INSTITUTE AND FACULTY OF ACTUARIES

EXAMINERS' REPORT

April 2017

Subject CT1 – Financial Mathematics
Core Technical

Introduction

The Examiners' Report is written by the Principal Examiner with the aim of helping candidates, both those who are sitting the examination for the first time and using past papers as a revision aid and also those who have previously failed the subject.

The Examiners are charged by Council with examining the published syllabus. The Examiners have access to the Core Reading, which is designed to interpret the syllabus, and will generally base questions around it but are not required to examine the content of Core Reading specifically or exclusively.

For numerical questions the Examiners' preferred approach to the solution is reproduced in this report; other valid approaches are given appropriate credit. For essay-style questions, particularly the open-ended questions in the later subjects, the report may contain more points than the Examiners will expect from a solution that scores full marks.

The report is written based on the legislative and regulatory context pertaining to the date that the examination was set. Candidates should take into account the possibility that circumstances may have changed if using these reports for revision.

Luke Hatter
Chair of the Board of Examiners
July 2017

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A. General comments on the *aims of this subject and how it is marked*

1. CT1 provides a grounding in financial mathematics and its simple applications. It introduces compound interest, the time value of money and discounted cashflow techniques which are fundamental building blocks for most actuarial work.

2. Please note that different answers may be obtained to those shown in these solutions depending on whether figures obtained from tables or from calculators are used in the calculations but candidates are not penalised for this. However, candidates may lose marks where excessive rounding has been used or where insufficient working is shown.

B. General comments on *student performance in this diet of the examination*

1. The comments that follow the questions concentrate on areas where candidates could have improved their performance. Where no comment is made, the question was generally answered well by most candidates. The examiners look most closely at the performance of the candidates close to the pass mark and the comments therefore often relate to those candidates.

2. Performance was of a similar standard to that of most recent examinations. As in previous examinations, the non-numerical questions were often answered poorly by marginal candidates.

C. Pass Mark

The Pass Mark for this exam was 60.

Solutions

Q1 (i)

$$i^{(4)} = 4\%$$

$$\left(1-\frac{d^{(12)}}{12}\right)^3 = v^{\frac{1}{4}}$$

$$= \left(1+\frac{i^{(4)}}{4}\right)^{-1}$$

$$\Rightarrow d^{(12)} = 12\left(1-\left(1+\frac{i^{(4)}}{4}\right)^{-\frac{1}{3}}\right) = 12(1-(1.01)^{-\frac{1}{3}})$$

$$\approx 3.9735\%$$ [2]

(ii)

$$1-\frac{d^{(12)}}{12} = e^{-\delta\frac{1}{12}}$$

$$\Rightarrow d^{(12)} = 12\left(1-e^{-0.05\frac{1}{12}}\right)$$

$$\approx 4.9896\%$$ [2]

(iii)

$$1-\frac{d^{(4)}}{4} = v^{\frac{1}{4}}$$

$$= \left(1-\frac{d^{(12)}}{12}\right)^3$$

$$\Rightarrow d^{(12)} = 12\left(1-\left(1-\frac{d^{(4)}}{4}\right)^{\frac{1}{3}}\right)$$

$$\approx 4.0134\%$$ [2]

[Total 6]

Generally well-answered.

Q2 Present value for 1st option:

$$7,800\, a_{\overline{16}|}^{(4)} = 7,800 \times 1.018559 \times 10.8378$$

$$= £86,103.52$$ [2]

Present value for 2nd option:

$$16,400 \times \left(v^2 + v^4 + ... + v^{16}\right)$$

$$16,400 v^2 \left(\frac{1-v^{16}}{1-v^2}\right)$$

$$16,400 \times 0.90703 \times \left(\frac{(1-0.45811)}{(1-0.90703)}\right)$$

$$= £86,702.94$$ [2]

(above uses factors from Formulae and Tables Book – exact answer is £86.702.16)

Therefore, 1st option is better for the borrower as the total value of the repayments is less than with the 2nd option. [1]

[Total 5]

> For the second option, a common mistake was to assume the annual rate of payment was £8,200. Some students omitted to give a final conclusion (and so did not actually answer the question).

