Relative growth optimal strategies in an asset market game

Yaroslav Drokin* Mikhail Zhitlukhin†

23 July 2020

Abstract

We consider a game-theoretic model of a market where investors compete for payoffs yielded by several assets. The main result consists in a proof of the existence and uniqueness of a strategy, called relative growth optimal, such that the logarithm of the share of its wealth in the total wealth of the market is a submartingale for any strategies of the other investors. It is also shown that this strategy is asymptotically optimal in the sense that it achieves the maximal capital growth rate when compared to competing strategies. Based on the results obtained, we study the asymptotic structure of the market when all the investors use the relative growth optimal strategy.

Keywords: relative growth optimal strategy, asset market game, evolutionary finance, martingale convergence.

MSC 2010: 91A25, 91B55. *JEL Classification:* C73, G11.

This paper was pubished in *Annals of Finance* (2020).
https://doi.org/10.1007/s10436-020-00360-6

1. Introduction

Growth optimal strategies are a well-studied topic in mathematical finance. However, the majority of models in the literature assume exogenously specified returns of assets and consider models with a single investor. In this setting growth optimal strategies arise as solutions of optimization problems (see Algoet and Cover (1988); Breiman (1961); Karatzas and Kardaras (2007); Platen (2006)). In the present paper we study growth optimality of investment strategies from a game-theoretic perspective and consider a model of a market (an *asset market game*), where several investors compete for random payoffs yielded by several assets at discrete moments of time on the infinite time interval. The payoffs are divided between the investors proportionally to shares of assets they buy at prices determined endogenously by a short-run equilibrium of supply and demand. As a result, the profit or loss of one investor depends not only on the realized payoffs, but also on actions of the competitors. In our model we assume that the assets are short-lived in the sense that they are traded at time t, yield payoffs at $t+1$, and then the cycle repeats. Thus, they can be viewed as some short-term investment projects rather than, e.g., common stock.

*National Research University Higher School of Economics, Faculty of Mathematics. 6 Usacheva St., Moscow 119048, Russia. E-mail: yadrokin@mail.ru.

†Steklov Mathematical Institute of the Russian Academy of Sciences. 8 Gubkina St., Moscow 119991, Russia. E-mail: mikhailzh@mi-ras.ru. Corresponding author.

The research was supported by the Russian Science Foundation, project 18-71-10097.

The goal of the paper is to identify an investment strategy, called relative growth optimal, such that the logarithm of the relative wealth of an investor who uses it is a submartingale no matter what the strategies of the other investors are (by relative wealth we mean the share of wealth of one investor in the total wealth of the market). In conventional (single-investor, i.e. non-game) market models, it is well-known that the submartingale property implies various asymptotic optimality properties for the growth rate of wealth (see, e.g., Algoet and Cover (1988); Karatzas and Kardaras (2007)). Results of a similar nature turn out to be true in our model as well, though their proofs use different ideas. In particular, we show that the relative wealth of an investor who uses the relative growth optimal strategy stays bounded away from zero with probability one. We also show that if the representative strategy of the other investors is asymptotically different, then such an investor will dominate in the market – the corresponding share of wealth will tend to one. In addition, the relative growth optimal strategy maximizes the growth rate of wealth and forms a symmetric Nash equilibrium in a game where all investors maximize their expected relative wealth.

Our model extends the model proposed by Amir et al. (2013), who also studied optimal strategies in an asset market game with short-lived assets. The main difference between our model and their model is that we assume the presence of a bank account (or a risk-free asset) with exogenous interest rate. In the simplest form, if the interest rate is zero, it just gives investors the possibility to put only a part of their wealth in the assets and to keep a part of wealth in cash. In the model of Amir et al. (2013), it is assumed that the whole wealth is reinvested in the assets in each time period, and that model can be obtained from ours if one let the interest rate be -1, so that it is not reasonable to keep money in the bank. The inclusion of a bank account in the model leads to a more difficult construction of the optimal strategy. But, at he same time, it also opens a series of new interesting questions regarding the asymptotic behavior of the absolute wealth of investors that do not arise in the model where the whole wealth is reinvested in assets.

Our paper can be reckoned among papers that study long-run performance of investment strategies from the point of view of evolutionary dynamics, i.e. a market is considered as a population of various strategies which compete for capital. This approach can be used to analyze forces that determine long-run market dynamics through a process of natural selection of investment strategies, see, e.g., the seminal paper by Blume and Easley (1992), where a model with a discrete probability space was considered, and its further development (Blume and Easley, 2006). In later works this field is called Evolutionary Finance; for recent literature reviews, see, e.g., Evstigneev et al. (2016); Holtfort (2019).

The paper is organized as follows. In Section 2, we formulate the model and introduce the notion of relative growth optimality of investment strategies. In Section 3, we construct a relative growth optimal strategy in an explicit form and show that it is unique in a certain sense. Sections 4 and 5 study further optimality properties of this strategy. Section 6 is devoted to the analysis of the asymptotics of the absolute wealth of investors when they use the relative growth optimal strategy.

2. The model

The market in the model consists of $M \geq 2$ investors, $N \geq 1$ risky assets, and a bank account (or cash). The assets yield payoffs which are distributed between the investors at discrete moments of time $t = 1, 2, \dots$ The investors choose, at every moment of time, proportions of their wealth they invest in the assets and proportions they keep in the bank account. The assets live for one period: they are traded at time t, yield payoffs at $t+1$, and then the cycle repeats. Asset prices are determined endogenously by a short-run equilibrium of supply and demand; in this model, without loss of generality, we assume that each asset is in unit supply.

Let $(\Omega, \mathcal{F}, \mathrm{P})$ be a probability space with a filtration $\mathbb{F} = (\mathcal{F}_t)_{t=0}^{\infty}$ on which all random variables will be defined. Payoffs of asset $n = 1, \dots, N$ are specified by a random sequence $X_t^n \geq 0$, $t \geq 1$, which is $\mathbb{F}$-adapted (X_t^n is $\mathcal{F}_t$-measurable for all $t \geq 1$). It is assumed that X_t^n are given exogenously, i.e. do not depend on actions of the investors. Return on the bank account is specified by an exogenous $\mathbb{F}$-predictable sequence $\rho_t \geq 0$ (i.e. ρ_t is $\mathcal{F}_{t-1}$-measurable), such that $\rho_t - 1$ is interpreted as spot interest rate between moments of time $t-1$ and t. Note that the interest rate may be negative. We will assume that

$$\rho_t + \sum_n X_t^n > 0 \ \text{ a.s. for all } t \geq 1 \tag{1}$$

(otherwise the model may degenerate as will become clear below).

The wealth of investor m is described by an adapted random sequence $Y_t^m \geq 0$. The quantity Y_t^m is the budget that this investor can allocate at time t for investment in the assets and the bank account. We assume that the initial budget Y_0^m of each investor is non-random and strictly positive. The wealth Y_t^m at moments of time $t \geq 1$ depends on investors' strategies and the asset payoffs.

A strategy of investor m consists of vectors of investment proportions $\lambda_t^m = (\lambda_t^{m,1}, \dots, \lambda_t^{m,N})$, $t \geq 1$, according to which this investor allocates available budget towards purchase of assets at time $t-1$. The proportion $1 - \sum_n \lambda_t^{m,n}$ is allocated in the bank account. We assume that short sales and borrowing from the bank account are not allowed, so vectors $\lambda_t = (\lambda_t^{m,n})$ belong to the set $\Delta = \{\lambda \in \mathbb{R}_+^{MN} : \sum_n \lambda^{m,n} \leq 1 \text{ for each } m\}$.

