Capital growth and survival strategies in a market with endogenous prices

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Abstract

We consider a multi-agent asset market model which includes assets of two types: long-lived assets with exogenous prices and short-lived assets with endogenous prices. The first main result consists of constructing a strategy which allows an agent to keep a non-vanishing share of market wealth over the infinite time horizon (a "survival" strategy). The second result shows that an agent who uses this strategy becomes the single surviving agent in the market with relative wealth converging to 1 if the representative strategy of the other agents is asymptotically different from it.

Keywords: survival strategies, capital growth, relative growth optimal strategies, endogenous prices, evolutionary finance, martingale convergence.

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1. Introduction

The main object of study of this paper is the asymptotic performance of investment strategies in stochastic market models. The mathematical theory of optimal capital growth originated with the works of Kelly (1956), Latané (1959) and Breiman (1961), and one of its central results consists of the fact that an agent who maximizes the expected logarithm of portfolio return achieves the fastest asymptotic growth of wealth over the infinite time horizon (see, e.g., Algoet and Cover (1988)). The standard assumption made in this theory is that an agent has negligible impact on a market, and hence, asset prices can be specified by exogenous random processes not depending on agents' strategies. The aim of this paper is to extend these results and study analogues of growth optimal strategies in a multi-agent market model which may contain assets with endogenously determined prices.

We consider a discrete-time model of a market with two types of assets. Assets of the first type, further called *exogenous*, have prices represented by exogenous random sequences and agents get profit or loss when the prices of these assets change. Assets of the second type, further called *endogenous*, pay exogenously defined dividends, while their prices are determined endogenously via a short-run equilibrium of supply and demand. Their supply volume is exogenous and positive, while the demand volume is determined by the agents' strategies. The dividends are distributed among the agents proportionally to the owned shares. Typically, an asset with larger dividends will be more attractive and there will be a higher demand for it.

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The important simplifying assumption that will be made in the paper is that the endogenous assets are short-lived (live for one period of time), i.e. they are bought at moment of time t, yield random payoffs at time $t+1$ and expire; then the agents buy new assets and the cycle repeats. Examples of such assets include contracts for production or delivery of a specific amount of some good, service and insurance contracts, etc. We assume that the sellers are exogenous to the market and are willing to accept any price determined by the buyers, the agents in the model. Short-lived assets have been used in various models in the literature. For example, Judd et al. (2003) studied asset trading volume in a general equilibrium model with long-lived and short-lived assets, where short-lived assets are in zero net supply; Araujo et al. (2002) showed the existence of equilibrium in an incomplete market model with short-lived assets protected by collateral; Carbone et al. (2021) provided an experimental comparison of exchange economies with long-lived and short-lived assets. Among models with short-lived assets which, like the present paper, do not employ the concept of general equilibrium and are closely related to our work, one can mention Amir et al. (2013); Blume and Easley (1992); Bottazzi and Dindo (2014); Hens and Schenk-Hoppé (2005), see the review below. It would be interesting to incorporate long-lived assets (e.g. common stock) with endogenous prices into our model, but this is a more difficult task and is left for future research.

In the present paper, we will be primarily interested in the asymptotic behavior of relative wealth of agents, i.e. their shares in total market wealth. We investigate it from a standpoint of evolutionary dynamics and view a market as a population of different strategies competing for capital. The central concept of the paper is the notion of a *survival strategy*. Such a strategy allows an agent to keep the relative wealth strictly bounded away from zero over the infinite time horizon. Our goal is to construct a survival strategy in an explicit form and to find what effect the presence of this strategy has on the asymptotic distribution of wealth among market agents. In particular, we are interested in conditions under which a strategy is asymptotically *dominating*, i.e. an agent using it becomes the single survivor in the market with the relative wealth converging to 1. In order to find a survival strategy, the notion of a *relative growth optimal* strategy will be useful. This is a strategy with the logarithm of its relative wealth being a submartingale. The fact that a non-positive submartingale converges implies that a relative growth optimal strategy is survival. The convergence of the compensator of this submartingale allows to obtain a sufficient condition for a survival strategy to be also dominating.

Note that, in contrast to the optimal growth theory for markets with exogenous prices, which deals with the absolute wealth of agents, we focus on the relative wealth, which turns out to be more amenable to asymptotic analysis in the case of endogenous prices. Actually, it is possible to provide an example showing that the goals of maximization of relative and absolute wealth in a model with endogenous prices may be incompatible (see Drokin and Zhitlukhin (2020, Section 6)).

Our first main result consists of constructing a relative growth optimal strategy in a tractable form, as a solution of a two-stage optimization problem. In the first stage, an agent determines the portfolio of exogenous assets by maximizing the expected logarithmic return (with some adjustments if it is not integrable); in the second stage the portfolio of endogenous assets is found via a solution of another maximization problem. An important property of this strategy is that it turns out to be relative growth optimal in any strategy profile, irrespectively of the strategies used by the other agents. In particular, an agent using this strategy does not need to know what strategies the other agents use, as the solutions of the two optimization problems depend only on the current total market wealth, the probability distribution of returns of the exogenous assets and payoffs of the endogenous assets, and a portfolio constraints set. These market characteristics are more easily observable compared to agents' individual wealth and their strategies, which may be useful for potential applications.

As a supplement to this result, we show that the constructed strategy can be obtained as a growth optimal strategy in a market with exogenous prices which are set equal to the endogenous prices induced by this strategy when all the agents in the market use it. In other words, it is an equilibrium strategy if the goal of each agent is to have a growth optimal portfolio. This fact can be regarded as a connection between the classical optimal growth theory with exogenous assets and our setting with endogenous assets.

Our second main result shows that the obtained strategy becomes the single surviving strategy in the market if the representative strategy of the other agents is asymptotically different from it. Consequently, if some agent uses this strategy, then any other agent who wants to survive in the market must use an asymptotically similar strategy. As a corollary, we show that this strategy asymptotically determines the prices of the endogenous assets.

The results we obtain are directly related to, and generalize the main results of Amir et al. (2013) and Drokin and Zhitlukhin (2020) (see Section 3.5 for a detailed comparison). Those papers also studied survival and growth optimal strategies in markets with short-lived assets and endogenous prices, however the models were less general. In the former paper, it was assumed that there are only assets with endogenous prices; the latter paper also included a risk-free bank account with an exogenous interest rate. The generalization pursued in this paper allows us to include exogenous assets in the model in a greater generality. This can be used to model markets where an individual agent has a negligible impact on the prices of some assets (represented by exogenous assets) and a non-negligible impact on the prices of other assets (endogenous assets). Note that the methods of the mentioned papers used to find the optimal strategies in the corresponding models, which essentially consist of "guessing" the optimal strategy and then verifying its optimality, cannot be simply extended to our model. On the other hand, the construction proposed in this paper is more general and direct, and presents interest by itself. In particular, the equilibrium characterization of the optimal strategy allows us to give it a more clear interpretation compared to earlier works in this field. Finally, our model allows us to include portfolio constraints on both endogenous and exogenous assets. Models with portfolio constraints have been widely studied in mathematical finance for exogenous assets (see, e.g., Cvitanić and Karatzas (1993); Cvitanić et al. (1999); Evstigneev et al. (2004); Jouini and Kallal (1995) among others), but questions of survival and growth optimality in markets with endogenous prices and portfolio constraints have not been studied yet.

Let us mention how this paper is related to other results in the literature. In models with exogenous prices, the asymptotic growth optimality of the log-optimal strategy was proved for a general discrete-time model by Algoet and Cover (1988); a review of other related results in discrete time can be found in, e.g., Cover and Thomas (2012, Chapter 16) or Hakansson and Ziemba (1995). For a treatment of a general model with continuous time and portfolio constraints, and a connection of growth optimal portfolios (numéraire portfolios) with absence of arbitrage, see Karatzas and Kardaras (2007).

Among various lines of research on markets with endogenous prices, our paper is most closely related to works in evolutionary finance on the stability and survival of investment strategies, which focus on evolutionary dynamics and properties like survival, extinction, dominance, and how they affect the structure of a market. Central to this direction are strategies that perform well, irrespectively of competitors' actions. One of the main results consists of that the strategy that splits its investment budget among risky assets proportionally to their expected dividends will survive in the market, provided that the agent's beliefs about the dividends are correct. See, for example, the papers of Amir et al. (2005, 2011); Blume and Easley (1992); Evstigneev et al. (2002, 2006); Hens and Schenk-Hoppé (2005), which establish this fact for different models and under different assumptions. A review of other results can be found in Evstigneev et al. (2016), see also Amir et al. (2020). Typically, this strategy turns out to be the only surviving strategy in the market, i.e. it dominates all other asymptotically different strategies. For results on market wealth evolution, when agents use other strategies, which may result in survival of several strategies, see, e.g., Bottazzi and Dindo (2014); Bottazzi et al. (2018).

Most of the above mentioned papers (including the present paper) consider agent-based models, where agents' strategies are specified directly as functions of a market state. Another large body of literature consists of results on market selection of investment strategies in the framework of general equilibrium, where agents maximize utility from consumption. Among those results one can mention, for example, Blume and Easley (2006); Borovička (2020); Sandroni (2000); Yan (2008). Holtfort (2019) provides a detailed survey of the literature in evolutionary finance over the last three decades, also including some earlier results.

The paper is organized as follows. Section 2 describes the model. The main results of the paper are stated in the three theorems included in Section 3. Section 4 contains their proofs.

2. The model

2.1. Notation

For vectors $x, y \in \mathbb{R}^N$, we will denote by $\langle x, y\rangle$ their scalar product, and by $|x| = \sum_n |x^n|$, $\|x\| = \sqrt{\langle x, x\rangle}$ the L^1 and L^2 norms. If $f\colon \mathbb{R} \to \mathbb{R}$ is a scalar function and x is a vector, then $f(x) = (f(x^1), \dots, f(x^N))$ denotes the coordinatewise application of f to x.

By e, we will denote the vector consisting of all unit coordinates, $e = (1, \dots, 1)$, which may be of different dimensions in different formulas. In particular, $\langle e, x\rangle$ is equal to the sum of coordinates of a vector x.

All equalities and inequalities for random variables are assumed to hold with probability 1 (almost surely), unless otherwise is stated.

2.2. Assets and agents

Let $(\Omega, \mathcal{F}, \mathrm{P})$ be a probability space with a discrete-time filtration $\mathbb{F} = (\mathcal{F}_t)_{t=0}^{\infty}$ on which all random variables will be defined. Without loss of generality, we will assume that $\mathcal{F}$ is P-complete and $\mathcal{F}_0$ contains all P-null events.

The market in the model consists of M agents (investors) and $N = N_1 + N_2$ assets of two types. The assets of the first type are available in unlimited supply and have exogenous prices; they are treated as in standard models of mathematical finance and, in particular, this class of assets may include a risk-free asset with an exogenous interest rate. Without loss of generality, we assume that these assets do not pay dividends (see Remark 2 below). The assets of the second type are available in limited supply; they yield payoffs (dividends) which are defined exogenously, but their prices are determined endogenously from an equilibrium of supply and demand in each time period. These assets are short-lived in the sense that they can be purchased by agents at time t, yield payoffs at $t+1$, and then get replaced with new assets; agents cannot sell them, and, in particular, short sales are not allowed (adding short sales would lead to conceptual difficulties which we prefer to avoid). We will call the assets of the first and second types, respectively, exogenous and endogenous.

The prices of the exogenous assets are represented by positive random sequences $(S_t^n)_{t=0}^{\infty}$, $n = 1, \dots, N_1$, which are $\mathbb{F}$-adapted (i.e. S_t^n is $\mathcal{F}_t$-measurable). By $X_t^n = S_t^n / S_{t-1}^n > 0$ we will denote the relative price changes. The payoffs of the endogenous assets (per one unit of an asset) are represented by non-negative adapted sequences $(Y_t^n)_{t=1}^{\infty}$, $n = 1, \dots, N_2$. Without loss of generality, we assume that the supply of each endogenous asset is equal to 1, so Y_t^n is the total payoff of asset n. Their prices will be defined later, as we first need to define the agents' strategies, on which they will depend.

