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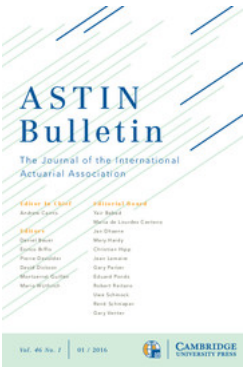

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Antonio Heras, Beatriz Balbás and José Luis Vilar

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CONDITIONAL TAIL EXPECTATION AND PREMIUM CALCULATION*

BY

ANTONIO HERAS, BEATRIZ BALBÁS AND JOSÉ LUIS VILAR

ABSTRACT

In this paper we calculate premiums which are based on the minimization of the Expected Tail Loss or Conditional Tail Expectation (CTE) of absolute loss functions. The methodology generalizes well known premium calculation procedures and gives sensible results in practical applications. The choice of the absolute loss becomes advisable in this context since its CTE is easy to calculate and to understand in intuitive terms. The methodology also can be applied to the calculation of the VaR and CTE of the loss associated with a given premium.

KEYWORDS

Premium Principles, Loss Functions, Risk Measures, Value at Risk, Conditional Tail Expectation.

1. INTRODUCTION

In insurance terminology, a ***premium*** is the price of the insurance coverage, that is, the payment that policyholders make in order to obtain protection from their risks. A ***premium principle*** is a rule for assigning premiums to the insurance risks. There are many different premium principles in the literature. The study of the properties and justification of the different premium principles is a classical issue in actuarial science (see, among many others, Goovaerts et al. (1984), Deprez & Gerber (1985), Wang et al. (1997), Kaas et al. (2001), Young (2004), Goovaerts et al. (2010)).

Among the large number of methodologies proposed in the literature for justifying different premium principles, we want to mention the application to this field of the so-called **risk measures**. In general, a risk measure is a functional assigning a number to each risk, defined in accordance with the intuitive

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principle that the more dangerous the risk, the higher the risk measure must be.

General risk measures are becoming more and more important in finance and insurance, and many classical financial and actuarial problems have been revisited taking into consideration modern risk measures beyond the variance. Focusing on the actuarial field, we can find applications of risk measures to several important issues, such as the problem of optimal reinsurance (see, for instance, Cai & Tan (2007), Cai et al. (2008), Balbás et al. (2009), Bernard & Tian (2009)), the calculation of capital requirements (Artzner (1999), Wirch & Hardy (1999), Panjer (2001), Denault (2001), Laeven & Goovaerts (2004), Dhaene et al. (2008), Furman & Zitikis (2008b)) and, of course, the definition of new premium principles (Wang (1995, 1996, 2000, 2002), Landsman & Sherris (2001), Goovaerts et al. (2003), Tsanakas & Desli (2003), Goovaerts et al. (2004b), Furman & Landsman (2006), Furman & Zitikis (2008a), Goovaerts & Laeven (2008)).

There are many different types of risk measures, such as coherent measures (Artzner et al. (1999)), deviation measures and expectation bounded risk measures (Rockafellar et al. (2006)), convex measures (Föllmer & Schied (2002)) and consistent measures (Goovaerts et al. (2004a)). Perhaps the most influential approach is the first one. A ***coherent measure*** ρ verifies the following properties:

Subadditivity: $\rho(Y+Z) \leq \rho(Y) + \rho(Z)$

Positive homogeneity: $\rho(cY) = c\rho(Y), \forall c > 0$

Translation invariance: $\rho(Y+c) = \rho(Y) + c, \forall c \in \mathbb{R}$

Monotonicity: $Y(\omega) \leq Z(\omega), \forall \omega \in \Omega \Rightarrow \rho(Y) \leq \rho(Z)$

(where Y, Z are random variables representing losses and Ω is their sample space).

Many important risk measures often used in actuarial problems are coherent risk measures, such as the Conditional Tail Expectation CTE (also called Conditional Value at Risk: see Rockafellar & Uryasev (2002), Rockafellar et al. (2006)), the expectation under distorted probabilities (Goovaerts & Laeven (2008), Wang 1995, 1996, 2000) (when the distortion function is concave: see Wirch & Hardy (1999)) and the spectral risk measures (Acerbi 2002) (when the spectrum verifies certain reasonable conditions). The famous Value at Risk (VaR) is not coherent, because it fails to fit the subadditivity property (Artzner et al. (1999)).

Let X be an insurance risk, that is, a non-negative random variable representing the total claim amount of an insurance policy in a given period of time. When a risk measure ρ is used for premium calculation, it is often assumed that the premium coincides with the risk measure of X:

$$P_X = \rho(X) \tag{1}$$

(Actually, the same assumption is usually made for the calculation of capital requirements). The translation invariance property is sometimes used for justifying

the assumption given in (1), because it is easy to prove that the risk measure is zero after adding the premium defined in (1):

$$\rho(X - P_X) = \rho(X) - P_X = \rho(X) - \rho(X) = 0$$

(Remember that our random variables represent losses, so that adding a positive amount of money is represented as subtracting that amount from the loss).

