INSTITUTE AND FACULTY OF ACTUARIES

EXAMINERS' REPORT

September 2016

Subject CT1 – Financial Mathematics Core Technical

Introduction

The Examiners' Report is written by the Principal Examiner with the aim of helping candidates, both those who are sitting the examination for the first time and using past papers as a revision aid and also those who have previously failed the subject.

The Examiners are charged by Council with examining the published syllabus. The Examiners have access to the Core Reading, which is designed to interpret the syllabus, and will generally base questions around it but are not required to examine the content of Core Reading specifically or exclusively.

For numerical questions the Examiners' preferred approach to the solution is reproduced in this report; other valid approaches are given appropriate credit. For essay-style questions, particularly the open-ended questions in the later subjects, the report may contain more points than the Examiners will expect from a solution that scores full marks.

The report is written based on the legislative and regulatory context pertaining to the date that the examination was set. Candidates should take into account the possibility that circumstances may have changed if using these reports for revision.

Luke Hatter
Chair of the Board of Examiners
December 2016

A. General comments on the *aims of this subject and how it is marked*

1. CT1 provides a grounding in financial mathematics and its simple applications. It introduces compound interest, the time value of money and discounted cashflow techniques which are fundamental building blocks for most actuarial work.

2. Please note that different answers may be obtained to those shown in these solutions depending on whether figures obtained from tables or from calculators are used in the calculations but candidates are not penalised for this. However, candidates may lose marks where excessive rounding has been used or where insufficient working is shown.

B. General comments on *student performance in this diet of the examination*

1. The comments that follow the questions concentrate on areas where candidates could have improved their performance. Where no comment is made, the question was generally answered well by most candidates. The examiners look most closely at the performance of the candidates close to the pass mark and the comments therefore often relate to those candidates.

2. Performance was very slightly weaker when compared with most recent diets. As in previous diets, the non-numerical questions were often answered poorly by marginal candidates.

C. Pass Mark

The Pass Mark for this exam was 60.

Solutions

Q1 (i) $e^{\delta/4} = 1.0125 \Rightarrow \delta = 4\times \ln 1.0125 = 0.0496901 = 4.969\%$ [1]

(ii) $1+i = 1.0125^4 \Rightarrow i = 0.0509453 = 5.095\%$ [1]

(iii) From (i) (though could be done in other ways)

$$\left(1-\frac{d^{(12)}}{12}\right) = e^{-\delta/12} = e^{-0.0041408} = 0.9958677 \Rightarrow d^{(12)} = 0.049587 = 4.959\%$$

[2]
[Total 4]

> This was generally answered well although many candidates ignored the specific rounding instructions or rounded incorrectly.

Q2 Various approaches (e.g. effective interest period can be changed etc.).
Work in quarters. Interest rate per quarter = 0.5%. Rate of payment per quarter = 25.
Number of quarters = 48.

$\text{PV} = 25\ddot{a}^{(3)}_{\overline{48}|} = 25\dfrac{i}{d^{(3)}}a_{\overline{48}|}$ [2]

$d^{(3)} = 3\left(1-1.005^{-1/3}\right) = 0.0049834,\ a_{\overline{48}|} = 42.5803$ [1]

Therefore, PV = 25 × (0.005/0.0049834) × 42.5803 = €1,068.05 [1]
[Total 4]

> Generally well answered

Q3 A loan repayable by a series of payments at fixed times set in advance.

Typically issued by banks and building societies

Typically long-term …
…e,g. used to fund house purchase
…and secured against the property

Each payment contains an element to pay interest on the loan with the remainder being used to repay capital

In its simplest form, the interest rate will be fixed …
….and the payments will be of fixed equal amounts.

The interest payment portion of the repayments will fall over time…
… and the capital payments will rise over time.

Risk that borrower defaults on loan

Complications might be added such as (i) allowing the loan to be repaid early or (ii) allowing the interest rate to vary

[½ mark for each point]
[Max 3]
[Total 3]

> Despite being a bookwork question, this was the worst answered question on the paper. It was not necessary to make all the above points for full marks.

Q4 (i) $X = (112 + 23) \times 1.1 = £148.5\text{m}$ [1]

(ii) TWRR is found from

$$(1+i)^2 = \frac{148.5}{135}\frac{160}{148.5+43} = 0.91906 \Rightarrow i = -0.04132 = -4.132\%$$ [3]

[Total 4]

> A significant number of marginal candidates failed to answer part (i) correctly.