The agents begin trading in the market at time $t = 0$ with non-random initial wealth $v_0^m > 0$, $m = 1, \dots, M$. Actions of an agent at time $t \geq 0$ are described by a pair of vectors $h_t = (\alpha_t, \beta_t)$, where $\alpha_t \in \mathbb{R}^{N_1}$, $\beta_t \in \mathbb{R}_+^{N_2}$ describe the proportions of the current wealth which this agent allocates to the assets of the two types (the wealth sequences are yet to be defined), i.e. the proportion α_t^n (respectively, β_t^n) of wealth is allocated to asset n at time t and held until $t+1$.

Since α_t, β_t are proportions, we require that $\langle e, \alpha_t \rangle + \langle e, \beta_t \rangle = 1$. It is assumed that the components of β_t are non-negative, because short sales of the endogenous assets are not allowed. Additionally, we will assume that the value of the part of the portfolio invested in the exogenous assets is non-negative, so $\langle e, \alpha_t \rangle \in [0, 1]$, where the upper bound holds because $\langle e, \alpha_t \rangle = 1 - \langle e, \beta_t \rangle$. As a result, we also have $\langle e, \beta_t \rangle \in [0, 1]$. Consequently, h_t is supposed to have values in the set

$$\mathcal{H} = \{(\alpha, \beta) \in \mathbb{R}^{N_1} \times \mathbb{R}_+^{N_2} : \langle e, \alpha \rangle \in [0, 1],\ \langle e, \beta \rangle = 1 - \langle e, \alpha \rangle\}.$$

For agent m, the vector of his/her investment proportions will be denoted by $h_t^m = (\alpha_t^m, \beta_t^m)$.

A strategy of an agent consists of investment proportions h_t^m selected at consecutive moments of time. It may (and, usually, does) depend on a random outcome $\omega \in \Omega$ and actions of other agents. In order to specify this dependence, introduce the measurable space $(\Theta, \mathcal{G})$ with

$$\Theta = \Omega \times \mathbb{R}_+^M \times (\mathcal{H}^M)^\infty, \qquad \mathcal{G} = \mathcal{F} \otimes \mathcal{B}(\mathbb{R}_+^M \times (\mathcal{H}^M)^\infty),$$

where $\mathcal{B}$ denotes the Borel σ-algebra, and $\chi = (\omega, v_0, h_0, h_1, \dots) \in \Theta$ consists of a random outcome ω, a vector of initial wealth $v_0 = (v_0^1, \dots, v_0^M) \in \mathbb{R}_+^M$, and vectors of investment proportions $h_t = (h_t^1, \dots, h_t^M) \in \mathcal{H}^M$ selected by the agents at each moment of time. An element $\chi \in \Theta$ will be called a *market history*.

Let $\mathbb{G} = (\mathcal{G}_t)_{t \geq 0}$ be the filtration on $(\Theta, \mathcal{G})$ defined by

$$\mathcal{G}_t = \sigma\big(A \times V_0 \times B_0 \times \dots \times B_t \times (\mathcal{H}^M)^\infty \mid A \in \mathcal{F}_t,\ V_0 \in \mathcal{B}(\mathbb{R}_+^M),\ B_s \in \mathcal{B}(\mathcal{H}^M), s \leq t\big),$$

i.e. $\mathcal{G}_t$ is the σ-algebra generated by sets $A \times V_0 \times B_0 \times \dots \times B_t \times (\mathcal{H}^M)^\infty$ with $A \in \mathcal{F}_t$ and Borel sets $V \subseteq \mathbb{R}_+^M$, $B_s \subseteq \mathcal{H}^M$. We define the strategy of an agent as a sequence of $\mathcal{G}_t$-measurable functions

$$\overline{H}_t(\chi) \colon \Theta \to \mathcal{H}, \qquad t \geq 0.$$

This assumption of measurability means that an agent at time t can choose his/her investment proportions based on the information conveyed by $\mathcal{F}_t$, the vector of initial wealth v_0, and the investment proportions $(h_0, \ldots, h_t)$ chosen by the agents before or at time t, so that the future information $\mathcal{F}_s$, $s > t$, and knowledge of future investment proportions h_s, $s > t$, cannot be used at t. Note that $\overline{H}_t$ may depend on the value of h_t, i.e. an agent may use information (partial or whole) about the actions of other agents at the same moment of time. This information may be available to an agent, for example, from asset prices.

In the notation below, a bar will be used to emphasize that a function depends on market history, while functions depending only on a random outcome ω will have no bars (cf. $\overline{H}(\chi)$ and $H(\omega)$ below).

We will call a vector of initial wealth v_0 and a strategy profile $(\overline{H}^1, \ldots, \overline{H}^M)$ *feasible* if there exists a sequence of $\mathcal{F}_t$-measurable functions $H_t(\omega) = (H_t^1(\omega), \ldots, H_t^M(\omega))$ with values in $\mathcal{H}^M$ such that for all ω, t, m

$$\overline{H}_t^m(\omega, v_0, H_0(\omega), H_1(\omega), \ldots) = H_t^m(\omega). \tag{1}$$

Such a sequence of functions H_t will be called a *realization* of the agents' strategies corresponding to the given strategy profile and initial wealth. Note that H_t are still random (depend on ω); the word *realization* will be used in the meaning that we fix initial wealth and a strategy profile, but not a random outcome.

Let us comment on the relation between $\overline{H}_t(\chi)$ and $H_t(\omega)$. The sequence of functions $\overline{H}_t^m$, $t \geq 0$, describes the strategy of agent m in the sense that it specifies a complete plan of his/her actions, for all random outcomes, initial wealth, and investment proportions selected by other agents, on which $\overline{H}_t^m$ depend through their arguments. The functions $H_t^m(\omega)$, $t \geq 0$, express investment proportions selected by this agent, depending on a random outcome ω, when he/she participates in a market described by a certain vector of initial wealth v_0 and a certain strategy profile $(\overline{H}^1, \ldots, \overline{H}^M)$. Note that different strategy profiles and vectors of initial wealth typically yield different realizations H_t.

Equation (1) means that for each ω the sequence $H_t(\omega)$ is a fixed point of the mapping $(h_0, h_1, \ldots) \mapsto \overline{H}_t(\omega, v_0, h_0, \ldots)$. In general, it is not guaranteed that such a fixed point exists for an arbitrary strategy profile and a vector of initial wealth, and we will not investigate in details the question of its existence and uniqueness, as this is not the primary goal of the paper. All the main results will be established for feasible vectors of initial wealth and strategy profiles, i.e. for which at least one solution of equation (1) exists. The uniqueness is not required, our results will hold for any realization of agents' strategies satisfying (1).

Nevertheless, in order to show that equation (1) is not unreasonable, let us provide a simple sufficient condition for the existence and uniqueness of its solution. This condition will be satisfied by the optimal strategy we propose in Section 3.2, and, in particular, the strategy profile in which all the agents use this strategy will be feasible with any vector of initial wealth.

Proposition 1. *Suppose that the strategy of each agent m is adapted to the filtration $\mathbb{G}^- = (\mathcal{G}_t^-)_{t \geq 0}$, where*

$$\begin{aligned}
\mathcal{G}_0^- &= \sigma\big(A \times V_0 \mid A \in \mathcal{F}_t,\ V_0 \in \mathcal{B}(\mathbb{R}_+^M)\big), \\
\mathcal{G}_t^- &= \sigma\big(A \times V_0 \times B_0 \times \ldots \times B_{t-1} \times (\mathcal{H}^M)^\infty \mid A \in \mathcal{F}_t, \\
&\qquad V_0 \in \mathcal{B}(\mathbb{R}_+^M),\ B_s \in \mathcal{B}(\mathcal{H}^M), s \leq t-1\big), \qquad t \geq 1.
\end{aligned}$$

Then for any vector of initial wealth $v_0 \in \mathbb{R}_+^M$, the strategy profile $(\overline{H}^1, \dots, \overline{H}^M)$ is feasible and there exists a unique sequence $H_t(\omega)$ satisfying (1) for all ω, t, m.

Proof. The measurability with respect to $\mathcal{G}_t^-$ implies that $\overline{H}_t^m$ can be represented as $\overline{H}_t^m(\omega, v_0, h_0, \dots) = \widetilde{H}_t^m(\omega, v_0, h_0, \dots, h_{t-1})$ with an appropriate unique function $\widetilde{H}_t^m$. Then the sequence $H_t(\omega)$ is defined by induction by the relations

$$H_0^m(\omega) = \widetilde{H}_0^m(\omega, v_0), \qquad H_t^m(\omega) = \widetilde{H}_t^m(\omega, v_0, H_0(\omega), \dots, H_{t-1}(\omega)),\ t \geq 1.$$

□

2.3. Agents' wealth sequences

We will define the prices of endogenous assets $\bar{p}_t^n$ and the wealth of agents $\overline{V}_t^m$ as functions of argument $\chi = (\omega, v_0, h_0, \dots) \in \Theta$ inductively in t. At this point, we do not need to fix a strategy profile (see Remark 1 below).

For $t = 0$, let $\overline{V}_0^m(\chi) = v_0^m$. Suppose for some $\chi \in \Theta$ the wealth sequences are defined up to a moment of time t, and $\overline{V}_t^m(\chi) \geq 0$ for all m. Then agent m can purchase $\bar{y}_t^{m,n}(\chi)$ units of asset n at this moment, where

$$\bar{y}_t^{m,n}(\chi) = \frac{\beta_t^{m,n} \overline{V}_t^m(\chi)}{\bar{p}_t^n(\chi)}.$$

The vector of investment proportions $\beta_t^m = (\beta_t^{m,1}, \dots, \beta_t^{m,N})$ appearing in this formula is the one from the component $h_t = (h_t^1, \dots, h_t^M)$, $h_t^m = (\alpha_t^m, \beta_t^m)$, of the market history $\chi = (\omega, v_0, h_0, \dots)$.

We define $\bar{p}_t^n$ as the market clearing prices, i.e. the prices which make the total demand for each endogenous asset being equal to its supply, where the latter is assumed to be 1. Thus, we have

$$\bar{p}_t^n(\chi) = \sum_{m=1}^{M} \beta_t^{m,n} \overline{V}_t^m(\chi). \tag{2}$$

If $\sum_m \beta_t^{m,n} = 0$ in (2) for some n, i.e. no agent invests in asset n, we put $\bar{y}_t^{m,n}(\chi) = 0$ for all m, and $\bar{p}_t^n(\chi) = 0$.

Thus, the portfolio of agent m between moments of time t and $t+1$ consists of $\bar{y}_t^{m,n}$ units of endogenous asset n, and $\bar{x}_t^{m,n}$ units of exogenous asset n, where

$$\bar{x}_t^{m,n}(\chi) = \frac{\alpha_t^{m,n} \overline{V}_t^m(\chi)}{S_t^n(\omega)}.$$

Like β_t^m above, the vector of investment proportions $\alpha_t^m = (\alpha_t^1, \dots, \alpha_t^M)$ and the random outcome ω are the elements of the market history χ.

Consequently, the wealth of agent m at time $t+1$ is determined by the relation

$$\begin{aligned}\overline{V}_{t+1}^m(\chi) &= \sum_{n=1}^{N_1} \bar{x}_t^{m,n}(\chi) S_{t+1}^n(\omega) + \sum_{n=1}^{N_2} \bar{y}_t^{m,n}(\chi) Y_{t+1}^n(\omega) \\ &= \left(\sum_{n=1}^{N_1} \alpha_t^{m,n} X_{t+1}^n(\omega) + \sum_{n=1}^{N_2} \frac{\beta_t^{m,n}}{\sum_k \beta_t^{k,n} \overline{V}_t^k(\chi)} Y_{t+1}^n(\omega) \right) \overline{V}_t^m(\chi),\end{aligned} \tag{3}$$

with $0/0 = 0$ in the right-hand side.

