Implied Vol Structure For LSV Model

HypHyp Case

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Outline

Stochastic Volatility Case

HypHyp model without local vol

$$
\begin{aligned}
dX_t &= \sigma g(y_t)\, dZ_t, \qquad & g(y) &= y + \sqrt{1+y^2} \\
dy_t &= -k y_t dt + \eta dY_t, \qquad & \eta &= \alpha\sqrt{2k}
\end{aligned}
$$

Stochastic volatility can be split in two parts:

- First is fully correlated with spot
- Second is uncorrelated with spot

$$
\begin{aligned}
dy_t = -k y_t dt + \rho\eta dZ_t & \qquad\qquad (first) \\
+ \eta\sqrt{1-\rho^2} dW_t & \qquad\qquad (second)
\end{aligned}
$$

We will see that:

- $\rho\eta$ defines skew of implied volatility
- $\eta\sqrt{1-\rho^2}$ - convexity of implied volatility

Stochastic Volatility Case

Implied volatility can be approximated by three components:

$$\sigma_{\text{imp}}(K, T)^2 \approx$$
$$\approx \sigma_{\text{ATM}}(T)^2 + \text{Skew}(T) \times (K - F_T) + \text{Convexity}(T) \times (K - F_T)^2$$

$\sigma_{\text{ATM}}(T)$ - ATM volatility curve
Skew(T) - linear component of strike
Convexity(T) - quadratic component of strike

Stochastic Volatility Case

This form is close to FX conventions in representation of main characteristics of implied vol. using standard instruments ATM, RR, BF

- ATM volatility - volatility level
- RR Volatility - skew
- BF Volatility - convexity

General properties for any SV model

Implied volatility smiles flattening with maturity T i.e.

- Skew(T)$\rightarrow$ 0 as T$\rightarrow \infty$
- Convexity(T)$\rightarrow$ 0 as T$\rightarrow \infty$
- Mean reversion parameter k controls speed of convergence to zero

Expansion for HypHyp Case

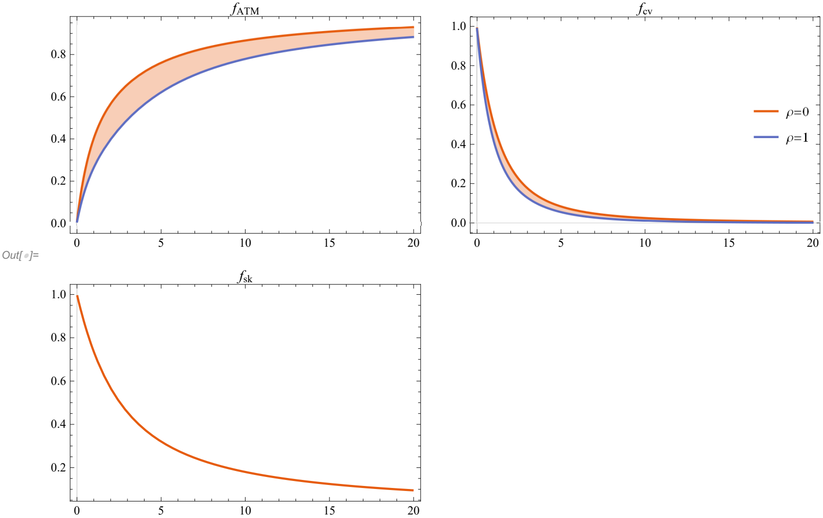
$$\sigma_{\mathrm{ATM}}(T)^2 = \sigma^2 + \frac{1}{k}\eta^2\sigma^2 f_{\mathrm{ATM}}(kT,\rho),$$
$$f_{\mathrm{ATM}}(0,\rho) = 0, f_{\mathrm{ATM}}(\infty,\rho) = 1$$

$$\mathrm{Skew}(T) = \eta\rho\sigma f_{\mathrm{sk}}(kT),$$
$$f_{\mathrm{sk}}(0) = 1, f_{\mathrm{sk}}(t) \sim \frac{2}{t} \text{ as } t \to \infty$$

$$\mathrm{Convexity}(T) = \frac{1}{4}\eta^2\left(\frac{1}{3} + 1 - \rho^2\right) f_{\mathrm{cv}}[kT,\rho]$$
$$f_{\mathrm{cv}}(0,\rho) = 1, f_{\mathrm{cv}}(t,\rho) \sim 4\frac{1-\rho^2}{\frac{1}{3}+1-\rho^2}\frac{1}{t^2} \text{ as } t \to \infty$$

Time structure of f_*

Dependence on ρ is week



Implied volatility

Previous formulas have been for ${\sigma_{\text{imp}}}^2$ they can be rewritten in terms of σ_{imp}

$$\sigma_{\text{imp}}(K, T) \approx \\ \sigma_{\text{ATM}}(T) + \text{Skew}(T) \times (K - F_T) + \text{Convexity}(T) \times (K - F_T)^2$$

$$\sigma_{\text{ATM}}(T) = \sigma + \sigma \frac{1}{2} \frac{1}{k} \eta^2 f_{\text{ATM}}(kT, \rho)$$

$$\text{Skew}(T) = \frac{1}{2} \eta \rho \; f_{\text{sk}}(kT)$$

$$\text{Convexity}(T) = \frac{1}{4} \frac{\eta^2}{\sigma} \left(1 - \rho^2 - \frac{1}{3}\right) \tilde{f}_{\text{cv}}[kT, \rho]$$