Q5 Let original price of zero coupon bond = P

$P = 100v^{80}$ at 2.5%

$\Rightarrow P = 100 \times 0.13870 = 13.87$ [2]

Equation of value for the purchaser:

$13.87e^{\delta t} = 80$ [1]

$$t = \frac{\ln(80/13.67)}{\delta} \Rightarrow t = \frac{\ln 5.7678}{0.048} = 36.506 \text{ years}$$ [1]

0.506 years is 185 days. There are 181 days to the end of June. Default is therefore on 4 July 2011. [1]

[Total 5]

This question was answered poorly with many candidates not able to formulate separate equations of value for:

- the original terms to determine the issue price.
- the revised terms to determine the date of default.

Q6 (i) Simple rate of return is (100 – 96)/96 = 0.041666 [1]

Expressed as an annual rate, this is: 0.041666 × (365/182) = 8.3562% [1]

(ii) Let the time in years = t

$(97.5 - 96)/96 = 0.035t$ [1]

$t = (97.5 - 96)/(0.035\times 96) = 0.44643$ years = 163 days [1]

(iii) Equation of value for the second investor:

$97.5(1+i)^{(182-163)/365} = 100$ [1]

$$(1+i)^{19/365} = \frac{100}{97.5} \Rightarrow i = \left(\frac{100}{97.5}\right)^{365/19} - 1 = 62.640\%$$ [1]

[Total 6]

Parts (i) and (ii) were very well answered. In part (iii), some candidates continued to assume a simple rate of return was required. Alternative answers to part (iii) based on different rounding of the answer in part (ii) were given full credit.

Q7 (i) Present value of dividends, I, is:

$$0.1\left(v^{1/4} + v^{1/2} + v^{3/4}\right)$$

Calculated at i'% when $1+i' = 1.04^2 = 1.0816$

So $I = 0.1\,(1.0816^{-0.25} + 1.0816^{-0.5} + 1.0816^{-0.75}) = 0.288499$ [2]

Hence, forward price, K, is:

$K = (1.1 - 0.288499) \times 1.0816^{0.75} = 0.86068 = 86.068p$ [2]

(ii) The price of the forward can be determined from the price of the share (for which it is a close substitute). The forward is like the share but with delayed settlement and without dividends. [Could also be said that the price of the share already takes into account expectations.] [2]

[Total 6]

Part (i) was answered well although some candidates ignored the fact that the interest rate given was a convertible half-yearly rate. Part (ii) was answered less well with the arguments of many marginal candidates being very unclear.

Q8 (i) $96 = 4a_{\overline{3}|} + 100v^3$ [1]

Try 6% RHS = 94.654 [½]

Try 5% RHS = 97.2768 [½]

Interpolation gives

$i \approx 0.05 + 0.01 \times \dfrac{97.2768 - 96}{97.2768 - 94.6540} = 0.0549 \approx 5.5\%$ [1]

(ii) Let i_n = spot rate for term n

Then $96 = 104v$ at $i_1 \Rightarrow 1 + i_1 = \dfrac{104}{96} \Rightarrow i_1 = 8.333\%$ [1]

$96 = 4v_{i_1} + 104v_{i_2}^2 \Rightarrow (1+i_2)^2 = \dfrac{104}{96 - 4v_{i_1}} = \dfrac{104}{96 - 3.69231} \Rightarrow i_2 = 6.145\%$ [2]

(iii) Let the forward rate be $f_{1,1}$

$(1+i_2)^2 = (1+i_1)(1+f_{1,1})$ [1]

$\Rightarrow 1.06145^2 = 1.08333 \times (1+f_{1,1}) \Rightarrow f_{1,1} = 0.04000 = 4\%$ [1]

(iv) The three year gross redemption yield is a complex form of weighted average of the three spot rates. [1]

The one-year spot rate is over 8%, the two-year rate is over 6% and the gross redemption yield is 5.5%. Therefore, the three-year rate must be less than 5.5% if the weighted average is 5.5%. [1]

[Total 10]

Part (i), (ii) and (iii) were generally answered well, although in part (iii) some candidates were not clear as to the forward rate required by the question. Part (iv) was very poorly answered. For this part no marks were available for calculation without explanation.