Observe that in equation (3) the value of $\overline{V}_{t+1}^m$ may become negative, which will make the right-hand side of the equation meaningless for the next time period. However, below we will introduce portfolio constraints which prohibit strategies that may lead to negative wealth. In view of this, we will restrict the domain of the functions $\overline{V}_t^m$ and define them on the sets which are smaller than Θ. Namely, introduce inductively the sets

$$\Theta_t = \{\chi \in \Theta : \overline{V}_s^m(\chi) \geq 0 \text{ for all } s \leq t,\ m = 1, \dots, M\}, \quad t \geq 0,$$

where $\overline{V}_s^m(\chi)$ are computed by (3). Note that $\Theta_0 = \Theta$, $\Theta_t \supseteq \Theta_{t+1}$, and $\Theta_t \in \mathcal{G}_t^-$. From now on, we will assume that the functions $\overline{V}_t^m$ are defined only for $\chi \in \Theta_t$ and, consequently, their values are non-negative for such elements χ.

It is also convenient to introduce the sets

$$\Theta_t' = \{\chi \in \Theta_t : \overline{V}_t^m(\chi) \neq 0 \text{ for some } m = 1, \dots, M\}, \quad t \geq 0.$$

Observe that agents' strategies at time t can be specified only on Θ_t', since for $\chi \in \Theta_t \setminus \Theta_t'$ (i.e. when the wealth of each agent at time t is zero) they can be defined in an arbitrary way because the wealth and endogenous asset prices will be zero for $s \geq t$ no matter what strategies are used. This observation will be used in the definition of the optimal strategy in order to avoid separate consideration of elements $\chi \in \Theta_t \setminus \Theta_t'$.

Remark 1. Note that in this subsection, it is not assumed that a strategy profile is fixed: the functions $\overline{V}_t$, $\bar{p}_t$, $\bar{x}_t$, $\bar{y}_t$ are defined for all market histories χ. If we fix a strategy profile $(\overline{H}^1, \dots, \overline{H}^M)$ and a vector of initial wealth v_0, we will obtain the realizations of these functions, i.e. $V_t^m(\omega) = \overline{V}_t^m(\omega, v_0, H_0(\omega), \dots)$, where $H_t(\omega)$ are the realizations of the agents' strategies defined in (1).

Remark 2. It is not difficult to extend the model to include dividend-paying exogenous assets. If asset $n \in \{1, \dots, N_1\}$ pays at time t dividend D_t^n per one share, then equation (3) modifies as follows:

$$\overline{V}_{t+1}^m(\chi) = \Biggl(\sum_{n=1}^{N_1} \alpha_t^{m,n}(X_{t+1}^n(\omega) + D_{t+1}^n(\omega)) + \sum_{n=1}^{N_2} \frac{\beta_t^{m,n}}{\sum_k \beta_t^{k,n} \overline{V}_t^k(\chi)} Y_{t+1}^n(\omega)\Biggr) \overline{V}_t^m(\chi).$$

Clearly, this model reduces to equation (3) if one introduces $\widetilde{X}_t^n = X_t^n + D_t^n$. Hence, consideration of the no-dividends model does not reduce the generality of our results.

2.4. Portfolio constraints

Portfolio constraints in the model are specified by a sequence of $\mathcal{G}_t^-$-measurable random non-empty closed convex sets[1] $\overline{C}_t(\chi) \subseteq \mathcal{H}$, $t \geq 0$. The constraints are the same for each agent.

We say that a strategy $\overline{H}$ satisfies the portfolio constraints if

$$\overline{H}_t(\chi) \in \overline{C}_t(\chi) \text{ for all } t \geq 0 \text{ and } \chi \in \Theta.$$

From now on, when writing "a strategy", we always mean a strategy satisfying the portfolio constraints.

We will consider portfolio constraints only of the following particular form: they are imposed on the exogenous and endogenous assets separately, and an agent can freely

[1] See Section 4.1 for details on random sets.

choose what proportion of wealth to invest in the assets of each of the two types. Namely, it will be assumed that

$$\overline{C}_t = (\overline{A}_t \times \overline{B}_t) \cap \mathcal{H}, \tag{4}$$

where $\overline{A}_t$ and $\overline{B}_t$ are $\mathcal{G}_t^-$-measurable closed convex sets in $\mathbb{R}^{N_1}$ and $\mathbb{R}_+^{N_2}$ such that $\langle e, \alpha\rangle \in [0,1]$, $\langle e,\beta\rangle \in [0,1]$ for any $\alpha \in \overline{A}_t(\chi)$, $\beta \in \overline{B}_t(\chi)$. We also require that

$$\text{if } \alpha \in \overline{A}_t(\chi), \text{ then } \lambda\alpha \in \overline{A}_t(\chi) \text{ for any } \lambda \in [0, 1/\langle e,\alpha\rangle], \tag{5}$$

$$\text{if } \beta \in \overline{B}_t(\chi), \text{ then } \lambda\beta \in \overline{B}_t(\chi) \text{ for any } \lambda \in [0, 1/\langle e,\beta\rangle] \tag{6}$$

(or $\lambda \in [0,\infty)$ if $\langle e,\alpha\rangle = 0$ or $\langle e,\beta\rangle = 0$); i.e. $\overline{A}_t$ and $\overline{B}_t$ can be represented as the intersections of some convex cones with the sets $\{\alpha \in \mathbb{R}^{N_1} : \langle e,\alpha\rangle \in [0,1]\}$ and $\{\beta \in \mathbb{R}_+^{N_2} : \langle e,\beta\rangle \in [0,1]\}$ respectively. Note that relation (4) implies that the sets $\overline{A}_t$, $\overline{B}_t$ cannot simultaneously (for the same t,χ) consist of only elements α or, respectively, β with zero sum of coordinates, since then the set $\overline{C}_t$ would be empty.

We will need to further restrict the class of portfolio constraints by introducing several assumptions on the structure of the sets $\overline{A}_t, \overline{B}_t$.

In what follows, let $K_t(\omega, d\widetilde{\omega})$ denote some fixed version of the regular conditional distribution with respect to $\mathcal{F}_t$. By P_t and E_t, we will denote, respectively, the regular probability and expectation computed with respect to K_t, i.e. for a random event $\Gamma \in \mathcal{F}$ and a random variable $\xi(\omega)$ we put

$$\mathrm{P}_t(\Gamma)(\omega) = K_t(\omega,\Gamma), \qquad \mathrm{E}_t(\xi)(\omega) = \int_\Omega \xi(\widetilde{\omega}) K_t(\omega, d\widetilde{\omega}). \tag{7}$$

(Actually, in all subsequent formulas we will apply E_t to random variables ξ which are functions of X_{t+1} and Y_{t+1}, i.e. $\xi = f(X_{t+1}, Y_{t+1})$ with some non-random functions $f(x,y)$. In this case, the expectation E_t can be defined by integration with respect to the regular conditional distribution of X_{t+1}, Y_{t+1}, but we prefer the above form since it is more concise.)

Let us introduce several random sets which will be needed to formulate the assumptions on the sets $\overline{A}_t, \overline{B}_t$:

- the sets of portfolios of exogenous assets which have non-negative values at the next moment of time:

$$D_t(\omega) = \{\alpha \in \mathbb{R}^{N_1} : \mathrm{P}_t(\langle \alpha, X_{t+1}\rangle \geq 0)(\omega) = 1\};$$

- the linear spaces of *null investments* (portfolios of exogenous assets with zero current and next value):

$$L_t(\omega) = \{\alpha \in \mathbb{R}^{N_1} : \langle e,\alpha\rangle = 0,\ \mathrm{P}_t(\langle \alpha, X_{t+1}\rangle = 0)(\omega) = 1\};$$

- the projection of $\overline{A}_t$ on the orthogonal space $L_t^\perp$:

$$\overline{A}_t^{\mathrm{p}}(\chi) = \{\alpha \in L_t^\perp(\omega) :\ \exists\, u \in L_t(\omega) \text{ such that } \alpha + u \in \overline{A}_t(\chi)\}.$$

Observe that the sets D_t, L_t are $\mathcal{F}_t$-measurable, and $\overline{A}_t^{\mathrm{p}}$ are $\mathcal{G}_t^-$-measurable. Indeed, we can represent $D_t(\omega) = \{\alpha : f(\omega,\alpha) = 0\}$, where $f(\omega,\alpha) = \mathrm{E}_t(\langle \alpha, X_t\rangle^- \wedge 1)(\omega)$ is a Carathéodory function, so D_t is measurable by Filippov's theorem (see Proposition 5 in Section 4.1). The set L_t is measurable since it is the intersection of the closed sets D_t,

$-D_t$ and $\{\alpha : \langle \alpha, e\rangle = 0\}$, and the intersection of closed measurable sets is measurable (see Proposition 4). The measurability of $\overline{A}_t^{\mathrm{p}}$ follows from Proposition 7.

Now we are ready to formulate the assumptions on the portfolio constraints. In the remaining part of the paper, we always assume that they are satisfied.

Assumptions. For all $t \geq 1$ and $\chi = (\omega, v_0, h_0, \ldots)$, it holds that

(A.1) $\bar{A}_t(\chi) \subseteq D_t(\omega)$;

(A.2) there exists $(\alpha, \beta) \in \bar{C}_t(\chi)$ such that $\mathrm{P}_t(\langle \alpha, X_{t+1}\rangle + \langle \beta, Y_{t+1}\rangle > 0)(\omega) = 1$;

(A.3) $\bar{A}_t^{\mathrm{p}}(\chi) \subseteq \bar{A}_t(\chi)$;

(A.4) $\bar{A}_t^{\mathrm{p}}(\chi)$ is a compact set.

Let us comment on the interpretation of these assumptions. (A.1) is imposed to ensure that any strategy which satisfies the portfolio constraints generates a non-negative wealth sequence. As a consequence, for the realization of any profile of strategies satisfying the portfolio constraints we have

$$(\omega, v_0, H_0(\omega), H_1(\omega), \ldots) \in \Theta_t \text{ a.s. for all } t \geq 0.$$

Since the underlying probability space and the filtration are complete, we can assume that the above inclusion holds for all $\omega \in \Omega$, as otherwise it is possible to modify the strategies on a set of zero probability to make it valid for all ω.

Assumption (A.2) implies that there exists a strategy with a strictly positive wealth sequence. Such a strategy can be found via a standard measurable selection argument, using that $\overline{C}_t$ are measurable sets. Observe that (A.2) is a very mild assumption. For example, it holds if there is a non-zero vector $\alpha \in \bar{A}_t$ with all non-negative coordinates (recall that $X_t^n > 0$ for all n), since then $(\alpha/|\alpha|, 0) \in \overline{C}_t$ by (5).

Assumption (A.3) means that the agents can add or subtract null investment vectors from their portfolios. Note that in the literature it is sometimes required that $L_t \subseteq \bar{A}_t$ (i.e. any investment that leads to no profit or loss is allowed). It is not difficult to see that in our model this requirement implies (A.3).

Assumption (A.4) will allow to reduce the problem of constructing the optimal strategy to an optimization problem on a compact set. One can see that this assumption is equivalent to the following version of the no-arbitrage condition.

Let $U_t(\omega)$ denote the cone of arbitrage opportunities in the exogenous assets at time $t \geq 0$, which consists of all $u \in \mathbb{R}^{N_1}$ such that

$$\langle e, u\rangle = 0, \qquad \mathrm{P}_t(\langle u, X_{t+1}\rangle \geq 0)(\omega) = 1, \qquad \mathrm{P}_t(\langle u, X_{t+1}\rangle > 0)(\omega) > 0.$$

We say that there are *no unbounded arbitrage opportunities* in the model if for all $\chi = (\omega, v_0, h_0, \ldots) \in \Theta$ and $t \geq 0$ the following holds:

(A.5) there is no $u \in U_t(\omega)$ such that $\lambda u \in \bar{A}_t(\chi)$ for any $\lambda > 0$.

In other words, an agent cannot infinitely multiply the profit from an arbitrage opportunity, but the set $\bar{A}_t$ may contain some of them. If there are no constraints on the exogenous assets (i.e. $\bar{A}_t = \{\alpha \in \mathbb{R}^{N_1} : \langle e, \alpha\rangle \in [0, 1]\}$), then (A.5) becomes the usual no-arbitrage condition $U_t = \emptyset$.

