\sum_{j=1}^N \lambda_{t,k}^j w_t^j} + \frac{\beta_{t+1,k}}{\sum_{j=1}^N w_t^j} \right). \tag{7}$$

Observe that the relative wealth (market share) w_t^i/W_t, which will play the key role in what follows, does not depend on the choice of the scaling factors Z_t.

In what follows, to avoid a degenerate model, we will assume that for all $t \geq 1$ and any values of s^t, $w_{t-1,1}, \ldots, w_{t-1,K}$ it holds that

$$\sum_{k=1}^{K} (\alpha_{t,k}(s^t) + \beta_{t,k}(s^t)) > 0, \qquad z_t(s^t, w_{t-1,1}, \ldots, w_{t-1,K}) > 0. \tag{8}$$

In particular, one can see that if the above condition holds, then a sufficient condition for admissibility of a strategy profile is that there is an agent who uses a strictly diversified strategy, i.e. $\lambda_{t,k}^i > 0$ for some i and all t, k.

3. Main results

3.1. Definitions

We will be interested in the behavior of the *relative wealth* or the *market shares* of the agents, which are defined by

$$r_t^i := \frac{w_t^i}{W_t}.$$

The following definitions introduce the two main concepts of the paper. Hereinafter, "a.s." means "almost surely", i.e. holding with probability 1.

Definition 1. We call a strategy Λ^i of agent i *survival*, if for any (admissible) vector of initial endowments w_0 and strategy profile $\Lambda = (\Lambda^1, \ldots, \Lambda^N)$ consisting of the given strategy Λ^i and arbitrary strategies Λ^j of agents $j \neq i$, it holds that $w_t^i > 0$ a.s. for all $t \geq 0$ and

$$\inf_{t \geq 0} r_t^i > 0 \text{ a.s.}$$

According to this definition, a survival strategy allows an agent to keep a.s. a strictly positive, bounded away from zero share of market wealth over an infinite time horizon irrespectively of strategies used by the other agents.

A comment is in order. One might think that the focus on survival substantially restricts the scope of the analysis, since one should care about survival only if "things go wrong". It turns out, however, that the class of survival strategies in most of the evolutionary finance models coincides with the class of *unbeatable strategies* performing in asymptotically not worse (or, even, strictly better) in terms of wealth accumulation than any other strategies competing in the market. Namely, as follows from the above definition, if agent i uses a survival strategy, then for the wealth of any other agent j it holds that $w_t^j \leq C w_t^j$ for all $t \geq 0$, where C is a random variable. This property expresses the fact that the wealth of any agent cannot grow asymptotically faster than the wealth of an agent who uses a survival strategy. Moreover, as we show in Section 3.3, under some additional conditions, a survival strategy actually outperforms asymptotically any other strategy in the market. For further details on the concept of unbeatable strategies, we refer to Section 6 in Amir et al. (2013) and Amir et al. (2023).

In order to find a survival strategy we will look for a *relatively growth-optimal* strategy, the concept of which is introduced in the next definition. To state it, recall that a random sequence ξ_t, which is adapted to the filtration generated by the state process s_t, is called a *submartingale* if $\mathrm{E}|\xi_t| < \infty$ and $\mathrm{E}_t \xi_{t+1} \geq \xi_t$ a.s. for all $t \geq 0$, where $\mathrm{E}_t(\cdot) = \mathrm{E}(\cdot \mid s^t)$ denotes the conditional expectation given $s^t = (s_1, \ldots, s_t)$. For $t = 0$, put $\mathrm{E}_0(\cdot) = \mathrm{E}(\cdot)$.

Definition 2. We call a strategy Λ^i *relatively growth-optimal*, if for any (admissible) vector of initial endowments w_0 and strategy profile $\Lambda = (\Lambda^1, \ldots, \Lambda^N)$, where Λ^i is the given strategy, it holds that $w_t^i > 0$ a.s. for all $t \geq 0$ and

$$\ln r_t^i \text{ is a submartingale.} \tag{9}$$

This concept is similar to the notion of a *growth-optimal* (or *log-optimal*) strategy in the classical capital growth theory for markets with exogenous asset prices, which explains the terminology. Recall that in this theory a strategy is called growth-optimal if no competing strategy can improve the expected logarithmic wealth relative to it (see, e.g., Algoet and Cover (1988)). Note that, however, in our model we focus on the *relative* wealth of an agent. Moreover, the relative wealth of an agent (as well as the absolute wealth) in our model depends also on the strategies of other agents, so the problem of constructing a relative growth-optimal strategy does not simply reduce to an optimization problem.

Proposition 1. *Any relatively growth-optimal strategy is a survival strategy.*

Proof. A non-positive submartingale has a finite limit with probability 1 as $t \to \infty$; see, e.g., Shiryaev (2019, Ch. 7.4). Hence, if Λ^i is a relatively growth-optimal strategy, then $\lim_{t\to\infty} \ln r_t^i$ is finite, which implies $\inf_{t\geq 0} r_t^i > 0$. $\square$

3.2. Construction of a relatively growth-optimal strategy

For $t \geq 1$, define the Δ^K-valued functions $g_t(\lambda, s^t)$, $\lambda \in \Delta^K$, by

$$g_{t,k}(\lambda, s^t) = \alpha_{t,k}(s^t) + \lambda_k \beta_{t,k}(s^t).$$

Let $\mathrm{P}_t(\cdot) = \mathrm{P}(\cdot \mid s^t)$ and $\mathrm{E}_t(\cdot) = \mathrm{E}(\cdot \mid s^t)$ denote the conditional probability and conditional expectation given s^t (where $\mathrm{P}_0(\cdot) = \mathrm{P}(\cdot)$, $\mathrm{E}_0(\cdot) = \mathrm{E}(\cdot)$). Introduce the functions $L_t = L_t(\lambda, s^t)$, $t \geq 0$, with values in Δ^K defined by

$$L_{t,k}(\lambda, s^t) = \mathrm{E}_t \left(\frac{g_{t+1,k}(\lambda, s^{t+1})}{\sum_{j=1}^K g_{t+1,j}(\lambda, s^{t+1})} \right).$$

We will assume that the conditional probabilities $\mathrm{P}_t(\cdot)$ and expectations $\mathrm{E}_t(\cdot)$ are computed with respect to some fixed variant of the regular conditional distribution of s^{t+1}, which implies that the functions $L_{t,k}$ are jointly measurable with respect to their arguments. For $t = 0$, the function $L_0 = L_0(\lambda)$ does not depend on the random state. The existence of the regular conditional distribution of s^{t+1} follows from the assumption that the measurable space S of possible states of the world is standard (for details, see, e.g., Appendix 2 in Arkin and Evstigneev (1987)).

In what follows, to alleviate notation, we will omit the arguments s^t and s^{t+1}. In particular, if $\Lambda_t = \Lambda_t(s^t)$ is a random function, then $g_{t+1}(\Lambda_t)$ and $L_t(\Lambda_t)$ will stand for $g_{t+1}(\Lambda(s^t), s^{t+1})$ and $L_t(\Lambda_t(s^t), s^t)$, respectively.

Proposition 2. *For each $t \geq 0$, there exists a measurable function $\Lambda_t^*(s^t)$ with values in Δ^K which has the following properties:*

(a) it holds that

$$\sum_{j=1}^K g_{t+1,j}(\Lambda_t^*) > 0 \ \textit{a.s.}, \tag{10}$$

$$\mathrm{E}_t \left(\frac{\beta_{t+1,k}}{\sum_{j=1}^K g_{t+1,j}(\Lambda_t^*)} \right) \leq 1 \ \textit{a.s.}, \quad k = 1, \ldots, K, \tag{11}$$

(b) Λ_t^* is a fixed point of L_t, i.e.

$$L_t(\Lambda_t^*) = \Lambda_t^* \ \textit{a.s.}, \tag{12}$$

where for $t = 0$, we assume that Λ_0^* is non-random.

