Applied Mathematical Finance I

Lecture 11: LIBOR Transition And New Risk-Free Rates

Vladimir Shangin

Vega Institute Foundation

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LIBOR Transition

- LIBOR has historically been the main interest rate benchmarks used in financial markets.
- Given that LIBORs were indicative rates which could be easily manipulated, the Financial Stability Board recommended developing alternative risk-free rates (RFRs) that are better suited as the reference rates for financial transactions.
- By now, new RFRs have been developed and adopted in all major economies
    - the US selected Treasuries repo financing rate called SOFR.
    - the Eurozone selected unsecured overnight rate called ESTR.
    - Russia selected unsecured overnight rate called Ruonia.

Daily Compounded Risk-Free Rate

- Current interest rate derivatives contracts reference daily compounded setting-in-arrears rates based on the corresponding overnight benchmark RFR.
- Consider a period $[T, T+\tau]$ and its partition

$$T = t_0 < t_1 < \cdots < t_n = T + \tau, \quad t_{i+1} - t_i = \tau_i,$$

  where $\{t_i\}_{i=0}^{n}$ is a set of business days in the period.
- Daily compounded rate $R(T, T+\tau)$ for that period is defined as

$$R(T, T+\tau) = \frac{1}{\tau}\left[\prod_{i=0}^{n-1}(1 + r_{t_i}\tau_i) - 1\right], \tag{1}$$

  where r_{t_i} is the overnight RFR fixing observed at t_i.

Recalling the Basics

- We now consider a continuous-time financial market with an instantaneous risk-free rate r.
- We assume that r is the rate paid on collateral and hence the only suitable discounting rate for (fully) collateralized transactions.
- The corresponding money-market account is

$$dB_t = r_t B_t \, dt, \quad B_0 = 1.$$

- The arbitrage-free price at time t of the risk-free zero-coupon bond with maturity $T \geq t$ is given by

$$p(t,T) = \mathbb{E}_t^{\mathbb{Q}} \left[\mathrm{e}^{-\int_t^T r_s \, ds} \right], \tag{2}$$

where $\mathbb{Q}$ is a risk-neutral measure.

Extended Zero-Coupon Bond

- We can extend the definition of zero-coupon bond price process $p(t, T)$ to times $t > T$ as follows. Using (2), we get

$$p(t, T) = \mathbb{E}_t^{\mathbb{Q}}\left[e^{\int_T^t r_s\,ds}\right] = e^{\int_T^t r_s\,ds} = \frac{B_t}{B_T} \tag{3}$$

since $\int_T^t r_s\,ds$ for $t > T$ is $\mathcal{F}_t$-measurable.

- In particular, we have that $p(t, 0) = B_t$, meaning that the money-market account can be viewed as a zero-coupon bond expiring immediately at $T = 0$.

- Extended T-bond can be seen as a self-financing strategy that consists of buying the zero-coupon bond with maturity T, and reinvesting the unit of cash received at T at the risk-free rate r from time T onwards.

Extended T-Forward Measure

- The extended zero-coupon bond price $p(t,T)$ is a valid numéraire since it is the value of a self-financing strategy and is strictly positive.
- We can then define the extended T-forward measure $\mathbb{Q}^T$ on $\mathcal{F}_t$ via specifying its Radon-Nikodym derivate

$$\left.\frac{d\mathbb{Q}^T}{d\mathbb{Q}}\right|_{\mathcal{F}_t} = \frac{p(t,T)}{p(0,T)\,B_t}.$$

- As opposed to the classic definition of forward measure, $\mathbb{Q}^T$-dynamics can now be defined for any time t, including times beyond maturity T.
- Extended T-forward measure is a hybrid measure that combines the classic T-forward measure up to the maturity time T with the risk-neutral money-market measure after T.

Continuously-Compounded Risk-Free Rate

- In our continuous-time setting, we approximate daily-compounded rate (1) as

$$R(T, T+\tau) = \frac{1}{\tau}\left[e^{\int_T^{T+\tau} r_t\,dt} - 1\right] \tag{4}$$

- Note that we can rewrite the expression for $R(T, T+\tau)$ in terms of the money-market account or extended T-bond as follows

$$R(T, T+\tau) = \frac{1}{\tau}\left[\frac{B_{T+\tau}}{B_T} - 1\right] = \frac{1}{\tau}\left[p(T+\tau, T) - 1\right]. \tag{5}$$

- In contrast to LIBORs, $R(T, T+\tau)$ is backward-looking in nature because one has to wait until the end of the accrual period to know the fixing value. In other words, $R(T, T+\tau)$ is $\mathcal{F}_{T+\tau}$-measurable.