Proposition 2. *Suppose the model satisfies assumptions (A.1), (A.3). Then assumptions (A.4) and (A.5) are equivalent.*

This result can be proved using standard arguments (see, e.g., Kabanov and Stricker (2001) or Karatzas and Kardaras (2007)).

3. Main results

3.1. The notion of optimality

We will be interested in the long-run behavior of relative wealth of agents, i.e. their shares in the total market wealth. We define the total market wealth and the relative wealth of agent m as the functions of market history $\chi = (\omega, v_0, h_0, \ldots)$

$$\overline{W}_t(\chi) = \sum_{m=1}^{M} \overline{V}_t^m(\chi), \qquad \bar{r}_t^m(\chi) = \frac{\overline{V}_t^m(\chi)}{\overline{W}_t(\chi)},$$

where the functions $\overline{V}_t^m$ are defined by equation (3), and we put $\bar{r}_t^m = 0$ when $\overline{W}_t = 0$. Recall that $\overline{V}_t^m$ is defined on the set Θ_t, hence we will assume that $\overline{W}_t$ and $\bar{r}_t^m$ are defined only on this set as well.

For a given feasible strategy profile and a vector of initial wealth, denote by $W_t(\omega)$, $V_t^m(\omega)$, $r_t^m(\omega)$ the corresponding realizations of the total market wealth, the agents' absolute and relative wealth, i.e. $W_t(\omega) = \overline{W}_t(\omega, v_0, H_0(\omega), \ldots)$, where $H_t(\omega)$ are the realizations of the agents' strategies, and $V_t^m(\omega)$, $r_t^m(\omega)$ are defined in a similar way. These functions are $\mathcal{F}_t$-measurable.

Definition 1. In a feasible strategy profile $(\overline{H}^1, \ldots, \overline{H}^M)$ with initial wealth $v_0 \in \mathbb{R}_+^M$ such that $v_0^m > 0$, we call a strategy $\overline{H}^m$ *survival*[2] if

$$\inf_{t \geq 0} r_t^m > 0 \text{ a.s.},$$

and call it *dominating* if

$$\lim_{t\to\infty} r_t^m = 1 \text{ a.s.}$$

Our main goal will be to show that the strategy $\widehat{H}$, which we construct in the next section, is survival in any strategy profile and dominating in a strategy profile in which the strategies of the other agents are different from it asymptotically (the precise meaning of this will be given in Section 3.4). Consequently, if some agents use $\widehat{H}$, then any other survival strategy should be asymptotically close to it. An important characteristic of $\widehat{H}$ will be that it essentially depends, as a function, only on a random outcome ω and the current total market wealth W_t, but not on the endogenous prices and relative wealth of agents[3].

Note that any survival strategy is asymptotically unbeatable in the following sense: if agent m uses a survival strategy then there exists a (finite-valued) random variable γ such that

$$r_t^k \leq \gamma r_t^m, \qquad k = 1, \ldots, M,\ t \geq 0,$$

which expresses the fact that the wealth of any other agent cannot grow asymptotically faster than the wealth of an agent who uses a survival strategy. For a discussion of unbeatable strategies as a game solution concept in related evolutionary finance models, see e.g. Amir et al. (2013).

[2] We use the terminology of Amir et al. (2013). Note that often a strategy is called survival if $\limsup_{t\to\infty} r_t^m > 0$, see, e.g., Blume and Easley (1992).

[3] Strictly speaking, $\widehat{H}$ may also depend on some additional information contained in market history χ, but only through the dependence of the portfolio constraints on such information.

We would like to emphasize that we do not insist on that all agents should use only survival strategies, as they may have other economic goals. We investigate what happens with a market *if* some agents use such strategies.

For construction of a survival strategy, the following notion will be useful.

Definition 2. For a given feasible strategy profile and initial wealth, we call a strategy $\overline{H}^m$ *relative growth optimal* if

$$V_t^m > 0 \text{ a.s. for all } t \geq 0 \text{ and } \ln r_t^m \text{ is a submartingale.}$$

Proposition 3. *A relative growth optimal strategy is survival.*

Proof. Since any non-positive submartingale has a finite limit with probability 1 (see, e.g., Shiryaev (2019, Chapter 7.4)), for a relative growth optimal strategy we have $\lim_{t\to\infty} \ln r_t^m > -\infty$, and therefore $r_\infty^m = \lim_{t\to\infty} r_t^m > 0$, which implies the claimed result. $\square$

3.2. Construction of a relative growth optimal strategy

We begin with a lemma which defines the components $\widehat{\alpha}, \widehat{\beta}$ of the strategy $\widehat{H}_t(\chi) = (\widehat{\alpha}_t(\chi), \widehat{\beta}_t(\chi))$ which will be shown to be relative growth optimal. Its statement is somewhat involved, but clarifying comments will be provided after Theorem 1 below.

Recall that we need to define $\widehat{\alpha}_t, \widehat{\beta}_t$ only on the set Θ_t' (see Section 2.3), while on its complement these functions can be defined in an arbitrary way (respecting the $\mathcal{G}_t$-measurability and the portfolio constraints), since this will not have any effect on realizations of wealth sequences.

In the statement of the lemma and subsequent results, we will use the following agreement to treat indeterminacy: $0/0 = 0$, $0 \cdot \ln 0 = 0$, $a \cdot \ln 0 = -\infty$ if $a > 0$.

Let us also introduce the notation $\overline{\mathrm{E}}_t$ for the conditional expectation of a function $\bar{\xi}$ depending on market history. Let K_t denote the regular conditional distribution defined in (7). For $\mathcal{G}$-measurable $\bar{\xi} = \bar{\xi}(\chi)$, put

$$\overline{\mathrm{E}}_t(\bar{\xi})(\chi) = \int_\Omega \bar{\xi}(\widetilde{\omega}, v_0, h_0, \ldots) K_t(\omega, d\widetilde{\omega}), \tag{8}$$

where the variables ω and $v_0, h_0, \ldots$ are the elements of $\chi = (\omega, v_0, h_0, \ldots)$, and $\widetilde{\omega}$ is the integration variable, provided that the integral is well-defined.

The argument χ of $\overline{\mathrm{E}}_t$ will be often omitted for brevity, but the dependence on it should be clear from the context.

Remark 3. Many formulas below will contain expectations of functions $\bar{\xi}$ of the form

$$\bar{\xi}(\omega, v_0, h_0, \ldots) = f(X_{t+1}(\omega), Y_{t+1}(\omega), \bar{\eta}(\omega, v_0, h_0, \ldots)),$$

where f is a $\mathcal{B}(\mathbb{R}^{N_1+N_2+d})$-measurable non-random function and $\bar{\eta}$ is a $\mathcal{G}_t$-measurable function with values in $\mathbb{R}^d$ for some d (for example, $\bar{\eta}$ can be $\overline{W}_t$). In order to see what (8) means in this case, it is useful to observe that $\overline{\mathrm{E}}_t(\bar{\xi})$ for such $\bar{\xi}$ can be computed as follows: for all fixed $(v_0, h_0, \ldots)$ we have, with probability 1 for all ω,

$$\overline{\mathrm{E}}_t(\bar{\xi})(\chi) = (\mathrm{E}_t\, f(X_{t+1}, Y_{t+1}, z))\big|_{z=\eta(\chi)} = \int_\Omega f(X_{t+1}(\widetilde{\omega}), Y_{t+1}(\widetilde{\omega}), z) K_t(\omega, d\widetilde{\omega})\Big|_{z=\eta(\chi)},$$

where $\chi = (\omega, v_0, h_0, \ldots)$ and the expectation E_t is defined in (7). This formula follows from that $\bar{\xi}(\omega, v_0, h_0, \ldots)$ becomes an $\mathcal{F}_t$-measurable random variable when $v_0, h_0, \ldots$ are fixed.

Lemma 1. *The following statements hold true for each $t \geq 0$.*

(a) Consider the $\mathcal{G}_{t+1}^-$-measurable vectors $\widetilde{Y}_{t+1}$ with values in $\mathbb{R}^{N_2}$ and components

$$\widetilde{Y}_{t+1}^n(\chi) = Y_{t+1}^n(\omega)\, \mathrm{I}(\exists\, \beta \in \overline{B}_t(\chi) : \beta^n > 0)$$

(where ω is an element of $\chi = (\omega, v_0, h_0, \dots)$), and the functions

$$g_i(x) = \frac{1}{i} + i \arctan\Big(\frac{x}{i}\Big), \qquad x \in \mathbb{R}_+,\ i = 1, 2, \dots \tag{9}$$

Then there exist $\mathcal{G}_t^-$-measurable functions $\widehat{\alpha}_{t,i}(\chi)$ such that for all $\chi \in \Theta_t'$

$$\widehat{\alpha}_{t,i} \in \operatorname*{argmax}_{\alpha \in \overline{A}_t^{\mathrm{p}}} \big\{ \overline{\mathrm{E}}_t \ln g_i(\langle \alpha, X_{t+1}\rangle \overline{W}_t + |\widetilde{Y}_{t+1}|) - \langle e, \alpha\rangle \big\}. \tag{10}$$

(b) There exists an increasing sequence of $\mathcal{G}_t^-$-measurable functions $i_j(\chi)$, $j \geq 1$, with positive integer values, and a $\mathcal{G}_t^-$-measurable function $\widehat{\alpha}_t(\chi)$ with values in $\overline{A}_t^{\mathrm{p}}$, such that on the set Θ_t'

$$\widehat{\alpha}_t = \lim_{j\to\infty} \widehat{\alpha}_{t,i_j}.$$

(c) The set $\widetilde{B}_t(\chi) = \{\beta \in \overline{B}_t(\chi) : |\beta| = 1 - \langle e, \widehat{\alpha}_t(\chi)\rangle\}$ is non-empty for $\chi \in \Theta_t'$ and there exists a $\mathcal{G}_t^-$-measurable function $\widehat{\beta}_t(\chi)$ with values in $\widetilde{B}_t$ such that for any $\chi \in \Theta_t'$

$$\widehat{\beta}_t \in \operatorname*{argmax}_{\beta \in \widetilde{B}_t} \bigg\{ \overline{\mathrm{E}}_t \frac{\langle \ln \beta, \widetilde{Y}_{t+1}\rangle}{\langle \widehat{\alpha}_t, X_{t+1}\rangle \overline{W}_t + |\widetilde{Y}_{t+1}|} \bigg\}. \tag{11}$$

Theorem 1. *In every feasible strategy profile, any strategy $\widehat{H} = (\widehat{\alpha}, \widehat{\beta})$ constructed as in Lemma 1 is relative growth optimal.*

Note that Lemma 1 defines $\widehat{\alpha}_t, \widehat{\beta}_t$ not necessarily in a unique way, hence we write "any strategy $\widehat{H}$" in the theorem. This may be so if, for example, some of the vectors X_t have linearly dependent components.

Observe that, since the functions $\widehat{\alpha}_t, \widehat{\beta}_t$ are $\mathcal{G}_t^-$-measurable, the strategy $\widehat{H}$ satisfies the sufficient condition of feasibility from Proposition 1, i.e. if the other agents use $\mathcal{G}_t^-$-measurable strategies, then the corresponding strategy profile is feasible. In particular, the strategy profile where all agents use $\widehat{H}$ is feasible.

Let us comment on technical aspects of the above results and explain why in Lemma 1 it is necessary to introduce the functions g_i and consider maximization problem (10). The idea behind the construction of the strategy $\widehat{H}$ is that we would like to find $(\widehat{\alpha}, \widehat{\beta})$ such that

$$\widehat{\alpha}_t \text{ maximizes } \overline{\mathrm{E}}_t \ln(\langle \alpha, X_{t+1}\rangle \overline{W}_t + |Y_{t+1}|) - \langle e, \alpha\rangle \text{ over } \alpha \in \overline{A}_t, \tag{12}$$

and, for this $\widehat{\alpha}_t$, define the component $\widehat{\beta}_t$ as in (11). Such a strategy would satisfy inequalities (21) and (29), which play the key role in the proofs.

