Survival investment strategies in a continuous-time market model with competition

Mikhail Zhitlukhin*

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Abstract

We consider a stochastic game-theoretic model of an investment market in continuous time with short-lived assets and study strategies, called survival, which guarantee that the relative wealth of an investor who uses such a strategy remains bounded away from zero. The main results consist in obtaining a sufficient condition for a strategy to be survival and showing that all survival strategies are asymptotically close to each other. It is also proved that a survival strategy allows an investor to accumulate wealth in a certain sense faster than competitors.

Keywords: survival strategies, market competition, relative wealth, growth optimal strategies, martingales.

1 Introduction

This paper proposes a stochastic game-theoretic model of an investment market in continuous time where investors compete for payoffs yielded by several assets. The main objective is to study questions about asymptotic optimality of investment strategies from *evolutionary* point of view: to describe what strategies *survive* in the competition for payoffs, what strategies *dominate* or *get extinct*, and how they affect the market structure in the long run. This circle of questions has been studied in a number of papers in the literature, but mostly in discrete time (see the reviews [7, 9], and Section 4.2 below). The model considered here is one of the few in continuous time.

A market in our model consists of several investors who invest their wealth in assets. Asset payoffs, which are specified by some exogenous stochastic processes and paid continuously, are distributed between the investors proportionally to the amount of wealth they allocate to each asset.

*Steklov Mathematical Institute of the Russian Academy of Sciences. 8 Gubkina St., Moscow, Russia. Email: mikhailzh@mi-ras.ru. The research was supported by the Russian Science Foundation, project no. 18-71-10097.

Naturally, a larger expected future payoff of an asset will attract more investors, which will reduce the share of the payoff received by each of them. Hence the investors face the problem how to allocate their wealth in an optimal way. In our model, assets are assumed to be short-lived in the sense that they are bought by investors, yield payoffs at the "next infinitesimal" moment of time, and then reappear again, but cannot be sold to capitalize on increased prices (such a model is simpler than a model of a stock market).

One of the main goals of the paper is to identify strategies that survive in the market in the sense that the relative wealth of an investor who uses such a strategy remains bounded away from zero on the whole time axis (by the relative wealth we mean the share of wealth of one investor in the total wealth of the market). It is not assumed that all the investors are rational, i.e. that their strategies are defined as solutions of some optimization problems. For example, they can use strategies that mimic other market participants, follow some empirical rules, etc. It is also not assumed that the investors know the strategies of their competitors. Therefore, a survival strategy should be robust in the sense that it guarantees a positive share of wealth no matter what strategies are used by the other investors.

The main results of the paper are as follows. First, we obtain a sufficient condition for a strategy to be survival. It is stated in an explicit form: we construct one particular survival strategy and show that any other strategy which is asymptotically close to it is survival as well. We also prove that such a strategy dominates in the market, i.e. the relative wealth of an investor who uses it tends to 1, when the representative strategy of the other investors is asymptotically different from it. Moreover, we show that using a survival strategy allows to achieve the highest asymptotic growth rate of wealth among the investors in the market.

The key idea to obtain these results is to find a strategy such that the process of the logarithm of its relative wealth is a submartingale. As it will be shown, its existence follows from Gibbs' inequality applied to a suitable representation of the relative wealth process. The survival property is then established using results on convergence of submartingales.

This approach was used for the first time in the paper [2], which studied a fairly general discrete-time model with short-lived assets. For particular instances of that model, similar results had been known before (see the review [7]), but they mainly used ideas based on the Law of Large Numbers, which limited them only to payoff sequences consisting of independent random variables. One can also mention the paper [1], where an approach similar to [2] was used in a model with long-lived assets, which describes a usual stock market. There, in order to obtain similar results, more subtle arguments were required and some restrictive assumptions were imposed on the model. From this point of view, our work is closer to [2], we also consider a model with only short-lived assets, but in continuous time.

Let us also mention that survival strategies are somewhat similar to growth optimal strategies in asset market models without competition (see [8, 11, 16]), as both arise from the problem of maximizing the logarithm of wealth. In particular, we show that survival strategies allow to achieve the highest growth rate of wealth, similarly to growth optimal strategies. However, an essential difference between these two classes of strategies is that survival strategies cannot be directly obtained from a single-agent wealth optimization problem because the evolution of wealth of one investor depends also on actions of the other investors.

The paper is organized as follows. In Section 2, we describe the model. In Section 3, we introduce the notion of a survival strategy and provide an explicit construction of one such strategy. The main results of the paper are stated in Section 4. Section 5 contains their proofs.

2 The market model

Before we study a general model in continuous time, let us consider a model in discrete time, in which the main objects and formulas have a clear interpretation. Based on it, we will formulate the general model.

2.1 Preliminary consideration: a model in discrete time

Let us fix a probability space $(\Omega, \mathcal{F}, \mathbb{F}, \mathrm{P})$ with a discrete-time filtration $\mathbb{F} = (\mathcal{F}_t)_{t=0}^{\infty}$, on which all the random variables will be defined.

The market in the model includes $M \geq 2$ investors and $N \geq 2$ assets, which yield non-negative random payoffs at moments of time $t = 1, 2, \dots$ The investors decide, simultaneously and independently of each other at every moment of time, what part of their wealth they invest in each of the assets, and the asset payoffs are split proportionally to the invested wealth amounts. The investment decision are made before the payoffs become known. We impose the assumption that at every moment of time the proportions of own wealth the investors allocate to the assets are the same for all the investors (however the distribution of invested wealth between the assets may be different and the investors are free to choose it); a model where they may differ would be more complicated and is not studied in this paper.

The payoffs of the assets are specified by random sequences $A_t^n(\omega) \geq 0$, which are adapted to the filtration (A_t^n is $\mathcal{F}_t$-measurable). These random sequences are exogenous, i.e. do not depend on actions of the investors.

The evolution of the investors' wealth is described by adapted sequences $Y_t^m(\omega) \geq 0$. The initial values $Y_0^m > 0$ are given, and further values depend on strategies used by the investors. A strategy of investor m is identified with a sequence $\lambda_t^m(\omega) \geq 0$ of random vectors in $\mathbb{R}^N$, which express the proportions of wealth invested in each of the assets. The sequences $\lambda^{m,n}$ are predictable ($\lambda_t^{m,n}$ if $\mathcal{F}_{t-1}$-measurable) and $\sum_n \lambda_t^{m,n} = 1$.

Given this, we state the equation which determines the evolution of investor m's wealth:

$$Y_t^m = (1-\delta_t)Y_{t-1}^m + \sum_n \frac{\lambda_t^{m,n} Y_{t-1}^m}{\sum_k \lambda_t^{k,n} Y_{t-1}^k} A_t^n, \qquad t \geq 1, \tag{1}$$

where $\delta_t(\omega)$ is the proportion of wealth each investor allocates for investment in the assets. The sequence of random variables $\delta_t \in [0,1)$ is predictable, given exogenously, and the same for all the investors.

Notice that the first term in the right-hand side of (1) is the amount of wealth not invested in the assets, and the second term is the received payoff. The fraction in the sum expresses the idea of division of payoffs proportionally to invested amounts of wealth. We treat the indeterminacy $0/0$, which happens when no one invests in asset n, as $0/0 = 1/M$, so in this case the payoff of the corresponding asset is split in the equal proportions. Note that always $Y_t^m > 0$ due to the assumption $\delta_t < 1$.

Let us emphasize that the components of the strategies $\lambda_t^{m,n}$ depend on a random outcome $\omega \in \Omega$, but do not depend on the investors' wealth or their strategies. This means that the investors, when deciding how to allocate their wealth, take into consideration only asset payoffs. Such strategies can be called *basic* (as, e.g., in the paper [2]). One could consider a more general model, where, for example, $\lambda_t^m = \lambda_t^m(\omega, Y_0, \ldots, Y_{t-1}, \lambda_0, \ldots, \lambda_{t-1})$, but this will not essentially increase the generality of the main results of our paper, see Remark 2 below.