The next theorem is the first main result of the paper.

Theorem 1. *A strategy $\Lambda^* = (\Lambda_t^*)_{t=0}^\infty$ which consists of functions satisfying properties (10)–(12) is relatively growth-optimal, and, consequently, survival.*

Note that such a relatively growth-optimal strategy Λ^* belongs to the class of basic strategies. Also observe that the functions z_t (see (6)) do not affect the form of Λ^*.

The idea of constructing a survival strategy as a fixed point of a random operator first appeared in the paper of Evstigneev et al. (2023). That paper considered a more general model, in which asset payoffs depend in a concave way on the fractions of market wealth allocated to the assets; however only local stability of a certain (constant) strategy was proved. In the case of i.i.d. affine payoffs, the strategy of Evstigneev et al. (2023) coincides with our strategy Λ^*, and the above proposition and theorem extend that construction to the non-i.i.d. case.

Theorem 1 does not provide a numerical algorithm for computing a survival strategy Λ^*. The problem of developing such algorithms might constitute an interesting topic for further research. It should be noted that in all the EF models with exogenous asset payoffs that have been considered up to now, it was possible to indicate efficient procedures for constructing Λ^*.

Remark 1. Let us make some comments regarding properties (10)–(12).

(a) As can be seen from the proof of Theorem 1, the main role in establishing the relative growth optimality of Λ^* is played by (12). Property (11) is needed to identify a "good" fixed point of L_t (see relations (28)–(29) in the proof of Theorem 1). Property (10) just ensures that the denominator in (11) is non-zero.

One can see that if a strategy Λ^* satisfies (10) and (12), then a simple sufficient condition for the validity of (11) is that for each $t \geq 0$

$$\mathrm{P}_t(\alpha_{t+1,k} > 0) > 0 \text{ a.s.}, \qquad k = 1, \dots, K$$

(this condition will be used below in Theorem 3 and in the example in Section 4). Indeed, in this case (11) is true since $L_{t,k}(\lambda^*, s^t) > 0$ for any $\lambda^* \in \Delta^K$, which implies $\Lambda^*_{t,k} > 0$ and therefore

$$\mathrm{E}_t \left(\frac{\beta_{t+1,k}}{\sum_{j=1}^K g_{t+1,j}(\Lambda^*_t)} \right) \leq \frac{L_{t,k}(\Lambda^*_t)}{\Lambda^*_{t,k}} = 1.$$

Here, the inequality is obtained by multiplying the numerator and denominator in the left-hand side by $\Lambda^*_{t,k}$ and then adding $\alpha_{t+1,k}$ to the numerator.

Finally, note that if the payoff coefficients $\alpha_{t,k}$ are strictly positive a.s., then any function Λ^*_t obviously satisfies (10).

(b) Functions Λ^*_t satisfying (10)–(12) are, in general, not unique. Theorem 1 states that any sequence of them constitutes a relatively growth-optimal strategy.

A simple example of the non-uniqueness of a relatively growth-optimal strategy is as follows. Assume that $\alpha_{t,k} \equiv 0$, $\beta_{t,k} \equiv 1$ and $Z_{t+1} = W_t$ for all t, k. It is easy to see that equation (5) in this case reads $w^i_{t+1} = w^i_t$. Consequently, the agents' wealth remain the same no matter what strategies they use.

Note that under additional assumptions on the payoff coefficients $\alpha_{t,k}$, $\beta_{t,k}$, it is possible to show that the functions satisfying (10)–(12) are unique. Related details can be found in Evstigneev et al. (2023).

3.3. A relatively growth-optimal strategy determines the aggregate market behavior

As was mentioned above, the fractions $\mu_{t,k}$ can be though of as the weighted strategy of the market agents (or the market portfolio). Our next result shows that, under an additional assumption, if at least one agent uses a relatively growth-optimal strategy, then $\mu_{t,k}$ approach this strategy in the limit as $t \to \infty$ with probability 1.

Theorem 2. *Suppose a strategy Λ^* satisfies conditions (10), (12), and the following stronger version of condition (11): there exists $\varepsilon > 0$ such that for any $t \geq 0$ it holds that*

$$\mathrm{E}_t \left(\frac{\beta_{t+1,k}}{\sum_{j=1}^K g_{t+1,j}(\Lambda_t^*)} \right) \leq 1 - \varepsilon \ a.s., \quad k = 1, \dots, K. \tag{13}$$

Then, if in a strategy profile $\Lambda = (\Lambda^1, \dots, \Lambda^N)$ agent i uses the strategy Λ^*, it holds that

$$\sum_{t=1}^{\infty} \|\lambda_t^i - \mu_t\|^2 < \infty \ a.s., \tag{14}$$

where $\lambda_t^i = \lambda_t^i(s^t)$ and $\mu_t = \mu_t(s^t)$ denote, respectively, the realization of the strategy of agent i and the realization of the weighted strategy of all agents in this strategy profile (see (1)). In particular, $\|\lambda_t^i - \mu_t\| \to 0$ as $t \to \infty$.

A sufficient condition for the validity of (13) is that all $\beta_{t,k}$ are uniformly bounded from above and $\alpha_{t,k}$ are uniformly bounded away from zero.

The above theorem implies that Λ^* is an asymptotically unique survival strategy in the class of basic strategies: any other basic survival strategy approaches it as $t \to \infty$ in the sense of (14).

In an i.i.d. case, we can also prove that Λ^* turns out to be a unique survival strategy in the class of all constant strategies (under mild additional assumptions). This is our third main result.

Theorem 3. *Suppose that the sequence of states of the world s_t, $t \geq 1$, consists of i.i.d. random elements and the payoff coefficient $\alpha_{t,k}$, $\beta_{t,k}$ do not depend on time, i.e. $\alpha_{t,k} = \alpha_k(s_t)$, $\beta_{t,k} = \beta_k(s_t)$. Then the following claims hold true.*

(a) There exists a constant relatively growth-optimal strategy $\Lambda_t^* \equiv \Lambda^* \in \Delta^K$.

(b) Assume that

$$\mathrm{P}(\alpha_k > 0) > 0 \text{ for each } k = 1, \dots, K. \tag{15}$$

Then the strategy Λ^* is a unique survival strategy in the class of all constant strategies and $\Lambda_k^* > 0$, $k = 1, \dots, K$. Moreover, Λ^* satisfies (13). In particular, in any strategy profile $\Lambda = (\Lambda^1, \dots, \Lambda^N)$, in which some agent uses the strategy Λ^*, it holds that $\mu_t \to \Lambda^*$ with probability 1 as $t \to \infty$.

(c) Assume that, in addition to (15), it holds that the random variables $\alpha_k + \beta_k \Lambda_k^*$ are linearly independent, i.e. if $\sum_{k=1}^K c_k(\alpha_k + \beta_k \Lambda_k^*) = 0$ a.s. for some constants c_k, then $c_k = 0$ for all $k = 1, \dots, K$.

Then in any strategy profile in which some agent uses the strategy Λ^* and other agents use constant strategies with strictly positive components ($\Lambda_k^i > 0$ for all i, k), it holds that $r_t^i \to 0$ a.s. as $t \to \infty$ for any agent i who uses a strategy $\Lambda^i \neq \Lambda^*$.