However, since problem (12) may have no solution, we find the solutions $\widehat{\alpha}_{t,i}$ of the maximization problems truncated by the functions g_i and select a convergent subsequence; then inequalities (21), (29) still remain satisfied. The purpose of the functions g_i is to obtain a sequence of approximations to the identity which are bounded from above and bounded away from zero (note that $g_i(x) \to x$ as $i \to \infty$ for any $x \in \mathbb{R}_+$). The particular form of such approximations we use in (9) is convenient since it satisfies some technical inequalities we need in the proof of Lemma 1 (see Section 4.2).

To ensure that such a subsequence exists, we use the observation that it is possible to maximize not over the whole set $\overline{A}_t$ but over its compact subset $\overline{A}_t^{\mathrm{p}}$. We also replace $\overline{Y}_t$ with $\widetilde{Y}_t$ to avoid the situation when an asset yields a positive payoff with positive conditional probability, but it is not possible to invest in it.

Note that when no portfolio constraints are imposed on the endogenous assets, i.e. $\overline{B}_t = \{\beta \in \mathbb{R}_+^{N_2} : |\beta| \le 1\}$ and hence $\widetilde{Y}_{t+1} = Y_{t+1}$, we can find $\widehat{\beta}_t$ explicitly:

$$\widehat{\beta}_t^n = \overline{\mathrm{E}}_t \frac{Y_{t+1}^n}{\langle \widehat{\alpha}_t, X_{t+1}\rangle \overline{W}_t + |Y_{t+1}|}. \tag{13}$$

Indeed, for $\widehat{\beta}_t$ defined by (13), we have $|\widehat{\beta}_t| = 1 - \langle e, \widehat{\alpha}_t\rangle$ as follows from equality (22) below, and for any $\beta \in \mathbb{R}_+^{N_2}$ with $|\beta| = 1 - \langle e, \widehat{\alpha}_t\rangle$ we have

$$\overline{\mathrm{E}}_t \frac{\langle \ln \beta, Y_{t+1}\rangle}{\langle \widehat{\alpha}_t, X_{t+1}\rangle \overline{W}_t + |Y_{t+1}|} = \langle \ln \beta, \widehat{\beta}_t\rangle \le \langle \ln \widehat{\beta}_t, \widehat{\beta}_t\rangle,$$

where the inequality follows from Gibb's inequality (see Proposition 9 below). Consequently, $\widehat{\beta}_t$ indeed delivers the maximum in (11).

3.3. The strategy $\widehat{H}$ as an equilibrium growth optimal strategy

Let us show that the strategy $\widehat{H}$ can be found as an equilibrium strategy of the representative agent who holds a growth optimal portfolio in a market with $N_1 + N_2$ exogenous assets, where the first N_1 assets are the same as in the original market, and the remaining N_2 assets are treated as exogenous with the prices being equal to the prices of the endogenous assets induced by $\widehat{H}$ in the original market. This notion of equilibrium is conceptually similar to the one in the Lucas model of an exchange economy (Lucas, 1978) with the logarithmic utility, though we do not consider consumption.

Recall that in a market with exogenous prices a strategy with portfolio value $\widehat{v}_t > 0$ is called *growth optimal* (or a *numéraire portfolio*) if for any other strategy with portfolio value $v_t \ge 0$ it holds that $v_t/\widehat{v}_t$ is a supermartingale. If $z_t = (v_t - v_{t-1})/v_{t-1}$ and $\widehat{z}_t = (\widehat{v}_t - \widehat{v}_{t-1})/\widehat{v}_{t-1}$ denote the one-period returns on the strategies' portfolios, then this supermartingality condition is equivalent to that for each $t \ge 0$

$$\mathrm{E}_t \frac{1 + z_{t+1}}{1 + \widehat{z}_{t+1}} \le 1. \tag{14}$$

To relate (14) to our model with exogenous and endogenous prices, denote by $\widehat{p}_t$ the endogenous prices that clear the market when all agents use the strategy $\widehat{H}$, i.e.

$$\widehat{p}_t^n(\chi) = \widehat{\beta}_t^n(\chi) \overline{W}_t(\chi).$$

Let $\overline{Z}$ denote the returns on the endogenous assets in this case,

$$\overline{Z}_{t+1}^n(\chi) = \frac{Y_{t+1}^n(\omega)}{\widehat{p}_t^n(\chi)} - 1,$$

where ω is an element of $\chi = (\omega, v_0, h_0, \ldots)$.

Now imagine there is an "infinitesimal" agent whose strategy does not affect the endogenous prices. Then, if this agents invests proportions (α_t, β_t) of his/her wealth in the assets, the one-period return on the portfolio will be

$$R_{t+1} = \langle \alpha_t, X_{t+1} - e\rangle + \langle \beta_t, \overline{Z}_{t+1}\rangle, \tag{15}$$

where $e = (1, \dots, 1)$ is subtracted because X_{t+1} is the vector of relative prices changes, so the returns are $X_{t+1}^n - 1$. The next theorem compares this quantity with the return on the portfolio $(\widehat{\alpha}_t, \widehat{\beta}_t)$ and proves a counterpart of inequality (14).

Theorem 2. *For any $t \geq 0$, $\chi \in \Theta_t'$, and $(\alpha, \beta) \in \overline{C}_t(\chi)$, we have (cf. (14))*

$$\overline{\mathrm{E}}_t \frac{1 + \langle \alpha, X_{t+1} - e \rangle + \langle \beta, \overline{Z}_{t+1} \rangle}{1 + \langle \widehat{\alpha}_t, X_{t+1} - e \rangle + \langle \widehat{\beta}_t, \overline{Z}_{t+1} \rangle} \leq 1. \tag{16}$$

If $\overline{\mathrm{E}}_t \,|\ln(\langle \widehat{\alpha}_t, X_{t+1} - e \rangle + \langle \widehat{\beta}_t, \overline{Z}_{t+1} \rangle)| < \infty$, then

$$(\widehat{\alpha}_t, \widehat{\beta}_t) \in \operatorname*{argmax}_{(\alpha,\beta)\in\overline{C}_t} \mathrm{E}_t \ln(\langle \alpha, X_{t+1} - e \rangle + \langle \beta, \overline{Z}_{t+1} \rangle). \tag{17}$$

Relation (16) expresses the above-mentioned idea of equilibrium. That is, if the market holds the portfolio $(\widehat{\alpha}_t, \widehat{\beta}_t)$, then any small agent who wants to have a growth optimal portfolio, must also choose the same investment proportions. Relation (17) is an analogue of the well-known fact that a growth optimal portfolio maximizes one-period expected logarithmic returns, under the respective integrability condition.

3.4. Asymptotic proximity of survival strategies

The next result describes the evolution of the relative wealth of strategies different from $\widehat{H}$. It will be stated for the case when there are no portfolio constraints on the endogenous assets, i.e $\overline{B}_t = \{\beta \in \mathbb{R}_+^{N_2} : |\beta| \leq 1\}$. This assumption is necessary because the proof relies on the explicit form of $\widehat{\beta}_t$ given by (13).

Given a feasible strategy profile and a vector of initial wealth, let $\alpha_t^m(\omega), \beta_t^m(\omega)$ be the components of the realization of agent m's strategy $H_t^m(\omega)$. By $H_t^R = (\alpha_t^R, \beta_t^R)$ we will denote the realization of the representative strategy of all the agents, which we define as the weighted sum of their strategies with r_t^m as the weights:

$$\alpha_t^R(\omega) = \sum_{m=1}^M r_t^m(\omega)\alpha_t^m(\omega), \qquad \beta_t^R(\omega) = \sum_{m=1}^M r_t^m(\omega)\beta_t^m(\omega).$$

In a similar way, by $H_t^r = (\alpha_t^r, \beta_t^r)$ we will denote the realization of the representative strategy of agents $m = 2, \dots, M$ weighted with their relative wealths excluding agent 1:

$$\alpha_t^r(\omega) = \sum_{m=2}^M \frac{r_t^m(\omega)}{1 - r_t^1(\omega)}\alpha_t^m(\omega), \qquad \beta_t^r(\omega) = \sum_{m=2}^M \frac{r_t^m(\omega)}{1 - r_t^1(\omega)}\beta_t^m(\omega),$$

where $0/0 = 0$. Note that $a_t^R, \beta_r^R, \alpha_t^r, \beta_t^r$ are $\mathcal{F}_t$-measurable.

The realization of the constraints sets $\overline{A}_t(\chi)$ will be denoted by A_t, where $A_t(\omega) = \overline{A}_t(\omega, v_0, H_0(\omega), \dots)$.

Theorem 3. *Suppose $\overline{B}_t = \{\beta \in \mathbb{R}_+^{N_2} : |\beta| \leq 1\}$, and agent 1 uses the strategy $\overline{H}^1 = \widehat{H}$. Considering the realizations of strategies, wealth sequences, and constraints sets, let*

$$Q_{t+1}(\omega) = \max_{\alpha \in A_t(\omega)} \langle \alpha, X_{t+1}(\omega) \rangle + \frac{|Y_{t+1}(\omega)|}{W_t(\omega)}.$$

Then, with probability 1,

$$\sum_{t=0}^{\infty}\Big(\frac{\langle \alpha_t^1-\alpha_t^R, X_{t+1}\rangle}{Q_{t+1}}\Big)^2+\|\beta_t^1-\beta_t^R\|^2<\infty, \tag{18}$$

and

$$\lim_{t\to\infty} r_t^1 = 1 \textit{ a.s. on the set } \Big\{\omega : \sum_{t=0}^{\infty}\Big(\frac{\langle \alpha_t^1-\alpha_t^r, X_{t+1}\rangle}{Q_{t+1}}\Big)^2+\|\beta_t^1-\beta_t^r\|^2=\infty\Big\}. \tag{19}$$

Note that the maximum in the definition of Q_{t+1} is attained because, according to Proposition 2, it can be taken over the compact set $A_t^{\mathrm{p}}(\omega)$. Furthermore, $Q_{t+1}>0$ by assumption (A.2).

Relation (18) essentially shows that if one agent uses the strategy $\widehat{H}$ then this agent asymptotically determines the representative strategy of the market so that h_t^R becomes close to h_t^1 in the sense that the series in (18) converges, and, consequently,

$$\frac{\langle \alpha_t^1-\alpha_t^R, X_{t+1}\rangle}{Q_{t+1}}\to 0,\quad \beta_t^1-\beta_t^R\to 0 \quad \text{as } t\to\infty$$

(note that, in general, we cannot say that $\alpha_t^1-\alpha_t^R\to 0$).

Relation (19) provides a sufficient condition for an agent using the strategy $\widehat{H}$ to dominate in the market, which happens when the realization of the representative strategy of the other agents is asymptotically different from the realization of $\widehat{H}$ in the sense that the series in (19) diverges.

Another corollary from Theorem 3 is that the presence of an agent who uses the strategy $\widehat{H}$ asymptotically determines the realizations of relative prices $\rho_t^n = p_t^n/W_t$ of the endogenous assets. It is not difficult to see that $\rho_t^n=\beta_t^{R,n}$, and hence (18) implies that for each n and $t\to\infty$ we have $\beta_t^{1,n}-\rho_t^n\to 0$.

3.5. Relation to other results in the literature

The above theorems generalize some known results on asymptotically optimal strategies. An immediate corollary from Theorem 2 is that in a market with only exogenous assets, the strategy $\widehat{\alpha}_t$ is a numéraire portfolio. Note that in this case $\widehat{\alpha}_t$ depends only on ω, but not on full market history χ (assuming that the constraints sets $\overline{C}_t$ also depend only on ω), in the sense that it can written as a function $\widehat{\alpha}_t(\omega, v_0, h_0,\dots)=a_t(\omega)$ for some $\mathcal{F}_t$-measurable a_t.

