Relatively growth optimal investment strategies in a market model with competition

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Abstract

We consider a game-theoretic model of a market where investors compete for payoffs yielded by several assets. The main result consists in a proof of existence and uniqueness of a strategy, called relatively growth optimal, such that the logarithm of the share of its wealth in the total wealth of the market is a submartingale for any strategies of the other investors. It is also shown that this strategy is asymptotically optimal in the sense that it achieves the maximal capital growth rate when compared to competing strategies. Based on the obtained results, we study the asymptotic structure of the market when all the investors use the relatively growth optimal strategy.

Keywords: market competition, relative optimality, growth optimal strategies, martingale convergence.

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1 Introduction

Growth optimal strategies are a well-studied topic in mathematical finance. However, the majority of models in the literature assume exogenously specified returns of assets and consider models with a single investor who "plays against the market". In this setting growth optimal strategies arise as solutions of optimization problems (see [1, 7, 11, 14]). In the present paper we study growth optimality of investment strategies from a different perspective. We consider a stochastic game-theoretic model of a market, where several players (investors) compete for random payoffs yielded by several assets at discrete moments of time on the infinite time interval. The payoffs are divided between the investors proportionally to invested amounts

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of capital, which are chosen by the investors simultaneously and independently. As a result, the profit or loss of one investor depends not only on the realized payoffs, but also on the actions of the competitors. In our model we will assume that the assets are short-lived in the sense that they exist for one period of time, make payoffs, and then reappear again, but cannot be traded between investors. Thus, they can be viewed more as some short-term investment projects rather than, e.g., common stock.

The goal of the paper is to identify an investment strategy, called relatively growth optimal, such that the logarithm of the relative wealth of an investor who uses it is a submartingale no matter what the strategies of the other investors are (by relative wealth we mean the share of wealth of one investor in the total wealth of the market). In models without competition, it is well-known that the submartingale property implies various asymptotic optimality properties for the rate of growth of wealth (see, e.g., [1, 11]). Results of a similar nature turns out to be true in our model as well, though their proofs use different ideas. In particular, we show that the relative wealth of an investor who uses the relatively growth optimal strategy stays bounded away from zero with probability one. We also show that if the representative strategy of the other investors is asymptotically different, then such an investor will dominate in the market – the corresponding share of wealth will tend to one. In addition, the relatively growth optimal strategy maximizes the capital growth rate, as well as maximizes the expected logarithmic utility of the wealth of an investor who uses it compared to the wealth of competitors. Moreover, this strategy forms a symmetric Nash equilibrium in a game where all the investors maximize their expected relative wealth.

Our model generalizes the model proposed by Amir, Evstigneev, and Schenk-Hoppé in [3], who also studied asymptotically optimal strategies in a market with competition consisting of short-lived assets. However, they assumed that investors reinvest all their wealth each moment of time, while in our model investors can choose what part of their wealth to reinvest and what part to keep. Essentially, the possibility to keep a part of wealth can be viewed as an investment in a risk-free asset with rate of return not depending on actions of investors. Since risk-free assets are included in the majority of models in mathematical finance, our paper presents a valuable generalization of [3] from the modeling point of view. Mathematically, it is not a trivial extension of the model of [3] (e.g. cannot be reduced to it by discounting) and, in particular leads to a more difficult construction of the optimal strategy. It also opens a series of new questions regarding the asymptotic behavior of the absolute wealth of investors that do not arise in the model where the whole wealth is reinvested.

Our paper can be reckoned among the group of papers that study long-run performance of investment strategies from the point of view of evolutionary dynamics, i.e. a market is considered as a population of various strategies which compete for capital. This approach can be used to analyze forces that determine long-run market dynamics through a process of natural selection of investment strategies, see e.g. the seminal paper [5], where a model with a discrete probability space was considered, and its further development in [6]. In later works this field is termed Evolutionary Finance. Recent literature reviews of it can be found in [8, 10].

The paper is organized as follows. In Section 2, we formulate the model and introduce the notion of relative growth optimality of investment strategies. In Section 3, we construct a relatively growth optimal strategy in an explicit form and show that it is unique in a certain sense. Sections 4 and 5 study further optimality properties of this strategy. Section 6 is devoted to analysis of the asymptotics of the absolute wealth of investors when they use the relatively growth optimal strategy.

2 The model

The market in the model includes $M \geq 2$ investors and $N \geq 1$ assets. The investors compete for payoffs yielded by the assets at discrete moments of time $t = 0, 1, \ldots$ The competition consists in that at each moment of time the investors decide (simultaneously and independently of each other) what part of their capital they invest in the assets. In return for investments, an investor receives a part of the payoff from each asset proportional to the invested capital in this asset. The payoffs are random and their amounts are not known beforehand. Not invested part of capital is kept in a risk-free account with return rate not affected by the competition (for example, cash).

The main objects of the model are random sequences specifying the assets payoffs and return on the risk-free account, functions specifying strategies of the investors, and random sequences expressing the wealth of the investors at each moment of time. All random variables and vectors will be defined on some probability space $(\Omega, \mathcal{F}, \mathrm{P})$ with a filtration $\mathbb{F} = (\mathcal{F}_t)_{t=0}^{\infty}$.

The assets payoffs are specified by random sequences $X_t^n \geq 0$, $t = 1, 2, \ldots$, which are $\mathbb{F}$-adapted (X_t^n are $\mathcal{F}_t$-measurable). It is assumed that X_t^n are given exogenously, i.e. do not depend on actions of the investors. Return on the risk-free account is specified by an exogenous $\mathbb{F}$-predictable sequence $\rho_t \geq 0$ (i.e. ρ_t is $\mathcal{F}_{t-1}$-measurable), such that $\rho_t - 1$ is equal to the interest rate between moments of time $t-1$ and t. It will be always assumed that

$$\rho_t + \sum_n X_t^n > 0 \quad \text{a.s. for all } t \geq 1. \tag{1}$$

As it will become clear below, if this inequality did not hold, we would be in a trivial situation when all the investors lose all of their wealth.

The wealth of investor m is specified by an adapted random sequence

$Y_t^m \geq 0$. We assume, without loss of generality, that the initial capital Y_0^m of each investor is non-random and strictly positive. The wealth Y_t^m at moments of time $t \geq 1$ depends on strategies used by the investors. In order to state an equation determining Y_t^m, we first define the notion of a strategy.

By a strategy of investor m we call a sequence of functions

$$\lambda_t^{m,n}(\omega, y_0, \dots, y_{t-1}) \colon \Omega \times \mathbb{R}_+^{tM} \to [0,1], \qquad t \geq 1,$$

which are $\mathcal{F}_{t-1} \otimes \mathcal{B}(\mathbb{R}_+^{tM})$-measurable and for all $\omega, t, y_0, \dots, y_{t-1}$ satisfy the condition

$$\sum_n \lambda_t^{m,n}(\omega, y_0, \dots, y_{t-1}) \leq 1.$$

The value of $\lambda_t^{m,n}$ is equal to the proportion of investor m's wealth invested in asset n at time $t-1$. The arguments $y_s = (y_s^1, \dots, y_s^M) \in \mathbb{R}_+^M$, $s \leq t-1$, correspond to possible amounts of wealth of the investors at past moments of time. The measurability in ω with respect to $\mathcal{F}_{t-1}$ means that future payoffs are not known to the investors at the moment when they decide upon invested proportions of wealth. We assume that the investors make their choice of $\lambda_t^{m,n}$ independently of each other and simultaneously, which results in that the functions $\lambda_t^{m,n}$ do not depend on the strategies of the other investors.

