Vega Summer School 2023

Stochastic models of prediction games

Lecture 2

Mikhail Zhitlukhin

The limit of a discrete-time prediction game

A game with two random events

Assume given a filtered probability space $(\Omega, \mathscr{F}, \mathbb{F}, \mathrm{P})$.

- Consider a game with two random events and identify outcomes with an adapted sequence $X = (X_t)_{t=1}^{\infty}$, $X_t \in \{0, 1\}$.
- One round in this game corresponds to small clock time Δt and

  $$\mathrm{P}(X_t = 1 \,|\, \mathscr{F}_{t-\Delta t}) = p_t \Delta t, \qquad \mathrm{P}(X_t = 0 \,|\, \mathscr{F}_{t-\Delta t}) = 1 - p_t \Delta t,$$

  where p_t are $\mathscr{F}_{t-\Delta t}$-measurable random variables.
- The reward for participation in the game is $\widetilde{c}_t = c_t \Delta t$.

- Assume the strategies of players are $\lambda^m = (\lambda^{m1}, \lambda^{m2})$,

$$\lambda^{m1} = \alpha_t^m \Delta t, \qquad \lambda^{m2} = \beta_t^m - \alpha^m \Delta t,$$

where α_t^m, β_t^m are $\mathscr{F}_{t-\Delta t}$-measurable, and $\alpha_t^m \geqslant 0$, $\beta_t^m \in (0,1]$.

- Then the wealth dynamics equation reads

$$\frac{W_t^m}{W_{t-\Delta t}^m} = \left(\bar{\beta}_t + \frac{c_t \Delta t}{\overline{W}_{t-\Delta t}}\right)\left(\frac{\lambda_t^{m1}}{\bar{\lambda}_t^1} X_t + \frac{\lambda_t^{m2}}{\bar{\lambda}_t^2}(1 - X_t)\right) + 1 - \beta_t^m,$$

where

$$\bar{\lambda}_t^1 = \bar{\alpha}_t \Delta t, \qquad \bar{\lambda}_t^2 = \bar{\beta}_t - \bar{\alpha}_t \Delta t,$$

and

$$\bar{\alpha}_t = \sum_{m=1}^{M} \alpha_t^m R_{t-\Delta t}^m, \qquad \bar{\beta}_t = \sum_{m=1}^{M} \beta_t^m R_{t-\Delta t}^m.$$

The limit as $\Delta t \to 0$

Expanding the above equation in Δt, we obtain the following approximate relations.

- If event $\{X_t = 1\}$ occurs, then

$$\frac{W_t^m - W_{t-\Delta t}^m}{W_{t-\Delta t}^m} \approx \frac{\alpha_t^m \bar{\beta}_t}{\bar{\alpha}_t} - \beta_t^m.$$

- If event $\{X_t = 1\}$ occurs, then

$$\frac{W_t^m - W_{t-\Delta t}^m}{W_{t-\Delta t}^m} \approx \left(\frac{\beta_t^m c_t}{\bar{\beta}_t \overline{W}_{t-\Delta t}} + \frac{\bar{\alpha}_t \beta_t^m}{\bar{\beta}_t} - \alpha_t^m \right) \Delta t.$$

- Consequently, the dynamics of the limiting wealth processes $W^m = (W_t^m)_{t \geqslant 0}$ can defined by the equations

$$
\begin{aligned}
\frac{dW_t^m}{W_{t-}^m} &= \left(\frac{\beta_t^m c_t}{\bar{\beta}_t \overline{W}_t} + \frac{\bar{\alpha}_t \beta_t^m}{\bar{\beta}_t} - \alpha_t^m \right) dt + \frac{\alpha_t^m \bar{\beta}_t}{\bar{\alpha}_t^n} - \beta_t^m dY_t, \\
d\overline{W}_t &= c_t dt,
\end{aligned}
$$

where $Y = (Y_t)_{t \geqslant 0}$ is a simple point process with compensator $A_t = \int_0^t p_s ds$, which counts the number of events $\{X_t = 1\}$ occurred by time t.

Reminder: point processes

Definition. A simple point process Y is a RCLL (right-continuous with left limits) adapted process with values in $\mathbb{N} \cup \{0\}$, such that $Y_0 = 0$ and $\Delta Y_t \in \{0, 1\}$.

Proposition. There exists a predictable process $A = (A_t)_{t \geqslant 0}$, called the compensator of A, such that $Y_t - A_t$ is a local martingale.

Example. A Poisson process with constant intensity p is a point process with compensator $A_t = pt$.

The model of a prediction game for a point process

Definition. A prediction game for a point process consists of

- $(\Omega, \mathscr{F}, \mathbb{F}, \mathrm{P})$: a filtered probability space with continuos-time filtration,
- $Y = (Y_t)_{t \geqslant 0}$: a simple point process with compensator $A_t = \int_0^t p_s ds$, where $p = (p_t)_{t \geqslant 0}$ is a continuous adapted process such that $0 < \underline{p} \leqslant p_t \leqslant \overline{p}$.
- $c = (c_t)_{t=1}^{\infty}$: a positive continuous function such that $0 < \underline{c} \leqslant c_t \leqslant \overline{c}$.

Definition. A strategy of a player is a predictable process $\lambda_t = (\alpha_t, \beta_t)$, where $\alpha_t \geqslant 0$, and $\beta_t \in [0,1]$.

We will consider only strategies with bounded component α_t.

