A continuous-time asset market game with short-lived assets

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Abstract

We consider a continuous-time game-theoretic model of an investment market with short-lived assets and endogenous asset prices. The first goal of the paper is to formulate a stochastic equation which determines wealth processes of investors and to provide conditions for the existence of its solution. The second goal is to show that there exists a strategy such that the logarithm of the relative wealth of an investor who uses it is a submartingale regardless of the strategies of the other investors, and the relative wealth of any other essentially different strategy vanishes asymptotically. This strategy can be considered as an optimal growth portfolio in the model.

Keywords: asset market game, relative growth optimal strategy, martingale convergence, evolutionary finance.

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1. Introduction

This paper proposes a dynamic game-theoretic model of an investment market – an *asset market game* – and study strategies that allow an investor to achieve faster growth of wealth compared to rival market participants. The model provides an outlook on growth optimal portfolios different from the well-known theory in a single-investor setting, which originated with Kelly [18] and Breiman [7] (see also [1, 16, 22] for a modern exposition of the subject). Our results belong to the strand of research on *evolutionary finance* – the field which studies financial markets from a point of view of evolutionary dynamics and investigates properties of investment strategies like survival, extinction, dominance, and how they affect the structure of a market. Reviews of recent results in this direction can be found in, e.g., [10, 13]. While the majority of models in evolutionary finance are discrete-time, the novelty and one of the goals of this paper consists in developing a continuous-time model.

The model considered here describes a market consisting of several assets and investors. The assets yield random payoffs which are divided between the investors proportionally to the number of shares of each asset held by an investor. One feature of the model, which makes it different from the classical optimal growth theory, is that the asset prices are determined endogenously by a short-run equilibrium of supply and demand and depend on the investors' strategies. As a result, the investor's wealth depends not only on their own strategies and realized assets' payoffs but also on strategies of the other investors in the market.

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One of the main results of the paper is a proof of the existence of a strategy such that the logarithm of the relative wealth of an investor who uses it is a submartingale, regardless of the strategies used by the other investors (by the relative wealth we mean the share of wealth of one investor in the total wealth of the market). In particular, we do not assume that investors are necessarily rational in the sense that their actions can be described as solutions of some well-posed optimization problems, and they need not be aware of strategies used by their rivals. Remarkably, the optimal strategy that we find needs only to know the current total market wealth and the probability distribution of future payoffs, but does not require the knowledge of the other investors' strategies or their individual wealth. Such a strategy can be attractive for possible applications, since quantitative information about individual market agents is always scarce. Besides this submartingale property, the strategy has other good characteristics (similar to those of growth optimal strategies in single-investor models), which will be also investigated in the paper.

The importance of these results consists in more than just the fact of existence of a "good" strategy – they also allow to describe the asymptotic structure of a market, i.e. the asymptotic distribution of wealth, the asset prices, and the representative strategy of all investors. We prove that if at least one investor uses the optimal strategy, then the relative wealth of the other investors, who use essentially different strategies, will vanish asymptotically and those investors will have a vanishing impact on the market. Consequently, the market becomes determined by investors who use the optimal strategy or strategies which are asymptotically close to it. Results of this type are well-known in the literature, beginning with the seminal paper of Blume and Easley [5]. It is worth noting that many of them do not employ standard game-theoretic concepts of a solution of a game, e.g. the Nash equilibrium, but rather seek for "unbeatable" or "winning" strategies (though sometimes one can show that an optimal strategy provides a Nash equilibrium for a particular payoff function). In some cases, this fact can be regarded as an advantage as it allows to model arbitrary behavior of market agents and does not rely on unobservable characteristics like individual utilities or beliefs.

In this paper we deal with a simplified market model and consider a market with only *short-lived* assets. Such assets can be viewed as short-term investment projects rather than, e.g., common stock – they are traded at time t, yield payoffs at the "next infinitesimal" moment of time, and then get replaced with new assets. Short-lived assets have no liquidation value, so investors can get profit (or loss) only from receiving asset payoffs and paying for buying new assets. Despite being a simplification of real stock markets, models with short-lived assets have been widely studied in the literature because they are more amenable to mathematical analysis and ideas developed for them may be transferred to advanced models (see a discussion in [10]).

This paper is tightly connected with paper [9], which considers the same model in discrete time. Regarding mathematical methods, both of the papers are based on the approach proposed by Amir et al. [4], which directly shows (in discrete time) that the logarithm of wealth of an investor who uses the optimal strategy is a submartingale regardless of the other investors' strategies (see also the paper of Amir et al. [3] where similar but technically more involved ideas were used for a model with long-lived assets). Then, using martingale convergence theorems, we can obtain results about the asymptotic structure of a market. This martingale approach is more general compared to methods used in earlier works, which were based on assumptions that payoff sequences and/or strategies are stationary (as in, e.g., [11, 12]). An essential difference of our model and the model of [4] (in addition to that we consider a continuous-time model) is that Amir et al. assume that market agents spend their whole wealth for purchasing assets in each time period, so the total market wealth is always equal the most recent total payoff of the assets. On the other hand, our model includes a risk-free asset (cash or a bank account with zero interest rate) that can be used by investors to store capital. This leads to more complicated wealth dynamics, but is necessary for consideration of a continuous-time model, where asset payoffs can be infinitesimal but yielded in a continuous way. Moreover, adding the possibility to store capital in cash opens interesting questions about the asymptotics of the total market wealth, which do not arise in models where the whole wealth is spend for purchasing assets with random payoffs. For example, as was observed in [9], greater uncertainty in asset payoffs may result in faster growth of investors' wealth – a fact which at first may seem counter-intuitive. In the present paper, we consider similar questions for the continuous-time model.

In evolutionary finance, there are few models with continuous time. One can mention the papers of Palczewski and Schenk-Hoppé [20, 21], in which a continuous-time model with long-lived assets is constructed. The paper [20] proves that the model can be obtained as a limit of discrete-time models, and [21] investigates questions of survival of investments strategies in it. However, their results are obtained only for time-independent strategies and under the assumption that cumulative dividend processes are pathwise absolutely continuous. In the present paper, we allow strategies to be time-dependent and asset payoffs to be represented by arbitrary processes. A continuous-time model with short-lived assets was also constructed in [24]. An essential limitation of that paper consists in the assumption that all investors spend the same proportion of their wealth for purchasing assets which is specified exogenously. This makes the mathematical analysis of the model considerably simpler compared to the present paper, both in showing that the wealth process is well-defined, and in construction of the optimal strategy.

The paper is organized as follows. In Section 2, we briefly describe a discrete-time model, which helps to explain the main ideas of the paper. The general model is formulated in Section 3. In Section 4, we define the notion of optimality of a strategy and construct a candidate optimal strategy. In Section 5, we formulate the main results, which state that this strategy is indeed optimal, and investigate some of its properties. Section 6 contains the proofs of the results. In the appendix, we formulate and prove several auxiliary facts about the Lebesgue decomposition and Lebesgue derivatives of non-decreasing random functions which are used in the paper.

2. Preliminary consideration: a discrete-time model

In this section, we describe the main ideas of the paper using a simple model with discrete time which avoids technical details of continuous time. Based on the discrete-time model, in Section 3 a general continuous-time model will be formulated. The model presented here is a slightly simplified version of the model from [9].

Let $(\Omega, \mathcal{F}, \mathrm{P})$ be a probability space with a filtration $\mathbb{F} = (\mathcal{F}_t)_{t=0}^{\infty}$. The model includes $M \geq 2$ investors and $N \geq 1$ assets which yield non-negative random payoffs at moments of time $t = 1, 2, \ldots$ The assets live for one period: they are purchased by the investors at time t, yield payoffs at $t+1$, and then the cycle repeats. The asset prices are determined endogenously by a short-run equilibrium of supply and demand. The supply of each asset is normalized to 1, and the demand depends on actions of the investors. The payoffs are specified in an exogenous way, i.e. do not depend on the investor's actions. Each investor receives a part of a payoff yielded by an asset which is proportional to the owned share of this asset.

The asset payoffs are specified by random sequences $A_t^n \geq 0$ adapted to the filtration. The wealth of investor m is described by an adapted random sequence $Y_t^m \geq 0$. The initial wealth Y_0^m of each investor is non-random and strictly positive. The wealth Y_t^m at subsequent moments of time $t \geq 1$ is determined by the investors' strategies and the asset payoffs.

A strategy of investor m is a plan according to which this investor allocates the available budget Y_t^m towards a purchase of assets. Such an allocation is specified by a sequence of vectors $l_t^m = (l_t^{m,1}, \dots, l_t^{m,N})$, where $l_t^{m,n}$ is a budget allocated towards a purchase of asset n at time $t-1$. At each moment of time, the vectors l_t^m are selected by the investors simultaneously and independently, so the model represents a simultaneous-move N-person dynamic game, and l_t^m represent the investors' actions. These actions may depend on a random outcome ω and current and past wealth of the investors, so we define a strategy $\boldsymbol{l}^m$ of investor m as a sequence of $\mathcal{F}_{t-1} \otimes \mathcal{B}(\mathbb{R}_+^{tM})$-measurable functions

$$\boldsymbol{l}_t^m(\omega, y_0, \dots, y_{t-1}) \colon \Omega \times \mathbb{R}_+^{tM} \to \mathbb{R}_+^N, \qquad t = 1, 2, \dots.$$

(We will use boldface letters to distinguish between strategies and their realizations, see below.) The arguments $y_s = (y_s^1, \dots, y_s^M) \in \mathbb{R}_+^M$, $s \leq t-1$, correspond to the wealth of the investors at the past moments of time. It is assumed that short sales are not allowed, so $l_t^{m,n} \geq 0$, and it is not possible to borrow money, so $\sum_n l_t^{m,n} \leq y_{t-1}^m$. The amount of wealth $y_{t-1}^m - \sum_n l_t^{m,n}$ is held in cash and carried forward to the next time period.

After selection of investment budgets l_t^m by the investors, the equilibrium asset prices p_{t-1}^n are determined from the market clearing condition that the aggregate demand for each asset is equal to the aggregate supply, which is assumed to be 1. At time $t-1$, investor m can buy $x_t^{m,n} = l_t^{m,n}/p_{t-1}^n$ units of asset n, so its price at time $t-1$ should be equal to the total amount of capital invested in this asset, $p_{t-1}^n = \sum_m l_t^{m,n}$. If $\sum_m l_t^{m,n} = 0$, i.e. no one buys asset n, we put $p_{t-1}^n = 0$ and $x_t^{m,n} = 0$ for all m.

Thus, investor m's portfolio between moments of time $t-1$ and t consists of $x_t^{m,n}$ units of asset n and $c_t^m := y_{t-1}^m - \sum_n l_t^{m,n}$ units of cash. At a moment of time t, the total payoff received by this investor from the assets in the portfolio is equal to $\sum_n x_t^{m,n} A_t^n$. In our model, the assets have no liquidation value, so the budgets used at time $t-1$ for buying assets are not returned to the investors. Consequently, investor m's wealth is described by the adapted sequence Y_t^m which is defined by the recursive relation

$$Y_t^m(\omega) = Y_{t-1}^m(\omega) - \sum_{n=1}^N l_t^{m,n}(\omega) + \sum_{n=1}^N \frac{l_t^{m,n}(\omega)}{\sum_k l_t^{k,n}(\omega)} A_t^n(\omega), \qquad t \geq 1, \tag{1}$$

where $l_t^{m,n}(\omega) = \boldsymbol{l}_t^{m,n}(\omega, Y_0, Y_1(\omega), \dots, Y_{t-1}(\omega))$ are the realizations of the investors' strategies, with $0/0 = 0$ in the right-hand side of (1).

Note that the investors' actions precede the asset prices, so they first "announce" the budgets they plan to allocate for buying the assets, and then the prices are adjusted to clear the market. This modeling approach is analogous to market games of Shapley–Shubik type. Its justification and details can be found in, e.g., Shapley and Shubik [23]. Also, one can see that the asset prices do not enter equation (1). They are needed to provide a financial interpretation of the equation, but we will not work with them directly.

We will be mainly interested in relative wealth of investors. For investor m, we define the relative wealth as the adapted sequence

$$r_t^m = \frac{Y_t^m}{\sum_k Y_t^k}.$$

Our goal will be to identify a strategy such that the relative wealth of an investor who uses it grows in the following sense: for any strategies of the other investors and any initial wealth, the sequence $\ln r_t^m$ is a submartingale (as a consequence, r_t^m will be a submartingale as well). Such a strategy will exhibit several asymptotic optimality properties, which we will consider in Sections 4 and 5.

3. The general model

In order to formulate a continuous-time counterpart of equation (1), observe that it can be written in the following form:

$$\Delta Y_t^m(\omega) = -\sum_{n=1}^{N} \Delta L_t^{m,n}(\omega) + \sum_{n=1}^{N} \frac{\Delta L_t^{m,n}(\omega)}{\sum_k \Delta L_t^{k,n}(\omega)} \Delta X_t^n(\omega), \tag{2}$$

where

$$L_t^{m,n}(\omega) = \sum_{s=1}^{t} l_s^{m,n}(\omega), \qquad X_t^n(\omega) = \sum_{s=1}^{t} A_s^n(\omega)$$

are, respectively, the process of the cumulative wealth invested by investor m in asset n and the cumulative payoff process of asset n. The symbol Δ denotes a one-step increment, e.g. $\Delta Y_t^m = Y_t^m - Y_{t-1}^m$.

The form of equation (2) suggests that an analogous model with continuous time can be obtained by considering continuous-time processes X_t, Y_t, L_t and replacing one-step increments with infinitesimal increments, e.g. ΔX_t with dX_t. Our next goal will be to define such a model properly. The model we are about to formulate includes the above discrete-time model as a particular case, but we do not investigate convergence of the discrete-time model to the general model.

Notation. We will work on a filtered probability space $(\Omega, \mathcal{F}, \mathbb{F}, \mathrm{P})$ with a continuous-time filtration $\mathbb{F} = (\mathcal{F}_t)_{t\in\mathbb{R}_+}$ satisfying the usual assumptions. By $\mathcal{P}$ we will denote the predictable σ-algebra on $\Omega \times \mathbb{R}_+$.

As usual, equalities and inequalities for random variables are assumed to hold with probability one. For random processes, an equality $X = Y$ is understood to hold up to P-indistinguishability, i.e. $\mathrm{P}(\exists\, t : X_t \neq Y_t) = 0$; in the same way we treat inequalities. Pathwise properties (continuity, monotonicity, etc.) are assumed to hold for all ω.

For vectors $x, y \in \mathbb{R}^N$, by $xy = \sum_n x^n y^n$ we denote the scalar product, by $|x| = \sum_n |x^n|$ the l_1-norm of a vector, and by $\|x\| = \sqrt{xx}$ the l_2-norm. For a scalar function $f\colon \mathbb{R} \to \mathbb{R}$ and a vector x the notation $f(x)$ means the application of the function to each coordinate of the vector, $f(x) = (f(x^1), \dots, f(x^N))$. If $x \in \mathbb{R}^{MN}$, we denote by x^m the vector $(x^{m,1}, \dots, x^{m,N}) \in \mathbb{R}^N$ and by $x^{\cdot,n}$ the vector $(x^{1,n}, \dots, x^{M,n}) \in \mathbb{R}^M$. The maximum of two numbers a, b is written as $a \vee b$, and the minimum as $a \wedge b$.

The notation $\xi \cdot G_t$ is used for the integral of a process ξ with respect to a process G. In what follows, all the integrators are non-decreasing càdlàg processes, so the integrals are understood in the pathwise Lebesgue-Stieltjes sense ($f \cdot G_t(\omega) = \int_0^t f_s(\omega) dG_s(\omega)$). If f, G are vector-valued, then $f \cdot G_t = \sum_n f^n \cdot G_t^n$.