For given initial wealths Y_0^m and strategies λ^m of the investors, we define the wealth sequence of investor m at moments of time $t \geq 1$ by the relation

$$Y_t^m = \rho_t \Big(1 - \sum_n \lambda_t^{m,n}(\bar{Y}_t)\Big) Y_{t-1}^m + \sum_n \frac{\lambda_t^{m,n}(\bar{Y}_t) Y_{t-1}^m}{\sum_k \lambda_t^{k,n}(\bar{Y}_t) Y_{t-1}^k} X_t^n, \qquad t \geq 1, \quad (2)$$

where $\bar{Y}_t = (Y_0, \dots, Y_{t-1})$. For brevity, the argument ω will be omitted where it does not lead to confusion. In the case of the indeterminacy $0/0$ in the second sum (which occurs if no one invests in asset n), the corresponding term is considered to be equal to zero.

The first term in the right-hand side of (2) represents the part of wealth not invested in the assets, but put in the risk-free account. As usual, the idea that it is free of risk is expressed by that ρ_t is $\mathcal{F}_{t-1}$-measurable, and, hence, known at the moment when λ_t are chosen. Note that the return on the risk-free account does not depend on the strategies of the other investors. However, we do not exclude the possibility that ρ_t may be zero.

The second term in (2) is the payoff received from the assets in exchange for invested capital; namely, the full payoff of each asset is split between the investors proportionally to the amounts of capital $\lambda_t^{k,n}(\bar{Y}_t) Y_{t-1}^k$ they invested in this asset (see Remark 1 for a justification of such a distribution of payoffs).

The wealth equation (2) is the main equation of our model and it describes the evolution of the market. But we will be mainly interested not in just the absolute wealth Y_t^m, but in the relative wealth, i.e. the proportion of wealth of one investor in the total wealth of all the investors. The total wealth is defined by

$$W_t = \sum_m Y_t^m,$$

and the relative wealth of investor m is defined by

$$r_t^m = \frac{Y_t^m}{W_t}$$

(when $W_t = 0$, we set $r_t^m = 0$).

Definition. We will call a strategy λ^m of investor m *relatively growth optimal* if for any initial capital $Y_0^m > 0$ and any initial capitals $Y_0^k \geq 0$ and strategies λ^k of the other investors $k \neq m$,

$$\ln r_t^m \text{ is a submartingale}^1.$$

Observe that if a strategy is relatively growth optimal, then also r_t^m is a submartingale by Jensen's inequality. As a corollary, it is not hard to see that a strategy profile in which every investor uses a relatively growth optimal strategy is a Nash equilibrium in the game where investors maximize $\mathrm{E} r_t^m$ at a fixed moment of time t. This follows from that if the strategies of investors $k \neq m$ are relatively growth optimal, then $r_t^m = 1 - \sum_{k \neq m} r_t^k$ is a supermartingale. As we will show in Section 3, a relatively growth optimal strategy is essentially unique, and so when every investor uses it, their relative wealth will remain constant.

Another simple corollary from the relative growth optimality is that such a strategy is a *survival strategy*:

$$\inf_{t \geq 0} r_t^m > 0 \text{ a.s.},$$

i.e. the relative wealth of an investor who follows it always stays bounded away from zero (we use the terminology of [3]; note that in, for example, [5], the term "survival" has a slightly different meaning). This fact readily follows from that $\ln r_t^m$ is a non-positive submartingale, and hence it has a finite limit $l = \lim_{t\to\infty} \ln r_t^m$ a.s. Therefore, $\lim_{t\to\infty} r_t^m = e^l > 0$ a.s.

Remark 1 (On the proportional distribution of payoffs and asset prices)**.** The distribution of payoffs proportionally to invested amounts of capital can be justified by introducing asset prices which are determined endogenously through the market clearing condition, i.e. the full cost (capitalization) of each asset should be equal to the amount of capital invested in it.

[1]Results from the theory of discrete-time martingales used in this paper can be found in, e.g., Chapter 7 of [15].

For instance, this approach was used in the papers [3, 5]. Without loss of generality, one can assume that each asset is in unit supply, and then the denominators $p_t^n = \sum_k \lambda_t^{k,n}(\bar{Y}_t)Y_{t-1}^k$ in the right-hand side of (2) become the prices of the assets at time $t-1$. It is worth emphasizing that in such a model the assets should be interpreted as *short-lived*, i.e. they exist for one period of time, yield payoffs, and then are identically re-born at the beginning of the next period. A major difference with standard models of stock markets is that investors cannot receive profit (or loss) from selling assets they hold.

Remark 2 (Relation to the Amir–Evstigneev–Schenk-Hoppé model)**.** Our model generalizes the model of Amir, Evstigneev, and Schenk-Hoppé [3], where it was assumed that investors reinvest their whole wealth in each time period. That is, the model of [3] can be obtained as a particular case of our model by taking $\rho_t \equiv 0$, so that it is never reasonable to reinvest not the whole wealth.

However, despite similarity, construction of the optimal strategy in our model turns out to be more difficult. In particular, the optimal strategy in [3] is *basic* in the sense that its investment proportions $\lambda_t^n(\omega)$ do not depend on current and past amounts of wealth Y_s^m, $s \le t$, while, as we will see in the next section, in our model they do depend on current wealth. Moreover, our model opens interesting questions about the asymptotic behavior of the total wealth of investors, which do not arise in the model of [3], since there the total wealth at each moment of time is simply equal to the current total payoff $\sum_n X_t^n$. We consider these questions in Section 6.

It is also worth mentioning that another model of this kind with short-lived assets and a risk-free asset was considered in the report [4], where the existence of a survival strategy was established, and also shown that all basic survival strategies are asymptotically equal. However, in that model asset payoffs depend on "money supply" (amount of capital not invested in assets) in a special way, which allows to reduce that model to the one of the paper [3].

3 Existence and uniqueness of a relatively growth optimal strategy

Let us first introduce some auxiliary notation and definitions.

We will use the following convenient notation for vectors. If $x, y \in \mathbb{R}^N$, we will denote their scalar product by $xy = \sum_n x^n y^n$, the L^1-norm by $|x| = \sum_n |x^n|$, and the L^2-norm by $\|x\| = \sqrt{xx}$. If $f\colon \mathbb{R} \to \mathbb{R}$ is a function, then $f(x)$ denotes the vector $(f(x^1), \dots, f(x^N))$. By $a \vee b$ we will denote the maximum of variables a, b, and by $a \wedge b$ the minimum.

Let us recall a usual notion of a realization of a strategy, which will be used below. If λ^m are strategies of investors and Y_0^m are their initial capitals, which define the wealth sequences Y_t^m, then by the *realization* of the strategy of investor m in this market we will call the predictable sequence $\lambda_t^m(\omega) = \lambda_t^m(\omega, Y_0, \dots, Y_{t-1}(\omega))$. We will denote it by the same symbol λ_t^m as the strategy, where it does not lead to confusion. When it is necessary to emphasize that a realization depends on initial capitals and strategies of investors we will use the notation $\lambda_t^m(\Lambda, Y_0)$, where $\Lambda = (\lambda^1, \dots, \lambda^M)$ stands for a strategy profile.

