An evolutionary finance model with short-lived assets and affine payoffs

1. Introduction

2. The model

Let $(\Omega, \mathcal{F}, \mathrm{P})$ be a probability space with a complete discrete-time filtration $\mathbb{F} = (\mathcal{F}_t)_{t\geq 0}$. The market in the model consists of $M \geq 2$ agents and $N \geq 2$ short-lived assets. Agent m, where $m = 1, \dots, M$, is characterized by her wealth W^m and her strategy Λ^m. The wealth $W^m = (W_t^m)_{t=0}^\infty$ is an $\mathbb{F}$-adapted random sequence, where $W_t^m = W_t^m(\omega)$ is the wealth held by the agent at time t. The variables W_t^m depend on the payoffs yielded by the assets and the strategies of the other agents through a certain equation which will be stated later. The total market wealth at time t will be denoted by $C_t = W_t^1 + \ldots + W_t^M$.

A strategy of agent m is a sequence $\Lambda^m = (\Lambda_t^m)_{t=0}^\infty$ of vector-valued functions $\Lambda_t^m = \Lambda_t^m(\omega, c)$ with values in the standard N-simplex $\Delta_N = \{\lambda \in \mathbb{R}_+^N : \lambda^1 + \ldots + \lambda^N = 1\}$ and measurable with respect to $\mathcal{F}_t \otimes \mathcal{B}(\mathbb{R}_+)$. For $n = 1, \dots, N$, the n-th coordinate $\Lambda_t^{m,n}$ specifies the proportion of wealth which agent m allocates for buying asset n at time t. Short sales are not allowed. A strategy may depend on a random state of the world ω and the total market wealth C_t (through the argument c of Λ_t^m), so that if the total market wealth at state ω is $C_t(\omega)$, then agent m allocates her wealth in proportions given by the vector $\Lambda_t^m(\omega, C_t(\omega))$.

It is possible to consider strategies of a more general form, for example depending on full market history, but this will not increase the generality of the main results. On the other hand, the dependence on the total market wealth is necessary and cannot be removed from the model because the survival strategy that we construct below needs it.

Each asset $n = 1, \dots, N$ at each moment of time $t \geq 1$ yields a random payoff which depends on the total amount of wealth invested by the agents in this asset at time $t-1$ according to the formula

$$P_t^n(\omega) = X_t^n(\omega) \sum_{m=1}^M \lambda_{t-1}^{m,n}(\omega) W_{t-1}^m(\omega) + Y_t^n(\omega), \tag{1}$$

where $X^n = (X_t^n)_{t=1}^\infty$ and $Y^n = (Y_t^n)_{t=1}^\infty$ are some exogenously given non-negative random sequence adapted to the filtration $\mathbb{F}$, and

$$\lambda_{t-1}^{m,n}(\omega) = \Lambda_{t-1}^{m,n}(\omega, C_{t-1}(\omega)), \qquad C_{t-1}(\omega) = \sum_{m=1}^M W_{t-1}^m(\omega).$$

We will assume that for each $t \geq 0$ and $n = 1, \dots, N$ it holds that

$$\mathrm{P}(Y_{t+1}^n > 0 \mid \mathcal{F}_t) > 0 \text{ a.s.} \tag{2}$$

The payoff of each asset is divided between the agents proportionally to the amount of wealth they allocated for investing in this asset at time $t-1$. As a result, the wealth sequence of agent m satisfies the recursive relation

$$W_{t+1}^m = \sum_{n=1}^N \frac{\lambda_t^{m,n} W_t^m}{\sum_{k=1}^M \lambda_t^{k,n} W_t^k} P_{t+1}^n = \sum_{n=1}^N \lambda_t^{m,n} W_t^m \biggl(X_{t+1}^n + \frac{Y_{t+1}^n}{\sum_{k=1}^M \lambda_t^{k,n} W_t^k} \biggr). \tag{3}$$