According to this model, if the insurance company charges an insufficient premium P strictly lower than P_X as defined in (1), then there still exists a remaining risk measured by $\rho(X-P) > 0$. This is intuitively correct, since insufficient premiums are associated with the possibility of high monetary losses for the company. On the other hand, the model predicts that if the company charges an excessive premium strictly higher than P_X, there should not be any risk at all, since $\rho(X-P) < 0$. This is not so clear in intuitive terms. After all, in this case it is the policyholder who incurs monetary losses. Notice also that excessive premiums can lead many policyholders to leave the company. To sum up, we think that cases of both charging insufficient (too low) and excessive (too large) premiums are risky practices, and their risk measure should be positive. We think that the premium calculation process based on risk measures should take into consideration the risk of both underestimation and overestimation losses (actually, a similar argument applies for the calculation of capital requirements, because its underestimation involves solvency problems and its overestimation is also associated with monetary losses due to the opportunity cost of the capital).

A possible way for solving this problem could be based on a methodology that should take into account both risk measures and loss functions. In order to apply this methodology, it is necessary to previously define a ***loss function*** for measuring the importance of the rating errors. These rating errors appear when the premium P does not coincide with the value x of the random variable X. We will denote by $L(P,x)$ the loss associated with both the premium P and the value $X = x$. This loss function assigns a numerical value to each possible rating error, representing the loss faced by the insurance company in that case. Once this loss function has been defined, as a second step, we could apply the well known risk measure methodology in order to calculate the global risk of the insurance company. Such a procedure, in which the insurance premium is derived by solving a minimization problem, fits the two level procedure for the evaluation of risk advocated by Goovaerts et al. (2010).

In this article we follow the previous suggestion and we propose that the premium should be chosen as the amount minimising the risk of the loss by the insurance company. We also offer a methodology for the calculation of these optimal premiums in some cases. In Section 2 we define the problem in precise mathematical terms and we show that it generalizes other well known premium calculation methodologies such as the Bayesian and the usual risk measures methodology. In Section 3 we explain a procedure for calculating the optimal premium when the risk measure is the important and well known

Conditional Tail Expectation, and in Section 4 we obtain the mathematical expression of the optimal premium when the loss function is the simple and easy to understand absolute loss. In this section we also prove that the premiums obtained have reasonable properties, and that the methodology can be applied to the calculation of the VaR and CTE of the loss for a given premium. Finally, Section 5 applies this methodology to a practical example.

2. The general formulation of the problem

As we mentioned in the Introduction, we propose calculating the optimal premium as the amount P_X that minimizes the risk measure of the loss function. In order to calculate that optimal premium, it is then necessary to previously select both a loss function L and a risk measure ρ. Once they have been chosen, the next step is the minimization of $\rho(L(P,X))$. That is, the optimal premium is calculated as the amount P_X minimizing

$$\rho\big[L(P,X)\big] \tag{2}$$

It is easy to check that this methodology generalizes both Bayesian and risk measure methodologies.

On the one hand, when the risk measure ρ is the mathematical expectation (which, in fact, is a coherent risk measure), (2) reduces to the following expression (3):

$$E\big[L(P,X)\big] \tag{3}$$

In this case, the optimal premium is the well known Bayesian premium, defined as the amount minimizing the expected loss (3). The Bayesian methodology for calculating premiums is a classical issue in actuarial science. For example, it is worth mentioning Heilmann (1989), who showed that some of the most famous premium principles defined in the literature (variance premium principle, exponential, Esscher, etc.) can be obtained by minimizing the expected loss associated with some particular loss functions.

On the other hand, the optimal premium obtained by minimizing (2) reduces to (1) when we select the loss function $L(P,x) = x - P$ (as long as we only consider non-negative values of the risk measure: notice that this loss function only takes positive values when the premium P underestimates the value $X = x$; in the case of overestimation, the loss is transformed into a gain!). In the following sections we will work with a more general loss function considering both underestimations and overestimations as losses.

As we said in the Introduction, the goal of this article is to propose a methodology for minimizing the expression (2) and to check that the solutions obtained have reasonable properties. In fact, we will study an important particular case of (2), where the loss function is the absolute loss function and the

risk measure is the so-called Conditional Tail Expectation. The proposed methodology, however, can also be applied to more general cases.

In the rest of the article we will assume that the total claim amount X is a continuous random variable taking non-negative values, and we will denote as $F(x)$ and $f(x)$ its distribution function and its density function, respectively.

3. The calculation of the premium minimizing the conditional tail expectation of the loss

As we said before, perhaps the most famous risk measures are the **coherent measures of risk** defined in Artzner et al. (1999). The most common choice among these coherent measures of risk is, undoubtedly, the **Conditional Tail Expectation** (**CTE**), also known as **Conditional Value at Risk (CVaR**), **Tail Value at Risk** (**TVaR**), **Average Value at Risk** (**AVaR**), **Expected Tail Loss**, etc. The definitions of these concepts proposed in the literature coincide for continuous random variables, although for some discrete random variables there could be differences among them and in some cases the risk measure may not be coherent (Hürlimann (2003)). Nevertheless, Rockafellar & Uryasev (2002) and Rockafellar et al. (2006) have proposed general definitions according to which the risk measure is always coherent and expectation bounded (remember, however, that we are dealing with continuous random variables in this article). CTE is also a member of other larger families of risk measures mentioned before, since it is a spectral measure in the sense of Acerbi (2002), it can be obtained as a distorted expectation in the sense of Wang (1995, 1996) (see Wirch & Hardy (1999)) and it can also be considered as a weighted premium calculation principle in the sense of Furman & Zitikis (2008a). Moreover, it shows suitable theoretical properties (such as, for instance, stochastic dominance consistency (Ogryczac & Ruszczynski (2002)), provides information about the degree of risk in monetary terms and is well-known and understood by many practitioners. For all these reasons, we have chosen CTE as our risk measure ρ: our proposal is to define the risk-adjusted optimal premium as the amount P^* minimizing the Conditional Tail Expectation of the loss