At each moment of time, investment proportions are selected by the investors simultaneously and independently, so the model represents a simultaneous-move N-person dynamic game, and the proportion vectors λ_t^m represent the investors' actions. These actions may depend on the game history, and we define a strategy Λ^m of investor m as a sequence of functions

$$\Lambda_t^m(\omega, y_0, \lambda_1, \dots, \lambda_{t-1}) \colon \Omega \times \mathbb{R}_+^M \times \Delta^{t-1} \to [0,1]^N, \qquad t \geq 1,$$

which are $\mathcal{F}_{t-1} \otimes \mathcal{B}(\mathbb{R}_+ \times \Delta^{t-1})$-measurable and for all $\omega, t, y_0, \lambda_1, \dots, \lambda_{t-1}$ satisfy the condition

$$\sum_n \Lambda_t^{m,n}(\omega, y_0, \lambda_1, \dots, \lambda_{t-1}) \leq 1.$$

The argument $y_0 \in \mathbb{R}_+^M$ corresponds to the vector of initial capital $Y_0 = (Y_0^1, \dots, Y_0^M)$. The arguments $\lambda_s = (\lambda_s^{m,n})$, $m = 1, \dots, M$, $n = 1, \dots, N$ are investment proportions selected by the investors at past moments of time (for $t = 1$, the function $\Lambda_t^m(\omega, y_0)$ does not depend on λ_s). The value of the function Λ_t^m corresponds to the vector of investment proportions λ_t^m. The measurability of Λ_t^m in ω with respect to $\mathcal{F}_{t-1}$ means that future payoffs are not known to the investors at the moment when they decide upon their actions.

After selection of investment proportions by the investors at time $t-1$, equilibrium asset prices p_{t-1}^n are determined from the market clearing condition that the aggregate demand of each asset is equal to the aggregate supply, which is assumed to be 1. Since investor m can purchase $x_t^{m,n} = \lambda_t^{m,n} Y_{t-1}^m / p_{t-1}^n$ units of asset n, the asset prices at time $t-1$ should be equal

$$p_{t-1}^n = \sum_m \lambda_t^{m,n} Y_{t-1}^m.$$

If $\sum_m \lambda_t^{m,n} = 0$, i.e. no one invests in asset n, we put $p_{t-1}^n = 0$ and $x_t^{m,n} = 0$ for all m.

Thus, investor m's portfolio between moments of time $t-1$ and t consists of $x_t^{m,n}$ units of asset n and $c_t^m := (1 - \sum_n \lambda_t^{m,n}) Y_{t-1}^m$ units of cash held in the bank account. At moment of time t, the total payoff received by this investor from the assets in her portfolio will be equal to $\sum_n x_t^{m,n} X_t^n$ and the (gross) return on the bank account will be $\rho_t c_t^m$. Consequently, her wealth is determined by the recursive relation

$$Y_t^m = \rho_t \Big(1 - \sum_n \lambda_t^{m,n}\Big) Y_{t-1}^m + \sum_n \frac{\lambda_t^{m,n} Y_{t-1}^m}{\sum_k \lambda_t^{k,n} Y_{t-1}^k} X_t^n, \qquad t \geq 1 \tag{2}$$

(with $0/0 = 0$ in the right-hand side). Here and in what follows, Y_t^m, Y_{t-1}^m, ρ_t, X_t^n are functions of ω only, and by $\lambda_t^{m,n}$ we denote the *realization* of investor m's strategy in this market, which is defined recursively as the predictable sequence

$$\lambda_t^{m,n}(\omega) = \Lambda_t^{m,n}(\omega, Y_0, \lambda_1(\omega), \ldots, \lambda_{t-1}(\omega)). \tag{3}$$

Note that in our model investors' actions precede asset prices, so investors first "announce" how much they allocate in each asset, and then the prices are adjusted to clear the market. This modeling approach is analogous to market games of Shapley–Shubik type. Although it is a simplification of a real market, such an approach is economically reasonable (see Shapley and Shubik (1977) for details and justification).

We call (2) the *wealth equation*, and it is the principal equation in our model. Mainly, we will be interested not in the absolute wealth Y_t^m, but in the relative wealth, i.e. the proportion of wealth of one investor in the total wealth of all investors. The total wealth is defined as

$$W_t = \sum_m Y_t^m,$$

and the relative wealth of investor m is defined as

$$r_t^m = \frac{Y_t^m}{W_t}$$

(when $W_t = 0$, we put $r_t^m = 0$).

Definition 1. We call a strategy Λ^m of investor m *relative growth optimal* if for any vector of initial capital Y_0 (with $Y_0^k > 0$ for all k) and strategies Λ^k of the other investors $k \neq m$,

$$\ln r_t^m \text{ is a submartingale.}$$

Such a strategy is optimal in several aspects. First, observe that if a strategy is relative growth optimal, then also r_t^m is a submartingale by Jensen's inequality. As a corollary, it is not hard to see that a strategy profile in which every investor uses a relative growth optimal strategy is a Nash equilibrium in the game where investors maximize $\mathrm{E} r_t^m$ at a fixed moment of time t. This follows from the fact that if the strategies of investors $k \neq m$ are relative growth optimal, then $r_t^m = 1 - \sum_{k \neq m} r_t^k$ is a supermartingale. As we will show in Section 3, a relative growth optimal strategy is unique, so when every investor uses it, their relative wealth will remain constant.

Second, as will be shown in Section 5, an investor who uses a relative growth optimal strategy achieves the highest growth rate of wealth compared to the other investors in the market. This property is analogous to the growth optimality property in single-investor market models (Algoet and Cover (1988); Breiman (1961); Kelly (1956); and others). However, it is essential that we require the logarithm of relative wealth $\ln r_t^m$ to be a submartingale; the logarithm of wealth $\ln Y_t^m$ may be not a submartingale (see Section 6).

Also, a relative growth optimal strategy belongs to the class of survival strategies, which plays the central role in Evolutionary Finance (see Evstigneev et al. (2016)) and is defined as follows.

Definition 2. A strategy Λ^m of investor m is called *survival* if for any strategies of the other investors

$$\inf_{t\geq 0} r_t^m > 0 \text{ a.s.}$$

An investor using a survival strategy cannot be driven out of the market (even asymptotically) in the sense that her relative wealth always stays bounded away from zero. In this definition, we use the terminology of Amir et al. (2013); note that, for example, Blume and Easley (1992) use the term "survival" in a somewhat different meaning. The fact that a relative growth optimal strategy is survival readily follows from that $\ln r_t^m$ is a non-positive submartingale, and hence it has a finite limit $l = \lim_{t\to\infty} \ln r_t^m$ a.s. (see, e.g., Chapter 7 of Shiryaev (2019) for this and other results from the theory of discrete-time martingales used in this paper). Therefore, $\lim_{t\to\infty} r_t^m = e^l > 0$ a.s.

Remark 1 (Relation to the Amir–Evstigneev–Schenk-Hoppé model)**.** Our model generalizes the model of Amir, Evstigneev, and Schenk-Hoppé (Amir et al., 2013), where it was assumed that investors reinvest their whole wealth in assets in each time period. That model can be obtained as a particular case of our model by taking $\rho_t \equiv 0$, so that it is never reasonable to keep money in the bank account.

However, despite similarity, construction of the optimal strategy in our model turns out to be more difficult. In particular, the optimal strategy of Amir et al. (2013) is *basic* in the sense that its investment proportions do not depend on past actions of investors, while, as we will see in the next section, in our model they depend on current wealth of all investors, which depends on their past actions. Moreover, our model opens interesting questions about the asymptotic behavior of the total wealth of investors, which do not arise when the whole wealth is reinvested in assets. We consider these questions in Section 6.