It is clear that given an initial condition $W_0 = (W_0^1, \dots, W_0^M)$ and a strategy profile $\Lambda = (\Lambda^1, \dots, \Lambda^M)$, the sequence $W_t = (W_t^1, \dots, W_t^M)$ is well-defined by the above relation, provided that for all $t \ge 0$ with probability 1 we have

$$\sum_{k=1}^M \lambda_t^{k,n} W_t^k \neq 0. \tag{4}$$

In what follows we will always assume that the objects defining the market model (i.e. agents' strategies and asset payoffs) are such that inequality (4) holds true. A sufficient condition for its validity consists in that for each $t \ge 1$ we have, with probability 1,

$$\sum_{n=1}^N (X_t^n + Y_t^n) > 0, \tag{5}$$

and there is at least one agent who uses a fully diversified strategy, i.e. for some $m = 1, \dots, M$ and all $t \ge 0$, $n = 1, \dots, N$ we have

$$\lambda_t^{m,n} > 0. \tag{6}$$

It is not difficult to see that if inequalities (6)–(6) are true, then agent m has strictly positive wealth at all moments of time, so (4) also holds. The survival strategy $\widehat{\Lambda}$ which we construct below will satisfy assumption (6).

It is easy to see that if one formally puts $Y_{t+1}^n = 0$ for all t, n (although our models does not permit to do so in view of assumption (2)), the model reduces to the standard mathematical finance model of an asset market with exogenous prices, where X_{t+1}^n are equal to relative changes of asset prices, i.e. $X_{t+1}^n = S_{t+1}^n / S_t^n$. When $X_{t+1}^n = 0$ for all t, n, it becomes the evolutionary finance model of a market with short-lived assets introduced by Amir et al. (2013).

3. Survival strategies

Definition 1. A strategy $\widehat{\Lambda}$ is called *survival* if for any strategy profile $\Lambda = (\Lambda^1, \dots, \Lambda^M)$ with $\Lambda^1 = \widehat{\Lambda}$ and any initial wealth vector $W_0 = (W_0^1, \dots, W_0^M)$ with $W_0^1 > 0$ it holds that

$$\inf_{t \ge 0} W_t^1 > 0 \text{ a.s.}$$

Denote by C_t the total market wealth, and r_t^m the relative wealth of agent m:

$$C_t = \sum_{m=1}^M W_t^m, \qquad r_t^m = \frac{W_t^m}{C_t}.$$

Definition 2. A strategy $\widehat{\Lambda}$ is called *relative growth optimal* if for any strategy profile $\Lambda = (\Lambda^1, \dots, \Lambda^M)$ with $\Lambda^1 = \widehat{\Lambda}$ and any initial wealth vector $W_0 = (W_0^1, \dots, W_0^M)$ with $W_0^1 > 0$ it holds that $\ln r_t^1$ is a submartingale.

Proposition 1. *A relative growth optimal strategy is survival.*

Proof. The result follows from that a non-positive submartingale converges to a finite limit with probability one. Hence, if agent 1 uses a relative growth optimal strategy, $\ln r_t^1$ has a finite limit as $t \to \infty$. $\square$

Now we will construct a relative growth optimal strategy. Let us introduce notation, which will be used in what follows.

For each $t \geq 0$, let $\mathrm{P}_t(\omega, d\widetilde{\omega})$ denote some variant of the regular conditional probability conditioned on $\mathcal{F}_t$, i.e. $\mathrm{P}_t(\omega, A) = \mathrm{P}(A \mid \mathcal{F}_t)(\omega)$ a.s. for any $A \in \mathcal{F}$. By E_t we will denote the conditional expectation computed with respect to P_t, i.e. $(\mathrm{E}_t Z)(\omega) = \int_\Omega Z(\widetilde{\omega})\mathrm{P}(\omega, d\widetilde{\omega})$ for any random variable Z for which the integral is well-defined.