Remark 2. The assumption of linear independence of the random variables $\alpha_k + \beta_k \Lambda_k^*$ can be interpreted as the *absence of redundant assets* when the weighted strategy coincides with Λ^*: one cannot construct a "synthetic asset", a portfolio with fixed weights consisting of assets $j \neq k$, that yields the same payoffs as a given asset k.

4. A numerical example

Let us illustrate the main results of the paper with simulations and consider the following simple model, in which only two assets are traded. Assume the random states of the world are modeled by a sequence of i.i.d. random vectors $s_t = (s_t^1, s_t^2)$ with values in the set $\{(1,0), (0,1), (1,1)\}$ and symmetric joint distribution

$$\mathrm{P}(s_t = (1,0)) = \mathrm{P}(s_t = (0,1)) = 1 - p, \qquad \mathrm{P}(s_t = (1,1)) = 2p - 1,$$

where $1/2 \leq p < 1/2$ is a parameter. Assume the payoff coefficients do not depend on time and are given by

$$\alpha_k(s_t) = \beta_k(s_t) = \mathrm{I}(s_t^k = 1), \quad k = 1, 2, \qquad Z_t = 1.$$

Thus, the payoff of each asset at time $t+1$ is either $1 + \mu_{t,k}$ with probability p or 0 with probability $1-p$. With probability $2p-1$, both of the assets yield payoffs simultaneously.

By symmetry, the relatively growth-optimal strategy in this model is $\Lambda^* = (1/2, 1/2)$; it is not difficult to check that it indeed satisfies the conditions of Proposition 2. By Theorem 3(b), it is a unique constant survival strategy. Note that it also satisfies the condition of part (c) of Theorem 3, i.e. $\alpha_k/\Lambda_k^* + \beta_k$ are linearly independent random variables.

Let us place Λ^* in a market environment. As an example, consider a market which consists of 9 agents who use the constant strategies $\Lambda^i = (i/10, 1 - i/10)$, where $i = 1, 2, \dots, 9$. In particular, agent $i = 5$ uses the strategy Λ^*. We do not include the strategies $\Lambda^0 = (0, 1)$ and $\Lambda^{10} = (1, 0)$ since their wealth vanishes in a finite number of time periods.

Figure 1 shows the evolution of the agents' wealth in one simulation of this market over 400 time periods with parameter $p = 2/3$. The first graph displays the relative wealth r_t^i of each agent, with r_t^i corresponding to the width of the corresponding colored area at time t. The second graph shows the relative wealth of the relatively growth-optimal strategy. As can be seen from these two graphs, the relatively growth-optimal strategy eventually dominates the market and its relative wealth converges to 1, as proved in Theorem 3(c). The third graph shows the fraction of the total market wealth invested in asset 1, i.e. $\mu_{t,1}$ (the fraction invested in the second asset is obviously $\mu_{t,2} = 1 - \mu_{t,1}$), which, as expected, converges to $\Lambda_1^* = 1/2$.

To show the convergence of the fractions $\mu_{t,k}$ to Λ^*, in Figure 2 we present multiple simulated trajectories of the sequence $\mu_{t,1}$ for the three different parameters $p = 1/2$, $p = 3/4$, $p = 19/20$.

5. Relation to existing models

To demonstrate the generality of our model, let us show how known results on log-optimal and survival strategies can be derived from it.

In the first part of this section, we will consider the evolutionary finance model with short-lived assets of Amir et al. (2013), in which the asset payoffs $A_{t,k}$ are exogenous, i.e. depend only on the random states s_t, but not on the agents' strategies. In the second part, we will consider the classical model of an asset market with exogenous asset prices, in which agents' actions do not affect the asset prices and the wealth of other agents.

5.1. The evolutionary finance model with exogenous asset payoffs

Suppose in our model $\beta_{t,k} = 0$, $Z_t = 1$. Then equation (5) describing the wealth dynamics reads

$$w_{t+1}^i = \sum_{k=1}^K \frac{\lambda_{t,k}^i w_t^i}{\sum_{j=1}^K \lambda_{t,k}^j w_t^j} \alpha_{t+1,k}.$$

If at least one agent uses a strictly diversified strategy ($\lambda_{t,k}^j > 0$ for all $k = 1, \dots, K$), then the wealth of this agent remains positive in view of (8), the denominator in the above formula does not vanish, and the total market wealth satisfies the relation

$$W_{t+1} = \sum_{k=1}^K \alpha_{t+1,k}.$$

As a result, the relative wealth of agents has the dynamics

$$r_t^i = \sum_{k=1}^K \frac{\lambda_{t,k}^i w_t^i}{\sum_{j=1}^K \lambda_{t,k}^j w_t^j} R_{t+1,k},$$

where $R_{t+1,k} := \alpha_{t+1,k} / \sum_{j=1}^K \alpha_{t+1,j}$ are the *relative payoffs* of the assets. The unique strategy which satisfies conditions (10)–(12) of Proposition 2 is

$$\Lambda_{t,k}^* = \mathrm{E}_t R_{t+1,k},$$

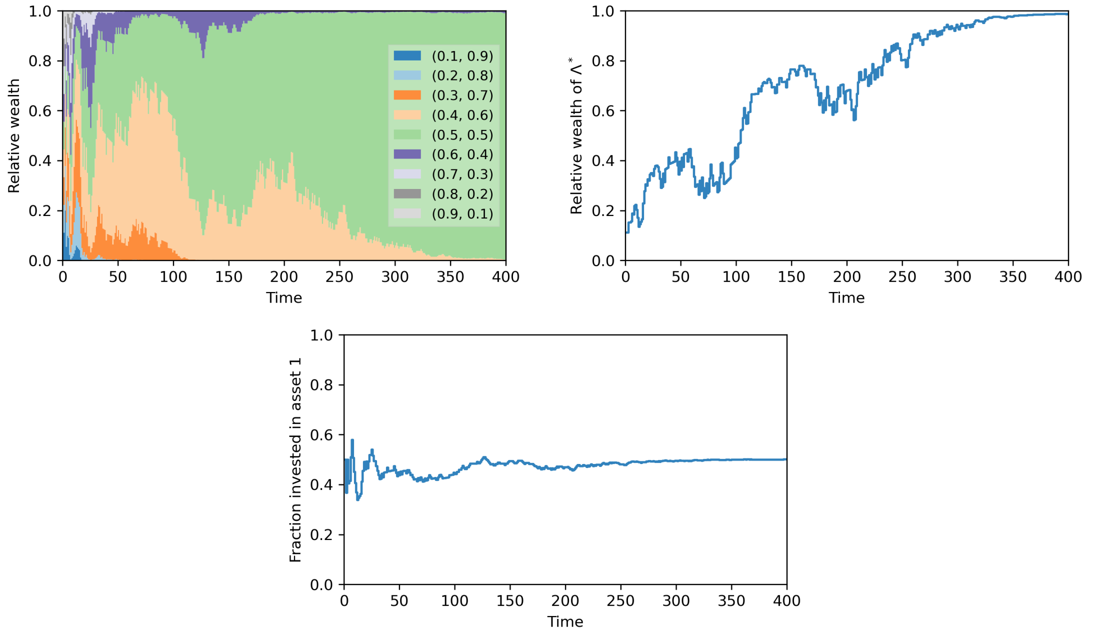

Figure 1: Evolution of agents' wealth in one simulation of the model. Top left: the relative wealth of each strategy $\Lambda^i = (i/10, 1 - i/10)$. Top right: the relative wealth of the relatively growth-optimal strategy Λ^*. Bottom: the fraction of the total market wealth invested in the first asset $\mu_{t,1}$.