3.1. Payoff processes and investment strategies

There are $N \geq 1$ assets yielding random payoffs which are distributed between $M \geq 2$ investors. The cumulative payoffs are represented by exogenous adapted non-decreasing càdlàg processes X_t with values in $\mathbb{R}_+^N$. Without loss of generality $X_0 = 0$.

A strategy of investor m is identified with a function $\boldsymbol{L}$ which represents the cumulative wealth invested in each asset and assuming values in $\mathbb{R}_+^N$. In order to specify how a strategy may depend on the past history of the market, let $(D, \mathcal{D}, (\mathcal{D}_t)_{t\geq 0})$ denote the filtered measurable space consisting of the space D of non-negative càdlàg functions $y\colon \mathbb{R}_+ \to \mathbb{R}_+^M$, the filtration $\mathcal{D}_t = \sigma(d_u, u \leq t)$, where d_u is the mapping $d_u(y) = y_u$ for $y \in D$, and $\mathcal{D} = \bigvee_{t\geq 0} \mathcal{D}_t$. Elements y of the space D represent possible paths of the wealth processes of the investors (which are yet to be defined) on the whole time axis $\mathbb{R}_+$. The wealth of each investor cannot become negative (this assumption will be imposed on a solution of the wealth equation in the next section), hence y assume values in $\mathbb{R}_+^N$.

Let $(E, \mathcal{E}, (\mathcal{E}_t)_{t\geq 0})$ be the filtered measurable space with

$$E = \Omega \times D, \qquad \mathcal{E}_t = \mathcal{F}_t \otimes \mathcal{D}_t, \qquad \mathcal{E} = \bigvee_{t\geq 0} \mathcal{E}_t.$$

Let $\mathcal{P}^E$ denote the predictable σ-algebra on $E \times \mathbb{R}_+$, i.e. $\mathcal{P}^E$ is generated by all measurable functions $\xi(\omega, y, t)\colon E \times \mathbb{R}_+ \to \mathbb{R}$ which are left-continuous in t for any fixed (ω, y) and $\mathcal{E}_t$-measurable for any fixed t. In what follows, functions $\xi(\omega, y, t)$ will be often written as $\xi_t(\omega, y)$, or $\xi_t(y)$ when omitting ω does not lead to confusion.

Definition 1. A strategy of an investor is a $\mathcal{P}^E$-measurable function $\boldsymbol{L}_t(\omega, y)$ with values in $\mathbb{R}_+^N$ and $\boldsymbol{L}_0(\omega, y) = 0$, which is non-decreasing and càdlàg in t.

The following lemma will be used further in the construction of the model.

Lemma 1. *Let $\boldsymbol{L}_t(y)$ be a $\mathcal{P}^E$-measurable function, and Y an adapted càdlàg process with values in $\mathbb{R}_+^M$. Then the process $L_t(\omega) = \boldsymbol{L}_t(\omega, Y(\omega))$ is predictable ($\mathcal{P}$-measurable).*

Proof. The σ-algebra $\mathcal{P}^E$ is generated by sets $C \times [s, \infty)$, where $s \geq 0$ and $C \in \mathcal{E}_{s-}$ (as usual, $\mathcal{E}_{s-} = \bigvee_{u<t} \mathcal{E}_u$ and $\mathcal{E}_{0-} = \mathcal{E}_0$), see [19, § 1.2]. Hence, approximating $\boldsymbol{L}_t(y)$ by simple $\mathcal{P}^E$-measurable functions, it is enough to prove the lemma for functions

$$\boldsymbol{L}_t(\omega, y) = \mathrm{I}((\omega, y) \in C)\, \mathrm{I}(t \geq s), \qquad C \in \mathcal{E}_{s-},\ s \geq 0. \tag{3}$$

Using that $\mathcal{D}_s$ is generated by sets

$$\{y \in D : y_{s_i} \in B_i,\ i = 1 \ldots, n\},$$

where $s_1 < \ldots < s_n \leq s$ and $B_i \in \mathcal{B}(\mathbb{R}_+^M)$, one can see that in (3) it is enough to consider only sets C of the form

$$C = A \times \{y \in D : y_{s_i} \in B_i,\ i = 1 \ldots, n\}, \quad A \in \mathcal{F}_{s_n},\ s_1 < \ldots < s_n < s.$$

For such sets, $\mathrm{I}((\omega, Y(\omega)) \in C) = \mathrm{I}(\omega \in A)\, \mathrm{I}(Y_{s_i}(\omega) \in B_i,\ i \leq n)$ is $\mathcal{F}_{s_n}$-measurable, and, hence, $\mathcal{F}_{s-}$-measurable. Therefore, $L_t(\omega) = \mathrm{I}((\omega, Y(\omega)) \in C)\, \mathrm{I}(t \geq s)$ is $\mathcal{P}$-measurable. $\square$

3.2. The wealth equation

The wealth of the investors will be described by a càdlàg adapted process Y with values in $\mathbb{R}_+^M$. In this section we state the equation which defines Y. We will always assume that the initial wealth Y_0^m of each investor is non-random and strictly positive. The set of vectors $y \in \mathbb{R}_+^M$ with all strictly positive coordinates will be denoted by $\mathbb{R}_{++}^M$.

Let X^c denote the continuous part of the payoff process X, i.e. the non-decreasing process with values in $\mathbb{R}_+^N$ defined as

$$X_t^c = X_t - \sum_{s \le t} \Delta X_t,$$

where $\Delta X_s = X_s - X_{s-}$ and for $s = 0$ we put $\Delta X_0 = 0$. Denote by μ the measure of jumps of X and by ν its compensator. Define the predictable scalar process G (the so-called operational time process) as

$$G_t = |X_t^c| + (|x| \wedge 1) * \nu_t,$$

where the star denotes integration with respect to ν, i.e. for a measurable function $f(\omega, t, x)$ on $\Omega \times \mathbb{R}_+ \times \mathbb{R}_+^N$

$$f * \nu_t(\omega) = \int_0^t f(\omega, s, x)\nu(\omega, ds, dx).$$

Note that X^c is not the continuous martingale part of X, as is usually denoted in stochastic calculus. Actually, all the martingales in our paper will have zero continuous part, so the notation X^c should not lead to confusion.

Let H be an arbitrary scalar predictable càdlàg non-decreasing process such that $G \ll H$ (i.e. for a.a. ω the measure on $\mathbb{R}_+$ generated by the function $G_t(\omega)$ is absolutely continuous with respect to the measure generated by $H_t(\omega)$).

Definition 2. We call a strategy profile $(\boldsymbol{L}^1, \dots, \boldsymbol{L}^M)$ and a vector of initial wealth $y_0 \in \mathbb{R}_{++}^M$ *feasible* if there exists a unique (up to P-indistinguishability) non-negative càdlàg adapted process Y, called the *wealth process*, which assumes values in $\mathbb{R}_+^M$ and satisfies the following conditions:

1) Y solves the *wealth equation*

$$dY_t^m = -d|L_t^m| + \sum_{n=1}^N \frac{l_t^{m,n}}{|l_t^{\cdot,n}|} dX_t^n, \qquad Y_0^m = y_0^m \tag{4}$$

for $m = 1, \dots, M$, where $L_t^m(\omega) = \boldsymbol{L}_t^m(\omega, Y(\omega))$, and l is any $\mathrm{P} \otimes H$-version of the $\mathbb{R}_+^{MN}$-valued process of predictable Lebesgue derivatives (see the appendix for details on Lebesgue derivatives; the measure $\mathrm{P} \otimes H$ is defined as in (65) there)

$$l_t^{m,n} = \frac{dL_t^{m,n}}{dH_t}; \tag{5}$$

2) if $Y_t^m(\omega) = 0$ or $Y_{t-}^m(\omega) = 0$, then $L_s^m(\omega) = L_{t-}^m(\omega)$ and $Y_s(\omega) = 0$ for all $s \ge t$.

When in (4) we have $|l_t^{\cdot,n}(\omega)| = 0$ for some ω, t, n, we put $l_t^{m,n}(\omega)/|l_t^{\cdot,n}(\omega)| = 0$. Observe that the derivatives l are well-defined, since if Y is an adapted càdlàg process, then $L^{m,n}$ is a predictable process according to Lemma 1.

As usual, equation (4) should be understood in the integral sense (a.s. for all t):

$$Y_t^m = Y_0^m - |L_t^m| + \sum_{n=1}^N \int_0^t \frac{l_s^{m,n}}{|l_s^{\cdot,n}|} dX_s^n, \tag{6}$$

where the integral is understood as a pathwise Lebesgue–Stieltjes integral. It is well-defined since the process X is càdlàg and non-decreasing, and the integrand is non-negative and bounded.

Let us clarify that we use Lebesgue derivatives in the wealth equation and not Radon-Nikodym derivatives (e.g. $dL_t^{m,n}/d|L_t^{\cdot,n}|$) for two reasons. First, this allows to differentiate with respect to a process H not depending on the solution of the equation, which is yet to be found. Second, it is natural to require that the solution should not depend on what particular version of the derivatives is used. This is so if $G \ll H$ (see Proposition 1 below). Thus, if one would like to use Radon–Nikodym derivatives, the process H should dominate both the processes $|L|$ and G, which would make formulas rather cumbersome.

Sufficient conditions for the existence and uniqueness of a solution of equation (4) will be provided in the next section. But now let us prove a result which shows that the solution, if it exists, does not depend on the choice of the process H and the versions of the derivatives l.

Proposition 1. *Suppose Y is a solution of (4), where the derivative process l is defined as in (5) with respect to some càdlàg non-decreasing predictable process H such that $G \ll H$. Then for any càdlàg non-decreasing predictable process $\widetilde{H}$ such that $G \ll \widetilde{H}$ and any $\mathrm{P} \otimes \widetilde{H}$-version of the derivative $\widetilde{l} = dL/d\widetilde{H}$, the process Y also solves (4) with $\widetilde{l}$ in place of l.*

Proof. Let $F\colon \mathbb{R}^{MN} \to \mathbb{R}^{MN}$ denote the function which specifies the distribution of payoffs in (4):

$$F(l)^{m,n} = \frac{l^{m,n}}{|l^{\cdot,n}|}, \tag{7}$$

where $F(l)^{m,n} = 0$ if $|l^{\cdot,n}| = 0$. As follows from (6), we have to show that for each m, n

$$F(l)^{m,n} \cdot X^n = F(\widetilde{l})^{m,n} \cdot X^n, \tag{8}$$

where $F(l)$ denotes the process $F(l_t(\omega))$, and $F(\widetilde{l})$ denotes $F(\widetilde{l}_t(\omega))$.

One can see that if $f, f' \geq 0$ are predictable processes such that $f = f'$ $\mathrm{P} \otimes G$-a.s., then $f \cdot X^n = f' \cdot X^n$. We have

$$l^{m,n} = \frac{dL^{m,n}}{dH} = \frac{dL^{m,n}}{d\widetilde{H}} \frac{d\widetilde{H}}{dH} = \widetilde{l}^{m,n} \frac{d\widetilde{H}}{dH} \quad \mathrm{P} \otimes G\text{-a.s.},$$

where the second equality holds in view of claim (b) of Proposition 4 from the appendix. Since $d\widetilde{H}/dH > 0$ $\mathrm{P} \otimes G$-a.s. by claim (c) of Proposition 4, we have $F(l)^{m,n} = F(\widetilde{l})^{m,n}$, so (8) holds, which finishes the proof. ☐

3.3. A sufficient condition for the existence and uniqueness of a solution of the wealth equation

The following theorem provides a sufficient condition for the existence and uniqueness of a solution of equation (4). Note that the main results of our paper, formulated in Section 5, do not require this condition to hold (they only require a unique solution to exist), and may be valid under less strict assumptions.

Theorem 1. *Suppose that for each m a strategy $\boldsymbol{L}^m$ of investor m satisfies the following two conditions.*

(C1) There exists a $\mathcal{P} \otimes \mathcal{B}(\mathbb{R}_+^M)$-measurable function $v_t^m(\omega, z)$ with values in $\mathbb{R}_+^N$ such that for all $\omega \in \Omega$, $y \in D$, $t \in \mathbb{R}_+$, $n = 1, \dots, N$,

$$\boldsymbol{L}_t^{m,n}(\omega, y) = \int_0^t v_s^{m,n}(\omega, y_{s-})\, \mathrm{I}(\inf_{u<s} y_u^m > 0) dG_s(\omega) \tag{9}$$

(for $s = 0$, we put $y_{0-} = y_0$), and for all $\omega \in \Omega$, $z \in \mathbb{R}_+^M$, $t \in \mathbb{R}_+$

$$|v_t^m(\omega, z)| \Delta G_t(\omega) \le z^m. \tag{10}$$

(C2) There exist sets $\Pi^{m,n} \in \mathcal{P}$, $n = 1, \dots, N$, a non-random function $C^m \colon \mathbb{R}_+^M \to (0, \infty)$, and a predictable càdlàg process $\delta^m > 0$ such that

$$v_t^{m,n}(\omega, z) = 0 \text{ for all } (\omega, t) \in \Pi^{m,n} \text{ and } z \in \mathbb{R}_+^N, \tag{11}$$

and for all $(\omega, t) \notin \Pi^{m,n}$ and $z, \widetilde{z}, a \in \mathbb{R}_+^M$ such that $z^k, \widetilde{z}^k \in [a^k/2, 2a^k]$ for all k, it holds that

$$v_t^{m,n}(\omega, z) \ge (C^m(a)\delta_t^m(\omega))^{-1} \text{ if } z^m > 0, \tag{12}$$

$$v_t^{m,n}(\omega, z) \le C^m(a)\delta_t^m(\omega), \tag{13}$$

$$|v_t^{m,n}(\omega, z) - v_t^{m,n}(\omega, \widetilde{z})| \le C^m(a)\delta_t^m(\omega)|z - \widetilde{z}|. \tag{14}$$

Then for any vector of initial wealth $y_0 \in \mathbb{R}_{++}^M$ and a predictable non-decreasing càdlàg process H such that $G \ll H$, equation (4) has a unique solution (up to P-indistinguishability).

The proof is provided in Section 6. Let us comment on the conditions imposed in the theorem.

In condition (C1), equation (9) restricts the class of strategies under consideration to strategies that from the whole information of investors' past wealth use only the knowledge of the current wealth y_{s-}, on which depend the "instantaneous" investment rates $v_t^{m,n}$. The indicator in the integrand appears for the purpose of ensuring that the process Y^m is non-negative: if Y_u^m or Y_{u-}^m become zero for some u, such a strategy stops investing afterwards. For the same reason we require (10) to hold, which means that an investor cannot spend more money than available. Note that (C1) implies that the realization of the strategy is absolutely continuous with respect to G, i.e. $L^m \ll G$, which is a reasonable requirement since if a strategy does not have this property, then it "wastes" money (invests in assets when the expected payoff is zero).

Condition (C2) is needed because the proof is based on a contraction mapping argument. Inequalities (12)–(13) are analogous to similar upper and lower bounds on equation coefficients in such proofs, while (14) is a Lipschitz continuity condition. Note that it would be too restrictive to require $v_t^{m,n}$ to be bounded away from zero globally in inequality (12). Indeed, if asset n does not yield a payoff "predictably" at time t, it would be natural to take $v_t^{m,n} = 0$. Therefore, we relax the lower bound on v^m by introducing the sets $\Pi^{m,n}$ where $v^{m,n}$ may vanish.