For an $\mathcal{F} \otimes \mathcal{B}(\mathbb{R}^d)$-measurable function $f(\omega, x)$, the conditional expectation $\mathrm{E}_t f$ will be interpreted as the $\mathcal{F}_t \otimes \mathcal{B}(\mathbb{R}^d)$-measurable function

$$(\mathrm{E}_t f)(\omega, x) = \int_\Omega f(\widetilde{\omega}, x)\mathrm{P}(\omega, d\widetilde{\omega}).$$

Lemma 1. *For each $t \geq 0$, consider the $\mathcal{F}_t \otimes \mathcal{B}(\mathbb{R}_+) \otimes \mathcal{B}(\Delta_N)$-measurable function $L_t \colon \Omega \times \mathbb{R}_+ \times \Delta_N \to \Delta_N$ defined by*

$$L_t^n(\omega, c, \lambda) = \mathrm{E}_t \left(\frac{c\lambda^n X_{t+1}^n + Y_{t+1}^n}{\sum_{i=1}^N (c\lambda^i X_{t+1}^i + Y_{t+1}^i)} \right).$$

Then there exists an $\mathcal{F}_t \otimes \mathcal{B}(\mathbb{R}_+)$-measurable function $\widehat{\Lambda}_t(\omega, c)$ with values in Δ_N such that

$$L(\omega, c, \widehat{\Lambda}_t(\omega, c)) = \widehat{\Lambda}_t(\omega, c) \text{ for all } \omega, c. \tag{7}$$

Proof. (Not finished). For fixed (w, c), the map $\lambda \mapsto L_t^n(\omega, c, \lambda)$ is a continuous map of the compact convex set Δ_N to itself. (Next need to apply Brouwer's fixed point theorem and prove the measurability.) $\square$

Theorem 1. *The strategy $\widehat{\Lambda}$ defined by (7) is relative growth optimal.*

The following auxiliary result will be used in the proof of Theorem 1.

Lemma 2. *Suppose $\alpha, \beta \in \mathbb{R}_+^N$ are two vectors such that $\sum_n \alpha^n \leq 1$, $\sum_n \beta^n \leq 1$ and for each $n = 1, \ldots, N$ it holds that if $\beta^n = 0$, then also $\alpha^n = 0$. Then*

$$\sum_{n=1}^N \alpha^n \ln \frac{\alpha^n}{\beta^n} \geq \frac{\|\alpha - \beta\|^2}{4} + \sum_{n=1}^N (\alpha^n - \beta^n), \tag{8}$$

where we define $\alpha^n \ln \frac{\alpha^n}{\beta^n} = 0$ if $\alpha^n = 0$ or $\beta^n = 0$.

Proof. We follow the lines of the proof of Lemma 2 in Amir et al. (2013), which establishes the above inequality in the case $\sum_n \alpha^n = \sum_n \beta^n = 1$. Using that $\ln x \leq 2(\sqrt{x} - 1)$ for any $x > 0$, we obtain

$$\begin{aligned} \sum_{n=1}^N \alpha^n \ln \frac{\alpha^n}{\beta^n} &= - \sum_{n \,:\, \alpha^n \neq 0} \alpha^n \ln \frac{\beta^n}{\alpha^n} \geq 2 \sum_{n=1}^N (\alpha^n - \sqrt{\alpha^n \beta^n}) \\ &= \sum_{n=1}^N (\sqrt{\alpha^n} - \sqrt{\beta^n})^2 + \sum_{n=1}^N (\alpha^n - \beta^n). \end{aligned}$$

Then we can use the inequality $(\sqrt{x} - \sqrt{y})^2 \geq (x-y)^2/4$, which is true for any $x, y \in [0, 1]$, and obtain (8). $\square$

Proof of Theorem 1. By the standard argument based on introduction of the representative agent, we can assume that the number of agents in the model $N = 2$.

Let r_t denote the relative wealth of agent 1, i.e.

$$r_t = \frac{W_t^1}{C_t},$$

where $C_t = W_t^1 + W_t^2$.