In order to get an idea how to state a continuous-time counterpart of equation (1), let us rewrite it in the following form:

$$\Delta Y_t^m = -Y_{t-1}^m \Delta V_t + \sum_n \frac{\lambda_t^{m,n} Y_{t-1}^m}{\sum_k \lambda_t^{k,n} Y_{t-1}^k} \Delta X_t^n, \qquad t \geq 1, \tag{2}$$

where X_t, V_t are the sequences of cumulative payoffs and cumulative investment proportions defined as

$$X_t = \sum_{s \leq t} A_s, \qquad V_t = \sum_{s \leq t} \delta_s, \tag{3}$$

and the symbol Δ denotes a one-period increment, e.g. $\Delta Y_t^m = Y_t^m - Y_{t-1}^m$.

The form of equation (2) suggests that an analogous model in continuous time can be obtained by considering continuous-time processes X_t, V_t, Y_t and "replacing" the one-step increments with the infinitesimal increments, e.g. ΔY_t with dY_t. Our goal for the rest of this section will be to define such a model in a proper way.

2.2 Notation

Let us introduce notation that will be used to formulate the continuous-time version of the above model.

From now on assume given a filtered probability space $(\Omega, \mathcal{F}, \mathbb{F}, \mathrm{P})$ with a continuous-time filtration $\mathbb{F} = (\mathcal{F}_t)_{t\in\mathbb{R}_+}$, which satisfies the usual assumptions, i.e. $\mathbb{F}$ is right-continuous ($\mathcal{F}_t = \bigcap_{s>t} \mathcal{F}_s$), the σ-algebra $\mathcal{F}$ is P-complete, and $\mathcal{F}_0$ contains all the P-null sets of $\mathcal{F}$.

For vectors $x, y \in \mathbb{R}^N$, by xy we will denote the scalar product, by $|x|$ the l_1-norm of a vector, and by $\|x\|$ the l_2-norm; for a scalar function $f\colon \mathbb{R} \to \mathbb{R}$ the notation $f(x)$ means the coordinatewise application of the function:

$$
\begin{aligned}
&xy = \sum_n x^n y^n, \quad |x| = \sum_n |x^n|, \quad \|x\| = \sqrt{xx},\\
&f(x) = (f(x^1), \dots, f(x^N)).
\end{aligned}
$$

If $G_t = G(t)$ is a non-decreasing function, then for a measurable function f_t denote

$$
f \cdot G_t = \int_0^t f_s dG_s,
$$

provided that the integral is well-defined (as a Lebesgue-Stieltjes integral). Functions f, G may be random, then $f \cdot G_t(\omega)$ is defined pathwise for each ω. If f is vector-valued and G is scalar-valued, then $f \cdot G_t = (f^1 \cdot G_t, \dots, f^N \cdot G_t)$; if both are vector-valued, then $f \cdot G_t = \sum_n f^n \cdot G_t^n$.

As usual, all equalities and inequalities between random variables are assumed to hold with probability one (almost surely). For random processes $X_t(\omega)$, $Y_t(\omega)$, the equality $X = Y$ is understood to hold up to P-indistinguishability, i.e. $\mathrm{P}(\exists\, t : X_t \neq Y_t) = 0$; in the same way we treat inequalities. Properties of trajectories (continuity, monotonicity, etc.) are assumed to hold for all ω, unless else is specified. If X, Y are right-continuous processes, then $X = Y$ if and only if $X_t = Y_t$ a.s. for all t.

By the predictable σ-algebra $\mathcal{P}$ on $\Omega \times \mathbb{R}_+$ we call, as usual, the σ-algebra generated by all left-continuous adapted processes. A process is predictable if it is measurable with respect to $\mathcal{P}$ as a map from $\Omega \times \mathbb{R}_+$ to $\mathbb{R}$ or to $\mathbb{R} \cup \{\pm\infty\}$.

2.3 The general model

As in the discrete-time model, there are $M \geq 2$ investors and $N \geq 2$ assets. The asset payoffs are specified by exogenous cumulative payoff processes X_t^n (cf. (3)), which are adapted to the filtration $\mathbb{F}$ and have non-decreasing càdlàg paths (right-continuous with left limits) with $X_0^n = 0$.

The cumulative proportion of wealth allocated by each investor to the assets is specified by means of an adapted non-decreasing càdlàg scalar process V_t with $V_0 = 0$ and jumps $\Delta V_t \in [0, 1)$ (as usual, $\Delta V_t = V_t - V_{t-}$, where $V_{t-} = \lim_{s\uparrow t} V_s$, and $\Delta V_0 = 0$). To avoid problems with non-integrability (see Section 3.2), we will assume that the jumps of V are uniformly bounded away from 1, i.e. there exists a constant $\gamma_V \in [0,1)$ such that for all $\omega \in \Omega$ and $t \geq 0$

$$\Delta V_t(\omega) \leq \gamma_V. \tag{4}$$

A strategy of investor m is identified with a predictable process λ_t^m of proportions of wealth invested in the assets, which assumes values in the standard simplex in $\mathbb{R}^N$, i.e. $\lambda_t^{m,n} \geq 0$ and $\sum_n \lambda_t^{m,n} = 1$. As was noted above, we consider only basic strategies, which means that λ_t^m does not depend on the "past history" of the market.

The wealth processes of the investors are defined as strictly positive càdlàg processes Y^m that satisfy the equation (a continuous-time counterpart of (2))

$$dY_t^m = -Y_{t-}^m dV_t + \sum_n \frac{\lambda_t^{m,n} Y_{t-}^m}{\sum_k \lambda_t^{k,n} Y_{t-}^k} dX_t^n \tag{5}$$

and such that $Y_- > 0$ (i.e. $Y_{t-}^m > 0$ for all $t \geq 0$ and m). Without loss of generality, we will always assume that the initial values $Y_0^m > 0$ are non-random. If $\lambda_t^{k,n} = 0$ for all k, then we assume that the value of the fraction in the right-hand side is equal to $1/M$ for corresponding n.

As usual, equation (5) should be understood in the integral form:

$$Y_t^m = Y_0^m + \sum_n \int_0^t \frac{\lambda_s^{m,n} Y_{s-}^m}{\sum_k \lambda_s^{k,n} Y_{s-}^k} dX_s^n - \int_0^t Y_{s-}^m dV_s, \qquad t \geq 0. \tag{6}$$

The integrals here are pathwise Lebesgue-Stieltjes integrals (which are well-defined since the processes X_t, V_t do not decrease, and the integrands are non-negative). It is not difficult to see that if Y^m satisfies (6), then it has finite variation on any interval $[0,t]$.

The next proposition shows that equation (5) has a unique solution, hence the wealth processes are well-defined.

Proposition 1. *For any non-random initial capitals $Y_0^m > 0$ and strategies λ^m, $m = 1, \dots, M$, there exists a unique adapted strictly positive càdlàg process $Y = (Y^1, \dots, Y^M)$ which satisfies (6) and $Y_- > 0$.*

3 Survival strategies

3.1 The notion of survival

For given initial capitals Y_0^m, investment strategies λ^m, and the corresponding wealth processes Y^m, define the process of total market wealth W and the relative wealth r^m of investor m:

$$W_t = |Y_t|, \qquad r_t^m = \frac{Y_t^m}{W_t}.$$

In the case when it is necessary to emphasize that the introduced processes depend on the initial capitals and the strategies, we will use the notation $Y_t^m(Y_0, \Lambda)$ and $r_t^m(Y_0, \Lambda)$, where $\Lambda = (\lambda^1, \dots, \lambda^M)$ denotes a profile of strategies.

The central definition of the present paper is the notion of a *survival strategy*. We call a strategy λ survival, if for any initial capitals $Y_0^m > 0$, $m = 1, \dots, M$, and a strategy profile $\Lambda = (\lambda^1, \dots, \lambda^M)$ with $\lambda^1 = \lambda$ and arbitrary strategies λ^m, $m = 2, \dots, M$, with probability one it holds that

$$\inf_{t \geq 0} r_t^1(Y_0, \Lambda) > 0,$$

i.e. a survival strategy guarantees that an investor who uses it will always have a share in the total wealth bonded away from zero.

As observed above, $Y > 0$ and $Y_- > 0$, hence the notion of survival can be equivalently stated as that $\liminf_{t\to\infty} r_t^1(Y_0, \Lambda) > 0$.

