A two-factor Cheyette model for commodities.

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1 General model framework.

We consider a two-factor general commodity futures model driven, under the risk-neutral measure, by the SDE

$$\frac{dF(t,T)}{F(t,T)} = \frac{g_1(T)}{g_1(t)} h_t^1 \, dW_t^1 + \frac{g_2(T)}{g_2(t)} h_t^2 \, dW_t^2 \tag{1}$$

where

$$g_i(t) = e^{-\int_0^t a_i(s) ds} \quad ,$$
$$\langle dW_t^1, dW_t^2 \rangle = \rho_{1,2}(t) \, dt$$

h_t^1 and h_t^2 are two generic adapted volatility processes and $(a_i(t))_{i \in (1,2)}$ are two deterministic mean-reversions. As part of the Cheyette model class [1], the above specification of futures volatilities results in commodity futures being Markovian functions of two diffusive mean-reverting state variables x_t^1 and x_t^2 whose dynamics are given by the SDEs

$$\begin{aligned} dx_t^1 &= -a_1(t) \, x_t^1 \, dt + h_t^1 \, dW_t^1 \\ dx_t^2 &= -a_2(t) \, x_t^2 \, dt + h_t^2 \, dW_t^2 \end{aligned} \tag{2}$$

and three non-diffusive state variables evolving according to

$$\begin{aligned} dy_t^1 &= \left(-2 \, a_1(t) \, y_t^1 + \left(h_t^1\right)^2\right) \, dt \\ dy_t^2 &= \left(-2 \, a_2(t) \, y_t^2 + \left(h_t^2\right)^2\right) \, dt \\ dy_t^3 &= \left(-\left(a_1(t) + a_2(t)\right) \, y_t^3 + h_t^1 \, \rho_{1,2}(t) \, h_t^2\right) \, dt \end{aligned} \tag{3}$$

Writing $g_1(t,T) = g_1(T)/g_1(t)$ and $g_2(t,T) = g_2(T)/g_2(t)$, the commodity futures read

$$\begin{aligned} F(t,T) &= F(0,T) \, \exp\Big[g_1(t,T) \, x_t^1 + g_2(t,T) \, x_t^2 \\ &- \frac{1}{2} \left(g_1^2(t,T) \, y_t^1 + g_2^2(t,T) \, y_t^2 + 2 \, g_1(t,T) \, g_2(t,T) \, y_t^3 \right) \Big] \end{aligned} \tag{4}$$

Qualitatively, one diffusive state variable is typically chosen to have faster mean reversion and higher instantaneous volatility than the other diffusive state variable in order for it to drive the more volatile front-end of the commodity futures curve whereas the other state variable controls the long-term dynamics of the curve.

2 Deterministic volatility.

The simple deterministic volatility case is important since it will constitute the backbone of the subsequent local volatility and stochastic-local volatility models. We consider deterministic volatility functions

$$h_t^1 = \sigma_1(t) \qquad h_t^2 = \sigma_2(t) \tag{5}$$

In this case, the convexity-carrying state variables $y_1(t)$, $y_2(t)$ and $y_3(t)$ are deterministic and correspond to the terminal variances and covariance of x_t^1 and x_t^2:

$$\begin{aligned} y_1(t) &= g_1^2(t) \int_0^t \left(\frac{\sigma_1(s)}{g_1(s)} \right)^2 ds \\ y_2(t) &= g_2^2(t) \int_0^t \left(\frac{\sigma_2(s)}{g_2(s)} \right)^2 ds \\ y_3(t) &= g_1(t)\, g_2(t) \int_0^t \frac{\sigma_1(s)\rho_{1,2}(s)\sigma_2(s)}{g_1(s)g_2(s)}\, ds \end{aligned} \tag{6}$$

Terminal volatilities of commodity futures can be easily calculated:

$$\Sigma^{\text{det}}(t,T) = \left[g_1^2(t,T)\, y_1(t) + g_2^2(t,T)\, y_2(t) + 2\, g_1(t,T)\, g_2(t,T)\, y_3 \right]^{1/2} \tag{7}$$