Continuously-Compounded Forward Rate

- We define the backward-looking forward rate $R(t, T, T+\tau)$ at time t as the value of the fixed rate K such that the cashflow $\tau\,[R(T, T+\tau) - K]$ paid at $T+\tau$ has zero value at t.

- By no-arbitrage, we have

$$R(t, T, T+\tau) = \mathbb{E}_t^{T+\tau}\left[R(T, T+\tau)\right], \tag{6}$$

  where the expectation is taken under the extended forward measure $\mathbb{Q}^{T+\tau}$. In particular, $R(t, T, T+\tau), t \geq 0$, is a martingale under $\mathbb{Q}^{T+\tau}$.

- To compute the right-hand side of the above formula, we switch to the risk-neutral measure $\mathbb{Q}$

$$\left.\frac{d\mathbb{Q}}{d\mathbb{Q}^{T+\tau}}\right|_{\mathcal{F}_{T+\tau}} = B_{T+\tau} \cdot p(0, T+\tau).$$

Continuously-Compounded Forward Rate (continued)

- We have

$$\mathbb{E}_t^{T+\tau}\left[e^{\int_T^{T+\tau} r_t\,dt}\right] = \mathbb{E}_t^{\mathbb{Q}}\left[\frac{e^{\int_T^{T+\tau} r_t\,dt}}{B_{T+\tau}\cdot p(0,T+\tau)}\right]\cdot \mathbb{E}_t^{T+\tau}\left[B_{T+\tau}\cdot p(0,T+\tau)\right]$$
$$= \mathbb{E}_t^{\mathbb{Q}}\left[e^{-\int_0^T r_t\,dt}\right]\cdot\frac{B_t}{p(t,T+\tau)} = \frac{p(t,T)}{p(t,T+\tau)}. \tag{7}$$

- Therefore, we get

$$R(t,T,T+\tau) = \frac{1}{\tau}\left[\frac{p(t,T)}{p(t,T+\tau)} - 1\right]. \tag{8}$$

- Notice that this is the classic, simply-compounded, forward-rate formula, which thanks to the definition of extended zero-coupon bond price, holds for each time t, even those after $T+\tau$.

Extended Instantaneous Forward Rates

- Recall that when $t \leq T$ we can express T-bond price in terms of instantaneous forward rates as

$$p(t,T) = \mathrm{e}^{-\int_t^T f(t,s)\,ds}.$$

- When time t is within the accrual period, $[T, T+\tau]$ forward rate $R(t,T,T+\tau)$ "aggregates" realized values of RFRs up to time t and instantaneous forward rates from t to $T+\tau$

$$1 + \tau R(t,T,T+\tau) = \frac{\mathrm{e}^{\int_T^t r_s\,ds}}{p(t,T+\tau)} = \mathrm{e}^{\int_T^t r_s\,ds + \int_t^{T+\tau} f(t,s)\,ds}.$$

- Given that $f(s,s) = r_s$, it is natural to define $f(t,s) = r_s$ for $t > s$, so we can write for all values of t

$$1 + \tau R(t,T,T+\tau) = \mathrm{e}^{\int_T^{T+\tau} f(t,s)\,ds}.$$

Extended HJM

- We assume one-factor dynamics for instantaneous forward curve under the risk-neutral measure $\mathbb{Q}$

$$df(t,T) = \{\dots\}\, dt + \sigma(t,T)\mathbb{1}_{\{t \le T\}}\, dW_t^{\mathbb{Q}},$$

  where the drift term is uniquely determined by HJM drift condition, $W^{\mathbb{Q}}$ is a standard $\mathbb{Q}$-Wiener process, and indicator function is introduced because $f(t,T) = r_T, t \ge T$, according to our definition above.

- As in the classic HJM framework, the application of Itô's lemma together with Fubini's theorem leads to the following risk-neutral dynamics of zero-coupon bond prices

$$\frac{dp(t,T)}{p(t,T)} = r_t\, dt - \int_t^T \sigma(t,u)\mathbb{1}_{\{t \le u\}}\, du\, dW_t^{\mathbb{Q}}. \tag{9}$$

Extended HJM (continued)

- In particular, it follows from (9) that for $t > T$

$$\frac{dp(t,T)}{p(t,T)} = r_t \, dt \tag{10}$$

which is consistent with formula (3).

