`vol` package

Mathematical formulas

Contents

1. Models based on a geometric Brownian motion

1.1. Black–Scholes model

The base asset in the model is stock which under the pricing measure has the dynamics

$$dS_t = rS_t dt + \sigma S_t dW_t, \qquad S_0 = s > 0,$$

where $r \in \mathbb{R}$ is the risk-free interest rate and $\sigma > 0$ is the volatility. We assume zero dividend rate.

Call option price (Black and Scholes, 1973). The price at time $t = 0$ of a call option with expiration time T and strike K:

$$C := e^{-rT}\,\mathrm{E}(S_T - K)^+ = s\Phi(d_1) - e^{-rT} K\Phi(d_2),$$

where

$$d_1 = \frac{1}{\sigma\sqrt{T}}\left(\ln\frac{s}{K} + \left(r + \frac{\sigma^2}{2}\right)T\right), \quad d_2 = d_1 - \sigma\sqrt{T} = \frac{1}{\sigma\sqrt{T}}\left(\ln\frac{s}{K} + \left(r - \frac{\sigma^2}{2}\right)T\right).$$

Put–call parity. The prices of a call option and a put option with the same strike and expiration time satisfy the relation

$$C - P = s - e^{-rT}K.$$

Greeks. Some common Greeks of a call option:

$$\begin{aligned}
\Delta &:= \frac{\partial C}{\partial s} = \Phi(d_1),\\
\mathcal{V} &:= \frac{\partial S}{\partial \sigma} = s\varphi(d_1)\sqrt{T},\\
\Theta &:= -\frac{\partial C}{\partial T} = -\frac{s\varphi(d_1)\sigma}{2\sqrt{T}} - rKe^{-rT}\Phi(d_2),\\
\Gamma &:= \frac{\partial^2 C}{\partial s^2} = \frac{\varphi(d_1)}{s\sigma\sqrt{T}}.
\end{aligned}$$

Implied volatility approximation. Brenner and Subrahmanyam (1988) approximation of implied volatility of a call options with strike $K \approx se^{rT}$:

$$\widehat{\sigma} \approx \frac{C}{s}\sqrt{\frac{2\pi}{T}} \approx \frac{2.5C}{s\sqrt{T}}.$$

It is obtained by setting $K = se^{rT}$, $\Phi(x) = \frac{1}{2} + \frac{1}{\sqrt{2\pi}}x$ in the Black–Scholes formula.

A more accurate approximation by Corrado and Miller (1996):

$$\widehat{\sigma} \approx \frac{\sqrt{2\pi}}{\sqrt{T}(s + e^{-rT}K)}\left(C - \frac{s - e^{-rT}K}{2} + \sqrt{\left(\left(C - \frac{s - e^{-rT}K}{2}\right)^2 - \frac{(s - e^{-rT}K)^2}{\pi}\right)^+}\right).$$

1.2. Black model

The base asset in the model is a forward or futures contract which under the pricing measure has the dynamics

$$dF_t = \sigma F_t dW_t, \qquad F_0 = f > 0,$$

where $\sigma > 0$ is the volatility.

In what follows, $r \in \mathbb{R}$ denotes the risk-free interest rate, T denotes the expiration time of an option, and T' denotes the expiration (delivery) time of a forward contract. For options on futures contracts, the expiration time of a futures contract does not enter the formulas.

Call option price (Black, 1976). The price of a call option with time to expiration T and strike K on a futures contract:

$$C_{\text{fut}} := e^{-rT}\,\mathrm{E}(F_T - K)^+ = e^{-rT}(f\Phi(d_1) - K\Phi(d_2)).$$

The price of a call option on a forward contract:

$$C_{\text{for}} := e^{-rT'}\,\mathrm{E}(F_T - K)^+ = e^{-rT'}(f\Phi(d_1) - K\Phi(d_2))$$

(the difference is in the discounting factor). In both formulas,

$$d_1 = \frac{1}{\sigma\sqrt{T}}\left(\ln\frac{f}{K} + \frac{\sigma^2}{2}T\right), \qquad d_2 = d_1 - \sigma\sqrt{T} = \frac{1}{\sigma\sqrt{T}}\left(\ln\frac{f}{K} - \frac{\sigma^2}{2}T\right).$$

Put–call parity. The prices of a call option and a put option with the same strike and expiration time satisfy the relations

$$C_{\text{fut}} - P_{\text{fut}} = e^{-rT}(f - K), \qquad C_{\text{for}} - P_{\text{for}} = e^{-rT'}(f - K).$$

Greeks. Some common Greeks of a call option:

$$\begin{aligned}
\Delta_{\text{fut}} &= e^{-rT}\Phi(d_1), & \Delta_{\text{for}} &= e^{-rT'}\Phi(d_1),\\
\mathcal{V}_{\text{fut}} &= e^{-rT} f\varphi(d_1)\sqrt{T}, & \mathcal{V}_{\text{for}} &= e^{-rT'} f\varphi(d_1)\sqrt{T},\\
\Theta_{\text{fut}} &= rC_{\text{fut}} - e^{-rT}\frac{f\varphi(d_1)\sigma}{2\sqrt{T}}, & \Theta_{\text{for}} &= -e^{-rT'}\frac{f\varphi(d_1)\sigma}{2\sqrt{T}},\\
\Gamma_{\text{fut}} &= e^{-rT}\frac{\varphi(d_1)}{f\sigma\sqrt{T}}, & \Gamma_{\text{for}} &= e^{-rT'}\frac{\varphi(d_1)}{f\sigma\sqrt{T}},
\end{aligned}$$

where $\Delta = \partial C/\partial f$, $\mathcal{V} = \partial C/\partial\sigma$, $\Theta = -\partial C/\partial T$, $\Gamma = \partial^2 C/\partial f^2$.