Implied volatility

In terms of original parameters with $\eta = \alpha\sqrt{2k}$

$$\sigma_{\mathrm{ATM}}(T) = \sigma + \sigma\alpha^2 f_{\mathrm{ATM}}(kT, \rho)$$

$$\mathrm{Skew}(T) = \sqrt{\frac{k}{2}}\alpha\rho \; f_{\mathrm{sk}}(kT)$$

$$\mathrm{Convexity}(T) = \frac{k}{2}\frac{\alpha^2}{\sigma}\left(1 - \rho^2 - \frac{1}{3}\right)\tilde{f}_{\mathrm{cv}}[kT, \rho]$$

Summary for HypHyp case

- Skew(T)$\rightarrow$ 0 as T$\rightarrow \infty$
- Convexity(T)$\rightarrow$ 0 as T$\rightarrow \infty$
- k defines speed of convergence to zero
- $\eta\rho$ defines skew
- $\frac{\eta^2}{\sigma}\left(1-\rho^2\right)$ defines convexity

Issue with SV model

Problem

Long term skew is close to zero!

Fix

Can be fixed by decreasing mean reversion parameter k and increasing vol of vol η.

But this will have impact on short end of skew curve.
In other words SV model doesn't have enough parameters to fit well short and long parts of skew curve.

Real Fix

Adding local volatility component

Rule of thumb for local volatility model

Take LV model with local volatility function $\sigma_{\text{loc}}(x)$

$$dX_t = \sigma_{\text{loc}}\left(X_t\right) dZ_t \tag{1}$$

Define skew ATM of local volatility as

$$\text{skew}_{\text{loc}}(T) = \partial_x \sigma_{\text{loc}}(x)|_{x=F_T} \tag{2}$$

Then skew of implied volatility is approximately half of skew of local volatility

$$\text{skew}(T) = \partial_x \sigma_{\text{imp}}(x)|_{x=F_T} \approx \frac{1}{2}\text{skew}_{\text{loc}}(T) \tag{3}$$

HypHyp LV

$$dX_t = \sigma f\left(X_t\right) dZ_t$$

$$\mathrm{Skew}_{\mathrm{loc}} = \sigma f'\left(x_0\right) = \frac{1}{2}\sigma\beta$$

$$\mathrm{Skew}_{\mathrm{imp}} \approx \frac{1}{4}\sigma\beta$$

$$f(x) = 1 + \frac{1}{2}\beta\left(\left(x - x_0\right) - 1 + \sqrt{1 + \left(x - x_0\right)^2}\right), \mathrm{for}\,\beta \geq 0$$

$$f(-\infty) = 1 - \frac{1}{2}\beta, \quad f(x) \sim \beta x \text{ as } x \to \infty$$

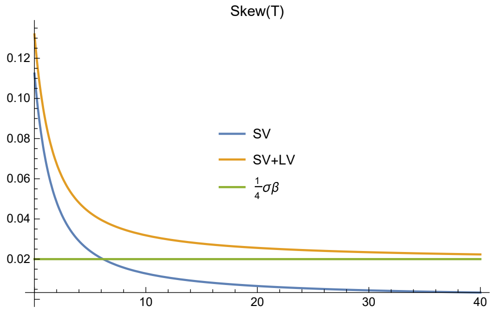
Combining SV and LV

$$dX_t = \sigma f\left(X_t\right) g\left(y_t\right) dZ_t$$
$$dy_t = -k y_t dt + \eta dY_t$$

By adding LV component to SV we add constant skew from LV component to *all smiles* of SV model

$$\text{Skew}(T) = \frac{1}{2}\eta\rho\ f_{\text{sk}}(kT) + \frac{1}{4}\sigma\beta \tag{4}$$

Out[•]=



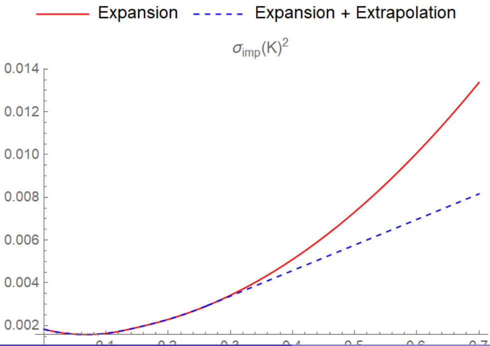
Wings of smile and implied volatility asymptotic

Problem

Implied volatility $\sigma_{\text{imp}}(T, K)^2$ is expressed in terms power series of K. Smile wings can be significantly overestimated or unstable for model parameters.

Solution

Find asymptotic of wings and extrapolate the expansion as linear from the point where skew given by expansion becomes larger than asymptotic skew.



Instead of using LSV model to generate non arbitrage implied volatility we could build implied volatility using results from

"Game of Vols" Peter Carr, Gregory Pelts

Arbitrage-free IV can always be generated by specifying a convex distortion function. They also propose a parametrization of this convex distortion function and an interpolation method in the time dimension. The option prices generated by this method are free of any arbitrage - the calendar arbitrage, the strike spread arbitrage, and the butterfly spread arbitrage.

Technical Details

A. Antonov, M. Konikov, M. Spector
A New Arbitrage-Free Parametric Volatility Surface
Available at SSRN:https://papers.ssrn.com/sol3/papers.cfm?abstract_id=3403708

Peter Carr and Gregory Pelts
Game of Vols (May 12, 2015).
Available at SSRN: https://ssrn.com/abstract=3422540 or http://dx.doi.org/10.2139/ssrn.3422540

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Price expansion for LSV model

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HypHyp asymptotic