The conditions of the theorem may look cumbersome, but it is possible to verify that certain strategies satisfy them. In particular, in Section 4.2 we do that for a candidate optimal strategy under mild additional assumptions.

4. Optimal strategies

4.1. Definition

If a strategy profile and a vector of initial wealth are feasible, we define the relative wealth of investor m as the process

$$r_t^m = \frac{Y_t^m}{|Y_t|},$$

where $r_t^m(\omega) = 0$ if $|Y_t(\omega)| = 0$.

We will be interested in finding strategies for which the relative wealth of an investor grows on average in the following sense.

Definition 3. For a given payoff process X, we call a strategy $\boldsymbol{L}$ *relative growth optimal* for investor m, if for any feasible initial wealth and a strategy profile where investor m uses this strategy, it holds that $Y_t^m > 0$ for all $t \geq 0$ and $\ln r^m$ is a submartingale.

Observe that if a strategy is relative growth optimal, then also r^m is a submartingale by Jensen's inequality. Another corollary from the relative growth optimality is that such a strategy is a *survival strategy* in the sense that the relative wealth of an investor who uses it always stays bounded away from zero,

$$\inf_{t\geq 0} r_t^m > 0 \text{ a.s.}, \tag{15}$$

(we use the terminology of [4]; note that, for example, in [5], the term "survival" has a somewhat different meaning). This follows from the fact that $\ln r^m$ is a non-positive submartingale, and hence it has a finite limit $z = \lim_{t\to\infty} \ln r_t^m$. Therefore, $\lim_{t\to\infty} r_t^m = e^z > 0$.

It is worth mentioning that the survival property (15) also implies that an investor who uses such a strategy achieves a not slower asymptotic growth rate of wealth than any other investor in the market, i.e. for any k

$$\limsup_{t\to\infty} \frac{1}{t} \ln Y_t^m \geq \limsup_{t\to\infty} \frac{1}{t} \ln Y_t^k \text{ a.s.} \tag{16}$$

This property is analogous to the notion of asymptotic growth optimality in single-investor market models (see, e.g., Section 3.10 in [17]). The validity of (16) follows from that $\sup_{t\geq 0} |Y_t|/Y_t^m < \infty$ by (15), so $\sup_{t\geq 0} Y_t^k/Y_t^m < \infty$ for any k. Hence $\limsup_{t\to\infty} t^{-1} \ln(Y_t^k/Y_t^m) \leq 0$, from which one can obtain (16).

4.2. A candidate relative growth optimal strategy

Denote by $\nu_{\{t\}}$ the predictable random measure on $\mathcal{B}(\mathbb{R}_+^N)$ defined by

$$\nu_{\{t\}}(\omega, A) = \nu(\omega, \{t\} \times A), \qquad A \in \mathcal{B}(\mathbb{R}_+^N),$$

and introduce the predictable process

$$\bar{\nu}_t = \nu_{\{t\}}(\mathbb{R}_+^N).$$

One can see that $\bar{\nu}_t$ is the conditional probability of a jump of the process X_t given the σ-algebra $\mathcal{F}_{t-}$ [14, Proposition II.1.17], i.e. $\bar{\nu}_t = \mathrm{P}(\Delta X_t \neq 0 \mid \mathcal{F}_{t-})$. We will always assume that a "good" version of the compensator is chosen – such that $\bar{\nu}_t(\omega) \in [0,1]$ for all ω, t.

The candidate relative growth optimal strategy, which we define below, will behave differently at points t where $\bar{\nu}_t = 0$ and where $\bar{\nu}_t > 0$. To deal with them, let us partition $\Omega \times \mathbb{R}_+ \times (0,\infty)$ into the following three sets belonging to $\mathcal{P} \otimes \mathcal{B}(\mathbb{R}_+)$:

$$\Gamma_0 = \{(\omega,t,c) : \bar{\nu}_t(\omega) = 0\},$$
$$\Gamma_1 = \Big\{(\omega,t,c) : 0 < \bar{\nu}_t(\omega) < 1, \text{ or } \bar{\nu}_t(\omega) = 1 \text{ and } \int_{\mathbb{R}_+^N} \frac{c}{|x|}\nu_{\{t\}}(\omega,dx) > 1\Big\},$$
$$\Gamma_2 = \Big\{(\omega,t,c) : \bar{\nu}_t(\omega) = 1 \ \text{ and } \ \int_{\mathbb{R}_+^N} \frac{c}{|x|}\nu_{\{t\}}(\omega,dx) \le 1\Big\}.$$

In the definition of the optimal strategy, the argument c in the triple (ω,t,c) will correspond to the value of the total wealth of all the investors right before time t, i.e. $|Y_{t-}|$ (points (ω,t,c) with $c=0$ are not included in any of the sets; it will be easier to deal with them separately). Roughly speaking, the sets Γ_i differ in the conditional size of possible jumps of the payoff process X. On Γ_0, the conditional probability of a jump is zero. On Γ_2, only "large" jumps of X occur (large relatively to the current total wealth), and Γ_1 is the remaining set where both "small" and "large" jumps can occur.

The next lemma defines an auxiliary function ζ which will be needed to specify what proportion of wealth the optimal strategy keeps in cash.

Lemma 2. *For each $(\omega,t,c) \in \Gamma_1$, there exists a unique solution $z^*(\omega,t,c) \in (0,c)$ of the equation*

$$\int_{\mathbb{R}_+^N} \frac{c}{z+|x|}\nu_{\{t\}}(\omega,dx) = 1 - \frac{c}{z}(1-\bar{\nu}_t(\omega)). \tag{17}$$

The function $\zeta(\omega,t,c)$ defined on $\Omega \times \mathbb{R}_+ \times \mathbb{R}_+$ by

$$\zeta = c\,\mathrm{I}(\Gamma_0) + z^*\,\mathrm{I}(\Gamma_1) \tag{18}$$

is $\mathcal{P} \otimes \mathcal{B}(\mathbb{R}_+)$-measurable.

Proof. For $(\omega,t,c) \in \Gamma_1$, the left-hand side of (17) is a strictly decreasing continuous function of z, while the right-hand side is a non-decreasing continuous function of z. The existence and uniqueness of the solution z^* then follows from comparison of their values at $z=c$ and $z \to 0$.

To prove the measurability of ζ, consider the function f defined on $\Omega \times \mathbb{R}_+^3 \to \mathbb{R}$ by

$$f(\omega,t,c,z) = \bigg(\int_{\mathbb{R}_+^N} \frac{c}{z+|x|}\nu_{\{t\}}(\omega,dx) - 1 + \frac{c}{z}(1-\bar{\nu}_t(\omega))\bigg)\,\mathrm{I}((\omega,t,c) \in \Gamma_1) \wedge 1.$$

Observe that f is a Carathéodory function, i.e. $\mathcal{P} \otimes \mathcal{B}(\mathbb{R}_+)$-measurable in (ω,t,c) and continuous in z. Then by Filippov's implicit function theorem (see, e.g., [2, Theorem 18.17]), the set-valued function

$$\phi(\omega,t,c) = \{z \in [0,c] : f(\omega,t,c,z) = 0\}$$

admits a measurable selector. Since ϕ on Γ_1 is single-valued ($\phi(\omega,t,c) = \{z^*(\omega,t,c)\}$), this implies the $\mathcal{P} \otimes \mathcal{B}(\mathbb{R}_+)$-measurability of ζ. □

It is known that there exists a predictable process b_t with values in $\mathbb{R}_+^N$ and a transition kernel $K_{\omega,t}(dx)$ from $(\Omega \times \mathbb{R}_+, \mathcal{P})$ to $(\mathbb{R}_+^N, \mathcal{B}(\mathbb{R}_+^N))$ such that up to P-indistinguishability

$$X_t^c(\omega) = b \cdot G_t(\omega), \qquad \nu(\omega, dt, dx) = K_{\omega,t}(dx)dG_t(\omega). \tag{19}$$

Since the filtration is complete, we can assume (19) holds for all ω, t. Also, it will be convenient to select "good" versions of b and K, which satisfy the following conditions for all (ω, t) (it is always possible to select such versions, see, e.g., [14, Proposition II.2.9]):

$$\begin{aligned} &|b_t(\omega)| = 0 \text{ if } \Delta G_t(\omega) > 0, \quad K_{\omega,t}(\{0\}) = 0, \\ &|b_t(\omega)| + \int_{\mathbb{R}_+^N} (1 \wedge |x|) K_{\omega,t}(dx) = 1. \end{aligned} \tag{20}$$

Define the $\mathcal{P} \otimes \mathcal{B}(\mathbb{R}_+)$-measurable function $\widehat{\lambda}(\omega, t, c)$ with values in $\mathbb{R}_+^N$:

$$\widehat{\lambda}_t(0) = 0, \qquad \widehat{\lambda}_t(c) = \frac{b_t}{c} + \int_{\mathbb{R}_+^N} \frac{x}{\zeta_t(c) + |x|} K_t(dx) \quad \text{for} \quad c > 0 \tag{21}$$

(the argument ω is omitted for brevity). Now we are in a position to introduce the strategy, which will be shown to be relative growth optimal. When used by investor m, its cumulative investment process is defined by

$$\widehat{\boldsymbol{L}}_t(y) = \int_0^t y_{s-}^m \widehat{\lambda}_s(|y_{s-}|) dG_s \tag{22}$$

(for $s = 0$, put $y_{0-} = y_0$). When it is necessary to emphasize that this strategy, as a function of y, depends on which investor uses it, we will use the notation $\widehat{\boldsymbol{L}}_t^m(y)$.

Generally speaking, the strategy $\widehat{\boldsymbol{L}}$ resembles optimal strategies in other models in evolutionary finance, as they all split investment budget between assets proportionally to expected asset payoffs (but quantitatively they differ in how these proportions are calculated). In the particular case when the payoff process X_t is discrete-time (as in Section 2), we obtain the same strategy that was found in [9]. Formally, the discrete-time case can be included in the general model by taking a process X_t such that $X_t = \sum_{s=0}^{\lfloor t \rfloor} \Delta X_s$; then $b = 0$ and $K_t(dx)$ is the conditional distribution of the jump ΔX_t for integer t.

To conclude this section, we state a proposition which provides sufficient conditions of feasibility of a strategy profile where one or several investors use the strategy $\widehat{\boldsymbol{L}}$. It is based on Theorem 1, but we show that the conditions of that theorem hold automatically for $\widehat{\boldsymbol{L}}$ under some mild additional assumptions on the payoff process. In particular, if these assumptions hold, then a strategy profile where all the investors use the strategy $\widehat{\boldsymbol{L}}$ is feasible (we will consider such profiles in Theorem 4 in the next section).

Define the predictable process with values in $\mathbb{R}_+^N$

$$h_t = b_t + \int_{\mathbb{R}_+^N} \frac{x}{1 + |x|} K_t(dx), \tag{23}$$

and define the scalar predictable process

$$p_t = \int_{\mathbb{R}_+^N} \frac{\nu_{\{t\}}(dx)}{(1 + |x|)^2}.$$

Proposition 2. *Suppose the process $(p_t \Delta G_t)^{-1} \mathrm{I}(\Delta G_t > 0)$ is locally bounded and for each n the process $(h_t^n)^{-1} \mathrm{I}(h_t^n > 0)$ is locally bounded (where $0/0 = 0$ for these processes). Then any strategy profile, in which every investor uses either the strategy $\widehat{\boldsymbol{L}}$ or a strategy which satisfies the conditions of Theorem 1, is feasible for any initial wealth $y_0 \in \mathbb{R}^M_{++}$.*

The proof is given in Section 6.

5. The main results

The following three theorems are the main results on relative growth optimal strategies. For convenience, we divide this section into three parts, each containing a theorem and comments. The proofs are in Section 6.

1. The first result establishes the existence of a relative growth optimal strategy ($\widehat{\boldsymbol{L}}$ is such a strategy) and shows that it is, in a certain sense, unique.

Theorem 2. *1. The strategy $\widehat{\boldsymbol{L}}$ is relative growth optimal.*

2. Suppose $\boldsymbol{L}$ is a strategy of investor M such that the profile $(\widehat{\boldsymbol{L}}^1, \dots, \widehat{\boldsymbol{L}}^{M-1}, \boldsymbol{L})$ and a vector of initial wealth $y_0 \in \mathbb{R}^M_{++}$ are feasible and r^M is a submartingale. Then $\boldsymbol{L}_t(Y) = \widehat{\boldsymbol{L}}_t^M(Y)$ for all $t \geq 0$, where Y is the solution of the wealth equation for this strategy profile and initial wealth.

Let us comment on the second part of the theorem. It can be regarded as a uniqueness result for a relative growth optimal strategy: if $M-1$ investors use the strategy $\widehat{\boldsymbol{L}}$, then the remaining investor, who wants the relative wealth to be a submartingale, has nothing to do but to act as using the strategy $\widehat{\boldsymbol{L}}$ as well. Here, "to act" means that the realization of the strategy of this investor, i.e. the cumulative investment process $L_t(\omega) = \boldsymbol{L}_t(\omega, Y(\omega))$ coincides (up to P-indistinguishability) with the process $\widehat{L}_t^M(\omega) = \widehat{\boldsymbol{L}}_t^M(\omega, Y(\omega))$. As a consequence, the relative wealth of each investor will stay constant.

However, note that the strategy $\boldsymbol{L}_t(\omega, y)$, as a function on $\Omega \times D \times \mathbb{R}_+$, may be different from $\widehat{\boldsymbol{L}}_t^M(\omega, y)$. Let us provide an example. Suppose there is only one asset with the non-random payoff process $X_t = t$ and two investors with initial wealth $y_0^1 = y_0^2 = 1$. In this case, $G_t = t$ and the strategy $\widehat{\boldsymbol{L}}$, if used by investor 2, has the form

$$\widehat{\boldsymbol{L}}_t(y) = \int_0^t \frac{y_{s-}^2}{y_{s-}^1 + y_{s-}^2} ds.$$

On the other hand, consider the strategy $\boldsymbol{L}$ for investor 2 defined as

$$\boldsymbol{L}_t(y) = \int_0^t \left(\frac{1}{3} \mathrm{I}(y_u^1 = 1 \text{ for all } u < s) + \frac{y_{s-}^2}{y_{s-}^1 + y_{s-}^2} \mathrm{I}(y_u^1 \neq 1 \text{ for some } u < s) \right) ds.$$

It is not hard to see that $\boldsymbol{L}$ is also relative growth optimal. However it leads to a different wealth process of investor 2 compared to $\widehat{\boldsymbol{L}}$ if, for example, $\boldsymbol{L}_t^1 \equiv 0$.

2. The second result shows that the strategy $\widehat{\boldsymbol{L}}$ asymptotically determines the structure of the market in the sense that if there is an investor who uses it, then the representative strategy of all the investors is asymptotically close to $\widehat{\boldsymbol{L}}$. (By the representative strategy we call the weighted sum of the investors' strategies with their relative wealth as the weights; see below.) Moreover, if the representative strategy of the other investors is asymptotically different from $\widehat{\boldsymbol{L}}$, they will be driven out of the market – their relative wealth will vanish as $t \to \infty$.

In order to state the theorem, let us introduce auxiliary processes. Suppose a unique solution of the wealth equation exists. Let $L_t^m(\omega) = \boldsymbol{L}_t^m(\omega, Y(\omega))$ be the realizations of the investors' strategies, and, as above, $l_t^m = dL_t^m/dG_t$. For each m, define the predictable process $L_t^{(s),m} = L_t^m - l^m \cdot G_t$, which is the singular part of the Lebesgue decomposition of L_t^m with respect to G_t (hence the superscript "(s)").