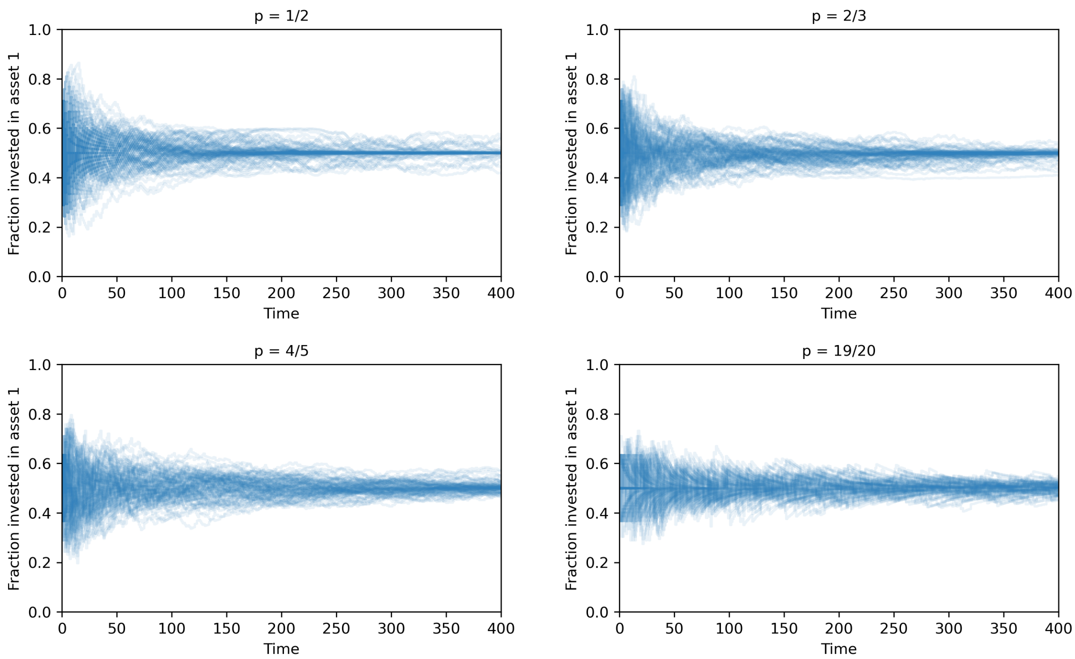

Figure 2: Trajectories of the fraction of the total market wealth invested in the first asset $\mu_{t,1}$ for different value of the probability p. Each graph contains 100 simulated trajectories.

i.e. it allocates the current available investment budget among the assets proportionally to the conditional expectations of their relative payoffs. The survival property of this strategy was first established by Amir et al. (2013). Further generalizations can be found in Drokin and Zhitlukhin (2020); Zhitlukhin (2023b).

5.2. The classical model with exogenous asset prices

Let us now consider a market with exogenous asset prices $S_{t,k}(s^t) > 0$, as in the standard textbook model without short sales (see, e.g., Föllmer and Schied (2011, Ch. 5)). The goal of this example is to show that such a market model is a particular case of our model, while the relatively growth-optimal strategy constructed in Theorem 1 maximizes the expected logarithm of an agent's wealth or, equivalently, the logarithmic return of portfolio. In particular, this provides a new characterization of an expected log-wealth maximizing strategy as a fixed point of the mapping defined in (12).

Denote by $X_{t+1,k} = S_{t+1,k}/S_{t,k}$ the asset returns. Then the evolution of wealth w_t of an agent who uses a strategy $\Lambda = \Lambda_t(s^t)$ is specified by the relation

$$w_{t+1} = w_t \langle \Lambda_t, X_{t+1} \rangle,$$

where $\langle \cdot, \cdot \rangle$ denotes the scalar product. This model can be obtained from our model if in equation (6) we put

$$\alpha_{t+1,k} = 0, \qquad \beta_{t+1,k} = X_{t+1,k}, \qquad Z_{t+1} = W_t \tag{16}$$

and assume that all the agents use the same strategy.

Recall that a log-optimal strategy $\Lambda^* = \Lambda^*_t(s^t)$ in the classical capital growth theory is a strategy which maximizes the expected log-return of a portfolio in each time period, i.e.

$$\Lambda^*_t \in \operatorname*{arg\,max}_{\lambda \in \Delta^K} \mathrm{E}_t \ln \langle \lambda, X_{t+1} \rangle. \tag{17}$$

Such a strategy is often referred to as the *Kelly portfolio rule*. Although the above optimization problem may not have a solution when the log-returns are not integrable, if we introduce the *relative returns*

$$R_{t,k} = \frac{X_{t,k}}{\sum_{j=1}^K X_{t,j}}, \qquad k = 1, \dots, K,$$

then it is not difficult to show that a log-optimal strategy can be characterized as a solution of the maximization problem for the logarithms of relative returns

$$\Lambda^*_t \in \operatorname*{arg\,max}_{\lambda \in \Delta^K} \mathrm{E}_t \ln \langle \lambda, R_{t+1} \rangle. \tag{18}$$

Namely, problem (18) always has a solution and if (17) has at least one solution, then the sets of solutions of (17) and (18) coincide.

Let us investigate the relation between relatively growth-optimal strategies, which satisfy conditions (10)–(12) in our model, and the classical notion of a log-optimal strategy (18). Observe that condition (10) is satisfied by any strategy, since it is equivalent to that

$$w_t\langle \Lambda_t^*, X_{t+1}\rangle > 0,$$

which holds because $X_{t+1,k} > 0$. Conditions (11) and (12) are equivalent to, respectively,

$$\mathrm{E}_t\left(\frac{R_{t+1,k}}{\langle \Lambda_t^*, R_{t+1}\rangle}\right) \le 1, \tag{19}$$

$$\mathrm{E}_t\left(\frac{\Lambda_{t,k}^* R_{t+1,k}}{\langle \Lambda_t^*, R_{t+1}\rangle}\right) = \Lambda_{t,k}^*. \tag{20}$$

It is easy to see that (19) implies (20). Indeed, multiplying the both sides of (19) by $\Lambda_{t,k}^*$, we get the inequality $\mathrm{E}_t(\Lambda_{t,k}^* R_{t+1,k}\langle \Lambda_t^*, R_{t+1}\rangle^{-1}) \le \Lambda_{t,k}^*$, which must actually be an equality with probability 1, since otherwise, by taking the sum of the both sides over $k = 1, \dots, K$, we would get the contradiction $1 < 1$ with positive probability.

Thus, in the particular case of our model under consideration, conditions (10)–(12) are equivalent to (19).

Proposition 3. *A strategy $\Lambda^* = \Lambda_t^*(s^t)$ satisfies conditions* (10)*–*(12) *of Proposition 2* (*or, equivalently, condition* (19)) *if and only if it is a* (*measurable*) *solution of maximization problem* (18).

Proof. If a strategy Λ^* satisfies condition (19), then for any other strategy $\Lambda = \Lambda_t(s^t)$ we have

$$\begin{aligned}\mathrm{E}_t \ln\langle \Lambda_t, R_{t+1}\rangle - \mathrm{E}_t \ln\langle \Lambda_t^*, R_{t+1}\rangle &= \mathrm{E}_t \ln\left(\frac{\langle \Lambda_t, R_{t+1}\rangle}{\langle \Lambda_t^*, R_{t+1}\rangle}\right)\\ &\le \left\langle \Lambda_t,\ \mathrm{E}_t\left(\frac{R_{t+1}}{\langle \Lambda_t^*, R_{t+1}\rangle}\right)\right\rangle - 1 \le \sum_{k=1}^K \Lambda_{t,k} - 1 = 0,\end{aligned}$$

where the first inequality holds because $\ln x \le x - 1$.