2.1 Reduction to the Gabillon model.

The standard Gabillon model [2]

$$\frac{dF(t,T)}{F(t,T)} = e^{-\int_t^T k(s)ds}\, \sigma_S(t)\, dW_t^S + \left(1 - e^{-\int_t^T k(s)ds} \right) \sigma_L(t)\, dW_t^L \tag{8}$$

is a special case of the two-factor deterministic volatility Cheyette model where

$$\begin{aligned} \sigma_1(t) &= \sigma_L(t) \\ \sigma_2(t) &= \left[\sigma_L^2(t) + \sigma_S^2(t) - 2\, \rho_G(t)\, \sigma_S(t)\, \sigma_L(t) \right]^{1/2} \\ a_1(t) &= 0 \\ a_2(t) &= k(t) \\ \rho_{1,2}(t) &= \frac{\sigma_S(t)\, \rho_G(t) - \sigma_L(t)}{\left[\sigma_L^2(t) + \sigma_S^2(t) - 2\, \rho_G(t)\, \sigma_S(t)\, \sigma_L(t) \right]^{1/2}} \end{aligned} \tag{9}$$

2.2 Calibration and marking of model parameters.

It should first be noted that, given the form (4) of the commodity futures, the model is automatically calibrated to the initial futures curve regardless of the model parameters.
Currently, the only liquidly-traded calibration instruments are options on commodity futures. Consequently, the model analytically calibrates the term structure of one of the instantaneous volatilities to options on futures once all the other model parameters have been marked according to the following procedure.

- In the model, the correlation C_{ij} between the instantaneous log-returns of two futures contracts with different tenors τ_i and τ_j is a function of the mean-reversion spread, the instantaneous volatilities ratio and the instantaneous correlation only:

$$\frac{\mathrm{Cov}\left(\frac{dF(t,t+\tau_i)}{F(t,t+\tau_i)}, \frac{dF(t,t+\tau_j)}{F(t,t+\tau_j)}\right)}{\sqrt{\mathrm{Var}\left(\frac{dF(t,t+\tau_i)}{F(t,t+\tau_i)}\right)\ \mathrm{Var}\left(\frac{dF(t,t+\tau_j)}{F(t,t+\tau_j)}\right)}} = C_{ij}(\alpha, \beta, \rho_{1,2}) \tag{10}$$

  where $\alpha = \sigma_2(t)/\sigma_1(t)$ and $\beta = a_2(t) - a_1(t)$.
  In this first marking step, we choose time-independent α, β and $\rho_{1,2}$ so that the model-implied instantaneous correlation matrix of log-returns over a set of liquidly-traded contracts matches best the historically-estimated correlation matrix of daily log-returns for that set of contracts.

- Similarly to the previous step, the ratio R_{ij} of the quadratic variations of two log-contracts is a function of the first mean-reversion and the previous three parameters only:

$$\frac{\int_{t_0}^{t_1}\left\langle \frac{dF(t,t+\tau_i)}{F(t,t+\tau_i)}, \frac{dF(t,t+\tau_i)}{F(t,t+\tau_i)}\right\rangle}{\int_{t_0}^{t_1}\left\langle \frac{dF(t,t+\tau_j)}{F(t,t+\tau_j)}, \frac{dF(t,t+\tau_j)}{F(t,t+\tau_j)}\right\rangle} = R_{ij}(a_1, \alpha, \beta, \rho_{1,2}) \tag{11}$$

  Having performed the previous step and fixed α, β and $\rho_{1,2}$, we now choose a time-independent value of a_1 so that the model-implied matrix of quadratic variation ratios matches best the corresponding matrix of historical realised variance ratios over a recent historical window.