Equivalently, it can be also reformulated as the property that there exists a strictly positive random variable C (generally, depending on the initial capitals Y_0^m and the strategy profile Λ) such that

$$Y_t^1 \geq C Y_t^m \text{ for all } m \text{ and } t \geq 0,$$

i.e. no strategy can provide the asymptotic growth of wealth faster than a survival strategy.

3.2 Construction of a survival strategy

We will now explicitly construct a candidate survival strategy. Its survival property, as well as other asymptotic optimality properties, will be established in Section 4. The exposition below relies on several known facts from stochastic calculus, which can be found, for example, in [10].

Let us split the process X_t into the continuous part X_t^c and the sum of jumps, i.e.

$$X_t = X_t^c + \sum_{s \leq t} \Delta X_s,$$

where X_t^c is a continuous non-decreasing process, $\Delta X_s = X_s - X_{s-}$, and for $s = 0$ we set $\Delta X_0 = 0$. It will be convenient to work with jumps ΔX_t and ΔV_t using the measure of jumps of the $(N+1)$-dimensional process (X_t, V_t). It is defined as the integer-valued random measure on $(\mathcal{S}, \mathcal{B}(\mathcal{S}))$, where $\mathcal{S} = \mathbb{R}_+ \times \mathbb{R}_+^{N+1}$ and $\mathcal{B}$ is the Borel σ-algebra, by the formula

$$\mu(\omega, A) = \sum_{t \geq 0} \mathrm{I}(\Delta(X_t, V_t)(\omega) \neq 0,\ (t, \Delta(X_t, V_t)(\omega)) \in A), \qquad A \in \mathcal{B}(\mathcal{S})$$

(actually, we can assume $\mathcal{S} = \mathbb{R}_+ \times \mathbb{R}_+^N \times [0, \gamma_V]$, where γ_V is the constant from bound (4)). For an integral of an $\mathcal{F} \otimes \mathcal{B}(\mathcal{S})$-measurable function with respect to a random measure we will use the notation

$$f * \mu_t(\omega) = \int_{(0,t]\times\mathbb{R}_+^{N+1}} f(\omega, s, x, v)\mu(\omega, ds, dx, dv), \tag{7}$$

assuming that the integral is well-defined (as a Lebesgue integral), possibly being $+\infty$ or $-\infty$. Henceforth, the variable $x \in \mathbb{R}_+^N$ corresponds to jumps of X, and $v \in \mathbb{R}_+$ to jumps of V. The integral (7) can be defined for a general random measure; in the particular case when μ is the measure of jumps of (X, V), it can be simply written as the sum

$$f * \mu_t(\omega) = \sum_{s\le t} f(\omega, s, \Delta X_s(\omega), \Delta V_s(\omega))\mathrm{I}(\Delta(X_s, V_s)(\omega) \neq 0).$$

In the case when f is a vector-valued measurable function, we treat the integral (7) as vector-valued and compute it coordinatewisely. In particular, the process X_t can be represented in the form

$$X_t = X_t^c + x * \mu_t,$$

where $x = (x^1, \ldots, x^N)$ and x^n stands for the function $(x, v) \mapsto x^n$.

Let $\widetilde{\mathcal{P}} = \mathcal{P} \otimes \mathcal{B}(\mathbb{R}_+^{N+1})$ be the predictable σ-algebra on $\Omega \times \mathbb{R}_+ \times \mathbb{R}_+^{N+1}$. Recall that a random measure ν is called predictable, if for any $\widetilde{\mathcal{P}}$-measurable non-negative function $f(\omega, t, x, v)$ the process $f * \nu_t$ is predictable ($\mathcal{P}$-measurable). From now on, let ν denote the compensator of the measure of jumps μ, i.e. a predictable random measure such that for any $\widetilde{\mathcal{P}}$-measurable non-negative function f holds the equality

$$\mathrm{E}(f * \mu_\infty) = \mathrm{E}(f * \nu_\infty),$$

or, equivalently, $f * (\mu - \nu)_t$ is a local martingale, provided that the process $|f| * \mu_t$ is locally integrable. The measure of jumps of an adapted càdlàg process always has a compensator, which is unique up to indistinguishability with respect to P [10, § II.1], hence in our model ν is well-defined. Since the processes X and V do not decrease, the inequality $(|x| \wedge 1 + v) * \nu_t < \infty$ holds a.s. for all t, see [13, § 4.1].

From the general theory, it is known that there exists a predictable càdlàg non-decreasing locally integrable scalar process G (an *operational time process*) such that, up P-indistinguishability,

$$X_t^c = b \cdot G_t, \qquad \nu(\omega, dt, dx, dv) = K_{\omega,t}(dx, dv)dG_t(\omega), \tag{8}$$

where b_t is a predictable process with values in $\mathbb{R}_+^N$, and $K_{\omega,t}(dx, dv)$ is a transition kernel from $(\Omega \times \mathbb{R}_+, \mathcal{P})$ to $(\mathbb{R}_+^{N+1}, \mathcal{B}(\mathbb{R}_+^{N+1}))$ which for all ω, t satisfies the properties

$$K_{\omega,t}(\{0\}) = 0, \qquad \int_{\mathbb{R}_+^{N+1}} (|x| \wedge 1 + v)K_{\omega,t}(dx, dv) < \infty.$$

Fore example, one can use the process

$$G_t = |X_t^c| + (|x| \wedge 1 + v) * \nu_t. \tag{9}$$

The possibility of representation (8) for this process can be proved similarly to Proposition II.2.9 in [10].

For b, K, G satisfying (8), define the predictable process a_t with values in $\mathbb{R}_+^N$ by the formula

$$a_t^n(\omega) = \int_{\mathbb{R}_+^{N+1}} \frac{x^n}{1 - v + |x|/W_{t-}(\omega)} K_{\omega,t}(dx, dv), \tag{10}$$

and define the strategy $\widehat{\lambda}$ by

$$\widehat{\lambda}_t = \frac{a_t + b_t}{|a_t + b_t|}, \tag{11}$$

where we put $\widehat{\lambda}_t^n = 1/N$ for all n whenever $|a_t + b_t| = 0$. This strategy will be a candidate for a survival strategy. Note that the continuous part of the process V is not involved in its construction.

Observe that the strategy $\widehat{\lambda}$ does not essentially depend on the choice of an operational time process in the following sense. Let the process G be defined by (9). Define the measure $\mathrm{Q} = \mathrm{P} \otimes G$ on $(\Omega \times \mathbb{R}_+, \mathcal{P})$, i.e. for $A \in \mathcal{P}$

$$\mathrm{Q}(A) = \mathrm{E}\biggl(\int_0^\infty \mathrm{I}((\omega, t) \in A) dG_t(\omega) \biggr).$$

Proposition 2. *Suppose G' is another operational time process satisfying (8) and let $\widehat{\lambda}, \widehat{\lambda}'$ be the strategies constructed with respect to G, G' as described above. Then $\widehat{\lambda} = \widehat{\lambda}'$ (Q-a.s.).*

Moreover, for any initial capitals $Y_0^m > 0$, $m = 1, \dots, M$, and strategy profiles $\Lambda = (\widehat{\lambda}, \lambda^2, \dots, \lambda^M)$, $\Lambda' = (\widehat{\lambda}', \lambda^2, \dots, \lambda^M)$ with arbitrary strategies λ^m, $m = 2, \dots, M$, the corresponding wealth processes are equal, i.e. $Y(Y_0, \Lambda) = Y(Y_0, \Lambda')$.