Another model of this kind with short-lived assets and a risk-free asset was considered by Belkov et al. (2017), where the existence of a survival strategy was established, and it was also shown that all basic survival strategies are asymptotically equal. However, in that model asset payoffs depend on "money supply" (amount of capital not invested in assets) in a special way, which allows to reduce that model to the one of Amir et al. (2013).

3. Existence and uniqueness of a relative growth optimal strategy

Let us first introduce auxiliary notation and definitions.

We will use the following convenient notation for vectors. If $x, y \in \mathbb{R}^N$, we will denote their scalar product by $xy = \sum_n x^n y^n$, the L^1-norm by $|x| = \sum_n |x^n|$, and the

L^2-norm by $\|x\| = \sqrt{xx}$. If $f\colon \mathbb{R} \to \mathbb{R}$ is a function, then $f(x)$ denotes the vector $(f(x^1), \dots, f(x^N))$. By $a \vee b$ we will denote the maximum of variables a, b, and by $a \wedge b$ the minimum.

The realization of investor m's strategy, defined in (3), will be denoted by $\lambda_t^m(\omega)$, and when it is necessary to emphasize that it depends on the initial capital and strategies of the other investors we will use the notation $\lambda_t^m(\omega; Y_0, L)$, where $L = (\Lambda^1, \dots, \Lambda^M)$ stands for a strategy profile; the argument ω will be often omitted for brevity.

Let us introduce the notion of equality of strategies that will be used to state that the relative growth optimal strategy is unique. Suppose $\tau(\omega; Y_0, L)$ denotes a family of random variables, i.e. for any fixed vector of initial capital Y_0 and a strategy profile $L = (\Lambda^1, \dots, \Lambda^M)$, the function $\omega \mapsto \tau(\omega; Y_0, L)$ is $\mathcal{F}$-measurable (and may assume the value $+\infty$).

Definition 3. We say that two strategies Λ^m and $\widetilde{\Lambda}^m$ of investor m are *equal in realization until* τ if for any vector of initial capital Y_0 and any strategies of the other investors Λ^k, $k \neq m$, we have the equality of realizations (a.s. for all $t \geq 1$)

$$(\lambda_t^m(\omega; Y_0, L) - \lambda_t^m(\omega; Y_0, \widetilde{L}))\mathrm{I}(t \leq \tau(\omega; Y_0, L) \wedge \tau(\omega; Y_0, \widetilde{L})) = 0,$$

where $L = (\Lambda^1, \dots, \Lambda^m, \dots, \Lambda^M)$, $\widetilde{L} = (\Lambda^1, \dots, \widetilde{\Lambda}^m, \dots, \Lambda^M)$ are the strategy profiles which differ only in the strategy of investor m.

For example, in Theorem 1 below, we will consider equality until $\tau = \inf\{t \geq 0 : r_t^m = 1\}$ – the first moment of time when the relative wealth of an investor reaches 1 (with $\tau(\omega) = \infty$ if $r_t^m(\omega) < 1$ for all t), which, considered as a function $\omega \mapsto \tau(\omega; Y_0, L)$, is a stopping time for any fixed Y_0 and L.

The reason why we need to work with equality until τ is that when the relative wealth of an investor becomes 1 (and the wealth of the competitors becomes zero), she may choose any investment proportions and her relative wealth will always remain 1 (provided that she does not invest in a "bad way" losing all her wealth). Hence, it is not possible to speak about uniqueness after this moment.

Now we can proceed to the construction of the relative growth optimal strategy. Let $K_t(\omega, A)\colon \Omega \times \mathcal{B}(\mathbb{R}_+^N) \to [0,1]$ denote the regular conditional distribution of the payoff vector $X_t = (X_t^1, \dots, X_t^N)$ with respect to $\mathcal{F}_{t-1}$, so that for each t and fixed ω the function $A \mapsto K_t(\omega, A)$ is a probability measure on the Borel σ-algebra $\mathcal{B}(\mathbb{R}_+^N)$, and for fixed $A \in \mathcal{B}(\mathbb{R}_+^N)$ the function $\omega \mapsto K_t(\omega, A)$ is a version of the conditional probability $\mathrm{P}(X_t \in A \mid \mathcal{F}_{t-1})$.

Define the sequence of sets $\Gamma_t \in \mathcal{F}_{t-1} \otimes \mathcal{B}(\mathbb{R}_+)$,

$$\Gamma_t = \left\{(\omega, c) : \int_{\mathbb{R}_+^N} \frac{c\rho_t(\omega)}{|x|} K_t(\omega, dx) > 1\right\}, \qquad t \geq 1,$$

with the following convention: $c\rho_t(\omega)/|x| = 0$ if $c\rho_t(\omega) = |x| = 0$ and $c\rho_t(\omega)/|x| = +\infty$ if $c\rho_t(\omega) > 0$ but $|x| = 0$.

The following lemma will play an auxiliary role in construction of the relative growth optimal strategy.

Lemma 1. *For all $t \geq 1$ and $(\omega, c) \in \Gamma_t$, there exists a unique solution $z \in (0, c]$ of the equation*

$$\int_{\mathbb{R}_+^N} \frac{c\rho_t(\omega)}{z\rho_t(\omega) + |x|} K_t(\omega, dx) = 1. \tag{4}$$

The function $\zeta_t(\omega, c)$ defined to be equal to this solution on Γ_t and equal to zero outside Γ_t is $\mathcal{F}_{t-1} \otimes \mathcal{B}(\mathbb{R}_+)$-measurable.

Proof. The existence and uniqueness of the solution for each $(\omega, c) \in \Gamma_t$ is straightforward: the left-hand side of (4) is a continuous and strictly decreasing function in $z \in (0, c]$ which assumes a value greater than 1 for $z = 0$ and a value not greater than 1 for $z = c$.

To prove the measurability, consider the function $f \colon \Omega \times \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}$,

$$f(\omega, c, z) = \left(2 \wedge \int_{\mathbb{R}_+^N} \frac{c\rho_t(\omega)}{z\rho_t(\omega) + |x|} K_t(\omega, dx) - 1\right) \mathrm{I}((\omega, c) \in \Gamma_t).$$

Observe that f is a Carathéodory function, i.e. $\mathcal{F}_{t-1} \otimes \mathcal{B}(\mathbb{R}_+)$-measurable in (ω, c) and continuous in z. Then by Filippov's implicit function theorem (see, e.g., Theorem 18.17 in Aliprantis and Border (2006)), the set-valued function

$$\phi(\omega, c) = \{z \in [0, c] : f(\omega, c, z) = 0\}$$

admits a measurable selector. Since ϕ on Γ_t is single-valued (we have $\phi(\omega, c) = \{\zeta_t(\omega, c)\}$), this implies that ζ_t is $\mathcal{F}_{t-1} \otimes \mathcal{B}(\mathbb{R}_+)$-measurable. $\square$

In what follows, we will use the notation $\chi_t = (y_0, \lambda_1, \dots, \lambda_t) \in \mathbb{R}_+^M \times \Delta^t$ for history of the market until time t. Denote by $C_t(\omega, \chi_t) = W_t(\omega) = |Y_t(\omega)|$ the total wealth of all investors at time t, where Y_t is defined recursively by relation (2) with the given initial wealth $Y_0 = y_0$ and investment proportions $\lambda_s^{m,n}$, which form the history χ_t.

Theorem 1. *The strategy $\widehat{\Lambda}$ with investment proportions defined by the relation*

$$\widehat{\Lambda}_t^n(\omega, \chi_{t-1}) = \int_{\mathbb{R}_+^N} \frac{x^n}{\zeta_t(\omega, C_{t-1}(\omega, \chi_{t-1}))\rho_t(\omega) + |x|} K_t(\omega, dx) \tag{5}$$

is relative growth optimal ($0/0 = 0$ in (5)).