$$CTE(L(P,X)) \tag{4}$$

In order to rigorously define CTE, it is necessary to define previously another well known risk measure, the so-called **Value at Risk** (**VaR**). Given a probability level $\beta \in (0,1)$, the associated Value at Risk is the lowest amount $\alpha \geq 0$ such that, with probability β, the loss will not exceed α, and the associated Conditional Tail Expectation is the conditional expectation of the losses above that amount α.

In our problem, given a probability level $\beta \in (0,1)$, a premium P and a loss function $L(P,x)$, VaR is defined as

$$VaR_\beta(P) = MIN\left\{\alpha \in \mathbb{R}^+ / \Pr\left[L(P,X) \leq \alpha\right] \geq \beta\right\}$$

And CTE is defined as

$$CTE_\beta(P) = \frac{1}{1-\beta} \int\limits_{L(P,x) \geq VaR_\beta(P)} L(P,x) f(x) dx$$

(see Rockafellar & Uryasev (2000, 2002) and Rockafellar et al. (2006). We must remind ourselves yet again that we are dealing with the continuous case).

Given the probability level $\beta \in (0,1)$ and the loss function, we define the optimal risk-adjusted premium as the amount P_β^* minimizing the Conditional Tail Expectation of the loss, $CTE_\beta(P)$. This seems to be a very difficult problem to solve, but Rockafellar & Uryasev (2000) have developed optimization techniques for the calculation of VaR and CTE that can be easily translated to our problem (these techniques have also been extended to discrete random variables in Rockafellar & Uryasev (2002)). In fact, Theorem 1 of Rockafellar & Uryasev (2000) implies that, given the premium P, $CTE_\beta(P)$ can be calculated as the minimum value of the following convex and continuously differentiable function of the parameter α:

$$U(\alpha) = \alpha + \frac{1}{1-\beta} \int_0^\infty \left[L(P,x) - \alpha\right]^+ f(x) dx \tag{4}$$

(where $\left[L(P,x) - \alpha\right]^+ = L(P,x) - \alpha$ if $L(P,x) \geq \alpha$ and $\left[L(P,x) - \alpha\right]^+ = 0$ if $L(P,x) < \alpha$). And the optimal solution α, if it is a strict (or single) solution, coincides with the Value at Risk $VaR_\beta(P)$.

Moreover, if we consider the previous function as a function V of the two variables P and α,

$$V(P,\alpha) = \alpha + \frac{1}{1-\beta} \int_0^\infty \left[L(P,x) - \alpha\right]^+ f(x) dx \tag{5}$$

then Theorem 2 of Rockafellar and Uryasev (2000) shows that V is a convex and continuously differentiable function with regard to both variables and that the minimization of this function produces a pair $(P_\beta^*, \alpha_\beta^*)$ such that P_β^* minimizes $CTE_\beta(P)$ and α_β^*, if unique, gives the corresponding $VaR_\beta(P_\beta^*)$.

When the selected risk measure is the Conditional Tail Expectation, it is not necessary to take into consideration sophisticated loss functions in order to obtain reasonable results. In the next section we will solve this optimization problem when the loss function is the simple absolute loss function, $L(P,x) = |P - x|$, and we will obtain premiums with reasonable properties (a similar solution was obtained by Laeven & Goovaerts (2004) in the context of optimal levels of solvency capital). Nevertheless, the methodology that we propose is suitable for any arbitrary loss function.

4. The calculation of the premium minimizing the conditional tail expectation of the absolute loss

The absolute loss is not used in Bayesian premium calculation models since it gives rise to a premium which is the median of the loss distribution (see Lemaire & Vandermeulen (1983)), and this is by no means admissible (unfortunately real claims distributions are asymmetric, therefore their medians are below their means and as a consequence insurance companies will have solvency problems if they use the absolute loss for Bayesian premium calculation, because they will lose money on average). Nevertheless, this objection may be overcome if we consider risk-adjusted premiums calculated according to our methodology with a high probability level β, as we will see below. In fact, as β increases, the risk adjusted premium will eventually exceed the net premium, and therefore the absolute loss may become attractive because it provides sensible risk adjusted premiums. Moreover, the absolute loss is easy to understand since, after all, it measures the loss in monetary units (remember, however, that it is not only concerned with the insurance company's loss, when $P < x$, but also with the loss of the policyholders, when $P > x$: therefore it measures a kind of "social loss"). Of course, the selection of this loss function can also be criticized. For example, in a real problem the cost associated with overestimations could be different from the cost of underestimations. This could happen in the premium calculation problem, when the loss of the insurance company by charging insufficient (too low) premiums does not coincide with the loss of the policyholders when the company charges excessive (too high) premiums. This could also happen in the calculation of capital requirements, when the cost of insolvency associated with underestimations of the capital does not coincide with the opportunity cost associated with its overestimation. Moreover, in real problems the loss could be nonlinear when the size of the estimation error is high, due to liquidity problems. In this article we will implicitly assume, therefore, that the size of the possible estimation errors is not very large, so that liquidity premiums are not required. In this section we will first study the symmetric case, when the loss function is the absolute value of the estimation error, and after that we will also study the more general asymmetric case.

