Applied Mathematical Finance I

Lecture 10: Cross-Currency Markets

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FX Rates

- Foreign exchange (FX) rate is a rate at which one can purchase a unit of one currency for an amount of another currency.
- FX rates refer to currency pairs where the first currency is the base (or foreign) currency and the second currency is the quote (or domestic) currency.
- For example, USDRUB quote of x implies that one unit of base currency (USD) can be exchanged for x units of quote currency (RUB).
- We distinguish between different kinds of quoted FX rates
  - TOD for today's settlement (T);
  - TOM for settlement on the next business day $(T + 1)$;
  - Spot for settlement on the spot date i.e. in two business days $(T + 2)$;
  - Forward for all settlements beyond spot date.
- TOD and TOM quotes are usually simply referred to as spot rates.

Example: Spot FX Rates Quotation

- We would like to buy 10m USD for EUR so we ask four banks for a quote. Note that the market will generally only quote in one way so we get quotes for EURUSD, not USDEUR
  - Bank A: 1.2405-16 (bid 1.2405, offer 1.2416);
  - Bank B: 1.2401-20;
  - Bank C: 1.2404-11;
  - Bank D: 1.2409-12.
- Buying USD means selling EUR so we only look at the bid quotes.
- We choose the 4-th quote as it gives the largest amount of USD per a unit of EUR.
- We make a deal so we will pay an amount of $10,000,000/1.2409 = 8,058,667.1$ EUR and receive 10m USD on the spot date i.e. two business day after the trade date.

Cross Rates

- A cross FX rate is a quote for a currency pair which does not involve USD, for instance EURRUB or GBPCHF.
- It is not uncommon that a cross pair is not directly traded so the cross rate is derived through a combination of quotes involving USD.
- Consider an example where a market user is given the following quotes
  - EURUSD: 1.2405-16;
  - GBPUSD: 1.5878-83.

  What are the implied bid/offer quotes for EURGBP cross rate?
- Given the above quotes, the market user can sell 1 EUR for 1.2405 USD and then sell dollars for 1.2405/1.5883 = 0,7810 GBP. Hence, EURGBP bid quote is 0,7810. The offer quote is 1.2416/1.5878 = 0,7820.
- Note that we assumed that both EURUSD and GBPUSD trades have the same settlement date.

FX Forward

- We assume there is a quoted FX rate S_t^{FORDOM} for a currency pair FORDOM for a spot transaction to be entered at t and settled at $t+\tau$, where τ is a settlement lag (typically 0-2 business days).

- Our aim is to determine arbitrage-free forward rate K for a transaction at a future time T with settlement at $T+\tau$ that can be entered at $t<T$ at zero cost.

- We first consider a classical replication approach (no collateralization involved) where we only assume that we can access both domestic and foreign risk-free financing.

- Namely, for now we assume there are two risk-free zero curves: a domestic one $p^{\text{DOM}}(t,T), T \geq t$ and a foreign one $p^{\text{FOR}}(t,T), T \geq t$.

FX Forward Replication

- Consider a trade where we sell at t FX forward for settlement at $T+\tau$ i.e. agree to deliver one unit of FOR for K units of DOM at $T+\tau$.
- To hedge this position, we can borrow cash in DOM currency, enter a spot FX transaction to convert it to FOR currency and then put cash in FOR currency on a deposit at FOR rate.
- Since the spot transaction settles at $t+\tau$, there is no need to borrow cash at t so we consider financing for a future period $[t+\tau, T+\tau]$ at the forward rate $F^{\text{DOM}}(t, t+\tau, T+\tau)$.
- Note that if we put x units of FOR currency on a deposit for a future period $[t+\tau, T+\tau]$ at forward rate $F^{\text{FOR}}(t, t+\tau, T+\tau)$, we will receive $x\left[1+F^{\text{FOR}}(t, t+\tau, T+\tau)(T-t)\right]$ units of FOR at $T+\tau$.