Remark 1. Obviously, the discrete-time model of Section 2.1 is a particular case of the general model. In discrete time, $\Delta X_t = A_t$, $\Delta V_t = \delta_t$, and one can take $G_t = [t]$ (the integer part). Then $X_t^c = 0$, and $K_{\omega,t}(dx, dv)$ is the regular conditional distribution of the pair (A_t, δ_t) with respect to $\mathcal{F}_{t-1}$. By straightforward computation, we find

$$a_t^n = W_{t-1} \mathrm{E}\biggl(\frac{A_t^n}{W_t} \biggm| \mathcal{F}_{t-1} \biggr), \qquad b_t = 0, \qquad \widehat{\lambda}_t = \frac{a_t}{|a_t|}. \tag{12}$$

4 Main results

4.1 Statements

In this section we assume given and fixed an operational time process G for which representation (8) holds, and $a, b, K, \widehat{\lambda}$ constructed from G as described in the previous section. To formulate the results, let us also introduce the predictable scalar process

$$H_t = \frac{|a+b|}{W_-} \cdot G_t.$$

Proposition 3. *The process H is finite: $H_t < \infty$ a.s. for all $t \geq 0$.*

For an adapted scalar process L_t we will use the notation $\mathcal{M}(L) = \{\tau_l(L),\ l \in \mathbb{R}_+\}$ for the class of stopping times when L exceeds a level l for the first time: $\tau_l(L) = \inf\{t \geq 0 : L_t \geq l\}$, where $\inf \emptyset = +\infty$.

Our first main result formulated in the following theorem states that if a strategy λ is close to $\widehat{\lambda}$ in a certain sense, then it is survival. In particular, $\widehat{\lambda}$ itself is survival.

Theorem 1. *Suppose a strategy λ satisfies the following conditions:*

(a) $\mathrm{P}(\exists\, t : \lambda_t^n = 0,\ \widehat{\lambda}_t^n \neq 0) = 0$ *for all n,*

(b) the process $U_t = \widehat{\lambda}_t(\ln \widehat{\lambda}_t - \ln \lambda_t)$ *satisfies* $U \cdot H_\infty < \infty$*,*

(c) $\mathrm{E}(U_\tau \Delta H_\tau \mathrm{I}(\tau < \infty)) < \infty$ *for any* $\tau \in \mathcal{M}(U \cdot H)$*.*

Then, if investor m uses the strategy λ, the limit $\lim_{t\to\infty} r_t^m > 0$ exists with probability one for any strategies λ^k of the other investors. In particular, the strategy λ is survival.

The proximity of a strategy λ to $\widehat{\lambda}$ is essentially determined by condition (b), while (a) and (c) are technical assumptions. Let us clarify that in conditions (b), (c) on the sets $\{\lambda_t^n = 0\}$ and $\{\widehat{\lambda}_t^n = 0\}$ the corresponding term in the scalar product in the definition of U_t is assumed to be zero. Also observe that the process U is non-negative as follows from Gibbs' inequality (see also Lemma 1 below). Therefore, the integral $U \cdot H_\infty$ and the expectation in condition (c) are always well-defined, though they may take on the value $+\infty$.

The next simple proposition can be useful for verification of conditions (a), (b) of Theorem 1 in particular models. Regarding (c), a simple sufficient condition for its validity is the continuity of the process G (and, hence, the continuity of H). For example, G is continuous when X and V are non-decreasing Lévy processes.

Proposition 4. *Suppose the processes X, V are such that the strategy $\widehat{\lambda}$ satisfies the inequality $\inf_{t\geq 0} \widehat{\lambda}_t^n > 0$ for all n. In that case, if a strategy λ satisfies the inequalities $\inf_{t\geq 0} \lambda_t^n > 0$ for all n and $\|\widehat{\lambda} - \lambda\|^2 \cdot H_\infty < \infty$, then it satisfies conditions (a), (b) of Theorem 1.*

The next result shows that all survival strategies are, in a certain sense, asymptotically close to the strategy $\widehat{\lambda}$.

Theorem 2. *If a strategy λ is survival, then $\|\widehat{\lambda} - \lambda\|^2 \cdot H_\infty < \infty$.*

The third theorem formulated below shows that a survival strategy dominates in the market, i.e. the relative wealth of an investor who uses it tends to 1 as $t \to \infty$, if the *representative strategy* of the other investors is essentially different from $\widehat{\lambda}$ in a certain sense. By the representative strategy of investors $k \neq m$ we call the predictable process $\widetilde{\lambda}$ which is the weighted sum of the strategies of these investors with their relative wealths as the weights:

$$\widetilde{\lambda}_t^n = \frac{1}{1 - r_{t-}^m} \sum_{k \neq m} \lambda_t^{k,n} r_{t-}^k.$$

Notice that $|\widetilde{\lambda}_t| = 1$.

Theorem 3. *Suppose investor m uses a strategy λ which satisfies the conditions of Theorem 1. Let $\widetilde{\lambda}$ be the representative strategy of the other investors. Then $\lim_{t\to\infty} r_t^m = 1$ a.s. on the set $\{\|\widehat{\lambda} - \widetilde{\lambda}\|^2 \cdot H_\infty = \infty\}$.*

The last result draws parallels between survival strategies in our model and growth optimal (or log-optimal) strategies in asset market models with exogenous asset prices (see the discussion in the next section). To state it, let us define the asymptotic growth rate of investor m's wealth Y_t^m as

$$\limsup_{t\to\infty} \frac{1}{t} \ln Y_t^m$$

(similarly to the definition of the asymptotic growth rate in asset market models with exogenous prices, see, e.g., [12, Chapter 3.10]), and define the growth rate of wealth Y_t^m between moments of time $s < t$ as

$$\frac{1}{t-s} \mathrm{E}\biggl(\ln \frac{Y_t^m}{Y_s^m} \biggm| \mathcal{F}_s\biggr).$$

Theorem 4. *1) If investor m uses a survival strategy, then this investor achieves the maximal asymptotic growth rate of wealth in the market: for any k*

$$\limsup_{t\to\infty} \frac{1}{t} \ln Y_t^m \geq \limsup_{t\to\infty} \frac{1}{t} \ln Y_t^k.$$

2) Suppose investor m uses the strategy $\widehat{\lambda}$ and let $\widetilde{Y}_t = \sum_{k\neq m} Y_t^k$ denote the total wealth of the other investors. Then Y_t^m grows faster than $\widetilde{Y}_t$ between any moments of time $s < t$ such that $\mathrm{E}(|\ln W_t| \mid \mathcal{F}_s) < \infty$, i.e.

$$\mathrm{E}\biggl(\ln \frac{Y_t^m}{Y_s^m} \biggm| \mathcal{F}_s\biggr) \geq \mathrm{E}\biggl(\ln \frac{\widetilde{Y}_t}{\widetilde{Y}_s} \biggm| \mathcal{F}_s\biggr).$$

Note that in the second claim of Theorem 4, it is generally not possible to say that $\mathrm{E}(\ln(Y_t^m/Y_s^m) \mid \mathcal{F}_s) \geq \mathrm{E}(\ln(Y_t^k/Y_s^k) \mid \mathcal{F}_s)$ for any k if the number of investors $M \geq 3$.

Remark 2. As was noted above, all investment strategies considered in the present paper are basic in the sense that their components are functions of t and ω only. One could also extend the model by allowing general strategies, where λ_t may depend on paths of the processes Y, λ up to time t in an appropriate non-anticipative way. However, most of the above results will remain valid in such an extended setting as well. Let us give an heuristic argument for that without entering into technical details.

For example, assume that the strategies $\lambda_t^m(\omega, Y_{t-}(\omega))$ can also depend on the current wealth in a way such that the wealth equation admits a unique solution Y. Then we can consider the realizations of the strategies $\bar{\lambda}_t^m(\omega) = \lambda_t^m(\omega, Y_{t-}(\omega))$ (provided that they are predictable processes), and by inspecting the proofs, one can see that Theorems 1, 3 and 4 and Proposition 4 will remain valid, if it is additionally required that a strategy λ in their statements is basic. In particular, Theorem 1 implies that in a model with general strategies a survival strategy exists and can be found among basic strategies ($\widehat{\lambda}$ is such a strategy). However, only basic survival strategies will be asymptotically close to $\widehat{\lambda}$, i.e. Theorem 2 does not hold if one allows λ to be a general survival strategy. A counterexample is provided in the paper [2] for a different model, but it can be carried to our setting as well.

4.2 Relation to other results in the literature

In general, works that study long-run dynamics of asset markets based on ideas of natural selection of investment strategies can be attributed to the field of *Evolutionary Finance*, which has been developed since the 1990-2000s. Recent reviews (mostly of discrete-time models) can be found in [7, 9].