On the other hand, if a strategy Λ_t^* solves problem (18), then it is known (see Algoet and Cover (1988, Th. 1)) that for any other strategy Λ

$$\mathrm{E}_t \frac{\langle \Lambda_t, R_{t+1}\rangle}{\langle \Lambda_t^*, R_{t+1}\rangle} \le 1.$$

Taking $\Lambda_t = (0, \dots, 0, 1, 0, \dots, 0)$, we obtain (19). □

6. Proofs of the main results

6.1. Auxiliary results

This section contains several simple lemmas needed to prove the main results.

Lemma 1. *Let $C \subset \mathbb{R}^K$ be a compact set and $(\Omega, \mathcal{F})$ be a measurable space. Suppose a function $L(x,\omega)\colon C \times \Omega \to C$ is continuous in x and measurable in ω. Then L has a measurable fixed point $\xi(\omega)$, i.e. $L(\xi(\omega),\omega) = \xi(\omega)$ for all $\omega \in \Omega$.*

This result follows from Brouwer's fixed point theorem and Aumann's measurable selection theorem (see, e.g., Castaing and Valadier (1977, Theorem III.22)).

Lemma 2. *Let $L^n(\omega)$, $n = 1, 2, \dots,$ be a sequence of measurable functions on a measurable space $(\Omega, \mathcal{F})$ with values in a compact set $C \subset \mathbb{R}^K$. Then there exists a measurable function $L^*(\omega)$ and a strictly increasing sequence of integer-valued measurable functions $n_i(\omega) \geq 1$, $i = 1, 2, \dots,$ such that $\lim_{i\to\infty} L^{n_i(\omega)}(\omega) = L^*(\omega)$ for any ω.*

The above lemma follows from Lemma 2.1.2 in Kabanov and Safarian (2009). The next lemma generalizes the Gibbs inequality. It will play a key role in the proof of Theorem 1. By $\|\cdot\|$ we will denote the Euclidean norm.

Lemma 3. *Suppose $x, y \in \mathbb{R}^K_+$ are two vectors such that $\sum_{k=1}^K x_k \leq 1$, $\sum_{k=1}^K y_k \leq 1$, and for each $k = 1, \dots, K$ it holds that if $y_k = 0$, then $x_k = 0$. Then*

$$\sum_{k=1}^K x_k \ln \frac{x_k}{y_k} \geq \frac{\|x-y\|^2}{4} + \sum_{k=1}^K (x_k - y_k), \tag{21}$$

where we define $x_k \ln \frac{x_k}{y_k} = 0$ if $x_k = 0$ or both $x_k = 0$ and $y_k = 0$.

Proof. Using that $\ln a \leq 2(\sqrt{a} - 1)$, we obtain

$$\begin{aligned}\sum_{k=1}^K x_k \ln \frac{x_k}{y_k} = -\sum_{k\,:\,x_k\neq 0} x_k \ln \frac{y_k}{x_k} \geq 2\sum_{k=1}^K (x_k - \sqrt{x_k y_k}) \\ = \sum_{k=1}^K (\sqrt{x_k} - \sqrt{y_k})^2 + \sum_{k=1}^K (x_k - y_k).\end{aligned}$$

Applying the inequality $(\sqrt{x}_k - \sqrt{y}_k)^2 \geq (x_k - y_k)^2/4$, which is true for any $x_k, y_k \in [0,1]$, we obtain (21). □

The final lemma in this section is a simple result from the theory of martingales.

Lemma 4. *Let $\zeta_t = \zeta_t(s^t)$, $t = 0, 1, \dots,$ be a random sequence (ζ_0 is constant) which is uniformly bounded from above (i.e. $\zeta_t \leq c$ a.s. for all t and some constant c) and $\mathrm{E}_{t-1}\zeta_t \geq \zeta_{t-1}$ a.s. for all $t \geq 1$. Then $\mathrm{E}|\zeta_t| < \infty$, so ζ_t is a submartingale.*

Proof. We have $\mathrm{E}_{t-1}\zeta_t^- \leq \mathrm{E}_{t-1}\zeta_t^+ - \zeta_{t-1} \leq c - \zeta_{t-1}$, and, consequently, $\mathrm{E}\zeta_t^- \leq c - \mathrm{E}\zeta_{t-1}$. From this relation, the claimed result follows by induction. □

6.2. Proof of Proposition 2

Fix $t \geq 0$. Define Δ^K-valued functions $g_t^n(\lambda^*, s^t)$ and $L_t^n(\lambda^*, s^t)$, $n = 1, 2, \ldots$, by

$$g_{t,k}^n = g_{t,k} + \frac{1}{n},$$
$$L_{t,k}^n(\lambda^*, s^t) = \mathrm{E}_t\left(\frac{g_{t+1,k}^n(\lambda^*, s^{t+1})}{\sum_{j=1}^K g_{t+1,j}^n(\lambda^*, s^{t+1})}\right).$$

By treating $\mathrm{E}_t(\cdot)$ as the conditional expectation with respect to a fixed variant of the regular conditional distribution of s^{t+1}, we can assume that the functions L_t^n are continuous in $\lambda^* \in \Delta^K$ and measurable in s^t. Hence by Lemma 1 they have measurable fixed points $\Lambda_t^n = \Lambda_t^n(s^t)$, i.e. for any s^t it holds that

$$L_t^n(\Lambda_t^n(s^t)) = \Lambda_t^n. \tag{22}$$

Let

$$\delta_{t,k}^n = \mathrm{E}_t\left(\frac{\beta_{t+1,k}}{\sum_{j=1}^K g_{t+1,j}^n(\Lambda_t^n)}\right).$$

Observe that

$$\delta_{t,k}^n \leq 1, \quad k = 1, \ldots, K, \tag{23}$$

since

$$(1 - \delta_{t,k}^n)\Lambda_{t,k}^n = \mathrm{E}_t\left(\frac{\alpha_{t+1,k} + 1/n}{\sum_{j=1}^K g_{t+1,j}^n(\Lambda_t^n)}\right) > 0.$$

By Lemma 2, it is possible to find an increasing sequence $n_i = n_i(s^t)$, $i = 1, 2, \ldots$, such that there exists the limit

$$\Lambda_t^* = \lim_{i \to \infty} \Lambda_t^{n_i}.$$

Now by passing to the limit $i \to \infty$ and $n_i \to \infty$ in (23) using Fatou's lemma and taking into account assumption (8), one can see that (10) holds. Then, by the dominated convergence theorem, we obtain (11) from (23), and (12) from (22).

6.3. Proof of Theorem 1

Fix an admissible vector of initial endowments and a strategy profile in which one agent uses the strategy Λ^*. Without loss of generality, assume that Λ^* is used by agent 1.

