Applied Mathematical Finance I

Lecture 3: Basic Concepts of Fixed-Income Markets

Vladimir Shangin

Vega Institute Foundation

October 5, 2023

Interest Rates

- A unit of cash today is (usually) worth more than a unit of cash tomorrow.
- As a consequence, lenders charge borrowers a cash amount, called the interest, above repayment of the original amount of the loan (principal).
- Interest rate is the rate at which interest is paid.
- Interest rates represent the “cost of money” - higher rates make borrowing more expensive.
- By a fixed-income market we mean a financial market where instruments on interest rates are traded.

Specifics of Interest Rates

- Interest rates are always quoted on an annual basis. Therefore, for shorter or longer tenors we need to calculate which fraction of a year (called day count fraction or coverage or simply year fraction) applies.
- Interest rate always comes attached with
  - Day count convention: a rule to convert a period of time to a year fraction.
  - Compounding style: determines how the interest amount is accrued over a certain period.
- The term-structure of rates: different rates apply to different maturities.
- Interest rates can be negative as opposed to equities or exchange rates. This can be seen as a "tax on holding money".

Rate Compounding Styles

- Rate compounding style defines how the interest amount accrued over a certain period is actually calculated given the interest rate r and the year fraction τ.
- Some common styles are
  - Precounted (Simple): a unit of cash invested becomes $1 + r\tau$.
  - Compounded n times a year: a unit of cash grows to $(1 + \frac{r}{n})^{n\tau}$.
  - Continuous compounding which is obtained in the limit $n \to \infty$: $1 \to e^{r\tau}$.
- It is usually more convenient to work with continuously compounded rates since the exponential function has nice analytic properties.

Discount Factors

- Discount factor represents the value of receiving a unit of cash at some future time.
- Given constant continuously compounded rate r, one needs $\mathrm{e}^{-r\tau}$ units of currency today to get 1 at a time horizon τ as $\mathrm{e}^{-r\tau} \cdot \mathrm{e}^{r\tau} = 1$. Therefore, the corresponding discount factor is $\mathrm{e}^{-r\tau}$. In case of a simple rate, discount factor becomes $\frac{1}{1+r\tau}$.
- Note that discount factor for a given time horizon is unique while interest rate depends on the choice of compounding style and day count convention.
- Given a discount factor, one can solve for rates under different conventions and styles.
- The inverse of discount factor is usually referred to as compounding factor.

Zero-Coupon Bonds

- Zero-coupon bond with maturity T (or simply T-bond) is a financial security that guarantees its holder one unit of cash at a future time T, with no intermediate payments.
- Let us denote by $p(t,T)$ the value of a T-bond at time $t \leq T$. Note that $p(t,T)$ is the discount factor between t and T.
- We will assume that
  - There exists a frictionless zero-coupon bond market: T-bonds are traded for every maturity $T \geq 0$. For fixed T, price process $p(t,T)$, $t \in [0,T]$, is a strictly positive adapted process.
  - T-bonds cannot default: $p(T,T) = 1$ for all $T \geq 0$.
  - Term-structure is smooth: for any fixed t, the T-bond price $p(t,T)$ is a differentiable function of T. The map $T \mapsto p(t,T)$, $T \geq t$ is called the discount curve (zero curve) observed at time t.

Zero Curve

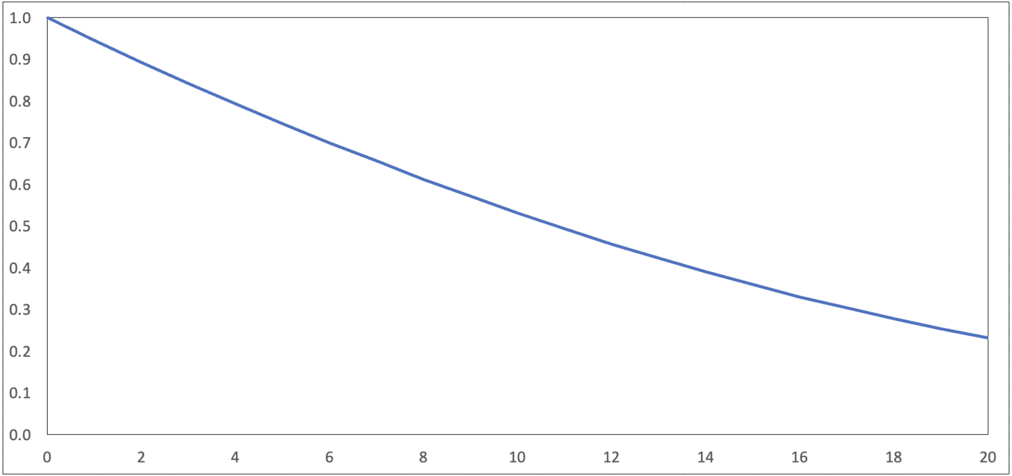

Figure: Example of a zero curve $T \mapsto p(0,T),\ T \in [0,20]$.