Moreover, $\widehat{\Lambda}$ is the unique relative growth optimal strategy in the sense that if Λ is another strategy for investor m such that its relative wealth r_t^m is a submartingale for any initial capital and strategies of the other investors, then $\widehat{\Lambda}$ and Λ are equal in realization until the time $\tau = \inf\{t \geq 0 : r_t^m = 1\}$.

It is easy to see that the proportion of wealth that $\widehat{\Lambda}$ keeps in the bank account is $1 - |\widehat{\Lambda}(\chi_{t-1})| = \zeta_t(C_{t-1}(\chi_{t-1}))/C_{t-1}(\chi_{t-1})$. In particular, if $\rho_t = 0$ for all t, then $\Gamma_t = \emptyset$ and $\zeta_t = 0$. In this case we obtain the same strategy that was found by Amir et al. (2013) – it divides the available budget between the assets proportionally to their expected *relative payoffs* $\int_{\mathbb{R}_+^N} x^n/|x| K_t(dx) = \mathrm{E}(X_t^n/|X_t| \mid \mathcal{F}_{t-1})$. When $\zeta_t \neq 0$, the strategy $\widehat{\Lambda}$ still divides the budget between the assets proportionally to their payoffs but the proportions are adjusted for the amount of capital kept in the bank account.

Before we proceed to the proof of Theorem 1, let us state one auxiliary inequality that we will use (it generalizes Gibbs' inequality).

Lemma 2. *Suppose $\alpha, \beta \in \mathbb{R}_+^N$ are two vectors such that $|\alpha|, |\beta| \leq 1$ and for each n it holds that if $\beta^n = 0$, then also $\alpha^n = 0$. Then*

$$\alpha(\ln \alpha - \ln \beta) \geq \frac{\|\alpha - \beta\|^2}{4} + |\alpha| - |\beta|, \tag{6}$$

where we put $\alpha^n(\ln \alpha^n - \ln \beta^n) = 0$ if $\alpha^n = 0$.

Proof. We follow the lines of the proof of Lemma 2 in Amir et al. (2013), which establishes the above inequality in the case $|\alpha| = |\beta| = 1$. Using that $\ln x \le 2(\sqrt{x} - 1)$ for any $x > 0$, we obtain

$$\begin{aligned}\alpha(\ln\alpha - \ln\beta) &= -\sum_{n\,:\,\alpha^n\neq 0} \alpha^n \ln(\beta^n/\alpha^n) \ge 2\sum_n (\alpha^n - \sqrt{\alpha^n\beta^n})\\ &= \sum_n (\sqrt{\alpha^n} - \sqrt{\beta^n})^2 + |\alpha| - |\beta|.\end{aligned}$$

Then we can use the inequality $(\sqrt{x}-\sqrt{y})^2 \ge (x-y)^2/4$, which is true for any $x, y \in [0,1]$, and obtain (6). □

Proof of Theorem 1. Without loss of generality, assume that the strategy $\widehat{\Lambda}$ is used by investor 1. Let λ_t denote the realization of this strategy, and $\widetilde{\lambda}_t$ the realization of the representative strategy of the other investors, which we define as the following weighted sum of the realizations of their strategies:

$$\widetilde{\lambda}_t^n = \sum_{m\ge 2} \frac{r_{t-1}^m}{1 - r_{t-1}^1} \lambda_t^{m,n}, \tag{7}$$

where $\widetilde{\lambda}_t = 0$ if $r_{t-1}^1 = 1$. By Y_t we will denote the wealth of investor 1, by $\widetilde{Y}_t := \sum_{m\ge 2} Y_t^m$ the total wealth of the other investors, and by $r_t = Y_t/(Y_t + \widetilde{Y}_t)$ the relative wealth of investor 1. Then Y_t satisfies the following relation, which follows from (2):

$$Y_t = \rho_t(1 - |\lambda_t|)Y_{t-1} + \sum_n \frac{\lambda_t^n Y_{t-1}}{\lambda_t^n Y_{t-1} + \widetilde{\lambda}_t^n \widetilde{Y}_{t-1}} X_t^n. \tag{8}$$

Observe that from the definition of $\widehat{\Lambda}$ and condition (1), it follows that for each t and almost all ω we have

$$K_t(\omega, \{x : \rho_t(\omega)(1 - |\lambda_t(\omega)|) + x\lambda_t(\omega) = 0\}) = 0. \tag{9}$$

In particular, this implies that $Y_t > 0$ a.s. for all t.

Introduce the predictable sequence of random vectors F_t with values in $\mathbb{R}_+^N$ which have the components

$$F_t^n = \frac{\lambda_t^n}{r_{t-1}\lambda_t^n + (1 - r_{t-1})\widetilde{\lambda}_t^n},$$

where $0/0 = 0$. Denoting the total wealth of the investors by $W_t = Y_t + \widetilde{Y}_t$, the equation (8) can be rewritten as

$$Y_t = \left(\rho_t(1 - |\lambda_t|) + \frac{F_t X_t}{W_{t-1}}\right) Y_{t-1}. \tag{10}$$

A similar equation is true for $\widetilde{Y}_t$, namely, $\widetilde{Y}_t = (\rho_t(1 - |\widetilde{\lambda}_t|) + \frac{\widetilde{F}_t X_t}{W_{t-1}})\widetilde{Y}_{t-1}$, where $\widetilde{F}_t^n = \widetilde{\lambda}_t^n/(r_{t-1}\lambda_t^n + (1 - r_{t-1})\widetilde{\lambda}_t^n)$. Using this, we obtain

$$W_t = \left(\rho_t(1 - r_{t-1}|\lambda_t| - (1 - r_{t-1})|\widetilde{\lambda}_t|) + \frac{|X_t|}{W_{t-1}}\right) W_{t-1}. \tag{11}$$

In this equation we used the equality $(r_{t-1}F_t^n + (1 - r_{r-1})\widetilde{F}_t^n)X_t^n = X_t^n$: on the set $\{\lambda_t^n > 0\}$ this is clear from the definition of F_t^n and $\widetilde{F}_t^n$, while on the set $\{\lambda_t^n = 0\}$ we have $X_t^n = 0$ a.s., which follows from the construction of $\widehat{\Lambda}$.

Let $\zeta_t(\omega)$ denote the predictable sequence $\zeta_t(\omega, W_{t-1}(\omega))$. As follows from the definition of $\widehat{\Lambda}$, we have $|\lambda_t| = 1-\zeta_t/W_{t-1}$. Let $\widetilde{\zeta}_t = (1-|\widetilde{\lambda}_t|)W_{t-1}$. Then dividing (10) by (11) we find that $\ln r_t - \ln r_{t-1} = f_t(X_t)$, where $f_t = f_t(\omega, x)$ is the $\mathcal{F}_{t-1}\otimes\mathcal{B}(\mathbb{R}^N_+)$-measurable function

$$f_t(x) = \ln\left(\frac{\zeta_t\rho_t + F_t x}{r_{t-1}\zeta_t\rho_t + (1-r_{t-1})\widetilde{\zeta}_t\rho_t + |x|}\right)$$

(for brevity, the argument ω will be omitted). Note that (9) implies $\zeta_t\rho_t + F_t X_t > 0$ a.s., hence we can define the value of $f_t(\omega, x)$ for x such that $\zeta_t(\omega)\rho_t(\omega) + F_t(\omega)x = 0$ in an arbitrary way. It will be convenient to put $f_t(\omega, x) = 0$ for such x.

