11.06.2022. Credit Risk Models Exam. Toraeva Aleksandra

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① $y_0(\tau) = c\cdot\log(1+\tau);\ \tau\in[0,T];\ c\in(0,\infty)$

a) Calculate the bond prices $B_0(\tau_i);\ \tau_i\in\{\tau_1\ldots\tau_5\}=\{1\ldots5\}$.

▨ $B_0(\tau_i) = e^{-y_0(\tau_i)\cdot\tau_i} = e^{-c\cdot\log(1+\tau_i)\cdot\tau_i} = \boxed{(1+\tau_i)^{-c\tau_i}};\ \tau_i\in\{1\ldots5\}$

б) Calculate the initial Libor rates $L_0(\tau_i;\tau_{i+1});\ \tau_i\in\{\tau_1\ldots\tau_4\}$

▨ $$L_t(\tau,\tau') = \frac{B_t(\tau)-B_t(\tau')}{(\tau'-\tau)\cdot B_t(\tau')}$$

$$L_0(\tau_i;\tau_{i+1}) = \frac{B_0(\tau_i)-B_0(\tau_{i+1})}{\underbrace{(\tau_{i+1}-\tau_i)}_{=1}\cdot B_0(\tau_{i+1})} = \frac{B_0(\tau_i)}{B_0(\tau_{i+1})}-1 = \boxed{\frac{(1+\tau_i)^{-c\tau_i}}{(1+\tau_{i+1})^{-c\tau_{i+1}}}-1}$$

c) Calculate continuously compounded forward rates $f_0(\tau_i);\ \tau_i\in\{\tau_1\ldots\tau_5\}$.

▨ $B_0(\tau) = e^{-\int_0^\tau f_0(s)ds}$

$\Rightarrow -\ln B_0(\tau) = \int_0^\tau f_0(s)ds$

$\Rightarrow f_0(\tau) = -\frac{\partial}{\partial\tau}\ln B_0(\tau) = -\frac{\partial}{\partial\tau}(-c\log(1+\tau)\cdot\tau) = c\log(1+\tau)+\frac{c\tau}{1+\tau}$

$\Rightarrow f_0(\tau_i) = c\left(\log(1+\tau_i)+\frac{\tau_i}{1+\tau_i}\right);\ \tau_i\in\{\tau_1\ldots\tau_5\}$

d) For a deterministic no-arbitrage time evolution, determine the short rate r_t which explains the initially observed yields.

▨ $$\underset{\overset{\parallel}{(1+\tau)^{-c\tau}}}{B_0(\tau)} = E^Q\left[e^{-\int_0^\tau r_s ds}\right] \underset{\text{т.к. deterministic evolution}}{\overset{\uparrow}{=}} e^{-\int_0^\tau r_s ds}$$

$\Rightarrow -\int_0^\tau r_s ds = \ln\left((1+\tau)^{-c\tau}\right) = -c\log(1+\tau)\tau$

$\Rightarrow \int_0^\tau r_s ds = \log(1+\tau)\cdot\tau$

$\Rightarrow r_\tau = (\log(1+\tau)\cdot\tau)' = \boxed{\log(1+\tau)+\frac{\tau}{1+\tau}}$

②

$$\begin{cases} dr_t = \theta dt + \sigma^{\sim} dW_t \\ r_0 = r_0^* \end{cases} \Rightarrow r_t = r_0^* + \theta t + \sigma^{\sim} W_t \sim N(r_0^* + \theta t; (\sigma^{\sim})^2 \cdot t)$$

$$S_T = e^{r_T}$$

а) Calculate the expectation $E^Q(S_T)$ of S_T with respect to the spot measure Q.

▶ $E^Q[S_T] = E^Q[e^{r_T}]$, где $r_T \sim N(\underbrace{r_0^* + \theta T}_{=\mu_\xi}; \underbrace{(\sigma^{\sim})^2 \cdot T}_{=\sigma_\xi^2})$

А $Ee^{\xi} = e^{\mu_\xi + \frac{\sigma_\xi^2}{2}}$

$$\Rightarrow \boxed{E^Q[S_T] = e^{r_0^* + \theta T + \frac{(\sigma^{\sim})^2 \cdot T}{2}}}$$ ▲

б) Calculate the expectation $E^{Q^T}(S_T)$ of S_T with respect to the forward measure Q^T.

▶ Надо найти плотность замены мер и поменять Винер процесс.

Имеем: $r_t = \theta dt + \sigma^{\sim} dW_t$

$$\Rightarrow B_t(\tau) = e^{\underbrace{-\theta \int_t^{\tau} (\tau - u) du}_{= \theta \frac{(t-\tau)^2}{2} \Big|_t^{\tau} = -\theta \frac{(t-\tau)^2}{2}} + \frac{\sigma^2}{2} \frac{(\tau - t)^3}{3} - (\tau - t) r_t}$$ — было в лекциях, пример.

$$\Rightarrow B_t(\tau) = e^{-\theta \frac{(t-\tau)^2}{2} + \frac{\sigma^2}{2} \cdot \frac{(\tau-t)^3}{3} - (\tau - t) r_t}$$

Хотим найти $dB_t(\tau)$:

Ну $f(t,x) = e^{-\theta \frac{(t-\tau)^2}{2} + \frac{\sigma^2}{2} \frac{(\tau-t)^3}{3} - (\tau-t)x}$

$$\Rightarrow dB_t(\tau) = \left(-\theta(t-\tau) - \frac{\sigma^2}{2}(\tau-t)^2 + r_t\right) B_t(\tau) dt - (\tau - t) B_t(\tau) \cdot \overset{\theta dt + \sigma^{\sim} dW_t}{\overset{\shortparallel}{dr_t}} + \frac{(\sigma^{\sim})^2}{2} \cdot (\tau-t)^2 B_t(\tau) dW_t =$$

$$= B_t(\tau) \left(\cancel{-\theta(t-\tau)} \cancel{- \frac{\sigma^2}{2}(\tau-t)^2} + r_t \cancel{-(\tau-t)\theta} + \cancel{\frac{(\sigma^{\sim})^2}{2} \cdot (\tau - t)^2}\right) dt - (\tau - t) B_t(\tau) \sigma^{\sim} dW_t =$$

$$= B_t(\tau) (r_t dt - (\tau - t) \sigma^{\sim} dW_t)$$

$$\Rightarrow \begin{cases} dB_t(\tau) = B_t(\tau)(r_t dt - (\tau - t)\sigma^{\sim} dW_t \\ dB_t = r_t B_t dt \end{cases}$$

$L_t = \frac{B_t(\tau)}{B_t}$; $dL_t = ?$

$f(\frac{x}{y}) = \frac{x}{y}$; $f_x' = \frac{1}{y}$ $\quad X_t = B_t(\tau)$

$f_y' = -\frac{x}{y^2}$ $\quad Y_t = B_t$

$$d\left(\frac{B_t(\tau)}{B_t}\right) = f'_x dX_t + f'_y dY_t + \frac{1}{2} \underset{0}{\underset{\parallel}{f''_{xx}(dX_t)^2}} + f''_{xy} \cdot \underset{0}{\underset{\parallel}{\underbrace{dX_t \cdot dY_t}}} + \frac{1}{2} \underset{0}{\underset{\parallel}{f''_{yy} \cdot (dY_t)^2}} =$$

$$= \frac{1}{B_t} \cdot B_t(\tau)\,(r_t dt - (\tau - t)\sigma^r dW_t) - \frac{B_t(\tau)}{B_t^2} \cdot r_t \cancel{B_t} dt =$$

$$= \frac{B_t(\tau)}{B_t}\left(\cancel{r_t dt} - (\tau - t)\sigma^r dW_t \cancel{- r_t dt}\right) =$$

$$= \frac{B_t(\tau)}{B_t}\left(-(\tau - t)\sigma^r\right) dW_t$$

$$\Rightarrow dL_t = \underset{-\mu_t}{\underset{\parallel}{\underbrace{-L_t \cdot (\tau - t)\sigma^r}}} dW_t$$

$\Rightarrow$ по т. Гирсанова: $\tilde{W}_t^{Q^T} = W_t + \int_0^t \mu_s ds$ ~~$= W_t + \sigma^r \int_0^t (\tau - s) ds =$~~

$$\Rightarrow d\tilde{W}_t^{Q^T} = dW_t + \underset{(\tau - t)\sigma^r}{\underset{\parallel}{\mu_t}} dt$$

$$\Rightarrow dW_t = d\tilde{W}_t^{Q^T} - (\tau - t)\sigma^r dt$$

$$\Rightarrow dr_t = \theta dt + \sigma^r dW_t = \theta dt + \sigma^r\left(d\tilde{W}_t^{Q^T} - (\tau - t)\sigma^r dt\right) =$$

$$= \left(\theta - (\tau - t)(\sigma^r)^2\right) dt + \sigma^r \cdot d\tilde{W}_t^{Q^T}$$

~~$\Rightarrow$ в мере Q^T: $r_T \sim N(r_0^* \ldots$~~

$$\Rightarrow r_t = r_0^* + \underbrace{\int_0^t (\theta - (\tau - s)(\sigma^r)^2) ds}_{} + \sigma^r \cdot \tilde{W}_t^{Q^T}$$

$$\parallel$$

$$\theta t + (\sigma^r)^2 \cdot \frac{(s - \tau)^2}{2}\Big|_0^t$$

$$\parallel \theta t + (\sigma^r)^2 ((t - \tau)^2 - \tau^2)$$

$$\parallel \theta t + (\sigma^r)^2 t (t - 2\tau)$$

$\Rightarrow$ в мере Q^T: $r_T \sim N\left(r_0^* + \theta T + (\sigma^r)^2 T(T - \overset{\tau = T}{2\tau})\,;\ (\sigma^r)^2 \cdot T\right)$

$$\Rightarrow \boxed{E^{Q^T}[S^T] = e^{\mu_S + \frac{\sigma_S^2}{2}} = e^{r_0^* + \theta T - (\sigma^r)^2 T^2 + \frac{(\sigma^r)^2 \cdot T}{2}}}$$

c) Calculate the forward-futures spread at time $t = 0$.

Spread $= E^{Q^T}[S_T] - ~~E~~^{Q}[S_T] = e^{r_0^* + \theta T + \frac{(\sigma^r)^2 T}{2}} \cdot \left(e^{-(\sigma^r)^2 T^2} - 1\right)$. this model (not HJM)

This spread is

③ $$\begin{cases} df_t(\tau) = \alpha_t(\tau)dt + \sigma_t(\tau)dW_t \\ f_0(\tau) = \dfrac{\lambda}{1+\tau} \end{cases}$$

$$\sigma_t(\tau) = \sigma^f(\tau-t)^3$$

a) Determine the expectation $E^Q(r_t)$ of the short rate with respect to Q.

From the HJM drift-condition:

$$\alpha_t(\tau) = \sigma_t(\tau)\int_t^\tau \sigma_t(s)ds = \sigma^f(\tau-t)^3 \cdot \int_t^\tau \sigma^f \cdot (s-t)^3 ds = (\sigma^f)^2 \cdot (\tau-t)^3 \cdot \frac{(s-t)^4}{4}\Big|_t^\tau = (\sigma^f)^2 \cdot \frac{(\tau-t)^7}{4}$$

$$\Rightarrow df_t(\tau) = (\sigma^f)^2 \cdot \frac{(\tau-t)^7}{4} dt + \sigma^f \cdot (\tau-t)^3 \cdot dW_t$$

$$\Rightarrow f_t(\tau) = f_0(\tau) + \frac{(\sigma^f)^2}{4} \cdot \int_0^t (\tau-s)^7 ds + \sigma^f \cdot \int_0^t (\tau-s)^4 dW_s$$

$$\Rightarrow f_t(\tau) = \frac{\lambda}{1+\tau} - \frac{(\sigma^f)^2}{32} \cdot (s-\tau)^8\Big|_0^t + \sigma^f \cdot \int_0^t (\tau-s)^4 dW_s$$

$$\Rightarrow f_t(\tau) = \frac{\lambda}{1+\tau} + \frac{(\sigma^f)^2}{32} \cdot \tau^8 - \frac{(\sigma^f)^2}{32} \cdot (t-\tau)^8 + \sigma^f \cdot \int_0^t (\tau-s)^4 dW_s$$

$$\Rightarrow r_t = f_t(t) = \frac{\lambda}{1+t} + \frac{(\sigma^f)^2}{32} \cdot t^8 + \underbrace{\sigma^f \int_0^t (t-s)^4 dW_s}_{\text{Martingale}} \quad (*)$$

$$\Rightarrow \boxed{E^Q[r_t] = \frac{\lambda}{1+t} + \frac{(\sigma^f)^2}{32} \cdot t^8}$$

б) Determine the Girsanov kernel φ_s from Q to Q^T.

Знаем, что если дано

$$df_t(\tau) = \alpha_t(\tau)dt + \sigma_t(\tau)dW_t,$$

то $$dB_t(\tau) = B_t(\tau)\Big(r_t dt - \overbrace{\Big(\int_t^\tau \sigma_t(s)ds\Big)}^{= \sigma^f \cdot \frac{(\tau-t)^4}{4}} dW_t\Big)$$

~~А из (*):~~

~~$dr_t = \left(-\frac{\lambda}{(1+t)^2} + \frac{2(\sigma^f)^2}{4} t^7\right)dt + \sigma^f (t-s)^4 dW_t\Big|_{s=t} + \sigma^f \int_0^t 4(t-s)^3 dW_s$~~

~~$\Rightarrow dr_t = \left(-\frac{\lambda}{(1+t)^2} + \frac{(\sigma^f)^2}{4} t^7\right)dt$~~

А ещё нам нужно $dB_t = ?$; $L_t = \dfrac{B_t(\tau)}{B_t}$

$$B_t = e^{-\int_0^t r_s ds}$$

$$\Rightarrow \begin{cases} dB_t(\tau) = B_t(\tau)\left(r_t dt - \frac{\sigma^f (\tau - t)^4}{4} dW_t\right) \\ dB_t = r_t B_t dt \end{cases}$$

$\Rightarrow$ по ф-ле Ито для $X_t = B_t(\tau)$
$Y_t = B_t$:

$$d\left(\frac{B_t(\tau)}{B_t}\right) = f'_x dX_t + f'_y dY_t + \frac{1}{2} f''_{xx} (dX_t)^2 + f''_{xy} \cdot dX_t \cdot dY_t + \frac{1}{2} \cdot f''_{yy} \cdot \underbrace{(dY_t)^2}_{0} =$$

$$= \frac{1}{B_t} B_t(\tau) \left(r_t dt - \frac{\sigma^f (\tau - t)^4}{4} dW_t\right) - \frac{B_t(\tau)}{B_t^2} \cdot r_t B_t dt =$$

$$= \frac{B_t(\tau)}{B_t}\left(\cancel{r_t dt - r_t dt}\right) - \frac{B_t(\tau)}{B_t} \cdot (\sigma^f) \cdot \frac{(\tau - t)^4}{4} dW_t$$

$$\Rightarrow dL_t = -L_t \cdot (\sigma^f) \cdot \frac{(\tau - t)^4}{4} dW_t$$

$$\Rightarrow \varphi_s = -\frac{\sigma^f \cdot (\tau - t)^4}{4}\Bigg|_{\tau = T} = \boxed{-\frac{\sigma^f (T - t)^4}{4}}$$

(т.к. Q^T так определялась)

④ $B_0^*(\tau) = \frac{1}{1+\tau^2}$

$\sigma_t(\tau) = \sigma^f \cdot (\tau - t + 1)^{-2}$

a) Calculate the drift $\alpha_t(\tau)$ from the HJM drift condition.

$$\alpha_t(\tau) = \sigma_t(\tau) \int_t^\tau \sigma_t(s)\,ds = \sigma^f (\tau - t + 1)^{-2} \cdot \int_t^\tau \sigma^f \cdot \frac{1}{(s-t+1)^2}\,ds = (\sigma^f)^2 \cdot \frac{1}{(\tau-t+1)^2} \cdot \frac{-1}{(s-t+1)}\Big|_t^\tau =$$

$$= -(\sigma^f)^2 \cdot \frac{1}{(\tau-t+1)^2}\left(\frac{1}{\tau-t+1} - 1\right) = \boxed{(\sigma^f)^2 \cdot \left((\tau-t+1)^{-2} - (\tau-t+1)^{-3}\right)}$$

б) Fit the initial forward rate $f_0^*(\tau)$ to the market data given by $B_0^*(\tau)$:

$$B_0^*(\tau) = e^{-\int_0^\tau f_0(s)\,ds}$$

$$\Rightarrow f_0^*(\tau) = -\frac{\partial}{\partial \tau} \ln B_0^*(\tau) = -\frac{\partial}{\partial \tau} \ln\left(\frac{1}{1+\tau^2}\right) = \frac{\partial}{\partial \tau} \ln(1+\tau^2) = \boxed{\frac{2\tau}{1+\tau^2}}$$

⑤ $\begin{cases} df_t(\tau) = \alpha_t(\tau)dt + \sigma_t(\tau)dW_t \\ f_0(\tau) = \lambda \end{cases}$

$$\sigma_t(\tau) = \sigma^f \cdot \frac{\tau}{1+t}$$

$dS_t = S_t(r_t dt + \sigma^S dW_t)$, with $r_t = f_t(t)$.

a) Is the process $r_t = f_t(t)$ Markovian?

Yes, it is, because $\sigma_t(\tau) = \sigma^f \cdot \tau \cdot \frac{1}{1+t} = \varphi(\tau)\cdot\xi(t)$ – представимо в такой форме

b) Calculate the volatility $\sigma_t^B(T)$ для

$$dB_t(\tau) = B_t(\tau)(r_t dt + \sigma_t^B(\tau)dW_t)$$

$$\sigma_t^B(\tau) = -\int_t^\tau \sigma_t(s)ds = -\int_t^\tau \sigma^f s \cdot \frac{1}{1+t}ds = -\sigma^f \cdot \frac{1}{1+t}\cdot\frac{s^2}{2}\Big|_t^\tau = \boxed{\frac{\sigma^f\cdot(t-\tau)^2}{2(1+t)}}$$

c) The fraction $Z_t = \frac{S_t}{B_t(T)}$ satisfies the following SDE:

$$dZ_t = Z_t(a(t)dt + m(t)dW_t).$$

Найти $a(t)$ и $m(t)$.