Define the proportion λ_t^m of wealth invested in the assets by investor m as the predictable process with values in $\mathbb{R}_+^N$ and the components

$$\lambda_t^{m,n} = \frac{l_t^{m,n}}{Y_{t-}^m},$$

where $0/0 = 0$. Note that by condition 2 of Definition 2, we have $l_t^{m,n} = 0$ on the set $\{(\omega,t) : Y_{t-}^m(\omega) = 0\}$ ($\mathrm{P} \otimes G$-a.s.). Introduce also the processes of "cumulative proportions" of invested wealth and their singular parts:

$$\Lambda_t^m = \frac{1}{Y_-^m} \cdot L_t^m, \qquad \Lambda_t^{(s),m} = \Lambda_t^m - \lambda^m \cdot G_t = \frac{1}{Y_-^m} \cdot L_t^{(s),m},$$

which are non-decreasing, predictable, càdlàg, and with values in $[0,+\infty]^N$.

For a set of investors $\mathbb{M} \subseteq \{1,\dots,M\}$, let us denote their total wealth by $Y_t^{\mathbb{M}} = \sum_{m\in\mathbb{M}} Y_t^m$, their relative wealth by $r_t^{\mathbb{M}} = \sum_{m\in\mathbb{M}} r_t^m$, and the processes associated with the realization of their representative strategy by $L_t^{\mathbb{M}} = \sum_{m\in\mathbb{M}} L_t^m$, $l_t^{\mathbb{M}} = dL_t^{\mathbb{M}}/dG_t = \sum_{m\in\mathbb{M}} l_t^m$, $L_t^{(s),\mathbb{M}} = \sum_{m\in\mathbb{M}} L_t^{(s),m}$, and

$$\begin{aligned}
\lambda_t^{\mathbb{M}} &= \frac{l_t^{\mathbb{M},n}}{Y_{t-}^{\mathbb{M}}} = \sum_{m\in\mathbb{M}} \frac{r_{t-}^m}{r_{t-}^{\mathbb{M}}} \lambda_t^m, \\
\Lambda_t^{\mathbb{M}} &= \frac{1}{Y_-^{\mathbb{M}}} \cdot L_t^{\mathbb{M}} = \sum_{m\in\mathbb{M}} \frac{r_-^m}{r_-^{\mathbb{M}}} \cdot \Lambda_t^m, \\
\Lambda_t^{(s),\mathbb{M}} &= \frac{1}{Y_-^{\mathbb{M}}} \cdot L_t^{(s),\mathbb{M}} = \sum_{m\in\mathbb{M}} \frac{r_-^m}{r_-^{\mathbb{M}}} \cdot \Lambda_t^{(s),m}.
\end{aligned}$$

To shorten the notation, for the set of all investors $\mathbb{M}_1 = \{1,\dots,M\}$ we will write $\bar\lambda_t^n = \lambda_t^{\mathbb{M}_1,n}$, and for the set $\mathbb{M}_2 = \{2,\dots,M\}$ write $\widetilde\lambda_t^n = \lambda_t^{\mathbb{M}_2,n}$, and similarly for the other processes.

Theorem 3. *Suppose investor 1 uses the strategy $\widehat{\boldsymbol{L}}$, the other investors use arbitrary strategies $\boldsymbol{L}^m$, and the strategy profile $(\widehat{\boldsymbol{L}}^1, \boldsymbol{L}^2, \dots, \boldsymbol{L}^M)$ is feasible for some initial wealth $y_0 \in \mathbb{R}_{++}^M$. Then*

$$\|\lambda^1 - \bar\lambda\|^2 \cdot G_\infty + |\bar\Lambda_\infty^{(s)}| < \infty \ \text{a.s.}, \tag{24}$$

and, as $t \to \infty$,

$$r_t^1 \to 1 \ \text{a.s. on } \{\omega : \|\lambda^1 - \widetilde\lambda\|^2 \cdot G_\infty(\omega) = \infty \text{ or } |\widetilde\Lambda_\infty^{(s)}(\omega)| = \infty\}. \tag{25}$$

Equation (24) expresses the idea that the investment proportions $\bar\lambda$ of the representative strategy of all the investors are close to $\lambda^1 = \widehat\lambda$ asymptotically in the sense that the integral $\int_0^t \|\widehat{\lambda}_s - \bar{\lambda}_s\| dG_s$ converges as $t \to \infty$ and the singular part $\bar{\Lambda}_t^{(s)}$ stays bounded. If $G_\infty = \infty$, this, roughly speaking, means that $\|\widehat{\lambda}_t - \bar{\lambda}_t\|$ is small asymptotically.

Equation (25) shows that the strategy $\widehat{\boldsymbol{L}}$ drives other strategies out of the market if they are asymptotically different from it. This result can be also regarded as asymptotic uniqueness of a survival strategy: if investors $m = 2, \dots, M$ want to survive against investor 1 who uses the strategy $\widehat{\boldsymbol{L}}$, they should also use (at least, collectively) a strategy asymptotically close to $\widehat{\boldsymbol{L}}$.

3. Theorems 2, 3 lead to the natural question: since the strategy $\widehat{\boldsymbol{L}}$ is s good, what will happen if all the investors decide to use it? Obviously, in this case their relative wealth will remain the same. However, it is interesting to look at the asymptotic behavior of the absolute wealth $W_t := |Y_t|$. A priori, it is even not obvious whether it will grow. Our third result partly answers this question: we prove that W does not decrease "on average" and provide a condition for $W_t \to \infty$ as $t \to \infty$.

Theorem 4. *Suppose all the investors use the strategy $\widehat{\boldsymbol{L}}$, and the initial wealth $y_0 \in \mathbb{R}^M_{++}$ and the strategy profile $(\widehat{\boldsymbol{L}}, \dots, \widehat{\boldsymbol{L}})$ are feasible.*

Then the process $V_t := 1/W_t$ is a supermartingale and there exists the limit $W_\infty := \lim_{t\to\infty} W_t \in (0, \infty]$ a.s. Moreover, if X is quasi-continuous (i.e. $\bar{\nu} \equiv 0$), then $\{W_\infty = \infty\} = \{(1 \wedge |x|^2) * \nu_\infty = \infty\}$ a.s.

If $\mathrm{E}\,|X_t| < \infty$ for all t, then also $\mathrm{E}\,W_t < \infty$ (since $W_t \le |y_0| + |X_t|$), and the process W_t will be a submartingale by Jensen's inequality. This is what we mean by the phrase that the total wealth does not decrease on average.

It is interesting to note that if one investor uses the strategy $\widehat{\boldsymbol{L}}$ and the other investors use arbitrary strategies, then it does not necessarily hold that the wealth of such an investor will grow. In particular, it may happen that $W_t \to 0$ as $t \to \infty$, which is remarkable because an investor always has a trivial strategy which guarantees that the wealth will not vanish – just keep all the wealth in cash. An example can be found in [9].

Another fact worth mentioning is that, as will become clear from the proof of the theorem, the continuous part of the payoff process X does not affect the process W if all the investors use the strategy $\widehat{\boldsymbol{L}}$, i.e. W will be the same for any payoff processes X and X' such that $X - X'$ is a continuous process. For example, if X is continuous, then $W_t = W_0$ for all $t \ge 0$ even if X is a strictly increasing process. In particular, observe that in the second claim of the theorem, the continuous part of X does not enter the condition for having $W_\infty = \infty$.

6. Proofs

6.1. Proof of Theorem 1

Without loss of generality we will assume that the functions C^m and the processes δ^m are the same for all the investors, since otherwise one can take $C(a) = \max_m C^m(a)$ and $\delta_t = \max_m \delta_t^m$. Moreover, we can assume that δ is a non-decreasing process or, otherwise, take $\delta'_t = \sup_{s\le t} \delta_t$ (δ'_t will be finite-valued since δ_t is predictable and càdlàg, and, hence, locally bounded; see, e.g., VII.32 in [8]). Proposition 1 implies that it is enough to prove the existence and uniqueness of a solution for some particular choice of the process H such that $G \ll H$. We will do this for $H = G$.

We are going to construct the process Y by induction on stochastic intervals $[0,\tau_{i,j}]$ with appropriately chosen stopping times $\tau_{i,j}$ ($i \in \{0,1,\dots,M\}$, $j \in \mathbb{Z}_+$) such that $\tau_{i,j} \le \tau_{i',j'}$ if $(i,j) \le (i',j')$ lexicographically (i.e. $i < i'$, or $i = i'$ and $j \le j'$), and $\sup_{i,j} \tau_{i,j} = \infty$. Here, "by induction" means that we will construct processes $Y^{i,j}$ such that on the set $\{(\omega,t) : t \le \tau_{i,j}(\omega)\}$ they satisfy equation (6) and on this set $Y^{i,j} = Y^{i',j'}$ for any $(i',j') \ge (i,j)$. From these processes we can form the single process Y satisfying (6) on the whole set $\Omega \times \mathbb{R}_+$.

Before providing an explicit construction, let us briefly explain the role that $\tau_{i,j}$ will play. The stopping times $\tau_{i,0}$ for $i \ge 1$ will be the moments when the wealth of one or several investors reaches zero "in a continuous way" (i.e. for some m we have $Y_t^m > 0$ for $t < \tau_{i,0}$ but $Y^m_{\tau_{i,0}-} = 0$). The index i will correspond to the i-th such event. Not necessarily all the investors will eventually have zero wealth; in that case we will put $\tau_{i,j}(\omega) = \infty$ for i starting from some i' and all j.

Between $\tau_{i,0}$ and $\tau_{i+1,0}$ we will construct a sequence of stopping times $\tau_{i,j} \to \tau_{i+1,0}$ as $j \to \infty$, such that on each interval $[\tau_{i,j}, \tau_{i,j+1})$ the wealth of all the investors, who have non-zero wealth at $\tau_{i,j}$, can be bounded away from zero by an $\mathcal{F}_{\tau_{i,j}}$-measurable variable. The wealth of some of those investors may become zero at $\tau_{i,j+1}$, but only "by a jump". If $\tau_{i,0}(\omega) < \infty$, it will also hold that $\tau_{i,j}(\omega) < \infty$ for all j.

The reason why we need to treat differently the moments when the wealth reaches zero in a continuous way and by a jump is that we do not assume that the function $C(a)$ is bounded in a neighborhood of zero (this is necessary, for example, to apply the theorem to the strategy $\widehat{\boldsymbol{L}}$ – see the proof of Proposition 2).

Now we will proceed to the construction of $\tau_{i,j}$ and $Y^{i,j}$. Let $\tau_{0,0} = 0$, and for all $t \ge 0$ put $Y_t^{0,0} = y_0$, where $y_0 \in \mathbb{R}^M_{++}$ is the given initial wealth. Suppose $\tau_{i,j}$ and $Y^{i,j}$ are constructed. We will now show how to construct $\tau_{i,j+1}, Y^{i,j+1}$. For brevity, i will be assumed fixed and omitted in the notation, so we will simply write τ_j, Y^j, while $Y^{j,m}$ will denote the m-th coordinate of Y^j.

Let $A(\omega) = \{m : Y^{j,m}_{\tau_j}(\omega) > 0\}$ denote the set of the investors who are still active (i.e. have positive wealth) at τ_j; for ω such that $\tau_j(\omega) = \infty$ we put $A(\omega) = \emptyset$. Observe that A is an $\mathcal{F}_{\tau_j}$-measurable random set (since it is finite, the measurability means that $\mathrm{I}(m \in A)$ are $\mathcal{F}_{\tau_j}$-measurable functions for each m).

On the set $\{\omega : A(\omega) = \emptyset\}$, define $\tau_{j+1} = \tau_j + 1$ (with $\tau_{j+1}(\omega) = \infty$ if $\tau_j(\omega) = \infty$), and on the set $\Omega' = \{\omega : A(\omega) \ne \emptyset\}$ define

$$\gamma = (\delta_{\tau_j} + 1) C(Y^j_{\tau_j})$$

and

$$\begin{aligned} \tau_{j+1} = \inf\Bigg\{ & t > \tau_j : |X_t - X_{\tau_j}| \ge \frac{1}{4M\gamma^2} \ \wedge \ \min_{m\in A} Y^{j,m}_{\tau_j}, && (26)\\ & \text{or } G_t - G_{\tau_j} \ge \frac{1}{2\gamma}\left(\frac{1}{2M} \wedge \min_{m\in A} Y^{j,m}_{\tau_j}\right), && (27)\\ & \text{or } \delta_t \ge \delta_{\tau_j} + 1, \ \text{or } t \ge \tau_j + 1 \Bigg\}. && (28) \end{aligned}$$

Observe that we have the strict inequality $\tau_{j+1} > \tau_j$ on Ω' since the processes X, G, δ are càdlàg. Also, $\tau_{j+1} \le \tau_j + 1$ by the condition in (28).

For each ω define the complete metric space $\mathbb{E}(\omega)$ consisting of càdlàg functions

$f\colon \mathbb{R}_+ \to \mathbb{R}_+^M$ satisfying the conditions

$$f_t = Y_t^j(\omega) \text{ for } t \le \tau_j(\omega), \tag{29}$$

$$f_t^m \in \left[\frac{1}{2}Y_{\tau_j}^{j,m}(\omega),\ 2Y_{\tau_j}^{j,m}(\omega)\right] \text{ for } t > \tau_j(\omega),\ m = 1, \dots, M, \tag{30}$$

and the metric

$$d(f, \widetilde{f}) = \sup_{t \ge 0} |f_t - \widetilde{f}_t|$$

(note that if $A(\omega) = \emptyset$, then $\mathbb{E}(\omega)$ consists of one element).

From now on, we will assume that ω is fixed and omit it in the notation. Consider the operator U on $\mathbb{E}$, which maps a function $f \in \mathbb{E}$ to the càdlàg function $g := U(f)\colon \mathbb{R}_+ \to \mathbb{R}_+^M$ defined by the formula

$$g_t^m = Y_{t\wedge\tau_j}^{j,m} - \int_0^t |v_s^m(f_{s-})|\,\mathrm{I}(\tau_j < s < \tau_{j+1},\ m \in A)dG_s \\ + \int_0^t F^m(l_s(f_{s-}))\,\mathrm{I}(\tau_j < s < \tau_{j+1})dX_s, \tag{31}$$

where $F\colon \mathbb{R}^{MN} \to \mathbb{R}^{MN}$ is the function defined in (7), and

$$l_s^{m,n}(\omega, z) = v_s^{m,n}(\omega, z)\,\mathrm{I}(m \in A(\omega)). \tag{32}$$

Let us show that U is a contraction mapping of $\mathbb{E}$ to itself. If $A(\omega) = \emptyset$, this is obvious, so consider ω such that $A(\omega) \ne \emptyset$. Suppose $f \in \mathbb{E}$, $g = U(f)$. First we will show that $g \in \mathbb{E}$. It is clear that g satisfies (29), and, if $m \notin A$, then g^m satisfies (30). To show that the lower bound in (30) is satisfied for $m \in A$, consider the first integral in (31). Since $f \in \mathbb{E}$, by condition (13) we have $v_s^{m,n}(f_{s-}) \le C(Y_{\tau_j}^j)\delta_s \le \gamma$, using that $\delta_s < \delta_{\tau_j} + 1$ for $s < \tau_{j+1}$. Hence the integral can be bounded from above by $\gamma(G_{\tau_{j+1}-} - G_{\tau_j})$, and this quantity does not exceed $\frac{1}{2}Y_{\tau_j}^{j,m}$ by the choice of τ_{j+1} (see (27)). Therefore, $g_t^m \ge \frac{1}{2}Y_{\tau_j}^{j,m}$ for $t \ge \tau_j$.