Zero-Coupon Bond Price Evolution

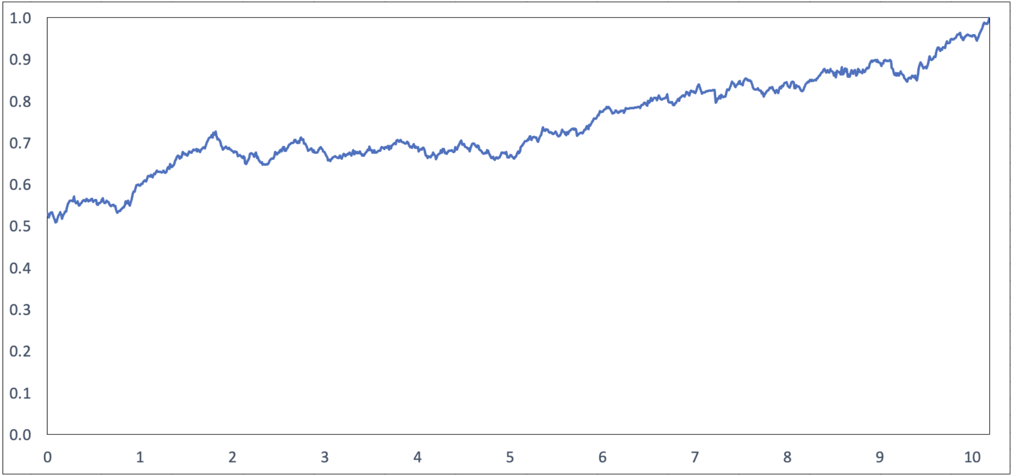

Figure: Zero-coupon bond price evolution $t \mapsto p(t, 10)$, $t \in [0, 10]$.

Spot Rate and Yield

- Buying T-bond at t and holding it to maturity T is equivalent to depositing money for term $\tau = T - t$ at some risk-free spot rate $L(t,T)$. Note that this rate is $\mathcal{F}_t$-measurable random variable.
- Assuming that $L(t,T)$ is quoted as a simple rate, it immediately follows that

$$\frac{1}{p(t,T)} = 1 + \tau L(t,T), \quad \text{or} \quad L(t,T) = \frac{1}{\tau}\left(\frac{1}{p(t,T)} - 1\right).$$

- Alternatively, we can define zero-coupon yield $r(t,T)$ as continuously compounded spot rate

$$\frac{1}{p(t,T)} = \mathrm{e}^{r(t,T)\tau}, \quad \text{or} \quad r(t,T) = -\frac{\ln p(t,T)}{\tau}.$$

London Inter-Bank Offered Rate (LIBOR)

- We will assume that simple spot rate $L(t, T)$ corresponds to London Inter-Bank Offered Rate (LIBOR) so we also refer to $L(t, T)$ as LIBOR or deposit rate.
- LIBOR represents the cost of funding for top tier banks in the London interbank market from their own perspective*.
- LIBORs have been seen as the benchmark rates for IR derivatives for the past four decades with the outstanding notional of OTC LIBOR-linked derivatives being about $400 trillion as of Dec 2007.
- In reality, LIBORs were subject to credit and liquidity risk as well as fixing manipulation which ultimately led to termination of publication on June 30, 2023. More on this later in the course.

*LIBOR fixing is computed daily as a trimmed average of unsecured borrowing rates committed by Contributor Panel banks.