FX Forward Replication (continued)

- At t, enter a spot FX transaction (settling at $t+\tau$) to buy x units of FOR for $x\,S_t^{\mathsf{FORDOM}}$ units of DOM, where $x = \frac{1}{1+F^{\mathsf{FOR}}(t,t+\tau,T+\tau)\,(T-t)}$.
- Also, we enter at t a forward loan over $[t+\tau, T+\tau]$ for an amount of $x\,S_t^{\mathsf{FORDOM}}$ in DOM currency and a forward deposit for an amount of x in FOR currency.
- At $t+\tau$, we receive an amount of $x\,S_t^{\mathsf{FORDOM}}$ in DOM currency, exchange it for x units of FOR and put that amount on a deposit at the rate pre-agreed at t.
- At $T+\tau$, we have exactly one unit of FOR on the account so we exchange it for K units in DOM currency as stipulated by the forward contract.
- Our net position at $T+\tau$ is then

$$K - S_t^{\mathsf{FORDOM}}\,\frac{1+F^{\mathsf{DOM}}(t,t+\tau,T+\tau)\,(T-t)}{1+F^{\mathsf{FOR}}(t,t+\tau,T+\tau)\,(T-t)} \text{ in DOM currency.}$$

Arbitrage-Free FX Forward Rate

- By no-arbitrage, we get the following expression for FX forward rate

$$K = F_{t,T}^{\text{FORDOM}} = S_t^{\text{FORDOM}} \frac{1 + F^{\text{DOM}}(t, t+\tau, T+\tau)\,(T-t)}{1 + F^{\text{FOR}}(t, t+\tau, T+\tau)\,(T-t)}.$$

- We can rewrite the above formula in terms of discount factors as

$$F_{t,T}^{\text{FORDOM}} = S_t^{\text{FORDOM}} \frac{p^{\text{FOR}}(t, T+\tau)}{p^{\text{FOR}}(t, t+\tau)} \frac{p^{\text{DOM}}(t, t+\tau)}{p^{\text{DOM}}(t, T+\tau)}. \tag{1}$$

- By introducing a notion of a forward discount factor for period $[t+\tau, T+\tau]$

$$p(t, t+\tau, T+\tau) = \frac{p(t, T+\tau)}{p(t, t+\tau)}$$

we can rewrite the formula for FX forward rate as

$$F_{t,T}^{\text{FORDOM}} = S_t^{\text{FORDOM}} \frac{p^{\text{FOR}}(t, t+\tau, T+\tau)}{p^{\text{DOM}}(t, t+\tau, T+\tau)}. \tag{2}$$

Spot PV

- Spot trade is typically a very short dated forward contract for delivery at $T+2$.
- Given a spot quote S_t^{FORDOM} for settlement at $t+\tau$ what is the actual value of one unit of FOR in terms of DOM today i.e. at t?
- The value of one unit of foreign currency in domestic currency is sometimes referred to as the Spot PV and this is nothing other than the TOD FX quote considered earlier.
- By similar no-arbitrage considerations we can derive that

$$\text{SpotPV}_t^{\text{FORDOM}} = S_t^{\text{FORDOM}} \frac{p^{\text{DOM}}(t, t+\tau)}{p^{\text{FOR}}(t, t+\tau)}. \tag{3}$$

- We use the concept of Spot PV when we want to report the value of a contract in a currency different to valuation currency.

No-Arbitrage Considerations For Cross FX Rates

- Consider a cross FX spot rate FORDOM which, by convention, has a spot settlement lag of τ.
- We assume that FORUSD pair is traded with the spot lag of θ, while for USDDOM pair the spot lag is δ.
- For example, for EURRUB and USDRUB pairs settlement lag is 1bd while EURUSD is traded "on the spot" i.e. with settlement at $T+2$ so in this case $\tau = \delta = 1\text{bd}$, $\theta = 2\text{bd}$.
- In order to exclude instantaneous arbitrage, the following relation must hold for TOD quotes (here we ignore the bid-offer spreads)

$$\text{SpotPV}_t^{\text{FORDOM}} = \text{SpotPV}_t^{\text{FORUSD}} \cdot \text{SpotPV}_t^{\text{USDDOM}}.$$

- Combining the above formula together with (1) and (3), we get

$$F_{t,T}^{\text{FORDOM}} = F_{t,T+\tau-\theta}^{\text{FORUSD}} \cdot F_{t,T+\tau-\delta}^{\text{USDDOM}}.$$