Q3 Forward price of the contract is $K_0 = (S_0 - I)e^{\delta T} = (78 - I)e^{0.14 \times 1}$ [1]

where I is the present value of income during the term of the contract.

$$\Rightarrow I = 3.2\left(e^{-0.14\times\frac{3}{12}} + e^{-0.14\times\frac{9}{12}}\right) = 5.97098$$ [1]

$$\Rightarrow K_0 = \left(78 - 5.97098\right)e^{\,0.14} = 82.85309$$ [½]

Forward price when new contract issued at time *r* (1 month) is

$$K_r = \left(S_r - I^*\right)e^{\delta(T-r)} = (80 - I^*)e^{0.11\times\frac{11}{12}}$$ [1]

where I^* is the present value of income during the term of the contract.

$$\Rightarrow I^* = 3.2\left(e^{-0.11\times\frac{2}{12}} + e^{-0.11\times\frac{8}{12}}\right) = 6.115599$$ [1]

$$\Rightarrow K_{\frac{1}{12}} = \left(80 - 6.115599\right)e^{0.11\times\frac{11}{12}}$$

$$= 81.72297$$ [½]

Value of original contract $= \left(K_r - K_0\right)e^{-\delta(T-r)}$ [½]

$$= \left(81.72297 - 82.85309\right)e^{-0.11\times\frac{11}{12}}$$

$$= -1.02172 = -£1.02172$$ [1½]

(above uses the rounded forward prices shown – exact answer is −£1.02174)

[Total 7]

Reasonably well-answered although a common mistake was not to deal with the change in the interest rate correctly. There are quicker ways to answer the question but candidates who took a methodical approach such as that outlined above were able to maximise the marks for working even if they made calculation errors.

Q4 (i) $v(t) = e^{-\int_0^t \delta(s)ds}$

For $0 \le t \le 2$

$$v(t) = e^{-\int_0^t 0.04ds} = e^{-0.04t}$$ [2]

For $t > 2$

$$v(t) = v(2).e^{-\int_2^t (0.02+k.s)ds}$$

$$= e^{-0.08} \times e^{-\left[0.02s+\frac{1}{2}ks^2\right]_2^t}$$

$$= e^{-0.08} \times e^{-\left[\left(0.02t+\frac{1}{2}kt^2\right)-\left(0.04+2k\right)\right]}$$

$$= e^{-0.02t-\frac{1}{2}kt^2-0.04+2k}$$ [3]

(ii) To calculate maximum value of k:

Now PV of outlay (in £000s)

$$= 500 + 250\left(e^{-0.02} + e^{-0.04}\right)$$

$$= 985.247$$ [1]

At t =10, PV of sale proceeds

$$= 2000\, e^{-0.08}.e^{-(0.2+50k-0.04-2k)}$$

$$= 2000e^{-0.24-48k}$$ [1]

So, for DPP = 10 years, we need PV of sale proceeds ≥ PV of outlay

$$\Rightarrow 2000\, e^{-0.24-48k} \geq 985.247$$

$$\Rightarrow \ln\left(0.4926235\right) \leq -0.24 - 48k$$

$$\Rightarrow k \leq 0.00975 \text{ so maximum value is } 0.00975$$ [2]

[Total 9]

In part (i), it was important to give the required expression for $t \leq 2$ explicitly.

Unfortunately in part (ii), there was a typographical error in the question paper. The intention was to ask for the maximum value of *k* rather than the minimum and the solution above obtains this maximum value. The answer to the question actually on the paper is that the minimum value would be $k = -\infty$. Students who gave this answer were given full credit as were students who obtained the maximum value above. Marginal candidates did not appear to have been disadvantaged as such candidates had typically been unable to calculate the PV of the sale proceeds in terms of *k*.

Q5 (i) No, as with only a single asset, the spread of the asset proceeds would be less than the spread of the liability outgo (at times 7 and 11). [1]
Thus, the convexity of the assets would be less than the convexity of the liabilities and the third condition of immunisation could not be satisfied. [1]

(ii) Redington's first condition states that the PV of the assets should equal the PV of the liabilities $\left(\text{using } v = \dfrac{1}{1.055} = 0.94787\right)$ and working in £ millions:

$$V_A = 15.363v^{7.5} + 3.787v^{14.25} = 12.048$$ [1]

$V_L = 11v^7 + 8.084v^{11} = 12.048$ [1]

Allowing for rounding (using three decimal places), Redington's first condition applies. [½]

Redington's second condition states that the discounted mean term (DMT) of the assets should be equal to the DMT of the liabilities, which equivalently can be written as

$V_A' = V_L'$ (where in the calculations below the derivatives are with respect to the force of interest)

$V_A' = 15.363 \times 7.5v^{7.5} + 3.787 \times 14.25v^{14.25} = 102.28$. [1]

$V_L' = 11 \times 7v^7 + 8.084 \times 11v^{11} = 102.28$ [1]

Allowing for rounding (using 2 decimal places), Redington's second condition applies. [½]

Since the spread of asset proceeds exceeds the spread of liability outgo (as asset proceeds are received at times 7.5 and 14.25, whereas liability outgo is paid at times 7 and 11), the convexity of the assets is greater than the convexity of the liabilities.