USD LIBORs

Maturity / rate 2020	first	last	high	low	average
USD LIBOR - overnight	1.541 %	0.078 %	1.577 %	0.051 %	0.366 %
USD LIBOR - 1 week	1.593 %	0.096 %	1.594 %	0.088 %	0.423 %
USD LIBOR - 2 weeks	-	-	-	-	-
USD LIBOR - 1 month	1.734 %	0.144 %	1.734 %	0.127 %	0.516 %
USD LIBOR - 2 months	1.821 %	0.190 %	1.841 %	0.166 %	0.596 %
USD LIBOR - 3 months	1.900 %	0.238 %	1.900 %	0.205 %	0.650 %
USD LIBOR - 4 months	-	-	-	-	-
USD LIBOR - 5 months	-	-	-	-	-
USD LIBOR - 6 months	1.910 %	0.258 %	1.910 %	0.234 %	0.686 %
USD LIBOR - 7 months	-	-	-	-	-
USD LIBOR - 8 months	-	-	-	-	-
USD LIBOR - 9 months	-	-	-	-	-
USD LIBOR - 10 months	-	-	-	-	-
USD LIBOR - 11 months	-	-	-	-	-
USD LIBOR - 12 months	1.995 %	0.342 %	1.995 %	0.328 %	0.766 %

Figure: USD LIBORs, all tenors.

Forward Rate

- Note that we can lock at t a rate spanning the future period $[T, T+\tau]$. Indeed,
  - at t: sell one T-bond and buy $\frac{p(t,T)}{p(t,T+\tau)}$ zero-coupon bonds with maturity $T+\tau$. Note that this results into a zero net investment.
  - at T: pay a unit of cash.
  - at $T+\tau$: receive $\frac{p(t,T)}{p(t,T+\tau)}$ units of cash.
- Effectively, we agree at t on a forward investment of a unit of cash at T yielding $\frac{p(t,T)}{p(t,T+\tau)}$ units of cash at $T+\tau$.
- Forward rate for $[T, T+\tau]$ prevailing at t is then given by

$$F(t,T,T+\tau) = \frac{1}{\tau}\left(\frac{p(t,T)}{p(t,T+\tau)} - 1\right).$$

- For fixed t and τ, the map $T \mapsto F(t,T,T+\tau)$ is called the forward curve for tenor τ observed at t.

Instantaneous Rates

- Instantaneous forward rate $f(t, T)$ is defined as

$$f(t, T) = \lim_{\tau \to 0} F(t, T, T + \tau) = -\frac{\partial \ln p(t, T)}{\partial T}.$$

  Interpretation: $f(t, T)$ is a forward rate for infinitesimal period $[T, T + dT]$ as seen at t.

- Given that $p(t, t) = 1$, the above formula is equivalent to

$$p(t, T) = \mathrm{e}^{-\int_t^T f(t,u)\,du}.$$

- Instantaneous short rate at t is defined by

$$r_t = f(t, t) = \lim_{T \to t} L(t, T).$$

  Loosely speaking, we can think of r_t as the overnight rate effective at date t.

"Rolling over" Strategy

- Note that since T-bond price process is pinned at maturity, $p(T,T) = 1$, the bond is locally risk-free over the infinitesimal period $[T - dT, T]$. We can use this fact to synthetically create the risk-free asset.
- At $t = 0$, invest a unit of cash in zero-coupon bonds maturing at Δt. We will get

$$\frac{1}{p(0,\Delta t)} = \mathrm{e}^{\int_0^{\Delta t} f(0,u)\,du} = 1 + r(0)\Delta t + o(\Delta t).$$

- Immediate reinvesting in zero-coupon bonds maturing at $2\Delta t$ gives

$$\frac{1}{p(0,\Delta t)}\frac{1}{p(\Delta t, 2\Delta t)} = (1 + r(0)\Delta t)(1 + r(\Delta)\Delta t) + o(\Delta t).$$

  at time $2\Delta t$, etc.
- In the limit $\Delta t \to 0$, the strategy of "rolling over" just-maturing bonds leads to the money-market account B_t.

Money-Market Account

- Money-market account (also called savings account or bank account) B_t grows instantaneously at short rate r_t

$$dB_t = r_t B_t dt, \quad B_0 = 1.$$

or

$$B_t = \mathrm{e}^{\int_0^t r_s \, ds}.$$

- Note that B_t is a locally risk-free asset in a sense that the return over the infinitesimal period $[t, t + dt]$ is known at time t.

Risk-Neutral Measure

- We assume that the market consisting of the money-market account and zero-coupon bonds for all maturities is free of arbitrage. Namely, we assume that there exists a probability measure $\mathbb{Q} \sim \mathbb{P}$ such that, for any $T \geq 0$, the process

$$\frac{p(t,T)}{B_t}, \quad t \in [0,T],$$

is a $\mathbb{Q}-$martingale. Measure $\mathbb{Q}$ is traditionally called a risk-neutral measure.