Let us also introduce the notion of equality of strategies that will be used to state that the relatively growth optimal strategy is unique. Suppose $\tau(\omega; \Lambda, Y_0)$ denotes a family of random variables, i.e. for any fixed vector of initial capitals Y_0 and strategies $\Lambda = (\lambda^1, \dots, \lambda^M)$, the function $\omega \mapsto \tau(\omega; \Lambda, Y_0)$ is $\mathcal{F}$-measurable (and may assume the value $+\infty$). We will say that two strategies λ^m and $\widetilde{\lambda}^m$ of investor m are *equal in realization until* τ if for any vector of initial capitals $Y_0 \geq 0$ with $Y_0^m > 0$ and any strategies of the other investors λ^k, $k \neq m$, we have the equality for realizations

$$(\lambda_t^m(\omega; \Lambda, Y_0) - \widetilde{\lambda}_t^m(\omega; \widetilde{\Lambda}, Y_0))\mathrm{I}(t \leq \tau(\omega) \wedge \widetilde{\tau}(\omega)) = 0 \text{ a.s. for all } t \geq 1,$$

where $\Lambda = (\lambda^1, \dots, \lambda^m, \dots, \lambda^M)$, $\widetilde{\Lambda} = (\lambda^1, \dots, \widetilde{\lambda}^m, \dots, \lambda^M)$ are the strategy profiles which differ only in the strategy of investor m, and $\tau(\omega), \widetilde{\tau}(\omega)$ denote $\tau(\omega; \Lambda, Y_0), \tau(\omega; \widetilde{\Lambda}, Y_0)$, respectively.

For example, in Theorem 1 below, we will consider equality until $\tau(\Lambda, Y_0) = \inf\{t \geq 0 : r_t^m = 1\}$ – the first moment of time when the relative wealth of an investor reaches 1, which is a stopping time for any fixed Y_0 and Λ.

Now we can proceed to the construction of the relatively growth optimal strategy. Let $K_t(\omega, A) \colon \Omega \times \mathcal{B}(\mathbb{R}_+^N) \to [0, 1]$ denote the regular conditional distribution of the payoff vector $X_t = (X_t^1, \dots, X_t^N)$ with respect to $\mathcal{F}_{t-1}$, so that for each t and fixed ω the function $A \mapsto K_t(\omega, A)$ is a probability measure on the Borel σ-algebra $\mathcal{B}(\mathbb{R}_+^N)$, and for fixed $A \in \mathcal{B}(\mathbb{R}_+^N)$ the function $\omega \mapsto K_t(\omega, A)$ is a version of the conditional probability $\mathrm{P}(X_t \in A \mid \mathcal{F}_{t-1})$.

Define the sequence of sets $\Gamma_t \in \mathcal{F}_{t-1} \otimes \mathcal{B}(\mathbb{R}_+)$,

$$\Gamma_t = \left\{(\omega, c) : \int_{\mathbb{R}_+^N} \frac{c\rho_t(\omega)}{|x|} K_t(\omega, dx) > 1\right\}, \qquad t \geq 1,$$

with the following convention: $c\rho_t(\omega)/|x| = 0$ if $c\rho_t(\omega) = |x| = 0$ and $c\rho_t(\omega)/|x| = +\infty$ if $c\rho_t(\omega) > 0$ but $|x| = 0$.

The following lemma will play an auxiliary role in construction of the relatively growth optimal strategy.

Lemma 1. *For all $t \geq 1$ and $(\omega, c) \in \Gamma_t$, there exists a unique solution $z \in (0, c]$ of the equation*

$$\int_{\mathbb{R}_+^N} \frac{c\rho_t(\omega)}{z\rho_t(\omega) + |x|} K_t(\omega, dx) = 1. \tag{3}$$

The function $\zeta_t(\omega, c)$ defined to be equal to this solution on Γ_t and equal to zero outside Γ_t is $\mathcal{F}_{t-1} \otimes \mathcal{B}(\mathbb{R}_+)$-measurable.

Proof. The existence and uniqueness of the solution for each $(\omega, c) \in \Gamma_t$ is straightforward: the left-hand side of (3) is a continuous and strictly decreasing function in $z \in (0, c]$ which assumes a value greater than 1 for $z = 0$ and a value not greater than 1 for $z = c$.

To prove the measurability, consider the function $f \colon \Omega \times \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}$,

$$f(\omega, c, z) = \left(2 \wedge \int_{\mathbb{R}_+^N} \frac{c\rho_t(\omega)}{z\rho_t(\omega) + |x|} K_t(\omega, dx) - 1\right) \mathrm{I}((\omega, c) \in \Gamma_t).$$

Observe that f is a Carathéodory function, i.e. $\mathcal{F}_{t-1} \otimes \mathcal{B}(\mathbb{R}_+)$-measurable in (ω, c) and continuous in z. Then by Filippov's implicit function theorem (see, e.g., [2, Theorem 18.17]), the set-valued function

$$\phi(\omega, c) = \{z \in [0, c] : f(\omega, c, z) = 0\}$$

admits a measurable selector. Since ϕ on Γ_t is single-valued ($\phi(\omega, c) = \{\zeta_t(\omega, c)\}$), this implies that ζ_t is $\mathcal{F}_{t-1} \otimes \mathcal{B}(\mathbb{R}_+)$-measurable. □

Theorem 1. *The strategy $\widehat{\lambda}$ defined by the relation*

$$\widehat{\lambda}_t^n(\omega, y_0, \ldots, y_{t-1}) = \int_{\mathbb{R}_+^N} \frac{x^n}{\zeta_t(\omega, |y_{t-1}|)\rho_t(\omega) + |x|} K_t(\omega, dx) \tag{4}$$

is relatively growth optimal.

Moreover, $\widehat{\lambda}$ is the unique relatively growth optimal strategy in the sense that if λ is another strategy for investor m such that its relative wealth r_t^m is a submartingale for any initial capitals and strategies of the other investors, then $\widehat{\lambda}$ and λ are equal in realization until the time $\tau = \inf\{t \geq 0 : r_t^m = 1\}$.

Observe that the strategy $\widehat{\lambda}$ does not depend on the individual wealths of investors, but only on the total wealth $W_{t-1} = |Y_{t-1}|$ (and the conditional distribution of the payoffs). Also, from the definition of $\zeta_t(c)$, it is not hard to see that the proportion of not reinvested wealth is equal to $\zeta_t(W_{t-1})/W_{t-1}$. In particular, if $(\omega, W_{t-1}) \notin \Gamma_t$, then the whole wealth is reinvested, while if $(\omega, W_{t-1}) \in \Gamma_t$, then only a part of it is reinvested.

Also observe that if $\rho_t = 0$ a.s. for all t, then $\Gamma_t = \emptyset$, and hence $\zeta_t(\omega, c) = 0$. In this case we obtain the same optimal strategy as was found in [3].

