Local Stochastic Volatility: Intro

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Dupire Equation: Recap

Stock price process

$$dS_t = S_t(\mu_t dt + \sigma(S_t, t)dZ_t) \tag{1}$$

where μ_t is risk-neutral drift and $\sigma(S_t, t)$ is local volatility.
Let $C(S_0, K, T)$ be the undiscounted risk-neutral value of a European option with strike K and expiration T.

Theorem (Dupire Equation)

Given the formulas above, the following differential equation holds:

$$\frac{\partial C}{\partial T} = \frac{\sigma^2 K^2}{2}\frac{\partial^2 C}{\partial K^2} + \mu\left(C - K\frac{\partial C}{\partial K}\right)$$

Dupire Equation: Recap

Corollary (Forward Dupire Equation)

Let $F_T = S_0 e^{\int_0^T \mu_t dt}$ (forward price). Then for $C(F_T, K, T)$ we get the following differential equation:

$$\frac{\partial C}{\partial T} = \frac{\sigma^2 K^2}{2} \frac{\partial^2 C}{\partial K^2}$$

So, *local volatility* can be defined as

$$\sigma^2(K, T, S_0) := \frac{\dfrac{\partial C}{\partial T}}{K^2 \dfrac{1}{2} \dfrac{\partial^2 C}{\partial K^2}}$$

Rhs of the equation above can be calculated from European options prices for all strikes and expirations.

Local Variance as a Conditional Expectation: "Elegant" Derivation, Recap

Let $F_{t,T} = S_t e^{\int_t^T \mu_t dt}$ (forward price). Then it satisfies the following SDE:

$$dF_{t,T} = \sqrt{v_t} F_{t,T} dZ_t \tag{2}$$

where v_t is local variance.
It can be shown that

$$\mathbb{E}\left[v_T | S_T = K \right] = \frac{\dfrac{\partial C}{\partial T}}{\dfrac{1}{2} K^2 \dfrac{\partial^2 C}{\partial^2 K}}$$

Therefore, *local variance* can be interpreted as risk-neutral expectation of instantaneous variance given the condition that terminal S_T equals to strike K.

Local Volatility Models: Pros and Cons

Pros:

- consistence with today's market prices
- calibration may not require numerical optimization

Cons:

- poor market dynamics for spot and volatility
- volatility of volatility is not tuned
- forward implied volatility flattens out

Stochastic Volatility Models: Pros and Cons

Pros:

- more in line with market dynamics
- forward implied volatility has more realistic behavior
- able to tune vol of vol

Cons:

- needed to guarantee the stability of calibrated parameters
- changing of vol of vol may require re-calibrating other parameters
- usually does not fit well short term skew/smile

LSV Models: Motivation

DNT options pay one unit of domestic currency at expiry if the spot FX rate never trades equal to or outside a lower barrier level nor an upper barrier level. If either the lower barrier level or the upper barrier level are touched prior to expiry, the option expires worthless.

It was observed (see [1]) that a LV model tends to under-price DNT options relative to the market prices. On the other hand, a SV model, calibrated to the market prices of vanilla options, tends to over-price DNT options relative to the market prices. In either case, the degree of mispricing was found to be well in excess of the bid-offer spread.

One wants to mix the two models into a LSV model, with a mixing parameter calibrated such that we match the market prices of e.g. DNT options.

LSV Models: What About Jumps?

Including jumps may be needed to fit the smiles/skew in short term. It implies various costs:

- augment calibration procedure to include jump parameters
- pricing is done by a PIDE rather than a PDE

Market prices of exotic equity options depending on forward vol (e.g., cliquet options) imply very steep forward skews, so the value of vol-of-vol parameter must be very large (see [2], [3]). Hence, local vol models are inconsistent in such cases and one way for the calibration to provide prices which match well both exotics and vanilla options is to add infrequent but large jumps.

Results from [2] indicate that the LSV model with jumps can fit both Equity and VIX option skews.

Lipton Model

$$\begin{cases} \dfrac{dS_t}{S_t} = (r_d - r_f - \lambda\varpi)dt + \sigma_L(t, S_t)\sqrt{V_t}dW_t^1 + (exp[J] - 1)dN_t \\ dV_t = \kappa(\theta - V_t)dt + \gamma\sqrt{V_t}dW_t^2 \\ \left\langle dW_t^1, dW_t^2 \right\rangle = \rho \end{cases}$$

where S_t is the spot FX rate, r_d and r_f are domestic and foreign interest rates, λ is the intensity of Poisson process N_t, W_t^1 and W_t^2 are brownian motions with correlation ρ, V_t is stochastic variance, $\varpi = \mathbb{E}\left[exp[J] - 1\right]$ and $\sigma_L(t, S_t)$ is local volatility function.
When $\sigma_L \equiv 1$, we get Heston SV model with jumps, when $\gamma = 0$ we get LV Dupire model with jumps.