Theorem 1. If $L(P, x) = |P - x|$, then $(P^*_\beta, \alpha^*_\beta)$ minimize the function V if and only if they are solutions of the following system of equations (6):

$$\begin{aligned} F(P - \alpha) &= \frac{1 - \beta}{2} \\ F(P + \alpha) &= \frac{1 + \beta}{2} \end{aligned} \tag{6}$$

(remember that F is the distribution function of the random variable X).

Proof:

We have to minimize the function

$$V(P,\alpha) = \alpha + \frac{1}{1-\beta}\int_0^{\infty}\left[\left|P-x\right|-\alpha\right]^{+} f(x)\,dx$$

or, equivalently,

$$V(P,\alpha) = \alpha + \frac{1}{1-\beta}\left[\int_{x>P+\alpha}(x-P-\alpha)\,f(x)\,dx + \int_{x<P-\alpha}(P-x-\alpha)\,f(x)\,dx\right]$$

Let us study the integrals inside the brackets:

$$\int_{x>P+\alpha}(x-P-\alpha)\,f(x)\,dx + \int_{x<P-\alpha}(P-x-\alpha)\,f(x)\,dx =$$

$$= \int_{x>P+\alpha} xf(x)\,dx - (P+\alpha)(1-F(P+\alpha)) + (P-\alpha)F(P-\alpha) - \int_{x<P-\alpha} xf(x)\,dx =$$

(defining $S(x) = 1 - F(x)$, and integrating by parts)

$$= (P+\alpha)S(P+\alpha) + \int_{P+\alpha}^{\infty} S(x)\,dx - (P+\alpha)S(P+\alpha) + (P-\alpha)F(P-\alpha) -$$

$$-(P-\alpha)F(P-\alpha) + \int_0^{P-\alpha} F(x)\,dx = \int_{P+\alpha}^{\infty} S(x)\,dx + \int_0^{P-\alpha} F(x)\,dx$$

Therefore the function V becomes

$$V(P,\alpha) = \alpha + \frac{1}{1-\beta}\left[\int_0^{P-\alpha} F(x)\,dx + \int_{P+\alpha}^{\infty} S(x)\,dx\right]$$

Taking the partial derivatives with respect to P and α,

$$\frac{\partial V}{\partial P} = \frac{1}{1-\beta}\left[F(P-\alpha) - S(P+\alpha)\right] = 0$$

$$\frac{\partial V}{\partial \alpha} = 1 - \frac{1}{1-\beta}\left[F(P-\alpha) + S(P+\alpha)\right] = 0$$

we obtain:

$$F(P-\alpha) = \frac{1-\beta}{2}$$

$$F(P+\alpha) = \frac{1+\beta}{2}$$

q.e.d.

Since the optimal solutions depend on both the probability level β and the random variable X, we should denote them as $(P^*_{X,\beta}, \alpha^*_{X,\beta})$. In order to simplify the notation, we will omit to mention X or β when the context is clear enough.

Remark 1: The solution of the system of equations (6) shows a reasonable behaviour: P^*_β approaches the median when $\beta \to 0$, and also $P^*_\beta \to \infty$ when $\beta \to 1$. As a consequence, if we take β sufficiently large then we will obtain a risk loaded premium, since the optimal premium P^*_β will eventually become higher than the net premium.

Remark 2: If it is possible to find the inverse function F^{-1} of the distribution function F, then system (6) can be easily solved:

$$\begin{aligned} P^*_\beta &= \frac{1}{2}\left[F^{-1}\left(\frac{1+\beta}{2}\right) + F^{-1}\left(\frac{1-\beta}{2}\right)\right] \\ \alpha^*_\beta &= \frac{1}{2}\left[F^{-1}\left(\frac{1+\beta}{2}\right) - F^{-1}\left(\frac{1-\beta}{2}\right)\right] \end{aligned} \tag{7}$$

For example, if X is an exponential random variable with parameter $a > 0$ and distribution function

$$F(x) = \begin{cases} 1 - e^{-\frac{x}{a}}, & x \geq 0 \\ 0, & x < 0 \end{cases}$$

Then we obtain that

$$\begin{aligned} P^*_\beta &= -\frac{a}{2}\left[\log\left(\frac{1+\beta}{2}\right) + \log\left(\frac{1-\beta}{2}\right)\right] \\ \alpha^*_\beta &= \frac{a}{2}\left[\log\left(\frac{1+\beta}{2}\right) - \log\left(\frac{1-\beta}{2}\right)\right] \end{aligned}$$

Remark 3: The optimal premiums obtained as solutions of system (6) have good properties, as shown below. Notice that the properties of no unjustified risk loading, no rip-off, translation and scale invariance, and subadditivity, hold for arbitrary coherent risk measures, not only for CTE.