Remark 3 (On the notion of uniqueness)**.** The notion of uniqueness we use may seem cumbersome, but, in general, it is not possible to say that the relatively growth optimal strategy is unique as a function $\lambda_t(\omega, y_0, \ldots, y_{t-1})$. First, it is easy to construct an example of a strategy which differs from $\widehat{\lambda}$ only on some impossible combinations of the arguments $\omega, y_0, \ldots, y_{t-1}$, i.e. such that $\mathrm{P}(Y_s(\omega) = y_s, s \leq t-1) = 0$ (it is easy to see that such combinations may exist; for example, suppose $|y_1| = |y_0| + 1$, but $|X_1| < 1$ a.s.). This observation means that we have to work with realizations of strategies, but not directly with functions $\lambda_t(\omega, y_0, \dots, y_{t-1})$.

Second, when the relative wealth of an investor becomes 1 (and the wealth of the competitors becomes zero), she may do anything and her relative wealth will always remain 1 (provided that she does not invest in a bad way losing all her wealth). Hence, it is not possible to speak about uniqueness after this moment.

Before we proceed to the proof of Theorem 1, let us state one auxiliary inequality that we will use (it generalizes Gibbs' inequality).

Lemma 2. *Suppose $\alpha, \beta \in \mathbb{R}_+^N$ are two vectors such that $|\alpha|, |\beta| \le 1$ and for each n it holds that if $\beta^n = 0$, then also $\alpha^n = 0$. Then*

$$\alpha(\ln\alpha - \ln\beta) \ge \frac{\|\alpha - \beta\|^2}{4} + |\alpha| - |\beta|, \tag{5}$$

where we put $\alpha^n(\ln\alpha^n - \ln\beta^n) = 0$ if $\alpha^n = 0$.

Proof. We follow the lines of the proof of Lemma 2 in [3], which establishes the above inequality in the case $|\alpha| = |\beta| = 1$. Using that $\ln x \le 2(\sqrt{x} - 1)$ for any $x > 0$, we obtain

$$\begin{aligned}\alpha(\ln\alpha - \ln\beta) &= -\sum_{n\,:\,\alpha^n \neq 0} \alpha^n \ln(\beta^n/\alpha^n) \ge 2\sum_n (\alpha^n - \sqrt{\alpha^n\beta^n}) \\ &= \sum_n (\sqrt{\alpha^n} - \sqrt{\beta^n})^2 + |\alpha| - |\beta|.\end{aligned}$$

Then we can use the inequality $(\sqrt{x} - \sqrt{y})^2 \ge (x-y)^2/4$, which is true for any $x, y \in [0,1]$, and obtain (5). □

Proof of Theorem 1. Without loss of generality, we will assume that the strategy $\widehat{\lambda}$ is used by investor 1. Let λ_t denote the realization of this strategy, and $\widetilde{\lambda}_t$ the realization of the representative strategy of the other investors, which we define as the following weighted sum of the realizations of their strategies:

$$\widetilde{\lambda}_t^n = \sum_{m\ge2} \frac{r_{t-1}^m}{1 - r_{t-1}^1} \lambda_t^{m,n}, \tag{6}$$

where $\widetilde{\lambda}_t = 0$ if $r_{t-1}^1 = 1$. By Y_t we will denote the wealth of investor 1, by $\widetilde{Y}_t := \sum_{m\ge2} Y_t^m$ the total wealth of the other investors, and by $r_t = Y_t/(Y_t + \widetilde{Y}_t)$ the relative wealth of investor 1. Then Y_t satisfies the following relation, which follows from (2):

$$Y_t = \rho_t(1 - |\lambda_t|)Y_{t-1} + \sum_n \frac{\lambda_t^n Y_{t-1}}{\lambda_t^n Y_{t-1} + \widetilde{\lambda}_t^n \widetilde{Y}_{t-1}} X_t^n. \tag{7}$$

Observe that from the definition of $\widehat{\lambda}$ and condition (1), it follows that for each t and almost all ω we have

$$K_t(\omega, \{x : \rho_t(\omega)(1 - |\lambda_t(\omega)|) + x\lambda_t(\omega) = 0\}) = 0. \tag{8}$$

In particular, this implies that $Y_t > 0$ a.s. for all t.

For convenience, introduce the predictable sequence of random vectors F_t with values in $\mathbb{R}_+^N$ which have the components

$$F_t^n = \frac{\lambda_t^n}{r_{t-1}\lambda_t^n + (1 - r_{t-1})\widetilde{\lambda}_t^n},$$

where $0/0 = 0$. Denoting the total wealth of the investors by $W_t = Y_t + \widetilde{Y}_t$, the equation (7) can be rewritten as

$$Y_t = \left(\rho_t(1 - |\lambda_t|) + \frac{F_t X_t}{W_{t-1}}\right) Y_{t-1}. \tag{9}$$

A similar equation is true for $\widetilde{Y}_t$, namely, $\widetilde{Y}_t = (\rho_t(1 - |\widetilde{\lambda}_t|) + \frac{\widetilde{F}_t X_t}{W_{t-1}})\widetilde{Y}_{t-1}$, where $\widetilde{F}_t^n = \widetilde{\lambda}_t^n/(r_{t-1}\lambda_t^n + (1 - r_{t-1})\widetilde{\lambda}_t^n)$. Using this, we obtain

$$W_t = \left(\rho_t(1 - r_{t-1}|\lambda_t| - (1 - r_{t-1})|\widetilde{\lambda}_t|) + \frac{|X_t|}{W_{t-1}}\right) W_{t-1}. \tag{10}$$

In this equation we used that $(r_{t-1}F_t^n + (1 - r_{r-1})\widetilde{F}_t^n)X_t^n = X_t^n$: on the set $\{\lambda_t^n > 0\}$ this is clear from the definition of F_t^n and $\widetilde{F}_t^n$, while on the set $\{\lambda_t^n = 0\}$ we have $X_t^n = 0$ a.s., which follows from the construction of $\widehat{\lambda}$.

Let $\zeta_t(\omega)$ denote the predictable sequence $\zeta_t(\omega, W_{t-1}(\omega))$. As follows from the definition of $\widehat{\lambda}$, we have $|\lambda_t| = 1 - \zeta_t/W_{t-1}$. Let $\widetilde{\zeta}_t = (1 - |\widetilde{\lambda}_t|)W_{t-1}$. Then dividing (9) by (10) we find that $\ln r_t - \ln r_{t-1} = f_t(X_t)$, where $f_t = f_t(\omega, x)$ is the $\mathcal{F}_{t-1} \otimes \mathcal{B}(\mathbb{R}_+^N)$-measurable function

$$f_t(x) = \ln\left(\frac{\zeta_t\rho_t + F_t x}{r_{t-1}\zeta_t\rho_t + (1 - r_{t-1})\widetilde{\zeta}_t\rho_t + |x|}\right)$$

(for brevity, the argument ω will be omitted). Note that (8) implies $\zeta_t\rho_t + F_tX_t > 0$ a.s., hence we can define the value of $f_t(\omega, x)$ for x such that $\zeta_t(\omega)\rho_t(\omega) + F_t(\omega)x = 0$ in an arbitrary way. It will be convenient to put $f_t(\omega, x) = 0$ for such x.