- Recall that the existence of an equivalent martingale measure excludes arbitrage among every finite selection of assets. To be more general one would have to consider self-financing strategies involving a continuum of zero-coupon bonds.

Arbitrage-Free Pricing Under Risk-Neutral Measure

- By the fundamental theorem of asset pricing, arbitrage-free price (present value) at time t of a single cash flow $\mathcal{X}_T$ paid at $T \geq t$ is given by

$$\mathrm{PV}_t = B_t \cdot \mathbb{E}_t^{\mathbb{Q}}\left[\frac{\mathcal{X}_T}{B_T}\right] = \mathbb{E}_t^{\mathbb{Q}}\left[\mathrm{e}^{-\int_t^T r(s)\,ds} \cdot \mathcal{X}_T\right]. \tag{1}$$

- Exceptionally important case is $\mathcal{X}_T = 1$ a.s. which corresponds to the T-bond. We then get

$$p(t,T) = B_t \cdot \mathbb{E}_t^{\mathbb{Q}}\left[\frac{1}{B_T}\right] = \mathbb{E}_t^{\mathbb{Q}}\left[\mathrm{e}^{-\int_t^T r(s)\,ds}\right].$$

- Generally, $\mathcal{X}_T$ and B_T are correlated random variables so the evaluation of conditional expectation (1) may be a hard task. Change of numéraire can help us to simplify the problem.

T-forward measure

- Let $(N, \mathbb{Q}^N)$ be a numéraire pair. Recall the general change of numéraire formula

$$\mathrm{PV}_t = B_t \cdot \mathbb{E}_t^{\mathbb{Q}}\left[\frac{\mathcal{X}_T}{B_T}\right] = N_t \cdot \mathbb{E}_t^{\mathbb{Q}^N}\left[\frac{\mathcal{X}_T}{N_T}\right].$$

- A special and exceptionally important example of a valid numéraire is the T-bond, $N_t = p(t, T)$. In this case, the corresponding martingale measure $\mathbb{Q}^T$, defined via its Radon-Nikodym derivative

$$\frac{d\mathbb{Q}^T}{d\mathbb{Q}} = \frac{p(T,T)}{B_T} \cdot \frac{B_0}{p(0,T)} = \frac{1}{B_T \cdot p(0,T)},$$

is called the T-forward measure. We denote expectation under $\mathbb{Q}^T$ simply as $\mathbb{E}^T$.

- The present value at t of a contingent claim $\mathcal{X}_T$ can be then written as

$$\mathrm{PV}_t = p(t,T) \cdot \mathbb{E}_t^T\left[\mathcal{X}_T\right]. \tag{2}$$

Connection Between Money-Market Account and Deposit Rate

- Let us assume that we have a unit of cash at t. Consider two alternative strategies
  - Invest the cash in the money-market at risk-free rate and get $\frac{B_T}{B_t}$ at T.
  - Put the money on the deposit for term $\tau = T - t$ at the prevailing LIBOR rate $L(t,T)$ and receive $1 + \tau \cdot L(t,T)$ at maturity T.
- Note that $\frac{B_T}{B_t}$ is $\mathcal{F}_T$-measurable while $1 + \tau \cdot L(t,T)$ is known at time t. According to (2), in the absence of arbitrage, we must have

$$\mathbb{E}_t^T\left[\frac{B_T}{B_t}\right] = \mathbb{E}_t^T\left[e^{\int_t^T r_s\,ds}\right] = 1 + \tau \cdot L(t,T) = \frac{1}{p(t,T)}.$$

Forward Rate as Expectation of Spot Rate

- Recall that the forward rate for period $[T, T+\tau]$ as seen at $t \leq T$ is given by

$$F(t,T,T+\tau) = \frac{1}{\tau}\left(\frac{p(t,T)}{p(t,T+\tau)} - 1\right) = \frac{\frac{1}{\tau}[p(t,T) - p(t,T+\tau)]}{p(t,T+\tau)}.$$

- Note that the numerator represents a value of a static (and hence self-financing) portfolio of two zero-coupon bonds while the denominator is the numéraire corresponding to $\mathbb{Q}^{T+\tau}$ martingale measure.
- It immediately follows that $F(t,T,T+\tau)$, $t \in [0,T]$, is a $\mathbb{Q}^{T+\tau}$-martingale.
- Given that $F(T,T,T+\tau) = L(T,T+\tau)$, we get

$$F(t,T,T+\tau) = \mathbb{E}_t^{T+\tau}\left[L(T,T+\tau)\right]. \tag{3}$$