Lipton Model

Consider Lipton model without jumps for forward price F_t. Then, introducing appropriate normalization, we get

$$\begin{cases} \dfrac{dF_t}{F_t} = \sigma_L(F_t)\sqrt{V_t}dW_t^1 \\ dV_t = \kappa(1 - V_t)dt + \gamma\sqrt{V_t}dW_t^2 \\ \left\langle dW_t^1, dW_t^2 \right\rangle = \rho \end{cases}$$

The corresponding PDE for option price:

$$\frac{\partial U}{\partial \tau} = \frac{V\sigma_L^2(F)}{2}\frac{\partial^2 U}{\partial F^2} + \rho\gamma V\sigma_L(F)\frac{\partial^2 U}{\partial F \partial V} + \frac{\gamma^2 V}{2}\frac{\partial^2 U}{\partial V^2} + \kappa(1 - V)\frac{\partial U}{\partial V}$$

Hesten-like Model

Heston-like Dynamics

$$
\begin{aligned}
dS_t &= (r_d - r_f)S_t dt + L(S_t, t)\sqrt{V_t} S_t dW_t^1, \quad S_0 = s, \\
dV_t &= \kappa(\theta - V_t)dt + \alpha\sqrt{V_t} dW_t^2, \quad V_0 = v, \\
dW_t^1 * dW_t^2 &= \rho dt,
\end{aligned}
$$

where r_d is the domestic interest rate and r_f the foreign interest rate in the context of FX market.

Existence and Uniqueness

Suppose that the initial values $S_0, V_0 > 0$, and $\kappa, \theta, \alpha > 0, -1 < \rho < 1$ and $r_d, r_f \in \mathbb{R}$, and the leverage function L is positive and bounded. Then there exists a unique solution (S_t, V_t) to the SLV model, given that $V_t >0$ for all $t > 0$. Moreover, there exists a function $p(S_t, V_t, t)$ it is the unique solution of the Fokker-Planck equation:

$$\frac{\partial p}{\partial t} = -\frac{\partial}{\partial S}(rSp) - \frac{\partial}{\partial V}(\kappa(\theta - V)p) + \frac{1}{2}\frac{\partial^2}{\partial S^2}(L^2 S^2 Vp) + \\ + \frac{\partial^2}{\partial S \partial V}(\alpha \rho LSVp) + \frac{1}{2}\frac{\partial^2}{\partial V^2}(\alpha^2 Vp)$$

Local Volatility

Using a local volatility model:

$$dS_t = (r_d - r_f)S_t dt + \sigma_{LV}(S_t, t)dW_t$$

we can derive $\sigma_{LV}(S, t)$ from the implied volatility $\sigma_{IV}(K, T|S_0)$.

Local volatility.

$\sigma_{LV}(S, t) =$

$$\sqrt{\frac{\sigma_{IV}^2 + 2\sigma_{IV} T \frac{\partial \sigma_{IV}}{\partial T} + 2(r_d - r_f)\sigma_{IV} K T \frac{\partial \sigma_{IV}}{\partial K}}{\left(1 + d_1(S, K) K \sqrt{T} \frac{\partial \sigma_{IV}}{\partial K}\right)^2 + \sigma_{IV} K^2 T \left(\frac{\partial^2 \sigma_{IV}}{\partial K^2} - d_1(S, K)\sqrt{T}\left(\frac{\partial \sigma_{IV}}{\partial K}\right)^2\right)}}\Bigg|_{K=S, T=t}$$

Leverage Function

To mimic the LV model, the diffusion term in the SLV model follows:

$$\sigma_{LV}(x,t)^2 = \mathbb{E}\left[L(S_t,t)^2 V_t | S_t = x\right] = L(x,t)^2 \mathbb{E}\left[V_t | S_t = x\right]$$

Furthermore, we have:

$$p_{LV}(S,t) = \int_{\mathbb{R}_+} p(S,V,t)\, dV$$

In conclusion, we have:

Leverage function.

$$L(x,t) = \frac{\sigma_{LV}(x,t)}{\sqrt{\mathbb{E}\left[V_t | S_t = x\right]}} = \sigma_{LV}(x,t) \sqrt{\frac{\int_{\mathbb{R}_+} p(S,V,t)\, dV}{\int_{\mathbb{R}_+} V p(S,V,t)\, dV}}$$

SABR model

Quick recap on SABR (stochastic alpha, beta, rho) model. Let F_t be the forward price, then:

$$\begin{cases} dF_t = \sigma_t (F_t)^\beta dW_t, \\ \frac{d\sigma_t}{\sigma_t} = \alpha dZ_t, \\ dW_t dZ_t = \rho dt. \end{cases}$$

Pros:

- Accurate closed form approximation for IV,
- Quick calibration.

Cons:

- Difficulties with negative interest rates,
- Arbitrage for high and low strike, when approximation is used.