$\begin{cases} dS_t = S_t(r_t dt + \sigma^S dW_t) \\ dB_t(\tau) = B_t(\tau)(r_t dt + \sigma_t^B(\tau)dW_t) \end{cases}$

$\Rightarrow$ по ф-ле Ито для $f(x,y) = \frac{x}{y}$: $f'_x = \frac{1}{y}$ $\quad f''_{xx} = 0$ $\quad$; $X_t = S_t$

$f'_y = -\frac{x}{y^2}$ $\quad f''_{xy} = -\frac{1}{y^2}$ $\quad Y_t = B_t(\tau)$

$f''_{yy} = \frac{2x}{y^3}$

$$\Rightarrow d\left(\frac{S_t}{B_t(\tau)}\right) = f'_x dX_t + f'_y dY_t + \frac{1}{2}\underset{\shortparallel \atop 0}{f''_{xx}}(dX_t)^2 + f''_{xy}\cdot dX_t\cdot dY_t + \frac{1}{2}\cdot f''_{yy}\cdot(dY_t)^2 =$$

$$= \frac{1}{B_t(\tau)}\cdot S_t(r_t dt + \sigma^S dW_t) - \frac{S_t}{(B_t(\tau))^{\cancel{2}}}\cdot\cancel{B_t(\tau)}\cdot(r_t dt + \sigma_t^B(\tau)dW_t) -$$

$$- \frac{1}{B_t(\tau)^{\cancel{2}}}\cdot S_t\cdot\sigma^S\cdot\cancel{B_t(\tau)}\cdot\sigma_t^B(\tau)dt + \frac{S_t}{(B_t(\tau))^3}\cdot\cancel{(B_t(\tau))^2}\cdot(\sigma_t^B(\tau))^2 dt =$$

$$= \frac{S_t}{B_t(\tau)}\left(\cancel{r_t dt} + \sigma^S dW_t - \cancel{r_t dt} - \sigma_t^B(\tau)dW_t - \sigma^S\cdot\sigma_t^B(\tau)dt + (\sigma_t^B)^2 dt\right) =$$

$$= \underset{\shortparallel \atop Z_t}{\boxed{\frac{S_t}{B_t(\tau)}}}\left(\underbrace{\sigma_t^B(\tau)(\sigma_t^B(\tau) - \sigma^S)}_{\shortparallel\, a(t)}dt + \underbrace{(\sigma^S - \sigma_t^B(\tau))}_{\shortparallel\, m(t)}dW_t\right)$$

$$\Rightarrow \boxed{\begin{cases} a(t) = \sigma_t^B(T)(\sigma_t^B(T) - \sigma^S) \\ m(t) = \sigma^S - \sigma_t^B(T) \end{cases}}$$

Exam 2021

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Question 1

Consider evolution of an interest rate market in continuous time $t = [0,T]$ over $T = 5$ years whose zero bonds mature at times $\{\tau_1, \tau_2, \dots, \tau_5\} = \{1, 2, \dots 5\}$. Suppose that the continuously compounded yields, observed at the initial time $t = 0$ are given by

$$y_0(\tau) = c \cdot \log(1+\tau), \qquad \tau \in [0,T]$$

with $c \in]0, \infty[$.

a) Calculate the bond prices $(B_0(\tau_i))_{\tau_i \in \{\tau_1, \dots, \tau_5\}}$. (2 marks)

b) Calculate the initial LIBOR rates $(L_0(\tau_i, \tau_{i+1}))_{\tau_i \in \{\tau_1, \dots, \tau_4\}}$. (2 marks)

c) Calculat continuously compounded forward rates $(f_0(\tau_i))_{\tau_i \in \{\tau_1, \dots, \tau_5\}}$. (2 marks)

d) For a deterministic no-arbitrage time evolution, determine the short rate $(r_t)_{t \in [0,T]}$ which explains the initially observed yields. (4 marks)

Question 2

Consider zero bond dynamics $(B_t(\tau))_{t \in [0,\tau]}$, $\tau \in [0,T]$ defined by the short rate model

$$dr_t = \theta dt + \sigma^r dW_t, \qquad r_0 = r_0^*$$

where the process $(W_t)_{t \in [0,T]}$ follows a Brownian motion with respect to the spot martingale measure $\mathbb{Q}$ and the parameters are given as

$$\theta, r_0^* \in \mathbb{R}, \quad \sigma^r \in]0, \infty[.$$

Consider a contingent claim whose terminal payoff at the time $t = T$ is given by $S_T = e^{r_T}$

a) Calculate the expectation $E_0(T) = \mathbb{E}^{\mathbb{Q}}(S_T)$ of S_T with respect to the spot measure $\mathbb{Q}$. (3 marks)

b) Calculate the expectation $F_0(T) = \mathbb{E}^{\mathbb{Q}^T}(S_T)$ of S_T with respect to the forward measure $\mathbb{Q}^T$. (3 marks)

c) What explains the difference between the forward and the futures prices on S_T? Calculate the forward-futures spread at the time $t = 0$. (4 marks)

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Question 3 (10 marks)

Consider a one-factor HJM model whose forward rate dynamics follows

$$df_t(\tau) = \alpha_t(\tau)dt + \sigma_t(\tau)dW_t, \qquad f_0(\tau) = \lambda/(1+\tau), \quad 0 \le t \le \tau \le T, \;\; \lambda \in]0,\infty[$$

with a Browinan Motion $(W_t)_{t\in[0,T]}$ under the spot martingale measure $\mathbb{Q}$. Assume that

$$\sigma_t(\tau) = \sigma^f(\tau - t)^3 \quad 0 \le t \le \tau \le T$$

with a pre-specified parameter $\sigma^f \in]0,\infty[$.

a) Determine the expectation $\mathbb{E}^{\mathbb{Q}}(r_t)$ of the short rate with respect to the spot martingale measure $\mathbb{Q}$ for all $t \in [0,T]$. (3 marks)

b) Determine the Girsanov kernel $(\varphi_s)_{s\in[0,T]}$ required for the density which performs the transformation from the spot martingale measure $\mathbb{Q}$ to the forward martingale measure $\mathbb{Q}^T$. (3 marks)

START EACH QUESTION ON A NEW PAGE

Question 4

Consider the time horizon $T = 1$ of a bond market and suppose that today's ($t = 0$) bond curve is given by

$$B_0^*(\tau) = \frac{1}{1+\tau^2} \qquad \text{for all } \tau \in [0, T].$$

For a one-factor HJM model with deterministic forward rate volatility

$$\sigma_t(\tau) = \sigma^f \cdot (\tau - t + 1)^{-2}, \qquad 0 \leq t \leq \tau \leq T, \ \sigma^f \in]0, \infty[$$

a) Calculate the drift $(\alpha_t(\tau))_{t\in[0,\tau]}$, $\tau \in [0, T]$ from the HJM drift condition. (5 marks)

b) Fit the initial forward rate $(f_0^*(\tau))_{\tau[0,T]}$ to the market data given by the bond curve $(B_0^*(\tau))_{\tau\in[0,T]}$. (5 marks)

Question 5

Consider a one-factor HJM model whose forward rate dynamics follows

$$df_t(\tau) = \alpha_t(\tau)dt + \sigma_t(\tau)dW_t, \qquad f_0(\tau) = \lambda, \quad 0 \leq t \leq \tau \leq T, \ \lambda \in]0, \infty[$$

with a Browinan Motion $(W_t)_{t\in[0,T]}$ under the spot martingale measure $\mathbb{Q}$. Assume that

$$\sigma_t(\tau) = \sigma^f \cdot \frac{\tau}{1+t} \quad 0 \leq t \leq \tau \leq T$$

with a pre-specified parameter $\sigma^f \in]0, \infty[$. Consider a risky asset following

$$dS_t = S_t(r_t dt + \sigma^S dW_t), \qquad S_0 = S_0^* \in]0, \infty[, \ \sigma^S \in]0, \infty[$$

with the short rate $(r_t = f_t(t))_{t\in[0,T]}$.

a) Is the process $(r_t = f_t(t))_{t\in[0,T]}$ Markovian? (2 marks)

b) Calculate the volatility $\sigma_t^B(T)$ for $0 \leq t \leq \tau \leq T$ defined by

$$dB_t(\tau) = B_t(T)(r_t dt + \sigma_t^B(\tau)dW_t)$$

(3 marks)

b) The fraction $(X_t = S_t/B_t(T))$ satisfies the following stochastic differential equation

$$dX_t = X_t(a(t)dt + m(t)dW_t).$$

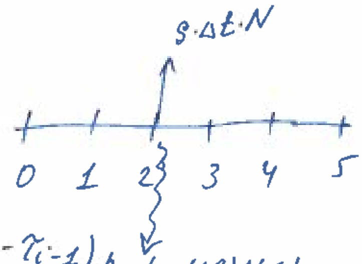
Determine the processes $(a(t))_{t\in[0,T]}$ and $(m(t))_{t\in[0,T]}$. (5 marks)

end of the exam paper

10.06.2022. Лапшин. Пробный экзамен.

① $L_0(\tau_i;\tau_{i+1})\Big|_{i=0}^{5}=\frac{1}{100+i}$

а) найти swap rate для $\tau_2\ldots\tau_5$.



$$\sum_{i=2}^{5} S\cdot\underset{\overset{\shortparallel}{1}}{\Delta t}\cdot\cancel{N}\cdot B_0(\tau_i)=\sum_{i=2}^{5}\cancel{N}\cdot\underbrace{L_{\tau_{i-1}}(\tau_{i-1},\tau_i)\cdot(\tau_i-\tau_{i-1})\cdot B_0(\tau_i)}_{\shortparallel}\quad L_1(1,2)N\cdot\Delta t$$

$$\underbrace{L_0(\tau_{i-1},\tau_i)\cdot(\tau_i-\tau_{i-1})\cdot B_0(\tau_i)}_{\shortparallel}$$

$$\underbrace{B_0(\tau_{i-1})-B_0(\tau_i)}_{\shortparallel}$$

$$B_0(1)-B_0(5)$$

$$\rightarrow \boxed{S=\frac{B_0(1)-B_0(5)}{\sum\limits_{i=2}^{5}B_0(\tau_i)}=\frac{B_0(1)-B_0(5)}{B_0(2)+B_0(3)+B_0(4)+B_0(5)}}$$

Но:

$$L_t(\tau,\tau')=\frac{B_t(\tau)-B_t(\tau')}{(\tau'-\tau)\cdot B_t(\tau')}$$

$$L_0(\tau_{i-1},\tau_i)=\frac{B_0(\tau_{i-1})-B_0(\tau_i)}{(\tau_i-\tau_{i-1})B_0(\tau_i)}=\frac{B_0(\tau_{i-1})}{B_0(\tau_i)}-1.$$

$$\Rightarrow \frac{B_0(\tau_{i-1})}{B_0(\tau_i)}=1+L_0(\tau_{i-1},\tau_i)$$

$$\Rightarrow \boxed{B_0(\tau_i)=\frac{B_0(\tau_{i-1})}{1+L_0(\tau_{i-1},\tau_i)}}$$

$$L_0(0,1)=\frac{\overset{=1}{B_0(0)}}{B_0(1)}-1\Rightarrow B_0(1)=\frac{B_0(0)}{1+L_0(0,1)}=\frac{1}{1+\frac{1}{100}}=\frac{1}{101/100}=\frac{100}{101}$$

$$L_0(1,2)=\frac{B_0(1)}{B_0(2)}-1=\cancel{\frac{100}{101}}\Rightarrow B_0(2)=\frac{B_0(1)}{1+\frac{1}{101}}=\frac{100/101}{102/101}=\frac{100}{102}$$

~~$L_0(2,3)=$~~ $B_0(3)=\frac{B_0(2)}{1+\frac{1}{102}}=\frac{100/102}{103/102}=\frac{100}{103}$

$B_0(4)=\frac{100}{104}$

$B_0(5)=\frac{100}{105}$

$$\Rightarrow S=\frac{B_0(1)-B_0(5)}{\sum\limits_{i=2}^{5}B_0(\tau_i)}=\frac{\frac{100}{101}-\frac{100}{105}}{\frac{100}{102}+\frac{100}{103}+\frac{100}{104}+\frac{100}{105}}=\boxed{\frac{4/101\cdot105}{\frac{1}{102}+\frac{1}{103}+\frac{1}{104}+\frac{1}{105}}}$$

б) Найти $y_0(\tau_k)$; $k = 1 \dots 5$.

$$B_0(t) = e^{-y_0(t) \cdot t}$$

$$\Rightarrow \ln B_0(t) = -y_0(t) \cdot t$$

$$\Rightarrow y_0(\tau) = -\frac{\ln B_0(\tau)}{\tau}$$

Но $B_0(\tau_k) = \frac{100}{100+k}$

$$\Rightarrow \boxed{y_0(\tau_k) = -\frac{\ln\left(\frac{100}{100+k}\right)}{k}}$$

в) $L_t(\tau_i; \tau_{i+1}) = ?$; $t = 3$
$i = 4$

Т.е $L_3(4;5) = ?$

$$\Rightarrow L_3(4,5) = L_0(4,5) = \boxed{\frac{1}{104}}$$

---

① $\begin{cases} dr_t = \theta dt + \tilde{\sigma} dW_t \\ r_0 = r_0^* \end{cases}$ $\qquad B_t(\tau) = E^Q\left[e^{-\int_t^\tau r_s ds}\right]$

а) $r_t = r_0^* + \theta t + \tilde{\sigma} W_t$

б) $B_0^*(\tau) = E^Q\left[e^{-\int_0^\tau r_s ds}\right] = e^{-\int_0^T f_0(s) ds}$

Ну $-\int_0^\tau r_s ds = -\int_0^\tau (r_0^* + \theta s + \tilde{\sigma} W_s) ds = -r_0^*\tau - \theta \cdot \frac{s^2}{2}\Big|_0^\tau - \tilde{\sigma}\underbrace{\int_0^\tau W_s ds}_{\tau \cdot W_\tau - \int_0^\tau s dW_s = \int_0^\tau (\tau - s) dW_s} = -r_0^*\tau - \theta \cdot \frac{\tau^2}{2} - \tilde{\sigma} \cdot \int_0^\tau (\tau - s) dW_s$

$$\Rightarrow \mu_\xi = -r_0^*\tau - \frac{\theta \cdot \tau^2}{2}$$

$$\sigma_\xi^2 = (\tilde{\sigma})\int_0^\tau \cdot (\tau - s)^2 ds = (\tilde{\sigma})^2 \cdot \frac{(s-\tau)^3}{3}\Big|_0^\tau = (\tilde{\sigma})^2 \cdot \frac{\tau^3}{3}$$

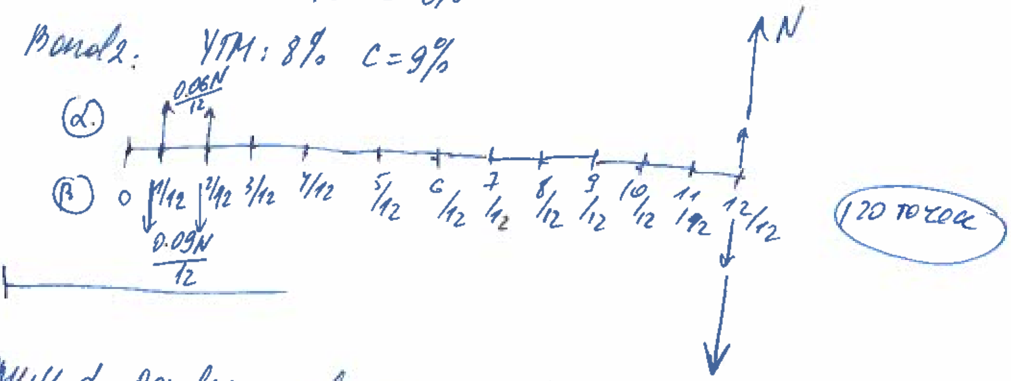
$$\Rightarrow -\int_0^\tau r_s ds \sim N(\mu_\xi; \sigma_\xi)$$

$$\Rightarrow E^Q\left[e^{-\int_0^\tau r_s ds}\right] = E e^{\mu_\xi} = e^{\mu_\xi + \frac{\sigma_\xi^2}{2}} = \boxed{e^{-r_0^*\tau - \frac{\theta \cdot \tau^2}{2} + \frac{(\tilde{\sigma})^2 \cdot \tau^3}{6}}}$$

c) $f_0^*(\tau) = -\frac{\partial}{\partial \tau} \ln(B_0^*(\tau)) = -\frac{\partial}{\partial \tau}\left(-r_0^*\tau - \frac{\theta\tau^2}{2} + \frac{(\tilde{\sigma})^2\tau^3}{6}\right) = \boxed{r_0^* + \theta\tau - \frac{(\tilde{\sigma})^2 \cdot \tau^2}{2}}$

③. Bond 1: YTM: 7% $c=6\%$

Bond 2: YTM: 8% $c=9\%$



купим α Bonds1 и продадим β Bonds 2.

$$\Rightarrow \begin{cases} \alpha \cdot \frac{0.06N}{12} = \beta \cdot \frac{0.09 \cdot N}{12} \Rightarrow 2\alpha = 3\beta \Rightarrow \cancel{\alpha} = \frac{3\beta}{2} \\ FV = (\alpha-\beta)N = 0.5\beta \end{cases}$$

$$PV = \underset{\overset{\parallel}{\frac{3\beta}{2}}}{\alpha} \cdot PV_1 - \beta \cdot PV_2$$

$$\Rightarrow \frac{PV}{FV} = \frac{\frac{3\beta}{2} PV_1 - \beta \cdot PV_2}{0.5\beta} = 3PV_1 - 2PV_2, \text{ где } PV_1 = N \cdot \left(e^{-\frac{0.07}{12}}\right)^{120} + \sum_{i=1}^{120} \left(\frac{0.06}{12} \cdot N\right) \cdot \left(e^{-\frac{0.07}{12}}\right)^i$$

$$PV_2 = N \cdot \left(e^{-\frac{0.08}{12}}\right)^{120} + \sum_{i=1}^{120} \left(\frac{0.09}{12} \cdot N\right) \cdot \left(e^{-\frac{0.08}{12}}\right)^i$$

---

④ $\begin{cases} dr_t = \theta dt + \tilde{\sigma} dW_t \\ r_0 = r_0^* \end{cases}$ $\qquad B_t = B_0 \cdot e$

$\Rightarrow r_t = r_0^* + \theta t + \tilde{\sigma} W_t$

a) $\begin{cases} df_t(\tau) = \alpha(t,\tau)dt + \sigma_t(\tau)dW_t \\ f_0(\tau) = f_0^*(\tau) \end{cases}$

Ну $B_t(\tau) = E_t^Q\left[e^{-\int_t^\tau r_s ds}\right]$

$$-\int_t^\tau r_s ds = -r_0^*(\tau-t) - \theta \cdot \frac{(\tau-t)^2}{2} - \tilde{\sigma} \underbrace{\int_t^\tau W_s ds}_{\overset{\parallel}{W_\tau \cdot \tau - W_t \cdot t - \int_t^\tau s dW_s}} =$$

$$B_t(\tau) = e^{-\theta \cdot \int_t^\tau (\tau-u)du + \frac{\sigma^2}{2}\frac{(\tau-t)^3}{3} - (\tau-t)r_t} \quad - \text{см. стр. 50}$$

$$\int_t^\tau (\tau-u)du \overset{\parallel}{=} \theta \frac{(t-\tau)^2}{2}\Big|_t^\tau \overset{\parallel}{=} -\theta\frac{(t-\tau)^2}{2}$$

$$B_t(\tau) = e^{-\theta\frac{(t-\tau)^2}{2} + \frac{\sigma^2}{2}\frac{(\tau-t)^3}{3} - (\tau-t)r_t}$$

$$f(t,x) = e^{-\theta\frac{(t-\tau)^2}{2} + \frac{\sigma^2}{2}\frac{(\tau-t)^3}{3} - (\tau-t)X}$$

$$dB_t(\tau) = \left(-\theta(t-\tau) - \frac{\sigma^2}{2}(\tau-t)^2 + r_t\right)B_t(\tau)dt - (\tau-t) \cdot B_t(\tau) \underset{\overset{\parallel}{\theta dt + \tilde{\sigma} dW_t}}{dr_t} + \frac{(\tilde{\sigma})^2}{2}(\tau-t)^2 B_t(\tau)dt =$$

$$= B_t(\tau)\left(-\theta(\cancel{t-\tau}) - \frac{\sigma^2}{2}(\tau-t)^2 + r_t + \frac{\tilde{\sigma}^2}{2}(\tau-t)^2 - (\tau-t)\theta\right)dt - (\tau-t)B_t(\tau)\tilde{\sigma} dW_t$$

$$\Rightarrow \boxed{dB_t(\tau) = B_t(\tau) r_t dt - (\tau - t) B_t(\tau) \tilde{\sigma} dW_t}$$

~~$B_t(\tau) = e^{-\underbrace{\int_t^\tau f_t(s)ds}_{= X_t(\tau)}}$~~

~~$dX_t(\tau) = -\underbrace{f_t(t)}_{= r_t}dt + \int_t^\tau \overbrace{df_t(s)}^{= \alpha_t(s)dt + \sigma_t(s)dW_t}ds = -r_t dt + \left(\int_t^\tau \alpha_t(s)ds\right)dt + \left(\int_t^\tau \sigma_t(s)ds\right)dW_t =$~~

$= -r_t dt$

Знаем:

Если дано $df_t(\tau) = \alpha_t(\tau)dt + \sigma_t(\tau)dW_t$,

то $\boxed{dB_t(\tau) = B_t^{(\tau)} r_t dt - B_t^{(\tau)}\left(\int_t^\tau \sigma_t(s)ds\right)dW_t}$

$$\Rightarrow \begin{cases} \boxed{\alpha_t(\tau) = r_t} \\ \int_t^\tau \sigma_t(s)ds = (\tau - t)\cdot\tilde{\sigma} \end{cases}$$

$$\Rightarrow \boxed{\sigma_t(\tau) = \tilde{\sigma}}$$

Ответ: $\alpha_t(\tau) = r_t$

$\sigma_t(\tau) = \tilde{\sigma}$.