The upper bound from (30) for $m \in A$ follows from that the second integral in (31) is bounded from above by $|X_{\tau_{j+1}-} - X_{\tau_j}|$ since $F^{m,n}(l) \le 1$ and by the choice of τ_{j+1} (see (26)) we have $|X_{\tau_{j+1}-} - X_{\tau_j}| \le Y_{\tau_j}^{j,m}$. Thus, g satisfies conditions (29)–(30), so $g \in \mathbb{E}$.

Now we will show that U is contracting. Consider $f, \widetilde{f} \in \mathbb{E}$ and $m \in A$. Then

$$|U(f)_t^m - U(\widetilde{f})_t^m| \le \int_{(\tau_j,\tau_{j+1})} |v_s^m(f_{s-}) - v_s^m(\widetilde{f}_{s-})|dG_s \\ + \sum_{n=1}^N \int_{(\tau_j,\tau_{j+1})} |F^{m,n}(l_s(f_{s-})) - F^{m,n}(l_s(\widetilde{f}_{s-}))|dX_s^n := \mathcal{I}_1^m + \mathcal{I}_2^m.$$

By conditions (14) and (28), we have $|v_s^{m,n}(f_{s-}) - v_s^{m,n}(\widetilde{f}_{s-})| \le \gamma d(f, \widetilde{f})$ for $s \in (\tau_j, \tau_{j+1})$. Hence

$$\mathcal{I}_1^m \le \gamma d(f, \widetilde{f})(G_{\tau_{j+1}-} - G_{\tau_j}) \le \frac{1}{4M} d(f, \widetilde{f}),$$

where the last inequality is due to (27).

To bound $\mathcal{I}_2^m$, observe that for each n and $(\omega,t)\in(\tau_j,\tau_{j+1})\setminus\Pi^{m,n}$ we have (by (12))

$$|l_s^{\cdot,n}(f_{s-})|\geq \min_{m\in A} v_s^{m,n}(f_{s-})\geq \frac{1}{\gamma}, \tag{33}$$

and a similar inequality is true for $|l_s^{\cdot,n}(\widetilde{f}_{s-})|$. It is straightforward to check that F satisfies the property

$$\left|\frac{\partial F^{m,n}}{\partial l^{p,q}}(l)\right|\leq \frac{1}{|l^{\cdot,n}|}\qquad \text{for any } m,n,p,q.$$

Hence, for any $l,\widetilde{l}\in\mathbb{R}_+^{MN}$ such that $|l^{\cdot,n}|\geq\alpha$ and $|\widetilde{l}^{\cdot,n}|\geq\alpha$ for all n with some $\alpha>0$, we have $|F^{m,n}(l)-F^{m,n}(\widetilde{l})|\leq\alpha^{-1}|l-\widetilde{l}|$. From this and (33), we find that on the set $(\tau_j,\tau_{j+1})\setminus\Pi^{m,n}$

$$|F^{m,n}(l_s(f_{s-}))-F^{m,n}(l_s(\widetilde{f}_{s-}))|\leq\gamma|l_s(f_{s-})-l_s(\widetilde{f}_{s-})|\leq\gamma^2 d(f,\widetilde{f}).$$

On $\Pi^{m,n}$ we have

$$F^{m,n}(l_s(f_{s-}))-F^{m,n}(l_s(\widetilde{f}_{s-}))=0,$$

and, consequently, obtain the bound

$$\mathcal{I}_2^m\leq\gamma^2 d(f,\widetilde{f})|X_{\tau_{j+1}-}-X_{\tau_j}|\leq\frac{1}{4M}d(f,\widetilde{f}).$$

Now we see that U is a contraction mapping: $d(U(f),U(\widetilde{f}))\leq\frac{1}{2}d(f,\widetilde{f})$.

As a result, $U(\omega)$ has a fixed point $f^*(\omega)$ for any ω. Observe that the operator U preserves adaptedness, i.e. if $f_t(\omega)$ is a càdlàg adapted process with values in $\mathbb{R}_+^M$ and satisfies conditions (29)–(30), then $U(\omega,f(\omega))$ is such a process as well. Hence f^* is a càdlàg adapted process since it can be obtained, for example, as the limit $U^{(n)}(Y^j_{t\wedge\tau_j})$ as $n\to\infty$ where n stands for the n-times application of U.

Now we can define the process Y^{j+1} as follows: for each m put

$$\begin{aligned}
Y_t^{j+1,m}&=f_t^{*,m}\quad\text{for } t<\tau_{j+1},\\
Y_t^{j+1,m}&=f^{*,m}_{\tau_{j+1}-}-|v^m_{\tau_{j+1}}(f^{*,m}_{\tau_{j+1}-})|\,\mathrm{I}(m\in A)\Delta G_{\tau_{j+1}}\\
&\quad+\sum_{n=1}^N F^{m,n}(v_{\tau_{j+1}}(f^{*,m}_{\tau_{j+1}-}))\,\mathrm{I}(m\in A)\Delta X^n_{\tau_{j+1}}\qquad\text{for } t\geq\tau_{j+1}
\end{aligned}$$

(note that $Y_t^{j+1}=Y^{j+1}_{\tau_{j+1}}$ for all $t\geq\tau_{j+1}$). Inserting $Y^{j+1,m}$ in (31), we obtain the equation for $t\in[\tau_j,\tau_{j+1}]$

$$Y_t^{j+1,m}=Y_{\tau_j}^{j,m}-\int_{(\tau_j,t]}|v_s^m(Y_{s-}^{j+1})|\,\mathrm{I}(Y_{\tau_j}^{j,m}>0)dG_s+\int_{(\tau_j,t]}F^m(l_s(Y^{j+1}_{s-}))dX_s. \tag{34}$$

The indicator here can be equivalently replaced by $\mathrm{I}(\inf_{u<s}Y_u^{j+1,m}>0)$, so the first integral becomes equal to $|\boldsymbol{L}_t^m(Y^{j+1})|-|\boldsymbol{L}^m_{\tau_j}(Y^{j+1})|$ by (9). In the second integral, on $(\tau_j,\tau_{j+1}]$ we have (as follows from (9))

$$l_t^{m,n}(Y_t^{j+1})=\frac{d\boldsymbol{L}_t^{m,n}(Y^{j+1})}{dG_t}.$$

Consequently, (34) implies that the process Y^{j+1} satisfies equation (6) for $t\leq\tau_{j+1}$.

Proceeding by induction, for fixed i we obtain the non-decreasing sequence of stopping times $\tau_{i,j}$ and the processes $Y^{i,j}$. Let $\tau_{i+1,0} = \lim_j \tau_{i,j} \in [0,\infty]$. On $[0, \tau_{i+1,0})$ define the process $Y^{i+1,0}$ by joining $Y^{i,j}$, i.e. for (ω, t) such that $t < \tau_{i+1,0}(\omega)$ put

$$Y_t^{i+1,0} = Y_t^{i,0}\,\mathrm{I}(t < \tau_{i,0}) + \sum_{j=1}^{\infty} Y_t^{i,j}\,\mathrm{I}(\tau_{i,j-1} \le t < \tau_{i,j}).$$

Observe that on the set $\{\tau_{i+1,0} < \infty\}$, the limit $Y^{i+1,0}_{\tau_{i+1,0}-}$ exists, since for $t < \tau_{i+1,0}$ the process $Y_t^{i+1,0}$ satisfies equation (6), in which the integral processes are non-decreasing and bounded by $X^n_{\tau_{i+1,0}}$, and the term $|L_t^m|$ is non-decreasing and bounded by $Y_0^m + |X_{\tau_{i+1,0}}|$. For $t \ge \tau_{i+1,0}$ put

$$Y_t^{i+1,0} = Y^{i+1,0}_{\tau_{i+1,0}-} - |l^m| + \sum_{n=1}^{N} F^{m,n}(l)\Delta X^n_{\tau_{i+1,0}}$$

with

$$l^{m,n} = v^{m,n}_{\tau_{i+1,0}}(Y^{i+1,0}_{\tau_{i+1,0}-})\,\mathrm{I}(\inf_{s<\tau_{i+1,0}} Y_s^{i+1,0,m} > 0)\Delta G_{\tau_{i+1,0}}$$

(the process $Y^{i+1,0}$ stays constant after $\tau_{i+1,0}$). One can see that now $Y^{i+i,0}$ satisfies equation (6) for $t \le \tau_{i+1,0}$. Then the proof of the existence of a solution is finished by induction. The uniqueness follows from the uniqueness of the fixed point of the operator U on each step of induction.

6.2. Proof of Proposition 2

As follows from Theorem 2 (see also the remark after its proof on p. 26), if a solution of the wealth equation exists and investor m uses the strategy $\widehat{\boldsymbol{L}}$, then the wealth of this investor does not vanish ($Y^m > 0$ and $Y^m_- > 0$). Therefore, it will be enough to prove Proposition 2 for a strategy profile $\mathcal{L}$ in which every investor uses either a strategy satisfying the conditions of Theorem 1, or the strategy $\widehat{\boldsymbol{L}}'$ such that, when used by investor m, its cumulative investment process is

$$\widehat{\boldsymbol{L}}'_t(m; y) = \int_0^t y^m_{s-}\widehat{\lambda}_s(|y_{s-}|)\,\mathrm{I}(\inf_{u<s} y_u^m > 0)dG_s$$

(it differs from the strategy $\widehat{\boldsymbol{L}}$ only by the presence of the indicator). In order to show that such a profile is feasible, we will verify conditions (C1), (C2) of Theorem 1 for $\widehat{\boldsymbol{L}}'(m)$.

Let $v_t^{m,n}(\omega, z) = z^m\widehat{\lambda}_t^n(\omega, |z|)$, so that $\widehat{\boldsymbol{L}}'(m)$ can be represented in the form (9). Inequality (10) is satisfied because if $\Delta G_t(\omega) > 0$ (and therefore $\bar\nu_t(\omega) > 0$), then $|\widehat{\lambda}_t(\omega, c)| = \int_{\mathbb{R}^N_+} |x|(\zeta_t(\omega, c) + |x|)^{-1}\nu_{\{t\}}(\omega, dx) \le 1$ as follows from the definition of $\widehat{\lambda}$. Hence condition (C1) holds.

In order to verify condition (C2), consider the sets

$$\Pi^{m,n} = \{(\omega, t) : h_t^n(\omega) = 0\}$$

and define the function $C(a)$ by

$$C(a) = \max\left(\frac{2|a| \vee 1}{a^m/2},\ 2|a| \vee 1,\ \frac{2 + 32|a|}{|a|^3 \wedge 1}\right) \text{ if } a^m > 0, \quad C(a) = 1 \text{ if } a^m = 0,$$

and the process δ_t by

$$\delta_t = \sup_{s\le t}\left(\max_n \frac{\mathrm{I}(h_s^n>0)}{h_s^n} \vee \frac{\mathrm{I}(\Delta G_s>0)}{p_s\Delta G_s}\right)\vee 1.$$

The local boundedness assumptions imply that δ_t is finite-valued.

Equality (11) clearly holds. To prove inequalities (12)–(13), consider $z,a\in\mathbb{R}_+^N$ such that $z^k\in[a^k/2,2a^k]$ for all k. Suppose $z^m>0$ (and, hence, $a^m>0$). Then (12) follows from that outside the set $\Pi^{m,n}$

$$v_t^{m,n}(z) = z^m\widehat{\lambda}_t^n(|z|) \ge \frac{z^m}{|z|\vee 1}h_t^n \ge \frac{a^m/2}{(2|a|\vee 1)\delta_t} \ge \frac{1}{C(a)\delta_t},$$

where in the first inequality we used the bound $\widehat{\lambda}_t(c)\ge h_t/(c\vee 1)$ for any $c>0$, which can be obtained from (21), (23) using that $\zeta_t(c)\in[0,c]$.

To prove (13), on the set Γ_0 we can use the estimate

$$\widehat{\lambda}_t^n(c) = \frac{b_t^n}{c} + \int_{\mathbb{R}_+^N}\frac{x^n}{c+|x|}K_t(dx) \le \frac{h_t^n}{c\wedge 1} \le \frac{1}{c\wedge 1}. \tag{35}$$

The last inequality here holds since $|h_t|\le |b_t| + \int_{\mathbb{R}_+^N}(1\wedge|x|)K_t(dx) = 1$ (see (20)). On the set $\Gamma_1\cup\Gamma_2$, we can use the estimate

$$\widehat{\lambda}_t^n(c) = \int_{\mathbb{R}_+^N}\frac{x^n}{\zeta_t(c)+|x|}K_t(dx) \le K_t(\mathbb{R}_+^N) = \frac{\bar{\nu}_t}{\Delta G_t} \le \frac{1}{\Delta G_t} \tag{36}$$

(note that if $(\omega,t,c)\in\Gamma_1\cup\Gamma_2$, then $\Delta G_t>0$ and $b_t=0$). Therefore, we obtain

$$v_t^{m,n}(z) \le z^m \max\left(\frac{1}{|z|\wedge 1}, \frac{\mathrm{I}(\Delta G_t>0)}{\Delta G_t}\right) \le (2|a|\vee 1)\delta_t \le C(a)\delta_t, \tag{37}$$

so (13) holds.

To prove (14), suppose $z,\widetilde{z},a\in\mathbb{R}_+^N$ and $z^k,\widetilde{z}^k\in[a^k/2,2a^k]$ for all k. If $z^m=\widetilde{z}^m=0$, then $v_t^{m,n}(z)=v_t^{m,n}(\widetilde{z})=0$, so (14) holds. If $\widetilde{z}^m=0$, but $z^m>0$, then, using (37), we obtain

$$|v_t^{m,n}(z)-v_t^{m,n}(\widetilde{z})| = v_t^{m,n}(z) \le (2|a|\vee 1)\delta_t \le \frac{(2|a|\vee 1)\delta_t}{a^m/2}|z-\widetilde{z}| \le C(a)\delta_t|z-\widetilde{z}|,$$

where we used the inequality $|z-\widetilde{z}|\ge z^m\ge a^m/2$. In a similar way, (14) is satisfied if $z^m=0$, but $\widetilde{z}^m>0$.