Domestic and Foreign Measures

- It is important to realize that there are two different T-forward measures depending on which economy we regard as "home".
- To be more precise, we must distinguish between domestic T-forward measure $\mathbb{Q}^{T^{\text{DOM}}}$ corresponding to numéraire $p^{\text{DOM}}(t,T)$ and the foreign T-forward measure $\mathbb{Q}^{T^{\text{FOR}}}$ corresponding to numéraire $p^{\text{FOR}}(t,T)$.
- Note that for any two payoffs $\mathcal{X}_T^{\text{DOM}}$ and $\mathcal{Y}_T^{\text{FOR}}$ denominated in domestic and foreign currencies respectively the following relations hold due to the fundamental theorem of asset pricing

$$\text{PV}_t^{\text{DOM}} = p^{\text{DOM}}(t,T)\,\mathbb{E}_t^{T^{\text{DOM}}}\left[\mathcal{X}_T^{\text{DOM}}\right],$$

$$\text{PV}_t^{\text{FOR}} = p^{\text{FOR}}(t,T)\,\mathbb{E}_t^{T^{\text{FOR}}}\left[\mathcal{Y}_T^{\text{FOR}}\right].$$

- Our aim now is to determine the connection between $\mathbb{Q}^{T^{\text{DOM}}}$ and $\mathbb{Q}^{T^{\text{FOR}}}$.

Connection Between DOM and FOR Measures

- We now consider a spot FX rate S_t^{FORDOM} and for simplicity assume that there is no spot settlement lag i.e. S_t^{FORDOM} is the true conversion rate between DOM and FOR currencies (TOD quote).
- Consider a payoff $\mathcal{X}_T^{\mathsf{DOM}}$ denominated in the domestic currency. Again,

$$\mathsf{PV}_t^{\mathsf{DOM}} = p^{\mathsf{DOM}}(t,T)\,\mathbb{E}_t^{T^{\mathsf{DOM}}}\left[\mathcal{X}_T^{\mathsf{DOM}}\right].$$

- On the other hand, we can look at this payoff in currency FOR so we define $\mathcal{Y}_T^{\mathsf{FOR}} = \mathcal{X}_T^{\mathsf{DOM}} / S_T^{\mathsf{FORDOM}}$. Since $\mathcal{Y}$ is in FOR, we have

$$\begin{aligned}\mathsf{PV}_t^{\mathsf{FOR}} &= p^{\mathsf{FOR}}(t,T)\,\mathbb{E}_t^{T^{\mathsf{FOR}}}\left[\mathcal{Y}_T^{\mathsf{FOR}}\right]\\ &= p^{\mathsf{FOR}}(t,T)\,\mathbb{E}_t^{T^{\mathsf{FOR}}}\left[\mathcal{X}_T^{\mathsf{DOM}} \Big/ S_T^{\mathsf{FORDOM}}\right].\end{aligned}$$

Connection Between DOM and FOR Measures (continued)

- Now, given that $\mathrm{PV}_t^{\mathsf{DOM}} = \mathrm{PV}_t^{\mathsf{FOR}} S_t^{\mathsf{FORDOM}}$, we obtain the following formula connecting forward measures for different currencies

$$\mathbb{E}_t^{T^{\mathsf{DOM}}}\left[\mathcal{X}_T^{\mathsf{DOM}}\right] = S_t^{\mathsf{FORDOM}} \frac{p^{\mathsf{FOR}}(t,T)}{p^{\mathsf{DOM}}(t,T)} \mathbb{E}_t^{T^{\mathsf{FOR}}}\left[\mathcal{X}_T^{\mathsf{DOM}} / S_T^{\mathsf{FORDOM}}\right].$$

- Using formula (2), we can rewrite the above relation as

$$\mathbb{E}_t^{T^{\mathsf{DOM}}}\left[\mathcal{X}_T^{\mathsf{DOM}}\right] = \mathbb{E}_t^{T^{\mathsf{FOR}}}\left[\mathcal{X}_T^{\mathsf{DOM}} \frac{F_{t,T}^{\mathsf{FORDOM}}}{S_T^{\mathsf{FORDOM}}}\right]. \tag{4}$$

- Taking $\mathcal{X}_T^{\mathsf{DOM}} = S_T^{\mathsf{FORDOM}}$, we get

$$\mathbb{E}_t^{T^{\mathsf{DOM}}}\left[S_T^{\mathsf{FORDOM}}\right] = F_{t,T}^{\mathsf{FORDOM}}.$$

- In particular, forward FX rate $F_{t,T}^{\mathsf{FORDOM}}$ is a martingale under $\mathbb{Q}^{T^{\mathsf{DOM}}}$.