To show that $\ln r_t$ is a submartingale, it will be enough to show that $\int_{\mathbb{R}_+^N} f_t(x)K_t(dx) \geq 0$, i.e. $\mathrm{E}(f_t(X_t) \mid \mathcal{F}_{t-1}) \geq 0$. Indeed, then $\ln r_t$ will be a generalized submartingale[2], but since it is bounded from above (by 0), this will also imply that $\ln r_t$ is a usual submartingale.

[2]Recall that a sequence S_t is called a generalized submartingale if $\mathrm{E}|S_0| < \infty$ and $\mathrm{E}(S_t \mid \mathcal{F}_{t-1}) \geq S_{t-1}$ for all $t \geq 1$ (but not necessarily $\mathrm{E}|S_t| < \infty$). It is easy to show that if $S_t \leq C_t$ for all t with some integrable random variables C_t, then S_t are integrable, and hence the sequence S_t is a usual submartingale.

Suppose for some t, ω, a vector x is such that $\zeta_t(\omega)\rho_t(\omega) + F_t(\omega)x > 0$, and, for each n, the equality $F_t^n(\omega) = 0$ implies $x^n = 0$. Then we have the bound

$$
\begin{aligned}
f_t(x) &= \ln\left(\frac{\zeta_t\rho_t + F_t x}{\zeta_t\rho_t + |x|}\right) + \ln\left(\frac{\zeta_t\rho_t + |x|}{r_{t-1}\zeta_t\rho_t + (1 - r_{r-1})\widetilde{\zeta}_t\rho_t + |x|}\right) \\
&\geq \frac{x \ln F_t}{\zeta_t\rho_t + |x|} + \frac{(1 - r_{t-1})(\zeta_t - \widetilde{\zeta}_t)\rho_t}{\zeta_t\rho_t + |x|} := g_t(x) + h_t(x),
\end{aligned}
$$

where we put $x^n \ln F_t^n = 0$ if $F_t^n = 0$. Here, for the first term in the second line we used the concavity of the logarithm, and for the second term the inequality $\ln a \geq 1 - a^{-1}$.

For each t, we have $K_t(\{x : \zeta_t\rho_t + xF_t = 0\}) = 0$ a.s. by (8), and also $K_t(\{x^n = 0\}) = 1$ a.s. on the set $\{F_t^n = 0\}$ by the definition of $\widehat{\lambda}$. Hence

$$
\int_{\mathbb{R}_+^N} f_t(x)K_t(dx) \geq \int_{\mathbb{R}_+^N} g_t(x)K_t(dx) + \int_{\mathbb{R}_+^N} h_t(x)K_t(dx) := I_t^g + I_t^h. \quad (11)
$$

For the integral I_t^g, using Lemma 2, we find

$$
\begin{aligned}
I_t^g &= \lambda_t \ln F_t = \lambda_t(\ln \lambda_t - \ln(r_{t-1}\lambda_t + (1 - r_{t-1})\widetilde{\lambda}_t)) \\
&\geq \frac{1}{4}(1 - r_{t-1})^2 \|\lambda_t - \widetilde{\lambda}_t\|^2 + (1 - r_{t-1})(|\lambda_t| - |\widetilde{\lambda}_t|).
\end{aligned} \quad (12)
$$

For the integral I_t^h, on the set $\{\omega : (\omega, W_{t-1}(\omega)) \in \Gamma_t\}$ we can use the equality $\int_{\mathbb{R}_+^N} \rho_t W_{t-1}/(\zeta_t\rho_t + |x|)K_t(dx) = 1$, and on its complement the equality $\zeta_t = 0$ and inequality $\int_{\mathbb{R}_+^N} \rho_t W_{t-1}/|x| K_t(dx) \leq 1$, which result in

$$
I_t^h \geq (1 - r_{t-1})(\zeta_t - \widetilde{\zeta}_t)/W_{t-1} = (1 - r_{t-1})(|\widetilde{\lambda}_t| - |\lambda_t|). \quad (13)
$$

Relations (11)–(13) imply $\int_{\mathbb{R}_+^N} f_t(x)K_t(dx) \geq 0$, so $\ln r_t$ is a submartingale, which proves that $\widehat{\lambda}$ is relatively growth optimal.

In order to prove the statement about uniqueness, suppose there exists another strategy λ' (without loss of generality, assume this is a strategy for investor 1) whose relative wealth r_t^1 is a submartingale for any initial capitals and strategies of the other investors, and there exist strategies $\lambda^2, \ldots, \lambda^M$ and a vector of initial capitals $Y_0 \geq 0$ with $Y_0^1 > 0$ such that the realizations of $\widehat{\lambda}$ and λ' are different in the markets $\Lambda = (\widehat{\lambda}, \lambda^2, \ldots, \lambda^M)$ and $\Lambda' = (\lambda', \lambda^2, \ldots, \lambda^M)$, i.e. $\mathrm{P}(\widehat{\lambda}_t \neq \lambda'_t) > 0$ for some t, where $\widehat{\lambda}_t(\omega)$, $\lambda'_t(\omega)$ are the realizations of the strategy of the first investor in the markets Λ and Λ', respectively.

Consider the predictable stopping time $\sigma = \inf\{t \geq 0 : \widehat{\lambda}_t \neq \lambda'_t\}$ and define the new strategies $\widetilde{\lambda}^m$, $m \geq 2$, by

$$
\begin{aligned}
\widetilde{\lambda}_t^m(\omega, y_0, \ldots, y_{t-1}) &= \lambda_t^m(\omega, y_0, \ldots, y_{t-1})\mathrm{I}(t < \sigma(\omega)) \\
&\quad + \widehat{\lambda}_t(\omega, y_0, \ldots, y_{t-1})\mathrm{I}(t \geq \sigma(\omega)).
\end{aligned}
$$

Observe that since $\{\sigma \le t\} \in \mathcal{F}_{t-1}$, the strategies are well-defined (i.e. $\mathcal{F}_{t-1} \otimes \mathcal{B}(\mathbb{R}_+^{tM})$-measurable).

Let $r_t(\omega)$ denote the realization $r_t^1(\omega, \widetilde{\Lambda}, Y_0)$ of the relative wealth of investor 1 when the investors use the strategies $\widetilde{\Lambda} = (\lambda', \widetilde{\lambda}^2, \ldots, \widetilde{\lambda}^M)$. Then, on one hand, on the set $\{\sigma < \infty\}$ we have $\mathrm{E}(r_\sigma \mid \mathcal{F}_{\sigma-1}) \ge r_{\sigma-1}$ by the choice of λ', and hence $\mathrm{E}(\ln(1-r_\sigma) \mid \mathcal{F}_{\sigma-1}) \le \ln(1-r_{\sigma-1})$ by Jensen's inequality. On the other hand, $\mathrm{E}(\ln(1-r_\sigma) \mid \mathcal{F}_{\sigma-1}) \ge \ln(1-r_{\sigma-1})$ since investors $m \ge 2$ use the strategy $\widehat{\lambda}$ after σ. Hence, on the set $\{\sigma < \infty,\ r_{\sigma-1} < 1\}$ we have

$$0 = \mathrm{E}\biggl(\ln \frac{1-r_\sigma}{1-r_{\sigma-1}} \biggm| \mathcal{F}_{\sigma-1}\biggr) \ge \frac{1}{4} r_{\sigma-1}^2 \|\lambda'_\sigma - \widetilde{\lambda}_\sigma\|^2, \tag{14}$$

where the inequality can be obtained from (12) and (13). In this formula, λ'_σ and $\widetilde{\lambda}_\sigma$ are the realizations of the strategy λ' and the representative strategy of investors $m \ge 2$ in the market $\widetilde{\Lambda}$ at time σ. It is not hard to see that on the set $\{\sigma < \infty\}$ they are equal, respectively, to $\lambda_\sigma^1(Y_0, \Lambda')$ and $\lambda_\sigma^1(Y_0, \Lambda)$. Consequently, (14) and the choice of σ imply that if the set $\{\sigma < \infty, r_{\sigma-1} < 1\}$ had positive probability, then $r_{\sigma-1} = 0$ a.s. on it. But this is impossible since up to $\sigma - 1$ the realization of the strategy λ' coincides with the realization of $\widehat{\lambda}$, and hence its relative wealth stays positive. Thus, $\mathrm{P}(\sigma < \infty, r_{\sigma-1} < 1) = 0$, which proves the claimed uniqueness. $\square$