б) $dS_t = S_t(r_t dt + \sigma^S dW_S)$

$dQ^S = \underbrace{\frac{S_T \cdot B_0}{B_T \cdot S_0}}_{= L_t} dQ$

$L_t = \frac{S_t \cdot B_0}{B_t \cdot S_0}$

$$\begin{cases} dS_t = S_t(r_t dt + \sigma^S dW_t) \\ dB_t = r_t B_t dt \end{cases} \Rightarrow d\left(\frac{S_t}{B_t}\right) = \left(\frac{S_t}{B_t}\right)\sigma_t^S dW_t$$

$$\Rightarrow dL_t = \sigma^S L_t dW_t$$

$$\to -\mu_S = \sigma^S$$

$$\Rightarrow \tilde{W}_t^S = W_t + \int_0^t \mu_s ds = W_t - \sigma^S \cdot t$$

$$\Rightarrow W_t = \tilde{W}_t^S + \sigma^S \cdot t$$

$$\Rightarrow dW_t = d\tilde{W}_t^S + \sigma^S dt$$

$$\Rightarrow dr_t = \theta dt + \tilde{\sigma} dW_t = \theta dt + \tilde{\sigma}(d\tilde{W}_t^S + \sigma^S dt) = (\theta + \tilde{\sigma}\sigma^S)dt + \tilde{\sigma} d\tilde{W}_t^S$$

~~$E^{Q^S}[r_t]$~~ $r_t = r_0^* + (\theta + \tilde{\sigma}\sigma^S)t + \tilde{\sigma}\cdot\tilde{W}_t^S$

$$\Rightarrow \boxed{E^{Q^S}[r_t] = r_0^* + (\theta + \tilde{\sigma}\sigma^S)t}$$

⑤ $B_0^*(\tau) = \frac{1}{1+c\tau}$

$\sigma_t(\tau) = \sigma\sqrt{\tau-t}$ – в НЗМ

б) $\alpha_t(\tau) = \sigma_t(\tau)\int_t^\tau \sigma_t(s)ds = \sigma^2\sqrt{\tau-t}\int_t^\tau \sqrt{s-t}\,ds = \sigma^2\sqrt{\tau-t}\cdot(s-t)^{3/2}\cdot\frac{2}{3}\Big|_t^\tau = \boxed{\frac{2}{3}\sigma^2\cdot(\tau-t)^2}$

а) $f_0^*(\tau) = ?$

ну $B_0^*(\tau) = e^{-\int_0^\tau f_0(s)ds}$

$\Rightarrow -\ln B_0^*(\tau) = \int_0^\tau f_0(s)ds$

$\Rightarrow f_0^*(\tau) = -\frac{\partial}{\partial\tau}\ln B_0^*(\tau) = -\frac{\partial}{\partial\tau}\ln\left(\frac{1}{1+c\tau}\right) = \frac{\partial}{\partial\tau}\ln(1+c\tau) = \boxed{\frac{c}{1+c\tau}}$

c) Найти $\sigma_t^B(\tau)$ для $dB_t(\tau) = B_t(\tau)(r_t dt + \sigma_t^B(\tau)dW_t)$

~~ну $B_t(\tau) = e^{-\int_t^\tau f_t(s)ds}$~~

ну $\sigma_t^B(\tau) = -\int_t^\tau \sigma_t(s)ds = -\int_t^\tau \sigma\sqrt{s-t}\,ds = -\sigma\cdot\frac{2}{3}\cdot(s-t)^{3/2}\Big|_t^\tau = \boxed{-\frac{2}{3}\sigma(\tau-t)^{3/2}}$

Sample Exam

START EACH QUESTION ON A NEW PAGE

Question 1 Consider LIBOR rates observed today ($t = \tau_0 = 0$) for times $\tau_i = i$, $i = 0, \dots 5$ measured in years

$$L_0(\tau_i, \tau_{i+1}))_{i=0}^{5} = \frac{1}{100 + i}.$$

a) Determine the at-the-money interest rate (swap rate) for an interest rate swap with dates $\tau_2, \dots, \tau_5$ ($m = 2$, $n = 5$). (3 marks)

b) Calculate the yields $y_0(\tau_k)$ $k = 1, \dots, 5$ (continuous compounding). (3 marks)

c) Assume that the market expectation hypothesis holds exactly. Determine the LIBOR rate $L_t(\tau_i, \tau_{i+1})$ for $t = 3$, $i = 4$. (4 marks)

Question 2

Consider zero bond dynamics $(B_t(\tau))_{t \in [0,\tau]}$, $\tau \in [0, T]$ defined by the short rate model

$$dr_t = \theta dt + \sigma^r dW_t, \qquad r_0 = r_0^*$$

where the process $(W_t)_{t \in [0,T]}$ follows a Brownian motion with respect to the spot martingale measure and the parameters are given as

$$\theta, r_0^* \in \mathbb{R}, \quad \sigma^r \in]0, \infty[.$$

a) Calculate the short rate evolution $(r_t)_{t \in [0,T]}$. (3 marks)

b) Calculate the initial bond curve $B_0^*(\tau) = \mathbb{E}^{\mathbb{Q}}(e^{-\int_0^\tau r_s ds})$. (3 marks)
Hint: $\int_0^t W_s ds$ follows normal distribution with mean zero and variance $\frac{t^3}{3}$.
Hint: If N is normally distributed then $\mathbb{E}(e^N) = e^{\mathbb{E}(N) + \frac{1}{2}\mathrm{Var}(N)}$.

c) Determine the initial forward rates $f_0^*(\tau) = -\frac{\partial}{\partial \tau} \ln(B_0^*(\tau))$. (4 marks)

Question 3 (10 marks)

Consider two coupon paying bonds (Bond 1 and Bond 2) with face value 10,000 AUD paying coupons monthly at the (annual) coupon rate of 6% (Bond 1) and 9% (Bond 2). Assume that the first coupon just has been paid at $\tau = 0$, the last coupon (in addition to the face value) will be paid at $\tau = 10$ and the bonds are traded now, at $\tau = 0$ at the yields 7% (Bond 1) and 8% (Bond 2). Is it possible to determine the price of a zero bond maturing at $\tau = 10$ using no-arbitrage arguments? If yes, calculate the yield of this bond (continuous compounding).

START EACH QUESTION ON A NEW PAGE

Question 4

Consider zero bond dynamics $(B_t(\tau))_{t\in[0,\tau]}$, $\tau \in [0,T]$ defined by the short rate model

$$dr_t = \theta dt + \sigma^r dW_t, \qquad r_0 = r_0^*$$

where the process $(W_t)_{t\in[0,T]}$ follows a Brownian motion with respect to the spot martingale measure $\mathbb{Q}$ and the parameters are given as

$$\theta, r_0^* \in \mathbb{R}, \quad \sigma^r \in]0,\infty[.$$

a) For the above short rate model, consider continuously compounded forward rates $(f_t(\tau))_{t\in[0,\tau]}$, $\tau \in [0,T]$ which follow

$$df_t(\tau) = \alpha(t,\tau)dt + \sigma_t(\tau)dW_t, \qquad f_0(\tau) = f_0^*(\tau).$$

Determine the functions $(\alpha_t(\tau))_{t\in[0,\tau]}$ and $(\sigma_t(\tau))_{t\in[0,\tau]}$. (5 marks)

b) Suppose that the price evolution $(S_t)_{t\in[0,T]}$ of a stock is given by the strong solution to the stochastic differential equation

$$dS_t = S_t(r_t dt + \sigma^S dW_t), \qquad S_0 = S_0^* \in]0,\infty[, \text{ with volatility } \sigma^S \in]0,\infty[.$$

Consider the the S-measure $\mathbb{Q}^S$ defined by

$$d\mathbb{Q}^S = \frac{S_T}{B_T}\frac{B_0}{S_0}d\mathbb{Q}$$

where $(B_t = e^{\int_0^t r_s ds})_{t\in[0,T]}$ denotes the price evolution of the standard savings account. Determine the distribution of r_T with respect to the measure $\mathbb{Q}^S$. (5 marks)

START EACH QUESTION ON A NEW PAGE

Question 5 Consider the horizon $T = 5$ of a bond market and suppose that today's $(t = 0)$ bond curve is given by

$$B_0^*(\tau) = \frac{1}{1 + c\tau} \qquad \text{for all } \tau \in [0, T] \text{ with } c \in]0, \infty[.$$

For a one-factor HJM model with deterministic forward rate volatilities

$$\sigma_t(\tau) = \sigma\sqrt{\tau - t}, \qquad 0 \leq t \leq \tau \leq T, \;\; \sigma \in]0, \infty[.$$

a) Calculate the initial forward rates $(f_0^*(\tau))_{\tau\in[0,T]}$. (3 marks)

b) Calculate the drift $(\alpha_t(\tau))_{t\in[0,\tau]}$, $\tau \in [0, T]$ from the HJM drift condition. (3 marks)

c) Determine the bond volatility $(\sigma_t^B(\tau))_{t\in[0,\tau]}$ for $\tau \in [0, T]$ defined by (4 marks)

$$dB_t(\tau) = B_t(\tau)(r_t dt + \sigma_t^B(\tau) dW_t), \qquad 0 \leq t \leq \tau \leq T.$$

end of the exam paper

11.05.2022. Assignment for Interest Rates and Credit Risk Models.

Question 1 Consider forward rates, based on continuous compounding, observed at $t=0$: $f_0^*(\tau)=0.1;\ \tau\in[0,6]$, where the time is measured in years.

a) Calculate the bond prices $B_0^*(\tau_i)$ for the times $\tau_i=i;\ i=1\ldots6$.

$$B_0^*(\tau)=e^{-\int_0^\tau \underbrace{f_0^*(s)}_{=0.1}ds}=e^{-0.1\tau}$$

$$\Rightarrow B_0^*(\tau_i)=e^{-0.1\cdot\tau_i}=\boxed{e^{-0.1\cdot i}},\ i=1\ldots6.$$

$i=1;\ B_0^*(\tau_1)\approx 0.9048$

$i=2;\ B_0^*(\tau_2)\approx 0.8187$

$i=3;\ B_0^*(\tau_3)\approx 0.7408$

$i=4;\ B_0^*(\tau_4)\approx 0.6703$

$i=5;\ B_0^*(\tau_5)\approx 0.6065$

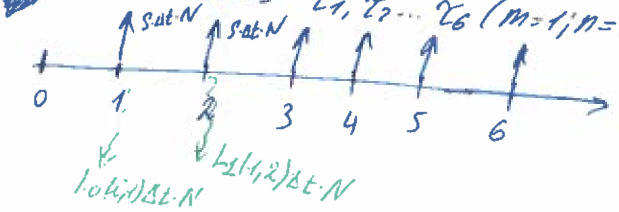
$i=6;\ B_0^*(\tau_6)\approx 0.5488$

b) Calculate the LIBOR rates $L_0(\tau_{i-1};\tau_i);\ i=1\ldots6$

$$\begin{cases} L_t(\tau;\tau')=\dfrac{B_t(\tau)-B_t(\tau')}{(\tau'-\tau)\,B_t(\tau')} \\ B_0(\tau)=e^{-0.1\tau} \end{cases}$$

$$\Rightarrow L_0(\tau_{i-1};\tau_i)=\frac{B_0(\tau_{i-1})-B_0(\tau_i)}{\underbrace{(\tau_i-\tau_{i-1})}_{=1}\cdot B_0(\tau_i)}=\frac{B_0(\tau_{i-1})}{B_0(\tau_i)}-1=\frac{e^{-0.1\tau_{i-1}}}{e^{-0.1\tau_i}}-1=e^{0.1\overbrace{(\tau_i-\tau_{i-1})}^{=1}}-1=e^{0.1}-1\approx\boxed{0.1052}$$

c) Calculate the at-the-money interest rate (swap rate) for an interest rate swap with dates $\tau_1;\tau_2\ldots\tau_6$ ($m=1;n=6$.



$$\Rightarrow\sum_{i=1}^{6} S\cdot\cancel{\Delta t\cdot N}\cdot B_{\tau_i}=\sum_{i=1}^{6} L_{\tau_{i-1}}(\tau_{i-1};\tau_i)\cdot\cancel{\Delta t\cdot N}\cdot B_{\tau_i}$$

$$\Rightarrow S=\frac{\sum_{i=1}^{6} L_{\tau_{i-1}}(\tau_{i-1};\tau_i)\,B_{\tau_i}}{\sum_{i=1}^{6} B_{\tau_i}}$$

но $$L_{\tau_{i-1}}(\tau_{i-1};\tau_i)=\frac{B_{\tau_{i-1}}(\tau_{i-1})-B_{\tau_{i-1}}(\tau_i)}{\underbrace{(\tau_i-\tau_{i-1})}_{=1}\cdot B_{\tau_{i-1}}(\tau_i)}=\frac{B_{\tau_{i-1}}(\tau_{i-1})}{B_{\tau_{i+1}}(\tau_i)}-1=\frac{1}{e^{-0.1\cdot1}}-1=e^{0.1}-1.$$

$$\Rightarrow S=(e^{0.1}-1)\cdot\frac{\sum_{i=1}^{6}B_{\tau_i}}{\sum_{i=1}^{6}B_{\tau_i}}=e^{0.1}-1\approx 0.1052$$

Question 2 Consider the time horizon $T=1$ of a bond market and suppose that today's $(t=0)$ bond curve is given by

$B_0^*(\tau) = e^{-c\sqrt{1+\tau}}$ for all $\tau \in [0,T]$ with $c \in (0,\infty)$

For a one-factor HJM model with deterministic forward rate volatility

$\sigma_t(\tau) = \sigma^f(\tau - t + 1)$; $0 \le t \le \tau \le T$; $\sigma^f \in (0,\infty)$

a) Calculate the drift $(\alpha_t(\tau))_{t\in[0,\tau]}$; $\tau \in [0,T]$ from the HJM drift condition.

▶ $\alpha_t(\tau) = \sigma_t(\tau) \cdot \int_t^\tau \sigma_t(s)ds = \sigma^f(\tau-t+1)\int_t^\tau \sigma^f(s-t+1)ds = (\sigma^f)^2 \cdot (\tau-t+1)\left(\frac{s^2}{2}\Big|_t^\tau - (t-1)(\tau-t)\right) =$

$= (\sigma^f)^2 \cdot (\tau-t+1)\left(\frac{\tau^2-t^2}{2} - (t-1)(\tau-t)\right) = (\sigma^f)^2 \cdot (\tau-t+1)\left((\tau-t)\left(\frac{\tau+t}{2} - t + 1\right)\right) =$

$= (\sigma^f)^2 \frac{(\tau-t+1)(\tau-t)}{2}(\tau+t-2t+2) = \boxed{\frac{(\sigma^f)^2 \cdot (\tau-t+1)(\tau-t)(\tau-t+2)}{2}}$ ◀

b) Fit the initial forward rate $(f_0^*(\tau))_{\tau\in[0,T]}$ to the market data given by the bond curve $(B_0^*(\tau))_{\tau\in[0,T]}$

▶ $B_0^*(\tau) = e^{-\int_0^\tau f_0^*(s)ds}$

$\Rightarrow \ln(B_0^*(\tau)) = -\int_0^\tau f_0^*(s)ds$

$\Rightarrow [\ln(B_0^*(\tau))]'_\tau = -f_0^*(\tau)$

$\Rightarrow f_0^*(\tau) = -(\ln B_0^*(\tau))'_\tau = (c\sqrt{1+\tau})'_\tau = \boxed{\frac{c}{2\sqrt{1+\tau}}}$ ◀

c) Is the factorization $\sigma_t(\tau) = \xi(t)\cdot\psi(\tau)$ satisfied?

▶ $\sigma_t(\tau) = \sigma^f(\tau-t+1) \ne \xi(t)\cdot\psi(\tau)$, where ξ, ψ - deterministic functions

$\Rightarrow$ <u>no</u> Markov property of the short rate ◀

d) Determine the short rate r_T.

▶ HJM model: $df_t(\tau) = \alpha_t(\tau)dt + \sigma_t(\tau)dW_t$

$\Rightarrow df_t(\tau) = (\sigma^f)^2 \cdot \frac{(\tau-t)(\tau-t+1)(\tau-t+2)}{2}dt + \sigma^f(\tau-t+1)dW_t$

$\Rightarrow f_t(\tau) = f_0(\tau) + \frac{(\sigma^f)^2}{2}\int_0^t [(\tau-s+1)((\tau-s+1)^2-1)]ds + \sigma^f\int_0^t(\tau-s+1)dW_s$

$\Rightarrow f_t(\tau) = \frac{c}{2\sqrt{1+\tau}} + \frac{(\sigma^f)^2}{2}\cdot\left[\int_0^t(\tau-s+1)^3ds - \int_0^t(\tau-s+1)ds\right] + \sigma^f\int_0^t(\tau-s+1)dW_s$

$\Rightarrow f_t(\tau) = \frac{c}{2\sqrt{1+\tau}} + \frac{(\sigma^f)^2}{2}\cdot\left[\frac{1}{4}(\tau+1)^4 - \frac{1}{4}(\tau-t+1)^4 + \frac{1}{2}(\tau-t+1)^2 - \frac{1}{2}(\tau+1)^2\right] + \sigma^f\int_0^t(t-s+1)dW_s$

$\Rightarrow r_t = f_t(t) = \frac{c}{2\sqrt{1+t}} + \frac{(\sigma^f)^2}{2}\left[\frac{1}{4}(t+1)^4 - \frac{1}{4} + \frac{1}{2} - \frac{1}{2}(t+1)^2\right] + \sigma^f\int_0^t(t-s+1)dW_s$

$\Rightarrow r_T = \frac{c}{2\sqrt{1+T}} + \frac{(\sigma^f)^2}{2}\cdot\left[\frac{1}{4}(T+1)^2-1)((T+1)^2+1) - \frac{1}{2}[(T+1)^2-1)]\right] + \sigma^f\cdot\int_0^T(T-s+1)dW_s$

$r_T = \frac{c}{2\sqrt{1+T}} + \frac{(\sigma^f)^2}{2}\cdot\frac{1}{4}T(T+2)(T^2+2T+\cancel{1+1-2}) + \sigma^f\int_0^T(T-s+1)dW_s$

$\Rightarrow \boxed{r_T = \frac{c}{2\sqrt{1+T}} + \frac{(\sigma^f)^2}{8}\cdot T^2(T+2)^2 + \sigma^f\cdot\int_0^T(T-s+1)dW_s}$ ◀

Question 3) Consider zero bond dynamics $(B_t(\tau))_{t\in[0,\tau]}$ $;\tau\in[0,T]$ defined by the Ho-Lee short rate model $\begin{cases} dr_t = (c+t)dt + \sigma^r dW_t \\ r_0 = r_0^* \end{cases}$,

where the process $(W_t)_{t\in[0,T]}$ follows a Brownian motion on $(\Omega, \mathcal{F}, Q, \mathcal{F}_t)$ with respect to the spot martingale measure Q and $\sigma^r, c, r_0^* \in (0,\infty)$ are fixed.

a) Calculate the expectation $E^Q(r_T)$ of the short rate with respect to the spot martingale measure Q in terms of the model parameters $\sigma^r, c, r_0^* \in (0,\infty)$.

▶ $\begin{cases} dr_t = (c+t)dt + \sigma^r dW_t \\ r_0 = r_0^* \end{cases}$

$\Rightarrow r_t = r_0^* + \int_0^t (c+s)ds + \sigma^r W_t$

$\Rightarrow r_t = r_0^* + ct + \frac{t^2}{2} + \sigma^r W_t$

$\Rightarrow \boxed{E^Q(r_T) = r_0^* + cT + \frac{T^2}{2}}$ ◄

b) Calculate the expectation $E^{Q^T}(r_T)$ of the short rate with respect to the forward martingale measure Q^T.

▶ $E^{Q^T}(r_T) = f_0(T)$

$f_0(T)$ можно найти из того, что $B_0(T) = e^{-\int_0^T f_0(s)ds}$ $\Rightarrow f_0(T) = -(\ln(B_0(T)))'_T$

А $B_0(T)$ мы найдём, зная r_t: $B_0(T) = E^Q[e^{-\int_0^T r_s ds}]$.