2.3 Pricing of Commodity Asian Options within the model.

The first instruments to be supported for indicative pricing under the deterministic volatility model are commodity Asian options. Asian options are priced analytically by matching the first three moments of the Asian average [3], calculated within the calibrated model, to that of the shifted log-normal distribution

$$S = (\mu - \delta)\, e^{-\frac{V}{2} + \sqrt{V}\, Z} + \delta \tag{12}$$

where μ, V and δ respectively denote the mean, log-variance and shift parameters and Z a standard $N(0,1)$ normal random variable. The price A of an Asian option struck at K is then given by the Black-Scholes-Merton formula with shifted forward and strike

$$A = D(0,T)\ \xi\ \left[(\mu - \delta)\ \Phi(\xi\, d_1) - (K - \delta)\ \Phi(\xi\, d_2)\right] \tag{13}$$

where

$$d_1 = \frac{\ln\left(\frac{\mu-\delta}{K-\delta}\right) + \frac{V}{2}}{\sqrt{V}}, \qquad d_2 = d_1 - \sqrt{V}\ , \tag{14}$$

$D(0,T)$ the discount factor to settlement of the Asian option, ξ the option parity and Φ the standard cumulative Gaussian distribution function.

3 Local Volatility.

3.1 Local volatility specification.

In order to capture volatility smile, we extend the deterministic volatility model to local volatility dynamics. Since the two-factor Cheyette futures model is Markovian in five state variables, the local volatility ought to have deterministic dependence in these. We also wish to preserve the covariance and volatility ratio structures of the backbone model in order to clearly separate the relative evolution of the different parts of the commodity futures curve from the local (and later local-stochastic) volatility dynamics.

The above considerations make it natural to equip the model with a multiplicative local volatility function A, common to the two volatility processes as follows

$$h_t^1 = A(t, \mathbf{x}_t, \mathbf{y}_t)\, \sigma_1(t) \qquad h_t^2 = A(t, \mathbf{x}_t, \mathbf{y}_t)\, \sigma_2(t) \tag{15}$$

with $\sigma_1(t)$ and $\sigma_2(t)$ are the same deterministic volatility functions as in the backbone model. In order to be consistent with the implied volatility interpolation used to value vanillas, we choose to use a Dupire local volatility, i.e. a non-parametric volatility function (cfr. [4]). The market inputs to the local volatility function calibration are the implied volatilities of liquidly-traded European options on futures. Different commodity futures are essentially different underlyings and only one option with a market-specified expiry per future is liquidly traded.

In the classical, single-underlying (e.g. FX, equities, ...), Dupire local volatility calibration procedure, the local volatility function is calibrated to an implied volatility surface obtained from time and strike interpolation of a set of liquid market-implied volatilities with fixed tenors. In contrast to the single-underlying case, we have a set commodity future underlyings each of which with a single market-implied marginal.

Since a Dupire local volatility requires a reasonably smooth implied volatility surface to be calibrated to, our approach will be to generate, for each future in turn, an early-expiry model-implied volatility surface that will be consistent with the available market-implied marginal for that given future. The generated early-expiry surfaces will then be used to calibrate a Dupire local volatility function for each future in turn, in order of increasing maturity. More specifically, if we denote by $T_1, ..., T_N$ the futures maturities in increasing order and $T_1^*, ..., T_N^*$ the corresponding option on futures expiries, the local volatility process has the form

$$A(t, \mathbf{x}_t, \mathbf{y}_t) = f_D\left(t, \log\left(\frac{F(t, T_i)}{F(0, T_i)}\right)\right) \qquad t \in \left]T_{i-1}^*, T_i^*\right] \tag{16}$$

where the dependency of $F(t, T_i)$ in $\mathbf{x}_t$ and $\mathbf{y}_t$ is given by (4). The Dupire local volatility function f can be expressed in terms of the early-expiry futures implied volatilities $\Sigma(t, z)$ as

$$f_D^2(t, z) = \frac{\Sigma^2 + 2\,\Sigma\, t\, \partial_t \Sigma}{1 - 2\, z\, \frac{\partial_z \Sigma}{\Sigma} + \Sigma\, t\, \partial_z^2 \Sigma + d_1\, d_2\, t\, \left(\partial_z \Sigma\right)^2}\; \frac{1}{\sigma_{t,T_i}^2} \tag{17}$$