European Vanilla FX Option Pricing

- Consider a European vanilla FX call option on S_t^{FORDOM} with strike price K given in DOM currency. The payoff of this contract at maturity T is given buy
$$\left[S_T^{\text{FORDOM}} - K\right]_+ .$$
- Given that $F_{T,T}^{\text{FORDOM}} = S_T^{\text{FORDOM}}$, it is more convenient to model forward price $F_{t,T}^{\text{FORDOM}}, t \leq T$, as it is a martingale under $\mathbb{Q}^{T^{\text{DOM}}}$.
- Let us assume for simplicity that we are given constant domestic risk-free rate r_d and foreign risk-free rate r_f such that
$$p^{\text{DOM}}(t, T) = \mathrm{e}^{-r_d\,(T-t)},$$
$$p^{\text{FOR}}(t, T) = \mathrm{e}^{-r_f\,(T-t)}.$$
- Recall that under the assumption of deterministic interest rates T-forward and risk-neutral measure coincide.

European Vanilla FX Option Pricing (continued)

- Let us assume that $F_{t,T}^{\mathsf{FORDOM}}$ has the following log-normal dynamics with constant volatility under domestic risk-neutral measure $\mathbb{Q}^{\mathsf{DOM}}$

$$dF_{t,T}^{\mathsf{FORDOM}} = F_{t,T}^{\mathsf{FORDOM}}\,\sigma\,dW_t^{\mathsf{DOM}}.$$

- We can now invoke Black's formula to get the value at t of the option expressed in DOM currency

$$\mathrm{PV}_t^{\mathsf{DOM}} = p^{\mathsf{DOM}}(t,T)\,\left[F\,\Phi\left(d_+\right) - K\,\Phi\left(d_-\right)\right],$$

where $F = F_{t,T}^{\mathsf{FORDOM}}$, $d_+ = \frac{\ln(F/K)}{\sigma\sqrt{T-t}} + \frac{\sigma\sqrt{T-t}}{2}$, $d_- = d_+ - \frac{\sigma\sqrt{T-t}}{2}$.

- Expressing the above formula in terms of r_d, r_f and FX spot $S_t = S_t^{\mathsf{FORDOM}}$, we get the Garman–Kohlhagen formula

$$\mathrm{PV}_t^{\mathsf{DOM}} = S_t\,\mathrm{e}^{-r_f\,(T-t)}\,\Phi\left(d_+\right) - K\,\mathrm{e}^{-r_d\,(T-t)}\,\Phi\left(d_-\right),$$

where $d_+ = \frac{\ln(S_t/K)+(r_d-r_f+\sigma^2/2)\cdot(T-t)}{\sigma\sqrt{T-t}}$, $d_- = d_+ - \sigma\sqrt{T-t}$.

Classical Cross-Currency Market

- We consider a model for cross-currency market given under the real-world probability measure $\mathbb{P}$. The market consists of
  - Domestic risk-free money market account $B_d(t) = \mathrm{e}^{r_d t}$.
  - Domestic risky asset with dynamics (we assume no repo and dividends) $dS(t) = \mu_S S(t)\,dt + \sigma_S S(t)\,dW_S^{\mathbb{P}}(t)$.
  - Foreign risk-free money market account $B_f(t) = \mathrm{e}^{r_f t}$.
  - FORDOM exchange rate $dX(t) = \mu_X X(t)\,dt + \sigma_X X(t)\,dW_X^{\mathbb{P}}(t)$, $d\langle W_S^{\mathbb{P}}, W_X^{\mathbb{P}}\rangle = \nu\,dt$.
- By the first fundamental theorem of asset pricing, there will be no arbitrage if and only if there is a measure $\mathbb{Q}^d \sim \mathbb{P}$ under which every domestic traded asset discounted by $B_d(t)$ is a $\mathbb{Q}^d$-martingale.