To show that $\ln r_t$ is a submartingale, it will be enough to show that $\int_{\mathbb{R}^N_+} f_t(x)K_t(dx) \geq 0$, i.e. $\mathrm{E}(f_t(X_t) \mid \mathcal{F}_{t-1}) \geq 0$. Indeed, then $\ln r_t$ will be a generalized submartingale[1], but since it is bounded from above (by 0), this will also imply that $\ln r_t$ is a usual submartingale (see Chapter 7.1 in Shiryaev (2019)).

Suppose for some t, ω, a vector x is such that $\zeta_t(\omega)\rho_t(\omega) + F_t(\omega)x > 0$, and, for each n, the equality $F^n_t(\omega) = 0$ implies $x^n = 0$. Then we have the bound

$$\begin{aligned} f_t(x) &= \ln\left(\frac{\zeta_t\rho_t + F_t x}{\zeta_t\rho_t + |x|}\right) + \ln\left(\frac{\zeta_t\rho_t + |x|}{r_{t-1}\zeta_t\rho_t + (1-r_{r-1})\widetilde{\zeta}_t\rho_t + |x|}\right) \\ &\geq \frac{x\ln F_t}{\zeta_t\rho_t + |x|} + \frac{(1-r_{t-1})(\zeta_t - \widetilde{\zeta}_t)\rho_t}{\zeta_t\rho_t + |x|} := g_t(x) + h_t(x), \end{aligned}$$

where we put $x^n \ln F^n_t = 0$ if $F^n_t = 0$. Here, for the first term in the second line we used the concavity of the logarithm, and for the second term the inequality $\ln a \geq 1 - a^{-1}$.

For each t, we have $K_t(\{x : \zeta_t\rho_t + xF_t = 0\}) = 0$ a.s. by (9), and also $K_t(\{x^n = 0\}) = 1$ a.s. on the set $\{F^n_t = 0\}$ by the definition of $\widehat{\Lambda}$. Hence

$$\int_{\mathbb{R}^N_+} f_t(x)K_t(dx) \geq \int_{\mathbb{R}^N_+} g_t(x)K_t(dx) + \int_{\mathbb{R}^N_+} h_t(x)K_t(dx) := I^g_t + I^h_t. \tag{12}$$

For the integral I^g_t, using Lemma 2, we find

$$\begin{aligned} I^g_t &= \lambda_t \ln F_t = \lambda_t(\ln\lambda_t - \ln(r_{t-1}\lambda_t + (1-r_{t-1})\widetilde{\lambda}_t)) \\ &\geq \frac{1}{4}(1-r_{t-1})^2\|\lambda_t - \widetilde{\lambda}_t\|^2 + (1-r_{t-1})(|\lambda_t| - |\widetilde{\lambda}_t|). \end{aligned} \tag{13}$$

For the integral I^h_t, on the set $\{\omega : (\omega, W_{t-1}(\omega)) \in \Gamma_t\}$ we can use the equality $\int_{\mathbb{R}^N_+} \rho_t W_{t-1}/(\zeta_t\rho_t + |x|)K_t(dx) = 1$, and on its complement the equality $\zeta_t = 0$ and inequality $\int_{\mathbb{R}^N_+} \rho_t W_{t-1}/|x| K_t(dx) \leq 1$, which result in

$$I^h_t \geq (1-r_{t-1})(\zeta_t - \widetilde{\zeta}_t)/W_{t-1} = (1-r_{t-1})(|\widetilde{\lambda}_t| - |\lambda_t|). \tag{14}$$

Relations (12)–(14) imply $\int_{\mathbb{R}^N_+} f_t(x)K_t(dx) \geq 0$, so $\ln r_t$ is a submartingale, which proves that $\widehat{\lambda}$ is relative growth optimal.

In order to prove the statement about uniqueness, suppose there exists another strategy Λ' (without loss of generality, assume this is a strategy of investor 1) with

[1] Recall that a sequence S_t is called a generalized submartingale if $\mathrm{E}|S_0| < \infty$ and $\mathrm{E}(S_t \mid \mathcal{F}_{t-1}) \geq S_{t-1}$ for all $t \geq 1$ (but not necessarily $\mathrm{E}|S_t| < \infty$). It is easy to show that if $S_t \leq C_t$ for all t with some integrable random variables C_t, then S_t is integrable, and hence a usual submartingale.

relative wealth r_t^1 being a submartingale for any initial capital and strategies of the other investors, and there exist strategies $\Lambda^2, \dots, \Lambda^M$ and a vector of initial capital Y_0 such that the realizations of $\widehat{\Lambda}$ and Λ' are different for the strategy profiles $\widehat{L} = (\widehat{\Lambda}, \Lambda^2, \dots, \Lambda^M)$ and $L' = (\Lambda', \Lambda^2, \dots, \Lambda^M)$, i.e. $\mathrm{P}(\widehat{\lambda}_t \neq \lambda'_t) > 0$ for some t, where $\widehat{\lambda}_t(\omega)$, $\lambda'_t(\omega)$ are the realizations of the strategy of the first investor in the markets $\widehat{L}$ and L', respectively.

Consider the predictable stopping time $\sigma = \inf\{t \geq 1 : \widehat{\lambda}_t \neq \lambda'_t\}$ and define the new strategies $\widetilde{\Lambda}^m$, $m \geq 2$, by

$$\widetilde{\Lambda}_t^m(\omega, \chi_{t-1}) = (\Lambda')_t^m(\omega, \chi_{t-1})\mathrm{I}(t < \sigma(\omega)) + \widehat{\Lambda}_t^m(\omega, \chi_{t-1})\mathrm{I}(t \geq \sigma(\omega)),$$

where $\chi_{t-1} = (y_0, \lambda_1, \dots, \lambda_{t-1})$ denotes history of the market until $t-1$. Observe that since $\{\sigma \leq t\} \in \mathcal{F}_{t-1}$, the strategies are well-defined (i.e. $\mathcal{F}_{t-1} \otimes \mathcal{B}(\mathbb{R}_+^M \times \Delta^{t-1})$-measurable).

Let $r_t(\omega)$ denote the realization $r_t^1(\omega, Y_0, \widetilde{L})$ of the relative wealth of investor 1 when the investors use the strategy profile $\widetilde{L} = (\Lambda', \widetilde{\Lambda}^2, \dots, \widetilde{\Lambda}^M)$. Then, on one hand, on the set $\{\sigma < \infty\}$ we have $\mathrm{E}(r_\sigma \mid \mathcal{F}_{\sigma-1}) \geq r_{\sigma-1}$ by the choice of Λ', and hence $\mathrm{E}(\ln(1 - r_\sigma) \mid \mathcal{F}_{\sigma-1}) \leq \ln(1 - r_{\sigma-1})$ by Jensen's inequality. On the other hand, $\mathrm{E}(\ln(1-r_\sigma) \mid \mathcal{F}_{\sigma-1}) \geq \ln(1 - r_{\sigma-1})$ since investors $m \geq 2$ use the strategy $\widehat{\Lambda}$ after σ. Hence, on the set $\{\sigma < \infty,\ r_{\sigma-1} < 1\}$ we have

$$0 = \mathrm{E}\left(\ln \frac{1 - r_\sigma}{1 - r_{\sigma-1}} \;\middle|\; \mathcal{F}_{\sigma-1}\right) \geq \frac{1}{4} r_{\sigma-1}^2 \|\lambda_\sigma - \widetilde{\lambda}_\sigma\|^2, \tag{15}$$

where the inequality can be obtained from (13) and (14). In this formula, λ_σ and $\widetilde{\lambda}_\sigma$ are the realizations of investor 1's strategy Λ' and the representative strategy of investors $m \geq 2$ at time σ in the market with the strategy profile $\widetilde{L}$. It is not hard to see that on the set $\{\sigma < \infty\}$ they are equal, respectively, to λ'_σ and $\widehat{\lambda}_\sigma$. Consequently, (15) and the choice of σ imply that if the set $\{\sigma < \infty, r_{\sigma-1} < 1\}$ had positive probability, then $r_{\sigma-1} = 0$ a.s. on it. But this is impossible since up to $\sigma - 1$ the realization of the strategy Λ' coincides with the realization of $\widehat{\Lambda}$, and hence its relative wealth stays positive. Thus, $\mathrm{P}(\sigma < \infty, r_{\sigma-1} < 1) = 0$, which proves the claimed uniqueness. □

4. The dominance property of a relative growth optimal strategy

The next short result shows that the relative wealth of the relative growth optimal strategy tends to 1 on the set of outcomes ω for which the realization $\widetilde{\lambda}$ of the representative strategy of the other investors is asymptotically different from the realization of $\widehat{\Lambda}$ in a certain sense ($\widetilde{\lambda}$ is defined in (7) above). This result can be viewed as asymptotic uniqueness of a survival strategy.