Alternatively:

$V_A'' = 15.363 \times 7.5^2 v^{7.5} + 3.787 \times 14.25^2 v^{14.25} = 936.94$.

$V_L'' = 11 \times 7^2 v^7 + 8.084 \times 11^2 v^{11} = 913.32 < V_A''$.

[1½]

Thus, the third condition is also satisfied and the company is immunised against small changes in the rate of interest. [½]

[Total 9]

In part (i), candidates tended to score 0 or 2 marks depending on whether they recognised the problem with using a single asset. Part (ii) was answered well.

Q6 The investor's proceeds in £ millions at the time of purchase can be calculated as:

$$\text{PV}_{\text{in}} = 1.25 \times (1-0.35) a_{\overline{5}|}^{(12)} [1 + 1.042^5 v^5 + 1.042^{10} v^{10} + \ldots + 1.042^{30} v^{30}] v^{\frac{9}{12}}$$

$$+11.5 v^{35+\frac{9}{12}} \quad @ i = 8\% \text{ p.a.}$$

[3]

$$= 0.8125 \times a_{\overline{5}|}^{(12)} \left(\frac{1-r^7}{1-r} \right) v^{0.75} + 11.5 v^{35.75} \; @ i = 8\% \text{ p.a.}$$ [1½]

$= 0.8125 \times 4.1371$ [1]

$\times 4.35767$ [1]

$\times 0.94391$ [½]

$+ 11.5 \times 0.06384$ [1]

$= 14.56039$ [½]

where we have:

$$r = \left(\frac{1.042}{1.08} \right)^5 = 0.836026 ; \quad a_{\overline{5}|}^{(12)} = \frac{1-0.68058}{0.07721} = 4.1371$$

At the same time, the investor's costs (in millions) are:

$$\text{PV}_{\text{out}} = 5.8 + 0.85 v^{\frac{6}{12}} \; @ i = 8\% \text{ p.a.}$$

$$= 5.8 + 0.85 \times 0.96225 = 6.6179$$ [1½]

Thus, the investor's net proceeds (in millions) are given by:

$\text{NPV} = \text{PV}_{\text{in}} - \text{PV}_{\text{out}} = 14.5604 - 6.6179 = £7.9425m$ [1]

[Total 11]

This was a question where it was beneficial to use a methodical approach. Common errors included not allowing for the three month period since purchase and assuming the rent increases were 4.2% every five years.

Q7 (i) Let i = money rate of return

Then

$$9{,}800 = 400\, a_{\overline{20}|}^{(2)} + 10{,}500 v^{20} - 0.30 \times 400 v^{5/12}\, a_{\overline{20}|} - 0.40\,(10{,}500 - 9{,}800) v^{20 5/12}$$

[4]

Try $i = 3\%$

RHS = 400 × 1.007445 × 14.8775 + 10,500 × 0.55368 – 120 × 0.98776 × 14.8775 – 280 × 0.546898 = 9,892.37 [1½]

Try $i = 4\%$

RHS = 400 ×1.009902 × 13.5903 + 10,500 ×0.45639 – 120 × 0.983791 × 13.5903 – 280 × 0.448989 = 8551.92 [1]

Since 8551.92 < 9800 < 9892.37

then $3\% < i < 4\%$ [½]

(ii) We can find i from:

$$i = 0.03 + \frac{9892.37 - 9800}{9892.37 - 8551.92} \times 0.01$$

$= 0.0307$ i.e. $i = 3.07\%$ p.a. [1½]

If inflation = 2% p.a. = e, then i' = net real yield can be found from

$$1 + i' = \frac{1+i}{1+e} = \frac{1.0307}{1.02}$$

$\Rightarrow i'$ = net real yield = 1.05% p.a. [1½]

(iii) If tax were collected on 1 April instead of 1 June each year then tax payments would be brought forward which would increase the present value of these payments. [1]

This would decrease both the net money yield and the net real yield. [1]
[Total 12]

In terms of average mark, this was the worst answered question on the paper. Many candidates simplified part (a) to assume that taxes were paid at the same time as the coupon/redemption payments (ignoring the 5-month time lag and/or assuming income tax was paid half-yearly) and they lost marks accordingly. It was possible to get full marks on part (b) even if part (a) was answered incorrectly. Part (c) was very poorly answered even though the points required were straightforward.