Implied volatility approximation. Similarly to Brenner and Subrahmanyam (1988), for a call option with strike $K \approx f$ we have

$$\widehat{\sigma} \approx e^{rT}\frac{C_{\text{fut}}}{f}\sqrt{\frac{2\pi}{T}} \approx e^{rT}\frac{2.5C}{f\sqrt{T}}, \qquad \widehat{\sigma} \approx e^{rT'}\frac{C_{\text{for}}}{f}\sqrt{\frac{2\pi}{T}} \approx e^{rT'}\frac{2.5C}{f\sqrt{T}}.$$

A remark on reduction of the Black model to the Black–Scholes model. When programming these models, it is be desirable to have a single set of functions (or a class) instead of two separate ones for the Black-Scholes and Black models. The formulas below show how computation in the Black model can be reduced to the Black–Scholes model.

Let $C^{(0)}$, $\Delta^{(0)}$, etc. denote the option price and Greeks in the Black–Scholes model with zero interest rate and initial price s being equal to the forward/futures price f; the volatility σ being the same as in the Black model.

Consider an option in the Black model with interest rate $r \in \mathbb{R}$ and let τ be the time to delivery of the base asset, which is $\tau = T$, the option's expiration time, if the base asset is a futures contract[1], or $\tau = T' \geq T$ if the base asset is a forward contract.

Then we have the following formulas:

$$\begin{aligned}
C &= e^{-r\tau} C^{(0)}, \\
\Delta &= e^{-r\tau} \Delta^{(0)}, \\
\mathcal{V} &= e^{-r\tau} \mathcal{V}^{(0)}, \\
\Theta &= \begin{cases} e^{-r\tau}\Theta^{(0)} & \text{for a forward contract,} \\ e^{-r\tau}\Theta^{(0)} + rC & \text{for a futures contract,} \end{cases} \\
\Gamma &= e^{-r\tau}\Gamma^{(0)}, \\
\widehat{\sigma} &= \widehat{\sigma}^{(0)}.
\end{aligned}$$

2. CEV (Constant Elasticity of Variance) model

(A concise exposition of the theory below can be found in Linetsky and Mendoza (2010)) The base asset has the following dynamics under the pricing measure:

$$dS_t = rS_t dt + \sigma S_t^\beta dW_t, \qquad S_0 = s > 0,$$

where $r \in \mathbb{R}_+$ is the risk-free interest rate, and $\sigma > 0$, $\beta \geq 0$ are parameters of the model. In what follows, assume that $\beta \neq 1$ (otherwise we have the Black–Scholes model). For convenience, denote

$$\nu = \frac{1}{2(\beta - 1)}.$$

If $\beta \in [0, 1)$, the process may reach zero (see below). We will assume that the process remains at zero forever after reaching it, e.g. the company bankrupts.

The case $r \neq 0$ can be reduced to $r = 0$ by the following transformation:

$$S_t^{(r)} \stackrel{\mathrm{d}}{=} e^{rt} S_{\tau(t)}^{(0)}, \qquad \text{where } \tau(t) = \frac{e^{2r(\beta-1)t} - 1}{2r(\beta - 1)}. \tag{1}$$

Another useful fact is that if $r = 0$, then the process

$$X_t = \frac{S_t^{2(1-\beta)}}{\sigma^2(1-\beta)^2}. \tag{2}$$

is a squared Bessel process of dimension $\delta = \frac{1-2\beta}{1-\beta}$, i.e. satisfies the SDE

$$dX_t = \delta dt + 2\sqrt{X_t} dW_t.$$

[1]Because futures contract are marked to market, effectively the base asset is delivered when the option is exercised.

Path and distributional properties. The following properties can be derived from the known transition distribution of the squared Bessel process (see section "Simulation" below).

Case $\beta \in [0, 1)$.

- The process S_t reaches zero in finite time with positive probability, and the distribution of the reaching time τ is

$$\mathrm{P}(\tau \le t) = G\left(|\nu|,\ \frac{rs^{2(1-\beta)}}{\sigma^2(1-\beta)(1-e^{2r(\beta-1)t})}\right),$$

  where $G(\nu, x) = \frac{1}{\Gamma(\nu)}\int_x^\infty u^{\nu-1}e^{-u}du$ is the complementary gamma distribution function (use `scipy.special.gammaincc(`ν`, `x`)`). For $r = 0$, the second argument of G should be understood in the limit $r \to 0$, i.e. we have

$$\mathrm{P}(\tau \le t) = G\left(|\nu|,\ \frac{s^{2(1-\beta)}}{2\sigma^2(1-\beta)^2 t}\right) \qquad (\text{if } r = 0).$$

- The density of the absolutely continuous part of the transition distribution) is given by (for $x > 0$)

$$\mathrm{P}(S_t^{(0)} \in dx \mid S_0 = s) = \frac{x^{-2\beta+\frac{1}{2}}\sqrt{s}}{\sigma^2|\beta-1|t} I_{|\nu|}\left(\frac{(sx)^{1-\beta}}{\sigma^2(\beta-1)^2 t}\right) \exp\left(-\frac{s^{2-2\beta}+2-2\beta}{2\sigma^2(\beta-1)^2 t}\right), \quad (3)$$

  where $I_\nu(z)$ is the modified Bessel function of the first kind of order ν (use `scipy.special.iv(`ν`, `z`)`). Note that the integral of this density over $\mathbb{R}_+$ is less than 1, and the remaining probability is the mass at zero.
- The process S_t is a martingale.

Case $\beta > 1$.