Let $\widehat{\lambda}_t(\omega) = \widehat{\Lambda}_t(\omega, C_{t-1}(\omega))$ and denote by $\bar{\lambda}_t = (\bar{\lambda}_t^1, \dots, \bar{\lambda}_t^N)$ the representative investment proportions of the two agents, i.e.

$$\bar{\lambda}_t^n(\omega) = r_t(\omega)\lambda_t^{1,n}(\omega) + (1 - r_t(\omega))\lambda_t^{2,n}(\omega).$$

Also introduce the notation

$$\mu_t^n = \frac{\widehat{\lambda}_t^n}{\bar{\lambda}_t^n}.$$

From equation (3), one can see that

$$W_{t+1}^1 = r_t \sum_{n=1}^N (C_t \widehat{\lambda}_t^n X_{t+1}^n + \mu_t^n Y_{t+1}^n), \qquad C_{t+1} = \sum_{n=1}^N (C_t \bar{\lambda}_t^n X_{t+1}^n + Y_{t+1}^n).$$

From these two relations, we find

$$\ln r_{t+1} - \ln r_t = \ln\biggl(\sum_{n=1}^N (C_t \widehat{\lambda}_t^n X_{t+1}^n + \mu_t^n Y_{t+1}^n)\biggr) - \ln\biggl(\sum_{n=1}^N (C_t \bar{\lambda}_t^n X_{t+1}^n + Y_{t+1}^n)\biggr).$$

Then we can write

$$\mathrm{E}_t \ln r_{r+1} - \ln r_t = F_t + G_t, \tag{9}$$

where

$$F_t = \mathrm{E}_t \ln\biggl(\frac{\sum_n (C_t \widehat{\lambda}_t^n X_{t+1}^n + \mu_t^n Y_{t+1}^n)}{\sum_i (C_t \widehat{\lambda}_t^i X_{t+1}^i + Y_{t+1}^i)}\biggr), \qquad G_t = \mathrm{E}_t \ln\biggl(\frac{\sum_i (C_t \widehat{\lambda}_t^i X_{t+1}^i + Y_{t+1}^i)}{\sum_n (C_t \bar{\lambda}_t^n X_{t+1}^n + Y_{t+1}^n)}\biggr).$$

Let us show that $F_t + G_t \geq 0$. Let

$$d_t^n = 1 - \mathrm{E}_t \frac{C_t X_{t+1}^n}{\sum_i (C_t \widehat{\lambda}_t^i X_{t+1}^i + Y_{t+1}^i)}, \tag{10}$$

so that from the definition of the strategy $\widehat{\Lambda}$ we have

$$d_t^n \widehat{\lambda}_t^n = \mathrm{E}_t\biggl(\frac{Y_{t+1}^n}{\sum_i (C_t \widehat{\lambda}_t^i X_{t+1}^i + Y_{t+1}^i)}\biggr).$$

Note that $d_t^n > 0$ due to assumption (2). Using the concavity of the logarithm and applying Lemma 2, we obtain

$$\begin{aligned} F_t &\geq \mathrm{E}_t\biggl(\sum_{n=1}^N \frac{Y_{t+1}^n}{\sum_i (C_t \widehat{\lambda}_t^i X_{t+1}^i + Y_{t+1}^i)} \ln \mu_t^n\biggr) = \sum_{n=1}^N d_t^n \widehat{\lambda}_t^n \ln \mu_t^n \\ &= \sum_{n=1}^n d_t^n \widehat{\lambda}_t^n \ln \frac{d_t^n \widehat{\lambda}_t^n}{d_t^n \bar{\lambda}_t^n} \geq \frac{1}{4} \sum_{n=1}^N (d_t^n (\widehat{\lambda}_t^n - \bar{\lambda}_t^n))^2 + \sum_{n=1}^N d_t^n (\widehat{\lambda}_t^n - \bar{\lambda}_t^n), \end{aligned} \tag{11}$$

where in the first inequality we considered the argument of the logarithm in the definition of F_t as the convex combination of the values

$$1,\ \mu_t^1,\ \ldots,\ \mu_t^N$$

with the coefficients

$$\frac{\sum_n C_t \widehat{\lambda}_t^n X_{t+1}^n}{\sum_i (C_t \widehat{\lambda}_t^i X_{t+1}^i + Y_{t+1}^i)},\quad \frac{Y_{t+1}^1}{\sum_i (C_t \widehat{\lambda}_t^i X_{t+1}^i + Y_{t+1}^i)},\quad \ldots,\quad \frac{Y_{t+1}^N}{\sum_i (C_t \widehat{\lambda}_t^i X_{t+1}^i + Y_{t+1}^i)}.$$