из $r_s = r_0^* + cs + \frac{s^2}{2} + \sigma^r W_s$

$\Rightarrow \int_0^T r_s ds = r_0^* T + \frac{cT^2}{2} + \frac{T^3}{6} + \sigma^r \cdot \underbrace{\int_0^T W_s ds}$

$\| \; TW_T - \int_0^T s\,dW_s$

$\| \; \int_0^T (T-s)dW_s$

$\Rightarrow \int_0^T r_s ds \sim N(r_0^* T + \frac{cT^2}{2} + \frac{T^3}{6}; (\sigma^r)^2 \cdot \underbrace{\int_0^T (T-s)^2 ds}_{\| \frac{(s-T)^3}{3}\Big|_0^T = \frac{T^3}{3}}) = N(r_0^* T + \frac{cT^2}{2} + \frac{T^3}{6}; (\sigma^r)^2 \cdot \frac{T^3}{3})$

$\Rightarrow -\int_0^T r_s ds \sim N(-r_0^* T - \frac{cT^2}{2} - \frac{T^3}{6}; (\sigma^r)^2 \cdot \frac{T^3}{3}) \simeq N(\mu_S; \sigma_S^2)$

$\Rightarrow E e^S = \int_{-\infty}^{+\infty} e^x \cdot \frac{1}{\sigma_S\sqrt{2\pi}} e^{-\frac{(x-\mu_S)^2}{2\sigma_S^2}} dx = \frac{1}{\sigma_S\sqrt{2\pi}} \int_{-\infty}^{+\infty} e^{\frac{-x^2 + 2x(\mu_S + \sigma_S^2) - \mu_S^2}{2\sigma_S^2}} dx = e^{\frac{(\mu_S + \sigma_S^2)^2 - \mu_S^2}{2\sigma_S^2}} = e^{\mu_S + \frac{\sigma_S^2}{2}}$

$\Rightarrow B_0(T) = E^Q[e^{-\int_0^T r_s ds}] = e^{\mu_S + \frac{\sigma_S^2}{2}} = e^{-r_0^* T - \frac{cT^2}{2} - \frac{T^3}{6} + \frac{(\sigma^r)^2 \cdot T^3}{6}}$

$\Rightarrow f_0(T) = -(\ln(B_0(T)))'_T = (r_0^* T + \frac{cT^2}{2} + \frac{T^3}{6} - \frac{(\sigma^r)^2 T^3}{6})'_T = r_0^* + cT + \frac{T^2}{2} - \frac{(\sigma^r)^2 T^2}{2}$

$\Rightarrow E^{Q^T}(r_T) = f_0(T) = \boxed{r_0^* + cT + \frac{T^2}{2} - \frac{(\sigma^r)^2 \cdot T^2}{2}}$ ◄

Question 4 Consider a one-factor HJM model whose forward rate dynamics follows

$$\begin{cases} df_t(\tau) = \alpha_t(\tau)dt + \sigma_t(\tau)dW_t; \\ f_0(\tau) = \lambda \end{cases} ; \quad 0 \le t \le \tau \le T; \ \lambda \in (0, \infty)$$

with a Brownian Motion $(W_t)_{t\in[0,T]}$ under the spot martingale measure Q.
Assume that $\sigma_t(\tau) = \sigma^f \cdot \tau \cdot t$; $0 \le t \le \tau \le T$ with a pre-specified parameter $\sigma^f \in (0, \infty)$.
Consider a risky asset following

$$\begin{cases} dS_t = S_t(r_t dt + \sigma^S dW_t) \\ S_0 = S_0^* \in (0,\infty); \sigma^S \in (0,\infty) \end{cases}$$

with the short rate $(r_t = f_t(t))_{t\in[0,T]}$.

a) Calculate the volatility $\sigma_t^B(\tau)$ for $0 \le t \le \tau \le T$ defined by $dB_t(\tau) = B_t(\tau)(r_t dt + \sigma_t^B(\tau)dW_t)$.

$$\sigma_t^B(\tau) = -\int_t^\tau \sigma_t(s)ds = -\int_t^\tau \sigma^f \cdot s \cdot t \cdot ds = -\sigma^f \cdot t \cdot \int_t^\tau s ds = -\sigma^f \cdot t \cdot \frac{s^2}{2}\Big|_t^\tau = -\frac{\sigma^f \cdot t \cdot (\tau^2 - t^2)}{2} = \boxed{\frac{\sigma^f \cdot t(t^2-\tau^2)}{2}}$$

b) With $z_u := \sigma^S - \sigma_u^B(T)$ calculate the quantity $\int_0^T |z_u|^2 du$.

$$|z_u|^2 = |\sigma^S - \sigma_u^B(T)|^2 = |\sigma^S - \frac{\sigma^f u}{2}(u^2-T^2)|^2 = (\sigma^S)^2 - \sigma^S \sigma^f \cdot u(u^2-T^2) + \frac{(\sigma^f)^2}{4}u^2(u^2-T^2)^2.$$

$$\Rightarrow \int_0^T |z_u|^2 du = \int_0^T \Big[(\sigma^S)^2 - \sigma^S \sigma^f \cdot u(u^2-T^2) + \frac{(\sigma^f)^2}{4} \cdot u^2(u^2-T^2)^2\Big]du =$$

$$= \int_0^T \Big[(\sigma^S)^2 - \sigma^S\sigma^f u^3 + \sigma^S\sigma^f \cdot u \cdot T^2 + \frac{(\sigma^f)^2}{4}u^6 - \frac{(\sigma^f)^2 \cdot u^4 T^2}{2} + \frac{(\sigma^f)^2}{4}u^2 \cdot T^4\Big]du =$$

$$= (\sigma^S)^2 \cdot T - \frac{\sigma^S\sigma^f T^4}{4} + \frac{\sigma^S\sigma^f \cdot T^4}{2} + \frac{(\sigma^f)^2 \cdot T^7}{28}^{|15} - \frac{(\sigma^f)^2 \cdot T^7}{10}^{|42} + \frac{(\sigma^f)^2 \cdot T^7}{12}^{|35} = \boxed{(\sigma^S)^2 \cdot T + \frac{\sigma^S\sigma^f T^4}{4} + \frac{2(\sigma^f)^2 \cdot T^7}{105}}$$

c) Find $E^Q\left(\frac{1}{B_T} \cdot (S_T - K)^+\right) = ?$

$$E^Q\Big[\frac{1}{B_T}\cdot(S_T-K)^+\Big] = E^Q\Big[\overset{S_0 \cdot B_0 = 1}{\frac{S_T}{B_T\,S_0}} \cdot \mathbb{1}(S_T \ge K)\Big] - K \cdot E^Q\Big[\overset{B_0(T) \quad B_T(T) = 1 \cdot B_0}{\frac{1}{B_T\,B_0(T)}} \cdot \mathbb{1}(S_T \ge K)\Big] =$$

$$= S_0 \cdot E^{Q^S}[\mathbb{1}\{S_T \ge K\}] - K \cdot B_0(T) \cdot E^{Q^T}[\mathbb{1}\{S_T \ge K\}] =$$

$$= \boxed{S_0 \cdot P_{Q^S}(S_T \ge K) - K \cdot B_0(T) \cdot P_{Q^T}(S_T \ge K)}$$

Чтобы посчитать $P_{Q^S}(S_T \ge K)$ и $P_{Q^T}(S_T \ge K)$, нужно узнать распределение S_T в мерах Q^S и Q^T.
А про меры Q^S и Q^T мы знаем только их плотности:

$$L_t^{Q^S} = \frac{S_t \cdot B_0}{B_t \cdot S_0}; \quad L_t^{Q^T} = \frac{B_t(T) \cdot B_0}{B_t \cdot B_0(T)}$$

Тогда из вида $L_t = \frac{B_t(T)}{B_t \cdot B_0(T)}$, применяя ф-лу Ито для $f(x,y) = \frac{x}{y}$ к $X_t = B_t(T)$; $Y_t = B_t$ получаем

$$\begin{cases} dB_t(T) = B_t(T) \cdot (r_t dt + \sigma_t^B dW_t) \\ dB_t = r_t B_t dt \end{cases} \Rightarrow d\left(\frac{B_t(T)}{B_t}\right) = \frac{B_t(T)}{B_t} \cdot \sigma_t^B(T) \Rightarrow \boxed{dL_t^{Q^T} = \sigma_t^B L_t dW_t} \quad dW_t^T = dW_t^Q - \sigma_t^B(t$$

Аналогично

$$\begin{cases} dS_t = S_t(r_t dt + \sigma_t^S dW_t) \\ dB_t = r_t B_t dt \end{cases} \Rightarrow d\left(\frac{S_t}{B_t}\right) = \left(\frac{S_t}{B_t}\right) \cdot \sigma_t^S dW_t \Rightarrow \boxed{dL_t^{Q^S} = \sigma_t^S L_t dW_t} \Rightarrow \boxed{dW_t^S = dW_t^Q - \sigma_t^S dt}$$

но мы сделаем по-другому:

- докажем, что $d\left(\frac{S_t}{B_t(T)}\right) = \left(\frac{S_t}{B_t(T)}\right)(\sigma^S - \sigma_t^B(T))\,dW_t^T$

как это док-ть: 1) $\begin{cases} dB_t(T) = B_t(T)(r_t dt + \sigma_t^B dW_t) \\ dB_t = r_t B_t dt \end{cases} \overset{?}{\Rightarrow} d\left(\frac{B_t(T)}{B_t}\right) = \frac{B_t(T)}{B_t}\cdot\sigma_t^B dW_t$

Ну $d\left(\frac{B_t(T)}{B_t}\right) = d\left(\frac{x}{y}\right) = \frac{1}{Y_t}dX_t - \frac{X_t dY_t}{Y_t^2} + \frac{1}{2}\underbrace{f''_{xx}}_{=0}(dX_t)^2 + f''_{xy}\underbrace{dX_t\cdot dY_t}_{=0} + \frac{1}{2}f''_{yy}\cdot\underbrace{(dY_t)^2}_{=0} =$

$= \frac{B_t(T)(r_t dt + \sigma_t^B dW_t)}{B_t} - \frac{B_t(T)\cdot r_t dt}{B_t} = \frac{B_t(T)}{B_t}\cdot\sigma_t^B dW_t$

2) $\begin{cases} dS_t = S_t(r_t dt + \sigma_t^S dW_t) \\ dB_t = r_t B_t dt \end{cases} \overset{?}{\Rightarrow} d\left(\frac{S_t}{B_t}\right) = \frac{S_t}{B_t}\cdot\sigma_t^S\cdot dW_t$

(аналогично, применяя ф-лу Ито для $f(x,y) = \frac{x}{y}$)

3) $\begin{cases} d\left(\frac{B_t(T)}{B_t}\right) = \frac{B_t(T)}{B_t}\cdot\sigma_t^B dW_t \quad Y_t \\ d\left(\frac{S_t}{B_t}\right) = \frac{S_t}{B_t}\cdot\sigma_t^S dW_t \quad X_t \end{cases} \overset{?}{\Rightarrow} d\left(\frac{S_t}{B_t(T)}\right) = \left(\frac{S_t}{B_t(T)}\right)\cdot(\sigma^S - \sigma_t^B(T))\,dW_t^T$

ну применяем ф-лу Ито для $f(x,y) = \frac{x}{y}$

$\Rightarrow d\left(\frac{S_t}{\cancel{B_t}}\cdot\frac{\cancel{B_t}}{B_t(T)}\right) = \frac{dX_t}{Y_t} - \frac{X_t\cdot dY_t}{Y_t^2} + \frac{1}{2}\cdot\underbrace{f''_{xx}}_{=0} + \underbrace{f''_{xy}}_{=\frac{1}{Y_t^2}}\cdot dX_t\cdot dY_t + \frac{1}{2}\cdot\underbrace{f''_{yy}}_{=\frac{X_t\cdot 2}{Y_t^3}}(dY_t)^2 =$

$= \frac{X_t\sigma_t^S dW_t}{Y_t} - \frac{X_t\cdot Y_t\sigma_t^B dW_t}{Y_t^2} + 0 - \frac{X_t\sigma_t^S\cdot Y_t\sigma_t^B dt}{Y_t^2} + \frac{1}{\cancel{2}}\cdot\frac{X_t\cdot\cancel{2}}{Y_t^3}\cdot Y_t^2(\sigma_t^B)^2 dt =$

$= \frac{X_t}{Y_t}\sigma_t^B(\sigma_t^B - \sigma_t^S)dt + \frac{X_t}{Y_t}(\sigma_t^S - \sigma_t^B)dW_t$

$\Rightarrow$ волатильность у $\frac{S_t}{B_t(T)}$ в мере Q равна $\frac{S_t}{B_t(T)}(\sigma_t^S - \sigma_t^B)$. $\Rightarrow$ она остается такой и в мере Q^T, т.к. волатильность при замене меры не меняется.

$\Rightarrow \boxed{d\left(\frac{S_t}{B_t(T)}\right) = \frac{S_t}{B_t(T)}\cdot\underbrace{(\sigma^S - \sigma_t^B(T))}_{=\Sigma_s}dW_t^T} \longleftarrow$ GBM с $\mu=0$; $\sigma_s = \Sigma_s$

$\Rightarrow S_T = \overset{=1}{B_T(T)}\cdot\frac{S_0}{B_0(T)}\cdot e^{\int_0^T\Sigma_s dW_s^T - \frac{1}{2}\int_0^T\Sigma_s^2 ds}$

$B_0(T) = e^{-\int_0^T \overset{=\lambda}{f_0(s)}ds} = e^{-\lambda T}$

$\Rightarrow P^{Q^T}(S_T \geq K) = P\left(\frac{S_0}{B_0(T)}\cdot e^{\int_0^T\Sigma_s dW_s^T - \frac{1}{2}\int_0^T\Sigma_s^2 ds} \geq K\right) = P\left(\int_0^T\Sigma_s dW_s^T - \frac{1}{2}\int_0^T\Sigma_s^2 ds \geq \ln\left(\frac{K\cdot B_0(T)}{S_0}\right)\right) =$

$= P\left(\underbrace{\int_0^T\Sigma_s dW_s^T}_{\sim N(0,\int_0^T\Sigma_s^2 ds)} \geq \ln\left(\frac{K\cdot B_0(T)}{S_0}\right) + \frac{1}{2}\int_0^T\Sigma_s^2 ds\right) = P\left(\xi\sim N(0,1) \geq \frac{\ln\left(\frac{K\cdot B_0(T)}{S_0}\right) + \frac{1}{2}\int_0^T\Sigma_s^2 ds}{\sqrt{\int_0^T\Sigma_s^2 ds}}\right) = \Phi\left(\frac{\ln\left(\frac{S_0\cdot e^{\lambda T}}{K}\right) - \frac{1}{2}\int_0^T\Sigma_s^2 ds}{\sqrt{\int_0^T\Sigma_s^2 ds}}\right) =: \Phi(d_2)$

Аналогично, $P^{Q^S}(S_T \geq K) = \Phi\left(\frac{\ln\left(\frac{S_0\cdot e^{\lambda T}}{K}\right) - \frac{1}{2}\int_0^T\Sigma_s^2 ds}{\sqrt{\int_0^T\Sigma_s^2 ds}}\right) =: \Phi(d_1)$

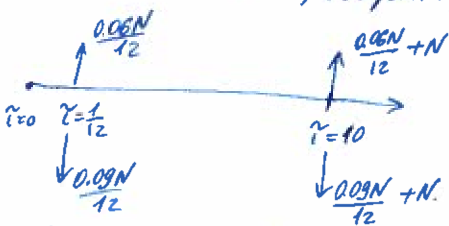
$\Rightarrow E^Q\left[\frac{1}{B_T}(S_T - K)^+\right] = S_0\cdot P_{Q^S}(S_T \geq K) - K\cdot B_0(T)\cdot P_{Q^T}(S_T \geq K) = \boxed{S_0\cdot\Phi(d_1) - K\cdot e^{-\lambda T}\cdot\Phi(d_2)}$ ◀

Question 5

Consider two coupon paying bonds (Bond 1 and Bond 2) with face value 10.000 AUD paying coupons monthly at the (annual) coupon rate of 6% (Bond 1) and 9% (Bond 2). Assume that the first coupon just has been paid at $\tau=0$, the last coupon (in addition to the face value) will be paid at $\tau=10$ and the bonds are traded now, at $\tau=0$, at the yields 7% (Bond 1) and 8% (Bond 2). Is it possible to determine the price of a zero bond maturing at $\tau=10$ using no-arbitrage arguments? If yes, calculate the yield of this bond (continuous compounding)

Bond 1: YTM = 7%; coupon = 6%; $PV_1 = N\cdot\left(e^{-\frac{0.07}{12}}\right)^{120} + \sum_{i=1}^{120}\left(\frac{0.06}{12}\right)\cdot N\cdot\left(e^{-\frac{0.07}{12}}\right)^{i} \approx 9268.26.$

Bond 2: YTM = 8%; coupon = 9%; $PV_2 = N\cdot\left(e^{-\frac{0.08}{12}}\right)^{120} + \sum_{i=1}^{120}\left(\frac{0.09}{12}\right)\cdot N\cdot\left(e^{-\frac{0.08}{12}}\right)^{i} \approx 10667.71.$



Пусть мы купили α шт. Bond 1 и продали β шт. Bond 2.
α и β должны удовл. соотношению, чтобы купоны все сократились, т.к у zero bond с номиналом FV и тек. стоимостью PV – нет купонов.

$$\Rightarrow \begin{cases} \frac{0.06N}{12}\cdot\alpha - \frac{0.09N}{12}\cdot\beta = 0 \\ \alpha N - \beta N = FV \\ PV = \alpha\cdot PV_1 - \beta\cdot PV_2 \end{cases} \Rightarrow \begin{cases} \alpha = \frac{9}{6}\beta = \frac{3}{2}\beta \\ \underbrace{(\alpha-\beta)}_{=0.5\beta}\cdot N = FV \\ PV = \frac{3}{2}\beta PV_1 - \beta\cdot PV_2 \end{cases} \Rightarrow \boxed{FV = 0.5\beta N}$$

$$\Rightarrow \frac{PV}{FV} = \frac{\frac{3}{2}\cancel{\beta}PV_1 - \cancel{\beta}PV_2}{0.5\cancel{\beta}\cdot N} = \frac{\frac{3}{2}PV_1 - PV_2}{0.5N} = \frac{3PV_1 - 2PV_2}{N} \approx \frac{6469.365898}{10000} \approx 0.646965\ldots$$

$\Rightarrow PV = 0.6469\cdot FV$

Ищем contin. yield of this bond:

$$e^{-10\cdot r} = 0.6469$$

$$\Rightarrow r = -\frac{\ln(0.6469)}{10}$$

$$\Rightarrow \boxed{r \approx 4.355\%}$$

Assignment

Assignment for Interest Rates and Credit Risk Models
START EACH QUESTION ON A NEW PAGE

Submit by 23:59 11 May 2022 by following the instructions below

Question 1 Consider forward rates, based on continuous compounding, observed at $t = 0$

$$f_0^*(\tau) = 0.1, \qquad \tau \in [0,6]$$

where the time is measured in years.

a) Calculate the bond prices $B_0^*(\tau_i)$ for the times $\tau_i = i$, $i = 1, \dots 6$, (3 marks)

b) Calculate the LIBOR rates $L_0(\tau_{i-1}, \tau_i)$, $i = 1, \dots, 6$. (4 marks)

c) Calculate the at-the-money interest rate (swap rate) for an interest rate swap with dates $\tau_1, \tau_2, \dots, \tau_6$ ($m = 1$, $n = 6$). (3 marks)

Question 2

Consider the time horizon $T = 1$ of a bond market and suppose that today's ($t = 0$) bond curve is given by

$$B_0^*(\tau) = e^{-c\sqrt{1+\tau}} \qquad \text{for all } \tau \in [0,T] \text{ with } c \in]0,\infty[.$$

For a one-factor HJM model with deterministic forward rate volatility

$$\sigma_t(\tau) = \sigma^f \cdot (\tau - t + 1), \qquad 0 \le t \le \tau \le T, \ \sigma^f \in]0,\infty[$$

a) Calculate the drift $(\alpha_t(\tau))_{t\in[0,\tau]}$, $\tau \in [0,T]$ from the HJM drift condition. (2 marks)

b) Fit the initial forward rate $(f_0^*(\tau))_{\tau[0,T]}$ to the market data given by the bond curve $(B_0^*(\tau))_{\tau\in[0,T]}$. (3 marks)

c) Is the factorization

$$\sigma_t(\tau) = \xi(t)\psi(\tau) \qquad 0 \le t \le \tau \le T \quad \psi, \xi \text{ deterministic functions}$$

satisfied? (required for Markov property of the short rate) (2 marks)

d) Determine the short rate r_T. (3 marks)

Question 3

Consider zero bond dynamics $(B_t(\tau))_{t\in[0,\tau]}$, $\tau \in [0,T]$ defined by the Ho-Lee short rate model

$$dr_t = (c + t)dt + \sigma^r dW_t, \qquad r_0 = r_0^*$$

where the process $(W_t)_{t\in[0,T]}$ follows a Brownian motion on $(\Omega, \mathcal{F}, \mathbb{Q}, (\mathcal{F}_t)_{t\in[0,T]})$ with respect to the spot martingale measure $\mathbb{Q}$ and $\sigma^r, c, r_0^* \in]0,\infty[$ are fixed.

a) Calculate the expectation $\mathbb{E}^{\mathbb{Q}}(r_T)$ of the short rate with respect to the spot martingale measure $\mathbb{Q}$ in terms of the model parameters $\sigma^r, c, r_0^* \in]0,\infty[$. (5 marks)

b) Calculate the expectation $\mathbb{E}^{\mathbb{Q}^T}(r_T)$ of the short rate with respect to the forward martingale measure $\mathbb{Q}^T$. (5 marks)

Question 4

Consider a one-factor HJM model whose forward rate dynamics follows

$$df_t(\tau) = \alpha_t(\tau)dt + \sigma_t(\tau)dW_t, \qquad f_0(\tau) = \lambda, \quad 0 \leq t \leq \tau \leq T, \;\; \lambda \in]0, \infty[$$

with a Browinan Motion $(W_t)_{t\in[0,T]}$ under the spot martingale measure $\mathbb{Q}$. Assume that

$$\sigma_t(\tau) = \sigma^f \cdot \tau \cdot t \quad 0 \leq t \leq \tau \leq T$$

with a pre-specified parameter $\sigma^f \in]0, \infty[$. Consider a risky asset following

$$dS_t = S_t(r_t dt + \sigma^S dW_t), \qquad S_0 = S_0^* \in]0, \infty[, \;\; \sigma^S \in]0, \infty[$$

with the short rate $(r_t = f_t(t))_{t\in[0,T]}$.

a) Calculate the volatility $\sigma_t^B(\tau)$ for $0 \leq t \leq \tau \leq T$ defined by

$$dB_t(\tau) = B_t(\tau)(r_t dt + \sigma_t^B(\tau)dW_t)$$

(3 marks)

b) With $(\Sigma_u = \sigma^S - \sigma_u^B(T))_{u\in[0,T]}$ calculate the quantity

$$\int_0^T |\Sigma_u|^2 du$$

(3 marks)

c) Calculate the price of the European Call on S_T

$$\mathbb{E}^{\mathbb{Q}}(e^{-\int_0^T r_s ds}(S_T - K)^+) \qquad K \geq 0$$

(4 marks)

Question 5 (10 marks)

Consider two coupon paying bonds (Bond 1 and Bond 2) with face value 10,000 AUD paying coupons monthly at the (annual) coupon rate of 6% (Bond 1) and 9% (Bond 2). Assume that the first coupon just has been paid at $\tau = 0$, the last coupon (in addition to the face value) will be paid at $\tau = 10$ and the bonds are traded now, at $\tau = 0$ at the yields 7% (Bond 1) and 8% (Bond 2). Is it possible to determine the price of a zero bond maturing at $\tau = 10$ using no-arbitrage arguments? If yes, calculate the yield of this bond (continuous compounding).