- The process never reaches zero.
- The transition density is given by the same formula (3), but now it integrates to 1.
- The process S_t is a strict local martingale and

$$\mathrm{E}\, S_t^{(r)} = e^{rt}s\left(1 - G\left(\nu,\ \frac{rs^{2-2\beta}}{\sigma^2(\beta-1)(e^{2r(\beta-1)t}-1)}\right)\right) \qquad \text{if } r > 0,$$
$$\mathrm{E}\, S_t^{(0)} = s\left(1 - G\left(\nu,\ \frac{s^{2-2\beta}}{2\sigma^2(\beta-1)^2 t}\right)\right).$$

Call option price. Let $C(T, K) = e^{-rT}\,\mathrm{E}(S_T - K)^+$. Then we have

$$\text{for } \beta \in (0,1): \quad C(T,K) = sQ(y; 2(1+|\nu|), \xi) - e^{-rT}KF(\xi; 2|\nu|, y),$$
$$\text{for } \beta > 1: \quad C(T,K) = sQ(\xi; 2\nu, y) - e^{-rT}KF(y; 2(1+\nu), \xi),$$

where $F(z; d, \lambda)$ and $Q(z; d, \lambda) = 1 - F(z; d, \lambda)$ are the distribution and survival functions of the non-central chi-square distribution with d degrees of freedom and non-centrality parameter λ (use `scipy.stats.ncx2(`d`, `λ`)`), and

$$\xi = \frac{2rs^{2(1-\beta)}}{\sigma^2(1-\beta)(1-e^{2r(\beta-1)T})}, \qquad y = \frac{2rK^{2(1-\beta)}}{\sigma^2(1-\beta)(e^{2r(1-\beta)T}-1)}.$$

For $r = 0$ we obtain ξ and y by passing to the limit $r \to 0$:

$$\xi = \frac{s^{2(1-\beta)}}{\sigma^2(1-\beta)^2 T}, \qquad y = \frac{K^{2(1-\beta)}}{\sigma^2(1-\beta)^2 T} \qquad (\text{if } r = 0).$$

Note that thanks to formula (1), we can reduce pricing of options with arbitrary $r > 0$ to the case $r = 0$:

$$C^{(r)}(T, K) = C^{(0)}(T', K'), \qquad T' = \tau(T) = \frac{e^{2r(\beta-1)T} - 1}{2r(\beta - 1)}, \qquad K' = e^{-rT}K. \tag{4}$$

Approximation of implied volatility. The Black–Scholes implied volatility produced by the CEV model can be approximated by the formula of Hagan and Woodward (1999). If $r = 0$, then

$$\widehat{\sigma}(T, K) = \frac{\sigma}{\widetilde{s}^{1-\beta}}\left(1 + \frac{(1-\beta)(2+\beta)}{24}\left(\frac{s-K}{\widetilde{s}}\right)^2 + \frac{(1-\beta)^2}{24}\frac{\sigma^2 T}{\widetilde{s}^{2(1-\beta)}} + \dots\right), \tag{5}$$

where $\widetilde{s} = \frac{1}{2}(s + K)$.

If $r > 0$, then with $T' = \tau(T)$, $K' = e^{-rT}K$ (as in (4)), we have

$$\widehat{\sigma}^{(r)}(T, K) = \widehat{\sigma}^{(0)}(T', K')\sqrt{\frac{T'}{T}}.$$

Simulation. Simulation can be done from the exact transition density. It is easier to simulate the squared Bessel process X_t from (2), then get $S^{(0)}$ and finally transform to $S_t^{(r)}$:

$$S_t^{(r)} = e^{rt}(\sigma^2(1-\beta)^2 X_{\tau(t)})^{\frac{1}{2(1-\beta)}}.$$

To simulate X_t, we use the following transition density:

$$\text{for } \beta > 1: \quad \mathrm{P}(X_{t+\Delta t} \le x \mid X_t = y) = \mathrm{P}\left(\chi'^2\left(\delta, \frac{y}{\Delta t}\right) \le \frac{x}{\Delta t}\right)$$
$$\text{for } \beta < 1: \quad \mathrm{P}(X_{t+\Delta t} \le x \mid X_t = y) = \mathrm{P}\left(\chi'^2\left(2-\delta, \frac{x}{\Delta t}\right) \ge \frac{y}{\Delta t}\right),$$

where $\chi'^2(d, \lambda)$ is the non-central chi-square distribution with d degrees of freedom and non-centrality parameter λ. In Python, if $\beta > 1$, it is convenient to use `scipy.stat.ncx2(df=`δ`, nc=`$\frac{x}{\Delta t}$`, scale=`Δt`).rvs(...)`. If $\beta < 1$, we have to simulate a uniform random variable and invert the distribution function.

3. Local volatility

Assume the base asset is stock. Let $S_0 = s$, and r be the (constant) risk-free interest rate. Denote by $\widehat{C}(T, K)$ the prices of call options observed in the market, which are assumed to be available for all $T \in [0, T_{\max}]$ and $K > 0$.

By *local volatility* we call a function $\sigma(t, s)$ such that the option prices $C(T, K) = e^{-rT}\,\mathrm{E}(S_T - K)^+$ produced by the model

$$dS_t = rS_t dt + \sigma(t, S_t)S_t dW_t, \qquad S_0 = s, \tag{6}$$

coincide with the market prices, i.e. $C(T, K) = \widehat{C}(T, K)$ for all $T \in [0, T_{\max}]$, $K > 0$. In what follows, assume that such $\sigma(t, s)$ exists, and equation (6) has a unique solution such that $e^{-rt}S_t$ is a martingale.

In a similar way, if the base asset is a futures of forward contract, by local volatility we call a function $\sigma(t,f)$ such that the model

$$dF_t = \sigma(t, F_t)F_t dW_t, \qquad F_0 = f,$$

produces option prices equal to the market prices.