Q9 (i) Let S_n = Accumulated value at time n of £1 invested at time 0

$$S_n = (1+i_1)(1+i_2)....(1+i_n)$$

$$\Rightarrow E[S_n] = E[(1+i_1)(1+i_2)....(1+i_n)]$$

$$= E(1+i_1).E(1+i_2).... .E(1+i_n) \text{ by independence}$$

and $E(1+i_t) = 1+E(i_t) = 1+j$

Hence $E(S_n)=(1+j)^n$

Now

$$\text{Var}[S_n] = E[S_n^2] - (E[S_n])^2$$

$$E[S_n^2] = E\left[(1+i_1)^2(1+i_2)^2....(1+i_n)^2\right]$$

$$= E\left[(1+i_1)^2\right].E\left[(1+i_2)^2\right].... .E\left[(1+i_n)^2\right]$$ [1]

by independence [½]

and

$$E\left[(1+i_t)^2\right] = E\left[\left(1+2i_t+i_t^2\right)\right] = 1+2E(i_t)+E\left(i_t^2\right)$$

and $\text{Var}[i_t] = s^2 = E\left(i_t^2\right) - [E(i_t)]^2 = E\left(i_t^2\right) - j^2$

$$\Rightarrow E\left(i_t^2\right) = s^2 + j^2$$ [1]

Hence

$$E\left[S_n^2\right] = \left(1+2j+j^2+s^2\right)^n$$

And $\text{Var}\left[S_n\right] = \left(1+2j+j^2+s^2\right)^n - (1+j)^{2n}$ [1]

Hence mean accumulation $= 8{,}000{,}000E\left(S_5\right)$ [½]

$$= 8{,}000{,}000(1.055)^5 = £10{,}455{,}680$$ [1]

Standard deviation of accumulation $= 8{,}000{,}000\sqrt{\text{Var}\left(S_5\right)}$ [½]

$$= 8{,}000{,}000\sqrt{\left(1+2\times 0.055+0.055^2+0.04^2\right)^5-(1.055)^{10}}$$

$$= 8{,}000{,}000\sqrt{1.7204573-1.7081445} = 8{,}000{,}000\times\sqrt{0.01231284}$$

$$= £887{,}706$$ [2]

Alternative Solution

$(1+i_t) \sim \text{lognormal}\ (\mu,\sigma^2)$

$\ln(1+i_t) \sim N(\mu,\sigma^2)$

$\ln(1+i_t)^5 = \ln(1+i_t)+\ln(1+i_t)+\cdots+\ln(1+i_t) \sim N(5\mu, 5\sigma^2)$

Given assumption that they are independent and identically distributed
$\therefore (1+i_t)^5 \sim \text{ lognormal}\,(5\mu,5\sigma^2)$ [2]

$$E(1+i_t) = \exp\left(\mu+\frac{\sigma^2}{2}\right) = 1.055$$ [1]

$$\text{Var}(1+i_t) = \exp(2\mu+\sigma^2)\left[\exp(\sigma^2)-1\right] = 0.04^2$$ [1]

$$\frac{0.04^2}{1.055^2} = \left[\exp\left(\sigma^2\right)-1\right] \Rightarrow \sigma^2 = 0.0014365$$ [1]

$$\exp\left(\mu+\frac{0.0014360}{2}\right) = 1.055 \Rightarrow$$

$$\mu = \ln 1.055 - \frac{0.0014365}{2} = 0.052823$$ [1]

$5\mu = 0.264113$

$5\sigma^2 = 0.0071825$.