Brief Submission Instructions

1. Please register with Gradescope (for free) with an email you like and use the class code `RW8E84` to join our class. Please choose `National Research University Higher School of Economics (HSE University)` as the school, otherwise you won't be able to register. Check out detailed instructions at `gradescope.com/get_started`. Please make sure to enter your real name and surname when registering.
2. Prepare a PDF file with your submission. Note that you should only submit a single PDF file, not individual images.
    - (a) Classic pen and paper (and no scanner).
        - i. Write your solutions with pen and paper as usual. Please solve each problem on a separate sheet of paper.
        - ii. Make sure there's a lot of light. The more light the better. Good lighting is very important for quality photos.
        - iii. Take photos of each sheet of paper.
        - iv. Use a mobile scanner app to crop and clean up the photos. Recommended free apps: Scannable by Evernote (iOS), Genius Scan (Android), Microsoft Office Lens (all platforms).
        - v. Use the scanning software to export all scanned pages to a single PDF file. Make sure you have one PDF with all pages in it.
    - (b) LaTeX, MS Word or other publishing systems.
        - i. Please insert page breaks so that each problem starts from a new page.
        - ii. Please make sure that you create a single PDF file.
    - (c) Case of tablet users (write-on-screen).
        - i. Please make sure that you can export a single PDF file from your writing software. If not, use a third-party tool to create a PDF file from several images, e.g. `smallpdf.com/jpg-to-pdf`.
        - ii. Make sure that each problem is solved on a separate page.
3. Log into your Gradescope account.
4. Submit the prepared PDF file. Don't submit individual images (the system may allow it, but doesn't process correctly internally).
5. Make sure to assign pages to problems. For each problem you turn in, assign the page or pages that contain its solution. This step is very important! If you skip this step, your submission will not be graded.
6. You can re-upload the submission before the deadline. The grader will only see the latest submission.
7. Wait for your assignment to be graded. You'll receive a notification via e-mail.
8. Log into your Gradescope account to view your grades.
9. You can submit a regrade request if you'd like to.

06.05.2022. Лапшин Семинар 10

$$\begin{cases} dr_t = \theta dt + \sigma^r dW_t \\ d\lambda_t = \sigma^\lambda (\rho dW_t + \sqrt{1-\rho^2} dW_t^\perp) \end{cases}; \quad corr(dr_t; d\lambda_t) = \rho$$

$d\lambda_t = \sigma^\lambda d\overline{W_t}$, $corr(dW_t; d\overline{W_t}) = \rho$

Risky (defaultable) bond: $\overline{B_t}(T)$

Есть exogenous random variable $E \sim Exp(1)$. Она $\perp$ от W_t и $\overline{W_t}$.

Есть случайный момент остановки τ: $\min\{t : \int_0^t \lambda_s ds \geq E\}$ – time of default

Тогда $Q(\tau \leq t) = 1 - e^{-\int_0^t \lambda_s ds}$

$Q(\tau \in [t; t+dt]) = \lambda_t dt$

Risky (defaultable) bond: $\overline{B_t}(T)$:

~~$\tau > T$ no default~~

В моделях интенсивности предполагается, что все облигации рано или поздно допустят дефолт.

Итак, если $\tau > T$ (т.е no default): get 1 at T

если $\tau \leq T$ (т.е произошел дефолт at τ): get $\boxed{Z_\tau}$ at τ.

↑ recovery value

Recovery specifications:

① Fraction of notional: $Z_t = const < 1$.

$Z_t = \xi$ – random variable $\in [0, 1]$

! при дефолте мы получаем сейчас ~~[illegible]~~ половину (например) от номинала облигации, а не через 15 лет, (если время погашения облигации было через 15 лет), т.е мы еще и заработали

② Fraction of ~~present market value~~ ~~$Z_t = \xi \cdot B_t$~~ present value: $Z_t = \xi \cdot B_t(T)$

③ Fraction of market value: $Z_t = \xi \cdot \overline{B_t(T)}$.

Мы можем сами решить, какую из 3-х моделей вообще мы используем

Loss Given Default (LGD): $LGD = 1 - RR$ ← измеряется в процентах

↑ recovery rate (ξ)

Предположим, что вып. 2 предположения:

Fraction of ~~[illegible]~~ market value (т.е акция стоила 500, стала 200),

+ multiple defaults.

Вот тут должно быть $\frac{spread}{LGD}$

$$\Rightarrow \overline{B_t}(T) = e^{-\int_t^T (r_s + \boxed{\lambda_s \ell}) ds},$$ где ℓ - Loss Given Default

т.е терять 100% денег с вер-тью 1% или 50% денег с вер-тью 2%,

т.к в ф-лы входит только $\lambda \cdot \ell$ — произведение этих величин

На практике это не так, но примерно так.

$\lambda_s \cdot \ell$ – наз. кредитным спредом.

На практике кредитный спред $\approx PD \cdot LGD$

↑ вероятность дефолта

↑ loss given default (он большой)

Т.е когда дефолт – у нас отнимают облигацию на 1000 руб, и вместо неё дают:

1) 500 руб наличными
2) безрисковую бескупонную облигацию на 500 руб, которая сейчас стоит 200 руб из-за дисконтирования
3) вместо рискованной облигации на 100% выдают рискованную облигацию на 500 руб, и эта новая облигация может тоже дефолтнуть, т.е будет 2-й дефолт.

<u>Условие кросс-дефолта</u>: если компания не заплатила по одной облигации, то те, кому она должна платить через месяц – тоже имеют право прийти к компании и требовать досрочного погашения.

$$P_{риск} = P_{б/р} \cdot ND + RR) \cdot P_{б/р} \cdot PD = P_{б/р} \cdot ND + RR) P_{б/р}(1 - ND) =$$

$$P = e^{-rT}$$

$$ND = e^{-\lambda T}$$

$$PD = 1 - e^{-\lambda T}$$

$\Rightarrow$

$$= ND(P - RR \cdot P) + R \cdot P =$$

$$= ND \cdot P(1 - RR) + R \cdot P$$

$ND = e^{-\lambda T}$

если $RR = 0$, то $\tilde{P} = NDP = e^{-(\lambda + r)T}$

если $RR \neq 0$ то

$$(1-\lambda)(1-r)(1-RR) + R(1-r) =$$

$$= (1 - \lambda - r) \cdot LGD$$

$$\boxed{\overbrace{(1-RR)}^{LGD} \cdot \underbrace{\lambda}_{PD} = S}$$

$$CVA \approx \underbrace{PD \cdot LGD}_{\approx S} \cdot EPE_{disc}$$

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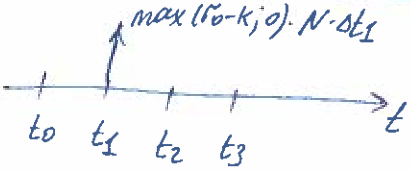
Swaption pricing

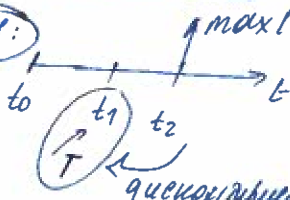


$\max(r_0-K;0)\cdot N\cdot\Delta t_1$

} floor

Caplets, Floors, Swaptions

Caplets:



$\max(L_{t}^{(t_1;t_2)}-K,0)\cdot\Delta t_2\cdot N$

это ф-ция от r_T.

дисконтируем: $\frac{\max(L_{t_1}(t_1;t_2)-K;0)\cdot(t_2-t_1)\cdot N}{1+L_t(t_1;t_2)\cdot(t_2-t_1)} = V_T(r_T)$

$V_0 = E^Q\left[\frac{V_T}{B_T}\right] ⊜$

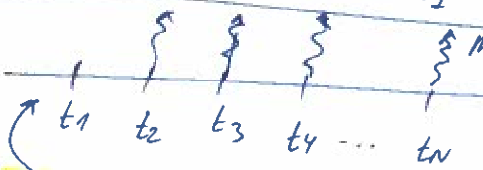
обе они зависят от проц. ставки

~~$V_0 = E^Q\left[\frac{V_{t_2}}{B_{t_2}}\right]$~~

Переходим в T-forward measure Q^T.

$$⊜ E^{Q^T}\left[\frac{V_T}{B_T}\cdot\frac{dQ}{dQ^T}\right] = E^{Q^T}\left[\frac{V_T}{\cancel{B_T}}\cdot\frac{\cancel{B_T}\cdot B_0(T)}{\underset{=1}{B_T(T)}\cdot\underset{=1}{B_0}}\right] = B_0(T)\cdot E^{Q^T}[V_T]$$

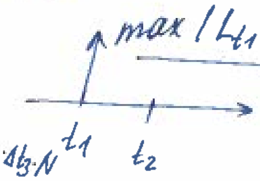
$$L_{t_1}(t_1;t_2) \leftarrow f(t_1;t_2) \leftarrow r_{t_1} \overset{Q^T}{\sim} N(\cdot,\cdot)$$



$\max(L_{t_{N-1}}(t_{N-1};t_N)\cdot(t_N-t_{N-1})\cdot N,0)$

это cap $= \sum_i(\text{caplet}_i)$ (т.е. каждый платёж по отдельности, либо да, либо нет)

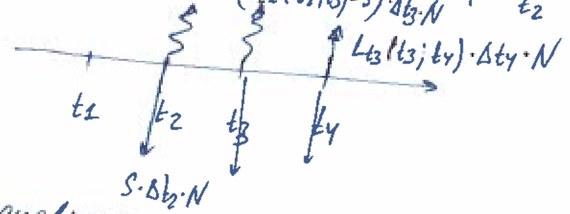
Дисконтируем каждый платёж:



$\frac{\max(L_{t_1}(t_2-t_1)-K;0)\cdot(t_2-t_1)\cdot N}{\underset{=P(t_1;t_2)}{1+L_{t_1}(t_1;t_2)\cdot(t_2-t_1)}}$

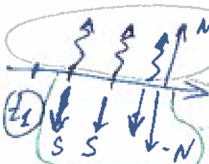
$= P(t_1;t_2)$ – по опр. (цена бескупонной облигации)

Swaption



В свопционе все платежи вместе: т.е в момент t_1 мы должны сразу сказать, берём мы всё или нет.

Swap price at t_1: $1-B(t_1;t_N)-\sum_{i=2}^{N}B(t_1;t_i)\cdot S$.



floating rate bond price $= 1\cdot N$ (let $N=1$) (она стоит 1 в любой момент)

fixed-rate bond discount to t_1.

Swaption payoff at $t=0$: $V_{t_1} = \max\left(1-\underset{=B(t_1;t_N;r_{t_1})}{B(t_1;t_N)} - S\cdot\sum_{i=2}^{N}\underset{=B(t_1;t_i;r_{t_1})}{B(t_1;t_i)}\Delta t_i, 0\right)$

Хотим посчитать: $\underbrace{E^{Q^{t_1}}}[V_{t_1}]\cdot B(0,t_1)$ – swaption price at $t=0$.

берём Q^{t_1}-форвардную меру, чтобы дисконтировать вне матожидания!

И как это считать?

Jamshidian trick: $B_{t_1}(t_N; r_{t_1}) - B(t_1; t_N; r_{t_1})$ is a determined and decreasing function of r_{t_1}.

Решаем ур-е: $\boxed{1 = B(t_1;t_N)^{r_{t_1}} + s\cdot\sum_{i=2}^{N} B(t_1;t_i)^{r_{t_1}}\cdot\Delta t_i}$ for r_{t_1}.

А правая часть этого ур-я – это монот. ф-ция от r_{t_1}

$\Rightarrow \exists!\ r^*_{t_1}$

$$\Rightarrow V_{t_1} = \max\Big(1 - B(t_1;t_N) - s\cdot\sum_{i=2}^{N} B(t_1;t_i)\Delta t_i; 0\Big) =$$

это $= \begin{cases} 0, & r_{t_1} \leq r^*_{t_1} \\ >0, & r_{t_1} > r^*_{t_1} \end{cases}$

$$= \max\Big(\underbrace{B(t_1;t_N;r^*_{t_1})}_{=:K_1} - B(t_1;t_N;r_{t_1}) + s\sum_{i=2}^{N}\big[\underbrace{B(t_1;t_i;r^*_{t_1})}_{=:K_i;\ i=2\ldots N} - B(t_1;t_i;r_{t_1})\big]\Delta t_i; 0\Big)$$

≤ 0, if $r_{t_1} \leq r^*_{t_1}$

вот max суммы, а стала сумма максимумов

$$= \underbrace{\max\big(K_1 - B(t_1;t_N;r_{t_1}); 0\big)}_{\substack{\leq 0 \text{ if } r_{t_1} \leq r^*_{t_1} \\ >0, \text{ if } r_{t_1} > r^*_{t_1}}} + s\cdot\sum_{i=2}^{N}\Delta t_i\cdot\max\big(\underbrace{K_i - B(t_1;t_i;r_{t_1})}_{\leq 0 \text{ if } r_{t_1} \leq r^*_{t_1}}; 0\big)$$

(т.к. B – монот. убыв. ф-ция)

а это ~~[illegible]~~ просто опцион на облигацию: zero-coupon bond option

и это тоже опцион на облигацию.

В модели Васичека были прямо формулы в лекции для опционов на облигаций.

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Квиз 4 Модель Васичека

① $\begin{cases} dr_t = k(\theta - r_t)dt + \sigma dW_t \\ r_0 = r_0 \end{cases}$; r_0, k, θ, σ – положит. констант.

Замена: $y_t = r_t - \theta$ $f(x) = x - \theta$

$\Rightarrow dy_t = 1 \cdot dr_t = -k y_t dt + \sigma dW_t$ $r_t = y_t - \theta$

Замена: $z_t = e^{kt} y_t$; $f(x) = e^{kt} \cdot x$

$f'_t = k \cdot e^{kt} \cdot x$

$f'_x = e^{kt}$

$f''_{xx} = 0.$

$\Rightarrow dz_t = k \cdot z_t dt + e^{kt} dy_t = k z_t dt + e^{kt}(-k y_t dt + \sigma dW_t) = k \cancel{z_t dt} - k \cancel{\underset{z_t}{e^{kt} y_t} dt} + \sigma e^{kt} d\ldots$

$\Rightarrow z_t = z_0 + \sigma \int_0^t e^{ks} dW_s$

обр. замена

$\Rightarrow e^{kt} y_t = e^{k \cdot 0} \cdot y_0 + \sigma \int_0^t e^{ks} dW_s$

$e^{kt}(r_t - \theta) = (r_0 - \theta) + \sigma \int_0^t e^{ks} dW_s$

$r_t \overset{Q}{\sim} N(\cdot, \cdot)$

$r_t \overset{Q^T}{\sim} N(?, ?)$

обр. замена

$\Rightarrow \boxed{r_t = \theta + e^{-kt}(r_0 - \theta) + \sigma \int_0^t e^{k(s-t)} dW_s} = r_0 e^{-kt} + \theta(1 - e^{-kt}) + \sigma \int_0^t e^{k(s-t)} dW_s$

② Убедимся, что эта модель явл. HJM-модели с $\boxed{\sigma_t(T) = \sigma \cdot e^{-k(T-t)}}$

Ну $\alpha_t(\tau) = \sigma_t(\tau) \int_t^\tau \sigma_t(s) ds = \sigma \cdot e^{-k(\tau - t)} \cdot \sigma \int_t^\tau e^{-k(s-t)} ds = \sigma^2 e^{-k\tau} \cdot e^{2kt} \cdot \int_t^\tau e^{-ks} ds =$

$= \sigma^2 e^{-k\tau} \cdot e^{2kt} (-\frac{1}{k}) \cdot e^{-ks} \Big|_t^\tau = \sigma^2 \cdot e^{-k\tau} \cdot e^{2kt} \cdot (-\frac{1}{k}) \cdot (e^{-k\tau} - e^{-kt}) =$

$= \frac{\sigma^2}{k} \cdot (e^{k(t-\tau)} - e^{2k(t-\tau)})$

$\Rightarrow df_t(\tau) = \alpha_t(\tau) dt + \sigma_t(\tau) dW_t.$ $\Rightarrow dr_t = \alpha_t(t) dt + \sigma_t(t) dW_t$

$\Rightarrow f_t(\tau) = f_0(\tau) + \frac{\sigma^2}{k} \boxed{\int_0^t (e^{k(s-\tau)} - e^{2k(s-\tau)}) ds} + \sigma \int_0^t e^{-k(\tau - s)} dW_s$

$\overset{\shortparallel}{e^{-k\tau}} \cdot \frac{1}{k} \cdot (e^{kt} - 1) - e^{-2k\tau} \cdot \frac{1}{2k} (e^{2kt} - 1)$

③ $r_t = f_t(t)$

$f_t(t) = f_0(t) + \frac{\sigma^2}{k} \cdot \left(\frac{e^{-kt}}{k}(e^{kt} - 1) - \frac{1}{2k} \cdot e^{-2kt}(e^{2kt} - 1)\right) + \sigma \int_0^t e^{k(s-t)} dW_s.$

$$\Rightarrow \underset{r_t}{f_t(t)} = \boxed{f_0(t) + \frac{\sigma^2}{k}\left(\frac{1}{k}(1-e^{-kt}) - \frac{1}{2k}(1-e^{-2kt})\right)} + \sigma\int_0^t e^{k(s-t)}dW_s$$

~~$f_t(t) = f_0(t) + \frac{\sigma^2}{2k^2}(2-2e^{-kt}-1+e^{-2kt}) + \sigma\int_0^t e^{k(s-t)}dW_s$~~

||

~~$r_t = f_0(t) + \frac{\sigma^2}{2k^2} + \frac{\sigma^2}{2k^2}e^{-kt}(e^{-kt}-2) + \sigma\int_0^t e^{k(s-t)}dW_s$~~

~~??~~

~~$r_t = \theta + e^{-kt}$~~

~~$f_0 = f_0(0) + \frac{\sigma^2}{2k^2} - \frac{\sigma^2}{2k^2} = f_0(0)$~~

?||

$$r_t = \boxed{\theta + e^{-kt}(\underset{f_0(0)}{r_0} - \theta)} + \sigma\int_0^t e^{k(s-t)}dW_s$$

~~$\Rightarrow f_0(t) + \frac{\sigma^2}{2k^2}(1-2e^{-kt}+e^{-2kt}) = \theta + e^{-kt}(\overset{=r_0}{f_0(0)} - \theta)$~~

$$\Rightarrow \boxed{f_0(t) + \frac{\sigma^2}{k^2}(1-e^{-kt}) - \frac{\sigma^2}{2k^2}(1-e^{-2kt}) = \theta + e^{-kt}(f_0(0)-\theta)}$$ — усл. на $f_0(t)$, при кот. модель HJM экв. модели Васичека

④ Зная HJM-модель, найти СДУ для цены облигации $B_t(T)$:

$$dB_t(T) = r_t B_t(T)dt + \sigma_t^3(T)B_t(T)dW_t.$$

Найти $\sigma_t^3(T)$.

Ну $\sigma_t^3(\tau) = -\int_t^\tau \sigma_t(s)ds = -\int_t^\tau \sigma e^{-k(s-t)}ds =$

$$= -\sigma\cdot e^{kt}\int_t^\tau e^{-ks}ds = \frac{\sigma}{k}\cdot e^{kt}\cdot e^{-ks}\Big|_t^\tau = \frac{\sigma}{k}e^{kt}(e^{-k\tau} - e^{-kt}) =$$

$$= \frac{\sigma}{k}(e^{-k(\tau-t)} - 1)$$

⑤ Теперь меняем меру на Q^T.

Для этого найти $L_t = \frac{dQ^T}{dQ}$, и найти φ_t: $\frac{dL_t}{L_t} = \varphi_t dW_t$

с/р 2

Ну $L_t = \frac{dQ^{\tau}}{dQ} = \frac{N_t^1}{N_t^2} \cdot \frac{N_0^2}{N_0^1} = \frac{B_t(\tau)}{B_t} \cdot \frac{B_0 \nearrow 1}{B_0(\tau)}$

$Q^{\tau} : N_t = B_t(\tau)$ ①

$Q : N_t = B_t$ ②

Хотим найти dL_t.

Мы знаем, что $dB_t = r_t B_t dt$

$dB_t(\tau) = r_t B_t(\tau) dt + \sigma_t^B(\tau) B_t(\tau) dW_t$.

$$\boxed{L_t = \frac{1}{B_0(\tau)} \cdot \frac{B_t(\tau)}{B_t}}$$

$f(x,y) = c\frac{x}{y}$

$X_t = B_t(\tau)$
$Y_t = B_t$

$\Rightarrow d\left(\frac{X_t}{Y_t}\right) = c\frac{1}{Y_t} dX_t - c\frac{X_t}{Y_t^2} dY_t$

$\Rightarrow d\left(\frac{1}{B_0(\tau)} \cdot \frac{B_t(\tau)}{B_t}\right) = \frac{1}{B_0(\tau)} \cdot \frac{1}{B_t} \cdot dB_t(\tau) - \frac{1}{B_0(\tau)} \cdot \frac{B_t(\tau)}{B_t^2} \cdot dB_t =$

$= \frac{1}{B_0(\tau)} \cdot \frac{1}{B_t} \cdot (\cancel{r_t B_t(\tau) dt} + \sigma_t^B(\tau) B_t(\tau) dW_t) \cancel{- \frac{1}{B_0(\tau)} \cdot \frac{B_t(\tau)}{B_t^2} \cdot r_t B_t dt} =$

$= \underbrace{\boxed{\frac{\sigma_t^B(\tau) B_t(\tau)}{B_0(\tau) \cdot B_t}}}_{L_t} dW_t$

$$\Rightarrow \boxed{dL_t = L_t \cdot \sigma_t^B(\tau) dW_t}$$

6.) Исп. т. Гирсанова, чтобы найти СДУ для r_t по мере Q^{τ}!