- Applying Itô's formula and using (8) and (9), we can derive the dynamics of forward rate $R(t,T,T+\tau)$ under extended forward measure $\mathbb{Q}^{T+\tau}$

$$\begin{aligned} dR(t,T,T+\tau) &= \left[R(t,T,T+\tau) + \frac{1}{\tau}\right] \int_T^{T+\tau} \sigma(t,u) \mathbb{1}_{\{t \le u\}} \, du \, dW_t^{T+\tau} \\ &= \left[R(t,T,T+\tau) + \frac{1}{\tau}\right] \int_{\max\{t,T\}}^{\max\{t,T+\tau\}} \sigma(t,u) \, du \, dW_t^{T+\tau}, \end{aligned} \tag{11}$$

where $W^{T+\tau}$ is a standard Wiener process under $\mathbb{Q}^{T+\tau}$.

Extended HJM (continued)

- Note that, in the accrual period $[T, T+\tau]$, the volatility of $R(t, T, T+\tau)$ decreases to zero and remains zero after $T+\tau$.
- This is inline with our expectations since compounded rate $R(T, T+\tau)$ should be less volatile as more RFR fixings are observed.
- As example, consider the case of a constant function $\sigma(t,u) \equiv \sigma$, which is equivalent to the Ho-Lee short-rate model. We have

$$dR(t, T, T+\tau) = [\tau R(t, T, T+\tau) + 1]\, \sigma\, g(t)\, dW_t^{T+\tau}, \tag{12}$$

where

$$g(t) = \frac{1}{\tau}\left[\max\{t, T+\tau\} - \max\{t, T\}\right] = \min\left\{\frac{(T+\tau-t)_+}{\tau}, 1\right\} \tag{13}$$

is the linear volatility decay function.

Classical LMM

- Let us now briefly recall the classical log-normal LIBOR Market Model.
- Given a tenor structure $T_0 < T_1 < \cdots < T_n$ and using the fact that forward LIBOR rate $F_t^i = F(t, T_{i-1}, T_i)$ is a martingale under (classical) T_i-forward measure, we postulate dynamics of each F_t^i under its native measure $\mathbb{Q}^{T_i}$ as

$$dF_t^i = F_t^i \, \sigma_i \, dW_t^{T_i}.$$

- To price exotic derivatives (for example via Monte Carlo techniques), one then needs to obtain the dynamics of a whole family of forward rates under a common probability measure which is typically a terminal measure $\mathbb{Q}^{T_n}$.

Classical LMM (continued)

- We showed that dynamics of $F^i, i < n$, when switching from $\mathbb{Q}^{T_i}$ to $\mathbb{Q}^{T_n}$, become

$$\frac{dF_t^i}{F_t^i} = -\sigma_i \cdot \sum_{k=i+1}^{n} \frac{\tau_k \cdot F_t^k \cdot \sigma_k \cdot \rho_{i,k}}{1 + \tau_k \cdot F_t^k} dt + \sigma_i \, dW_t^{T_n},$$

  where $\rho_{i,k}$ is the correlation between W^{T_i} and W^{T_k}.

- One of the drawbacks of LMM is that $\mathbb{Q}$-dynamics of forward LIBOR rates are not directly available. This is because we started by modelling a discrete forward rates without making any assumptions about short-rate r and the corresponding money-market account B_t.

- As a consequence, pricing of LIBOR futures contract in LMM is not straightforward as futures rate $f(t, T, T + \tau)$ is given by

$$f(t, T, T + \tau) = \mathbb{E}_t^{\mathbb{Q}} \left[L(T, T + \tau) \right], \tag{14}$$

  where $L(T, T + \tau)$ is the LIBOR fixing for tenor τ observed at T.

Extended LMM (FMM)

- Our aim now is to extend LMM to the case of continuously-compounded RFR rates. This extension is called generalized Forward Market Model (FMM).

- Again, we are given a tenor structure $0 = T_0 < T_1 < \cdots < T_n, \tau_i = T_i - T_{i-1}$. Being inspired by (12)-(13) and using the fact that $R_t^i = R(t, T_{i-1}, T_i)$ is a martingale under the extended forward measure $\mathbb{Q}^{T_i}$, we assume that

$$dR_t^i = R_t^i \sigma_i g_i(t)\, dW_t^{T_i}, \tag{15}$$

  where $g_i(t) = \min\left\{\frac{(T_i - t)_+}{\tau_i}, 1\right\}$ and $dW^{T_i}\, dW^{T_j} = \rho_{i,j}\, dt$.