Define (omitting the argument s^t for brevity)

$$\theta_{t,k} = \frac{\lambda_{t,k}^1}{\mu_{t,k}}.$$

Then equation (5) defining the wealth dynamics can be written as

$$w_{t+1}^1 = r_t^1 \sum_{k=1}^{K} \theta_{t,k} A_{t+1,k} = r_t^1 \sum_{k=1}^{K} (\theta_{t,k}\alpha_{t+1,k} + \lambda_{t,k}^1 \beta_{t+1,k}) Z_{t+1}.$$

From (12), we see that if $\mathrm{P}_t(\alpha_{t+1,k} + \beta_{t+1,k} = 0)(s^t) = 1$, then $\Lambda_{t,k}^*(s^t) = 0$. This and assumption (8) implies that $w_{t+1}^1 > 0$. Therefore, the total market wealth satisfies the relation

$$W_{t+1} = \sum_{k=1}^{K} A_{t+1,k} = \sum_{k=1}^{K} (\alpha_{t+1,k} + \mu_{t,k}\beta_{t+1,k}) Z_{t+1}.$$

From the above two relations, we find

$$\ln r_{t+1}^1 - \ln r_t^1 = \ln\left(\frac{\sum_{k=1}^{K} (\theta_{t,k}\alpha_{t+1,k} + \lambda_{t,k}^1 \beta_{t+1,k})}{\sum_{j=1}^{K} (\alpha_{t+1,j} + \mu_{t,j}\beta_{t+1,j})} \right). \tag{24}$$

Consequently, we can represent

$$\mathrm{E}_t \ln r_{r+1}^1 - \ln r_t^1 = \mathrm{E}_t(F_{t+1} + G_{t+1}),$$

where

$$F_{t+1} = \ln\left(\frac{\sum_{k=1}^{K} (\theta_{t,k}\alpha_{t+1,k} + \lambda_{t,k}^1 \beta_{t+1,k})}{\sum_{j=1}^{K} (\alpha_{t+1,j} + \lambda_{t,j}^1 \beta_{t+1,j})} \right), \tag{25}$$

$$G_{t+1} = \ln\left(\frac{\sum_{k=1}^{K} (\alpha_{t+1,k} + \lambda_{t,k}^1 \beta_{t+1,k})}{\sum_{j=1}^{K} (\alpha_{t+1,j} + \mu_{t,j}\beta_{t+1,j})} \right). \tag{26}$$

Let us show that $\mathrm{E}_t(F_{t+1} + G_{t+1}) \geq 0$. Consider the argument of the logarithm in (25) as the convex combination of the values

$$\theta_{t,1}, \ \ldots, \ \theta_{t,K}, 1$$

with the coefficients

$$\frac{\alpha_{t+1,k}}{\sum_{j=1}^{K} (\alpha_{t+1,j} + \lambda_{t,j}^1 \beta_{t+1,j})}, \ k = 1, \ldots, K, \qquad \frac{\sum_{k=1}^{K} \lambda_{t,k}^1 \beta_{t+1,k}}{\sum_{j=1}^{K} (\alpha_{t+1,j} + \lambda_{t,j}^1 \beta_{t+1,j})}.$$

By the concavity of the logarithm, we obtain

$$F_{t+1} \geq \sum_{k=1}^{K} \frac{\alpha_{t+1,k}}{\sum_{j=1}^{K} (\alpha_{t+1,j} + \lambda_{t,j}^1 \beta_{t+1,j})} \ln \theta_{t,k}. \tag{27}$$

Let

$$\gamma_{t,k} = 1 - \mathrm{E}_t \left(\frac{\beta_{t+1,k}}{\sum_{j=1}^{K} (\alpha_{t+1,j} + \lambda_{t,j}^1 \beta_{t+1,j})} \right), \qquad k = 1, \ldots, K. \tag{28}$$

By (11), we have $\gamma_{t,k} \in [0,1]$, and from (12) it follows that

$$\gamma_{t,k}\lambda^1_{t,k} = \mathrm{E}_t\left(\frac{\alpha_{t+1,k}}{\sum_{j=1}^K(\alpha_{t+1,j} + \lambda^1_{t,j}\beta_{t+1,j})}\right). \tag{29}$$

Taking the expectation in (27), we find

$$\begin{aligned}\mathrm{E}_t F_{t+1} \geq \sum_{k=1}^K \gamma_{t,k}\lambda^1_{t,k}\ln\theta_{t,k} &= \sum_{k=1}^K \gamma_{t,k}\lambda^1_{t,k}\ln\frac{\gamma_{t,k}\lambda^1_{t,k}}{\gamma_{t,k}\mu_{t,k}}\\ &\geq \frac{1}{4}\sum_{k=1}^K(\gamma_{t,k}(\lambda^1_{t,k} - \mu_{t,k}))^2 + \sum_{k=1}^K \gamma_{t,k}(\lambda^1_{t,k} - \mu_{t,k}),\end{aligned} \tag{30}$$

where in the second inequality we applied Lemma 3 to the vectors x, y with coordinates

$$x_k = \gamma_{t,k}\lambda^1_{t,k}, \qquad y_k = \gamma_{t,k}\mu_{t,k}.$$

Observe that the conditions of the lemma $\sum_{k=1}^K x_k \leq 1$, $\sum_{k=1}^K y_k \leq 1$ are met because the vectors λ^1_t and μ_t have this property and $\gamma_{t,k} \in [0,1]$.

In order to bound $\mathrm{E}_t G_{t+1}$, by using the inequality $\ln a \geq 1 - a^{-1}$, we find

$$\begin{aligned}\mathrm{E}_t G_{t+1} &\geq \mathrm{E}_t\left(\frac{\sum_{k=1}^K(\lambda^1_{t,k} - \mu_{t,k})\beta_{t+1,k}}{\sum_{j=1}^K(\alpha_{t+1,j} + \lambda^1_{t,j}\beta_{t+1,j})}\right)\\ &= \sum_{k=1}^K(1-\gamma_{t,k})(\lambda^1_{t,k} - \mu_{t,k}) = \sum_{k=1}^K \gamma_{t,k}(\mu_{t,k} - \lambda^1_{t,k}),\end{aligned} \tag{31}$$

where the last equality holds because $\sum_{k=1}^K \lambda^1_{t,k} = \sum_{k=1}^K \mu_{t,k} = 1$.

From (30) and (31), we obtain

$$\mathrm{E}_t(F_{t+1} + G_{t+1}) \geq \frac{1}{4}\sum_{k=1}^K(\gamma_{t,k}(\lambda^1_{t,k} - \mu_{t,k}))^2, \tag{32}$$

so $\mathrm{E}_t(F_{t+1} + G_{t+1}) \geq 0$. By Lemma 4, we conclude that $\ln r^1_t$ is a submartingale.

6.4. Proof of Theorem 2

Assume that the strategy Λ^* is used by agent 1. In the course of proof of Theorem 1, we have shown that $\zeta_t := \ln r^1_t$ is a submartingale. By Doob's decomposition, it can be represented as $\zeta_t = \zeta_0 + M_t + A_t$, where M_t is a martingale, A_t is a predictable non-decreasing sequence (the *compensator* of ζ_t), and $M_0 = A_0 = 0$. Since ζ_t has a finite limit as $t \to \infty$, the compensator converges as well, i.e. $\lim_{t\to\infty} A_t < \infty$ a.s. Inequality

(32) implies that

$$A_t := \sum_{u=0}^{t-1} (\mathrm{E}_u \zeta_{u+1} - \zeta_u) \geq \frac{1}{4} \sum_{u=0}^{t-1} \sum_{k=1}^{K} (\gamma_{u,k}(\lambda_{u,k}^1 - \mu_{u,k}))^2.$$

From condition (13) and relation (28), it follows that $\gamma_{t,k} \geq \varepsilon > 0$. Then relation (14) follows from the convergence of A_t. This proves the first claim of the theorem.

In order to see that inequality (13) holds if $\beta_{t,k}$ are uniformly bounded from above and $\alpha_{t,k}$ are uniformly bounded away from zero, observe that in this case the right-hand side of (29) is uniformly bounded away from zero. Consequently, $\gamma_{t,k}$ are also uniformly bounded away from zero, which implies (13).

6.5. Proof of Theorem 3

(a) The existence of a constant strategy satisfying conditions (10)–(12), and hence being a relatively growth-optimal strategy, easily follows from inspecting the proof of Proposition 2.