Ну Если $\begin{cases} dZ_t = -\mu_t Z_t dW_t \\ Z_0 = 1, \end{cases}$ $\leftarrow$ $dL_t = \underset{-\mu_t}{\underbrace{\sigma_t^B(\tau)}} \cdot L_t dW_t$ $\Rightarrow \mu_t = -\sigma_t^B(\tau)$

то по мере $dQ = Z_t dP$,

процесс $\tilde{W}_t = W_t + \int_0^t \mu_s ds$ - явл. броун. движ. i.e. $dW_t = d\tilde{W}_t - \mu_t dt$

Ну $dr_t = k(\theta - r_t) dt + \sigma dW_t$; $\tilde{W}_t = W_t - \int_0^t \sigma_s^B(\tau) ds$

$\| \frac{\sigma}{k}(e^{-k(\tau-s)} - 1)$

$\Rightarrow W_t = \tilde W_t + \frac{\sigma}{k}\int_0^t (e^{ks}\cdot e^{-k\tau} - 1)ds = \tilde W_t + \frac{\sigma}{k}\cdot e^{-k\tau}\cdot\frac{1}{k}\cdot(e^{kt}-1) - \frac{\sigma}{k}t =$

$= \tilde W_t + \frac{\sigma}{k^2}\cdot e^{-k\tau}(e^{kt}-1) - \frac{\sigma}{k}t$

$\Rightarrow dW_t = d\tilde W_t + \left(\frac{\sigma}{k}e^{-k(\tau-t)} - \frac{\sigma}{k}\right)dt = d\tilde W_t + \frac{\sigma}{k}(e^{-k(\tau-t)}-1)dt$

$\Rightarrow dr_t = k(\theta - r_t)dt + \sigma\cdot \underset{\| \; d\tilde W_t^T + \frac{\sigma}{k}(e^{-k(T-t)}-1)dt}{dW_t} =$

$= k\theta - kr_t + \sigma d\tilde W_t^T + \frac{\sigma^2}{k}(e^{-k(T-t)}-1)dt =$

$= k(\underbrace{\theta + \frac{\sigma^2}{k}(e^{-k(T-t)}-1)}_{\| \; \tilde\theta(t)} - r_t)dt + \sigma d\tilde W_t^T$

$\Rightarrow \boxed{dr_t = k([\underbrace{\theta+\frac{\sigma^2}{k}(e^{-k(T-t)}-1)}_{\| \; \tilde\theta(t)}] - r_t)dt + \sigma d\tilde W_t^T}$

⑦ Примените подход из п.1, чтобы это проинтегрировал. Должно получиться, что r_t имеет норм. распр. отн. Q^T. Найти $E_{Q^T}[r_t]$ и $Var_{Q^T}[r_t]$.

Решение: 1) $y_t = r_t - \theta$

~~$\Rightarrow dy_t = dr_t = k([\theta+\frac{\sigma^2}{k}(e^{-k(T-t)}-1)] - \overset{y_t+\theta}{r_t})dt + \sigma d\tilde W_t^T = k(-y_t + \frac{\sigma^2}{k}(e^{-k(T-t)}-1))dt + \sigma d\tilde W_t^T$~~

~~2) $z_t = e^{kt}y_t$; $f(x) = e^{kt}\cdot x$~~

~~$f'_t = k\cdot e^{kt}x$~~

~~$f'_x = e^{kt}$~~

~~$\Rightarrow dz_t = k\cdot\underbrace{e^{kt}\cdot y_t}_{\| \; z_t}dt + e^{kt}k(-y_t + \frac{\sigma^2}{k}(e^{-k(T-t)}-1))dt + e^{kt}\cdot\sigma\cdot d\tilde W_t^T =$~~

~~$= kz_t dt - k\underbrace{e^{kt}y_t}_{z_t}dt + e^{kt}\cdot\frac{\sigma^2}{k}(e^{-k(T-t)}-1)dt + e^{kt}\sigma d\tilde W_t^T$~~

1) $y_t = r_t - \tilde\theta(t) = r_t - \theta - \frac{\sigma^2}{k}(e^{-k(T-t)}-1)$

$f(x) = x - \theta - \frac{\sigma^2}{k}(e^{-k(T-t)}-1)$

$f'_x = 1$

$f'_t = -\frac{\sigma^2}{k}\cdot e^{-k(T-t)}\cdot k = -\sigma^2 e^{-k(T-t)}$

$\Rightarrow z_t = z_0 + \int_0^t e^{ks}\cdot\frac{\sigma^2}{k}\cdot(e^{-k(T-s)}-1)ds + \int_0^t e^{ks}\sigma\cdot d\tilde W_s^T$

$\Rightarrow e^{kt}y_t = y_0 + \int_0^t e^{ks}\cdot\frac{\sigma^2}{k}(e^{-k(T-s)}-1)ds + \int_0^t e^{ks}\cdot\sigma\cdot d\tilde W_s^T$

$e^{kt}(r_t-\theta) = r_0 - \theta + \int_0^t e^{ks}\cdot\frac{\sigma^2}{k}(e^{-k(T-s)}-1)ds + \int_0^t e^{ks}\cdot\sigma\cdot d\tilde W_s^T$

$\Rightarrow \boxed{r_t = \theta + e^{\text{[illegible]}}(r_0-\theta) + \int_0^t \frac{\sigma^2}{k} e^{k(s-t)}(e^{-k(T-s)}-1)ds + \int_0^t e^{k(s-t)}\sigma d\tilde W_s^T}$

$$\Rightarrow dy_t = f'_t dt + f'_x d\tilde{r}_t = -\sigma^2 \cdot e^{-k(T-t)} dt + k\left(\theta + \frac{\sigma^2}{k}\left(e^{-k(T-t)} - 1\right) - \tilde{r}_t\right)dt + \sigma d\tilde{W}_t^T =$$

$$= \left(\cancel{-\sigma^2 \cdot e^{-k(T-t)}} + k\theta + \cancel{\sigma^2 \cdot e^{-k(T-t)}} - \sigma^2 - k\boxed{\tilde{r}_t}\right)dt + \sigma d\tilde{W}_t^T =$$

$$\tilde{r}_t = y_t + \theta + \frac{\sigma^2}{k}\left(e^{-k(T-t)} - 1\right)$$

$$= \left(\cancel{k\theta} - \sigma^2 - ky_t \cancel{-k\theta} - \sigma^2\left(e^{-k(T-t)} - 1\right)\right)dt + \sigma d\tilde{W}_t^T =$$

$$= \left(\cancel{-\sigma^2} - ky_t - \sigma^2 \cdot e^{-k(T-t)} \cancel{+\sigma^2}\right)dt + \sigma d\tilde{W}_t^T$$

$$\frac{\sigma^2}{2k}$$

$$\Rightarrow \tilde{r}_t = \theta + e^{-kt}(r_0 - \theta) + \frac{\sigma^2}{k}\underbrace{\int_0^t e^{k(s-t)}\left(e^{-k(T-s)} - 1\right)ds}_{} + \sigma\int_0^t e^{k(s-t)} d\tilde{W}_s^T$$

$$\int_0^t \left(e^{k(2s-t-T)} - e^{k(s-t)}\right)ds$$

$$= e^{-k(t+T)} \cdot \frac{1}{2k} \cdot e^{2ks}\Big|_0^t - e^{-kt} \cdot \frac{1}{k} \cdot e^{ks}\Big|_0^t$$

$$= \frac{1}{2k} \cdot e^{-k(t+T)}\left(e^{2kt} - 1\right) - e^{-kt} \cdot \frac{1}{k}\left(e^{kt} - 1\right) =$$

$$= \frac{1}{2k}\left(e^{k(t-T)} - e^{-k(t+T)}\right) - \frac{1}{k}\left(1 - e^{-kt}\right)$$

$$\Rightarrow r_t = \theta + e^{-kt}(r_0 - \theta) + \frac{\sigma^2}{k}\left(\frac{1}{2k}\left(e^{k(t-T)} - e^{-k(t+T)}\right) - \frac{1}{k}\left(1 - e^{-kt}\right)\right) + \sigma\int_0^t e^{k(s-t)} d\tilde{W}_s^T$$

$$r_t = \theta + (r_0 - \theta)e^{-kt} - \frac{\sigma^2}{k^2}\left(1 - e^{-kt}\right) + \frac{\sigma^2}{2k^2}\left(e^{2kt} - 1\right)e^{-k(T-t)}$$

$$\boxed{\Rightarrow E^{Q^T}(r_t) = \theta + e^{-kt}(r_0 - \theta) + \frac{\sigma^2}{k}\left(\frac{1}{2k}\left(e^{k(t-T)} - e^{-k(t+T)}\right) - \frac{1}{k}\left(1 - e^{-kt}\right)\right)}$$

$$\boxed{D^{Q^T}(r_t) = \sigma^2 \cdot \int_0^t e^{2k(s-t)} ds = \sigma^2 \cdot e^{-2kt} \int_0^t e^{2ks} ds = \sigma^2 \cdot e^{-2kt} \cdot \frac{1}{2k} \cdot \left(e^{2kt} - 1\right) = \boxed{\frac{\sigma^2}{2k}\left(1 - e^{-2kt}\right)}}$$

(Кау)z и HJM model.

13.04.2022. [illegible]. семинар 7.

дано:

$$\begin{cases} B_0(\tau) = e^{-\frac{c\tau^2}{2}} \\ \sigma_t(\tau) = \sigma\tau e^{-t} \\ \text{где } df_t(\tau) = d_t(\tau)dt + \sigma_t(\tau)dW_t \end{cases}$$

1) найти $d_t(\tau)$, $0 \le t \le \tau \le T$

ну $d_t(\tau) = \sigma_t(\tau)\int_t^\tau \sigma_t(s)ds = \sigma\tau e^{-t}\cdot\int_t^\tau \sigma s e^{-t}ds = \sigma\tau\cdot e^{-t}\cdot\sigma\cdot e^{-t}\cdot\frac{s^2}{2}\Big|_t^\tau = \boxed{\sigma^2\tau e^{-2t}\frac{(\tau^2-t^2)}{2}}$

2) найти $f_0^*(\tau)$, соотв. zero-coupon yeild curve.

ну $B_0^*(\tau) = e^{-\int_0^\tau f_0(s)ds} \overset{?}{=} e^{-\frac{c\tau^2}{2}}$

=> $\int_0^\tau f_0(s)ds = \frac{c\tau^2}{2}$

=> $\boxed{f_0(\tau) = c\tau}$

3) написать стохаст. динамику для $f_t(\tau)$:

$$\boxed{df_t(\tau) = \sigma^2\tau e^{-2t}\frac{(\tau^2-t^2)}{2}dt + \sigma\tau e^{-t}dW_t}$$

~~4) Явл. ли r_t – марковским?~~

~~=> $f_t(\tau) = f_0(\tau) + \frac{\sigma^2\tau}{2}\cdot\int_0^t e^{-2s}(\tau^2-s^2)ds + \sigma\tau\int_0^t e^{-s}dW_s$~~

~~=> $r_t = f_t(t) = ct + \frac{\sigma^2 t}{2}\cdot\int_0^t e^{-2s}(t^2-s^2)ds + \sigma t\int_0^t e^{-s}dW_s$~~

~~=> $dr_t = c + \frac{\sigma^2}{2}\cdot\left(\int_0^t e^{-2s}(t^2-s^2)ds\right)dt + \frac{\sigma^2 t}{2}\cdot e^{-2t}\overset{\to 0}{(t^2-t^2)}dt +$~~

~~$+ \frac{\sigma^2 t}{2}\left(\int_0^t e^{-2s}2t\,ds\right)dt + \sigma\left(\int_0^t e^{-s}dW_s\right)dt$~~

4) Явл. ли r_t – марковским?

$\sigma_t(\tau) = \sigma\tau e^{-t} = g(t)\cdot h(\tau) \Rightarrow \boxed{да}$

5) найти $\sigma_t^B(\tau)$ для $dB_t(\tau) = B_t(\tau)(r_t dt + \sigma_t^B(\tau)dW_t)$

ну $\sigma_t^B(\tau) = -\int_t^\tau \sigma_t(s)ds = -\int_t^\tau \sigma s e^{-t}ds = -\sigma e^{-t}\cdot\int_t^\tau s\,ds = \boxed{-\sigma e^{-t}\cdot\frac{(\tau^2-t^2)}{2}}$

Связь между фьючерсами и форвардами.

] τ – futures expiration

futures price $F_t(\tau) = E^Q[S_\tau|\mathcal{F}_t]$ – безо всякого дисконтирования = future value [illegible]

$S_t = B_t(T)$ – облигация, которая истекает в момент T. ; $0 \le t \le \tau < T$.

↑ futures maturity ↖ bond maturity

Если $\tau = T$, то $S_T = B_T(T) = 1$ – неинтересно.

На практике $\tau << T$, где τ = 3мес, T = 50лет.

Мы нам нужно $\overset{E_t(\tau)=}{E^Q}[S_\tau | \mathcal{F}_t] = E^Q[B_\tau(T) | \mathcal{F}_t] = ?$ ← $S_t = B_t(T)$

Мы $B_\tau(T) = E^Q[e^{-\int_\tau^T r_s ds} | \mathcal{F}_\tau]$

$E_0(\tau) = E^Q[e^{-\int_\tau^T r_s ds}]$ – т.к. $E[E[\xi|F]] = E[\xi]$.

Мы считаем: 1) $r_s = \dots$

2) $-\int_\tau^T r_s ds = \dots$ #

3) $B_\tau(T) = \dots$

4) $E_0(\tau) = E^Q[B_\tau(T)]$

Мы 1) $\begin{cases} dr_t = \theta dt + \sigma dW_t \\ r_0 = r_0 \end{cases} \Rightarrow \boxed{r_s = r_0 + \theta s + \sigma W_s}$

2) $-\int_\tau^T r_s ds = -\int_\tau^T (r_0 + \theta s + \sigma W_s) ds = -r_0(T-\tau) - \theta \cdot \frac{(T^2-\tau^2)}{2} - \sigma \overbrace{\int_\tau^T W_s ds}$ ∼

$\sim N\left(-r_0(T-\tau) - \frac{\theta}{2}(T^2 - \tau^2) + \sigma(T-\tau)W_\tau ; \frac{\sigma^2}{3}(T-\tau)^3\right) =$

$= N\left(-(T-\tau)(r_0 + \theta\tau + \sigma W_\tau) - \theta \underbrace{\left(\frac{T^2-\tau^2}{2} - \tau(T-\tau)\right)}_{= \frac{\theta(T-\tau)^2}{2}}\right)$.

$\sim N(\cdot,\cdot)$

$\boxed{\int_\tau^T (T-s) dW_s} + (T-\tau)W_\tau$ интегр. опое W_τ.

$\|$

$= sW_s \Big|_\tau^T - \int_\tau^T s dW_s$

$\Rightarrow \boxed{-\int_\tau^T r_s ds \sim N\left(-r_\tau(T-\tau) - \frac{\theta(T-\tau)^2}{2} ; \frac{\sigma^2}{3}(T-\tau)^3\right)}$

3) Если $\xi \sim N(\mu, \sigma^2)$,

то $E e^\xi = e^{\mu + \frac{\sigma^2}{2}}$

$\Rightarrow B_\tau(T) = e^{-(T-\tau) r_\tau - \frac{\theta}{2}(T-\tau)^2 + \frac{1}{2}\frac{\sigma^2 (T-\tau)^3}{3}}$

4) $E_0(T) =$ ~~$e^{-r_0(T-\tau) - \frac{\theta}{2}(T-\tau)^2}$~~ $e^{-(T-\tau)(r_0 + \theta\tau + \frac{\theta}{2}(T-\tau) - \frac{1}{2}\frac{\sigma^2(T-\tau)^2}{3}) + \frac{\sigma^2\tau}{2}}$

---

② $F_0(\tau) = E^{Q^T}[B_\tau(T)]$

$L_t = \frac{dQ^{new}}{dQ^{old}} = \frac{N_t^{new}}{\dots} \cdot \frac{N_0^{old}}{}$ ✓ change of numeraire

(ч2)

$\Rightarrow F_0(\tau) = E^{Q^T}[B_\tau(T)] = \cancel{\;} \;?$

$\Rightarrow ? = E^Q\left[\frac{B_\tau}{\cancel{B_\tau(T)}} \cdot \frac{B_0(T)}{\underset{\parallel\, 1}{B_0}} \cdot \cancel{B_\tau(T)}\right] = \boxed{B_0(T)\, E^Q[B_\tau]}$

$\boxed{E^{Q^{old}}[L_t \cdot f] = E^{Q^{new}}[f]}$

$! \quad \boxed{\begin{array}{l} Q^T: N_t = B_t(T) \\ Q: N_t = B_t \end{array}}$

$\boxed{\begin{array}{l} B_0 = 1 \\ B_t = e^{\int_0^t r_s ds} \end{array}}$

ну $F_0(\tau) = E^{Q^T}[B_\tau(T)] = E^Q\left[\frac{dQ^T}{dQ} B_\tau(T)\right] =$

$F_0(\tau) = E^{Q^\tau}[B_t(\tau)]$

$\parallel$

$E^Q\left[\frac{dQ^\tau}{dQ} \cdot B_t(T)\right].$

$\parallel$

$E^Q\left[\frac{\cancel{B_\tau(\tau)}^{\,1}}{B_\tau} \cdot \frac{\cancel{B_0}^{\,1}}{B_0(\tau)} \cdot B_\tau(T)\right]$

$\parallel$

$\frac{1}{B_0(\tau)} E^Q\left[\frac{B_\tau(T)}{B_\tau}\right] = \frac{1}{B_0(\tau)} \cdot \frac{B_0(T)}{B_0} = \frac{B_0(T)}{B_0(\tau)} = e^{-r_0(T-\tau) - \frac{\theta}{2}(T^2-\tau^2) + \frac{1}{2}\frac{\sigma^2}{3}(T^3-\tau^3)}$

март. по Q

(по опр. марг. меры Q)

Change of Numeraire

$L_t = \frac{dQ^1}{dQ^2} = \frac{N_t^1}{N_t^2} \cdot \frac{N_0^2}{N_0^1}; \quad E^{Q^2}[L_t \cdot f] = E^{Q^1}[f].$

$Q^\tau: N_t = B_t(\tau)$ ①

$Q: N_t = B_t$ ②

08.04.2022. Лапшин. Семинар 6.

Разбор квиза 2

Дано: $\begin{cases} dr_t = \theta(t)dt + \sigma dW_t \\ r_0 = r_0. \end{cases}$

$y_0(\tau) = c_1 \ln(\tau + c_2);$

1) Найти $B_0(\tau)$

ну $B_0(\tau) = e^{-y_0(\tau)\tau} = (c_2+\tau)^{-c_1\tau}$

2) найти $B_t(\tau)$.

$B_t(\tau) = E^Q\left[\frac{1}{B_\tau} / F_t\right]$

надо это найти

А дано: $\begin{cases} dr_t = \theta(t)dt + \sigma dW_t \\ r_0 = r_0 \end{cases}$

$\Rightarrow r_t = r_0 + \int_0^t \theta(s)ds + \sigma W_t$

Далее, $dB_t = r_t B_t dt$

Это неслучайно!

$\Rightarrow \frac{1}{B_\tau} = e^{-\int_0^\tau r_u du} = e^{-r_0\tau - \int_0^\tau\int_0^u \theta(s)ds - \sigma\int_0^\tau W_u du}$

$\| \ \tau W_\tau - \int_0^\tau u\, dW_u = \int_0^\tau (\tau-u)dW_u$

$\| \ \int_0^t (\tau-u)dW_u + \int_t^\tau (\tau-u)dW_u$

$\Rightarrow B_t(\tau) = E^Q\left[\frac{1}{B_\tau} / F_t\right] = e^{-r_0\tau - \int_0^t\int_0^u \theta(s)ds - \sigma\int_0^t (\tau-u)dW_u + \frac{\sigma^2}{2}\frac{(\tau-t)^3}{3}} \ominus$

$^2 N(\dots, \int_t^\tau (\tau-u)^2 du) = N(0, \frac{(\tau-t)^3}{3})$

$\xi \sim N(\mu, \sigma^2)$

$E[e^\xi] = e^{\mu + \frac{\sigma^2}{2}}$

$\ominus \left(e^{-r_0 t - \int_0^t\int_0^u \theta(s)ds du - \sigma\int_0^t (\tau-u)dW_u}\right) \cdot e^{-r_0(\tau-t) - \sigma\int_t^\tau\int_0^u \theta(s)ds + \frac{\sigma^2}{2}\frac{(\tau-t)^3}{3}} =$

$\| \ \frac{1}{B_t}$

$$= \frac{e^{-r_0(\tau-t) - \sigma\int_t^\tau\int_0^u \theta(s)ds du + \frac{\sigma^2}{2}\frac{(\tau-t)^3}{3}}}{B_t}$$

(где B_t - считается из того, что $dB_t = r_t B_t dt$)

3) Найти $\theta(t)$:

ну $B_0(\tau) = (c_2+\tau)^{-c_1\tau}$ — from yield curve

$B_t(\tau) = e^{-r_0(\tau-t) - \int_t^\tau\int_0^u \theta(s)duds + \frac{\sigma^2}{2}\frac{(\tau-t)^3}{3}}$ — from SDE.