No unjustified risk loading: If the random variable X always takes a constant value $c(X(\omega) = c \geq 0, \forall \omega \in \Omega)$, then the optimal premium also takes that value $(P^*_X = c)$.

This is evident, since only in this case does the risk measure take its lowest value, $\rho = 0$.

No rip-off: If $X(\omega) \leq c, \forall \omega \in \Omega$, then $P_X^* \leq c$.

Suppose that $P_X^* > c$. Then $\left|P_X^* - X(\omega)\right| = \left|c - X(\omega)\right| + (P_X^* - c), \forall \omega \in \Omega$, and by the translation invariance property of the risk measures we have $\rho(\left|P_X^* - X\right|) = \rho(\left|c - X\right| + (P_X^* - c)) = \rho(\left|c - X\right|) + (P_X^* - c) > \rho(\left|c - X\right|)$ but this contradicts the hypothesis that P_X^* minimizes (2).

Translation invariance: If the risk X increases by a fixed amount c, then the premium also increases by that fixed amount $(P_{X+c}^* = P_X^* + c)$.

Since $F_{X+c}(x) = F_X(x - c)$, if (P_X^*, α_X^*) is the solution of system (6), then $(P_X^* + c, \alpha_X^*)$ is the optimal solution of the new system

$$F_{X+c}(P - \alpha) = \frac{1-\beta}{2}$$
$$F_{X+c}(P + \alpha) = \frac{1+\beta}{2}$$

This property also holds for any coherent risk measure ρ, as long as Problem (2) has unique solutions. This is a consequence of the following inequalities:

$$\rho[L(P_{X+c}^*, X + c)] \leq \rho[L(P_X^* + c, X + c)] = \rho[L(P_X^*, X)] \leq$$
$$\leq \rho[L(P_{X+c}^* - c, X)] = \rho[L(P_{X+c}^* - c, X + c - c)] = \rho[L(P_{X+c}^*, X + c)]$$

Scale invariance: $P_{\lambda X}^* = \lambda P_X^*$

Since $F_{\lambda X}(x) = F_X(\frac{x}{\lambda})$, if (P_X^*, α_X^*) is the solution of system (6), then $(\lambda P_X^*, \lambda \alpha_X^*)$ is the optimal solution of the new system

$$F_{\lambda X}(P - \alpha) = \frac{1-\beta}{2}$$
$$F_{\lambda X}(P + \alpha) = \frac{1+\beta}{2}$$

As in the previous case, it is also possible to generalize this property for any coherent risk measure:

$$\rho[L(P_{\lambda X}^*, \lambda X)] \leq \rho[L(\lambda P_X^*, \lambda X)] = \rho[\lambda L(P_X^*, X)] = \lambda \rho[L(P_X^*, X)] \leq$$
$$\leq \lambda \rho[L(\tfrac{1}{\lambda} P_{\lambda X}^*, X)] = \lambda \rho[L(\tfrac{1}{\lambda} P_{\lambda X}^*, \tfrac{1}{\lambda} \lambda X)] = \rho[L(P_{\lambda X}^*, \lambda X)]$$

Monotonicity: $X(\omega) \leq Y(\omega), \forall \omega \in \Omega \Rightarrow P_X^* \leq P_Y^*$

Let us assume the existence of the inverse distribution functions. Then we have that

$$X(\omega) \leq Y(\omega),\ \forall \omega \in \Omega \Rightarrow F_Y(x) \leq F_X(x), \forall x \Rightarrow F_X^{-1}(\beta) \leq F_Y^{-1}(\beta), \forall \beta$$

Hence we have, according to the first formula of (7), that $P_X^* \leq P_Y^*$.

Subadditivity: The global risk decreases when we add different risks.

The subadditivity and monotonicity properties of the coherent risk measures guarantee that

$$\rho\big(L\big(P_{X+Y}^*, X+Y\big)\big) \leq \rho\big(L\big(P_X^* + P_Y^*, X+Y\big)\big) = \rho\big(\big|P_X^* + P_Y^* - (X+Y)\big|\big) \leq$$

$$\leq \rho\big(\big|P_X^* - X\big| + \big|P_Y^* - Y\big|\big) \leq \rho\big(\big|P_X^* - X\big|\big) + \rho\big(\big|P_Y^* - Y\big|\big)$$

Remark 4: Instead of calculating the optimal premium that guarantees the minimum risk, we can take the premium as given, and then calculate the risk we run in such a situation. In order to do that, we have to remember that, given the premium P, $CTE_\beta(P)$ can be calculated as the minimum value of the function $U(\alpha)$ given in (4). Moreover, according to Rockafellar & Uryasev (2000), the optimal solution α^* minimizing $U(\alpha)$ coincides with the Value at Risk $VaR_\beta(P)$. Following similar arguments to those applied in the proof of Theorem 1, we can then conclude that $VaR_\beta(P)$ can be calculated as the solution α^* of the equation (8):

$$F(P+\alpha) - F(P-\alpha) = \beta \tag{8}$$

Remark 5: So far we have assumed a loss function which is symmetric around the premium. For the reasons commented before, this may be an unacceptable assumption in many real cases. In fact, the methodology also works when we take into consideration asymmetric loss functions assigning different weights to underestimations and overestimations of the losses. Consider, for example, the following asymmetric loss function:

$$L(P,x) = \begin{cases} \omega_1(P-x), & P > x \\ \omega_2(x-P), & P \leq x \end{cases}$$