Let us first mention other works related to the results of Theorems 1, 2 and 3. A similar model in discrete time was studied in the paper [2]. Its main difference is that at every moment of time the whole wealth is reinvested, i.e. the wealth equation, instead of (1), is the following one:

$$Y_t^m = \sum_n \frac{\lambda_t^{m,n} Y_{t-1}^m}{\sum_k \lambda_t^{k,n} Y_{t-1}^k} A_t^n. \tag{13}$$

Notice that it can be formally obtained from (1) by taking $\delta_t = 1$. The main results of the paper [2] also consist in finding a survival strategy in an explicit form and proving that all survival strategies are asymptotically close to it. In that model, a survival strategy is defined by the same formula as (12) with $\delta_t = 1$ (one should put $v \equiv 1$ in (10)). As extension of the model was considered in the paper [6], where investors can also decide what part of their wealth they allocate for investment in the assets, and what part they keep in a risk-free account. This is expressed in that $\sum_n \lambda_t^{m,n}$ may be less than 1.

Similar results were also obtained in the paper [1] for a (more difficult) model in discrete time which assumes that investors can sell their assets at subsequent moments of time for the price determined by the market (through the balance of supply and demand) – such an assumption is natural for a model of a stock market. Quite remarkably, a survival strategy in that model also exists and can be found in the class of basic strategies which depend only on the structure of dividend sequences, but not on actions of investors.

Let us also mention the paper [3] (see also the subsequent paper [4]) – one of the first in this direction – where a result similar to our Theorem 1 was obtained (among other results). The model considered in that paper is a simple particular case of (13), where at each moment of time only one asset yields a payoff and its amount, if paid, is known in advance.

Note that the model (13) cannot be straightforwardly generalized to the case of continuous time, since a continuous-time model should allow that during an "infinitesimally short" period of time the payoff A_t can be "infinitesimally small" – but then equation (13) makes no sense. There is no such problem in our model due to the assumption $\delta_t < 1$.

Among (few) other results for the case of continuous time, let us mention the paper [14], where convergence of a discrete-time model to a continuous-time model was studied, and the paper [15], where questions of survival and dominance of investment strategies were investigated in the case when payoffs are specified by absolutely continuous non-decreasing processes.

Finally, with regard to Theorem 4, one can see that survival strategies, and in particular $\widehat{\lambda}$, are similar to growth optimal strategies (also called log-optimal strategies, benchmark strategies, numéraire portfolios) in models of mathematical finance without competition, in the sense that they lead to the fastest growth of wealth. An account on growth optimal strategies can be found, for example, in [5, Chapter 5], [8] for discrete-time models, and in [11, 16] for general continuous-time models. However, note that there is an essential difference between survival strategies and growth optimal strategies: the latter are obtained as solutions of single-agent optimization problems for wealth processes, while the former cannot be obtained in such a way because competitors' strategies are unknown to an investor.

5 Proofs

Before we proceed to the proofs, let us briefly recall, for the reader's convenience, the notion of the stochastic exponent, which will be used in several places below. If Z is a scalar semimartingale, then the stochastic exponent $\mathcal{E}(Z)$ of Z is the semimartingale that solves the stochastic differential equation (which always has a unique strong solution, see [10, § I.4f])

$$d\mathcal{E}(Z)_t = \mathcal{E}(Z)_{t-}dZ_t, \qquad \mathcal{E}(Z)_0 = 1.$$

In all the cases we are going to consider, only adapted càdlàg processes with finite variation will be used as Z, so this equation should be understood in the sense of pathwise Lebesgue–Stieltjes integration. The Dolean–Dade formula implies that in this case

$$\mathcal{E}(Z)_t = e^{Z_t}\prod_{s\le t}(1+\Delta Z_s)e^{-\Delta Z_s}. \tag{14}$$

In particular, if $\Delta Z > -1$, then $\mathcal{E}(Z) > 0$ and $\mathcal{E}(Z)_- > 0$.

Proof of Proposition 1. Introduce the function $F\colon \mathbb{R}_+^{MN+M} \to \mathbb{R}_+^{MN}$ which specifies the distribution of payoffs in equation (6):

$$[F(\lambda, y)]^{m,n} = \frac{\lambda^{m,n}y^m}{\sum_k \lambda^{k,n}y^k}.$$

When the denominator equals 0 for some n, we define $[F(\lambda, y)]^{m,n} = 1/M$. It is straightforward to check that $|\partial[F(\lambda, y)]^{m,n}/\partial y^k| \le 1/y^k$. Hence, F is Lipschitz continuous in y on any set $\{y : y^m \ge a \text{ for all } m\}$, where a is a positive constant. Let $l(a)$ be a function, defined for $a \in (0,\infty)$, such that $|F(\lambda, y) - F(\lambda, \widetilde{y})| \le l(a)|y - \widetilde{y}|$ for any $\lambda \in \mathbb{R}_+^{MN}$ and $y, \widetilde{y} \in \mathbb{R}_+^M$ with $y^m, \widetilde{y}^m \ge a$ for all m. We can assume that $l(a) \ge 1$ for any a and $l(a)$ is bounded on any compact set in $\mathbb{R}_+ \setminus \{0\}$.

Let $y^* = \min_m Y_0^m$. Define the sequence of stopping times τ_i with $\tau_0 = 0$ and, for $i \ge 1$,

$$\begin{aligned}\tau_i = \inf\biggl\{ t \ge \tau_{i-1} : |X_t| \ge |X_{\tau_{i-1}}| + \frac{1}{4l(y^*\mathcal{E}(-V)_{\tau_{i-1}}/2)} \quad \text{or} \\ V_t \ge V_{\tau_{i-1}} + \frac{1}{4} \wedge \frac{y^*\mathcal{E}(-V)_{\tau_{i-1}}}{2(y^* + |X_{\tau_{i-1}}| + 1/4)} \biggr\} \wedge (\tau_{i-1}+1),\end{aligned}$$

where $\inf\emptyset = \infty$. It is not hard to see that $\tau_i \le i$ and $\tau_i \to \infty$ as $i \to \infty$.

We will construct a solution of (6) by induction on the intervals $[0, \tau_i]$. Namely, we will define a sequence of adapted càdlàg processes $Y^{(i)}$ with finite variation on any interval $[0, t]$ that satisfy equation (6) on $[0, \tau_i]$ and have the property

$$Y_t^{(i)} = Y_t^{(i-1)} \text{ for } t \le \tau_{i-1}.$$

For $i = 0$, let $Y_t^{(0)} = Y_0$ for all $t \ge 0$. Suppose the process $Y^{(i-1)}$ is constructed. Observe that equation (6) implies that for each m

$$Y_0^m \mathcal{E}(-V)_t \le Y_t^{(i-1),m} \le Y_0^m + |X_t| \text{ for } t \le \tau_{i-1}. \tag{15}$$

Here, the right inequality is clear, and the left one follows from that the process $Y^{(i-1),m}/\mathcal{E}(-V)$ is non-decreasing, which can be seen by computing its stochastic differential.

Let us now construct $Y^{(i)}$. Consider the Banach space of bounded càdlàg functions $f\colon \mathbb{R}_+ \to \mathbb{R}^M$ with the norm $\|f\| = \sup_{t\ge 0}|f_t|$ and denote by $\mathbb{D}$ is closed subset consisting of f with values in $\mathbb{R}^M_+$. For each ω, consider the operator H which maps $f \in \mathbb{D}$ to

$$\begin{aligned} H(\omega, f)_t = {} & Y^{(i-1)}_{t\wedge\tau_{i-1}} \\ & + \mathrm{I}(t > \tau_{i-1}) \int_{(\tau_{i-1},t]} \mathrm{I}(u < \tau_i)(F(\lambda_u, f_{u-})dX_u - f_{u-}dV_u), \end{aligned} \tag{16}$$

where the random variables in the right-hand side are evaluated for a given ω. Notice that H preserves the adaptedness of processes in the sense that if Y is an adapted process, then so is $H(Y)$.