Using the inequality $\ln a \geq 1 - a^{-1}$, which is valid for any $a > 0$, we find

$$G_t \geq \mathrm{E}_t \frac{\sum_n C_t (\widehat{\lambda}_t^n - \bar{\lambda}_t^n) X_{t+1}^n}{\sum_i (C_t \widehat{\lambda}_t^i X_{t+1}^i + Y_{t+1}^i)} = \sum_{n=1}^N d_t^n (\bar{\lambda}_t^n - \widehat{\lambda}_t^n). \tag{12}$$

Thus, $F_t + G_t \geq 0$. Therefore, from (9), one can see that $\ln r_t$ is a generalized submartingale[1], and hence a true submartingale since it is bounded from above. $\square$

To state the second theorem, let us introduce the notation $\nu_t = (\nu_t^1, \ldots, \nu_t^N)$ for the representative strategy of agents $m = 2, \ldots, M$, i.e.

$$\nu_t^n(\omega) = \frac{1}{1 - r_t^1(\omega)} \sum_{m=2}^m r_t^m(\omega) \lambda_t^{m,n}(\omega).$$

Theorem 2. *Suppose that agent 1 uses the strategy $\widehat{\Lambda}$. Then she is a single survivor, i.e. $\lim_{t\to\infty} r_t^1 = 1$, a.s. on the set*

$$\left\{\omega : \sum_{t=0}^\infty \sum_{n=1}^N (d_t^n(\omega)(\lambda_t^{1,n}(\omega) - \nu_t^n(\omega)))^2 = \infty\right\}, \tag{13}$$

where d_t^n are defined as in (10).

Proof. In the proof of Theorem 1 we showed that $\ln r_t^1$ is a convergent submartingale, and as follows from (9), (11), (12), its compensator (i.e. a predictable non-decreasing sequence A_t such that $\ln r_t - A_t$ is a martingale)

$$A_t = \sum_{s=1}^t (\mathrm{E}_{s-1} \ln r_s^1 - \ln r_{s-1}^1)$$

can be bounded from below by

$$A_{t+1} \geq \frac{1}{4} \sum_{s=1}^t \sum_{n=1}^N (d_s^n(\lambda_s^{1,n} - \bar{\lambda}_s^n))^2 = \frac{1}{4} \sum_{s=1}^t (1 - r_s)^2 \sum_{n=1}^N (d_s^n(\lambda_s^{1,n} - \nu_s^n))^2.$$

Since the compensator of a convergent submartingale converges with probability 1, on the set (13) we necessarily have $\lim_{t\to\infty} r_t = 1$ a.s., which proves the theorem. $\square$

[1] Recall that a sequence S_t is called a generalized submartingale if $\mathrm{E}\,|S_0| < \infty$ and $\mathrm{E}(S_t \mid \mathcal{F}_{t-1}) \geq S_{t-1}$ for all $t \geq 1$ (but not necessarily $\mathrm{E}\,|S_t| < \infty$). One can show that if $S_t \leq C_t$ for all t with some integrable random variables C_t, then S_t is integrable, and hence a true submartingale.

4. Relation to other results

1. The modeld with exogenous returns. Assume that in our model $Y_t^n \equiv 0$. Then equation (3) becomes

$$W_{t+1}^m = \sum_{n=1}^N \lambda_t^{m,n} W_t^m X_{t+1}^n.$$

This is the familiar equation which defines the wealth of a self-financing strategy in a market with exogenous returns X_t^n. For example, if the asset prices are S_t^n, then it is natural to put $X_t^n = S_t^n/S_{t-1}^n$. Clearly, in this case the wealth of an agent depends only on his/her strategy and does not depend on the strategy of the other agents. Let us denote the corresponding wealth sequence by $W_t(\lambda)$.

A strategy $\widehat{\lambda}$ is called *growth optimal* (or a *numeraire portfolio*), if for any other strategy λ

$$\frac{W_t(\lambda)}{W_t(\widehat{\lambda})} \text{ is a supermartingale,}$$

which is equivalent to our definition of a relative growth optimal strategy when $Y_t^n \equiv 0$.