- The main property that distinguishes FMM from LMM is its completeness in a sense that we can obtain forward rate dynamics not only under forward measure but also under the risk-neutral measure $\mathbb{Q}$.

Forward Rate Dynamics Under $\mathbb{Q}$

- The change of drift in (15) due to a change of measure from $\mathbb{Q}^{T_i}$ to $\mathbb{Q}$ is given by $d\langle \ln R^i, \ln Z\rangle_t$, where Z is given by

$$Z_t = \left.\frac{d\mathbb{Q}}{d\mathbb{Q}^{T_i}}\right|_{\mathcal{F}_t} = \frac{B_t\, p(0, T_i)}{p(t, T_i)}.$$

- We have

$$\begin{aligned} \ln \frac{B_t}{p(t, T_i)} &= \ln \frac{p(t, 0)}{p(t, T_i)} = \ln \prod_{j=1}^{i} \frac{p(t, T_{j-1})}{p(t, T_j)} \\ &= \ln \prod_{j=1}^{i} (1 + \tau_i R_t^i) = \sum_{j=1}^{i} \ln(1 + \tau_i R_t^i). \end{aligned} \tag{16}$$

Forward Rate Dynamics Under $\mathbb{Q}$ (continued)

- Therefore,

$$d\langle \ln R^i, \ln Z\rangle_t = d\ln R^i_t \, dZ_t = \sum_{j=1}^{i} d\ln R^i_t \, d\ln(1+\tau_j R^j_t)$$

$$= \sum_{j=1}^{i} \frac{1}{R^i_t} \frac{\tau_j}{1+\tau_j R^j_t} dR^i_t \, dR^j_t$$

$$= \sigma_i g_i(t) \sum_{j=1}^{i} \frac{\tau_j R^j_t \sigma_j g_j(t)}{1+\tau_j R^j_t} \rho_{i,j} \, dt.$$

- The $\mathbb{Q}$-dynamics of R^i_t then become

$$dR^i_t = \sigma_i g_i(t) \sum_{j=1}^{i} \frac{\tau_j R^j_t \sigma_j g_j(t)}{1+\tau_j R^j_t} \rho_{i,j} \, dt + R^i_t \sigma_i g_i(t) \, dW^{\mathbb{Q}}_t. \tag{17}$$

Valuation of RFR Caplet

- Standard RFR caplet with strike K pays $\tau_i \left(R(T_{i-1}, T_i) - K \right)_+$ at time T_i.
- Its price at time t is then given by

$$\mathrm{PV}_t = p(t, T_i)\, \mathbb{E}_t^{T_i} \left[\tau_i \left(R(T_{i-1}, T_i) - K \right)_+ \right]. \tag{18}$$

- Log-normal dynamics (15) leads to Black-like price expression

$$\mathrm{PV}_t = R_t^i \, \Phi \left(\frac{\ln(R_t^i / K) + \frac{1}{2} v}{\sqrt{v}} \right) - K \, \Phi \left(\frac{\ln(R_t^i / K) - \frac{1}{2} v}{\sqrt{v}} \right), \tag{19}$$

where Φ is the standard normal CDF and

$$v = \sigma_i^2 \int_t^{T_i} g_i(s)^2 \, ds = \sigma_i^2 \left[(T_{i-1} - t)_+ + \frac{1}{3} \frac{\left(T_i - \max\{t, T_{i-1}\} \right)^3}{\tau_i^2} \right].$$

Adjusted Variance

- Note that when $t \leq T_{i-1}$ the expression for variance v in (19) simplifies to

$$v = \sigma_i^2 \left(T_{i-1} + \frac{\tau_i}{3} \right).$$

- Alternatively, we can obtain similar result by assuming that

$$dr_t = \sigma_i \, dW_t. \tag{20}$$

- We approximate

$$R(T_{i-1}, T_i) = \frac{1}{\tau_i} \left[\mathrm{e}^{\int_{T_{i-1}}^{T_i} r_t \, dt} - 1 \right] \approx \frac{1}{\tau_i} \int_{T_{i-1}}^{T_i} r_t \, dt. \tag{21}$$

- The variance of $R(T_{i-1}, T_i)$ is then given by

$$\mathrm{Var}(R(T_{i-1}, T_i)) = \mathbb{E} \left[\left(\frac{1}{\tau_i} \int_{T_{i-1}}^{T_i} \sigma_i \, W_t \, dt \right)^2 \right] = \sigma_i^2 \left(T_{i-1} + \frac{\tau_i}{3} \right).$$