Butterfly arbitrage

From the derivation of the Dupire's formula $\frac{\partial^2 C}{\partial K^2} = Q$, where Q is the price distribution at the expiration. This is equivalent to the convexity of the option price. Common approximation of the IV in SABR model:

$$\sigma_{impl} = \alpha \frac{\log(F_0/K)}{D(\zeta)} \Bigg\{ 1 + \left[\frac{2\gamma_2 - \gamma_1^2 + 1/F^2}{24} \left(\frac{\sigma_0 F^\beta}{\alpha}\right)^2 + \frac{\rho\gamma_1}{4} \frac{\sigma_0 F^\beta}{\alpha} + \frac{2 - 3\rho}{2} 4 \right] T\alpha^2 \Bigg\},$$

where

$$\gamma_1 = \beta/F, \quad \gamma_2 = -\beta(1-\beta)/F^2, \quad \zeta = \alpha \frac{F_0^{1-\beta} - K^{1-\beta}}{\sigma_0(1-\beta)},$$

$$D(\zeta) = \log\left(\frac{\sqrt{1 - 2\rho\zeta + \zeta^2} + \zeta - \rho}{1 - \rho} \right).$$

Butterfly arbitrage

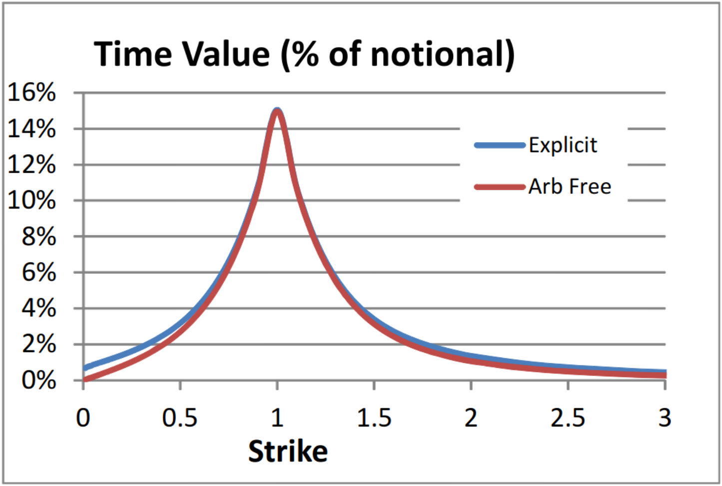

Figure: Price for SABR

Butterfly arbitrage

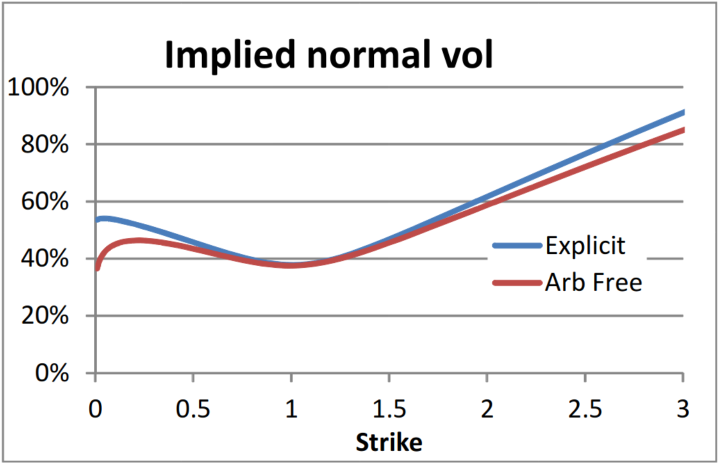

Figure: IV for SABR

Butterfly arbitrage

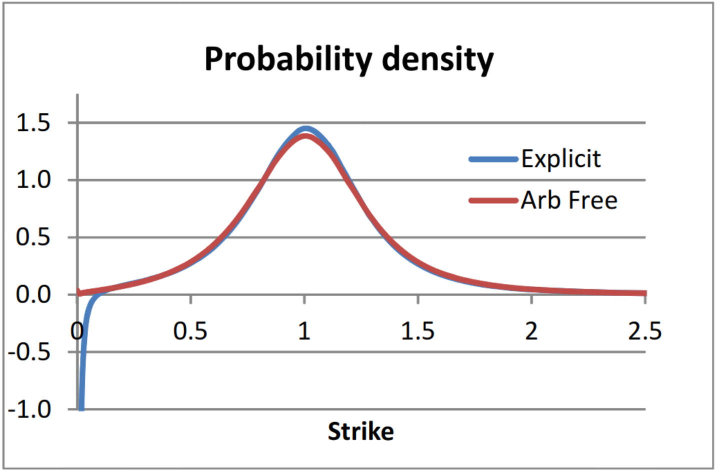

Figure: PDF for SABR

SABR-like model

Adding local volatility function we obtain:

$$\begin{cases} \frac{dF_t}{F_t} = \sigma_t f(t, F_t) dW_t, \\ \frac{d\sigma_t}{\sigma_t} = \gamma\alpha dZ_r, \\ dW_t dZ_t = \gamma\rho dt. \end{cases} \tag{3}$$

γ is mixing coefficient betwen 0 and 1, which allow us to chose any model between vanilla LV and vanilla SABR. The latter assumes the choice $f(t, F_t) = (F_t)^{\beta-1}$.

Calibration scheme:

1. Calibrate pure stochastic volatility model with $\gamma = 1$ and $f(t, F_t) = 1$.
2. Set $\gamma < 1$ and compute f through Dupire's generalized formula.

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