Let us consider the case $z^m>0$, $\widetilde{z}^m>0$. Denote $c=|z|$, $\widetilde{c}=|\widetilde{z}|$. Then

$$|v_t^{m,n}(z)-v_t^{m,n}(\widetilde{z})| \le \widehat{\lambda}_t^n(\widetilde{c})|z^m-\widetilde{z}^m| + z^m|\widehat{\lambda}_t^n(c)-\widehat{\lambda}_t^n(\widetilde{c})|. \tag{38}$$

Using (35)–(36), the first term in the right-hand side can be bounded as follows:

$$\widehat{\lambda}_t^n(\widetilde{c})|z^m-\widetilde{z}^m| \le \frac{2\delta_t}{|a|\wedge 1}|z-\widetilde{z}|.$$

For the second term in the right-hand side of (38) we have

$$\begin{aligned}
&z^m|\widehat{\lambda}_t^n(c)-\widehat{\lambda}_t^n(\widetilde{c})| \le 2a^m\biggl\{\frac{|c-\widetilde{c}|}{c\widetilde{c}}b_t^n \\
&+|c-\widetilde{c}|\left(\int_{\mathbb{R}_+^N}\frac{x^n}{(c+|x|)(\widetilde{c}+|x|)}K_t(dx)\right)\mathrm{I}(\Delta G_t=0) \\
&+|\zeta_t(c)-\zeta_t(\widetilde{c})|\left(\int_{\mathbb{R}_+^N}\frac{x^n}{(\zeta_t(c)+|x|)(\zeta_t(\widetilde{c})+|x|)}K_t(dx)\right)\mathrm{I}(\Delta G_t>0)\biggr\} \\
&:= 2a^m\{A_1+A_2+A_3\},
\end{aligned}$$

where, for brevity, A_i denote the three terms in the braces. Using that $|c-\widetilde{c}| \le |z-\widetilde{z}|$ and $|b_t| \le 1$, $|h_t| \le 1$, we obtain

$$A_1 \le \frac{4|c-\widetilde{c}|}{|a|^2} b_t^n \le \frac{4|z-\widetilde{z}|}{|a|^2},$$

and

$$A_2 \le \frac{|c-\widetilde{c}|}{c(\widetilde{c}\wedge 1)} \int_{\mathbb{R}_+^N} \frac{x^n}{1+|x|} K_t(dx) \le \frac{4|z-\widetilde{z}|}{|a|(|a|\wedge 1)} h_t^n \le \frac{4|z-\widetilde{z}|}{|a|^2 \wedge 1}.$$

Let us bound A_3. Assume $c \ge \widetilde{c}$ (hence also $\zeta_t(c) \ge \zeta_t(\widetilde{c})$) and $\zeta_t(c) > 0$. Then we have

$$\begin{aligned} A_3 &\le (\zeta_t(c) - \zeta_t(\widetilde{c})) \left(\int_{\mathbb{R}_+^N} \frac{1}{\zeta_t(c)+|x|} K_t(dx) \right) \mathrm{I}(\Delta G_t > 0) \\ &\le \frac{\zeta_t(c)-\zeta_t(\widetilde{c})}{c\Delta G_t} \mathrm{I}(\Delta G_t > 0) \le (\zeta_t(c) - \zeta_t(\widetilde{c})) \frac{2\,\mathrm{I}(\Delta G_t > 0)}{|a|\Delta G_t}, \end{aligned} \tag{39}$$

where in the second inequality we used the bound

$$\begin{aligned} \int_{\mathbb{R}_+^N} \frac{1}{\zeta_t(c)+|x|} K_t(dx) &= \frac{1}{c\Delta G_t} \int_{\mathbb{R}_+^N} \frac{c}{\zeta_t(c)+|x|} \nu_{\{t\}}(dx) \\ &= \frac{1}{c\Delta G_t} \left(1 - \frac{c}{\zeta_t(c)} (1-\bar{\nu}_t) \right) \le \frac{1}{c\Delta G_t}. \end{aligned}$$

Here the second equality follows from (17) – notice that $(\omega, t, c) \in \Gamma_1$ because we assume $\zeta_t(c) > 0$.

Now we need to bound $\zeta_t(c) - \zeta_t(\widetilde{c})$ in (39). Let Q_t be the random measure on $\mathbb{R}_+^N$ defined by $\mathrm{Q}_t(A) = \nu_{\{t\}}(A) + (1-\bar{\nu}_t)\,\mathrm{I}(0 \in A)$. Observe that $\mathrm{Q}_t(\mathbb{R}_+^N) = 1$. Since $(\omega, t, c) \in \Gamma_1$ and $(\omega, t, \widetilde{c}) \in \Gamma_1 \cup \Gamma_2$, from (17) and (18) we find that

$$\int_{\mathbb{R}_+^N} \frac{1}{\zeta_t(c)+|x|} \mathrm{Q}_t(dx) = \frac{1}{c}, \qquad \int_{\mathbb{R}_+^N} \frac{1}{\zeta_t(\widetilde{c})+|x|} \mathrm{Q}_t(dx) \le \frac{1}{\widetilde{c}}.$$

From this, we obtain

$$\begin{aligned} \frac{1}{\widetilde{c}} - \frac{1}{c} &\ge (\zeta_t(c) - \zeta_t(\widetilde{c})) \int_{\mathbb{R}_+^N} \frac{\mathrm{Q}_t(dx)}{(\zeta_t(c)+|x|)(\zeta_t(\widetilde{c})+|x|)} \\ &\ge (\zeta_t(c) - \zeta_t(\widetilde{c})) \int_{\mathbb{R}_+^N} \frac{\mathrm{Q}_t(dx)}{(c+|x|)^2} \\ &\ge (\zeta_t(c) - \zeta_t(\widetilde{c})) \int_{\mathbb{R}_+^N} \frac{\nu_{\{t\}}(dx)}{(c+|x|)^2} \ge \frac{\zeta_t(c) - \zeta_t(\widetilde{c})}{c^2 \vee 1} p_t. \end{aligned}$$

Hence, we conclude that

$$\zeta_t(c) - \zeta_t(\widetilde{c}) \le \frac{(c-\widetilde{c})(c^2 \vee 1)}{c\widetilde{c} p_t} \le \frac{4|z-\widetilde{z}|}{(|a|^2 \wedge 1)p_t}. \tag{40}$$

From (39) and (40), we find

$$A_3 \le \frac{8|z-\widetilde{z}|\,\mathrm{I}(\Delta G_t > 0)}{(|a|^3 \wedge 1)p_t \Delta G_t}.$$

This implies that inequality (14) is satisfied when $z^m > 0$ and $\widetilde{z}^m > 0$:

$$|v(z)_t^{m,n} - v(\widetilde{z})_t^{m,n}| \le \left(\frac{2}{|a| \wedge 1} + 2|a|\left(\frac{4}{|a|^2} + \frac{4}{|a|^2 \wedge 1} + \frac{8}{|a|^3 \wedge 1}\right)\right)\delta_t|z - \widetilde{z}|$$
$$\le \frac{2 + 32|a|}{|a|^3 \wedge 1}\delta_t|z - \widetilde{z}| \le C(a)\delta_t|z - \widetilde{z}|.$$

Thus, condition (C2) holds, which finishes the proof.

6.3. Proof of Theorem 2

The key idea of the proof of the first claim of the theorem is to show that $\ln r_t$ is a σ-submartingale by showing that its *drift rate* is non-negative. Since $\ln r_t$ is a non-positive process, it will be then a usual submartingale [15, Proposition 3.1]. For the reader's convenience, let us briefly recall the related notions and known results; details can be found in, e.g., [15].

A scalar semimartingale Z with $Z_0 = 0$ is called a σ-submartingale if there exists a non-decreasing sequence of predictable sets $\Pi_n \in \mathcal{P}$ such that $Z_t^{\Pi_n} := \int_0^t \mathrm{I}_s(\Pi_n)dZ_s$ is a submartingale for each n and $\bigcup_n \Pi_n = \Omega \times \mathbb{R}_+$. Suppose the triplet (B^h, C, ν) of predictable characteristics of Z with respect to a truncation function $h(z)$ admits the representation $B^h = b^h \cdot G$, $C = c \cdot G$, $\nu = K \otimes G$ with predictable processes b_t^h, c_t, a transition kernel $K_t(dz)$, and a non-decreasing predictable càdlàg process G_t. Then Z is a σ-submartingale if and only if $\mathrm{P} \otimes G$-a.e. on $\Omega \times \mathbb{R}_+$

$$\int_{|z|>1} |z|K_t(dz) < \infty \quad \text{and} \quad \mathfrak{d}_t := b_t^h + \int_{\mathbb{R}} (z - h(z))K_t(dz) \ge 0$$

(see [16, Proposition 11.2] and [15, Lemma 3.1]). The predictable process $\mathfrak{d}$ is called the drift rate of Z with respect to G. One can see that it does not depend on the choice of the truncation function h (see [14, Proposition II.2.24]).

Observe that if

$$\int_{\mathbb{R}} |z|K(dz) < \infty,$$

then $\mathfrak{d}_t = b_t^0 + \int_{\mathbb{R}} zK_t(dz)$, where $b_t^0 = b_t^h - \int_{\mathbb{R}} h(z)K_t(dz)$ is a well-defined predictable process, which does not depend on the choice of h. From this we obtain the corollary that will be used further in the proof: if Z is a non-positive semimartingale, then it will be a submartingale if $\mathrm{P} \otimes G$-a.s.

$$\int_{z<0} zK_t(dz) > -\infty \qquad \text{and} \qquad \mathfrak{d}_t = b_t^0 + \int_{\mathbb{R}} zK_t(dz) \ge 0. \tag{41}$$

In particular, observe that for a non-positive semimartingale it holds that $\int_{z>0} zK(dz) < \infty$ since $K_t(\{z : z > -Z_{t-}\}) = 0$. As a consequence, if (41) is satisfied, then the process $\mathfrak{d}$ is G-integrable and the compensator of Z_t is

$$A_t = \mathfrak{d} \cdot G_t. \tag{42}$$

Let us also state one auxiliary inequality, which generalizes well-known Gibbs' inequality, and will play an important role in the proof. Suppose $\alpha, \beta \in \mathbb{R}_+^N$ are two vectors such that $|\alpha|, |\beta| \le 1$ and for each n it holds that if $\beta^n = 0$, then also $\alpha^n = 0$. Then

$$\alpha(\ln \alpha - \ln \beta) \ge \frac{\|\alpha - \beta\|^2}{4} + |\alpha| - |\beta|, \tag{43}$$

where $\alpha^n(\ln\alpha^n - \ln\beta^n) = 0$ if $\alpha^n = 0$. A short direct proof can be found in [9, Lemma 2].

Now we can proceed to the proof of the first claim of the theorem. Assume that the strategy $\widehat{\boldsymbol{L}}$ is used by investor $m = 1$, and the wealth equation has a unique solution Y_t. We will use the notation of Section 5 and introduce the predictable $\mathbb{R}^N_+$-valued processes λ^1_t, Λ^1_t for investor 1 and $\widetilde{\lambda}_t, \widetilde{\Lambda}_t, \widetilde{\Lambda}^{(s)}_t$ for the other investors. To keep the notation concise, from now on the superscript "1" for investor 1 will be omitted, so we will simply write λ_t, Λ_t. It will be also convenient to assume that the particular version of λ_t is selected: $\lambda_t(\omega) = \widehat{\lambda}_t(\omega, W_{t-}(\omega))$ for all (ω, t) with the function $\widehat{\lambda}_t(c)$ defined in (21).

Let $W_t = Y_t + \widetilde{Y}_t$ denote the total market wealth, and $r_t = Y_t/W_t$ the relative wealth of investor 1. Define the predictable process F with values in $\mathbb{R}^N_+$ by

$$F^n_t = \frac{\lambda^n_t}{r_{t-}\lambda^n_t + (1-r_{t-})\widetilde{\lambda}^n_t},$$

where $0/0 = 0$. Then Y and W can be written as stochastic exponents

$$\begin{aligned} Y &= Y_0\mathcal{E}\Big(-|\Lambda| + \frac{F}{W_-}\cdot X\Big),\\ W &= W_0\mathcal{E}\Big(-r_-\cdot|\Lambda| - (1-r_-)\cdot|\widetilde{\Lambda}| + \frac{1}{W_-}\cdot|X|\Big). \end{aligned} \tag{44}$$

Recall that the stochastic exponent of a semimartingale S is the process $\mathcal{E}(S)$ which solves the equation $d\mathcal{E}(S)_t = \mathcal{E}(S)_t dS_t$ with $\mathcal{E}(S)_0 = 1$. It is known that $\mathcal{E}(S) > 0$ and $\mathcal{E}(S)_- > 0$ if and only if $\Delta S \neq -1$, see [14, § II.8a]. From the definition of $\widehat{\lambda}$, one can check that $\Delta(-|\Lambda| + (F/W_-)\cdot X) > -1$ up to an evanescent set, hence $Y > 0$ and $Y_- > 0$.

Let $\zeta_t(\omega)$ denote the predictable process $\zeta_t(\omega, W_{t-}(\omega))$. As follows from the definition of $\widehat{\boldsymbol{L}}$ and ζ, we have $\zeta_t = (1-|\Delta\Lambda_t|)W_{t-}$. Let $\widetilde{\zeta}_t = (1-|\Delta\widetilde{\Lambda}_t|)W_{t-}$. Define the predictable function $f(\omega, t, x)$ by

$$f_t(x) = \ln\left(\frac{\zeta_t + F_t x}{r_{t-}\zeta_t + (1-r_{t-})\widetilde{\zeta}_t + |x|}\right).$$

Using the Doléans–Dade formula, which for a process of bounded variation S takes the form $\mathcal{E}(S)_t = \exp(S^c_t + \sum_{u\le t}\ln(1+\Delta S_u))$, we obtain

$$\ln r_t = \ln r_0 + (1-r_-)\cdot(|\widetilde{\Lambda}^c_t| - |\Lambda^c_t|) + \frac{F-1}{W_-}\cdot X^c_t + \sum_{s\le t} f_s(\Delta X_s).$$

For the further analysis, it will be convenient to split the process $\ln r_t$ into several parts. Let $f_t(x) = f^1_t(x) + f^2_t(x) + f^3_t(x)$, where

$$\begin{aligned} f^1_t(x) &= f_t(x)\,\mathrm{I}(\Delta G_t = 0, \Delta\widetilde{\Lambda}_t = 0),\\ f^2_t(x) &= f_t(x)\,\mathrm{I}(\Delta G_t > 0),\\ f^3_t(x) &= f_t(x)\,\mathrm{I}(\Delta G_t = 0, \Delta\widetilde{\Lambda}_t > 0). \end{aligned}$$

Then

$$\ln r_t = \ln r_0 + Z_t + \widetilde{Z}_t \tag{45}$$

with the processes

$$Z_t = (1-r_-)\cdot(|\widetilde{\Lambda}^c_t| - |\widetilde{\Lambda}^{(s)c}_t| - |\Lambda^c_t|) + \frac{F-1}{W_-}\cdot X^c_t + \sum_{s\le t}(f^1_s + f^2_s)(\Delta X_s), \tag{46}$$

$$\widetilde{Z}_t = (1-r_-)\cdot|\widetilde{\Lambda}^{(s)c}_t| + \sum_{s\le t} f^3_s(\Delta X_s), \tag{47}$$

where $\widetilde{\Lambda}^{(s)c}_t = \widetilde{\Lambda}^{(s)}_t - \sum_{u\le t}\Delta\widetilde{\Lambda}^{(s)}_u$ is the continuous part of the singular part of the Lebesgue decomposition of $\widetilde{\Lambda}$ with respect to G.

Observe that $\mathrm{I}(\Delta X\ne 0,\ \Delta G = 0,\ \Delta\widetilde{\Lambda}\ne 0) = 0$ since the set $\{\Delta X \ne 0,\ \Delta G = 0\}$ is totally inaccessible and the process $\widetilde{\Lambda}$ is predictable. Therefore,

$$\sum_{s\le t} f^3_s(\Delta X_s) = \sum_{s\le t} f^3_s(0) = -\sum_{s\le t}\ln(1-(1-r_{s-})|\Delta\widetilde{\Lambda}^s_s|). \tag{48}$$

From this formula and (47), it follows that $\widetilde{Z}_t$ is a non-decreasing predictable càdlàg process, so in order to show that $\ln r_t$ is a σ-submartingale, it is enough to show that Z_t is a σ-submartingale.