In a market with only endogenous assets and no portfolio constraints, as follows from (13), the optimal strategy is given by

$$\widehat{\beta}_t^n = \mathrm{E}_t\,\frac{Y_{t+1}^n}{|Y_{t+1}|}$$

(here $\widehat{\beta}_t^n$ also depends on ω only). This strategy was found by Amir et al. (2013). They also proved that it is asymptotically unique in the class of survival strategies depending only on ω, which is a particular case of our Theorem 3. For earlier results in this direction, see the papers of Amir et al. (2005); Evstigneev et al. (2002); Hens and Schenk-Hoppé (2005), which study models with short-lived assets with additional assumptions on admissible strategies or on asset payoffs.

Finally, suppose that there is only one exogenous asset, short sales of this asset are not allowed, and there are no other portfolio constraints, i.e. $\overline{C}_t = \mathbb{R}_+ \times \mathbb{R}_+^{N_2}$. Then $\widehat{\alpha}_t$ is defined as follows: if $\overline{\mathrm{E}}_t(X_{t+1}\overline{W}_t/|Y_{t+1}|)(\chi) \le 1$, then $\widehat{\alpha}_t(\chi) = 0$; otherwise $\widehat{\alpha}_t(\chi)$ is the unique solution of the equation

$$\overline{\mathrm{E}}_t \frac{X_{t+1}\overline{W}_t}{\alpha + |Y_{t+1}|}(\chi) = 1.$$

This can be seen from relations (21)–(22) below. Indeed, if $\overline{\mathrm{E}}_t(X_{t+1}\overline{W}_t/|Y_{t+1}|) \le 1$, then equality (22) can be true only if $\widehat{\alpha}_t = 0$. In the case $\overline{\mathrm{E}}_t(X_{t+1}\overline{W}_t/|Y_{t+1}|) > 1$, equality (22) has two solutions, the zero one and a non-zero one. But if $\widehat{\alpha}_t = 0$, then (21) cannot hold true for $\alpha > 0$, hence we are left only with the non-zero solution.

After $\widehat{\alpha}_t$ has been defined as above, the component $\widehat{\beta}_t$ can be found from (13), which gives

$$\widehat{\beta}_t^n = \overline{\mathrm{E}}_t \frac{Y_{t+1}^n}{\widehat{\alpha}_t X_{t+1}\overline{W}_t + |Y_{t+1}|}.$$

This strategy was obtained by Drokin and Zhitlukhin (2020) in a somewhat more general case, whee the sequence X_t is predictable (e.g. the exogenous asset is a risk-free bond).

4. Proofs of the main results

4.1. Auxiliary results on random sets

This section contains results from the theory of random sets which will be used in the proofs for dealing with portfolio constraints.

By a *random set* (or a *measurable correspondence*) in $\mathbb{R}^N$ defined on a measurable space $(S, \mathcal{S})$ we call a set-valued function $\phi\colon S \to 2^{\mathbb{R}^N}$ such that for any open set $A \subseteq \mathbb{R}^N$ it holds that $\phi^{-1}(A) \in \mathcal{S}$, where $\phi^{-1}(A) = \{s : \phi(s) \cap A \neq \emptyset\}$ is the lower inverse of A. An equivalent definition is that the distance function $d(x, \phi(s))$ is $\mathcal{S}$-measurable for any $x \in \mathbb{R}^N$ (where $d(x, \emptyset) = \infty$). In what follows, the role of $(S, \mathcal{S})$ will be played by $(\Omega, \mathcal{F}_t)$, $(\Theta, \mathcal{G}_t)$, or $(\Theta, \mathcal{G}_t^-)$.

A random set is called closed (respectively, compact, non-empty) if $\phi(s)$ is closed (compact, non-empty) for any $s \in \mathcal{S}$. A *measurable selector* is an $\mathcal{S}$-measurable function ξ such that $\xi(s) \in \phi(s)$ for any s. A function $f(s, x)\colon S \times \mathbb{R}^N \to \mathbb{R}$ is called a *Carathéodory function* if it is measurable in s and continuous in x.

The following results are known for random sets in $\mathbb{R}^N$.

Proposition 4. *If ϕ_n, $n = 1, 2, \dots,$ are random sets, then $\cup_n \phi_n$ is a random set; if ϕ_n are also closed, then $\cap_n \phi_n$ is a closed random set.*

Proposition 5 (Filippov's theorem)**.** *Suppose ϕ is a non-empty compact random set, f is a Carathéodory function, and π is a measurable function. Then the correspondence*

$$\psi(s) = \{x \in \phi(s) : f(s, x) = \pi(s)\}$$

is measurable and compact. Moreover, if ψ is non-empty, then it has a measurable selector ξ, and hence $f(s, \xi(s)) = \pi(s)$.

Proposition 6 (Measurable maximum theorem)**.** *For a non-empty compact random set ϕ and a Carathéodory function f, let μ be the maximum function and ψ be the argmax correspondence defined by*

$$\mu(s) = \max_{x \in \phi(s)} f(s, x), \qquad \psi(s) = \operatorname*{argmax}_{x \in \phi(s)} f(s, x).$$

Then μ is measurable and ψ is non-empty, compact, measurable, and has a measurable selector.

Proofs of the above results can be found in the book of Aliprantis and Border (2006, Chapter 18) for random sets in general metric spaces, except the result about $\cap_n \phi_n$, which holds (in a metric space) if ϕ_n are compact. For $\mathbb{R}^N$, it can be extended to closed sets using that $\mathbb{R}^N$ is σ-compact.

For convenience, the following results are provided with proofs (they are not included in the above-mentioned book).

Proposition 7. *Let L be a random linear subspace of $\mathbb{R}^N$ (i.e. for each s the set $L(s)$ is a linear space and the correspondence L is measurable), $L^\perp$ be the orthogonal space, and ϕ be a closed random set in $\mathbb{R}^N$. Then the projection correspondence*

$$\operatorname{pr}_L \phi(s) = \{x \in L(s) : \exists\, y \in L^\perp(s) \text{ such that } x + y \in \phi(s)\}$$

is measurable.

Proof. By Castaing's theorem (see Corollary 18.14 in Aliprantis and Border (2006)), a non-empty closed correspondence is measurable if and only if it can be represented as the closure of a countable family of measurable selectors from it. Hence, we can find measurable ξ_i such that $\phi(s) = \operatorname{cl}\{\xi_i(s), i \geq 1\}$ on the set $\{s : \phi(s) \neq \emptyset\}$. Using that

$$\operatorname{cl}(\operatorname{pr}_L \phi(s)) = \begin{cases} \operatorname{cl}\{\operatorname{pr}_L \xi_i(s), i \geq 1\}, & \text{if } \phi(s) \neq \emptyset, \\ \emptyset, & \text{if } \phi(s) = \emptyset, \end{cases}$$

one can see that $\operatorname{cl}(\operatorname{pr}_L \phi)$ is measurable. Since the measurability of a correspondence is equivalent to the measurability of its closure (Aliprantis and Border, 2006, Lemma 18.3), $\operatorname{pr}_L \phi$ is measurable. □

Proposition 8. *Let ϕ be a non-empty compact random set and ξ_n be a sequence of measurable selectors from it. Then there exists a measurable selector ξ from ϕ and a sequence of measurable functions $1 \leq i_1(s) < i_2(s) < \dots$ with integer values such that $\lim_{j\to\infty} \xi_{i_j(s)}(s) = \xi(s)$ for all s.*

Proof. The set $\psi(s) = \cap_n \operatorname{cl}\{\xi_k(s),\ k \geq n\}$ is measurable, non-empty, and closed, so there exists a measurable selector $\xi \in \psi$ (by Castaing's theorem mentioned above). Then the sequence i_j can be constructed by induction as follows. Put $i_1 = 1$. If i_j is defined, consider the random set $\eta_j(s) = \{k > i_j(s) : |\xi_k(s) - \xi(s)| \leq j^{-1}\} \subset \mathbb{N}$, which is measurable, non-empty, and closed. Let i_{j+1} be a measurable selector from η_j. Then $|\xi_{i_{j+1}} - \xi| < j^{-1}$, which gives the desired convergence. □

4.2. Proof of Lemma 1

Proof of claim (a). Fix any $t \geq 0$. Let $f_i(\chi, \alpha)$ be the function which is maximized in the definition of $\widehat{\alpha}_{t,i}$ in (10), i.e. for $\chi \in \Theta'_t$ put

$$f_i(\chi, \alpha) = \overline{\mathrm{E}}_t \ln g_i(\langle \alpha, X_{t+1}\rangle \overline{W}_t + |\widetilde{Y}_{t+1}|)(\chi) - \langle e, \alpha\rangle,$$

while for $\chi \notin \Theta'_t$, put $f_i(\chi, \alpha) = 0$ for any α. The function f_i is a Carathéodory function, and the set $\overline{A}^{\mathrm{p}}_t$, over which it is maximized, is compact by assumption (A.4). Hence the measurable maximum theorem implies the existence of a measurable selector $\widehat{\alpha}_{t,i}$ from the argmax in (10).

Proof of claim (b) readily follows from Proposition 8. Before we continue with the proof of claim (c), let us show that $\widehat{\alpha}_t$ satisfies three relations that will be used in its proof, as well as in the proof of Theorem 1.

Lemma 2. *For any $t \geq 0$, $\chi \in \Theta'_t$, and $\alpha \in \bar{A}_t(\chi)$ we have*

$$\bar{\mathrm{P}}_t(\langle \widehat{\alpha}_t, X_{t+1}\rangle \overline{W}_t + |\widetilde{Y}_{t+1}| > 0) = 1, \tag{20}$$

$$\bar{\mathrm{E}}_t\left(\frac{\langle \widehat{\alpha}_t - \alpha, X_{t+1}\rangle \overline{W}_t}{\langle \widehat{\alpha}_t, X_{t+1}\rangle \overline{W}_t + |\widetilde{Y}_{t+1}|}\right) \geq \langle e, \widehat{\alpha}_t - \alpha\rangle, \tag{21}$$

$$\bar{\mathrm{E}}_t\left(\frac{\langle \widehat{\alpha}_t, X_{t+1}\rangle \overline{W}_t}{\langle \widehat{\alpha}_t, X_{t+1}\rangle \overline{W}_t + |\widetilde{Y}_{t+1}|}\right) = \langle e, \widehat{\alpha}_t\rangle. \tag{22}$$

Proof. Fix t, χ, α, and for each $i \geq 1$ consider the function $u_i(\varepsilon) = f_i((1-\varepsilon)\widehat{\alpha}_t + \varepsilon\alpha)$, $\varepsilon \in [0,1]$, i.e.

$$u_i(\varepsilon) = \bar{\mathrm{E}}_t \ln g_i\big(\langle (1-\varepsilon)\widehat{\alpha}_{t,i} + \varepsilon\alpha, X_{t+1}\rangle \overline{W}_t + |\widetilde{Y}_{t+1}|\big)(\chi) - \langle e, (1-\varepsilon)\widehat{\alpha}_{t,i}(\chi) + \varepsilon\alpha\rangle. \tag{23}$$

Since $\ln g_i(x)$ is concave for $x \geq 0$, the function $u_i(\varepsilon)$ is also concave. As it attains the maximum value at $\varepsilon = 0$, the right derivative $u'_i(0) \leq 0$. From this inequality we will obtain relations (20) and (21) by interchanging the order of differentiation and taking expectation, and then passing to the limit $i \to \infty$. In order to show that the derivative and expectation can be interchanged, let us write the expectation in (23) as $\bar{\mathrm{E}}_t(\ln g_i(q(\widetilde{\omega}, \varepsilon)))(\chi) = \int_\Omega \ln g_i(q(\widetilde{\omega}, \varepsilon)) K_t(\omega, d\widetilde{\omega})$ with the function

$$q(\widetilde{\omega}, \varepsilon) = \langle (1-\varepsilon)\widehat{\alpha}_{t,i}(\widetilde{\chi}) + \varepsilon\alpha, X_{t+1}(\widetilde{\omega})\rangle \overline{W}_t(\widetilde{\chi}) + |\widetilde{Y}_{t+1}(\widetilde{\chi})|,$$

where $\widetilde{\chi} = (\widetilde{\omega}, v_0, h_0, \ldots)$ and ω, v_0, $(h_s)_{s\geq 0}$ are taken from the fixed $\chi = (\omega, v_0, h_0, \ldots)$. Treating the right derivative as the corresponding limit and applying Fatou's lemma, we obtain ($\widetilde{\chi}$ and $\widetilde{\omega}$ will be omitted for brevity)

$$(\bar{\mathrm{E}}_t \ln g_i(q(\varepsilon)))'_{\varepsilon=0} \geq \bar{\mathrm{E}}_t\left(\frac{g'_i(q(0))}{g_i(q(0))}\langle \alpha - \widehat{\alpha}_{t,i}, X_{t+1}\rangle \overline{W}_t\right). \tag{24}$$

Fatou's lemma can be applied since for $\varepsilon \in (0,1)$ we have the lower bound (P_t-a.s. in $\widetilde{\omega}$)

$$\begin{aligned}\frac{\ln g_i(q(\varepsilon)) - \ln g_i(q(0))}{\varepsilon} \geq (\ln g_i(q(\varepsilon)))' = \frac{g'_i(q(\varepsilon))}{g_i(q(\varepsilon))}\langle \alpha - \widehat{\alpha}_{t,i}, X_{t+1}\rangle \overline{W}_t \\ \geq -i g'_i((1-\varepsilon)\langle \widehat{\alpha}_{t,i}, X_{t+1}\rangle \overline{W}_t)\langle \widehat{\alpha}_{t,i}, X_{t+1}\rangle \overline{W}_t \geq -\frac{i^3}{1-\varepsilon}.\end{aligned}$$

Here, in the first inequality, we used the concavity of $\ln g_i(q(\varepsilon))$. In the second inequality, we used the relation $\mathrm{P}_t(\langle \alpha, X_{t+1}\rangle \geq 0) = 1$, the bound $g_i(x) \geq 1/i$, and that $g'_i(x)$ is non-increasing for $x \geq 0$. The last inequality holds because $g'_i(x)x \leq i^2$.