Theorem 2. *Suppose investor 1 uses the relative growth optimal strategy. Let $\widehat{\lambda}_t$ denote its realization, $\widetilde{\lambda}_t$ denote the realization of the representative strategy of the other investors, and $\Omega' \in \mathcal{F}$ be the set*

$$\Omega' = \Big\{\omega : \sum_{t \geq 1} \|\widehat{\lambda}_t(\omega) - \widetilde{\lambda}_t(\omega)\|^2 = \infty\Big\}.$$

Then $r_t^1(\omega) \to 1$ a.s. on Ω'. In particular, $\|\widehat{\lambda}_t(\omega) - \widetilde{\lambda}_t(\omega)\| \to 0$ a.s. on the set $\{\omega : \lim_{t\to\infty} r_t^1(\omega) < 1\}$.

Proof. We will use the same notation as in the proof of Theorem 1. Since $\ln r_t$ is a non-positive submartingale, it converges a.s. and its compensator C_t converges a.s. as well. From the proof of Theorem 1, it follows that

$$C_t - C_{t-1} = \int_{\mathbb{R}_+^N} f_t(x) K_t(dx) \geq \frac{1}{4}(1 - r_{t-1})^2 \|\widehat{\lambda}_t - \widetilde{\lambda}_t\|^2.$$

Then on the set Ω' we necessarily have $r_t \to 1$ a.s., since otherwise C_t would diverge. $\square$

5. Maximization of the growth rate of wealth

Recall that in a single-investor market model a *numéraire portfolio* is a strategy such that the ratio of the wealth of any other strategy to the wealth of this strategy is a supermartingale. The term "numéraire portfolio" was introduced by Long (1990); often it is also called a growth optimal strategy, or a benchmark portfolio (see Hakansson and Ziemba (1995); Karatzas and Kardaras (2007); Platen (2006)). It is well-known that numéraire portfolios have a number of optimality properties: they maximize the asymptotic growth rate of wealth, maximize the expected logarithmic utility, minimize the time to reach a given wealth level, etc. (see, e.g., Algoet and Cover (1988); Breiman (1961) for results in discrete time, and Karatzas and Kardaras (2007) for results in a general semimartingale model, including a connection with the arbitrage theory). In this section we will show that the relative growth optimal strategy in our model has similar properties.

By the asymptotic growth rate of the wealth Y_t we call $\limsup_{t\to\infty} \frac{1}{t} \ln Y_t$ (see, e.g., Chapter 3.10 in Karatzas and Shreve (1998)), and the t-step growth rate at time s can be defined as $\frac{1}{t}\mathrm{E}(\ln \frac{Y_{s+t}}{Y_s} \mid \mathcal{F}_s)$. These notions have especially clear interpretation in a single-investor model with i.i.d. asset returns: then the log-returns of a growth optimal strategy are i.i.d. as well, and the asymptotic growth rate and the t-step growth rate are equal and non-random.

Theorem 3. *Suppose investor 1 uses the relative growth optimal strategy and the other investors use arbitrary strategies. Then for any vector of initial capital Y_0 the following claims are true.*
1) Investor 1 maximizes the asymptotic growth rate of wealth: for any m

$$\limsup_{t\to\infty} \frac{1}{t} \ln Y_t^1 \geq \limsup_{t\to\infty} \frac{1}{t} \ln Y_t^m \quad a.s. \tag{16}$$

2) Suppose there are only two investors ($M = 2$) and $\mathrm{E}|X_t| < \infty$ for all t. Then investor 1 maximizes the t-step growth rate of wealth: for any $t, s \geq 0$

$$\mathrm{E}\left(\ln \frac{Y_{s+t}^1}{Y_s^1} \;\middle|\; \mathcal{F}_s\right) \geq \mathrm{E}\left(\ln \frac{Y_{s+t}^2}{Y_s^2} \;\middle|\; \mathcal{F}_s\right) \quad a.s. \tag{17}$$

(In inequalities (16), (17), the both sides may assume the values $\pm\infty$; in (17) we put $\ln 0/0 = -\infty$.)

Proof. 1) As was noted above, if investor 1 uses the relative growth optimal strategy, then $\inf_t r_t^1 > 0$ a.s., and hence $\sup_t W_t/Y_t^1 < \infty$. Therefore, $\sup_t Y_t^m/Y_t^1 < \infty$ for any m. This implies that for any sequence T_t such that $\lim_{t\to\infty} T_t = +\infty$ (in particular, for $T_t = t$) we have the inequality

$$\limsup_{t\to\infty} \frac{1}{T_t} \ln \frac{Y_t^m}{Y_t^1} \leq 0 \quad \text{a.s.}$$

From here, one can obtain (16).

2) From the condition $\mathrm{E}|X_t| < \infty$, it follows that $\mathrm{E}\ln W_t < +\infty$ for all t. On the set $\{\omega : \mathrm{E}(\ln W_{s+t} \mid \mathcal{F}_s)(\omega) = -\infty\}$ we have $\mathrm{E}(\ln Y^1_{s+t} \mid \mathcal{F}_s) = \mathrm{E}(\ln Y^2_{s+t} \mid \mathcal{F}_s) = -\infty$, so inequality (17) holds on this set. On the set $\{\omega : \mathrm{E}(\ln W_{s+t} \mid \mathcal{F}_s)(\omega) > -\infty\}$, use the submartingale property of $\ln r^1_t$, which implies

$$\mathrm{E}\Bigl(\ln \frac{Y^1_{s+t}}{Y^1_s} \Bigm| \mathcal{F}_s\Bigr) \ge \mathrm{E}\Bigl(\ln \frac{W_{s+t}}{W_s} \Bigm| \mathcal{F}_s\Bigr).$$

Since $r^2_t = 1 - r^1_t$ is a supermartingale, $\ln r^2_t$ is a generalized supermartingale, so, in a similar way,

$$\mathrm{E}\Bigl(\ln \frac{Y^2_{s+t}}{Y^2_s} \Bigm| \mathcal{F}_s\Bigr) \le \mathrm{E}\Bigl(\ln \frac{W_{s+t}}{W_s} \Bigm| \mathcal{F}_s\Bigr),$$

which proves (17). □

Remark 2. 1. It is clear from the proof that (16) holds if investor 1 uses any survival strategy.

2. Note that the second claim of Theorem 3 generally does not hold in the case $M \ge 3$. For example, it can happen that investor 1 uses the strategy $\widehat{\Lambda}$, investor 2 acts in an unoptimal way, and investor 3 manages to find a strategy which is better than $\widehat{\Lambda}$.