We will make use of condition (41). Since the process Z is of bounded variation, it is not difficult to see (from, e.g., the canonical representation of a semimartingale) that its continuous part can be represented as $Z^c_t = b^0\cdot G_t$, where b_0 is the predictable process from (41). From (46), we find

$$b^0_t = (1-r_{t-})(|\widetilde{\lambda}_t| - |\lambda_t|)\,\mathrm{I}(\Delta G_t = 0) + \frac{(F_t-1)b_t}{W_{t-}}.$$

The measure of jumps μ^Z of Z is such that for a function $g(\omega,t,z)$ with $g(\omega,t,0)=0$ we have

$$g*\mu^Z_t = g(f^1+f^2)*\mu_t + \sum_{s\le t} g(f^2_s(0))\,\mathrm{I}(\Delta X_s = 0),$$

so its compensator can be represented in the form $\nu^Z = K^Z dG$ with the kernel K^Z such that

$$\int_{\mathbb{R}} g_t(z)K^Z_t(dz) = \int_{\mathbb{R}^N_+} g_t(f^1_t(x) + f^2_t(x))K_t(dx) + \frac{1-\bar{\nu}_t}{\Delta G_t} g_t(f^2_t(0))$$

(when $\Delta G_t(\omega) = 0$, we have $f^2_t(\omega,x) = 0$, so we treat the last term in the right-hand side as zero). Consequently, the drift rate of Z with respect to G is $\mathfrak{d}_t = b^0_t + \int_{\mathbb{R}} zK^Z_t(dz) = h^1_t + h^2_t$ with the predictable processes

$$h^1_t = (1-r_{t-})(|\widetilde{\lambda}_t| - |\lambda_t|)\,\mathrm{I}(\Delta G_t = 0) + \frac{(F_t-1)b_t}{W_{t-}} + \int_{\mathbb{R}^N_+} f^1_t(x)K_t(dx),$$

$$h^2_t = \int_{\mathbb{R}^n_+} f^2_t(x)K_t(dx) + \frac{1-\bar{\nu}_t}{\Delta G_t} f^2_t(0). \tag{49}$$

We need to show that $h^1, h^2 \ge 0$. For h^1, using the inequality $x-1\ge \ln x$ for $x>0$, we find that

$$(F_t - 1)b_t \ge b_t\ln(F_t), \tag{50}$$

where we put $b^n_t\ln(F^n_t) = 0$ if $F^n_t = 0$ (notice that if $F^n_t = 0$, then $\lambda^n_t = 0$, so also $b^n_t = 0$).

Introduce the set $\mathcal{X}_t(\omega) = \{x \in \mathbb{R}^N_+ : x^n = 0 \text{ if } F^n_t(\omega) = 0,\ n = 1, \dots, N\}$. On the set $\{(\omega, t, x) : \Delta G_t(\omega) = 0,\ x \in \mathcal{X}_t(\omega)\}$, using the concavity of the logarithm, the equality $\Delta\Lambda_t = 0$ if $\Delta G_t = 0$, and the inequality $\widetilde{\zeta}_t \le W_{t-}$ we obtain

$$f^1_t(x) \ge \ln\left(\frac{W_{t-} + F_t x}{W_{t-} + |x|}\right) \ge \frac{x \ln F_t}{W_{t-} + |x|}, \tag{51}$$

where we put $x^n \ln(F^n_t) = 0$ if $F^n_t = x^n = 0$. Denote

$$a_t = \int_{\mathbb{R}^N_+} \frac{x W_{t-}}{W_{t-} + |x|} K_t(dx). \tag{52}$$

As follows from (21), we have $K_t(\omega, \mathbb{R}^N_+ \setminus \mathcal{X}_t(\omega)) = 0$. Then from (51)–(52) we obtain

$$\int_{\mathbb{R}^N_+} f^1_t(x) K_t(dx) = \int_{\mathcal{X}_t} f^1_t(x) K_t(dx) \ge \frac{a_t \ln F_t}{W_{t-}} \mathrm{I}(\Delta G_t = 0).$$

Together with (50), this implies

$$h^1_t \ge \left((1 - r_{t-})(|\widetilde{\lambda}_t| - |\lambda_t|) + \frac{(a_t + b_t) \ln F_t}{W_{t-}}\right) \mathrm{I}(\Delta G_t = 0).$$

From (21), it follows that we have $\lambda_t = (a_t + b_t)/W_{t-}$ when $\Delta G_t = 0$, so on the set $\{\Delta G = 0\}$

$$\begin{aligned} h^1_t &\ge (1 - r_{t-})(|\widetilde{\lambda}_t| - |\lambda_t|) + \lambda_t \ln F_t \\ &= (1 - r_{t-})(|\widetilde{\lambda}_t| - |\lambda_t|) + \lambda_t(\ln \lambda_t - \ln(r_{t-}\lambda_t + (1 - r_{t-})\widetilde{\lambda}_t)). \end{aligned}$$

Applying inequality (43), we obtain

$$h^1_t \ge \frac{1}{4}(1 - r_{t-})^2 \|\lambda_t - \widetilde{\lambda}_t\|^2 \, \mathrm{I}(\Delta G_t = 0) \ge 0. \tag{53}$$

Let us prove that $h^2 \ge 0$. Consider the set $\{\Delta G > 0\}$, on which we have

$$\begin{aligned} f^2_t(x) &= \ln\left(\frac{\zeta_t + F_t x}{r_{t-}\zeta_t + (1 - r_{t-})\widetilde{\zeta}_t + |x|}\right) \\ &= \ln\left(\frac{\zeta_t + F_t x}{\zeta_t + |x|}\right) + \ln\left(\frac{\zeta_t + |x|}{r_{t-}\zeta_t + (1 - r_{t-})\widetilde{\zeta}_t + |x|}\right), \end{aligned}$$

and, using the concavity of the logarithm, we find that for $x \in \mathcal{X}_t(\omega)$

$$f^2_t(x) \ge \frac{x \ln F_t}{\zeta_t + |x|} + \ln\left(\frac{\zeta_t + |x|}{r_{t-}\zeta_t + (1 - r_{t-})\widetilde{\zeta}_t + |x|}\right) := A_t(x) + B_t(x). \tag{54}$$

For the term $A_t(x)$, applying inequality (43), we get

$$\begin{aligned} \int_{\mathbb{R}^N_+} A_t(x) K_t(dx) &= \lambda_t \ln(F_t) = \lambda_t(\ln \lambda_t - \ln(r_{t-}\lambda_t + (1 - r_{t-})\widetilde{\lambda}_t)) \\ &\ge \frac{1}{4}(1 - r_{t-})^2 \|\lambda_t - \widetilde{\lambda}_t\|^2 + (1 - r_{t-})(|\lambda_t| - |\widetilde{\lambda}_t|). \end{aligned}$$

For the term $B_t(x)$, using the inequality $\ln x \geq 1 - x^{-1}$, we obtain

$$B_t(x) \geq \frac{(1-r_{t-})(\zeta_t - \widetilde{\zeta}_t)}{\zeta_t + |x|}.$$

From the definition of ζ_t (see (17)), it follows that

$$\int_{\mathbb{R}_+^N} \frac{1}{\zeta_t + |x|} K_t(dx) \geq \frac{1}{W_{t-}\Delta G_t} - \frac{1-\bar{\nu}_t}{\zeta_t \Delta G_t},$$

so we have

$$\int_{\mathbb{R}_+^N} B_t(x) K_t(dx) \geq (1-r_{t-})(|\widetilde{\lambda}_t| - |\lambda_t|) - \frac{(1-r_{t-})(1-\bar{\nu}_t)(\zeta_t - \widetilde{\zeta}_t)}{\zeta_t \Delta G_t}, \tag{55}$$

where for the first term in the right-hand side we used that $\zeta_t - \widetilde{\zeta}_t = (|\widetilde{\lambda}_t| - |\lambda_t|) W_{t-}\Delta G_t$. Thus, using (54)–(55) and that $K_t(\omega, \mathbb{R}_+^N \setminus \mathcal{X}_t(\omega)) = 0$, we find

$$\int_{\mathbb{R}_+^N} f_t^2(x) K_t(dx) \geq \frac{1}{4}(1-r_{t-})^2 \|\lambda_t - \widetilde{\lambda}_t\|^2 - \frac{(1-r_{t-})(1-\bar{\nu}_t)(\zeta_t - \widetilde{\zeta}_t)}{\zeta_t \Delta G_t}. \tag{56}$$

Also, using again the inequality $\ln x \geq 1 - x^{-1}$, we obtain

$$f_t^2(0) = \ln\left(\frac{\zeta_t}{r_{t-}\zeta_t + (1-r_{t-})\widetilde{\zeta}_t}\right) \geq \frac{(1-r_{t-})(\zeta_t - \widetilde{\zeta}_t)}{\zeta_t}. \tag{57}$$

Hence, from (49), (56), and (57), we find that

$$h_t^2 \geq \frac{1}{4}(1-r_{t-})^2 \|\lambda_t - \widetilde{\lambda}_t\|^2 \,\mathrm{I}(\Delta G_t > 0) \geq 0. \tag{58}$$

Thus, we have proved that $h^1, h^2 \geq 0$, so $\ln r_t$ is a submartingale, which finishes the proof of the first claim of the theorem.

To prove the second claim, suppose investors $m = 1, \dots, M-1$ use the strategy $\widehat{\boldsymbol{L}}$ and investor M use some strategy $\boldsymbol{L}$. If r_t^M is a submartingale, then $\ln r_t^1$ is a supermartingale by Jensen's inequality, and hence a martingale by the first claim of the theorem. Consequently, we find from (45) (with the same notation as above)

$$\widetilde{Z}_t = 0 \text{ a.s. for all } t \geq 0, \qquad h^1 + h^2 = 0 \;\; \mathrm{P} \otimes G\text{-a.s.}$$

The first equality implies that $L^{(s),M} = 0$, so $L^M \ll G$. The second equality, together with (53) and (58), implies that $\widetilde{\lambda}_t = \widehat{\lambda}_t(W_{t-})$ $\mathrm{P} \otimes G$-a.s., and therefore $\lambda_t^M = \widehat{\lambda}_t(W_{t-})$ $\mathrm{P} \otimes G$-a.s. Then from (22) we obtain $L^M = \widehat{\boldsymbol{L}}^M(Y)$, which finishes the proof.

Remark. As can be seen from the proof, the wealth of an investor who uses the strategy $\widehat{\boldsymbol{L}}$ does not vanish ($Y^m > 0$ and $Y_-^m > 0$) on any solution of the wealth equation (if it exists). This fact is needed in the proof of Proposition 2.

6.4. Proof of Theorem 3

We will use the same notation as in the proof of Theorem 2. Since $\ln r_t$ is a non-positive submartingale, there exists the limit $r_\infty = \lim_{t\to\infty} r_t$. As we have shown, $\ln r_t = \ln r_0 + Z_t + \widetilde{Z}_t$, where Z_t is a submartingale with drift rate

$$\mathfrak{d}_t = h_t^1 + h_t^2 \ge \frac{1}{4}(1-r_{t-})^2\|\lambda_t - \widetilde{\lambda}_t\|^2 = \frac{1}{4}\|\lambda_t - \bar{\lambda}_t\|^2.$$

Hence the compensator $A_t = \mathfrak{d}\cdot G_t$ of Z_t (see (42)) satisfies the inequality

$$A_t \ge \frac{1}{4}\|\lambda - \bar{\lambda}\|^2 \cdot G_t.$$

Since Z_t is bounded from above ($Z_t \le -\ln r_0$), A_t converges to a finite limit A_∞, so $\|\lambda - \bar{\lambda}\|^2 \cdot G_\infty < \infty$. Moreover, on the set $\{\|\lambda - \widetilde{\lambda}\|^2 \cdot G_\infty = \infty\}$ we necessarily have $r_\infty = 1$, because otherwise we would have $A_\infty = \infty$ on this set.

From the inequality $\ln(1-(1-r_{s-})|\Delta\widetilde{\Lambda}_s^{(s)}|) \le -(1-r_{s-})|\Delta\widetilde{\Lambda}_s^{(s)}|$ and (47), (48), we obtain

$$\widetilde{Z}_t \ge (1-r_-)\cdot|\widetilde{\Lambda}_t^{(s)}| = |\bar{\Lambda}_t^{(s)}|.$$

Since $\widetilde{Z}$ converges, we have $|\bar{\Lambda}_\infty^{(s)}| < \infty$, and on the set $\{|\widetilde{\Lambda}_\infty^{(s)}| = \infty\}$ we have $r_\infty = 1$.

6.5. Proof of Theorem 4

Suppose all the investors use the strategy $\widehat{\boldsymbol{L}}$. By virtue of (44), $W_t = W_0\mathcal{E}(S)_t$ with the process

$$S_t = -|\widehat{\lambda}(W_-)|\cdot G_t + \frac{1}{W_-}\cdot|X_t| = -\frac{|x|}{\zeta+|x|} * \nu_t + \sum_{s\le t}\frac{|\Delta X_s|}{W_{s-}},$$

where ζ denotes the predictable process $\zeta_t(W_{t-})$. In particular, the continuous part S_t^c and the jumps ΔS_t are given by

$$\begin{aligned} S_t^c &= -\frac{|x|\,\mathrm{I}(\bar{\nu}=0)}{W_- + |x|} * \nu_t,\\ \Delta S_t &= -\int_{\mathbb{R}_+^N}\frac{|x|}{\zeta_t+|x|}\nu_{\{t\}}(dx) + \frac{|\Delta X_t|}{W_{t-}} = \frac{\zeta_t + |\Delta X_t|}{W_{t-}} - 1. \end{aligned}$$

From the formula $\mathcal{E}(S)_t = \exp(S_t^c + \sum_{u\le t}\ln(1+\Delta S_u))$, we find $V_t = V_0\mathcal{E}(U)_t$ with the process

$$U_t = -S_t^c - \sum_{s\le t}\frac{\Delta S_s}{1+\Delta S_s} = -S_t^c + \sum_{s\le t}\left(\frac{W_{s-}}{\zeta_s + |\Delta X_s|} - 1\right). \tag{59}$$

The continuous part of U_t is $U_t^c = -S_t^c = b^U\cdot G_t$ with the predictable process

$$b_t^U = \int_{\mathbb{R}_+^N}\frac{|x|}{W_{t-}+|x|}K_t(dx)\,\mathrm{I}(\Delta G_t = 0),$$

and the measure of jumps μ^U acts on functions $f(\omega,t,u)$ with $f(\omega,t,0)=0$ as

$$f * \mu_t^U = f\left(\frac{W_-}{\zeta+|x|} - 1\right) * \mu_t + \sum_{s\le t} f_s\left(\frac{W_{s-}}{\zeta_s} - 1\right)\mathrm{I}(\Delta X_s = 0,\ \bar{\nu}_s > 0),$$

so its compensator ν^U is such that

$$f * \nu^U_t = f\Big(\frac{W_-}{\zeta + |x|} - 1\Big) * \nu_t + \sum_{s\le t} f_s\Big(\frac{W_{s-}}{\zeta_s} - 1\Big)(1-\bar\nu_s)\,\mathrm{I}(\bar\nu_s > 0).$$

In particular, $\nu^U = K^U \otimes G$ with the transition kernel K^U such that

$$\int_{\mathbb{R}} f_t(u) K^U_t(du) = \int_{\mathbb{R}^N_+} f_t\Big(\frac{W_{t-}}{\zeta_t + |x|} - 1\Big) K_t(dx) + f\Big(\frac{W_{t-}}{\zeta_t} - 1\Big)\frac{(1-\bar\nu_t)}{\Delta G_t}\,\mathrm{I}(\Delta G_t > 0).$$

From the definition of ζ in Lemma 2, it follows that $\int_{\mathbb{R}} |u| K^U_t(du) < \infty$, and hence the drift rate of U with respect to G_t is given by

$$\mathfrak{d}^U_t = b^U_t + \int_{\mathbb{R}} u K^U_t(du) \le 0,$$

where the inequality follows from that on the set $\{\Delta G = 0\}$ we have $\zeta_t = W_{t-}$, and on the set $\{\Delta G > 0\}$ we have

$$\int_{\mathbb{R}^N_+}\Big(\frac{W_{t-}}{\zeta_t + |x|} - 1\Big) K_t(dx) \le \Big(1 - \frac{W_{t-}}{\zeta_t}\Big)\frac{(1-\bar\nu_t)}{\Delta G_t}$$

in view of that $K_t(dx) = (\Delta G_t)^{-1}\nu_{\{t\}}(dx)$ and the definition of ζ.