Therefore, from (23) and (24), we obtain

$$0 \geq u'(0) \geq \bar{\mathrm{E}}_t(\bar{\xi}_i\langle \alpha, X_{t+1}\rangle \overline{W}_t) - \bar{\mathrm{E}}_t(\bar{\xi}_i\langle \widehat{\alpha}_{t,i}, X_{t+1}\rangle \overline{W}_t) - \langle e, \alpha - \widehat{\alpha}_{t,i}\rangle, \tag{25}$$

where

$$\bar{\xi}_i = \frac{g'_i(q(0))}{g_i(q(0))} = \frac{g'_i(\langle \widehat{\alpha}_{t,i}, X_{t+1}\rangle \overline{W}_t + |\widetilde{Y}_{t+1}|)}{g_i(\langle \widehat{\alpha}_{t,i}, X_{t+1}\rangle \overline{W}_t + |\widetilde{Y}_{t+1}|)}.$$

Now take $i \to \infty$ in (25). To the first expectation in the right-hand side of (25), we can apply Fatou's lemma, which yields

$$\liminf_{i\to\infty} \overline{\mathrm{E}}_t(\bar{\xi}_i \langle \alpha, X_{t+1}\rangle \overline{W}_t) \geq \overline{\mathrm{E}}_t\biggl(\frac{\langle \alpha, X_{t+1}\rangle \overline{W}_t}{\langle \widehat{\alpha}_t, X_{t+1}\rangle \overline{W}_t + |\widetilde{Y}_{t+1}|}\biggr). \tag{26}$$

To the second expectation, we can apply the dominated convergence theorem by observing that for all $x \geq 0$

$$0 \leq \frac{x g_i'(x)}{g_i(x)} \leq 2, \qquad \lim_{i\to\infty} \frac{x g_i'(x)}{g_i(x)} = \mathrm{I}(x > 0). \tag{27}$$

The inequality above can be obtained by using that $\arctan(x/i) \geq x/(2i)$ if $x \leq i$ and $\arctan(x/i) \geq \pi/4$ if $x \geq i$; the computation of the limit is straightforward. Then by the dominated convergence theorem

$$\lim_{i\to\infty} \overline{\mathrm{E}}_t(\bar{\xi}_i \langle \widehat{\alpha}_{t,i}, X_{t+1}\rangle \overline{W}_t) = \overline{\mathrm{E}}_t\biggl(\frac{\langle \widehat{\alpha}_t, X_{t+1}\rangle \overline{W}_t}{\langle \widehat{\alpha}_t, X_{t+1}\rangle \overline{W}_t + |\widetilde{Y}_{t+1}|}\biggr), \tag{28}$$

where $0/0 = 0$ (according to the expression for the limit in (27)). In particular, this limit is finite, which implies that the limit in (26) is also finite (otherwise inequality (25) would not hold as $i \to \infty$).

By assumption (A.2), there exists $\widetilde{\alpha} \in \overline{A}_t(\chi)$ such that $\overline{\mathrm{P}}_t(\langle \widetilde{\alpha}, X_{t+1}\rangle \overline{W}_t + |\widetilde{Y}_{t+1}| > 0) = 1$. Taking $\alpha = \widetilde{\alpha}$ in (26), we see that (20) must hold, since otherwise the limit in (26) would be infinite. Thus, (20) is proved. Then relation (21) can be obtained by passing to the limit in (25) using (26), (28).

It remains to prove (22). If $\langle e, \widehat{\alpha}_t(\chi)\rangle = 1$, it clearly follows from (21) with $\alpha = 0$. If $\langle e, \widehat{\alpha}_t(\chi)\rangle < 1$, we can consider small $\varepsilon > 0$ and take as α in (21)

$$\alpha^{(\pm\varepsilon)} := (1 \pm \varepsilon)\widehat{\alpha}_t(\chi) \in \overline{A}_t(\chi),$$

which gives (22) after simple transformations. □

Proof of claim (c) of Lemma 1. If $\overline{B}_t(\chi) \neq \{0\}$, then $\widetilde{B}_t(\chi) \neq \emptyset$ in view of (6). If $\overline{B}_t(\chi) = \{0\}$, then $\widetilde{Y}_t(\chi) = 0$, and (22) implies that $\langle e, \widehat{\alpha}_t(\chi)\rangle = 1$, so $\widetilde{B}_t(\chi) = \{0\}$. Thus, $\widetilde{B}_t(\chi)$ is non-empty, which proves the first part of the claim.

Let $f(\chi, \beta)$ denote the function being maximized in (11):

$$f(\chi, \beta) = \sum_{n=1}^{N_2} \ln \beta^n \, \overline{\mathrm{E}}_t\biggl(\frac{\widetilde{Y}_{t+1}^n}{\langle \widehat{\alpha}_t, X_{t+1}\rangle \overline{W}_t + |\widetilde{Y}_{t+1}|}\biggr)(\chi).$$

We would like to apply the measurable maximum theorem (see Proposition 6), but the function f may be discontinuous in β. To overcome this problem, take $\mathcal{G}_{t-1}$-measurable $\widetilde{\beta}(\chi) \in \overline{B}_t(\chi)$ such that $|\widetilde{\beta}(\chi)| = 1 - \widehat{\alpha}_t(\chi)$ and $\widetilde{\beta}^n(\chi) > 0$ if $\overline{\mathrm{P}}_t(\widetilde{Y}_{t+1}^n > 0)(\chi) > 0$. Then we can consider the function

$$\widetilde{f}(\chi, \beta) = \max(f(\chi, \beta), f(\chi, \widetilde{\beta}(\chi))),$$

which is a Carathéodory function and satisfies the relation

$$\operatorname*{argmax}_{\beta \in \widetilde{B}_t} f(\chi, \beta) = \operatorname*{argmax}_{\beta \in \widetilde{B}_t} \widetilde{f}(\chi, \beta).$$

Hence the measurable maximum theorem can be applied to $\widetilde{f}$, giving $\widehat{\beta}_t$ which also maximizes f.

4.3. Proofs of Theorems 1 and 2

Let us prove two more inequalities which together with (21) will be used in the proofs.

Lemma 3. *For any $t \geq 0$, $\chi \in \Theta'_t$, and $\beta \in \overline{B}_t(\chi)$ we have*

$$\overline{\mathrm{E}}_t\left(\frac{\langle \ln \widehat{\beta}_t - \ln \beta, \widetilde{Y}_{t+1}\rangle}{\langle \widehat{\alpha}_t, X_{t+1}\rangle \overline{W}_t + |\widetilde{Y}_{t+1}|}\right) \geq |\widehat{\beta}_t| - |\beta|, \tag{29}$$

$$\overline{\mathrm{E}}_t\left(\frac{|\widetilde{Y}_{t+1}| - \sum_n \beta^n \widetilde{Y}^n_{t+1}/\widehat{\beta}^n_t}{\langle \widehat{\alpha}_t, X_{t+1}\rangle \overline{W}_t + |\widetilde{Y}_{t+1}|}\right) \geq |\widehat{\beta}_t| - |\beta|, \tag{30}$$

where in (30) we let $\beta^n \widetilde{Y}^n_{t+1}(\chi)/\widehat{\beta}^n_t(\chi) = 0$ if $\widehat{\beta}^n_t(\chi) = 0$ (then $\overline{\mathrm{P}}_t(\widetilde{Y}^n_{t+1} = 0)(\chi) = 1$ as follows from (11)).

Proof. Clearly, (29) holds if $|\beta| = |\widehat{\beta}_t(\chi)|$, as follows from the definition of $\widehat{\beta}_t$. If $|\beta| \neq |\widehat{\beta}_t(\chi)|$, we have

$$\begin{aligned}\overline{\mathrm{E}}_t\left(\frac{\langle \ln \widehat{\beta}_t - \ln \beta, \widetilde{Y}_{t+1}\rangle}{\langle \widehat{\alpha}_t, X_{t+1}\rangle \overline{W}_t + |\widetilde{Y}_{t+1}|}\right) &\geq \overline{\mathrm{E}}_t\left(\frac{|\widetilde{Y}_{t+1}| \ln(|\widehat{\beta}_t|/|\beta|)}{\langle \widehat{\alpha}_t, X_{t+1}\rangle \overline{W}_t + |\widetilde{Y}_{t+1}|}\right) \\ &\geq \overline{\mathrm{E}}_t\left(\frac{|\widetilde{Y}_{t+1}|}{\langle \widehat{\alpha}_t, X_{t+1}\rangle \overline{W}_t + |\widetilde{Y}_{t+1}|}\right)\frac{|\widehat{\beta}_t| - |\beta|}{|\widehat{\beta}_t|} = |\widehat{\beta}_t| - |\beta|,\end{aligned}$$

where in the first inequality we represented $\ln \beta = \ln(\beta|\widehat{\beta}_t|/|\beta|) - \ln(|\widehat{\beta}_t|/|\beta|)$ and applied (29) to $\beta|\widehat{\beta}_t|/|\beta|$ instead of β; in the second inequality we used the estimate $\ln a \geq 1 - a^{-1}$; and in the equality applied (22), using that

$$\overline{\mathrm{E}}_t\left(\frac{|\widetilde{Y}_{t+1}|}{\langle \widehat{\alpha}_t, X_{t+1}\rangle \overline{W}_t + |\widetilde{Y}_{t+1}|}\right) = 1 - \overline{\mathrm{E}}_t\left(\frac{\langle \widehat{\alpha}_t, X_{t+1}\rangle \overline{W}_t}{\langle \widehat{\alpha}_t, X_{t+1}\rangle \overline{W}_t + |\widetilde{Y}_{t+1}|}\right) = 1 - \langle e, \widehat{\alpha}_t\rangle = |\widehat{\beta}_t|.$$

This proves (29).