приравниваем (при t=0): $c_1 \ln(c_2+\tau)\tau = r_0(\tau - 0) + \int_0^\tau\int_0^u \theta(s)dsdu + \frac{\sigma^2}{2}\frac{\tau^3}{3}.$

$\Rightarrow \frac{\partial}{\partial\tau}: c_1\ln(c_2+\tau) + \frac{c_1\tau}{c_2+\tau} = \int_0^\tau \theta(s)ds + r_0 - \frac{\sigma^2}{2}\cdot\tau^2$

$\frac{\partial}{\partial\tau}: \frac{C_1}{\dots} + c_1\frac{(c_2+\tau)-\tau}{\dots}$ $\theta(\tau)$ $\sigma^2\tau$ $\frac{c_1(2c_2+\tau)}{\dots}$ $\sigma^2\tau$

01.04.2022. Калинин. Семинар 5.

Кейс 1 ① $B_t = 1-0.05t$ — это zero-coupon bond price; $t=1\ldots 5$.

Найти continuously comp. zero-coupon rate r_3.

②

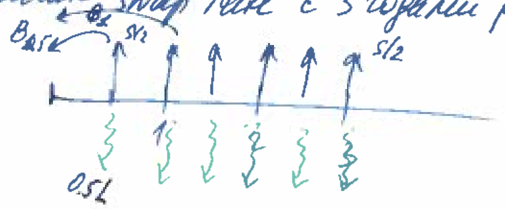
③ Ставку $Libor_0(2;2.5)$:

начало через 2 года

длится 0.5 года

$$L_t(\tau,\tau') = \frac{B_t(\tau)-B_t(\tau')}{(\tau'-\tau)B_t(\tau')} \Rightarrow L_0(2;2.5) = \frac{B_0(2)-B_0(2.5)}{(2.5-2)B_0(2.5)} = 0.05417\ldots$$

④ Calculate swap rate с 3 годами до погашения и платежами pay в полгода.



Задача 1d из прошлого листка:

1d) $E^Q\left[\left(e^{-\int_0^1 r_s ds}\right)\left((S_T-K)^+\right)\right]$

Short rate!

$= B_t$

$= X_T$ — price at time T = payoff

цена ~~опциона~~ опциона-колл,

A zero-coupon rate $y(t)$:

$$e^{-y(t)t} = E^Q\left[e^{-\int_0^t r_s ds}\right]$$

zero-coupon rate

short rate

по 1-й фунд. теореме: $\forall$ notional asset N_t $\exists Q$: $\forall$ traded asset X_t вып: $\frac{X_t}{N_t}$ is a martingale отн. Q.

Risk-neutral measure: $N_t - B_t$.

$$\Rightarrow \frac{X_0}{\underset{1}{\underset{\parallel}{B_0}}} = E^Q\left[\frac{X_T}{B_T}\right]$$

$$\Rightarrow X_0 = E^Q\left[\frac{X_T}{B_T}\right].$$

Актив(акция): $dS_t = S_t(\mu dt + \sigma^S dW_t)$; $dB_t = r_t dt$

переход в риск-нейтр. меру

$$dS_t = S_t(r dt + \sigma^S dW_t^Q)$$

$$\Rightarrow \begin{cases} dB_t = B_t r_t dt \\ dS_t = S_t(rdt + \sigma^S dW_t^Q) \end{cases}$$

$$f(x,y) = \frac{x}{y} \Rightarrow d\left(\frac{X_t}{Y_t}\right) = \frac{1}{Y_t} dX_t - \frac{X_t}{Y_t^2} dY_t .$$

$$\Rightarrow d\left(\frac{S_t}{B_t}\right) = \frac{dS_t}{B_t} - \frac{S_t}{B_t^2} \cdot dB_t = \frac{S_t}{B_t}(rdt + \sigma^S dW_t^Q) - S_t \cdot \frac{\cancel{B_t} r_t dt}{B_t^2} = \frac{S_t}{B_t} \sigma^S dW_t^Q$$

$$\Rightarrow d\left(\frac{S_t}{B_t}\right) = \frac{S_t}{B_t} \sigma^S \cdot dW_t^Q \Rightarrow \frac{S_t}{B_t}$$ – действительно мартингал

Теперь возьмём в качестве $N_t := B(t,T)$

Мы в прошлом году посчитали, какое СДУ для облигации $B(t,T)$:

$$dN_t = N_t(rdt + \sigma_t^B(T) dW_t^Q)$$

Дрейф у всех торгуемых активов в риск-нейтральной мере $= r$.

А вот σ_t^B – мы в прошлом году считали.

Хотим: $Q \rightsquigarrow Q^T$

$N_t = B_t$ $\quad N_t = B(t,T)$

Итак, знаем

$$d\left(\frac{S_t}{N_t}\right) = d\left(\frac{S_t}{B_t} \cdot \frac{B_t}{B(t,T)}\right)$$

Ещё знаем, что $dN_t = N_t(rdt + \sigma_t^B(T) dW_t^Q)$

$\Rightarrow$ по ф-ле Ито: $d\left(\frac{B(t,T)}{B_t}\right) = \frac{B(t,T)}{B_t} \cdot \sigma_t^B(T) dW_t^Q$

$$\Rightarrow d\left(\left(\frac{B(t,T)}{B_t}\right)^{-1}\right) = \dots \text{ф-ла Ито} \dots = (\sigma_t^B(T)^2 dt - \sigma_t^B(T) dW_t^Q)$$

$$\Rightarrow d\left(\frac{S_t}{B(t,T)}\right) \overset{\text{ф-ла Ито}}{=} = \frac{S_t}{B(t,T)} \cdot \left[\sigma_t^B(T)(\sigma_t^B(T) - \sigma^S)dt + (\sigma_t^S - \sigma_t^B(T))dW_t^Q\right] =$$

$$\overset{!}{=} \frac{S_t}{B(t,T)} \cdot (\sigma_t^S - \sigma_t^B(T)) dW_t^T$$

– так должно быть, т.к. $\frac{S_t}{B(t,T)}$ – мартингал относ. Q^T. А коэф. $(\sigma_t^S\ \sigma_t^B(T))$ – волатильность – при замене меры не меняется

$$\Rightarrow dW_t^T = dW_t^Q - \sigma_t^B(T) dt$$

А это же теорема Гирсанова

$$\Rightarrow W_t^T = W_t^Q - \int_0^t \underbrace{\sigma_s}_{\text{это } \varphi_s \text{ from т. Гирсанова}}^B(T) ds$$

Girsanov: $L_t = \frac{dQ^{new}}{dQ^{old}} = e^{\int_0^t \varphi_s dW_t - \frac{1}{2}\int_0^t \varphi_s^2 ds}$ – плотность ~~[illegible]~~ замены меры,

$W_t^{new} = W_t^{old} - \int_0^t \varphi_s ds.$

$L_t = \frac{dQ^{new}}{dQ^{old}}$: $dL_t = L_t \varphi_t dW_t$ – по опр. L_t

В нашем случае: $dL_t = \sigma_t^B(T) L_t \, dW_t$; $L_0 = 1$.

Заметим, что $d\left(\frac{B(t,T)}{B_t} \cdot \frac{B_0}{B(0,T)} \right) = \sigma_t^B(T) \left(\frac{B(t,T)}{B_t} \cdot \frac{B_0}{B(0,T)} \right) dW_t$

$$\frac{B(0,T)}{B_0} \cdot \frac{B_0}{B(0,T)} = 1.$$

нормализация

$$\Rightarrow L_t = \frac{B(t,T)}{B_t} \cdot \frac{B_0}{B(0,T)} = \frac{N_t^{new}}{N_t^{old}} \cdot \frac{N_0^{old}}{N_0^{new}}$$

Т.е. мы нашли плотность меры Q^T.

А нам-то эта плотность нужна, т.к.:

$$E^Q\left[\frac{(S_T - K)^+}{B_T} \right] = E^Q\left[\frac{(S_T - K)}{B_T} \mathbb{1}_{S_T > K} \right] =$$

$$= E^Q\left[\frac{S_T}{B_T} \cdot \mathbb{1}_{S_T > K} \right] - \underbrace{K \cdot E^Q\left[\frac{1}{B_T} \cdot \mathbb{1}_{S_T > K} \right]}_{Q \rightsquigarrow Q^T} = K \cdot B(0,T) \, E^Q\left[\frac{1}{B(0,T) B_T} \cdot \mathbb{1}_{S_T > K} \right]$$

$$\| \; E^{Q^T}\left[\mathbb{1}_{S_T > K} \right] = \underline{\underline{Q^T(S_T > K)}}$$

$\| \; E^{Q^T}\left[L_T \cdot \frac{1}{B_T} \cdot \mathbb{1}_{S_T > K} \right]$, где $L_T = \frac{B(T,T)}{B_T} \cdot \frac{B_0}{B(0,T)} = \frac{1}{B(0,T)} \cdot \frac{1}{B_T}$

$$E^{Q^T}[f(x)] = E^Q\left[\frac{dQ^T}{dQ} = L_T \cdot f(x) \right].$$

В мере Q^T: $dS_t = S_t(r\,dt + \sigma^S dW_t^Q) = S_t\left(r\,dt + \sigma^S\left(dW_t^T + \sigma_t^B(T)\,dt\right)\right) = S_t\left(r + \sigma^S \sigma_t^B(T)\right)dt + \sigma^S dW_t^T$

заменили dW_t^Q на dW_t^T

Т.е. мы знаем распр. S_t в мере Q^T. $\longrightarrow$ посчитаем $Q^T(S_T > K)$

Или по-другому: $d\left(\frac{S_t}{B_t(T)} \right) = \frac{S_t}{B_t(T)} \left(\sigma^S - \sigma_t^B(T) \right) dW_t^T$

$$\Rightarrow S_T = \overset{=1}{B(T,T)} \cdot \frac{S_0}{B(0,T)} \cdot e^{\int_0^T \Sigma_s dW_s^T - \frac{1}{2} \int_0^T \Sigma_s^2 ds}.$$

$\Rightarrow \ln S_T \overset{Q^T}{\sim} N(\ldots, \ldots)$ – норм. распр.

$\Rightarrow Q^T(\ln S_T > \ln K)$

$\| \; \ldots$

$N(\ldots)$ – стандартное норм. распр.

Аналогично, $E^Q\left[\frac{S_T}{B_T} \cdot \mathbb{1}_{S_T > K} \right] \rightsquigarrow Q^S$: $N_t = S_t$.

29.04.2022. Лапшин. Семинар 4

Теорема Гирсанова и замена меры.

1.б) Волатильность облигации – либо вспомнить ф-лу из лекций, либо вывести ф-лу из ф-лы Ито.

Calculate the volatility $\sigma_t^B(\tau)$ for $0 \le t \le \tau \le T$, defined by

$$dB_t(\tau) = B_t(\tau)(r_t dt + \sigma_t^B(\tau) dW_t)$$

Нам дана модель HJM:

$$df_t(\tau) = \alpha_t(\tau)dt + \sigma_t(\tau)dW_t$$

$\sigma_t(\tau)$ – дано; $\alpha_t(\tau)$ – можем найти из HJM-$\overset{drift}{\text{condition}}$ – делали в прошлый раз

по опр. цены облигации:

$$B_t(\tau) = e^{-\int_t^\tau \overset{\text{случ. величина}}{f_t(s)} ds}$$

Честно считаем по ф-ле Ито, в два шага:

Сначала обозн. $X_t(\tau) = \int_t^\tau f_t(s)ds$

$$\Rightarrow dX_t(\tau) = -f_t(t)dt + \int_t^\tau \underbrace{df_t(s)}_{\alpha_t(s)dt + \sigma_t(s)dW_s} ds =$$

$$= -r_t dt + \left(\int_t^\tau \alpha_t(s)ds\right)dt + \left(\int_t^\tau \sigma_t(s)ds\right)dW_t =$$

Вспомним HJM drift-condition:

$$\boxed{\alpha_t(\tau) = \sigma_t(\tau)\int_t^\tau \sigma_t(s)ds}$$

$$= -r_t dt + \frac{1}{2}\left(\int_t^\tau \sigma_t(s)ds\right)^2 dt + \left(\int_t^\tau \sigma_t(s)ds\right)dW_t$$

Далее, $B_t(\tau) = e^{-X_t}$

→ берём $f(x,t) = e^{-x}$

$f'_x = -e^{-x}$;

$f''_{xx} = e^{-x}$

$$\Rightarrow dB_t(\tau) = -e^{-X_t}dX_t + \frac{1}{2}e^{-X_t}\cdot(dX_t)^2 =$$

$$= -\underbrace{e^{-X_t}}_{B_t(\tau)}\left(-r_t dt + \cancel{\frac{1}{2}\left(\int_t^\tau \sigma_t(s)ds\right)^2 dt} + \left(\int_t^\tau \sigma_t(s)ds\right)dW_t\right) + \frac{1}{2}e^{-X_t}\cdot\cancel{\left(\int_t^\tau \sigma_t(s)ds\right)^2} dt =$$

$$= \boxed{B_t(\tau)\left(r_t dt - \left(\int_t^\tau \sigma_t(s)ds\right)dW_t\right)}$$

⇒ искомая сигма (т.е волатильность облигации [illegible]) $\boxed{\sigma_t^B(\tau) = -\int_t^\tau \sigma_t(s)ds}$

итак, $\sigma_t^{3}(\tau) = -\int_t^\tau \sigma_t(s)ds \overset{\nearrow}{=} -\int_t^\tau \sigma^f \cdot s ds = -\sigma^f \cdot \frac{s^2}{2}\Big|_t^\tau = \boxed{\frac{\sigma^f}{2}(t^2-\tau^2)}$

$\sigma_t(\tau) = \sigma^f \cdot \tau$; $0 \le t \le \tau \le T$.

(1c) ~~Просто посчитать~~ обозн. $\Sigma_u := \sigma^S - \underset{\sim}{\sigma_u^B(T)}$; посчитать $\int_0^T |\Sigma_u|^2 du$.

σ^S ↑ волатильность акции (дана)

$\sigma_u^B(T)$ — волатильность цены фикс. облигации со сроком до погашения T.

Для акции: $dS_t = S_t(r_t dt + \sigma^S dW_t)$ — в риск-нейтральной мере.

ну $|\Sigma_u|^2 = |\sigma^S - \sigma_u^B(T)|^2 = |\sigma^S - \frac{\sigma^f}{2}(u^2-T^2)|^2 = (\sigma^S)^2 - \sigma^S \sigma^f(u^2-T^2) + \frac{(\sigma^f)^2}{4}(u^2-T^2)^2$

$$\Rightarrow \int_0^T |\Sigma_u|^2 du = \int_0^T \left[(\sigma^S)^2 - \sigma^S \cdot \sigma^f \cdot (u^2-T^2) + \frac{(\sigma^f)^2}{4}(u^2-T^2)^2\right] du =$$

$$= (\sigma^S)^2 T - \sigma^S \cdot \sigma^f \cdot \frac{T^3}{3} + \sigma^S \cdot \sigma^f \cdot T^3 + \frac{(\sigma^f)^2}{4} \cdot \left(\frac{T^5}{5} - \frac{2T^5}{3} + T^5\right) =$$

$$= \left(\sigma^S + \frac{\sigma^f T^2}{2}\right)^2 T - \sigma^f\left(\sigma^S + \frac{\sigma^f T^2}{2}\right)\frac{T^3}{3} + \frac{(\sigma^f)^2}{4} \cdot \frac{T^5}{5}$$

(1d) $E^Q\left(\frac{1}{B_T}(S_T-K)^+\right) = ?$ ← это по н-й фунд. теореме, это и есть тек. стоимость,

т.к. $E^Q\left(\frac{V_T}{N_T}\right) = \frac{V_0}{N_0 = 1}$ ← current price ⟹ $\boxed{V_0 = E^Q\left(\frac{V_T}{N_T}\right)!}$

↑ они коррелированы, т.к. зависят от одного и того же Винер. процесса!

И что делать?

Трюк: замена меры, или замена numeraire.

у теории: $E^Q\left(\frac{V_T}{N_T}\right) = \frac{V_0}{N_0}$

↖ это мартингал, если правильно выбрать меру Q!

Т.е. неважно, какое B_T — для любого B_T будет ∃ своя правильная мера Q

Для $N_0 = B_t = e^{\int_0^t r_s ds}$ — т.е. B_t — это банковский счет. —

Эта Q наз. риск-нейтральной мерой.

стоимость в момент t облигации с исполнением в T ~~[illegible]~~ t.

$\boxed{B_t(\tau) = e^{-\int_t^\tau f_t(s)ds}}$

Но в качестве N_t можно выбирать и другие активы.

В этом упр. нам надо выбрать $\overset{N_t}{B_t(T)}$ стоимость в t облигации с исп. в T. $\to$ Q — наз. T-forward measure.

Т.е. мог ~~[illegible]~~ остаток в банк. вклад / облигацию / акцию.

А можем выбрать $N_t = S_t$. ⟹ Q обозн. Q^S

И фишка в том, что вне зависимости от того, $N_t = B_t$; $N_t = B_t(T)$; $N_t = S_t$ –

всё равно ~~[illegible]~~ $\frac{V_T}{N_T}$ будет мартингалом для нужного Q

$$\Rightarrow E^Q\left(\underset{\uparrow}{\frac{1}{B_T}}(S_T-K)^+\right) = E^Q\left(\frac{(S_T-K)\cdot \mathbb{1}_{\{S_T\geq K\}}}{B_T}\right) = S_0 E^Q\left[\frac{S_T\cdot \overset{B_0=1}{}}{B_T S_0}\mathbb{1}(S_T\geq K)\right] - K E^Q\left[\frac{\overset{B_T(T)=1,\ B_0=1}{1}}{B_T\cdot B_0(T)}\mathbb{1}_{\{S_T\geq K\}}\right]\cdot B_0(T)$$

облигация должна исполняться в тот же момент, что и опцион.

ф-ла замены меры

$$= \left\{ E^P(X) = E^Q\left(X\cdot \frac{dP}{dQ}\right);\ \frac{dP}{dQ}\geq 0;\ E\left(\frac{dP}{dQ}\right) = 1 \right\} ⊜$$

плотность Радона-Никодима

(!) $E^Q\left(\frac{S_T\cdot B_0}{B_T\cdot S_0}\right) = 1$, т.к. $E^Q\left(\frac{S_T}{B_T}\right) = \frac{S_0}{B_0}$, т.к. $\frac{S_T}{B_T}$ – мартингал по мере Q.

$\Rightarrow \frac{S_T\cdot B_0}{B_T\cdot S_0} = \frac{dQ^S}{dQ}$.

(!) $\frac{B_t(T)}{B_t}$ – тоже мартингал, как дисконтированная стоимость облигации $\Rightarrow E^Q\left[\frac{B_T(T)}{B_T}\right] = \frac{B_0(T)}{B_0}$

⊜ $S_0\cdot E^{Q^S}[\mathbb{1}\{S_T\geq K\}] - K\cdot B_0(T)\cdot E^{Q^T}[\mathbb{1}\{S_K\geq K\}] = S_0\cdot P_{Q^S}(S_T\geq K) - K\cdot B_0(T)\cdot P_{Q^T}(S_K\geq K)$.

Т.е. $E^Q\left(\frac{1}{B^T}(S_T-K)^+\right)$ = сумме двух вер-тей, но по сложным мерам

Надо понять, что это за меры.

Для этого есть т. Гирсанова.

По н-й фунд. теореме: $\frac{S_t}{B_t(T)}$ – мартингал относ. Q – т.к. ∀ знаменателя $B_t(T)$ ∃ Q такая, что ∀ числителя S_t: $\frac{S_t}{B_t(T)}$ – будет мартингалом

Есть change of numeraire density: $L_t = \frac{N_t^{new}}{N_t^{old}}\cdot\frac{N_0^{old}}{N_0^{new}}$ – вот такой вид имеет плотность при замене меры с плотностью $\varphi_s = \Sigma_u = \sigma_t^S - \sigma_t^B(T$,

По т. Гирсанова, имея эту плотность, получаем:

$$W_t^{new} = W_t^{old} - \int_0^t \varphi_s ds,\quad L_t = e^{\int_0^t \varphi_s dW_t - \frac{1}{2}\int_0^t \varphi_s^2 ds}$$

$$\text{«}\frac{dQ^{new}}{dQ^{old}}\text{»} = \underbrace{\frac{N_t^{new}}{N_t^{old}}}_{=B_t}\cdot\underbrace{\frac{N_0^{old}}{N_0^{new}}}_{=B_t(T)}$$

Т.е. Винер процесс для новой меры имеет такой вид

Хотим найти распр. S_T в мере Q^T (и в мере Q^S).

~~[illegible]~~

Но мы знаем, что: $\frac{S_t}{B_t(T)}$ – это мартингал относ. Q.