Q8 (i) The TWRR for fund A and B results from the annual rates achieved for 2015 and 2016:

$$\text{TWRR}_A: \quad (1+i)^2 = 1.42 \times 1.03 \quad \Rightarrow \quad i = 20.94\%$$

$$\text{TWRR}_B: \quad (1+i)^2 = 1.36 \times 1.02 \quad \Rightarrow \quad i = 17.78\%$$

[3]

(ii) In order to calculate the MWRR, first we need to calculate the values of the funds at the beginning and at the end of 2016. Working in £m, we have for fund A and B where $F_{t,A}$ and $F_{t,B}$ are the fund values at the end of 2014 + t for Funds A and B respectively:

$$F_{1,A} = (1.5 + 0.3) \times 1.42 = 2.556 \quad \text{and}$$

$$F_{2,A} = (F_{1,A} + 1.7) \times 1.03 = 4.38368$$

$$F_{1,B} = (2.3 + 2) \times 1.36 = 5.848 \quad \text{and}$$

$$F_{2,B} = (F_{1,B} + 0.2) \times 1.02 = 6.1689\,6$$

[2]

Then the MWRR result from the EV at the end of 2016:

$$\text{MWRR}_A: (1.5 + 0.3) \times (1+i)^2 + 1.7 \times (1 \times i) = 4.38368$$

$$\therefore 1.8x^2 + 1.7x - 4.38368 = 0$$

$$\Rightarrow \quad x = 1.158229 \quad \Rightarrow \quad i_A = 15.823\%$$

[3]

$$\text{MWRR}_B: (2.3 + 2) \times (1+i)^2 + 0.2 \times (1+i) = 6.16896$$

$$\therefore 4.3x^2 + 0.2x - 6.16896 = 0$$

$$\Rightarrow \quad x = 1.174735 \quad \Rightarrow \quad i_B = 17.474\%$$

[3]

(iii) The TWRR is a more reliable indicator of the manager's performance since it is independent of the size of the amounts and the time at which investments are made … [½]
…both of which are outside the manager's control. [½]

In this case, manager A performed better than manager B for both 2015 and 2016 by achieving a higher TWRR for each of those years (i.e. $42\% > 36\%$ and $3\% > 2\%$). [1]

It should be noted that manager A had a worse MWRR for the 2 year period than manager B because manager A had so few funds invested during the best period for investment which was 2015/manager A received a large cashflow just before a period of poor performance. [1]

[Total 14]

> Many candidates failed to notice the quick way that part (i) could be solved although much of the extra working that they undertook was needed for part (ii) anyway.
> Part (iii) was poorly answered although other approaches to those given above could be used to gain full credit. It is important in this type of question to refer to the actual results obtained and the actual data given. Unsubstantiated answers to this part were given no credit.

Q9 (i) Let $p(t)$ = Price of t-year bond

$g_1 = i_1 = 0.071 = 7.100\%$ p.a. [½]

$p(2) = 5a_{\overline{2|}} + 100v^2 \ @ \ g_2 = 7.2\%$ p.a. [1]

$= 5 \times 1.8030185 + 100 \times 0.8701827$

$= 96.0334$ [1]

and $96.0334 = \dfrac{5}{1.071} + \dfrac{105}{(1+i_2)^2}$ [1]

$\Rightarrow \ i_2 = 7.203\%$ p.a. [1]

$p(3) = 5a_{\overline{3|}} + 100v^3 \ @ \ g_3 = 7.3\%$ p.a.