($\omega_1, \omega_2 > 0$ are the different weights associated with overestimations and underestimations, respectively). In this case, it is easy to modify the proof of Theorem 1 in order to show that the new optimal solution $(P_\beta^*, \alpha_\beta^*)$ coincides with the solution of the following system of equations:

$$\begin{aligned} F\left(P - \frac{\alpha}{\omega_1}\right) &= \frac{\omega_2 - \beta\omega_2}{\omega_1 + \omega_2} \\ F\left(P + \frac{\alpha}{\omega_2}\right) &= \frac{\omega_2 + \beta\omega_1}{\omega_1 + \omega_2} \end{aligned} \tag{9}$$

Moreover, in this case the equivalent of Equation (8) is the next one:

$$F\left(P+\frac{\alpha}{\omega_2}\right)-F\left(P-\frac{\alpha}{\omega_1}\right)=\beta \tag{10}$$

The properties of the optimal premiums obtained from System (9) are similar to those obtained in the symmetric case: no unjustified risk loading, no rip-off, translation and scale invariance, subadditivity and monotonicity. Again, the first five properties hold for any arbitrary coherent risk measure, not only CTE. Also, if it is possible to find the inverse function F^{-1} of the distribution function F, then System (9) can be easily solved:

$$\begin{aligned} P_\beta^* &= \frac{\omega_1}{\omega_1+\omega_2}F^{-1}\left(\frac{\omega_2-\beta\omega_2}{\omega_1+\omega_2}\right)+\frac{\omega_2}{\omega_1+\omega_2}F^{-1}\left(\frac{\omega_2+\beta\omega_1}{\omega_1+\omega_2}\right) \\ \alpha_\beta^* &= \frac{\omega_1\omega_2}{\omega_1+\omega_2}\left[F^{-1}\left(\frac{\omega_2+\beta\omega_1}{\omega_1+\omega_2}\right)-F^{-1}\left(\frac{\omega_2-\beta\omega_2}{\omega_1+\omega_2}\right)\right] \end{aligned} \tag{11}$$

For example, when X is an exponential random variable with parameter $\alpha>0$, we have:

$$\begin{aligned} P_\beta^* &= -\frac{a}{\omega_1+\omega_2}\left[\omega_1\log\left(\frac{\omega_1+\beta\omega_2}{\omega_1+\omega_2}\right)+\omega_2\log\left(\frac{\omega_1-\beta\omega_1}{\omega_1+\omega_2}\right)\right] \\ \alpha_\beta^* &= \frac{a\omega_1\omega_2}{\omega_1+\omega_2}\left[\log\left(\frac{\omega_1+\beta\omega_2}{\omega_1-\beta\omega_1}\right)\right] \end{aligned}$$

It may be interesting to note that, in the extreme cases when one of the weights is zero, we obtain extreme solutions. If, for example, the weights are $\omega_2=1$, $\omega_1\to 0$, then $P_\beta^*\to\infty$ (which is sensible, because if the insurance company is not worried at all about the overestimation errors, it is rational to charge very high premiums). Of course, in the opposite case when $\omega_1=1, \omega_2\to 0$, then $P_\beta^*\to 0$. As we commented before, in real problems the insurance company should choose positive values of the weights, corresponding to the different valuations of under and overestimation errors.

5. Exemplification

In this section we will perform some numerical calculations, applying our results to a real world example. For that purpose we will consider a structure function which has been useful in motor insurance (see Panjer & Willmot (1992) pp. 301, 322). X stands here for a random variable distributed by means of an inverse Gaussian distribution with density

$$f(x)=\sqrt{\frac{\vartheta}{\pi x^3}}\,e^{-\frac{1}{2}\frac{\vartheta(x-\mu)^2}{\mu^2 x}},\quad x>0,\ \mu>0,\ \vartheta>0$$

The mean and variance of this random variable are $\mu, \frac{\mu^3}{\vartheta}$, respectively, and the parameters are $\mu = 0.15514$, $\vartheta = 0.15582$. The net premium coincides with the mathematical expectation (0.1554, in this case). Assuming some reasonable (close to 1) values for β, we can calculate the optimal values of P_β^* (and the corresponding α_β^*) following (6). Table 1 summarizes these results. Notice that the premiums obtained are greater than the net premium. Notice also that their values are lower than the premiums calculated applying the usual CTE criterion (which are shown in the last column). For example, the premium minimizing the CTE of the loss when $\beta = 0.9$ is 0.24069, whereas CTE(X) for the same value of β is 0.51875.

TABLE 1

SUMMARY OF THE OPTIMAL P_β^* AND THE CORRESPONDING VaR_β VALUES FOR DIFFERENT β'S. THE FOURTH COLUMN SHOWS THE OPTIMAL VALUES OF THE OBJECTIVE FUNCTION. $CTE_\beta(X)$ IS GIVEN FOR COMPARISON.