~~$d\left(\frac{S_t}{B_t(T)}\right)$~~ = Ещё знаем:

$$\begin{cases} d\left(\frac{S_t}{B_t}\right) = \left(\frac{S_t}{B_t}\right)\sigma_t^S dW_t \\ d\left(\frac{B_t(T)}{B_t}\right) = \left(\frac{B_t(T)}{B_t}\right)\sigma_t^B(T) dW_t \end{cases}$$

– т.к. $\begin{cases} dS_t = S_t(r_t dt + \sigma^S dW_t) \\ dB_t = r_t B_t dt \end{cases}$ по усл. (и применим ф-лу Ито)

– это такая же ф-ла, т.к. B_t – тоже GBM, но у него ещё σ_t^B вводили в начале.

⇒ честно делим $\frac{S_t}{B_t}$ на $\frac{B_t(T)}{B_t}$, получим (опять применяя ф-лу Ито):

$$\boxed{d\left(\frac{S_t}{B_t(T)}\right) = \left(\frac{S_t}{B_t(T)}\right)\underbrace{(\sigma_t^S - \sigma_t^B(T))}_{\substack{\text{а это неслучайная ф-ция!} \\ \| \\ \Sigma_t}} dW_t^T}$$

⇒ мы это проинтегрировать можем:

⇒ $$\boxed{\frac{S_t}{B_t(T)} = \frac{S_0}{B_0(T)} e^{\int_0^t \Sigma_u dW_u^T}}$$ ← по т. Гирсанова: $W_u^T = W_u - \int_0^u \Sigma_s ds$

~~А еще мы знаем, что по т. Гирсанова: $W_u = W_u$~~

возьмем плотность $L_t = e^{\int_0^t \varphi_s dW_s \underbrace{- \frac{1}{2}\int_0^t \varphi_s^2 ds}_{\substack{\text{просто константа нормализации,} \\ \text{чтоб это мо = 1 имело.}}}}$

(т.е мы про Q_T только знаем, что она определяется плотностью $L_t = \frac{B_t(T)}{B_t} \cdot \frac{B_0}{B_0(T)}$)

И хотим найти распр. S_t в мере Q^T.

Мы знаем, что $\frac{S_t}{B_t(T)}$ - мартингал в мере Q^T.

И её дифференциал: $d\left(\frac{S_t}{B_t(T)}\right) = \left(\frac{S_t}{B_t(T)}\right)(\sigma_t^3 - \sigma_t^B(T))dW_t^T$ ← нет члена при dt.

19.04.2022. [illegible] семинар 3.

Модели HJM

Раньше ~~[illegible]~~ мы моделировали short rate r_t: ставка от t до $t+dt$.

Если B_t – bank account, то $dB_t = r_t B_t dt$

$$B_t = B_0 \cdot e^{\int_0^t r_\tau d\tau}; \quad B_0 = 1.$$

Т.е. если r_t – и через нее все пересчитывается: $B_t = B_0 \cdot e^{\int_0^t r_\tau d\tau}$

$\hookrightarrow B_0(t)$ – discount factor (= bond price)

$B_0(t) = E^Q[\frac{1}{B_t}]$.

$\hookrightarrow f_0(t)$: $B_0(t) = e^{-\int_0^t f_0(\tau)d\tau}$

instantaneous forward rate

r_t можно моделировать:

$$\begin{cases} dr_t = K(\theta - r_t)dt + \sigma dW_t \\ r_0 = z_0. \end{cases}$$

Но такие модели недостаточно гибкие.

Теперь будем моделировать все $f_0(t)$ эконом.

ставка на каждый срок эволюционирует сама по себе.

! В моделях HJM: если задать коэф. диффузии, то коэф. сноса определяется автоматически.

① $df_t(\tau) = \alpha_t(\tau)dt + \overset{const}{\boxed{\sigma^f(\tau-t)}}dW_t$ – дано

$\sigma_t(\tau)$

а) $\alpha_t(\tau) = \sigma_t(\tau)\int_t^\tau \sigma_t(s)ds = \sigma^f(\tau-t)\cdot\int_t^\tau \sigma^f(\tau-s)ds = (\sigma^f)^2\cdot(\tau-t)\left(-\frac{(\tau-s)^2}{2}\right)\Big|_t^\tau = \frac{(\sigma^f)^2\cdot(\tau-t)}{2}$

б) $\boxed{df_t(\tau) = \frac{(\sigma^f)^2(\tau-t)^3}{2}dt + \sigma^f(\tau-t)dW_t.}$

напоминание: $r_t = f_t(t)$

СДУ, его надо проинтегрировать.

$$\Rightarrow f_t(\tau) = \underbrace{}_{\boxed{f_0(\tau)+}}(\sigma^f)^2\cdot\int_0^t\frac{(\tau-t)^3}{2}dt + \int_0^t \sigma^f(\tau-t)dW_t = f_0(\tau) + (\sigma^f)^2\frac{(\tau^4-(\tau-t)^4)}{8} + \sigma^f\cdot\tau\cdot\boxed{W_t} - \boxed{\sigma^f\int_0^t s\,dW_s}$$

$\sim N(0, (\sigma^f)^2\cdot\frac{t^3}{3})$

это дано не явно, а в форме $B_0^*(\tau) = e^{-\lambda^* t} = e^{-\int_0^\tau f_0(s)ds}$

Они коррелированы! Поэтому надо дисперсию этих штук вместе считать

⇒ можно было оставить в виде $\sigma^f\cdot\int_0^t(\tau-s)dW_s \sim N(0, \frac{\tau^3-(\tau-t)^3}{3}\cdot(\sigma^f)^2)$

то есть $f_t(\tau)$ – мы типа нашли.

Но при [illegible]

Итак, $f_t(\tau)=f_0(\tau)+(\sigma^f)^2\frac{(\tau^4-(\tau-t)^4)}{8}+\sigma^f\int_0^t(\tau-s)dW_s$

$\Rightarrow r_t=f_t(t)=f_0(t)+(\sigma^f)^2\frac{t^4}{8}+\underbrace{\sigma^f\int_0^t(t-s)dW_s}_{\| \ \sigma^f t W_t-\sigma^f\int_0^t s dW_s}$

Хотим ур-е на r_t:

$$\boxed{dr_t=\left(f_0'(t)+(\sigma^f)^2\cdot\frac{t^3}{2}+\sigma^f W_t\right)dt}+\cancel{\sigma^f t dW_t}-\cancel{\sigma^f t dW_t}$$

↑а это не марковская динамика

$\Rightarrow dr_t$ – не представимо в виде $dr_t=\mu dt+\sigma dW_t$ – т.к. W_t не измеримо отн. F^{r_t}

т.е. r_t – не марковское.

Критерий марковости: $\sigma_t(\tau)=g(t)\cdot h(\tau)$ – если так, то r_t – марковское

c) $B_0^*(\tau)=e^{-\lambda^*\tau}\rightsquigarrow f_0^*(\tau)=\lambda^*$ – initial condition

~~ну~~ $B_t^*(\tau)=e^{-\lambda(\tau-t)}\rightsquigarrow f_t(\tau)$.

ну $B_t(\tau)=e^{-\int_t^\tau f_t(s)ds}\Rightarrow f_t(\tau)=(-\ln B_t(\tau))'_\tau\in F_t$

Дано: $B_t^*(\tau)=e^{-\lambda(\tau-t)}$

$\Rightarrow f_t(\tau)=(-\ln B_t(\tau))'_\tau=(\lambda(\tau-t))'_\tau=\lambda$

А в нашей модели может быть, что:

$$\underbrace{f_t(\tau)=\underset{\lambda^*}{\underset{\|}{f_0(\tau)}}+(\sigma^f)^2\frac{(\tau^4-(\tau-t)^4)}{8}+\sigma^f\cdot\int_0^t(\tau-s)dW_s}_{f_t(\tau)\text{ evolved from }f_0(\tau)}\overset{?}{=}\lambda$$

↑observed $f_t(\tau)$.

ну $\lambda^*-\lambda^0+(\sigma^f)^2\frac{(\tau^4-(\tau-t)^4)}{8}\overset{?}{=}\sigma^f\cdot\int_0^t(\tau-s)dW_s;\ \forall\tau\in[t,T]$

нет, так не бывает, т.к. левая часть нелин. по τ, а правая – линейна по τ.

! Нач. усл. для HJM – это кривая $f_0(\tau)$

Мораль: модель, что $f_0(\tau)=\lambda^*$ – одна для всех τ

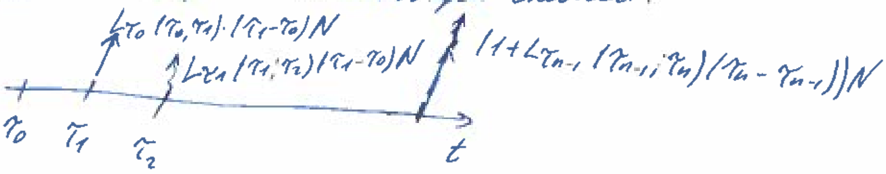
$f_t(\tau)=\lambda$ – одна для всех τ – не согласуется с 1-мерной HJM-моделью

d) $$\frac{dB_t(\tau)}{B_t(\tau)} = \underset{\substack{\uparrow \\ \text{доходность}}}{r_t}\,dt + \underbrace{\sigma_t^{3}(\tau)}_{\substack{\uparrow \\ \text{найти это!}}}\,dW_t$$ — ну это GBM.

и любой торгуемый актив будет иметь такую компоненту сноса.

09.03.2022. [illegible]. Семинар 2

Задача 1 Облигация с плавающей ставкой.



$$PV_t = N(B_t(\tau_n) + \sum_{k:\tau_{k+1}>t} B_t(\tau_k)\cdot \underbrace{L_t(\tau_k;\tau_{k+1})(\tau_{k+1}-\tau_k)}_{\frac{B_t(\tau_k)-B_t(\tau_{k+1})}{B_t(\tau)}}$$

$$\Rightarrow PV_t = N(B_t(\tau_n) + \sum_{k:\tau_{k+1}>t}(B_t(\tau_k)-B_t(\tau_{k+1}))) = NB_t(\tau_{k^*}),\text{ где } k^*=\min\{k:\tau_{k+1}>t\}$$

У нас $t=\tau_m$. $\Rightarrow k^*=m$

$$\Rightarrow PV_t = N\cdot \underbrace{B_{\tau_m}(\tau_m)}_{1} = N.$$

Т.е такую облигацию можно купить за номинал!

Задача 2 a) Мы ~~фикс~~. swap-payer или swap-receiver?

~~payer~~ Мы фикс. платим – значит, мы swap-payer.

б) Цена свопа:

фикс. нога стоит $N\cdot\sum_{i=m+1}^{n} B_t(\tau_i)\cdot K(\tau_i-\tau_{i-1})$

Плавающая нога – стоит $N\cdot\sum_{i=m+1}^{n} B_t(\tau_i)L_t(\tau_{i-1},\tau_i)(\tau_i-\tau_{i-1})$

$$\Rightarrow \text{цена свопа} = N\cdot\sum_{i=m+1}^{n} B_t(\tau_i)(L_t(\tau_{i-1},\tau_i)-K)(\tau_i-\tau_{i-1})$$

c) Swaption – это опцион на то, чтобы ~~войти~~ в своп.

At time τ_m: можем либо получить своп бесплатно, либо ~~решить~~ его не получать.

(если своп сколько-нибудь стоит – то конечно хотим)

$$\Rightarrow \text{Swaption-price at time } \tau_m = \left(N\cdot\sum_{i=m+1}^{n} B_{\tau_m}(t_i)(L_{\tau_m}(\tau_{i-1},\tau_i)-K)(\tau_i-\tau_{i-1})\right)^+$$

d) Cap: $N(L_{\tau_{i-1}}(\tau_{i-1},\tau_i)-K)^+(\tau_i-\tau_{i-1})$

$$\boxed{\text{Swap}\le\text{Swaption}\le\text{cap}}$$

(т.к макс(суммы) ≤ сумма макс)

Задача 3 Short-rate models

$$\begin{cases} dr_t = \theta dt + \sigma^r dW_t \\ r_0 = r_0^* \end{cases}$$ ← индекс! не степень!

Найти $B_0(\tau)$; $E^Q(e^{\int_0^\tau r_s ds})$

a) $B_0(\tau) = E^Q(e^{-\int_0^\tau r_s ds})$ – ну по опр.

Ну решаем СДУ: $dr_t = \theta dt + \sigma^r dW_t$

$$\Rightarrow r_t = r_0 + \theta t + \sigma^r W_t$$

ну $d(t\cdot W_t) = tdW_t + W_tdt$

Теперь считаем $\int_0^\tau r_s ds = r_0\tau + \frac{\theta\tau^2}{2} + \sigma^r \int_0^\tau W_s ds$, $\int_0^\tau W_s ds = \tau W_\tau - \int_0^\tau s dW_s$ это интеграл Ито, = это ясно, что такое

$$\Rightarrow \int_0^{\tau} r_s ds \sim N\left(r_0\tau + \frac{\theta\tau^2}{2}; (\sigma^r)^2 \cdot \frac{\tau^3}{3}\right) \sim N(\mu_\xi; \sigma_\xi^2)$$

$$\Rightarrow E^Q\left[e^{-\int_0^{\tau} r_s ds}\right] = e^{-\mu_\xi + \frac{\sigma_\xi^2}{2}} = \boxed{e^{-r_0\tau - \frac{\theta\tau^2}{2} + \frac{(\sigma^r)^2\tau^3}{6}}}$$

$$\overset{E^Q[e^{-\int_0^t r_s ds}]}{\parallel}$$

б) $B_0(t) = e^{-\int_0^t f_0(\tau)d\tau}$ ← instanteneous forward rate $= e^{-r_0(t)\cdot t}$ ↖ spot rate

$\boxed{r_t = r_{~~t~~}(t) = f_{~~t~~}(t) = \text{short rate}}$

Найдём отсюда $f_0(\tau)$, т.к. $B_0(t)$ – у нас есть.

$r_0(t) = \frac{1}{t}\int_0^t f_0(t)dt$ – это <u>ОДНА</u> ставка сейчас на 2 года!

$$\boxed{E^Q\left[e^{-\int_0^t r_s ds}\right] = B_0(t) = e^{-\int_0^t f_0(\tau)d\tau} = e^{-r_0(t)t}}$$

↑ Short rate ↑ instant. forward rate ↑ Spot rate

$$B_0(\tau) = e^{-r_0\tau - \frac{\theta\tau^2}{2} + \frac{(\sigma^r)^2\tau^3}{6}}$$

$$\Rightarrow r_0(\tau) = r_0 + \frac{\theta\tau}{2} - \frac{(\sigma^r)^2\tau^2}{6}.$$

$$f_0(\tau) = r_0 + \theta\tau - \frac{(\sigma^r)^2\tau^2}{2} \qquad \Big\rfloor\; f_0(\tau) = \frac{\int_0^t r_0(t)dt}{\tau}$$

c) $E^Q[B_T] = E^Q\left[e^{\int_0^T r_s ds}\right] = e^{r_0\tau + \frac{\theta\tau^2}{2} + \frac{(\sigma^r)^2\tau^3}{6}}$

02.03.2022. Ничишин. Семинар 1.

Лист 1, задача 1

$\longleftarrow$ Spot-rate!

$y_0^*(\tau) = 0.1\cdot 7;\ \tau \in [0,6]$

а) найти $B_0^*(\tau_i), \tau_i = i$ — коэф. дисконтирования

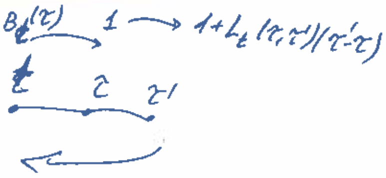
$B_0(t)\cdot e^{y_0(t)\cdot t} = 1.$

$\Rightarrow B(t) = \frac{1}{e^{y_0(t)\cdot t}}$

$$\boxed{B(t) = e^{-r(t)\cdot t}}$$

б) Найти $L_0(\tau_{i-1};\tau_i)$

НУ $\boxed{L_t(\tau,\tau') = \frac{B_t(\tau)-B_t(\tau')}{(\tau'-\tau)B_t(\tau')}}$



$B_t(\tau')\cdot(1+L_t(\tau,\tau')(\tau'-\tau))$

$\Rightarrow 1+L_t(\tau,\tau')(\tau'-\tau) = \frac{B_t(\tau)}{B_t(\tau')}$

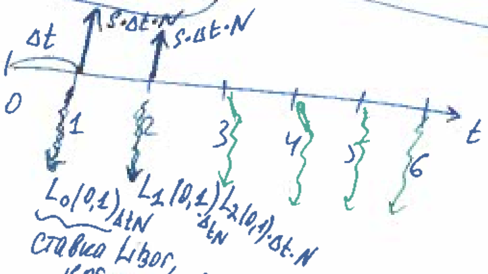
$\Rightarrow L_t(\tau,\tau') = \frac{B_t(\tau)-B_t(\tau')}{(\tau'-\tau)B_t(\tau')}$

в) Swap rate для свопа с датами $\tau_1, \tau_2 \dots \tau_6$.

$m=1;\ n=6.$

Инвестируем 1 в момент τ_i; получаем $1+L(\tau_i;\tau_{i+1})(\tau_{i+1}-\tau_i)$ в момент τ_{i+1}.

Смысл LIBOR:



ставка Libor, кот. изв. в момент $t=0$ на срок от $t=0$ до $t=1$.

term − Δt
time − t.

Ищем ставку S: $\sum_{i=1}^{6} S\cdot\Delta t\cdot N\cdot B_{t_i} = \sum_{i=1}^{6} L_{t_{i-1}}(t_{i-1};t_i)\cdot\Delta t\cdot N\cdot B_{t_i} \Rightarrow$ $\boxed{S = \frac{\sum_{i=1}^{6} L_{t_{i-1}}(t_{i-1};t_i)B_{t_i}}{\sum_{i=1}^{6} B_{t_i}}}$

Assume that forward rates are realised, i.e. $L_{t_i}(t_i;t_{i+1}) = L_0(t_i;t_{i+1})$

Задача 2

Есть 5% (— купон.) 20-летняя облигация.

YTM = 7%.

а) пусть после 1 купона она продаётся с YTM = 6%.

Найти доходность за 1 год.

НУ цена облигации при $t=0$ ~~$PV_1 = \frac{F}{(1+0.07)^{20}} + \sum_{k=1}^{20}\frac{0.05F}{(1+0.07)^k}$~~

~~PV_2~~

$$PV_1\Big|_{t=0} = \frac{F}{e^{20\cdot 0.07}} + \sum_{k=1}^{20}\frac{0.05F}{e^{k\cdot 0.07}}$$

$$PV_2\Big|_{t=1} = \frac{F}{e^{19\cdot 0.06}} + \sum_{k=1}^{19}\frac{0.05F}{e^{k\cdot 0.06}}$$

$$\boxed{\text{И доходность} = \frac{PV_2 - PV_1 + 0.05F}{PV_1}}$$

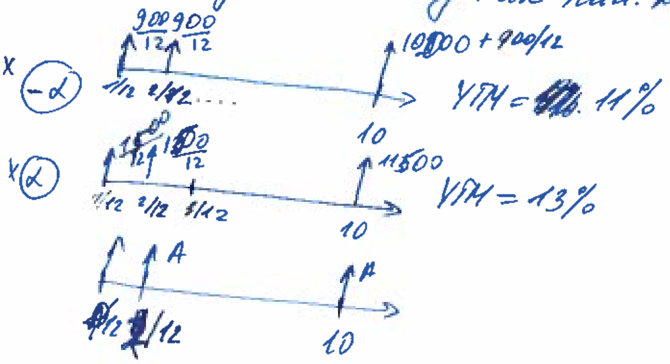
б) пусть мы её продали после 8 платежа, YTM=6%.

Найти the realized compound yeild if coupon can be reinvested на 1 год под 3%.

$y_{realised}$: $(1+y_{realized})^n = \frac{V}{P}$

Задача 3

Получим ипотеку как лин. комб. 2-х облигаций.



$N=10000$

$$\begin{cases}\frac{1500}{12}\alpha - \frac{900}{12}\alpha = A. \\ P = \alpha P_2 - \alpha P_1.\end{cases} \Rightarrow \alpha = \frac{12A}{600} = \frac{A}{50}.$$

$$\Downarrow\ P = \alpha(P_2-P_1) = \overset{\text{знаем}}{(P_2-P_1)}\cdot\frac{A}{50} \Rightarrow \boxed{A = \frac{50\cdot P}{P_2-P_1}} \leftarrow \text{Ответ.}$$

$$P_1 = \sum_{k=0}^{119} \frac{900/12}{e^{k\cdot 0.11}} + \frac{10.000}{e^{119\cdot 0.11}}$$

$$P_2 = \sum_{k=0}^{119} \frac{1500/12}{e^{k\cdot 0.13}} + \frac{10.000}{e^{119\cdot 0.13}}$$

Задача 4

$r_t = r_t(0)$; $r_t = \frac{0.1}{(1+t)^2}$

$B(0,t) = e^{-\int_0^t r_\tau d\tau}$. | вообще-то $B(0,t)\ E^Q[e^{-\int_0^t r_\tau d\tau}] = e^{-r_0(t)\cdot t}$

a) найти $L_0(\tau_i;\tau_{i+1})$

б) найти swap rate

c) найти цену swaption

d) Доходность по всем инструментам одинаковая! Не важно, в какой пропорции мы свои деньги распределяем.