To prove (30), observe that for any $\beta \in \overline{B}_t(\chi)$ the function

$$f(\chi, \varepsilon) = \overline{\mathrm{E}}_t\left(\frac{\langle \ln((1-\varepsilon)\widehat{\beta}_t + \varepsilon\beta), \widetilde{Y}_{t+1}\rangle}{\langle \widehat{\alpha}_t, X_{t+1}\rangle \overline{W}_t + |\widetilde{Y}_{t+1}|}\right)(\chi) - |(1-\varepsilon)\widehat{\beta}_t(\chi) + \varepsilon\beta|, \qquad \varepsilon \in [0,1],$$

is differentiable in ε on $[0,1)$. Also, it attains its maximum at $\varepsilon = 0$, since we have $f(0) - f(\varepsilon) \geq 0$ by applying (29) with $\beta' = (1-\varepsilon)\widehat{\beta}_t + \varepsilon\beta \in \overline{B}_t(\chi)$ instead of β. Consequently, the derivative in ε

$$f'(\chi, 0) = \overline{\mathrm{E}}_t\left(\sum_{n=1}^{N_2} \frac{(\beta^n - \widehat{\beta}^n_t)\widetilde{Y}^n_{t+1}}{\widehat{\beta}^n_t(\langle \widehat{\alpha}_t, X_{t+1}\rangle \overline{W}_t + |\widetilde{Y}_{t+1}|)}\right)(\chi) + |\widehat{\beta}_t(\chi)| - |\beta|$$

should be non-positive, which gives (30). □

Proof of Theorem 1. Assume that the strategy $\widehat{H}$ is used by agent $m = 1$. Let us fix the initial wealth and the strategies of the other agents, and pass on to the realization of the strategies $H^m_t = (\alpha^m_t, \beta^m_t)$, wealth V^m_t, and relative wealth r^m_t as functions of ω only.

Introduce the predictable sequence of random vectors $F_t \in \mathbb{R}^{N_2}_+$ with the components

$$F^n_t = \frac{\beta^{1,n}_t}{\sum_m r^m_t \beta^{m,n}_t},$$

where $0/0 = 0$. From (3), we obtain the relations

$$V^1_{t+1} = \left(\langle \alpha^1_t, X_{t+1}\rangle + \frac{\langle F_t, \widetilde{Y}_{t+1}\rangle}{W_t}\right) V^1_t, \qquad W_{t+1} = \left(\sum_{m=1}^M r^m_t \langle \alpha^m_t, X_{t+1}\rangle + \frac{|\widetilde{Y}_{t+1}|}{W_t}\right) W_t.$$

Consequently, we find $\ln r^1_{t+1} - \ln r^1_t = f_t(X_{t+1}, \widetilde{Y}_{t+1})$, where $f_t = f_t(\omega, x, y)$ is the $\mathcal{F}_t \otimes \mathcal{B}(\mathbb{R}^{N_1+N_2})$-measurable function

$$f_t(x,y) = \ln\left(\frac{\langle \alpha^1_t, x\rangle W_t + \langle F_t, y\rangle}{W_t \sum_m r^m_t \langle \alpha^m_t, x\rangle + |y|}\right)$$

(the argument ω is omitted for brevity).

We need to show that $\mathrm{E}_t\, f_t(X_{t+1}, \widetilde{Y}_{t+1}) \ge 0$. Rewrite the function $f_t(x,y)$ as

$$\begin{aligned} f_t(x,y) &= \ln\left(\frac{\langle \alpha^1_t, x\rangle W_t + |y|}{W_t \sum_m r^m_t \langle \alpha^m_t, x\rangle + |y|}\right) + \ln\left(\frac{\langle \alpha^1_t, x\rangle W_t + \langle F_t, y\rangle}{\langle \alpha^1_t, x\rangle W_t + |y|}\right) \\ &:= f^{(1)}_t(x,y) + f^{(2)}_t(x,y). \end{aligned} \tag{31}$$

For the first term, we can use the inequality $\ln x \ge 1 - x^{-1}$ and apply (21), which yields

$$\mathrm{E}_t\, f^{(1)}_t(X_{t+1}, \widetilde{Y}_{t+1}) \ge \mathrm{E}_t \frac{\langle \alpha^1_t - \sum_m r^m_t \alpha^m_t, X_{t+1}\rangle W_t}{\langle \alpha^1_t, X_{t+1}\rangle W_t + |\widetilde{Y}_{t+1}|} \ge \left\langle e,\ \alpha^1_t - \sum_{m=1}^M r^m_t \alpha^m_t \right\rangle. \tag{32}$$

For the second term in (31), we have

$$\mathrm{E}_t\, f^{(2)}_t(X_{t+1}, \widetilde{Y}_{t+1}) \ge \mathrm{E}_t \frac{\langle \ln F_t, \widetilde{Y}_{t+1}\rangle}{\langle \widehat{\alpha}_t, X_{t+1}\rangle W_t + |\widetilde{Y}_{t+1}|} \ge |\beta^1_t| - \sum_{m=1}^M r^m_t |\beta^m_t|, \tag{33}$$

where the first inequality follows from the concavity of the logarithm if we consider the $(M+1)$-dimensional vector $(\langle \alpha^1_t, X_t\rangle W_t + |\widetilde{Y}_t|)^{-1}(\langle \alpha^1_t, X_t\rangle W_t,\ \widetilde{Y}^1_t, \dots, \widetilde{Y}^M_t)$ as the coefficients of the convex combination of values $(1, F^1_t, \dots, F^M_t)$, and the second inequality follows from that $\ln F_t = \ln \beta^1_t - \ln \sum_m r^m_t \beta^m_t$ and inequality (29).

Using that $|\beta^m_t| + \langle e, \alpha^m_t\rangle = 1$, we see that $\mathrm{E}_t\, f_t(X_{t+1}, \widetilde{Y}_{t+1}) \ge 0$, hence $\mathrm{E}_t \ln r^1_{t+1} \ge \ln r^1_t$. Since $\ln r^1_t$ is a non-positive sequence, this inequality also implies the integrability of $\ln r^1_t$ (by induction, beginning with $\ln r^1_0$), so it is a submartingale. □

Proof of Theorem 2. When all agents use $\widehat{H}$, from (2) we find $\bar{p}^n_t = \widehat{\beta}^n_t \overline{W}_t$, and hence $\langle \widehat{\beta}_t, \overline{Z}_{t+1}\rangle = \widetilde{Y}_{t+1}/\overline{W}_t$. Adding (21) and (30), we obtain (16). Then (17) follows by Jensen's inequality. □

4.4. Proof of Theorem 3

We will need the following proposition which provides two inequalities of a general nature.

Proposition 9. *1) For any $a, b \in (0,1]$*

$$\ln\frac{a+b}{2} - \frac{\ln a + \ln b}{2} \ge \frac{(a-b)^2}{8}. \tag{34}$$

2) Suppose $x, y \in \mathbb{R}^N_+$ are two vectors such that $|x| \le 1$, $|y| \le 1$, and for each n it holds that if $y^n = 0$, then also $x^n = 0$. Then

$$\langle x,\, \ln x - \ln y\rangle \ge \frac{\|x-y\|^2}{4} + |x| - |y|. \tag{35}$$

Proof. 1) Assume $a \le b$ (the case $b \le a$ is considered in the same way). The inequality clearly holds if $a = b$. Let $f(a)$ be the difference of its left-hand side and right-hand side, with b fixed. It is enough to show that $f'(a) \le 0$ for $a \in (0, b]$. After differentiation, this becomes equivalent to $a(a+b) \le 2$. The latter inequality is clearly true, provided that $a, b \in (0, 1]$.

2) Using that $\ln x \le 2(\sqrt{x} - 1)$ for any $x > 0$, and $(\sqrt{x} - \sqrt{y})^2 \ge (x-y)^2/4$ for any $x, y \in [0, 1]$, we obtain

$$\begin{aligned}\langle x, \ln x - \ln y\rangle &= -\sum_{n\,:\,x^n \neq 0} x^n \ln(y^n/x^n) \ge 2\sum_{n=1}^{N}(x^n - \sqrt{x^n y^n}) \\ &= \sum_{n=1}^{N}(\sqrt{x^n} - \sqrt{y^n})^2 + |x| - |y| \ge \frac{\|x-y\|^2}{4} + |x| - |y|.\end{aligned}$$

□

Proof of Theorem 3. We will use the same notation for realizations of strategies as in the proof of Theorem 1. It was shown that $\ln r_t^1$ is a submartingale. Let c_t be its compensator, i.e. the predictable non-decreasing sequence such that $\ln r_t^1 - c_t$ is a martingale; in the explicit form

$$c_t = \sum_{s \le t}(\mathrm{E}_{s-1} \ln r_s^1 - \ln r_{s-1}^1).$$

As was shown in the proof of Theorem 1,

$$c_{t+1} - c_t = \mathrm{E}_t\, f_t(X_{t+1}, Y_{t+1}) = \mathrm{E}_t(f_t^{(1)}(X_{t+1}, Y_{t+1}) + f_t^{(2)}(X_{t+1}, Y_{t+1}))$$

with $f^{(1)}, f^{(2)}$ defined in (31). Since $\ln r_t^1$ is non-positive and converges, we have $c_\infty < \infty$ with probability 1. Let us consider again inequalities (32)–(33) and strengthen them using Proposition 9. Fix $t \ge 1$ and let

$$a = \frac{\langle \alpha_t^1, X_{t+1}\rangle + |Y_{t+1}|/W_t}{Q_{t+1}}, \qquad b = \frac{\langle \widehat{\alpha}_t^1, X_{t+1}\rangle + |Y_{t+1}|/W_t}{Q_{t+1}}.$$

Note that $a, b \in (0, 1]$. Then

$$\begin{aligned}\mathrm{E}_t\, f_t^{(1)}(X_{t+1}, Y_{t+1}) &= 2\,\mathrm{E}_t\biggl(\ln a - \frac{\ln a + \ln b}{2}\biggr) \\ &\ge 2\,\mathrm{E}_t\biggl(\ln\frac{a+b}{2} - \frac{\ln a + \ln b}{2}\biggr) + \langle e, \widehat{\alpha}_t - \alpha_t^R\rangle \qquad (36) \\ &\ge \biggl(\frac{\langle \alpha^1 - \alpha^R, X_{t+1}\rangle}{2Q_{t+1}}\biggr)^2 + \langle e, \alpha_t^1 - \alpha_t^R\rangle.\end{aligned}$$

To obtain the first inequality we subtracted and added $2\ln((a+b)/2)$ and used the estimate

$$\mathrm{E}_t\biggl(\ln a - \ln\frac{a+b}{2}\biggr) \ge \frac{1}{2}\,\mathrm{E}_t\,\frac{\langle \alpha_t^1 - \alpha_t^R, X_{t+1}\rangle W_t}{\langle \alpha_t^1, X_{t+1}\rangle W_t + |Y_{t+1}|} \ge \frac{1}{2}\langle e, \alpha_t^1 - \alpha_t^R\rangle,$$

which is obtained similarly to (32). In the second inequality of (36) we applied (34).

For the function $f^{(2)}$, using that there are no portfolio constraints on the endogenous assets, so $\widehat{\beta}_t$ is given by (13), we find

$$\begin{aligned} \mathrm{E}_t\, f_t^{(2)}(X_{t+1}, Y_{t+1}) &\geq \mathrm{E}_t\, \frac{\langle \ln F_t, Y_{t+1}\rangle}{\langle \alpha_t^1, X_{t+1}\rangle W_t + |Y_{t+1}|} = \langle \ln F_t, \beta_t^1 \rangle = \langle \beta_t^1, \ln \beta_t^1 - \ln \beta_t^R \rangle \\ &\geq \frac{\|\beta_t^1 - \beta_t^R\|^2}{4} + |\beta_t^1| - |\beta_t^R|, \end{aligned} \tag{37}$$

where the first inequality is obtained similarly to (33), and in the second one we applied (35). Consequently, from (36), (37), we obtain

$$c_{t+1} - c_t \geq \left(\frac{\langle \alpha_t^1 - \alpha_t^R, X_{t+1}\rangle}{2Q_{t+1}} \right)^2 + \frac{\|\beta_t^1 - \beta_t^R\|^2}{4}.$$

From here, using that $c_\infty < \infty$, we get (18). Moreover, $\alpha_t^1 - \alpha_t^R = (1 - r_t^1)(\alpha_t^1 - \alpha_t^r)$ and $\beta_t^1 - \beta_t^R = (1 - r_t^1)(\beta_t^1 - \beta_t^r)$, so on the set (19) we necessarily have $\lim_{t\to\infty} r_t^1 = 1$. $\square$

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