For each ω, introduce the set

$$\mathbb{D}^{(i)}(\omega) = \Big\{ f \in \mathbb{D} : \frac{1}{2} Y_0^m \mathcal{E}(-V)_{\tau_{i-1}}(\omega) \le f_t^m \le Y_0^m + |X_{\tau_{i-1}}(\omega)| + \frac{1}{4} \\ \text{for all } m \text{ and } t \ge \tau_{i-1}(\omega) \Big\}.$$

Let us show that $H(\omega)$ maps $\mathbb{D}^{(i)}(\omega)$ into itself. Indeed, a function $H(f)$ is càdlàg. The upper bound for $H(f)$ in the definition of $\mathbb{D}^{(i)}$ follows from the inequalities (for $t \ge \tau_{i-1}$)

$$H(f)_t^m \le Y^m_{\tau_{i-1}} + |X_{\tau_i -}| - |X_{\tau_{i-1}}| \le Y_0^m + |X_{\tau_i-}| + \frac{1}{4},$$

where the first inequality here follows from (16) with the bound $|F(\lambda, y)| \le 1$, and the second one follows from the right inequality in (15) and the estimate $|X_{\tau_i-}| - |X_{\tau_{i-1}}| \le 1/4$ which holds by the choice of τ_i.

The lower bound for $H(f)$ follows from that for $t \ge \tau_{i-1}$

$$\begin{aligned} H(f)_t^m &\ge Y^m_{\tau_{i-1}} - \int_{(\tau_{i-1},\tau_i)} f_{u-}dV_u \\ &\ge Y_0^m \mathcal{E}(-V)_{\tau_{i-1}} - \Big(Y_0^m + |X_{\tau_{i-1}}(\omega)| + \frac{1}{4}\Big)(V_{\tau_i-} - V_{\tau_{i-1}}) \\ &\ge \frac{1}{2} Y_0^m \mathcal{E}(-V)_{\tau_{i-1}}, \end{aligned}$$

where the second inequality holds due to (15) and the upper bound for $f \in \mathbb{D}^{(i)}$, while the third inequality is valid according to the choice of τ_i.

Thus H maps $\mathbb{D}^{(i)}$ into itself. Moreover, it is a contraction mapping since for any $f, \widetilde{f} \in \mathbb{D}^{(i)}$

$$\begin{aligned} \|H(f) - H(\widetilde{f})\| &\le \int_{(\tau_{i-1},\tau_i)} (|F(\lambda_t, f_{t-}) - F(\lambda_t, \widetilde{f}_{t-})|dX_t + |f_{t-} - \widetilde{f}_{t-}|dV_t) \\ &\le \frac{1}{2}\|f - \widetilde{f}\|. \end{aligned}$$

Here, in order to bound the integral with respect to dX_t, we use that $f_{t-}, \widetilde{f}_{t-} \geq y^*\mathcal{E}(-V)_{\tau_{i-1}}/2$ by the definition of $\mathbb{D}^{(i)}$, hence the integrand can be bounded by $|f_{t-} - \widetilde{f}_{t-}| l(y^*\mathcal{E}(-V)_{\tau_{i-1}}/2)$, and so the value of the integral does not exceed $\|f - \widetilde{f}\|/4$ because $|X_{\tau_i-}| - |X_{\tau_{i-1}}| \leq 1/(4l(y^*\mathcal{E}(-V)_{\tau_{i-1}}/2))$. In a similar way, the integral with respect to dV_t is also not greater than $\|f - \widetilde{f}\|/4$ because $V_{\tau_i-} - V_{\tau_{i-1}} \leq 1/4$.

Consequently, H has a fixed point $\widetilde{Y}$, which satisfies equation (6) on the half-interval $[0, \tau_i)$. The process $\widetilde{Y}$ is adapted, since it can be obtained as the limit (for each ω and t) of the adapted processes $H^n(Y^{(i-1)})$ as $n \to \infty$, where n stands for n-times application of H. Define

$$Y_t^{(i)} = \widetilde{Y}_t \mathrm{I}(t < \tau_i) + \Big[\widetilde{Y}_{\tau_i-} + F(\lambda_{\tau_i}, \widetilde{Y}_{\tau_i-})\Delta X_{\tau_i} - \widetilde{Y}_{\tau_i-}\Delta V_t\Big] \mathrm{I}(t \geq \tau_i).$$

Then $Y^{(i)}$ is the sought-for process which satisfies (6) on the whole interval $[0, \tau_i]$. The strict positivity of $Y^{(i)}$ and $Y_-^{(i)}$ follows from the left inequality in (15).

The uniqueness of the solution of (6) follows from the uniqueness of the fixed point of the operator H on each step of the induction. □

Proof of Proposition 2. Suppose G' is a predictable process satisfying (8) with some process b' and transition kernel K'. Then the random measure generated by G on $\mathbb{R}_+$ is a.s. absolutely continuous with respect to the measure generated by G', and, according to [10, Proposition I.3.13], there exists a non-negative predictable process ρ such that $G = \rho \cdot G'$. Hence $X^c = b \cdot G = (\rho b) \cdot G'$, while, at the same time, $X^c = b' \cdot G'$, so we have $\rho b = b'$ ($\mathrm{P} \otimes G'$-a.s. and, hence, Q-a.s.). In a similar way, $\rho K = K'$ (Q-a.s.), which implies $\rho a = a'$ (Q-a.s.), where a, a' are the processes defined by (10) with respect to K and K'. Then, by (11), we have $\widehat{\lambda} = \widehat{\lambda}'$ (Q-a.s.), which proves the first claim of the proposition.

To prove the second claim, observe that if f_t, f'_t are predictable non-negative processes such that $f = f'$ (Q-a.s.), then $f \cdot X = f' \cdot X$. Hence, if in the wealth equation (5) we replace the strategy $\lambda^1 = \widehat{\lambda}$ by $\widehat{\lambda}'$, the wealth process $Y_t(Y_0, \Lambda)$ will remain its solution (up to P-indistinguishability). □

Proof of Proposition 3. Define the sequence of stopping times τ_i, $i \geq 0$, with $\tau_0 = 0$ and

$$\tau_i = \inf\{t \geq \tau_{i-1} : W_t \leq 1/i \text{ or } W_t \geq i\} \wedge (\tau_{i-1} + 1), \qquad i \geq 1.$$

Then we have

$$H_{\tau_i-} \leq i(|a| + |b|) \cdot G_{\tau_i-} \leq i\left(i \wedge \frac{|x|}{1 - \gamma_V}\right) * \nu_{\tau_i-} + i|X^c_{\tau_i-}| < \infty,$$

where in the second inequality we used the bound $|a_t| \leq \int_{\mathbb{R}_+^{N+1}} (i \wedge |x|/(1 - \gamma_V)) K_t(dx, dv)$ for $t < \tau_i$, which follows from (10), and in the last inequality used the property $(|x| \wedge 1) * \nu_t < \infty$. Since $W > 0$ and $W_- > 0$, we have $\tau_i \to \infty$ as $i \to \infty$, and, hence, $H_t < \infty$ for all $t \geq 0$. $\square$

In the following lemma we establish an auxiliary inequality that will be used in the subsequent proofs. To state it, let us introduce the function

$$\overline{\ln}\, x = \begin{cases} \ln x, & \text{if } x > 0, \\ -1, & \text{if } x \leq 0. \end{cases}$$

Lemma 1. *Suppose vectors $\alpha, \beta \in \mathbb{R}^N_+$ are such that $|\alpha| = |\beta| = 1$, and, for each n, we have $\alpha^n = 0$ if $\beta^n = 0$. Then*

$$\alpha(\overline{\ln}\,\alpha - \overline{\ln}\,\beta) \geq \frac{\|\alpha - \beta\|^2}{4}. \tag{17}$$

Proof. For vectors with strictly positive coordinates this inequality follows from the inequality for the Kullback–Leibler and Hellinger–Kakutani distances (a direct proof can be found, for example, in [2], Lemma 2): it is sufficient to consider α, β as probability distributions on a set of N elements. For vectors which may have null coordinates, instead of α, β one can take $c\alpha + (1-c)u$, $c\beta + (1-c)u$, where $c \in (0,1)$ and u is a vector with strictly positive coordinates and $|u| = 1$. Then let $c \to 1$ and use the continuity of the function $x\,\overline{\ln}\,x$ and the norm $\|\cdot\|$ to obtain (17). $\square$

Proof of Theorem 1. Suppose investor m uses a strategy λ which satisfies conditions (a)–(c). First we are going to prove that the process $S_t = \ln r_t^m + U \cdot H_t$ is a local submartingale.