Dupire's formula (Dupire, 1994). The function $\sigma(t,s)$ can be found from the equation

$$\sigma^2(t,s) = \frac{2C'_T(t,s) + rsC'_K(t,s)}{s^2 C''_{KK}(t,s)}.$$

If the base asset is a forward contract, we have the same formula as for $r = 0$, i.e.

$$\sigma^2_{\text{for}}(t,f) = \frac{2C'_T(t,f)}{f^2 C''_{KK}(t,f)}.$$

If the base asset is a futures contract, then

$$\sigma^2_{\text{fut}}(t,f) = \frac{2(C'_T(t,f) + rC(t,f))}{f^2 C''_{KK}(t,f)}.$$

In practical applications, one need to use market prices $\widehat{C}$ in place of C and find the derivatives of the function $\widehat{C}(T,K)$ numerically (perhaps, first interpolating the price surface with some smooth or piecewise smooth function).
Remark 1. It is necessary to impose some technical conditions for the validity of Dupire's formula. For example it is sufficient to require that the solution S_t of (6) has a continuous density $f(t,s)$ for each $t \in (0, T_{\max}]$ and the function $C(T,K)$ is in the class $C^{1,2}$.

Approximation of implied volatility. Assume the interest rate is zero and the base asset price has the dynamics

$$dS_t = A(S_t)dW_t, \qquad S_0 = s > 0.$$

Then the Black–Scholes implied volatility produced by this model can be approximated by the following formula (Hagan and Woodward (1999); formula (5) for the CEV model is a particular case of it):

$$\begin{aligned}\widehat{\sigma}(T,K) = \frac{A(\widetilde{s})}{\widetilde{s}}\biggl\{1 + \frac{1}{24}\biggl[\frac{A''(\widetilde{s})}{A(\widetilde{s})} - 2\biggl(\frac{A'(\widetilde{s})}{A(\widetilde{s})}\biggr)^2 + \frac{2}{\widetilde{s}^2}\biggr](s-K)^2 \\ + \frac{1}{24}\biggl[2\frac{A''(\widetilde{s})}{A(\widetilde{s})} - \biggl(\frac{A'(\widetilde{s})}{A(\widetilde{s})}\biggr)^2 + \frac{1}{\widetilde{s}^2}\biggr]A^2(\widetilde{s})T + \dots\biggr\}\end{aligned}$$

where $\widetilde{s} = \frac{1}{2}(s+K)$.

4. Heston model

The base asset has the following dynamics under the pricing measure:

$$\begin{aligned}&dS_t = rS_t dt + \sqrt{V_t}dW^1_t, \qquad S_0 = s > 0,\\ &dV_t = \kappa(\theta - V_t)dt + \sigma\sqrt{V_t}dW^2_t, \qquad V_0 = v > 0,\\ &dW^1_t dW^2_t = \rho dt,\end{aligned}$$

where $r \in \mathbb{R}$ is the risk-free interest rate, and $\kappa, \theta, \sigma > 0$, $\rho \in (-1, 1)$ are parameters.

The variance process V_t is strictly positive for all $t \geq 0$ if Feller's condition hold:

$$2\kappa\theta \geq \sigma^2;$$

otherwise it is non-negative and reflects at zero.

4.1. Heston's and Lewis' semi-closed formulas for option prices

Heston's formula (Heston, 1993). The price of a call option with expiration time T and strike K:

$$C = \frac{s - e^{-rTK}}{2} + \frac{1}{\pi}\int_0^\pi \mathrm{Re}\left(\frac{e^{-u\ln K} s\widetilde{\varphi}(u) - e^{-rT}K\varphi(u)}{iu}\right) du,$$

where $\varphi(u) = \mathrm{E}\, e^{iu\ln X_T}$ is the characteristic function of the log-price $X_T = \ln S_T$ at expiration time under the martingale measure, and $\widetilde{\varphi}(u)$ is the same characteristic function but under the measure such that e^{rt}/S_t is a martingale. These characteristic functions are related as follows:

$$\widetilde{\varphi}(u) = \frac{\varphi(u-i)}{se^{rT}}.$$

The function $\varphi(u)$ is given by the formula (with $i = \sqrt{-1}$ everywhere)

$$\varphi(u) = \exp(C(u) + D(u)v + iu\ln s),$$

where

$$C(u) = irTu + \frac{\kappa\theta}{\sigma^2}\left((\kappa - i\rho\sigma u - d(u))T - 2\ln\left(\frac{1 - g(u)e^{-d(u)T}}{1 - g(u)}\right)\right),$$

$$D(u) = \frac{\kappa - i\rho\sigma u - d(u)}{\sigma^2}\left(\frac{1 - e^{-d(u)T}}{1 - g(u)e^{-d(u)T}}\right),$$

and

$$d(u) = \sqrt{(i\rho\sigma u - \kappa)^2 + \sigma^2(iu + u^2)}, \qquad g(u) = \frac{i\rho\sigma u - \kappa + d(u)}{i\rho\sigma u - \kappa - d(u)}.$$

Remark 2. The above formulas are obtained by choosing the good (stable) solution of the Riccati equation arising in the derivation of Heston's formula, see Albrecher et al. (2007).

Lewis' formula (Lewis, 2000). The price of a call option with expiration time T and strike K:

$$C = s - \frac{Ke^{-rT}}{2\pi}\int_{-\infty+i/2}^{+\infty+i/2} e^{-ixu}\frac{\widehat{H}(u)}{u^2 - iu} du,$$

where $x = \ln(s/K) + rT$ and $\widehat{H}(u)$ is the *fundamental transform* for the Heston model, which is given by

$$\widehat{H}(u) = \exp(f_1(u) + vf_2(u)),$$

where

$$f_1(u) = \frac{2\kappa\theta}{\sigma^2}\left[qg(u) - \ln\left(\frac{1-h(u)e^{-q\xi(u)}}{1-h(u)}\right)\right],$$
$$f_2(u) = \left(\frac{1-e^{-q\xi(u)}}{1-h(u)e^{-q\xi(u)}}\right)g(u),$$

and

$$g(u) = \frac{b(u)-\xi(u)}{2}, \quad h(u) = \frac{b(u)-\xi(u)}{b(u)+\xi(u)}, \quad q = \frac{\sigma^2 T}{2},$$
$$\xi(u) = \sqrt{b(u)^2 + \frac{4(u^2-iu)}{\sigma^2}}, \qquad b(u) = \frac{2(i\rho\sigma u+\kappa)}{\sigma^2}.$$

Remark 3. The above formulas are from Baustian et al. (2017, Ch. 2). They can be also found in Lewis (2000), but in a somewhat less compact form.