$= 5 \times 2.609998 + 100 \times 0.8094701$

$= 93.9970$ [1]

$\Rightarrow \ 93.9970 = \dfrac{5}{1.071} + \dfrac{5}{(1.07203)^2} + \dfrac{105}{(1+i_3)^3}$ [½]

$\Rightarrow \ i_3 = 7.307\%$ p.a. [1]

(ii) $f_0 = i_1 = 7.1\% = 7.100\%$ p.a. [½]

$(1+f_0)(1+f_1) = (1+i_2)^2$ [1]

$$\Rightarrow 1+f_1 = \frac{(1.07203)^2}{1.071}$$

$$\Rightarrow f_1 = 7.306\% \text{ p.a.}$$ [1]

(*Above answer is based on rounded answer for* i_2. *Exact answers is 7.305%*).

$$(1+i_2)^2(1+f_2) = (1+i_3)^3$$

$$\Rightarrow \quad 1+f_2 = \frac{(1.07307)^3}{(1.07203)^2}$$

$$\Rightarrow \quad f_2 = 7.515\% \text{ p.a.}$$ [1½]

(*Above answer is based on rounded answers for* i_2 *and* i_3. *Exact answers is 7.516%*).

(iii) The spot rate for a term is the geometric average of the forward rates making up that term. [1]
Since the spot rates increase with term, the forward rates must increase at a faster rate than the spot rates to ensure that the geometric average of the forward rates is itself increasing with term. [1]
[Total 13]

Many marginal candidates answered part (i) as if the gross redemption yields given were actually spot yields. Others assumed the price of the bonds all to be par. Part (ii) was answered well even by candidates who had struggled with part (i). The examiners recognised that part (iii) would stretch many candidates and indeed this part was found to be challenging.

Q10 (i) Value of annuity $= 20,000\, \ddot{a}_{\overline{1}|}^{(12)}\left(1+1.03v+1.03^2v^2+\ldots+1.03^{19}v^{19}\right)$ [2]

$$= 20,000 \times 1.037525 \times 0.93458 \times \left(\frac{1-\left(\frac{1.03}{1.07}\right)^{20}}{1-\frac{1.03}{1.07}}\right)$$ [1]

$= 19,393.4173 \times 14.26488$

$= £276,645.$ [1]

(above uses factors from Formulae and Tables Book – exact answer is £276,639)

(ii) Let S_5 = Accumulation of £1 after 5 years and let i_t = investment return for year *t*.

$$[\,E(S_5) = E\left(\prod_{t=1}^{5} 1+i_t\right)$$

$$= \prod_{t=1}^{5} E(1+i_t) \text{ using independence}$$

$$= \prod_{t=1}^{5} (1+E(i_t))\,]$$

Now $E(i_t) = 0.6\times 0.04 + 0.4\times 0.07$

$= 0.052$ for $t = 1, 2, \ldots 5$ [1]

$\Rightarrow$ Expected accumulation $= 200{,}000E(S_5)$

$= 200{,}000\times(1.052)^5$ [1]

$= 200{,}000\times 1.288483$

$= £257{,}696.60$ [1]

(iii) The variance of the accumulation is

$$200{,}000^2\times\left(E\left(S_5^2\right) - E\left(S_5\right)^2\right)$$ [1]

$$[\text{where } E\left(S_5^2\right) = E\left(\prod_{t=1}^{5}(1+i_t)^2\right)$$

$$= E\left(\prod_{t=1}^{5}\left(1+2i_t+i_t^2\right)\right)$$

$$= \prod_{t=1}^{5}\left(1+2E(i_t)+E(i_t^2)\right) \text{from independence}]$$

Now $E(i_t^2) = 0.6\times 0.04^2 + 0.4\times 0.07^2$

$= 0.00292$ for $t = 1, 2, \ldots 5$ [1]

Hence, $E(\mathrm{S}_5^2) = (1+2\times 0.052+0.00292)^5$

$= 1.661809$ [1]

$\Rightarrow$ Standard deviation of accumulated fund is

$$200,000\times\left(1.661809-1.288483^2\right)^{\frac{1}{2}}$$

= £8,051.23 [1]

(above uses factors from Formulae and Tables Book – exact answer is £8,051.74)

(iv) Note that $200,000\times(1.07)^4(1.04)$ = 272,645.57 < 276,639
and $200,000\times(1.07)^5$ = 280,510.35 > 276,639

Hence, the individual would require the annual return to be 7% p.a. for each of the 5 years in order to reach the required fund. [2]

The probability of this happening is

$(0.4)^5 = 0.01024$ [1]

[Total 14]

Parts (i) and (ii) were answered reasonably well although a few candidates appeared to be under time pressure if this was the last question to be attempted. Part (iii) was less well answered with many marginal candidates confusing $E(i_t^2)$ and $\mathrm{Var}(i_t)$. Part (iv) was generally only answered by the strongest candidates with many candidates incorrectly applying a lognormal distribution to the problem.

END OF EXAMINERS' REPORT