It will be convenient to represent the wealth process Y^m and the process of total wealth W as stochastic exponents. Put, for brevity,

$$\theta_t = \frac{1}{W_{t-}},$$

and associate with the strategy $\lambda^m = \lambda$ of investor m the N-dimensional predictable process F_t with the components

$$F_t^n = \frac{\lambda_t^{m,n}\theta_t}{\sum_k \lambda_t^{k,n} r_{t-}^k},$$

where in the case of the indeterminacy $0/0$ we define $F_t^n = (MW_{t-}r_{t-}^m)^{-1}$. Then the processes Y_t^m and W_t satisfy the equations

$$dY_t^m = Y_{t-}^m(F_t dX_t - dV_t), \qquad dW_t = d|X_t| - W_{t-}dV_t.$$

Therefore, they can be represented in the form

$$Y_t^m = Y_0^m \mathcal{E}(F \cdot X - V)_t, \qquad W_t = W_0 \mathcal{E}(\theta \cdot |X| - V)_t.$$

Let $Z_t = \ln r^m_t = \ln \mathcal{E}(F \cdot X - V)_t - \ln \mathcal{E}(\theta \cdot |X| - V)_t + Z_0$. As follows from (14),

$$Z_t = (F-\theta)\cdot X^c_t + \sum_{s\le t} \ln\left(\frac{1-\Delta V_s + F_s \Delta X_s}{1-\Delta V_s + \theta_s|\Delta X_s|}\right) + Z_0$$

(in the first term, θ is subtracted from each coordinate of F). Define the predictable function $f\colon \Omega \times \mathbb{R}_+ \times \mathbb{R}^{N+1}_+ \to \mathbb{R}$ by

$$f(\omega,t,x,v) = \ln\left(\frac{1-v+F_t(\omega)x}{1-v+\theta_t(\omega)|x|}\right).$$

Using this function, it is possible to write

$$Z_t = (F-\theta)\cdot X^c_t + f * \mu_t + Z_0.$$

Let us prove the representation

$$Z_t = g\cdot G_t + f*(\mu-\nu)_t + Z_0 \tag{18}$$

with the function

$$g_t = (F_t-\theta_t)b_t + \int_{\mathbb{R}^{N+1}_+} f_t(x,v)K_t(dx,dv).$$

To prove (18), it is sufficient to show that $f*\nu_t < +\infty$ and $g\cdot G_t > -\infty$ for all t (and then we will also have $|f|*\nu_t < \infty$).

Consider the stopping times $\tau_i = \inf\{t\ge 0 : r^m_t \le 1/i \text{ or } W_t \le 1/i\}$ with $\inf\emptyset = +\infty$. It is not difficult to see that $F^n_t \le \theta_t(r^m_{t-})^{-1}$ for all n. Then $f \le \frac{i^2|x|}{1-\gamma_V+i|x|}$ on the set $\{t<\tau_i(\omega)\}$. Since $(|x|\wedge 1)*\nu_t < \infty$ for all t, we have $f*\nu_{\tau_i-} < +\infty$. Because $\tau_i\to\infty$ as $i\to\infty$ (due to the strict positivity of W, W_-, r^m, and r^m_-), passing to the limit $i\to\infty$, we obtain $f*\nu_t<+\infty$ for all t.

Let us prove that $g\cdot G_t > -\infty$ for all t. Define the set $\mathcal{X}(\omega,t) = \{(x,v)\in \mathbb{R}^{N+1}_+\setminus\{0\} : x^n = 0 \text{ if } F^n_t(\omega)=0,\ n=1,\dots,N\}$. Using Jensen's inequality and the concavity of the logarithm, we find that for any $(x,v)\in\mathcal{X}(\omega,t)$

$$\begin{aligned} f_t(x,v) &= \ln\left(\frac{1-v}{1-v+\theta_t|x|} + \frac{\theta_t|x|}{1-v+\theta_t|x|}\frac{F_tx}{\theta_t|x|}\right)\\ &\ge \frac{\theta_t|x|}{1-v+\theta_t|x|}\ln\left(\frac{F_tx}{\theta_t|x|}\right) \ge \frac{\theta_t x\,\overline{\ln}(F_t/\theta_t)}{1-v+\theta_t|x|}.\end{aligned}$$

This implies that for each t

$$\int_{\mathbb{R}^{N+1}_+} f_t(x,v)K_t(dx,dv) = \int_{\mathcal{X}_t} f_t(x,v)K_t(dx,dv) \ge \theta_t a_t\,\overline{\ln}(F_t/\theta_t),$$

where we use that $K_t(\mathbb{R}_+^{N+1}\backslash\mathcal{X}_t) = 0$. Indeed, the set $\mathbb{R}_+^{N+1}\backslash\mathcal{X}(\omega,t)$ consists of (x,v) such that $F_t^n(\omega) = 0$ but $x^n > 0$ for some n. On the set $\{F_t^n = 0\}$ we have $\lambda_t^{m,n} = 0$, so, by condition (a) of the theorem, $\widehat{\lambda}_t^n = 0$ a.s. on this set, and therefore $K_t(\{x^n > 0\}) = 0$.

Then we can write

$$\begin{aligned} g_t &\geq (F_t - \theta_t)b_t + \theta_t a_t \,\overline{\ln}(F_t/\theta_t) \geq \theta_t(a_t + b_t)\,\overline{\ln}(F_t/\theta_t) \\ &\geq \widehat{\lambda}_t(\overline{\ln}\,\lambda_t - \overline{\ln}\,\pi_t)|a_t + b_t|\theta_t \end{aligned} \tag{19}$$

(in the second inequality we used that $F_t^n/\theta_t - 1 \geq \overline{\ln}(F_t^n/\theta_t)$), where

$$\pi_t^n = \sum_k \lambda_t^{k,n} r_{t-}^k. \tag{20}$$

Note that $|\pi_t| = 1$. Applying Lemma 1 to the vectors $\alpha = \widehat{\lambda}_t$ and $\beta = \pi_t$, from formula (19) we find

$$g_t \geq \widehat{\lambda}_t(\overline{\ln}\,\lambda_t - \overline{\ln}\,\widehat{\lambda}_t)|a_t + b_t|\theta_t = -U_t|a_t + b_t|\theta_t,$$

where, to obtain the equality, we changed $\overline{\ln}$ to $\ln$, which is possible due to condition (a). Then, by condition (b), $g \cdot G_t \geq -U \cdot H_t > -\infty$, which proves representation (18). In particular, $|f| * \nu_t < \infty$ for any t. Since a predictable non-decreasing finite-valued process is locally integrable [13, Lemma 1.6.1], the process $|f| * \nu_t$ is locally integrable, and, hence, $f * (\mu - \nu)_t$ is a local martingale. Therefore, $S_t = Z_t + U \cdot H_t$ is a local submartingale. Following a standard technique, let us show that this fact and condition (c) imply that Z_t has an a.s.-finite limit as $t \to \infty$.

Consider the sequence of stopping times

$$\tau_i = \inf\{t \geq 0 : U \cdot H_t \geq i\}, \qquad i \in \mathbb{N}, \tag{21}$$

where $\inf\emptyset = +\infty$. By condition (b), for a.a. ω we have $\tau_i(\omega) = \infty$ starting from some i. For each i, the process $S_t^{(i)} = S_{t\wedge\tau_i}$, $t \geq 0$, is a local submartingale and, moreover, for all $t \geq 0$

$$S_t^{(i)} \leq U \cdot H_{\tau_i} \leq i + U_{\tau_i}\Delta H_{\tau_i}\mathrm{I}(\tau_i < \infty). \tag{22}$$

From condition (c), it follows that the random variable in the right-hand side of the above inequality is integrable. Consequently, $S_t^{(i)}$ is a usual submartingale and there exists the a.s.-finite limit $\lim_{t\to\infty} S_t^{(i)} = S_{\tau_i}$ (by Doob's martingale convergence theorem, see Theorem I.1.39 in [10]). Letting $i \to \infty$, we obtain the existence of the a.s.-finite limit $S_\infty = \lim_{t\to\infty} S_t$ and $Z_\infty = S_\infty - U \cdot H_\infty$. This implies $\lim_{t\to\infty} r_t^m = \exp(Z_\infty) > 0$, which proves the theorem.