Let S_5 be the accumulation of one unit after five years:

$$E(S_5) = \exp\left(5\times\mu + \frac{5\sigma^2}{2}\right) = \exp\left(0.264113 + \frac{0.0071825}{2}\right)$$

$=1.30696$ [½]

$$\text{Var}(S_5) = \exp(2\times 5\mu + 5\sigma^2)\left[\exp\left(5\sigma^2\right) - 1\right]$$

$= \exp(2 \times 0.264113 + 0.0071825)\ .\ (\exp 0.0071825 - 1)$

$= \exp 0.53541\ (\exp 0.0071825 - 1)$

$= 0.01231284$ [½]

Mean value of the accumulation of premiums is

$8{,}000{,}000 \times 1.30696 =$ £10,455,680. [1]

Standard deviation of the accumulated value of the premiums is

$8{,}000{,}000 \times 0.012312849^{0.5} =$ £887,706 [1]

(ii) If the company invested in fixed-interest securities, it would obtain a guaranteed accumulation of £8,000,000 * $(1.04)^5$ = £9,733,223. In one sense, there is a 100% probability that a loss will be made and therefore the policy is unwise. [1]

The "risky" investment strategy leads to an expected profit. [1]

On the other hand, the standard deviation of the accumulation from the risky investment strategy will be higher than investing in the fixed-interest securities. Whilst there is a chance of an even greater profit from this strategy, there is also a chance of a more considerable loss than from investing in fixed-interest securities. [1]

[Total 12]

> Many candidates either ignored the requirement in part (i) to derive the necessary formulae or had difficulty in performing the derivation often trying to combine the two methods above without success. Part (ii) was better answered than the other explanation questions on the paper.

Q10 (i) The payback period simply tells an investor when the total cash inflows from the investment have exceeded the total cash outflows. This tells the investor nothing about the overall profitability of the project. [2]

(ii) The present value of outgoing cash flows at a rate of return of 6% per annum effective is as follows (in £m):

$$\ddot{a}_{\overline{3}|} + 0.05\left(a_{\overline{13}|} - a_{\overline{3}|}\right)$$

$= 1.06\times2.6730 + 0.05(8.8527 - 2.6730) = 3.14238$ [2]

The present value of the incoming cash flows is as follows (in £m):

$$= 0.495v^3\ddot{a}^{(12)}_{\overline{1}|}\left(1+1.01v+1.01^2v^2+...+1.01^9v^9\right)$$

$$= 0.495v^3\frac{d}{d^{(12)}}\left(1+1.01v+1.01^2v^2+...+1.01^9v^9\right)$$

$= 0.495\times0.83962 \times 0.973784 \times (1 - 1.01^{10}/1.06^{10}) / (1 - 1.01/1.06)$

$= 0.404716 \times 8.12352 = 3.2877$ [3½]

NPV of cash flows at 6% = 3.2877 – 3.1424 = £0.1453m = £145,300 [1]

The project has a positive NPV at 6% and therefore an IRR higher than 6% and the first criteria is met. [½]

By the end of the 10th year, the total outgoing cash flows will have been: £3,000,000 plus 7 × £50,000 or £3,350,000. [1]

Total incoming cash flows are:

$495{,}000\times (1 + 1.01 + 1.01^2 +...+ 1.01^6)$ (i.e. rate of payment of £495,000 rising by 1% per year for seven years). [1]

Geometric progression with common ratio 1.01 and seven terms

$= 495{,}000(1 - 1.01^7)/(1 - 1.01) = £3{,}570{,}700$ [1]

This is greater than total outgoing cash flows and therefore second criterion is met. [½]

There is clearly a positive cash flow in the fifth year as the incoming cash flows will be greater than £495,000 and the outgoing cash flows will be £50,000. [1]

Therefore final criterion is met. [½]

[Total 14]

> This was the worst answered of the longer questions. The examiners strongly recommend that candidates take a systematic approach to the question and e.g. derive the PVs of the outgo and income separately. Marginal candidates would have benefited from showing their intermediate working in greater depth and/or with greater clarity, explaining all steps.
>
> Candidates who assumed that the repayments continued for 11 years, rather than 10, were not penalised.

Q11 (i) Duration = $\sum_t tC_t v^t \Big/ \sum_t C_t v^t$

$$= \frac{\left(\sum_{t=1}^{3} 4\times 0.04\times tv^t\right)+\left(\sum_{t=1}^{10} 5\times 0.04\times tv^t\right)+4\times 1\times 3v^3+5\times 1\times 10v^{10}}{\left(\sum_{t=1}^{3} 4\times 0.04\times v^t\right)+\left(\sum_{t=1}^{10} 5\times 0.04\times v^t\right)+4\times 1\times v^3+5\times 1\times v^{10}}$$

$$= \frac{0.16(Ia)_{\overline{3|}}+0.20(Ia)_{\overline{10|}}+12v^3+50v^{10}}{0.16a_{\overline{3|}}+0.20a_{\overline{10|}}+4v^3+5v^{10}}$$ [4]