where

$$
\begin{aligned}
z &= \log\left(\frac{F(t,T_i)}{F(0,T_i)}\right) \quad , \\
d_1 &= \frac{-z}{\Sigma\sqrt{t}} + \frac{1}{2}\Sigma\sqrt{t} \quad , \\
d_2 &= d_1 - \Sigma\sqrt{t}
\end{aligned}
\tag{18}
$$

and

$$
\sigma_{t,T_i} = \left[g_1^2(t,T_i)\,\sigma_1^2(t) + g_2^2(t,T_i)\,\sigma_2^2(t) + 2\,g_1(t,T_i)\,g_2(t,T_i)\,\sigma_1(t)\,\rho_{1,2}(t)\,\sigma_2(t)\right]^{1/2} \tag{19}
$$

is the instantaneous log-volatility of the future $F(t,T_i)$ in the backbone deterministic volatility model.

3.2 Numerical calibration procedure.

The local volatility calibration procedure will involve a Monte-Carlo diffusion of the state variables up to the penultimate calibration expiry.
The first step in the calibration is to mark the parameters of the deterministic volatility backbone model and calibrate it to ATM future implied volatilities exactly as described in section 2.2. As already mentioned above, the local volatility factor being multiplicative and common to the two diffusive driving variables preserve the instantaneous correlations and instantaneous volatility ratios of the future contracts so the parameter marking procedure of the backbone model still holds in the local volatility extension. Once the backbone model is calibrated, the local volatility is calibrated as follows:

- The i-th ($i = 1, ..., N$) local volatility grid spans the time interval $]T^*_{i-1}, T^*_i]$, with $T^*_0 = 0$. On each local volatility grid, we write the early-expiry volatilities as

$$
\Sigma_i(t,z) = \Sigma^{\det}(t,T_i) + \delta_i(t,z) \tag{20}
$$

  where $\delta(t,z)$ is the smile volatility offset to the terminal volatility of the deterministic volatility backbone model, given by (7).

- We start the calibration of the local volatility model with the first local volatility grid, that is the time interval $]0, T^*_1]$. Smile offsets $\delta(T^*_1, z)$ for options on the front-line future are obtained from market quotes (using the same strike-direction interpolation as that used to value vanillas) whereas we choose $\delta(0,z) = 0$ as initial offsets. We then time-interpolate the offsets $\delta(t,z)$ between 0 and T^*_1, hence constructing the early-expiry surface $\Sigma(t,z)$.
  From the early-expiry surface, we compute local volatilities on the first grid using the Dupire formula (17). Once local volatilities have been computed, the model is fully calibrated in the first time interval and we perform a Monte-Carlo diffusion of the five state variables $\mathbf{x}$ and $\mathbf{y}$ up to T^*_1. The purpose of the Monte-Carlo diffusion is to compute the early expiry volatilities $\Sigma_2(T^*_1, z)$ of options on the second future contract expiring at T^*_1. We can now move on to the next local volatility grid.

- We repeat the previous step for all the time intervals in turn but unlike in the case of the first grid, for $i > 1$, initial smile offsets $\delta_i(T^*_{i-1}, z)$ are obtained from the early-expiry volatilities $\Sigma_i(T^*_{i-1}, z)$ computed by means of Monte-Carlo diffusion over the previous grid:

$$\delta_i(T^*_{i-1}, z) = \Sigma_i(T^*_{i-1}, z) - \Sigma^{\text{det}}(T^*_{i-1}, T_i) \tag{21}$$

- Finally, the last grid does not require any Monte-Carlo diffusion since its smile offsets construction only requires the early-expiry volatilities $\delta_N(T^*_{N-1}, z)$ calculated by means of Monte-Carlo diffusion over the penultimate grid.

3.3 Derivative pricing in the local volatility model.

Once the model is fully calibrated, that is when the backbone model has been calibrated and the local volatilities have been computed to the last model calibration expiry following the procedure outlined in the previous section, a Monte-Carlo diffusion of the five state variables can be performed and, at any future time, the value of the state variables on any Monte-Carlo paths gives access via (4) to all the values of the simulated commodity futures. We can therefore perform Monte-Carlo pricing of commodity derivatives using the local volatility model, subject to the general limitations of Monte-Carlo techniques.