4 The dominance property of a relatively growth optimal strategy

The next short result will show that the relative wealth of the relatively growth optimal strategy tends to 1 (i.e. the strategy *dominates* in the market according to the terminology of [3]) on the set of outcomes ω for which the realization of the representative strategy $\widetilde{\lambda}$ of the other investors is asymptotically different from $\widehat{\lambda}$ in a certain sense ($\widetilde{\lambda}$ is defined in (6) above). This result can be viewed as asymptotic uniqueness of a survival strategy.

Theorem 2. *Suppose investor 1 uses the relatively growth optimal strategy. Let $\widehat{\lambda}_t$ denote its realization, $\widetilde{\lambda}_t$ denote the realization of the representative strategy of the other investors, and $\Omega' \in \mathcal{F}$ be the set*

$$\Omega' = \Bigl\{\omega : \sum_{t \ge 0} \|\widehat{\lambda}_t(\omega) - \widetilde{\lambda}_t(\omega)\|^2 = \infty\Bigr\}.$$

Then $r_t^1(\omega) \to 1$ a.s. on the set Ω'. In particular, $\|\widehat{\lambda}_t(\omega) - \widetilde{\lambda}_t(\omega)\| \to 0$ a.s. on the set $\{\omega : \lim_{t\to\infty} r_t^1(\omega) < 1\}$.

Proof. We will use the same notation as in the proof of Theorem 1. Since $\ln r_t$ is a non-positive submartingale, it converges a.s. and its compensator

C_t converges a.s. as well. From the proof of Theorem 1, it follows that

$$\Delta C_t := C_t - C_{t-1} = \int_{\mathbb{R}_+^N} f_t(x) K_t(dx) \geq \frac{1}{4}(1 - r_{t-1})^2 \|\widehat{\lambda}_t - \widetilde{\lambda}_t\|^2.$$

Then on the set Ω' we necessarily have $r_t \to 1$ a.s., since otherwise C_t would diverge. $\square$

5 Other optimality properties

Recall that in a market model without competition a numéraire portfolio is a strategy such that the ratio of the wealth of any other strategy to the wealth of this strategy is a supermartingale. The term "numéraire portfolio" was introduced in [13]; often it is also called a growth optimal strategy or a benchmark portfolio (see [9, 11, 14]). It is well-known that numéraire portfolios have a number of optimality properties: they maximize the asymptotic capital growth rate, maximize the expected logarithmic utility, minimize the time to reach a given capital level, etc. (see e.g. [1, 7] for results in discrete time, and [11] for results in a general semimartingale model, including a connection with the arbitrage theory). In this section we will show that the relatively growth optimal strategy in our model has similar properties.

By the asymptotic growth rate of the wealth Y_t of a strategy we will call $\limsup_{t\to\infty} \frac{1}{t} \ln Y_t$ (see, e.g., [12, Chapter 3.10]), and the t-step growth rate at time s can be defined as $\frac{1}{t}\mathrm{E}(\ln \frac{Y_{s+t}}{Y_s} \mid \mathcal{F}_s)$. These notions have especially clear interpretation in a model without competition when asset returns are i.i.d., so that the log-returns of a growth optimal strategy are i.i.d. as well, and the asymptotic growth rate and t-step growth rate are equal and non-random.

Theorem 3. *Suppose investor 1 uses the relatively growth optimal strategy and the other investors use arbitrary strategies. Then for any initial capitals $Y_0^m > 0$ the following claims are true.*
1) Investor 1 maximizes the asymptotic growth rate of wealth: for any m

$$\limsup_{t\to\infty} \frac{1}{t} \ln Y_t^1 \geq \limsup_{t\to\infty} \frac{1}{t} \ln Y_t^m \quad a.s. \tag{15}$$

2) Suppose there are only two investors ($M = 2$) and $\mathrm{E}|X_t| < \infty$ for all t. Then investor 1 maximizes the t-step growth rate of wealth: for any $t, s \geq 0$

$$\mathrm{E}\left(\ln \frac{Y_{s+t}^1}{Y_s^1} \,\middle|\, \mathcal{F}_s\right) \geq \mathrm{E}\left(\ln \frac{Y_{s+t}^2}{Y_s^2} \,\middle|\, \mathcal{F}_s\right) \quad a.s. \tag{16}$$

(In inequalities (15), (16), the both sides may assume the values $\pm\infty$; in (16) we put $\ln 0/0 = -\infty$.)

Proof. 1) As was noted above, if investor 1 uses the relatively growth optimal strategy, then $\inf_t r_t^1 > 0$ a.s., and hence $\sup_t W_t/Y_t^1 < \infty$. Therefore, $\sup_t Y_t^m/Y_t^1 < \infty$ for any m. This implies that for any sequence T_t such that $\lim_{t\to\infty} T_t = +\infty$ (in particular, for $T_t = t$) we have the inequality

$$\limsup_{t\to\infty} \frac{1}{T_t} \ln \frac{Y_t^m}{Y_t^1} \le 0 \quad \text{a.s.}$$

From here, one can obtain (15).

2) From the condition $\mathrm{E}|X_t| < \infty$, it follows that $\mathrm{E} \ln W_t < +\infty$ for all t. On the set $\{\omega : \mathrm{E}(\ln W_{s+t} \mid \mathcal{F}_s)(\omega) = -\infty\}$ we have $\mathrm{E}(\ln Y_{s+t}^1 \mid \mathcal{F}_s) = \mathrm{E}(\ln Y_{s+t}^2 \mid \mathcal{F}_s) = -\infty$, so inequality (16) holds on this set. On the set $\{\omega : \mathrm{E}(\ln W_{s+t} \mid \mathcal{F}_s)(\omega) > -\infty\}$, use the submartingale property of $\ln r_t^1$, which implies

$$\mathrm{E}\biggl(\ln \frac{Y_{s+t}^1}{Y_s^1} \biggm| \mathcal{F}_s\biggr) \ge \mathrm{E}\biggl(\ln \frac{W_{s+t}}{W_s} \biggm| \mathcal{F}_s\biggr).$$

Since $r_t^2 = 1 - r_t^1$ is a supermartingale, $\ln r_t^2$ is a generalized supermartingale, so, in a similar way,

$$\mathrm{E}\biggl(\ln \frac{Y_{s+t}^2}{Y_s^2} \biggm| \mathcal{F}_s\biggr) \le \mathrm{E}\biggl(\ln \frac{W_{s+t}}{W_s} \biggm| \mathcal{F}_s\biggr),$$

which proves (16). □

Remark 4. Note that the second claim of Theorem 3 generally does not hold in the case $M \ge 3$ (but still it can be used to compare a strategy of one investor with the representative strategy of the other investors, since such comparison can be reduced to the case of two investors). For example, it can happen that investor 1 uses the strategy $\widehat{\lambda}$, investor 2 acts in an unoptimal way, and investor 3 manages to find a strategy which is be better than $\widehat{\lambda}$.