The classic result (see, e.g, **?**) is that if the log-returns are integrable (i.e. $\mathrm{E} \ln X_{t+1}^n < \infty$), then the growth optimal strategy can be found by maximizing the expected log-return of the portfolio:

$$\widehat{\lambda}_t \in \arg\max_\lambda \mathrm{E}\left(\ln \frac{W_{t+1}(\lambda)}{W_t(\lambda)} \mid \mathcal{F}_t \right) = \arg\max_\lambda \mathrm{E}\left(\ln \sum_{n=1}^N \lambda_t^n X_{t+1}^n \mid \mathcal{F}_t \right).$$

If the log-returns are not integrable, this problem may be have no solution, however it is easy to see that if one introduces the relative returns $R_t^n = X_t^n / \sum_{i=1}^N X_t^i$, then the growth-optimal strategy can be found as

$$\widehat{\lambda}_t \in \arg\max_\lambda \mathrm{E}\left(\ln \sum_{n=1}^N \lambda_t^n R_{t+1}^n \mid \mathcal{F}_t \right). \tag{14}$$

This maximization problem always has a solution, provided that $\sum_{n=1}^N R_t^n > 0$ a.s. However, note that the solution may be not unique if, e.g., the R_t^n are linearly dependent.

Let us show that our strategy $\widehat{\lambda}_t$ solves (14). Show that our strategy is a solution of (14) (or, it will be better to show that any fixed point defined in Lemma 1 is a solution of (14)).

2. The model with short-lived assets and endogenous prices. Now suppose that $X_t^n \equiv 0$. In this case we get the evolutionary finance model with short-lived assets of Amir et al. (2013). They found the optimal strategy

$$\widehat{\lambda}_t^n = \mathrm{E}\left(\frac{Y_{t+1}^n}{\sum_{k=1}^N Y_t^n} \mid \mathcal{F}_t \right),$$

which clearly agrees with formula (7).

3. The model with endogenous short-lived assets with exogenous assets. Consider the model of a market with $N = N_1 + N_2$ assets of two types: assets of the first type ($n = 1, \dots, N_1$) have exogenous prices and returns like in the first example, while assets of the second type ($n = N_1 + 1, \dots, N$) have endogenous prices and are short-lived like in the second example. For simplicity, assume that the log-returns of the assets of the first type are integrable, i.e. $\mathrm{E}\,|\ln X_t^n| < \infty$ for $n = 1, \dots, N_1$.

This model was considered by **?** in a somewhat more general form and it was found that the relative growth optimal strategy $\widehat{\lambda}_t = (\widehat{\alpha}_t^1, \dots, \widehat{\alpha}_t^{N_1}, \widehat{\beta}_t^1, \dots, \widehat{\beta}_t^{N^2})$, can be found as follows. First, one finds $\widehat{\alpha}_t^n$ by solving the optimization problem

$$\widehat{\alpha}_t = \arg\max_\alpha \left\{ \mathrm{E}\left(\ln\left(C_t \sum_{n=1}^{N_1} \alpha^n X_{t+1}^n + \sum_{n=N_1+1}^{N} Y_t^n \right) \mid \mathcal{F}_t \right) - \sum_{n=1}^{N_1} \alpha \right\},$$

where $C_t = \sum_{m=1}^M W_t^m$ denotes the total market wealth, and the maximum in the above formula is taken over the set $\{\alpha \in \mathbb{R}_+^{N_1} : a^1 + \dots + a^{N_1} \leq 1\}$. Then, $\widehat{\beta}_t^n$, $n = 1, \dots, N_2$, are defined by

$$\widehat{\beta}_t^n = \mathrm{E}\left(\frac{Y_{t+1}^{N_1+n}}{C_t \sum_{n=1}^{N_1} \alpha_t^n X_{t+1}^n + \sum_{n=1}^{N_2} Y_{t+1}^{N_1+n}} \mid \mathcal{F}_t \right).$$

Show how to obtain this formula from our formula.

References

Amir, R., Evstigneev, I. V., and Schenk-Hoppé, K. R. (2013). Asset market games of survival: a synthesis of evolutionary and dynamic games. *Annals of Finance*, 9(2):121–144.