4 Local-stochastic volatility.

4.1 Stochastic volatility specification.

We now extend the local volatility model to mixed local-stochastic volatility dynamics by equipping it with a common multiplicative exponential Ornstein-Uhlenbeck stochastic volatility function:

$$\frac{dF(t,T)}{F(t,T)} = e^{Y_t}\, A(t, \mathbf{x}_t, \mathbf{y}_t) \left(\frac{g_1(T)}{g_1(t)}\, \sigma_1(t)\, dW^1_t + \frac{g_2(T)}{g_2(t)}\, \sigma_2(t)\, dW^2_t \right) \tag{22}$$

with

$$dY_t = -\lambda(t)\, Y_t\, dt + \nu(t)\, dW^3_t \tag{23}$$

The choice of instantaneous correlations between dW^3 and the Brownian drivers of the commodity curve will be discussed in the next section. Without loss of generality, we can choose $Y_0 = 0$ since a non-zero value is simply absorbed into the local volatility as a deterministic shift. For the same reason, there is no need for a mean-reversion level.

4.2 Stochastic volatility parameters calibration.

In order to calibrate a local-stochastic volatility model, we must first estimate or calibrate the parameters of the stochastic volatility process and then as a second step calibrate the local volatility function in presence of the stochastic volatility component. Note that since we use a Dupire local volatility specification, the local volatility will in principle allow the model to hit the target market implied volatilities exacty, regardless of the values of the stochastic volatility parameters.

This is an important feature since it means that the stochastic volatility parameters are genuine exotic parameters whose values do not affect one-dimensional marginals of the commodity futures and therefore vanilla prices. We now concentrate on the estimation of the stochastic volatility parameters. In general, we try to estimate model parameters from market-implied information as much as possible and then resort to historics or ad-hoc marking to assign values to the remaining parameters in the model. One of the popular methods (in particular in FX) to calibrate the stochastic volatilty parameters of an LSV model is to first calibrate the term-structure of stochastic volatility parameters so that the pure stochastic volatility model delivers market implied volatilities at a small number of select strikes at each expiry and then re-scale those pre-calibrated stochastic volatility parameters by a scaling factor $\theta \in [0, 1]$. This is the approach we adopt for the local-stochastic Cheyette model. First, we turn to the calibration of the pure stochastic volatility model.
From the standpoint of the way stochastic volatility generates smile/skew, each underlying commodity future $F(t, T)$ evolves with the effective one-factor driver

$$d\tilde{W}_t = \frac{g_1(t, T_i)\,\sigma_1(t)}{\tilde{\sigma}(t)}\,dW_t^1 + \frac{g_2(t, T_i)\,\sigma_2(t)}{\tilde{\sigma}(t)}\,dW_t^2 \qquad t \in \left]T_{i-1}^*, T_i^*\right] \tag{24}$$

where

$$\tilde{\sigma} = \sqrt{g_1^2(t, T_i)\,\sigma_1^2(t) + g_2^2(t, T_i)\,\sigma_2^2(t) + 2\,g_1(t, T_i)\,g_2(t, T_i)\,\sigma_1(t)\,\sigma_2(t)\,\rho_{1,2}(t)} \tag{25}$$

and

$$\left\langle d\tilde{W}_t, dW_t^3 \right\rangle = \tilde{\rho}(t)\,dt \tag{26}$$

In order to calibrate the pure stochastic volatility parameters, we compute the values of ν and $\tilde{\rho}$ such that the model-delivered implied volatilities match the reference "risk-reversal" and "butterfly"

$$\begin{aligned} \text{RR} &= \Sigma|_{d=d_{OOTM}} - \Sigma|_{d=-d_{OOTM}} \\ \text{BF} &= \frac{1}{2}\left(\Sigma|_{d=d_{OOTM}} - \Sigma|_{d=-d_{OOTM}}\right) - \Sigma|_{d=-d_{ATM}} \end{aligned} \tag{27}$$

where d_{OOTM} and d_{ATM} respectively correspond to reference "out-of-the-money" and "at-the-money" points. Note that the stochastic volatility mean-reversion speed λ is marked by the trading desk before ν and $\tilde{\rho}$ are calibrated.
Once the above stochastic volatility parameters calibration has been carried out and before the local volatility is calibrated, the instantaneous volatility-of-volatility ν and correlation $\tilde{\rho}$ are re-scaled by the exotic parameter $\theta \in [0, 1]$. Up to numerical and Monte-Carlo errors, target market vanilla prices in the fully-calibrated local-stochastic model do not depend on the value of the exotic parameter.