4.2. Simulation methods

Suppose we need to simulate the values of the processes (S_t, V_t) at points $t_i = i\Delta t$, $i = 0, \dots, n$, where $\Delta t > 0$ is a time step.

Euler's scheme. In order to avoid negative values of the variance process, standard Euler's scheme can be modified as follows: denote $X_t = \ln S_t$ and simulate

$$X_{t_0} = \ln s, \quad V_{t_0} = v,$$
$$X_{t_{i+1}} = X_{t_i} + \left(r - \frac{V_{t_i}^+}{2}\right)\Delta t + \sqrt{V_{t_i}^+}\left(\rho Z_{i+1} + \sqrt{1-\rho^2}Z'_{i+1}\right)\sqrt{\Delta t},$$
$$V_{t_{t+i}} = V_{t_i} + \kappa(\theta - V_{t_i}^+)\Delta t + \sigma\sqrt{V_{t_i}^+}Z'_{i+1}\sqrt{\Delta t},$$

where Z_i and Z'_i are independent sequences of i.i.d. standard normal variables. Then we recover $S_{t_i} = \exp(X_{t_i})$,

Exact scheme (Broadie and Kaya, 2006). The idea of this scheme is based on the representation

$$S_{t_{i+1}} = S_{t_i}\exp\left(r\Delta t - \frac{1}{2}\int_{t_i}^{t_{i+1}} V_s ds + \rho\int_{t_i}^{t_{i+1}}\sqrt{V_s}dW_s^1 + \sqrt{1-\rho^2}\int_{t_i}^{t_{i+1}}\sqrt{V_s}dW_s^2\right), \tag{7}$$

$$V_{t_{i+1}} = V_{t_i} + \kappa\theta\Delta t - \kappa\int_{t_i}^{t_{i+1}} V_s ds + \sigma\int_{t_i}^{t_{i+1}}\sqrt{V_s}dW_s^1, \tag{8}$$

where W_t^1 and W_t^2 are independent Brownian motions. To pass from t_i to t_{i+1}, perform the following steps:

1. simulate $V_{t_{i+1}}$ given the value of V_{t_i},
2. simulate $I_{i+1} := \int_{t_i}^{t_{i+1}} V_s ds$ given the values of V_{t_i} and $V_{t_{i+1}}$,
3. simulate $J_{i+1}^1 := \int_{t_i}^{t_{i+1}}\sqrt{V_s}dW_s^1$ and $J_{i+1}^2 := \int_{t_i}^{t_{i+1}}\sqrt{V_s}dW_s^2$,
4. express $S_{t_{i+1}}$ through the simulated variables.

Step 1. It is known that

$$\mathrm{Law}(V_{t_{i+1}} \mid V_{t_i}) = \frac{\sigma^2(1-e^{-\kappa\Delta t})}{4\kappa}\chi_d'^2\left(\frac{4\kappa e^{-\kappa\Delta t}}{\sigma^2(1-e^{-\kappa\Delta t})}V_{t_i}\right), \qquad d = \frac{4\theta\kappa}{\sigma^2},$$

where $\chi_d'^2(\lambda)$ is the non-central chi-square distribution with d degrees of freedom and non-centrality parameter λ. Hence, we can sample from this distribution to get $V_{t_{i+1}}$.

Step 2 (the most difficult). Let $F(x)$ denote the conditional distribution of I_{i+1} (yet to be found):

$$F(x) = \mathrm{P}(I_{i+1} \le x \mid V_{t_i}, V_{t_{i+1}}).$$

We can simulate U_{i+1} from a sequence of i.i.d. uniform random variables and numerically solve the equation $F(x) = U_{i+1}$. Then let $I_{i+1} = x^*$ for the solution (i.e. use Smirnov's transform).

To compute $F(x)$, we invert the conditional characteristic function, which is known is a closed form. Namely, let

$$\gamma(u) = \sqrt{\kappa^2 - 2\sigma^2 iu}, \qquad c_1 = e^{-\kappa\Delta t}, \qquad c_2(u) = e^{-\gamma(u)\Delta t}.$$

Then

$$\begin{aligned}\varphi(u) := \mathrm{E}(e^{iuI_{i+1}} \mid V_{t_i}, V_{t_{i+1}}) = {} & \frac{\gamma(u)\sqrt{c_2(u)/c_1}(1-c_1)}{\kappa(1-c_2(u))} \\ & \times \exp\left(\frac{V_{t_{i+1}}+V_{t_i}}{\sigma^2}\left[\frac{\kappa(1+c_1)}{1-c_1} - \frac{\gamma(u)(1+c_2(u))}{1-c_2(u)}\right]\right) \\ & \times \frac{I_{0.5d-1}\left(\sqrt{V_{t_i}V_{t_{i+1}}c_2(u)}\frac{4\gamma(u)}{\sigma^2(1-c_2(u))}\right)}{I_{0.5d-1}\left(\sqrt{V_{t_i}V_{t_{i+1}}c_1}\frac{4\kappa}{\sigma^2(1-c_1)}\right)},\end{aligned}$$

where $d = 4\theta\kappa/\sigma^2$ is as above, and $I_\nu(x)$ is the modified Bessel function of the first kind (use `scipy.special.iv(`ν`, `x`)`).