(b) Suppose (15) holds. Let

$$\gamma_k = 1 - \mathrm{E}\left(\frac{\beta_k}{\sum_{j=1}^K (\alpha_j + \Lambda_k^* \beta_j)} \right), \qquad k = 1, \dots, K. \tag{33}$$

Then (12) implies (cf. (28)–(29))

$$\gamma_k \Lambda_k^* = \mathrm{E}\left(\frac{\alpha_k}{\sum_{j=1}^K (\alpha_j + \Lambda_j^* \beta_j)} \right), \tag{34}$$

hence $\gamma_k \Lambda_k^* > 0$, so $\gamma_k > 0$, which means that condition (13) is satisfied. From Theorem 2, we obtain the convergence $\mu_t \to \Lambda^*$.

If $\widetilde{\Lambda} \in \Delta^K$ is another constant survival strategy, then it must survive in the strategy profile $(\widetilde{\Lambda}, \Lambda^*, \dots, \Lambda^*)$. This means $\inf_{t \geq 0} r_t^1 > 0$ a.s. But then the convergence $\mu_t = r_t^1 \widetilde{\Lambda} + (1 - r_t^1)\Lambda^* \to \Lambda^*$ takes place only if $\widetilde{\Lambda} = \Lambda^*$. Hence, Λ^* is a unique survival strategy.

(c) Consider a strategy profile in which some agent, say agent 1, uses the strategy Λ^*. Let agent i use a constant strategy $\Lambda^i \neq \Lambda^*$. To prove the theorem, we need to show that $r_t^1/r_t^i \to \infty$ with probability 1 as $t \to \infty$. For this end, we will show that

$$\liminf_{t \to \infty} t^{-1} \ln \frac{r_t^1}{r_t^i} > 0. \tag{35}$$

From equation (24) in the proof of Theorem 1, it follows that

$$D_{t+1} := \ln \frac{r_{t+1}^1}{r_{t+1}^i} - \ln \frac{r_t^1}{r_t^i} = \ln \left(\frac{\sum_{k=1}^K (\theta_{t,k}^1 \alpha_{t+1,k} + \Lambda_k^* \beta_{t+1,k})}{\sum_{k=1}^K (\theta_{t,k}^i \alpha_{t+1,k} + \Lambda_k^i \beta_{t+1,k})} \right),$$

where in the model under consideration $\alpha_{t,k} = \alpha_k(s^t)$, $\beta_{t,k} = \beta_k(s^t)$, and $\theta^1_{t,k} = \Lambda^*_k/\mu_{t,k}$, $\theta^i_{t,k} = \Lambda^i_k/\mu_{t,k}$. Then we have

$$t^{-1}\ln\frac{r^1_t}{r^i_t} = t^{-1}\ln\frac{r^1_0}{r^i_0} + t^{-1}\sum_{u=0}^{t-1}\mathrm{E}_u D_{u+1} + t^{-1}\sum_{u=0}^{t-1}(D_{u+1} - \mathrm{E}_u D_{u+1}).$$

It is easy to see that the sequence D_t is uniformly bounded:

$$\frac{1}{c} \le D_t < c,$$

where $c = \max_{i,k}\Lambda^i_k/\min_{i,k}\Lambda^i_k$. Then from the SLLN for martingales, we have $\xi_t := t^{-1}\sum_{u=0}^{t-1}(D_{u+1} - \mathrm{E}_u D_{u+1}) \to 0$ a.s. Thus, to establish (35), it will be sufficient to show that there exists $\varepsilon > 0$ and a random time τ such that for $t \ge \tau$ it holds that

$$\mathrm{E}_t D_{t+1} \ge \varepsilon. \tag{36}$$

By Theorem 1, we have $\mu_{t,k} \to \Lambda^*_k$, so $\theta^1_{t,k} \to 1$ and $\theta^i_{t,k} \to \Lambda^i_k/\Lambda^*_k$. Consequently, with probability 1 we have

$$\lim_{t\to\infty}\mathrm{E}_t D_{t+1} = \mathrm{E}\ln\left(\frac{\sum_{k=1}^K(\alpha_k + \Lambda^*_k\beta_k)}{\sum_{k=1}^K(\Lambda^i_k\alpha_k/\Lambda^*_k + \Lambda^i_k\beta_k)}\right) =: \mathrm{E}\ln\zeta.$$

Then, in order to prove (36), we need to show that $\mathrm{E}\ln\zeta > 0$, or equivalently $\mathrm{E}\ln\zeta^{-1} < 0$. In view of the strict concavity of the logarithm and Jensen's inequality, it is enough to show that $\mathrm{E}\zeta^{-1} = 1$ and ζ is not constant a.s. To prove the former, use relations (33) and (34), which yield

$$\mathrm{E}\zeta^{-1} = \mathrm{E}\left(\frac{\sum_{k=1}^K(\Lambda^i_k\alpha_k/\Lambda^*_k + \Lambda^i_k\beta_k)}{\sum_{k=1}^K(\alpha_k + \Lambda^*_k\beta_k)}\right) = \sum_{k=1}^K(\gamma_k\Lambda^i_k + (1-\gamma_k)\Lambda^i_k) = 1.$$

The fact that ζ is not constant follows from the assumption that the random variables $\alpha_k + \beta_k\Lambda^*_k$ (and, hence, $\alpha_k/\Lambda^*_k + \beta_k$) are linearly independent. Indeed, if $\zeta = c$, then

$$\sum_{k=1}^K(c\Lambda^*_k - \Lambda^i_k)\left(\frac{\alpha_k}{\Lambda^*_k} + \beta_k\right) = 0,$$

which implies $c\Lambda^*_k = \Lambda^i_k$ for all k, so $\Lambda^i = \Lambda^*$, which contradicts our assumption.

Compliance with ethical standards

Funding. The work of M. V. Zhitlukhin was performed at the Steklov International Mathematical Center and supported by the Ministry of Science and Higher Education of the Russian Federation (agreement no. 075-15-2022-265).

Conflict of interests. The authors declare that they have no conflict of interests.

Ethical approval. This article does not contain any studies with human participants or animals performed by any of the authors.

References

Algoet, P. H. and Cover, T. M. (1988). Asymptotic optimality and asymptotic equipartition properties of log-optimum investment. *The Annals of Probability*, 16(2):876–898.

Amir, R., Evstigneev, I. V., Hens, T., Potapova, V., and Schenk-Hoppé, K. R. (2021). Evolution in pecunia. *Proceedings of the National Academy of Sciences*, 118(26):e2016514118.

Amir, R., Evstigneev, I. V., Hens, T., and Xu, L. (2011). Evolutionary finance and dynamic games. *Mathematics and Financial Economics*, 5:161–184.

Amir, R., Evstigneev, I. V., and Potapova, V. (2023). Unbeatable strategies. *Economic Theory*, pages 1–30.

Amir, R., Evstigneev, I. V., and Schenk-Hoppé, K. R. (2013). Asset market games of survival: a synthesis of evolutionary and dynamic games. *Annals of Finance*, 9(2):121–144.

Anderson, P. W., Arrow, K. J., and Pines, D., editors (1988). *The Economy As An Evolving Complex System.* CRC Press, London.

Arkin, V. I. and Evstigneev, I. V. (1987). *Stochastic Models of Control and Economic Dynamics.* Academic Press.

Arthur, W. B., Durlauf, S. N., and Lane, D. A. (1997). *The Economy As An Evolving Complex System II.* CRC Press, London.

Blume, L. and Easley, D. (1992). Evolution and market behavior. *Journal of Economic Theory*, 58(1):9–40.