4.3 Instantaneous correlations

The instantaneous correlation between the stochastic volatility driver and the equivalent one-factor driver of the underlying needs to be projected onto the two factors driving the commodity futures. We have

$$\tilde{\sigma}(t)\,\tilde{\rho}(t) = g_1(t, T_i)\,\sigma_1(t)\,\rho_{1,3}(t) + g_2(t, T_i)\,\sigma_2(t)\,\rho_{2,3}(t) \qquad t \in \left]T_{i-1}^*, T_i^*\right] \tag{28}$$

We distribute $\tilde{\rho}$ into $\rho_{1,3}$ and $\rho_{2,3}$ so that the above constraint is satisfied and the semi-definite-positive character of the resulting $(\rho_{1,2}, \rho_{1,3}, \rho_{2,3})$ correlation matrix is sufficiently robust.
Assuming that $\alpha > 1$ and the first Cheyette factor drives the long end of the commodity futures curve, we choose to couple the first Cheyette factor to the stochastic volatility driver directly with correlation $\tilde{\rho}$ as the short-term dynamics of the model is dominated by the local volatility. The correlation between the stochastic volatility driver and the second Cheyette factor then follows from (28) so that

$$\begin{aligned} \rho_{1,3} &= \tilde{\rho} \\ \rho_{2,3} &= \frac{\tilde{\rho}}{\alpha\, e^{-\beta\,(T_i-t)}} \left(\sqrt{1+\alpha^2\, e^{-2\beta\,(T_i-t)} + 2\alpha\, e^{-\beta\,(T_i-t)}\,\rho_{1,2}} - 1 \right) \end{aligned} \quad (29)$$

where α and β are respectively the Cheyette instantaneous volatility ratio and mean-reversion spread as defined in section 2.2 .

4.4 Local volatility calibration in presence of stochastic volatility

Owing to the work of Gyongy [5], Derman and Kani [6], it is well known that in presence of stochastic volatility, the local voltility function that is going to result in our model delivering the market implied volatilities, satisfies

$$A^2(t, \mathbf{x}_t, \mathbf{y}_t)\, E[e^{2Y_t} | F(t,T_i)] = f_D^2 \left(t, \log\left(\frac{F(t,T_i)}{F(0,T_i)} \right) \right) \qquad t \in \left]T^*_{i-1}, T^*_i\right] \quad (30)$$

We efficiently compute the above conditional expectation within our Monte-Carlo framework at every local volatility calibration time step to calibrate the local volatility so that we re-price the benchmark target vanilla options accurately.

4.5 Derivative pricing in the local-stochastic volatility model.

As for the pure local volatility model, commodity derivatives are priced and risk-managed through Monte-Carlo diffusion of the full calibrated local-stochastic volatility model, subject to the general limitations of Monte-Carlo techniques.

5 References

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[3] Turnbull, S., Wakeman, L. (1991). "A Quick Algorithm for Pricing European Average Options". *Journal of Financial and Quantitative Analysis, 26, 377-389.*

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[5] Gyongy, I. (1986). "Mimicking the one-dimensional marginal distributions of processes having an Ito differential." *Probab. Theory Related Fields 71, 501-516.*

[6] Derman, E. and Kani, I. (1994). "Riding on a smile." *Risk 7, 32-39.*

A Dated Brent and other products based on averaged-price futures in the Cheyette Commodity Model.

For products based on average-price futures, e.g. Dated Brent and most Platts products, there is a lack (or absence) of liquidity of options on average-price futures. However, the above products are almost always closely related to future-based products with liquid European option markets.