Classical Cross-Currency Market (continued)

- So we switch to a measure $\mathbb{Q}^d$ where the dynamics of S become

$$dS(t) = r_d\,S(t)\,dt + \sigma_S\,S(t)\,dW_S^{\mathbb{Q}^d}(t). \tag{5}$$

- We now need to identify the risk-neutral drift of X by no-arbitrage arguments.
- Note that the foreign money market account converted to the domestic currency is a valid domestic asset and hence the process given by $B_f(t)X(t)$ must grow at rate r_d under $\mathbb{Q}^d$.
- Itô's formula then implies that

$$dX(t) = (r_d - r_f)\,X(t)\,dt + \sigma_X\,X(t)\,dW_X^{\mathbb{Q}^d}(t).$$

Adding a Foreign Risky Asset

- Now, consider a foreign risky asset Z with the dynamics under $\mathbb{Q}^d$ given by

$$dZ(t) = \mu_Z Z(t)\,dt + \sigma_Z Z(t)\,dW_Z^{\mathbb{Q}^d}(t),$$

where $d\langle W_Z^{\mathbb{Q}^d}, W_X^{\mathbb{Q}^d}\rangle = \rho\,dt$.

- Again, in order to avoid arbitrage, its domestic price (composite price) $Y = Z(t)\,X(t)$ must grow at rate r_d. By Itô's lemma,

$$\begin{aligned} dY &= X\,dZ + Z\,dX + dZ\,dX \\ &= Y(\mu_Z + r_d - r_f + \rho\,\sigma_X\sigma_Z)dt + Y(\sigma_X\,dW_X^{\mathbb{Q}^d} + \sigma_Z\,dW_Z^{\mathbb{Q}^d}). \end{aligned} \tag{6}$$

- By no-arbitrage, the drift of the foreign risky asset under the domestic measure[*] is given by

$$\mu_Z = r_f - \rho\,\sigma_X\sigma_Z. \tag{7}$$

[*] Of course, under the foreign risk-neutral measure $\mathbb{Q}^f$ the drift must be equal to r_f.

Composite Options

- We can rewrite (6) as

$$dY(t) = r_d Y(t)\,dt + \sigma Y(t)\,dW^{\mathbb{Q}^d}(t), \tag{8}$$

where $\sigma = \sqrt{\sigma_X^2 + 2\rho\,\sigma_X\sigma_Z + \sigma_Z^2}$ and W is a standard $\mathbb{Q}^d$-Brownian motion.

- Composite call option with strike K expressed in domestic currency pays at maturity time T an amount of

$$(Z(T)\,X(T) - K)_+ = (Y(T) - K)_+.$$

- In view of (8), we can value this kind of payoff by using the Black-Scholes formula with volatility σ as in (8).

Quanto options

- Quanto call option with strike K given in foreign currency pays at maturity T an amount of

$$(Z(T) - K)_+$$

directly in domestic currency.

- By the fundamental pricing theorem, the option's present value at t is

$$\mathrm{PV}_t^{\mathrm{Quanto}} = \mathrm{e}^{-r_d(T-t)}\, \mathbb{E}_t^{\mathbb{Q}^d} (Z(T) - K)_+.$$

Note that here we take expectation under the domestic measure, while the underlying price Z is denominated in foreign currency.

- In view of (7), we can price such options by using the Black-Scholes formula with riskless rate r_d and dividend yield $r_d - r_f + \rho\, \sigma_X \sigma_Z$.

Collateralization

- Let us recall the mechanics of collateralization. Consider a fully collateralized transaction where a party A buys at time t an asset worth $V(t)$ from a party B.
- There is no actual cash exchange at t: A buys an asset worth $V(t)$, and B then have to immediately post cash collateral for an amount of $V(t)$ to A in order to secure the liability just sold.
- At $t + dt$, A returns the collateral amount $V(t)$ to B and also pays the interest $c(t)\, V(t)\, dt$. On the other hand, the new mark-to-market value of the asset is $V(t + dt)$ so B, in turn, posts this amount to A.
- Of course, only net amount is actually paid at $t + dt$. Summarizing the above, fully collateralized asset generates continuous stream of payments

$$V(t + dt) - V(t)\,(1 + c(t)\, dt) = dV(t) - c(t)\, V(t)\, dt. \tag{9}$$

Cross-Currency Swaps

- Recall that in a cross-currency swap counterparties exchange payments in different currencies.
- To price a cross-currency basis swap (that is a swap with both legs paying floating amounts) one generally needs four zero curves, namely discounting and forwarding curves for each of the legs.
- Given these four curves, we price each leg separately using forwarding curve for forecasting future cashflows and then discounting them to the value date. We can then convert leg PVs to a single currency using a concept of FX SpotPV (3).
- Important question we have not answered so far: how do we discount non USD cashflows collateralized in USD?