A simple example can be constructed even for a non-random market. Let $M = 3$, $N = 1$, $Y^m_0 = 1$ for $m = 1, 2, 3$, and $\rho_1 = 1$, $X_1 = 1$. Then we have $\widehat{\lambda}_1 = 1/3$. However, if $\lambda^1_1 = \widehat{\lambda}_1$ and $\lambda^2_1 = 1$, then the strategy $\lambda^3_1 = 0$ turns out to be better than λ^1_1 after one step: $Y^1_1 = 11/12$, but $Y^3_1 = 1$.

6. Growth of wealth when investors use the relative growth optimal strategy

It is interesting to observe that using the relative growth optimal strategy does not necessarily imply “favorable” asymptotics of the absolute wealth.

We begin with an example which shows that the wealth of an investor who uses the strategy $\widehat{\Lambda}$ may vanish asymptotically because the other investors use “bad” strategies such that the total wealth W_t vanishes. At the same time, there is a strategy the wealth of which does not vanish, but it is not relative growth optimal. In the second part of this section, we investigate the case when all the investors use the relative growth optimal strategy; in that case their wealth will normally grow.

Example 1. We consider a non-random model with two investors and one asset. Suppose the investors have the initial capital $Y^1_0 = Y^2_0 = 1$ and use the strategies that invest the proportions $\lambda^1_{t+1} = \frac{1}{2}$ and $\lambda^2_{t+1} = \frac{1}{2} + \frac{1}{2t}$ in the asset in each time period. Suppose $\rho_t = 1$ for all t, and the (non-random) payoff sequence X_t is defined by

$$X_{t+1} = \frac{Y^1_t + Y^2_t}{2}.$$

This equation together with the wealth equation (2) uniquely define the sequences Y^1_t, Y^2_t, and $W_t = Y^1_t + Y^2_t$:

$$\begin{aligned} &Y^1_{t+1} = Y^1_t\biggl(\frac{1}{2} + \frac{X_{t+1}}{Y^1_t + (1+\frac{1}{t})Y^2_t}\biggr), \qquad Y^2_{t+1} = Y^2_t\biggl(\frac{t-1}{2t} + \frac{(1+\frac{1}{t})X_{t+1}}{Y^1_t + (1+\frac{1}{t})Y^2_t}\biggr), \\ &W_{t+1} = W_t - \frac{Y^2_t}{2t}. \end{aligned} \tag{18}$$

It is easy to see that we have $\zeta_t(W_{t-1}) = W_{t-1}/2$, and hence $\widehat{\lambda}_t = 1/2$. Thus, the strategy of the first investor is relative growth optimal.

Let, as always, $r_t^m = Y_t^m/W_t$, $m = 1, 2$. According to Theorem 1, there exists the limit $r_\infty^2 = \lim_{t\to\infty} r_t^2 \in [0, 1)$. We will now show that $r_\infty^2 > 0$ and $W_\infty = 0$. From (18), we find

$$r_{t+1}^2 = r_t^2(1-\alpha_t), \qquad \text{where } \alpha_t = \frac{r_t^2(1-r_t^2)}{2t^2 + tr_t^2 - (r_t^2)^2}.$$

It is easy to see that $\alpha_t \in (0,1)$ and $\alpha_t = O(t^{-2})$ as $t \to \infty$. Hence, there exists the limit $r_\infty^2 > 0$. Also, from (18) we have $W_{t+1} = W_t(1 - r_t^2/(2t))$. Since $\sum_t r_t^2/t = \infty$, we have $W_\infty = 0$.

However, there is a trivial strategy that guarantees that the wealth does not vanish: $\lambda_t = 0$ for all t. ☐

Now we turn to analysis of the situation when all the investors use the relative growth optimal strategy. Obviously, in this case the relative wealth of the investors will stay constant. Our goal will be to investigate the asymptotics of the total wealth W_t.

To avoid uninteresting complications, let us assume from now on that $\rho_t(\omega) > 0$ for all t, ω. Introduce the discounting sequence

$$D_t = \rho_1 \cdot \ldots \cdot \rho_t, \qquad D_0 = 1,$$

and denote by $W_t' = W_t/D_t$ the discounted wealth of the investors, and by $X_t' = X_t/D_t$ the discounted payoffs.

First, we will show that W_t' does not decrease in the sense that W_t' is a generalized submartingale. But then one can ask the question: will W_t' asymptotically grow to infinity (provided that $\sum_t |X_t| = \infty$)? The answer turns out to be quite interesting. We consider it only in the case when the discounted payoffs X_t' are i.i.d., and show that $W_t' \to \infty$ if X_t are truly random (i.e. the support of the distribution of X_t contains more than one point), while W_t' stays bounded if X_t' are non-random.

Theorem 4. *Suppose all the investors use the strategy $\widehat{\Lambda}$. Then the following claims are true.*

1) The sequence $1/W_t'$ is a supermartingale, W_t' is a generalized submartingale, and there exists the limit $W_\infty' := \lim_{t\to\infty} W_t' \in (0,\infty]$ a.s.

2) Assume additionally that $\rho_t = \rho > 0$ for all t, where ρ is a constant; X_t' is a sequence of i.i.d. random vectors; and the filtration $\mathbb{F}$ is generated by X_t, i.e. $\mathcal{F}_t = \sigma(X_s, s \le t)$. If X_t are not equal to a constant vector a.s., then $W_\infty' = \infty$ a.s.; otherwise $W_t' = W_0 \vee |X_1|/\rho$ for all $t \ge 1$.

Proof. From (11) we find that W_t' satisfies the equation

$$W_t' = (1 - |\widehat{\lambda}_t|)W_{t-1}' + |X_t'| \tag{19}$$

(here $\widehat{\lambda}_t$ denotes the realization of the relative growth optimal strategy).

To simplify the proof, let us first show that it can be reduced to the case when $\rho_t = 1$ for all t. Indeed, consider the two markets: the first one is defined by the sequences $X_t^{(1)}$, $\rho_t^{(1)}$ and initial capital $Y_0^{(1)} > 0$, while the second one by the sequences $X_t^{(2)} = X_t^{(1)}/D_t^{(1)}$, $\rho_t^{(2)} = 1$, and $Y_0^{(2)} = Y_0^{(1)}$. Assume all the investors in the both markets use the relative growth optimal strategy. Denote the total wealth in these markets by $W_t^{(1)}$ and $W_t^{(2)}$, respectively.

It is not hard to see that $\Gamma_t^{(2)} = \{(\omega, c/D_{t-1}^{(1)}(\omega)) : (\omega, c) \in \Gamma_t^{(1)}\}$, and $\zeta_t^{(2)}(\omega, c) = \zeta_t^{(1)}(\omega, cD_{t-1}^{(1)}(\omega))/D_{t-1}^{(1)}(\omega)$. From this and (5), by induction, we find that $\widehat{\lambda}_t^{(2)} = \widehat{\lambda}_t^{(1)}$ and $W_t^{(2)} = W_t^{(1)}/D_t^{(1)}$. Thus, the discounted wealth in the original market specified by X_t and ρ_t will be the same as the wealth in the market specified by X_t' and $\rho_t' = 1$. So, from now on we may assume $\rho_t = 1$ and $X_t' = X_t$, $W_t' = W_t$ for all t.

From (9), it follows that $W_t > 0$ for all t. Let $V_t = 1/W_t$. From (19), we find

$$\frac{V_t}{V_{t-1}} = \frac{1}{1 - |\widehat{\lambda}_t| + V_{t-1}|X_t|} = \frac{W_{t-1}}{\zeta_t(W_{t-1}) + |X_t|},$$

where we used that $1 - |\widehat{\lambda}_t| = V_{t-1}\zeta_t(W_{t-1})$. By the construction of ζ_t, we have $\mathrm{E}(V_t/V_{t-1} \mid \mathcal{F}_{t-1}) \le 1$, hence V_t is a generalized supermartingale, and, hence, a usual supermartingale since it is non-negative. This also implies that W_t is a generalized submartingale (via the identity $W_t = \exp(-\ln V_t)$ and Jensen's inequality). Moreover, since a non-negative supermartingale has a finite limit, there exists $V_\infty = \lim_t V_t \in [0, \infty)$ a.s., and consequently there exists $W_\infty \in (0, \infty]$ a.s. This finishes the proof of the first claim of the theorem.