The inversion procedure is based on the formula

$$F(x) = \frac{2}{\pi}\int_0^\infty \frac{\sin(ux)}{u}\mathrm{Re}(\varphi(u))du.$$

To compute the integral (note: `scipy.integrate.quad` works pretty bad here), use the approximation

$$F(x) \approx \frac{hx}{\pi} + \frac{2}{\pi}\sum_{j=1}^{N}\frac{\sin(hjx)}{j}\mathrm{Re}(\varphi(hj)).$$

In order to guarantee precision ε, the parameters $h > 0$ and $N \in \mathbb{N}$ should be chosen such that

$$\begin{aligned} h &\ge \frac{\pi}{u_\varepsilon}, \qquad \text{where } 1 - F(u_\varepsilon) = \varepsilon, \\ \frac{|\varphi(hN)|}{N} &< \frac{\pi\varepsilon}{2}. \end{aligned}$$

The second condition here poses no problem when the summation in the integral approximation is performed in a loop – we stop at first j such that $N = j$ satisfies the second condition (and, additionally, to avoid a huge loop, stop when j becomes quite large, e.g. $j = 1000$). The first condition is tough, Broadie and Kaya suggest to find u_ε simply large enough, for example $u_\varepsilon = m + 5s$, where m, s are the mean and standard deviation of the distribution F, which can be found by numerical differentiation of the characteristic function (use `scipy.misc.derivative`). That is, we put

$$h = \frac{\pi}{m + 5s}, \qquad m = \frac{\varphi'(0)}{i}, \qquad s = \sqrt{-\varphi''(0)}.$$

Step 3. Put

$$\begin{aligned} J^1_{i+1} &= \frac{1}{\sigma}(V_{t_{i+1}} - V_{t_i} - \kappa\theta\Delta t + \kappa I_{i+1}), \\ J^2_{i+1} &= \sqrt{I_{i+1}} Z_{i+1}, \end{aligned}$$

where Z_{i+1} is a sequence of i.i.d. standard normal variables independent of V_t. Note that the formula for J^2_{i+1} follows from the observation that W^2_t is independent of V_t, hence J_{i+1} has the normal distribution with variance $\int_{t_i}^{y_{i+1}} V_s ds = I_{i+1}$.

Step 4. Finally, put

$$S_{t_{i+1}} = S_{t_i} \exp\left(r\Delta t - \frac{1}{2} I_{i+1} + \rho J^1_{i+1} + \sqrt{1-\rho^2} J^2_{i+1} \right).$$

QE scheme (Andersen, 2008). (QE means "quadratic-exponential") This scheme has the same Steps 3–4 as in the exact scheme, but modifies Steps 1–2 as follows.

Step 1. We simulate $V_{t_{i+1}}$ from an approximation of its conditional distribution given the value V_{t_i}. Define

$$\begin{aligned} m &= \theta + (V_{t_i} - \theta) e^{-\kappa\Delta t}, \\ s^2 &= \frac{V_{t_i} \sigma^2 e^{-\kappa\Delta t}}{\kappa}(1 - e^{-\kappa\Delta t}) + \frac{\theta\sigma^2}{2\kappa}(1 - e^{-\kappa\Delta t})^2 \\ \psi &= \frac{s^2}{m^2}, \\ b^2 &= \frac{2}{\psi} - 1 + \sqrt{4 - 2\psi}, \\ a &= \frac{m}{a + b^2}, \\ p &= \frac{\psi - 1}{\psi + 1}, \\ \beta &= \frac{1-p}{m}, \end{aligned}$$

where actually $m = \mathrm{E}(V_{t_{i+1}} \mid V_{t_i})$, $s^2 = \mathrm{Var}(V_{t_{i+1}} \mid V_{t_i})$. Then consider two cases.

1. If $\psi \le \frac{3}{2}$, simulate
$$V_{t_{i+1}} = a(b + Z_{i+1})^2,$$
where Z_i is a sequence of i.i.d. standard normal variables.
2. If $\psi > \frac{3}{2}$, simulate $V_{t_{i+1}}$ from the exponential distribution with mass at zero:

$$\mathrm{P}(V_{t_{i+1}} = 0) = p, \qquad \mathrm{P}(V_{t_{i+1}} \in dx) = \beta(1-p)e^{-\beta x} \text{ for } x > 0.$$

This case be implemented by defining

$$V_{t_{i+1}} = \begin{cases} 0, & \text{if } U_{i+1} \le p, \\ \frac{1}{\beta} \ln \frac{1-p}{1-U_{i+1}}, & \text{if } U_{i+1} > p. \end{cases}$$

Here U_i is a sequence of i.i.d. uniform random variables on $[0,1]$.

Remark 4. The idea of the above procedure is that if V_{t_i} is "large", then its conditional distribution can be approximated by a squared normal distribution, while if V_{t_i} is "small", it is better to approximate $V_{t_{i+1}}$ by an exponential distribution with mass at zero. The parameters a, b, p, β are chosen to match the conditional mean and variance of the approximation with the true values. It turns out that such a match is possible if $\psi \le 2$ for the squared normal approximation and $\psi \ge 1$ for the exponential-with-mass approximation. Hence we use the threshold $\psi = \frac{3}{2}$ in the above method (but, in principle, any value in $[1,2]$ will work).