Cross-Currency Framework With Collateralization

- We start by considering the economy under collaterization in domestic currency DOM at rate $c_d(t)$. This economy consists of
  - Domestic zero-coupon bond $P_{d,d}(t,T)$ collateralized at $c_d(t)$.
  - Foreign zero-coupon bond $P_{f,d}(t,T)$ denominated in currency FOR and collateralized at $c_d(t)$.
  - Spot* FX rate $X(t)$ representing a number of units of DOM per a unit of FOR.
- By (9), domestic zero-coupon bond generates the following cashflow over a period $[t, t+dt]$

$$dP_{d,d}(t,T) - c_d(t)\,P_{d,d}(t,T)\,dt.$$

- Next step is to identify a cashflow generated by trading $P_{f,d}(t,T)$.

*For simplicity, we assume there is no spot settlement lag.

Foreign Zero-Coupon Bond Under Domestic Collateral

- Consider a trade where party A buys from party B a foreign zero-coupon bond collateralized in DOM currency.
- The cashflows are
  - At t, A pays an amount of $P_{f,d}(t,T)$ in FOR currency to B.
  - B then must post a collateral to A. Since the bond is collateralized in DOM currency, the actual amount to be posted is $P_{f,d}(t,T)X(t)$.
  - At $t+dt$, A returns the collateral $P_{f,d}(t,T)X(t)$ to B and also pays the interest for an amount of $c_d(t)P_{f,d}(t,T)X(t)dt$.
  - The new mark-to-market price of the bond is $P_{f,d}(t+dt,T)$ so B posts $P_{f,d}(t+dt,T)X(t+dt)$ as new collateral to A.
- Therefore, from the viewpoint of A, the net cashflow in DOM currency at $t+dt$ is given by

$$d(P_{f,d}(t,T)X(t)) - c_d(t)P_{f,d}(t,T)X(t)dt.$$

Drift of FX Rate

- Our aim now is to identify a domestic cashflow we can generate by holding a unit of FOR currency.
- Having a unit of FOR currency, we can enter a repo trade. Namely
    - At t, borrow an amount of X(t) in DOM currency and post the unit of foreign currency as collateral.
    - At $t + dt$, get back the unit of FOR and pay back $X(t)$ plus the interest of $r_{d,f}(t)X(t)dt$, where $r_{d,f}(t)$ is a rate agreed on a domestic loan collateralized in foreign currency.
- This is in fact an instantaneous FX swap transaction.
- Given that we can sell a unit of FOR at $t + dt$ for $X(t + dt)$ units of DOM currency, the net cashflow at $t + dt$ is

$$dX(t) - r_{d,f}(t)X(t)dt.$$

Cross-Currency Model Under Domestic Collateral

- Let us summarize the instruments and the cashflow they generate over $[t, t+dt]$
  - Domestic bonds allows us to generate $dP_{d,d}(t,T) - c_d(t)\, P_{d,d}(t,T)\, dt$.
  - Foreign bonds yield $d(P_{f,d}(t,T)X(t)) - c_d(t)P_{f,d}(t,T)X(t)dt$.
  - Instantaneous FX swaps give us $dX(t) - r_{d,f}(t)X(t)dt$.
- By Girsanov theorem, we can switch to a "Domestic risk-neutral" measure $\mathbb{Q}^d$ under which the dynamics become

$$\begin{pmatrix} dX/X \\ dP_{d,d}/P_{d,d} \\ d(P_{f,d}X)/(P_{f,d}X) \end{pmatrix} = \begin{pmatrix} r_{d,f} \\ c_d \\ c_d \end{pmatrix} dt + \Sigma\, dW^d, \tag{10}$$

  where Σ is a 3×3 matrix and W^d is a standard 3-dimensional $\mathbb{Q}^d$-Wiener process.
- Note the analogy with the classical model: c_d plays a role of the risk-free rate r_d and $r_{d,f}$ is the analogue to $r_d - r_f$.