Proof of Proposition 4. It is clear that if the conditions of the proposition are satisfied then the strategy λ satisfies condition (a). Denote $\widehat{\xi} = \inf_{t,n} \widehat{\lambda}_t^n$ and $\xi = \inf_{t,n} \lambda_t^n$. Then we have the inequalities

$$(\lambda_t^n - \widehat{\lambda}_t^n)(\ln \lambda_t^n - \ln \widehat{\lambda}_t^n) \leq \frac{(\lambda_t^n - \widehat{\lambda}_t^n)^2}{\widehat{\lambda}_t^n} \quad \text{if } \lambda_t^n \geq \widehat{\lambda}_t^n,$$

$$(\lambda_t^n - \widehat{\lambda}_t^n)(\ln \lambda_t^n - \ln \widehat{\lambda}_t^n) \leq \frac{\ln(\xi)(\lambda_t^n - \widehat{\lambda}_t^n)^2}{(\xi - 1)\widehat{\lambda}_t^n} \quad \text{if } \lambda_t^n < \widehat{\lambda}_t^n,$$

where we used the inequalities $\ln(1+x) \leq x$ if $x \geq 0$ and $\ln(1+x) \geq x\varepsilon^{-1}\ln(1+\varepsilon)$ if $x \in [\varepsilon, 0]$, $\varepsilon > -1$, applied to $x = (\lambda_t^n - \widehat{\lambda}_t^n)/\widehat{\lambda}_t^n$ and $\varepsilon = \xi - 1$. Since $\lambda_t(\ln \lambda_t - \ln \widehat{\lambda}_t) \geq 0$ (by Lemma 1), we find

$$U_t \leq \frac{\ln(\xi)\|\lambda_t - \widehat{\lambda}_t\|^2}{(\xi - 1)\widehat{\xi}}.$$

Consequently, $U \cdot H_\infty < \infty$, so λ satisfies condition (b). $\square$

In order to prove Theorems 2 and 3, we will need the following auxiliary result.

Lemma 2. *Suppose investor m uses a strategy λ satisfying conditions (a)–(c) of Theorem 1, and let π_t be the process defined by (20). Then $\|\widehat{\lambda} - \pi\|^2 \cdot H_\infty < \infty$.*

Proof. In the course of proof of Theorem 1 we have established inequality (19). Together with (17), it implies

$$\|\widehat{\lambda} - \pi\|^2 \cdot H_\infty \leq 4\widehat{\lambda}(\overline{\ln}\,\widehat{\lambda} - \overline{\ln}\,\pi) \cdot H_\infty \leq 4(g \cdot G_\infty + U \cdot H_\infty).$$

It remains to show that $g \cdot G_\infty + U \cdot H_\infty < \infty$. Consider the stopping times τ_i defined in (21). Then for any $t \geq 0$

$$\mathrm{E}(g \cdot G_{t\wedge\tau_i} + U \cdot H_{t\wedge\tau_i}) = \mathrm{E}S_t^{(i)} + S_0 \leq i + \mathrm{E}(U_{\tau_i}\Delta H_{\tau_i}\mathrm{I}(\tau_i < \infty)),$$

where the equality holds because $g \cdot G_{t\wedge\tau_i} + U \cdot H_{t\wedge\tau_i}$ is the compensator of the submartingale $S^{(i)}$ defined in the proof of Theorem 1, and the inequality holds in view of (22) and that $S_0 = Z_0 \leq 0$. Passing to the limit $t \to \infty$, by the monotone convergence theorem $\mathrm{E}(g \cdot G_{\tau_i} + U \cdot H_{\tau_i}) < \infty$, and hence $g \cdot G_{\tau_i} + U \cdot H_{\tau_i} < \infty$. Passing to the limit $i \to \infty$, we obtain $g \cdot G_\infty + U \cdot H_\infty < \infty$, since $\tau_i(\omega) = \infty$ starting from some i for a.a. ω. $\square$

Proof of Theorem 2. Consider the market where investor 1 uses the strategy $\lambda^1 = \lambda$, and the other investors use the strategy $\widehat{\lambda}$, i.e. $\lambda^m = \widehat{\lambda}$, $m = 2, \dots, M$. In this case $\pi_t = \lambda_t r_{t-}^1 + \widehat{\lambda}_t(1 - r_{t-}^1)$ and $\|\widehat{\lambda} - \pi_t\| = r_{t-}^1\|\widehat{\lambda}_t - \lambda_t\|$.

Then from Lemma 2 we obtain $(r^1_-\|\widehat{\lambda}-\lambda\|)^2 \cdot H_\infty < \infty$. Since λ is a survival strategy, we have $\inf_t r^1_t > 0$. Therefore, $\|\widehat{\lambda}-\lambda\|^2 \cdot H_\infty < \infty$. $\square$

Proof of Theorem 3. We have $\pi_t = \lambda_t r^m_{t-} + (1-r^m_{t-})\widetilde{\lambda}_t$. By virtue of Lemma 2, we obtain

$$\|\widehat{\lambda}-\lambda+(1-r^m_-)(\lambda-\widetilde{\lambda})\|^2 \cdot H_\infty = \|\widehat{\lambda}-\pi\|^2 \cdot H_\infty < \infty.$$

Since the strategy λ is survival, $\|\widehat{\lambda}-\lambda\|^2 \cdot H_\infty < \infty$ by Theorem 2. From these inequalities, it follows that $(1-r^m_-)^2\|\lambda-\widetilde{\lambda}\|^2 \cdot H_\infty < \infty$. According to Theorem 1, there exists the a.s.-finite limit $r^m_\infty = \lim_{t\to\infty} r^m_t$. Then necessarily $r^m_\infty = 1$ a.s. on the set $\{\|\lambda-\widetilde{\lambda}\|^2 \cdot H_\infty = \infty\}$, which a.s. coincides with the set $\{\|\widehat{\lambda}-\widetilde{\lambda}\|^2 \cdot H_\infty = \infty\}$ as follows from Theorem 2. $\square$

Proof of Theorem 4. 1) If investor m uses a survival strategy, then for any strategies of the other investors the inequality $\inf_t r^m_t > 0$ holds with probability one. Then $\sup_t W_t/Y^m_t < \infty$ and therefore $\sup_t Y^k_t/Y^m_t < \infty$ for any k. Hence we obtain the inequality

$$\limsup_{t\to\infty} \frac{1}{t} \ln \frac{Y^k_t}{Y^m_t} \le 0,$$

which easily implies the first claim of the theorem.

2) From the proof of Theorem 1, it follows that if investor m uses the strategy $\widehat{\lambda}$, then $\ln r^m_t$ is a submartingale, so for any $s \le t$

$$\mathrm{E}(\ln r^m_t \mid \mathcal{F}_s) \ge \ln r^m_s. \tag{23}$$

Using that $\mathrm{E}(\ln W_t \mid \mathcal{F}_s)$ is finite-valued due to the assumption of the theorem, and adding to the both sides of above the inequality $\mathrm{E}(\ln(W_t/Y^m_s) \mid \mathcal{F}_s)$ we obtain

$$\mathrm{E}\biggl(\ln \frac{Y^m_t}{Y^m_s} \biggm| \mathcal{F}_s\biggr) \ge \mathrm{E}\biggl(\ln \frac{W_t}{W_s} \biggm| \mathcal{F}_s\biggr). \tag{24}$$

Let $\widetilde{r}_t = \widetilde{Y}_t/W_t = 1 - r^m_t$. From (23), by Jensen's inequality, we find

$$\mathrm{E}(\ln \widetilde{r}_t \mid \mathcal{F}_s) \le \ln \widetilde{r}_s,$$

where the conditional expectation may assume the value $-\infty$. Then, similarly to (24), we have

$$\mathrm{E}\biggl(\ln \frac{\widetilde{Y}_t}{\widetilde{Y}_s} \biggm| \mathcal{F}_s\biggr) \le \mathrm{E}\biggl(\ln \frac{W_t}{W_s} \biggm| \mathcal{F}_s\biggr),$$

which together with (24) proves the second claim of the theorem. $\square$

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