Step 2. We simply put

$$I_{i+1} = \frac{1}{2}(V_{t_i} + V_{t_{i+1}}).$$

Steps 3 and 4. The formulas here are the same as in Broadie–Kaya's scheme, but note that the final formula for $S_{t_{i+1}}$ can be explicitly written as follows:

$$S_{t_{i+1}} = S_{t_i} \exp\left(r\Delta t + K_0 + K_1 V_{t_i} + K_2 V_{t_{i+1}} + \sqrt{K_3(V_{t_i} + V_{t_{i+1}})} Z'_{i+1} \right),$$

where Z'_i is a sequence of i.i.d. standard normal variables (independent of Z_i and U_i), and K_i are constants:

$$\begin{aligned} K_0 &= -\frac{\rho\kappa\theta}{\sigma}\Delta t, \\ K_1 &= \frac{1}{2}\left(\frac{\kappa\rho}{\sigma} - \frac{1}{2}\right)\Delta t - \frac{\rho}{\sigma}, \\ K_2 &= \frac{1}{2}\left(\frac{\kappa\rho}{\sigma} - \frac{1}{2}\right)\Delta t + \frac{\rho}{\sigma}, \\ K_3 &= \frac{1}{2}(1-\rho^2)\Delta t. \end{aligned}$$

E+M scheme (Mrázek and Pospíšil, 2017). (E+M means "exact + Milstein".) This scheme uses the same Steps 2–4 as the QE scheme, but in Step 1, to simulate the process V_t, it uses the Milstein scheme as follows:

$$V_{t_{i+1}} = V_{t_i} + \kappa(\theta - V_{t_i}^+)\Delta t + \sigma\sqrt{V_{t_i}^+} Z_{i+1}\sqrt{\Delta t} + \frac{1}{4}\sigma^2(Z_{i+1}^2 - 1)\Delta t.$$

5. SABR (Stochastic Alpha, Beta, Rho) model

Assume the dynamics of the base asset F_t (e.g., a futures of a forward contract) is defined by the equations

$$\begin{aligned} dF_t &= \alpha_t F_t^\beta dW_t^1, \qquad F_0 = f > 0, \\ d\alpha_t &= \nu\alpha_t dW_t^2, \qquad \alpha_0 = \alpha > 0, \\ dW_t^1 dW_t^2 &= \rho dt, \end{aligned}$$

where $\alpha > 0$, $\beta > 0$, $\rho \in (-1,1)$, $\nu > 0$ are the model parameters. If $\beta < 1$, the process F_t may reach zero; in that case we will assume that it gets trapped at zero.

Approximation of implied volatility (Hagan et al., 2002). The implied volatility produced by the model can be approximated by the formula

$$\widehat{\sigma}(T,K) = \frac{\alpha}{(fK)^{(1-\beta)/2}\left\{1+\frac{(1-\beta)^2}{24}\ln^2\frac{f}{K}+\frac{(1-\beta)^4}{1920}\ln^4\frac{f}{K}+\dots\right\}}\cdot\left(\frac{z}{x(z)}\right)\cdot$$
$$\cdot\left\{1+\left[\frac{(1-\beta)^2}{24}\frac{\alpha^2}{(fK)^{1-\beta}}+\frac{\rho\beta\nu\alpha}{4(fK)^{(1-\beta)/2}}+\frac{2-3\rho^2}{24}\nu^2\right]T+\dots\right\},$$

where

$$z=\frac{\nu}{\alpha}(fK)^{(1-\beta)/2}\ln\frac{f}{K},\qquad x(z)=\ln\left\{\frac{\sqrt{1-2\rho z+z^2}+z-\rho}{1-\rho}\right\}.$$

For at-the-money options ($K=f$) we have

$$\widehat{\sigma}(T,f)=\frac{\alpha}{f^{1-\beta}}\left\{1+\left[\frac{(1-\beta)^2}{24}\frac{\alpha^2}{f^{2-2\beta}}+\frac{\rho\beta\alpha\nu}{4f^{1-\beta}}+\frac{(2-3\rho^2)\nu^2}{24}\right]T+\dots\right\},$$

i.e. in the above general formula we let $z/x(z)=1$, its limit value as $z\to 0$ (we have $z=0$ when $K=f$).

For $\beta=0$, there is a more accurate formula (with an expansion of the 4th order, rather than the 2nd order in the general formula):

$$\widehat{\sigma}(T,f)=\alpha\frac{\ln(f/K)}{f-K}\cdot\left(\frac{z}{x(z)}\right)\cdot\left\{1+\left[\frac{\alpha^2}{24fK}+\frac{2-3\rho^2}{24}\nu^2\right]T+\dots\right\},$$

where

$$z=\frac{\nu}{\alpha}\sqrt{fK}\ln\frac{f}{K},\qquad x(z)=\ln\left\{\frac{\sqrt{1-2\rho z+z^2}+z-\rho}{1-\rho}\right\}.$$

For $\beta=1$, a more accurate formula (also, with a 4th order expansion) is

$$\widehat{\sigma}(T,f)=\alpha\cdot\left(\frac{z}{x(z)}\right)\cdot\left\{1+\left[\frac{\rho\alpha\nu}{4}+\frac{(2-3\rho^2)\nu^2}{24}\right]T+\dots\right\},$$

where

$$z=\frac{\nu}{\alpha}\ln\frac{f}{K},\qquad x(z)=\ln\left\{\frac{\sqrt{1-2\rho z+z^2}+z-\rho}{1-\rho}\right\}.$$

6. SVI (Stochastic Volatility Inspired) model

The SVI model (Gatheral, 2004) approximates the volatility curve without making explicit assumptions about dynamics of the asset price process. Fix an expiration time $T>0$ and let

$$x=\ln\frac{K}{F_T}$$

denote the log-moneyness of an option with expiration time T and strike $K(x)=F_Te^x$, where $F_t=\mathrm{E}(S_t)=e^{rt}S_0$ is the forward price. By $w(x)$ denote the total implied variance of the option, i.e.

$$w(x)=\widehat{\sigma}^2(T,K(x))T,$$

where $\widehat{\sigma}(T,K)$ is, as usual, the implied volatility.

Three parametrizations of the SVI model. The following three parametrizations can be found in the literature (see Gatheral and Jacquier (2014)).

Raw parametrization. The SVI model in the *raw parametrization* approximates the total implied variance (for fixed T) by the function

$$w(x) = a + b\left(\rho(x-m) + \sqrt{(x-m)^2 + \sigma^2}\right),$$

where $a \in \mathbb{R}$, $b \geq 0$, $|\rho| < 1$, $m \in \mathbb{R}$, $\sigma > 0$ are parameters of the model. Note that, of course, the same formula can used to approximate the implied volatility curve in $(\ln K, \widehat{\sigma}^2)$ coordinates, after a change of parameters.