A simple example can be constructed even for a non-random market. Let $M = 3$, $N = 1$, $Y_0^m = 1$ for $m = 1, 2, 3$, and $\rho_1 = 1$, $X_1 = 1$. Then we have $\widehat{\lambda}_1 = 1/3$. However, if $\lambda_1^1 = \widehat{\lambda}_1$ and $\lambda_1^2 = 1$, then the strategy $\lambda_1^3 = 0$ turns out to be better than λ_1^1 after one step: $Y_1^1 = 11/12$, but $Y_1^3 = 1$.

6 Growth of the total wealth when investors use the relatively growth optimal strategy

It is interesting to observe that using the relatively growth optimal strategy does not necessarily imply "favorable" asymptotics of the absolute wealth Y_t^m.

We begin with an example which shows that the wealth of an investor who uses the strategy $\widehat{\lambda}$ may vanish asymptotically, because the other investors use "bad" strategies such that the total wealth W_t vanishes. At the same time, there is a strategy the wealth of which does not vanish, but it is not relatively growth optimal. In the second part of this section, we investigate the case when all the investors use the relatively growth optimal strategy; in that case their wealth will normally grow.

Example. We consider a non-random model with two investors and one asset. Suppose the investors have the initial capitals $Y_0^1 = Y_0^2 = 1$ and use the strategies that invest the proportions $\lambda_{t+1}^1 = \frac{1}{2}$ and $\lambda_{t+1}^2 = \frac{1}{2} + \frac{1}{2t}$ in the asset in each time period. Suppose $\rho_t = 1$ for all t, and the (non-random) payoff sequence X_t is defined by

$$X_{t+1} = \frac{Y_t^1 + Y_t^2}{2}.$$

This equation together with the wealth equation (2) uniquely define the sequences Y_t^1, Y_t^2, and $W_t = Y_t^1 + Y_t^2$:

$$Y_{t+1}^1 = Y_t^1\left(\frac{1}{2} + \frac{X_{t+1}}{Y_t^1 + (1+\frac{1}{t})Y_t^2}\right), \quad Y_{t+1}^2 = Y_t^2\left(\frac{t-1}{2t} + \frac{(1+\frac{1}{t})X_{t+1}}{Y_t^1 + (1+\frac{1}{t})Y_t^2}\right),$$
$$W_{t+1} = W_t - \frac{Y_t^2}{2t}. \tag{17}$$

It is easy to see that we have $\zeta_t(W_{t-1}) = W_{t-1}/2$, and hence $\widehat{\lambda}_t = 1/2$. Thus, the strategy of the first investor is relatively growth optimal.

Let, as always, $r_t^m = Y_t^m/W_t$, $m = 1, 2$. According to Theorem 1, there exists the limit $r_\infty^2 = \lim_{t\to\infty} r_t^2 \in [0, 1)$. We will now show that $r_\infty^2 > 0$ and $W_\infty = 0$. From (17), we find

$$r_{t+1}^2 = r_t^2(1 - \alpha_t), \qquad \text{where } \alpha_t = \frac{r_t^2(1 - r_t^2)}{2t^2 + tr_t^2 - (r_t^2)^2}.$$

It is easy to see that $\alpha_t \in (0, 1)$ and $\alpha_t = O(t^{-2})$ as $t \to \infty$. Hence, there exists the limit $r_\infty^2 > 0$. Also, from (17) we have $W_{t+1} = W_t(1 - r_t^2/(2t))$. Since $\sum_t r_t^2/t = \infty$, we have $W_\infty = 0$.

However, there is a trivial strategy that guarantees that the wealth does not vanish: $\lambda_t = 0$ for all t. □

Now we turn to analysis of the situation when all the investors use the relatively growth optimal strategy. Obviously, in this case the relative wealth of the investors r_t^m will stay constant. Our goal will be to investigate the asymptotics of the total wealth W_t.

To avoid uninteresting complications, let us assume from now on that $\rho_t(\omega) > 0$ for all t, ω. Introduce the discounting sequence

$$D_t = \rho_1 \cdot \ldots \cdot \rho_t, \qquad D_0 = 1,$$

and denote by $W'_t = W_t/D_t$ the discounted wealth of the investors, and by $X'_t = X_t/D_t$ the discounted payoffs.

First, we will show that W'_t does not decrease in the sense that W'_t is a generalized submartingale. But then one can ask the question: will W'_t asymptotically grow to infinity (provided that $\sum_t |X_t| = \infty$)? The answer turns out to be quite interesting. We consider it only in the case when the discounted payoffs X'_t are i.i.d., and show that $W'_t \to \infty$ if X_t are truly random (not equal to a constant vector), while W'_t stays bounded if X'_t are non-random.

Theorem 4. *Suppose all the investors use the strategy $\widehat{\lambda}_t$. Then the following claims are true.*

1) We have $W'_t > 0$ a.s. for all t, the sequence $1/W'_t$ is a supermartingale, and W'_t is a generalized submartingale. In particular, there exists the limit $W'_\infty := \lim_{t\to\infty} W'_t \in (0, \infty]$ a.s.

2) Assume additionally that $\rho_t = \rho > 0$ for all t, where ρ is a constant; X'_t is a sequence of i.i.d. random vectors; and the filtration $\mathbb{F}$ is generated by X_t, i.e. $\mathcal{F}_t = \sigma(X_s, s \le t)$. If X_t are not equal to a constant vector a.s., then $W'_\infty = \infty$ a.s.; otherwise $W'_t = W_0 \vee |X_1|/\rho$ for all $t \ge 1$.

Proof. From (10) we find that W'_t satisfies the equation

$$W'_t = (1 - |\widehat{\lambda}_t|)W'_{t-1} + |X'_t| \tag{18}$$

(here $\widehat{\lambda}_t$ denotes the realization of the relatively growth optimal strategy).

To simplify the proofs, let us first show that they can be reduced to the case when $\rho_t = 1$ for all t. Indeed, consider the two markets: the first one is defined by the sequences $X_t^{(1)}$, $\rho_t^{(1)}$ and initial capitals $Y_0^{(1)} > 0$, while the second one by the sequences $X_t^{(2)} = X_t^{(1)}/D_t^{(1)}$, $\rho_t^{(2)} = 1$, and $Y_0^{(2)} = Y_0^{(1)}$. Assume all the investors in the both markets use the relatively growth optimal strategy. Denote the total wealth in these markets by $W_t^{(1)}$ and $W_t^{(2)}$, respectively. Let us show that $W_t^{(2)} = W_t^{(1)}/D_t^{(1)}$ a.s. for all t.