To prove the second claim, we will need the following auxiliary result on convergence of positive supermartingales, which is a corollary from Proposition 7.1 in Karatzas and Kardaras (2007). Suppose S_t is a strictly positive scalar supermartingale, and consider the generalized supermartingale Z_t defined by

$$\Delta Z_t := Z_t - Z_{t-1} = \frac{S_t}{S_{t-1}} - 1, \qquad Z_0 = 0.$$

Denote by A_t the compensator of Z_t. Let $h(x) = x^2 \wedge |x|$ and introduce the predictable sequence H_t by

$$\Delta H_t = \mathrm{E}(h(\Delta Z_t) \mid \mathcal{F}_{t-1}), \qquad H_0 = 0.$$

Then we have

$$\{\omega : \lim_{t\to\infty} S_t(\omega) = 0\} = \{\omega : \lim_{t\to\infty} (A_t(\omega) + H_t(\omega)) = \infty\} \text{ a.s.}$$

We will apply this result to $S_t = V_t$, so that

$$\Delta Z_t = \frac{W_{t-1}}{\zeta_t(W_{t-1}) + |X_t|} - 1.$$

Observe that since X_t are i.i.d. random vectors, the function $\zeta_t(\omega, c)$ and the conditional distribution $K_t(\omega, dx)$ can be chosen not depending on ω, t, hence we will write them simply as $\zeta(c)$ and $K(dx)$. Then $\Delta H_t = g(W_{t-1})$ with the function

$$g(c) = \int_{\mathbb{R}_+^N} h\left(\frac{c}{\zeta(c) + |x|} - 1\right) K(dx), \qquad c > 0.$$

It is not hard to check that $g(c)$ is continuous on $(0, \infty)$. Moreover, if X_t are non-constant (so the support of $K(dx)$ contains more than one point), then $g(c) > 0$ for all $c > 0$. Therefore, $g(c)$ is separated from zero on any compact set $B \not\ni \{0\}$. So, on the set $\{\omega : \lim_t W_t(\omega) < \infty\}$ we have $H_\infty = \infty$ a.s., which implies $\mathrm{P}(\lim_t W_t < \infty) = 0$ in the case when X_t are non-constant.

When X_t are constant, $X_t \equiv X \in \mathbb{R}_+^N$, it is easy to see from the definition of ζ that

$$(1 - |\widehat{\lambda}_t|)W_{t-1} = \zeta(W_{t-1}) = (W_{t-1} - |X|)^+.$$

Then (19) implies $W_t = W_{t-1} \vee |X|$, and, hence, $W_t = W_0 \vee |X|$. $\square$

Example 2. The proved result leads to an observation, which at first seems counter-intuitive: if X_t and $\widetilde{X}_t$ are two sequences of payoffs such that $X_t \geq \widetilde{X}_t$ for all t, it may happen that the wealth of investors will grow faster under the smaller sequence $\widetilde{X}_t$.

As an example, let $\rho_t \equiv 1$ and $X_t \equiv X \in \mathbb{R}_+^N \setminus \{0\}$ be the same constant vector, while $\widetilde{X}_t = X\xi_t$, where $\xi_t \in [0,1]$ are i.i.d. non-constant random variables. Then, if all the investors use the relative growth optimal strategy, under the sequence X_t the wealth becomes $W_0 \wedge |X|$ after $t = 1$ and stop growing, but it grows to infinity under $\widetilde{X}_t$.

This can be explained by that the presence of randomness in $\widetilde{X}_t$ prevents the investors from "betting too much".

7. Conclusion

We studied a model of a market where several investors compete for payoffs yielded by short-lived assets. The main result of the paper consists in proving that there exists (and is unique) an investment strategy – the relative growth optimal strategy – such that the sequence of its relative wealth is a submartingale for any strategies of competing investors.

This strategy has a number of other optimality properties. It forms a symmetric Nash equilibrium when all the investors maximize their expected relative wealth. It is also a survival strategy in the sense that its relative wealth always stays separated from zero with probability one on the whole infinite time interval. Moreover, its relative wealth tends to 1 if the representative strategy of the other investors is asymptotically different from it. It is also shown that the relative growth optimal strategy posses properties similar to growth optimal strategies (numéraires) in single-investor market models, in particular, it maximizes the asymptotic and t-step growth rate of wealth.

Our paper extends the model of Amir et al. (2013) to a market with a bank account (or a risk-free asset). Inclusion of a bank account in the model leads to interesting analysis of the asymptotics of the absolute wealth of investors. In particular, it turns out that the relative growth optimality (or survival) property of a strategy does not necessarily imply that its absolute wealth will grow if the competitors use "bad" strategies.

References

Algoet, P. H. and Cover, T. M. (1988). Asymptotic optimality and asymptotic equipartition properties of log-optimum investment. *The Annals of Probability*, 16(2):876–898.

Aliprantis, C. D. and Border, K. C. (2006). *Infinite Dimensional Analysis: A Hitchhiker's Guide*. Springer, 3rd edition.

Amir, R., Evstigneev, I. V., and Schenk-Hoppé, K. R. (2013). Asset market games of survival: a synthesis of evolutionary and dynamic games. *Annals of Finance*, 9(2):121–144.

Belkov, S., Evstigneev, I. V., and Hens, T. (2017). An evolutionary finance model with a risk-free asset. Swiss Finance Institute Research Paper 17-28, Swiss Finance Institute.

Blume, L. and Easley, D. (1992). Evolution and market behavior. *Journal of Economic Theory*, 58(1):9–40.

Blume, L. and Easley, D. (2006). If you're so smart, why aren't you rich? Belief selection in complete and incomplete markets. *Econometrica*, 74(4):929–966.

Breiman, L. (1961). Optimal gambling systems for favorable games. In *Proceedings of the 4th Berkeley Symposium on Mathematical Statistics and Probability*, volume 1, pages 63–68.

Evstigneev, I., Hens, T., and Schenk-Hoppé, K. R. (2016). Evolutionary behavioral finance. In Haven, E. et al., editors, *The Handbook of Post Crisis Financial Modelling*, pages 214–234. Palgrave Macmillan UK.

Hakansson, N. H. and Ziemba, W. T. (1995). Capital growth theory. In *Finance*, volume 9 of *Handbooks in Operations Research and Management Science*, pages 65–86. Elsevier.

Holtfort, T. (2019). From standard to evolutionary finance: a literature survey. *Management Review Quarterly*, 69(2):207–232.

Karatzas, I. and Kardaras, C. (2007). The numéraire portfolio in semimartingale financial models. *Finance and Stochastics*, 11(4):447–493.

Karatzas, I. and Shreve, S. E. (1998). *Methods of Mathematical Finance.* Springer.

Kelly, Jr, J. L. (1956). A new interpretation of information rate. *Bell System Technical Journal*, 35(4):917–926.

Long, J. B. (1990). The numeraire portfolio. *Journal of Financial Economics*, 26(1):29–69.

Platen, E. (2006). A benchmark approach to finance. *Mathematical Finance*, 16(1):131–151.

Shapley, L. and Shubik, M. (1977). Trade using one commodity as a means of payment. *Journal of political economy*, 85(5):937–968.

Shiryaev, A. N. (2019). *Probability–2.* Springer, 3rd edition.