Definition. The wealth of the players is defined by the process

$$\frac{dW_t^m}{W_{t-}^m} = \left(\frac{\beta_t^m c_t}{\bar{\beta}_t \overline{W}_t} + \frac{\bar{\alpha}_t \beta_t^m}{\bar{\beta}_t} - \alpha_t^m \right) dt + \left(\frac{\alpha_t^m \bar{\beta}_t}{\bar{\alpha}_t} - \beta_t^m \right) dY_t,$$

where

$$\overline{W}_t = \sum_{m=1}^{M} W_t^m, \qquad \bar{\alpha}_t = \frac{1}{\overline{W}_{t-}} \sum_{m=1}^{M} \alpha_t^m W_{t-}^m, \qquad \bar{\beta}_t = \frac{1}{\overline{W}_{t-}} \sum_{m=1}^{M} \beta_t^m W_{t-}^m.$$

The solution of the above equation is understood in the integral form:

$$W_t^m = W_0^m + \int_0^t \left(\frac{\beta_s^m c_s}{\bar{\beta}_s \overline{W}_s} + \frac{\bar{\alpha}_s \beta_s^m}{\bar{\beta}_s} - \alpha_s^m \right) W_{s-}^m ds + \sum_{s \leqslant t} \left(\frac{\alpha_s^m \bar{\beta}_s}{\bar{\alpha}_s} - \beta_s^m \right) W_{s-}^m \, \mathrm{I}(\Delta Y_s \neq 0).$$

It can be shown that under our assumptions this equation has a unique solution.

Survival strategies

Existence of a survival strategy

Theorem. The strategy $\widehat{\lambda} = (\widehat{\alpha}, \widehat{\beta})$ with $\alpha_t = p_t$, $\beta_t = 1$ is survival.

Proof. Suppose player m uses $\widehat{\lambda}$. Let $Z_t = \ln R_t^m$, where $R_t^m = W_t^m / W_t$. We are going to show that Z_t is a submartingale.

- We find

$$dZ_t = \left(\frac{c_t}{\bar{W}_t} \left(\frac{1}{\bar{\beta}_t} - 1 \right) + \frac{\bar{\alpha}_t}{\bar{\beta}_t} - p_t \right) dt + \left(\frac{p_t \bar{\beta}_t}{\bar{\alpha}_t} - 1 \right) dY_t.$$

- Let $M_t = Y_t - \int_0^t p_s ds$. Then $dZ_t = a_t dt + b_t dM_t$, where

$$a_t = \frac{c_t}{\bar{W}_t} \left(\frac{1}{\bar{\beta}_t} - 1 \right) + \frac{\bar{\alpha}_t}{\bar{\beta}_t} + \frac{p_t^2 \bar{\beta}_t}{\bar{\alpha}_t} - 2p_t, \qquad b_t = \frac{p_t \bar{\beta}_t}{\bar{\alpha}_t} - 1.$$

- One can see that $a_t \geqslant 0$ because

$$a_t = \frac{c_t}{\overline{\overline{W}}_t} \left(\frac{1}{\bar{\bar{\beta}}_t} - 1 \right) + \frac{(\pi_t - p_t)^2}{\pi_t}.$$

  This implies that Z_t is a submartingale, so it has a finite limit.

Convergence of predictions

Introduce the following equivalence relation for functions $f, g \colon \mathbb{R}_+ \to \mathbb{R}$:

$$f \sim g \iff \int_0^\infty \theta(f_t - g_t)dt < \infty,$$

where

$$\theta(x) = x^2\,\mathrm{I}(|x| < 1) + |x|\,\mathrm{I}(|x| \geqslant 1).$$

Theorem 2. If some player uses the strategy $\widehat{\lambda}$, then $\pi \sim p$ a.s.

Proof. From the proof of the previous theorem, we have $\int_0^\infty a_t dt < \infty$, so

$$\int_0^\infty \frac{(\pi_t - p_y)^2}{\pi_t} dt < \infty.$$

The rest of the proof follows from the next lemma.

Lemma. Suppose $f(t)$ is bounded away from zero and $g(t)$ is non-negative and bounded. Then

$$f \sim g \text{ if and only if } \int_0^\infty \frac{(f_t - g_t)^2}{f_t}\, dt < \infty.$$

A necessary condition for survival

Theorem 3. Suppose $\inf_{t \geqslant 0}(c_t/\overline{W}_t) > 0$. Then for any survival strategy $\lambda = (\alpha, \beta)$ it holds that

$$\int_0^\infty (1 - \beta_t)dt < \infty \quad \text{and} \quad \alpha \sim p\beta.$$

Proof. Consider the strategy profile $\Lambda = (\widehat{\lambda}, \lambda)$.

- From the proof of Theorem 1, we obtain

$$\int_0^\infty \left(\frac{1}{\bar{\beta}_t} - 1 \right) dt < \infty.$$

  Since $\bar{\beta}_t = R_t^1 + R_t^2 \beta_t$, this implies

$$\int_0^\infty R_t^2 (1 - \beta_t)/\bar{\beta}_t dt < \infty.$$

  Therefore, if λ is survival, then $\int_0^\infty (1 - \beta_t) dt < \infty$.

- We have $\pi_t - p_t = R_t^2(\alpha_t - p_t \beta_t)/(R_t^1 + R_t^2 \beta_t)$. Survival implies that the trajectories of the process $R_t^2/(R_t^1 + R_t^2 \beta_t)$ are bounded from above and away from zero. Also, $\pi \sim p$ by Theorem 2. This implies $\alpha \sim p\beta$.