β	P_β^*	α_β^*	$CTE_\beta(L(X, P_\beta^*))$	$CTE_\beta(X)$
0.9	0.24069	0.21204	0.31518	0.51875
0.925	0.26612	0.23994	0.34515	0.57328
0.95	0.30373	0.28041	0.38838	0.65291
0.975	0.37213	0.35251	0.46472	0.79574

In the next two tables we change the formulation of the problem. Instead of calculating the optimal premium that minimizes the risk measure of the loss, we now suppose that the premium P is given (it may be due to market forces) and we calculate the associated values of $VaR_\beta(L(P, X))$ and $CTE_\beta(L(P, X))$, following the instructions given in remark 4. We will represent the premiums with the help of a security loading θ, so that $P = (1 + \theta)E(X)$. Tables 2 and 3 summarize the numerical results. For example, a security loading of 30% ($\theta = 0.3$) gives a premium $P = 0.201682$ and the corresponding values for VaR and CTE when $\beta = 0.9$ are 0.178438 and 0.32438, respectively. The results in Table 2 can also be used to approximate the premium that minimizes VaR. If, for example, $\beta = 0.9$, then the third column of Table 2 shows that the premium with minimum VaR is (close to) 0.178411 (or, equivalently, the security loading is approximately 15%). For the same reasons, the last three columns of Table 2 show that the optimal loadings are (close to) 30%, 45% and 90% when β = 0.925, 0.95 and 0.975, respectively (of course, we can obtain more accurate approximations by considering a thinner mesh for the security loading). A similar analysis can be made in Table 3 in relation to CTE. For example, when $\beta = 0.9$ the third column shows that the premium that minimizes CTE is close to 0.248224. Of course, this analysis is redundant, since we already know how to calculate the exact premiums minimizing the CTE (Table 1 shows that the optimal premium in this case is 0.24069).

TABLE 2

$VaR_\beta(L(P,X))$ FOR GIVEN PREMIUMS P EXPRESSED BY MEANS OF A SECURITY LOADING θ. THE NET PREMIUM IS EQUAL TO 0.15514.

θ	P	β			
		0.9	0.925	0.95	0.975
0	0.15514	0.177084	0.225655	0.297603	0.429012
0.15	0.178411	0.162251	0.202384	0.274332	0.405741
0.3	0.201682	0.178438	0.18675	0.251061	0.38247
0.45	0.224953	0.198046	0.203592	0.22779	0.359199
0.6	0.248224	0.218872	0.223706	0.231066	0.335928
0.9	0.294766	0.262365	0.266697	0.272046	0.289388
1.05	0.318037	0.284654	0.288878	0.293916	0.302393
1.2	0.341308	0.307173	0.311326	0.316174	0.323394
1.35	0.364579	0.329862	0.333965	0.338688	0.345301
1.5	0.38785	0.352678	0.356747	0.361383	0.367636
1.65	0.411121	0.375592	0.379635	0.384209	0.390228

TABLE 3

$CTE_\beta(L(P,X))$ FOR GIVEN PREMIUMS P EXPRESSED BY MEANS OF A SECURITY LOADING θ.

θ	P	β			
		0.9	0.925	0.95	0.975
0	0.15514	0.363615	0.418144	0.497777	0.6406
0.15	0.178411	0.340671	0.394873	0.474506	0.617329
0.3	0.201682	0.32438	0.371856	0.451235	0.594058
0.45	0.224953	0.316558	0.355192	0.427964	0.570787
0.6	0.248224	0.315474	0.346899	0.406873	0.547516
0.75	0.271495	0.319622	0.345302	0.394123	0.524245
0.9	0.294766	0.327825	0.34894	0.388792	0.500974
1.05	0.318037	0.339176	0.35666	0.389343	0.481206
1.2	0.341308	0.352981	0.36757	0.394523	0.469666
1.35	0.364579	0.368706	0.380982	0.403347	0.464997
1.5	0.38785	0.385942	0.396362	0.415045	0.465825
1.65	0.411121	0.40437	0.413299	0.42902	0.471032

In Tables 4, 5 and 6 we illustrate numerically the case of an asymmetric loss function. We still work with the same example, but this time the loss function assigns different weights to the underestimations and overestimations of the value of the random variable. Following the notation given in remark 5, we consider an asymmetric loss function with weights $\omega_1 = 1$, $\omega_2 = 2$. Table 4

shows the new optimal values of P_β^* (and the corresponding α_β^*) calculated according to (9). Notice that we are now giving more importance to the overestimation errors than to the underestimations. The consequence is that we obtain larger premiums in Table 4 than in Table 1. The interpretation of Tables 5 and 6 is similar to that of Tables 2 and 3. For example, a security loading of 30% ($\theta = 0.3$) gives the premium $P = 0.201682$ and the corresponding values for VaR and CTE when $\beta = 0.9$ are 0.261084 and 0.634146, respectively (these values are bigger than those obtained in Tables 2 and 3, for the reason already explained). Again, Table 5 can be used to approximate the value of the premium that minimizes VaR (and also Table 6 can be used to do the same job in relation to CTE). If, for example, $\beta = 0.9$, then the third column of Table 5 shows that the premium with minimum VaR is (close to) 0.224953 (or, equivalently, the security loading is approximately 45%).