Natural parametrization. This parametrization assumes

$$w(x) = \Delta + \frac{\omega}{2}\left(1 + \zeta\rho(x-\mu) + \sqrt{(\zeta(x-\mu)+\rho)^2 + 1 - \rho^2}\right).$$

where $\omega \geq 0$, $\Delta, \mu \in R$, $|\rho| < 1$, $\zeta > 0$ are parameters.

The following formulas can be used for conversion between the raw and natural parametrizations (note that the parameter ρ is the same):

$$\begin{aligned}
\omega &= \frac{2b\sigma}{\sqrt{1-\rho^2}}, & a &= \delta + \frac{1}{2}\omega(1-\rho^2),\\
\delta &= a - \frac{1}{2}\omega(1-\rho^2), & b &= \frac{\omega\zeta}{2},\\
\mu &= m + \frac{\rho\sigma}{\sqrt{1-\rho^2}}, & m &= \mu - \frac{\rho}{\zeta},\\
\zeta &= \frac{\sqrt{1-\rho^2}}{\sigma}, & \sigma &= \frac{\sqrt{1-\rho^2}}{\zeta}.
\end{aligned}$$

Jump-wing parametrization. The parameters are $v(T)$, $\psi(T)$, $p(T)$, $c(T)$, $\widetilde{v}(T)$ (this parametrization specifies dependence on T), which are obtained from the raw parameters as follows:

$$\begin{aligned}
v &= \frac{a + b(-\rho m + \sqrt{m^2+\sigma^2})}{T},\\
\psi &= \frac{b}{2\sqrt{Tv}}\left(-\frac{m}{\sqrt{m^2+\sigma^2}} + \rho\right),\\
p &= \frac{1}{\sqrt{Tv}}b(1-\rho),\\
c &= \frac{1}{\sqrt{Tv}}b(1+\rho),\\
\widetilde{v} &= \frac{1}{T}\left(a + b\sigma\sqrt{1-\rho^2}\right).
\end{aligned}$$

To construct a volatility surface from this parametrization, we can keep the jump-wing parameters fixed, and for each T find the corresponding raw parameters to get a volatility curve for this T. The raw parameters are obtained as follows. First, define

$$b = \frac{1}{2}\sqrt{Tv}(c+p), \qquad \rho = 1 - \frac{2p}{c+p}, \qquad \beta = \rho - \frac{2\psi\sqrt{Tv}}{b}.$$

Then, if $\beta = 0$, the remaining raw parameters are

$$
\begin{aligned}
m &= 0, \\
\sigma &= \frac{(v - \widetilde{v})T}{b(1 - \sqrt{1 - \rho^2})}, \\
a &= T\widetilde{v} - b\sigma\sqrt{1 - \rho^2}.
\end{aligned}
$$

If $\beta \neq 0$ and $|\beta| \leq 1$, then

$$
\begin{aligned}
m &= \frac{(v - \widetilde{v})T}{b(-\rho + \operatorname{sgn}(\alpha)\sqrt{1 + \alpha^2} - \alpha\sqrt{1 - \rho^2})}, \qquad \text{where } \alpha = \operatorname{sgn}(\beta)\sqrt{\frac{1}{\beta^2} - 1}, \\
\sigma &= \alpha m, \\
a &= T\widetilde{v} - b\sigma\sqrt{1 - \rho^2}.
\end{aligned}
$$

If $|\beta| > 1$, we get a non-convex volatility curve; this case should be excluded.

Parameter calibration. The following two-step procedure was proposed by Martini and De Marco (2012) (see also Aurell (2014)). Let us work with the raw parametrization. Introduce the variable

$$
y(x) = \frac{x - m}{\sigma},
$$

so that the raw parametrization can be rewritten in the form

$$
w(x) = a + dy(x) + cz(x), \qquad z(x) = \sqrt{y(x)^2 + 1},
$$

where

$$
d = \rho b\sigma, \qquad c = b\sigma.
$$

For each m, σ we can solve the *inner* optimization problem

$$
\sum_{i=1}^{n} (a + dy(x_i) + cz(x_i) - w_m(x_i))^2 \xrightarrow[(a,d,c)\in\mathcal{D}]{} \min,
$$

where $w_m(x_i)$ are observed market total implied variances, and $\mathcal{D}$ is the domain defined by the constraints

$$
\begin{aligned}
&0 \leq c \leq 4\sigma, \\
&|d| \leq c, \\
&|d| \leq 4\sigma - c, \\
&0 \leq a \leq \max_i w(x_i).
\end{aligned}
$$

This constraints are derived from the model well-posedness conditions and conditions for absence of arbitrage (see Aurell (2014) for details). The inner optimization problem is a convex minimization problem on a compact domain, hence it has a unique solution (Python's `scipy.optimize.minimize(method="SLSQP")` works quite well here).

Then we solve the *outer* minimization problem and find optimal m, σ:

$$
\sum_{i=1}^{n} (w(x_i \mid m, \sigma, a^*(m,\sigma), b^*(m,\sigma), \rho^*(m,\sigma)) - w_m(x_i))^2 \xrightarrow[(m,\sigma)\in\mathcal{E}]{} \min,
$$

where a^*, b^*, σ^* is the solution of the inner minimization problem for given m, σ, and $\mathcal{E}$ is the domain

$$\min_i x_i \leq m \leq \max_i x_i,$$
$$\sigma_{\min} \leq \sigma \leq \max_\sigma .$$

The boundaries $\sigma_{\min}$, $\sigma_{\max}$ are specified by the user, a reasonable choice is $\sigma_{\min} = 10^{-4}$, $\sigma_{\max} = 10$. The objective function in the outer problem is not nice, hence some global optimization algorithm should be used.

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