FX Forward Under Domestic Collateral

- Let us now consider FX forward paying $X(T) - K$ units of domestic currency at T.
- Under collateralization with domestic currency, the price process becomes

$$\mathrm{PV}_t = \mathbb{E}_t^{\mathbb{Q}^d}\left(\mathrm{e}^{-\int_t^T c_d(s)} \cdot (X(T) - K)\right). \tag{11}$$

- The third equation in (10) implies that

$$P_{f,d}(t,T)X(t) = \mathbb{E}_t^{\mathbb{Q}^d}\left(\mathrm{e}^{-\int_t^T c_d(s)} \cdot X(T)\right).$$

- So we can rewrite (11) as

$$\mathrm{PV}_t = X(t)P_{f,d}(t,T) - K\,P_{d,d}(t,T).$$

- Therefore, FX forward par strike under domestic collateral is given by

$$K_d = X(t) \cdot \frac{P_{f,d}(t,T)}{P_{d,d}(t,T)}. \tag{12}$$

- This is the analogue of formula (2) under collateralization with c_d.

Cross-Currency Model Under Foreign Collateral

- Repeating the arguments above, we can switch to a "Foreign risk-neutral" measure $\mathbb{Q}^f$ where

$$\begin{pmatrix} d(1/X)/(1/X) \\ dP_{f,f}/P_{f,f} \\ d(P_{d,f}/X)/(P_{d,f}/X) \end{pmatrix} = \begin{pmatrix} -r_{d,f} \\ c_f \\ c_f \end{pmatrix} dt + \hat{\Sigma} \cdot dW^f. \tag{13}$$

- Note that we can view an FX forward as a contract paying $1 - \frac{K}{X_T}$ in FOR.
- Under collateralization if FOR, the value of the forward becomes

$$\begin{aligned} \mathrm{PV}_t &= \mathbb{E}_t^{\mathbb{Q}^d}\left(\mathrm{e}^{-\int_t^T c_f(s)} \cdot \left(1 - \frac{K}{X(T)}\right)\right). \\ &= P_{f,f}(t,T) - K\, P_{d,f}(t,T)/X(t). \end{aligned}$$

- Therefore, FX forward par strike under collateralization in FOR is given by

$$K_f = X(t) \cdot \frac{P_{f,f}(t,T)}{P_{d,f}(t,T)}. \tag{14}$$

Independence of FX Forward From Collateral Currency

- FX forward rate remains unchanged when switching collateral currency from DOM to FOR.
- Therefore

$$\frac{P_{f,d}(t,T)}{P_{d,d}(t,T)} = \frac{P_{f,f}(t,T)}{P_{d,f}(t,T)}. \tag{15}$$

- This is a very important fact as it allows us to imply the discounting curve under collateralization in foreign currency in case if there are no market instruments to calibrate this curve to.
- The procedure is called the curve triangulation.

Cross-Currency Swaps Under Collateralization

- Consider, for instance, a USDRUB cross-currency swap where the USD leg pays compounded SOFR rate while the RUB leg pays fix amounts. Liquid cross-currency swaps are collateralized in USD so let us first assume that this swap is collateralized with USD SOFR.

- To value the RUB leg, we need the RUB discounting curve under the collateralization in USD. In our notations above this is $P_{d,f}(t,T), T \geq t$.

- The good thing is that we can calibrate $P_{d,f}(t,T)$ to market quotes of liquid USDRUB cross-currency swaps and then price the RUB leg of the considered swap.

- We usually refer to the curve $P_{d,f}$ as the offshore curve. In our example, it is RUB offshore USD curve.

Curve Triangulation

- How do we price USDRUB swap collateralized with RUB Ruonia?
- To value the USD leg, we now need the foreign discounting curve under RUB collateralization $P_{f,d}(t,T), T \geq t$. There is no liquidity in trading USDRUB swaps collateralized in RUB so we cannot simply imply this curve from the quotes of market instruments.
- This is where formula (15) comes into the play. We first build the curve $P_{d,f}(t,T), T \geq t$ from the quotes of liquid swaps (collateralized in USD) and then imply $P_{f,d}(t,T)$ by

$$P_{f,d}(t,T) = P_{f,f}(t,T) \cdot \frac{P_{d,d}(t,T)}{P_{d,f}(t,T)}.$$

- Now, we can value the USD leg of the swap collateralized in RUB by discounting USD cashflows with $P_{f,d}(t,T)$.