Bottazzi, G. and Dindo, P. (2014). Evolution and market behavior with endogenous investment rules. *Journal of Economic Dynamics and Control*, 48:121–146.

Bottazzi, G., Dindo, P., and Giachini, D. (2018). Long-run heterogeneity in an exchange economy with fixed-mix traders. *Economic Theory*, 66:407–447.

Bottazzi, G., Dosi, G., and Rebesco, I. (2005). Institutional architectures and behavioral ecologies in the dynamics of financial markets. *Journal of Mathematical Economics*, 41(1-2):197–228.

Breiman, L. (1961). Optimal gambling systems for favorable games. In *Proceedings of the 4th Berkeley Symposium on Mathematical Statistics and Probability*, volume 1, pages 63–68.

Brock, W. A., Hommes, C. H., and Wagener, F. O. (2005). Evolutionary dynamics in markets with many trader types. *Journal of Mathematical Economics*, 41(1-2):7–42.

Castaing, C. and Valadier, M. (1977). *Convex Analysis and Measurable Multifunctions*. Springer.

Coury, T. and Sciubba, E. (2012). Belief heterogeneity and survival in incomplete markets. *Economic Theory*, 49:37–58.

Drokin, Y. and Zhitlukhin, M. (2020). Relative growth optimal strategies in an asset market game. *Annals of Finance*, 16:529–546.

Evstigneev, I., Hens, T., Potapova, V., and Schenk-Hoppé, K. R. (2020). Behavioral equilibrium and evolutionary dynamics in asset markets. *Journal of Mathematical Economics*, 91:121–135.

Evstigneev, I., Hens, T., and Schenk-Hoppé, K. R. (2016). Evolutionary behavioral finance. In Haven, E. et al., editors, *The Handbook of Post Crisis Financial Modelling*, pages 214–234. Palgrave Macmillan UK.

Evstigneev, I. V., Hens, T., and Schenk-Hoppé, K. R. (2002). Market selection of financial trading strategies: Global stability. *Mathematical Finance*, 12(4):329–339.

Evstigneev, I. V., Hens, T., Schenk-Hoppé, K. R., et al. (2015). *Mathematical Financial Economics: A Basic Introduction*. Springer International Publishing, Switzerland.

Evstigneev, I. V., Hens, T., and Vanaei, M. J. (2023). Evolutionary finance: A model with endogenous asset payoffs. *Journal of Bioeconomics*, 25:117–143.

Evstigneev, I. V. and Vanaei, M. J. (2022). Evolutionary behavioral finance: A model with endogenous asset payoffs. Economics Discussion Paper Series, EDP-2202, April 2022, http://hummedia.manchester.ac.uk/schools/soss/economics/discussionpapers/EDP-2202.pdf.

Farmer, J. D. (2002). Market force, ecology and evolution. *Industrial and Corporate Change*, 11(5):895–953.

Farmer, J. D. and Lo, A. W. (1999). Frontiers of finance: Evolution and efficient markets. *Proceedings of the National Academy of Sciences*, 96(18):9991–9992.

Föllmer, H. and Schied, A. (2011). *Stochastic Finance: An Introduction in Discrete Time*. Walter de Gruyter, Berlin/New York, 3rd edition.

Hakansson, N. H. and Ziemba, W. T. (1995). Capital growth theory. In *Finance*, volume 9 of *Handbooks in Operations Research and Management Science*, pages 65–86. Elsevier.

Hens, T. and Naebi, F. (2022). Behavioral heterogeneity in the capm with evolutionary dynamics. *Journal of Evolutionary Economics*, 32(5):1499–1521.

Holtfort, T. (2019). From standard to evolutionary finance: a literature survey. *Management Review Quarterly*, 69(2):207–232.

Kabanov, Y. and Safarian, M. (2009). *Markets with Transaction Costs: Mathematical Theory*. Springer.

Kelly, Jr, J. L. (1956). A new interpretation of information rate. *Bell System Technical Journal*, 35(4):917–926.

Li, E. X., Livdan, D., and Zhang, L. (2009). Anomalies. *The Review of Financial Studies*, 22(11):4301–4334.

Lintner, J. (1965). The valuation of risk assets and the selection of risky investments in stock portfolios and capital budgets. *Review of Economics and Statistics*, 47:13–37.

Lo, A. W. (2004). The adaptive markets hypothesis. *The Journal of Portfolio Management*, 30(5):15–29.

Lo, A. W. (2005). Reconciling efficient markets with behavioral finance: the adaptive markets hypothesis. *Journal of Investment Consulting*, 7(2):21–44.

Lo, A. W. (2012). Adaptive markets and the new world order. *Financial Analysts Journal*, 68(2):18–29.

Lo, A. W. (2017). *Adaptive markets*. Princeton University Press.

Lo, A. W., Orr, H. A., and Zhang, R. (2018). The growth of relative wealth and the kelly criterion. *Journal of Bioeconomics*, 20:49–67.

MacLean, L. C., Thorp, E. O., and Ziemba, W. T. (2010). Long-term capital growth: the good and bad properties of the Kelly and fractional Kelly capital growth criteria. *Quantitative Finance*, 10(7):681–687.

MacLean, L. C., Thorp, E. O., and Ziemba, W. T., editors (2011). *The Kelly Capital Growth Investment Criterion: Theory and practice*, volume 3 of *World Scientific Handbook in Financial Economic Series*. World Scientific.

Maynard Smith, J. and Price, G. R. (1973). The logic of animal conflict. *Nature*, 246(5427):15–18.

Radner, R. (1972). Existence of equilibrium of plans, prices, and price expectations in a sequence of markets. *Econometrica*, 40(2):289–303.

Radner, R. (1982). Equilibrium under uncertainty. In *Handbook of Mathematical Eeconomics*, volume 2, pages 923–1006. Elsevier.

Schaffer, M. E. (1988). Evolutionarily stable strategies for a finite population and a variable contest size. *Journal of Theoretical Biology*, 132(4):469–478.

Schnetzer, M. and Hens, T. (2022). Evolutionary finance for multi-asset investors. *Financial Analysts Journal*, 78(3):115–127.

Sciubba, E. (2005). Asymmetric information and survival in financial markets. *Economic Theory*, 25:353–379.

Shiryaev, A. N. (2019). *Probability–2*. Springer, 3rd edition.

Tobin, J. (1969). A general equilibrium approach to monetary theory. *Journal of money, credit and banking*, 1(1):15–29.

Tobin, J. and Brainard, W. (1977). Asset market and cost of capital. In *Economic Progress, Private Values and Public Policy, Essays in Honor of William Fellner*, pages 235–262. North-Holland, Amsterdam.

Zhang, R., Brennan, T. J., and Lo, A. W. (2014). Group selection as behavioral adaptation to systematic risk. *PloS One*, 9(10):e110848.

Zhitlukhin, M. (2021). Survival investment strategies in a continuous-time market model with competition. *International Journal of Theoretical and Applied Finance*, 24(01):2150001.

Zhitlukhin, M. (2022). A continuous-time asset market game with short-lived assets. *Finance and Stochastics*, 26(3):587–630.

Zhitlukhin, M. (2023a). Asymptotic minimization of expected time to reach a large wealth level in an asset market game. *Stochastics*, 95(1):67–78.

Zhitlukhin, M. (2023b). Capital growth and survival strategies in a market with endogenous prices. *SIAM Journal on Financial Mathematics*, 14(3):812–837.

Ziemba, W. T. (2015). A response to Professor Paul A. Samuelson's objections to Kelly capital growth investing. *The Journal of Portfolio Management*, 42(1):153–167.