TABLE 4

SUMMARY OF OPTIMAL P_β^* AND α_β^* VALUES FOR DIFFERENT β'S, TAKING WEIGHTS $\omega_1 = 1$, $\omega_2 = 2$. THE FOURTH COLUMN SHOWS THE OPTIMAL VALUES OF THE OBJECTIVE FUNCTION. $CTE_\beta(X)$ IS GIVEN FOR COMPARISON.

β	P_β^*	α_β^*	$CTE_\beta(L(X, P_\beta^*))$	$CTE_\beta(X)$
0.9	0.36281	0.33119	0.47403	0.51875
0.925	0.39898	0.37033	0.51546	0.57328
0.95	0.45198	0.42670	0.57494	0.65291
0.975	0.54729	0.52629	0.67942	0.79574

TABLE 5

OPTIMAL $VaR_\beta(L(P,X))$ FOR GIVEN PREMIUMS P EXPRESSED BY MEANS OF A SECURITY LOADING θ, WHEN THE LOSS FUNCTION IS ASYMMETRIC WITH WEIGHTS $\omega_1 = 1$, $\omega_2 = 2$.

θ	P	β			
		0.9	0.925	0.95	0.975
0	0.15514	0.354168	0.451309	0.595205	0.858024
0.15	0.178411	0.307626	0.404767	0.548663	0.811482
0.3	0.201682	0.261084	0.358225	0.502121	0.76494
0.45	0.224953	0.215136	0.311683	0.455579	0.718398
0.6	0.248224	0.22735	0.265141	0.409037	0.671856
0.75	0.271495	0.246774	0.254059	0.362495	0.625314
0.9	0.294766	0.267499	0.273145	0.315953	0.578772
1.05	0.318037	0.288901	0.293925	0.302315	0.53223
1.2	0.341308	0.310737	0.315433	0.32197	0.485688
1.35	0.364579	0.33288	0.337376	0.343172	0.439146
1.5	0.38785	0.355248	0.359613	0.365001	0.392604
1.65	0.411121	0.377789	0.382062	0.387193	0.396399

TABLE 6

OPTIMAL $CTE_\beta(L(P,X))$ FOR GIVEN PREMIUMS P EXPRESSED BY MEANS OF A SECURITY LOADING θ, WHEN THE LOSS FUNCTION IS ASYMMETRIC WITH WEIGHTS $\omega_1 = 1$, $\omega_2 = 2$.

θ	P	β			
		0.9	0.925	0.95	0.975
0	0.15514	0.72723	0.836288	0.995554	1.2812
0.15	0.178411	0.680688	0.789746	0.949012	1.234658
0.3	0.201682	0.634146	0.743204	0.90247	1.188116
0.45	0.224953	0.587606	0.696662	0.855928	1.141574
0.6	0.248224	0.546633	0.65012	0.809386	1.095032
0.75	0.271495	0.516691	0.605588	0.762844	1.04849
0.9	0.294766	0.496006	0.571291	0.716302	1.001948
1.05	0.318037	0.482888	0.546747	0.671387	0.955406
1.2	0.341308	0.475944	0.530255	0.636175	0.908864
1.35	0.364579	0.474051	0.520379	0.610524	0.862322
1.5	0.38785	0.476304	0.515946	0.592834	0.81578
1.65	0.411121	0.481971	0.516001	0.581743	0.772552

6. CONCLUSIONS

It is well known that a great deal of research has been carried out recently on the subject of risk measurement in both financial and actuarial contexts. In this paper we have shown that modern risk measures such as VaR and CTE can help us to deal with an important actuarial issue, the problem of premium calculation. It is possible to find applications of VaR and CTE to premium calculation in the literature, where the premium is calculated as the VaR or CTE of the random variable representing the total claim amount, for a given probability level. We suggest an alternative two-step methodology for obtaining the premium. The first step is to select a loss function representing the loss associated with underestimations and overestimations of the total claim amount. In the second step the premium is calculated as the amount minimizing the risk measure of that loss. We have shown in this paper that it is easy to calculate premiums that minimize the CTE of simple loss functions such as the absolute loss. Besides, these premiums verify reasonable properties. We also consider asymmetric loss functions where the loss associated with underestimations of the true claim amount is different from the loss associated with overestimations. We also show a procedure for calculating the VaR and CTE of the loss when the premium is considered as given, maybe due to market forces. Finally, this procedure can help us to approximate the value of the premium that minimizes the VaR of the loss.

This methodology can be applied to other loss functions different from the absolute loss, although in these cases it may be necessary to rely on more complex numerical calculations.

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ANTONIO HERAS (corresponding author)
Departamento de Economía Financiera y Contabilidad I
(Economía Financiera y Actuarial)
Facultad de Ciencias Económicas, Universidad Complutense de Madrid,
(Campus de Somosaguas), 28223 Pozuelo de Alarcón,
Madrid (Spain)
E-Mail: aheras@ccee.ucm.es

BEATRIZ BALBÁS
Departamento de Análisis Económico y Finanzas
Universidad de Castilla – La Mancha
Talavera, Toledo (Spain)
E-Mail: beatriz.balbas@uclm.es

JOSÉ LUIS VILAR
Departamento de Economía Financiera y Contabilidad 1
(Economía Financiera y Actuarial)
Universidad Complutense de Madrid
Madrid (Spain)
and
Actuaris Ibérica
E-Mail: jlvilarz@ccee.ucm.es