Consequently, U_t is a σ-supermartingale. This implies that V_t is also a σ-supermartingale, and, hence, a usual supermartingale because it is non-negative. In particular, it has an a.s.-limit $V_\infty = \lim_{t\to\infty} V_t \in [0,\infty)$, and therefore $W_\infty = 1/V_\infty \in (0,\infty]$, which proves the first claim of the theorem.

If $\bar\nu \equiv 0$, we have $\zeta_t = W_{t-}$ for all t, so equation (59) becomes

$$U_t = -\frac{|x|}{W_- + |x|} * (\mu_t - \nu_t),$$

and, hence, U_t is a purely discontinuous local martingale with bounded jumps, $\Delta U_t \in (-1, 0]$. Consequently, according to Proposition 7.1 in [16], we have $\{V_\infty = 0\} = \{|u|^2 * \nu^U_\infty = \infty\}$ a.s., or equivalently $\{W_\infty = \infty\} = \{(\frac{|x|}{W_- + |x|})^2 * \nu_\infty = \infty\}$ a.s. From this and the existence of the limit W_∞ follows the second claim of the theorem.

7. Appendix: Lebesgue derivatives

In this appendix we assemble several known facts about the Lebesgue decomposition and Lebesgue derivatives of σ-finite measures, and prove auxiliary results for random measures generated by predictable non-decreasing càdlàg processes.

The Lebesgue decomposition of σ-finite measures. Let $(\Omega, \mathcal{F})$ be a measurable space. First recall the following known result, which can be found (in a slightly different form), e.g., in Chapter 3.2 of [6].

Proposition 3. *Let $\mathrm{P}, \widetilde{\mathrm{P}}$ be σ-finite measures on $(\Omega, \mathcal{F})$. Then there exists a measurable function $Z \ge 0$ (P-a.s. and $\widetilde{\mathrm{P}}$-a.s.) and a set $\Gamma \in \mathcal{F}$ such that*

$$\widetilde{\mathrm{P}}(A) = \int_A Z\, d\mathrm{P} + \widetilde{\mathrm{P}}(A \cap \Gamma) \quad \textit{for any } A \in \mathcal{F}, \tag{60}$$

and
$$\mathrm{P}(\Gamma) = 0. \tag{61}$$
Such Z is P*-a.s. unique and Γ is* $\widetilde{\mathrm{P}}$*-a.s. unique, i.e. if Z' and Γ' also satisfy the above properties, then $Z = Z'$* P*-a.s., and* $\widetilde{\mathrm{P}}(\Gamma \triangle \Gamma') = 0$ *(where $\Gamma\triangle\Gamma' = \Gamma\setminus\Gamma' \cup \Gamma'\setminus\Gamma$ denotes the symmetric difference).*

The function Z – the Lebesgue derivative of $\widetilde{\mathrm{P}}$ with respect to P – is denoted in this paper by $d\widetilde{\mathrm{P}}/d\mathrm{P}$. If $\widetilde{\mathrm{P}} \ll \mathrm{P}$, the Lebesgue derivative coincides with the Radon–Nikodym derivative and one can take $\Gamma = \emptyset$. When it is necessary to emphasize that the set Γ is related to $\widetilde{\mathrm{P}}$ and P, we use the notation $\Gamma_{\widetilde{\mathrm{P}}/\mathrm{P}}$.

In an explicit form, Z and Γ can be constructed as follows. Let Q be any σ-finite measure on $(\Omega, \mathcal{F})$ such that $\mathrm{P} \ll \mathrm{Q}$, $\widetilde{\mathrm{P}} \ll \mathrm{Q}$ (for example, $\mathrm{Q} = \mathrm{P} + \widetilde{\mathrm{P}}$). Then
$$Z = \frac{d\widetilde{\mathrm{P}}}{d\mathrm{Q}}\left(\frac{d\mathrm{P}}{d\mathrm{Q}}\right)^{-1} \mathrm{I}\left(\frac{d\mathrm{P}}{d\mathrm{Q}} > 0\right), \qquad \Gamma = \left\{\omega : \frac{d\mathrm{P}}{d\mathrm{Q}}(\omega) = 0\right\},$$
where the derivatives are in the Radon–Nikodym sense.

By approximating a measurable function with simple functions, from (60), it follows that for any $\mathcal{F}$-measurable function $f \geq 0$
$$\int_\Omega f d\widetilde{\mathrm{P}} = \int_\Omega f \frac{d\widetilde{\mathrm{P}}}{d\mathrm{P}} d\mathrm{P} + \int_\Omega f\, \mathrm{I}(\Gamma) d\widetilde{\mathrm{P}} \tag{62}$$
(where the integrals may assume the value $+\infty$).

The following proposition contains facts about Lebesgue derivatives that are used in the paper.

Proposition 4. *Let $\mathrm{P}, \widetilde{\mathrm{P}}, \mathrm{Q}$ be σ-finite measures on $(\Omega, \mathcal{F})$. Then the following statements are true.*

(a) Suppose Q is representable in the form $\mathrm{Q}(A) = \int_A f d\mathrm{P} + \int_A \widetilde{f} d\widetilde{\mathrm{P}}$, where $f, \widetilde{f} \geq 0$ are measurable functions, and $\widetilde{f} = 0$ P*-a.s. Then*
$$\frac{d\mathrm{P}}{d\mathrm{Q}} = \frac{1}{f}\mathrm{I}(f > 0, \widetilde{f} = 0), \quad \Gamma_{\mathrm{P}/\mathrm{Q}} = \{f = 0, \widetilde{f} = 0\}.$$

(b) If R is a σ-finite measure such that $\mathrm{R} \ll \mathrm{P}$ and $\mathrm{R} \ll \mathrm{Q}$, then
$$\frac{d\widetilde{\mathrm{P}}}{d\mathrm{P}} = \frac{d\widetilde{\mathrm{P}}}{d\mathrm{Q}}\frac{d\mathrm{Q}}{d\mathrm{P}} \quad \mathrm{R}\text{-}a.s. \tag{63}$$

(c) If R is as in (b), then $d\mathrm{Q}/d\mathrm{P} > 0$ and $d\mathrm{P}/d\mathrm{Q} > 0$ R*-a.s.*

Proof. (a) is obtained by straightforward verification of (60)–(61).

(b) Observe that for any $A \in \mathcal{F}$ we have
$$\begin{aligned}\widetilde{\mathrm{P}}(A) &= \int_A \frac{d\widetilde{\mathrm{P}}}{d\mathrm{Q}} d\mathrm{Q} + \widetilde{\mathrm{P}}(A \cap \Gamma_{\widetilde{\mathrm{P}}/\mathrm{Q}}) \\ &= \int_A \frac{d\widetilde{\mathrm{P}}}{d\mathrm{Q}}\frac{d\mathrm{Q}}{d\mathrm{P}} d\mathrm{P} + \int_\Omega \mathrm{I}(A \cap \Gamma_{\mathrm{Q}/\mathrm{P}})\frac{d\widetilde{\mathrm{P}}}{d\mathrm{Q}} d\mathrm{Q} + \widetilde{\mathrm{P}}(A \cap \Gamma_{\widetilde{\mathrm{P}}/\mathrm{Q}}) \\ &= \int_A \frac{d\widetilde{\mathrm{P}}}{d\mathrm{Q}}\frac{d\mathrm{Q}}{d\mathrm{P}} d\mathrm{P} + \widetilde{\mathrm{P}}(A \cap (\Gamma_{\mathrm{Q}/\mathrm{P}} \cup \Gamma_{\widetilde{\mathrm{P}}/\mathrm{Q}})),\end{aligned} \tag{64}$$

where to obtain the second equality we applied (62), and to obtain the third one we expressed the second integral in the second line from the equality

$$\widetilde{\mathrm{P}}(A\cap\Gamma_{\mathrm{Q}/\mathrm{P}})=\int_{\Omega}\mathrm{I}(A\cap\Gamma_{\mathrm{Q}/\mathrm{P}})\frac{d\widetilde{\mathrm{P}}}{d\mathrm{Q}}d\mathrm{Q}+\widetilde{\mathrm{P}}(A\cap\Gamma_{\mathrm{Q}/\mathrm{P}}\cap\Gamma_{\widetilde{\mathrm{P}}/\mathrm{Q}}).$$

Suppose for $A=\{\frac{d\widetilde{\mathrm{P}}}{d\mathrm{P}}>\frac{d\widetilde{\mathrm{P}}}{d\mathrm{Q}}\frac{d\mathrm{Q}}{d\mathrm{P}}\}$ we have $\mathrm{R}(A)>0$. Then also $\mathrm{R}(A')>0$ for $A'=A\cap(\Gamma_{\mathrm{Q}/\mathrm{P}}\cup\Gamma_{\widetilde{\mathrm{P}}/\mathrm{Q}}\cup\Gamma_{\widetilde{\mathrm{P}}/\mathrm{P}})^c$ because $\mathrm{R}(\Gamma_{\mathrm{Q}/\mathrm{P}})=\mathrm{R}(\Gamma_{\widetilde{\mathrm{P}}/\mathrm{Q}})=\mathrm{R}(\Gamma_{\widetilde{\mathrm{P}}/\mathrm{P}})=0$. Consequently, $\mathrm{P}(A')>0$. But this leads to a contradiction between decomposition (60) and equality (64) for $\widetilde{\mathrm{P}}(A')$, since according to them we would have

$$\int_{A'}\frac{d\widetilde{\mathrm{P}}}{d\mathrm{P}}d\mathrm{P}=\int_{A'}\frac{d\widetilde{\mathrm{P}}}{d\mathrm{Q}}\frac{d\mathrm{Q}}{d\mathrm{P}}d\mathrm{P},$$

which is impossible due to the choice of A. Hence $\mathrm{R}(\frac{d\widetilde{\mathrm{P}}}{d\mathrm{P}}>\frac{d\widetilde{\mathrm{P}}}{d\mathrm{Q}}\frac{d\mathrm{Q}}{d\mathrm{P}})=0$. In the same way we show that $\mathrm{R}(\frac{d\widetilde{\mathrm{P}}}{d\mathrm{P}}<\frac{d\widetilde{\mathrm{P}}}{d\mathrm{Q}}\frac{d\mathrm{Q}}{d\mathrm{P}})=0$.

(c) follows from (63) if one takes $\widetilde{\mathrm{P}}=\mathrm{P}$. □

The Lebesgue decomposition of non-decreasing predictable processes. Let $(\Omega,\mathcal{F},(\mathcal{F}_t)_{t\geq 0},\mathrm{P})$ be a filtered probability space satisfying the usual assumptions, and $\mathcal{P}$ be the predictable σ-algebra on $\Omega\times\mathbb{R}_+$. For a scalar non-decreasing càdlàg predictable process G, denote by $\mathrm{P}\otimes G$ the measure on $\mathcal{P}$ defined as

$$\mathrm{P}\otimes G(A)=\mathrm{E}(I(A)\cdot G_\infty),\qquad A\in\mathcal{P}. \tag{65}$$

Observe that $\mathrm{P}\otimes G$ is σ-finite on $\mathcal{P}$. Indeed, this can be shown by considering the predictable stopping times $\tau_n=\inf\{t\geq 0: G_t\geq n\}$. The stochastic intervals $A_n=[0,\tau_n):=\{(\omega,t):t<\tau_n(\omega)\}$ are predictable, i.e. $A_n\in\mathcal{P}$, while $\mathrm{P}\otimes G(A_n)\leq n$ and $\bigcup_n A_n=\Omega\times\mathbb{R}_+$.

Proposition 5. *(a) For any scalar non-decreasing càdlàg predictable processes $G,\widetilde{G}$ there exists a predictable process $\xi\geq 0$ and a set $\Gamma\in\mathcal{P}$ such that up to* P*-indistinguishability*

$$\widetilde{G}=\widetilde{G}_0+\xi\cdot G+\mathrm{I}(\Gamma)\cdot\widetilde{G}\quad\text{and}\quad\mathrm{I}(\Gamma)\cdot G=0. \tag{66}$$

(b) A predictable process $\xi\geq 0$ and a set $\Gamma\in\mathcal{P}$ satisfy (66) if and only if ξ is a version of the Lebesgue derivative $d(\mathrm{P}\otimes\widetilde{G})/d(\mathrm{P}\otimes G)$ and Γ is the corresponding set from the Lebesgue decomposition.

We denote any $\mathrm{P}\otimes G$-version of such a process ξ by $d\widetilde{G}/dG$ or $d\widetilde{G}_t/dG_t$, and call it a predictable Lebesgue derivative of $\widetilde{G}$ with respect to G. When it is necessary to emphasize that the set Γ is related to $\widetilde{G}$ and G, we use the notation $\Gamma_{\widetilde{G}/G}$.

Proof. Without loss of generality assume $\widetilde{G}_0=0$.

(a) Let $\xi=d(\mathrm{P}\otimes\widetilde{G})/d(\mathrm{P}\otimes G)$ and Γ be the corresponding set from the Lebesgue decomposition. Define the process

$$\widetilde{G}'=\xi\cdot G+\mathrm{I}(\Gamma)\cdot\widetilde{G}.$$

We have to show that $\widetilde{G}'=\widetilde{G}$. Since $\widetilde{G}'$ and $\widetilde{G}$ are càdlàg, it is enough to show that $\widetilde{G}'_t=\widetilde{G}_t$ a.s. for any $t\geq 0$, and this is equivalent to that

$$\mathrm{E}(\widetilde{G}'_t\,\mathrm{I}(B))=\mathrm{E}(\widetilde{G}_t\,\mathrm{I}(B))\quad\text{for any }B\in\mathcal{F}_t. \tag{67}$$

Let M be the bounded càdlàg martingale such that $M_u = \mathrm{E}(\mathrm{I}(B) \mid \mathcal{F}_u)$. We have

$$\mathrm{E}(\widetilde{G}'_t \, \mathrm{I}(B)) = \mathrm{E}(\widetilde{G}'_t M_t) = \mathrm{E}(M_- \cdot \widetilde{G}'_t),$$

and, similarly,

$$\mathrm{E}(\widetilde{G}_t \, \mathrm{I}(B)) = \mathrm{E}(M_- \cdot \widetilde{G}_t), \tag{68}$$

where we used the following fact: if A_t is a non-decreasing càdlàg predictable process and M_t is a bounded càdlàg martingale, then for any stopping time τ we have $\mathrm{E}(M_\tau A_\tau) = \mathrm{E}(M_- \cdot A_\tau)$. This result is proved in [14, Lemma I.3.12] in the case $\mathrm{E}\, A_\infty < \infty$, from which our case follows by a localization procedure.

Finally, from the definition of $\widetilde{G}'$ and the Lebesgue decomposition of the measure $\mathrm{P} \otimes \widetilde{G}$, it follows that the measures $\mathrm{P} \otimes \widetilde{G}$ and $\mathrm{P} \otimes \widetilde{G}'$ coincide. Hence, for any non-negative $\mathcal{P}$-measurable function f we have $\mathrm{E}(f \cdot \widetilde{G}_t) = \mathrm{E}(f \cdot \widetilde{G}'_t)$, which finishes the proof by (67)–(68).

(b) In view of the construction in (a), it only remains to show that if ξ, Γ satisfy (66), then ξ is the Lebesgue derivative and Γ is the corresponding predictable set. This follows from straightforward verification of properties (60)–(61). □

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