It is not hard to see that $\Gamma_t^{(2)} = \{(\omega, c/D_{t-1}^{(1)}(\omega)) : (\omega, c) \in \Gamma_t^{(1)}\}$, and $\zeta_t^{(2)}(\omega, c) = \zeta_t^{(1)}(\omega, cD_{t-1}^{(1)}(\omega))/D_{t-1}^{(1)}(\omega)$, from which we find that $\widehat{\lambda}_t^{(2)}(\omega, c) = \widehat{\lambda}_t^{(1)}(\omega, cD_{t-1}^{(1)}(\omega))$, where c stands for $|y_{t-1}|$ from the arguments of $\widehat{\lambda}_t$ as defined in (4) (recall that $\widehat{\lambda}_t$ does not actually depend on $y_0, \dots, y_{t-2}$). Using these relations and (18), the equality $W_t^{(2)} = W_t^{(1)}/D_t^{(1)}$ can be established by induction.

Thus, the discounted wealth in the original market specified by X_t and ρ_t will be the same as the wealth in the market specified by X'_t and $\rho'_t = 1$. So, from now on we may assume $\rho_t = 1$ and $X'_t = X_t$, $W'_t = W_t$ for all t.

Observe that from the definition of $\widehat{\lambda}$, it follows that on the set $\{\omega : |\widehat{\lambda}_t(\omega)| = 1\}$ we have $\mathrm{P}(|X_t| > 0 \mid \mathcal{F}_{t-1}) = 1$ a.s. Then equation (18) implies $W_t > 0$ a.s. for all $t \ge 0$.

Let $V_t = 1/W_t$. From (18), we also find

$$\frac{V_t}{V_{t-1}} = \frac{1}{1 - |\widehat{\lambda}_t| + V_{t-1}|X_t|} = \frac{W_{t-1}}{\zeta_t(W_{t-1}) + |X_t|},$$

where we used that $1 - |\widehat{\lambda}_t| = V_{t-1}\zeta_t(W_{t-1})$. By the choice of ζ_t, we have $\mathrm{E}(V_t/V_{t-1} \mid \mathcal{F}_{t-1}) \leq 1$, hence V_t is a generalized supermartingale, and, hence, a usual supermartingale since it is non-negative. This also implies that W_t is a generalized submartingale (via the identity $W_t = \exp(-\ln V_t)$ and Jensen's inequality). Moreover, since a non-negative supermartingale has a finite limit, there exists $V_\infty = \lim_t V_t \in [0, \infty)$ a.s., and consequently there exists $W_\infty \in (0, \infty]$ a.s. This finishes the proof of the first claim of the theorem.

To prove the second claim, we will need the following auxiliary result on convergence of positive supermartingales, which is a corollary from Proposition 7.1 in [11]. Suppose S_t is a strictly positive scalar supermartingale, and consider the generalized supermartingale Z_t defined by

$$\Delta Z_t := Z_t - Z_{t-1} = \frac{S_t}{S_{t-1}} - 1, \qquad Z_0 = 0.$$

Denote by A_t the compensator of Z_t. Let $h(x) = x^2 \wedge |x|$ and introduce the predictable sequence H_t by

$$\Delta H_t = \mathrm{E}(h(\Delta Z_t) \mid \mathcal{F}_{t-1}), \qquad H_0 = 0.$$

Then we have

$$\{\omega : \lim_{t\to\infty} S_t(\omega) = 0\} = \{\omega : \lim_{t\to\infty} (A_t(\omega) + H_t(\omega)) = \infty\} \text{ a.s.}$$

We will apply this result to $S_t = V_t$, so that

$$\Delta Z_t = \frac{W_{t-1}}{\zeta_t(W_{t-1}) + |X_t|} - 1.$$

Observe that since X_t are i.i.d. random vectors, the function $\zeta_t(\omega, c)$ and the conditional distribution $K_t(\omega, dx)$ can be chosen not depending on ω, t, hence we will write them simply as $\zeta(c)$ and $K(dx)$. Then $\Delta H_t = g(W_{t-1})$ with the function

$$g(c) = \int_{\mathbb{R}_+^N} h\left(\frac{c}{\zeta(c) + |x|} - 1\right) K(dx), \qquad c > 0.$$

It is not hard to check that $g(c)$ is continuous on $(0, \infty)$. Moreover, if X_t are non-constant (so $K(dx)$ is not concentrated at one point), then $g(c) > 0$ for all $c > 0$. Therefore, $g(c)$ is separated from zero on any compact set

$B \not\ni \{0\}$. So, on the set $\{\omega : \lim_t W_t(\omega) < \infty\}$ we have $H_\infty = \infty$ a.s., which implies $\mathrm{P}(\lim_t W_t < \infty) = 0$ in the case when X_t are non-constant.

When X_t are constant, $X_t \equiv X \in \mathbb{R}_+^N$, it is easy to see from the definition of ζ that

$$(1 - |\widehat{\lambda}_t|)W_{t-1} = \zeta(W_{t-1}) = (W_{t-1} - |X|)^+.$$

Then (18) implies $W_t = W_{t-1} \vee |X|$, and, hence, $W_t = W_0 \vee |X|$. □

Remark 5 (Randomness stimulates growth)**.** The proved result leads to an observation, which at first seems counter-intuitive: if X_t and $\widetilde{X}_t$ are two sequences of payoffs such that $X_t \geq \widetilde{X}_t$ for all t, it may happen that the wealth of investors will grow faster under the smaller sequence $\widetilde{X}_t$.

As an example, let $\rho_t \equiv 1$ and $X_t \equiv X \in \mathbb{R}_+^N \setminus \{0\}$ be the same constant vector, while $\widetilde{X}_t = X\xi_t$, where $\xi_t \in [0, 1]$ are i.i.d. non-constant random variables. Then, if all the investors use the relatively growth optimal strategy, under the sequence X_t the wealth becomes $W_0 \wedge |X|$ after $t = 1$ and stop growing, but it grows to infinity under $\widetilde{X}_t$.

This can be explained by that the presence of randomness in $\widetilde{X}_t$ prevents the investors from "betting too much".

7 Conclusion

We studied a model of a market where several investors compete for payoffs yielded by short-lived assets. The main result of the paper consists in proving that there exists (and is unique) an investment strategy – the relatively growth optimal strategy – such that the sequence of its relative wealth is a submartingale for any strategies of competing investors.

This strategy forms a symmetric Nash equilibrium when all the investors maximize their expected relative wealth. It is also a survival strategy in the sense that its relative wealth always stays separated from zero with probability one on the whole infinite time interval. Moreover, its relative wealth tends to 1 if the representative strategy of the other investors is asymptotically different from it. It is also shown that the relatively growth optimal strategy posses properties similar to growth optimal strategies (numéraires) in markets without competition: it maximizes the asymptotic and one-step growth rate of wealth, as well as maximizes the expected logarithmic utility of wealth compared to the competing investors.

Our paper generalizes the results of [3] to a market with a risk-free asset, which gives investors a possibility to reinvest only a part of their wealth. Inclusion of a risk-free asset in the model also leads to interesting analysis of the asymptotics of the absolute wealth of the investors. In particular, it turns out that the relative growth optimality (or survival) property of a strategy does not necessarily imply that its absolute wealth will grow if the competitors use "bad" strategies.

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