Therefore, duration

$$= \frac{0.16\times 5.2422+0.20\times 36.9624+12\times 0.83962+50\times 0.55839}{0.16\times 2.6730+0.20\times 7.3601+4\times 0.83962+5\times 0.55839} = \frac{46.2264}{8.05015}$$

= 5.742 years [3]

(ii) Present value of new portfolio per unit nominal $=\dfrac{1}{i}$ [1]

Volatility of new portfolio per unit nominal $=-\dfrac{\frac{d}{di}\left(\frac{1}{i}\right)}{\frac{1}{i}}=\dfrac{1}{i}$ [1]

Duration of new portfolio, applying equation above is
$\text{volatility}\times(1+i)=\dfrac{1+i}{i}=\dfrac{1.06}{0.06}$

= 17.666 years [2]

(iii) Present value of existing bonds:

$=$ £8.05013bn [½]

Let the nominal amount of new bonds issued = X

Present value of new bonds $=0.05Xa_{\overline{\infty|}}=\dfrac{0.05X}{0.06}$

$$\Rightarrow\frac{0.05X}{0.06}=0.8\times 8.05013\Rightarrow X=£7.728\text{bn}$$

[1½]
[Total 13]

In part (i), many candidates incorrectly calculated DMTs separately for the two bonds which simplified the question and did not produce the DMT of the whole portfolio. Part (ii) was poorly answered with many candidates not recognising the relationship between DMT and volatility. It is also important in such questions to state the time units in the final answer. Part (iii) did seem to act as a differentiator between candidates, with the strongest candidates able to proceed clearly through the question.

Q12 (i) $50=100v(20)$

where $v(20)=\exp\left(-\int_0^{10}0.03dt\right)\exp\left(-\int_{10}^{20}at\,dt\right)$ [3]

$$=\exp\left[-0.03t\right]_0^{10}\exp\left[-\frac{at^2}{2}\right]_{10}^{20}$$

$$= e^{-0.3}e^{-150a} = e^{-0.3-150a} = 0.5 \qquad [1]$$

$$\Rightarrow a = \frac{\ln 2 - 0.3}{150} = 0.0026210 \qquad [1]$$

(ii) $40 = 100v(28)$

where $v(28) = v(20)\exp\left(-\int_{20}^{28} bt\,dt\right)$ [1]

$$= -0.5\exp\left[\frac{-bt^2}{2}\right]_{20}^{28}$$

$$= 0.5e^{-192b} = 0.4 \qquad [2]$$

$$\Rightarrow b = \frac{-\ln 0.8}{192} = 0.0011622 \qquad [1]$$

(iii) Equivalent annual effective rate of discount can be found from:

$$100(1-d)^{28} = 40 \Rightarrow (1-d)^{28} = 0.4 \qquad [1]$$

$$\Rightarrow d = 0.032195 = 3.220\% \qquad [1]$$

(iv) (a) We require: $\int_3^7 \rho(t)v(t)\,dt = \int_3^7 e^{-0.04t}e^{-0.03t}dt = \int_3^7 e^{-0.07t}dt$ [1½]

$$= \left[\frac{e^{-0.07t}}{-0.07}\right]_3^7 \qquad [1]$$

$$= -8.751806 + 11.579775 = 2.827969 \qquad [1½]$$

(b) The present value of the payment stream $2.827969 = X\left(\bar{a}_{\overline{7}|} - \bar{a}_{\overline{3}|}\right)$ where X is the continuous payment stream using $\delta = 0.03$. [2]

$$X = \frac{2.827969}{\bar{a}_{\overline{7}|} - \bar{a}_{\overline{3}|}} = \frac{2.827969}{\left(\dfrac{1-e^{-0.03\times 7}}{0.03}\right) - \left(\dfrac{1-e^{-0.03\times 3}}{0.03}\right)}$$

$$= \frac{2.827969}{6.31386 - 2.86896} = 0.82092 \qquad [2]$$

[Total 19]

This was well answered apart from part (iv)(b). It was pleasing to see many candidates using good exam technique to leave enough time for this question which proved to be more straightforward than the other longer questions.

END OF EXAMINERS' REPORT