Analytical approach to Cheyette local volatility model

Alexander Gairat
Quantitative Research *

Local volatility formula for rolling maturity forward rate

In this paper, we present an analytical approach to valuing local volatility for the Cheyette model when implied volatilities for rolling forward rates are given.

A Dupire-like formula was derived in [2], [4], and [3]. However, some terms in these formulas are not valued analytically and require numerical calculation, typically using a Monte Carlo approach.

Cheyette local volatility model

The Cheyette local volatility model is defined by the following stochastic differential equations (SDEs):

$$dx_t = (y_t - \mu x_t)\, dt + \sigma_r\, (x_t, y_t, t)\, dW_t \tag{1}$$

$$dy_t = \left(\sigma_r\, (x_t, y_t, t)^{\,2} - 2\mu y_t\right) dt \tag{2}$$

$$x_0 = y_0 = 0 \tag{3}$$

local volatility σ_r is a function of time and process state (x_t, y_t)
short rate in this model is defined as

$$r_t = f(0,t) + x_t \tag{4}$$

where $f(0,t)$ is instantaneous forward rate defined by initial yield curve. To simplify formulas we will assume that

$$f(0,t) = 0 \tag{5}$$

Rolling forward rate

We are given implied volatilities $\sigma_{\text{imp}}(T,K)$ for rolling forward rates $F_T(T+\Delta\mathrm{T})$ in terms of Bachelier formula. In risk neutral measure value of call option on forward rate with maturity T and strike K is

$$C(T,K) = Ee^{-\int_0^{T+\Delta\mathrm{T}} r_t\,dt}\left(F_T(T,T+\Delta\mathrm{T}) - K\right)_+ \quad (6)$$

$$= \mathrm{BH}\left(F_0(T,T+\Delta\mathrm{T}), K, T, \sigma_{\mathrm{imp}}(T,K)\right) \quad (7)$$

See details in Cheyette1F_withSmile.pdf

To simplify for small $\Delta\mathrm{T}$ we can replace $F_T(T+\Delta\mathrm{T})$ by instantaneous forward rate $f_t(T)$ for time T observed at time t

$$f_t(T) \approx F_t(T,T+\Delta\mathrm{T}) \quad (8)$$

And hence we will consider call options on instantaneous forward rate

$$C(T,K) = Ee^{-\int_0^{T} r_t\,dt}\left(f_T(T) - K\right)_+ \quad (9)$$

$$= \mathrm{BH}\left(f_0(T), K, T, \sigma_{\mathrm{imp}}(T,K)\right) \quad (10)$$

Fixed horizon forward rate

For T measure where numeraire is

$$N_T = e^{\int_0^T r_t\,dt} \quad (11)$$

Denote by E^T expectation under measure associated with numeraire N_T

$$Ee^{-\int_0^{T} r_t\,dt}\left(f_T(T) - K\right)_+ = E^T\left(f_T(T) - K\right)_+ \quad (12)$$

and $f_t(T)$ is martingale in this T measure i.e. we have SDE without drift

$$df_t(T) = \sigma_{\mathrm{loc}}\left(t, f_t(T)\right) dZ^T{}_t \quad (13)$$

If we would have values of options with maturities $t \leq T$ on $f_t(T)$

$$C^T(t,K) = Ee^{-\int_0^{T} r_t\,dt}\left(f_t(T) - K\right)_+ = E^T\left(f_t(T) - K\right)_+ \quad (14)$$

$$= \mathrm{BH}\left(f_0(T), K, T, \sigma^T{}_{\mathrm{imp}}(t,K)\right) \quad (15)$$

where $C^T(t,K)$ - is the value of option on non rolling forward rate with fixed horizon T and $C(T,K) = C^T(T,K)$ price for rolling forward case.

$\sigma^T{}_{\mathrm{imp}}(t,K)$ are implied volatilities for these options. Upper index T indicates that these implied volatilities are related to $f_t(T)$.

Then we could just apply Dupire formula to $\sigma^T{}_{\mathrm{imp}}(t,K)$ to calculate $\sigma_{\mathrm{loc}}(t,K)$.

$$\sigma_{\mathrm{loc}}(t,K)^2 = 2\frac{\partial_t C^T}{\partial_{K,K} C^T} \quad (16)$$

Note that

$$\partial_{K,K} C^T(T,K) = \partial_{K,K} C(T,K) \quad (17)$$

Hence we need to find only nominator $\partial_t C^T$

Nominator valuation

$$\partial_t C^T(t,K)|_{t=T} = \partial_t C^{T+dT}(t,K)|_{t=T} + O(dT) \tag{18}$$
$$= \frac{C^{T+dT}(T+dT,K) - C^{T+dT}(T,K)}{dT} + O(dT) \tag{19}$$
$$= \frac{C^{T+dT}(T+dT,K) - C^T(T,K) + C^T(T,K) - C^{T+dT}(T,K)}{dT} + O(dT) \tag{20}$$
$$= \frac{C(T+dT,K) - C(T,K)}{dT} - \frac{C^{T+dT}(T,K) - C^T(T,K)}{dT} \tag{21}$$
$$= \partial_T C(T,K) - A + O(dT) \tag{22}$$

We get that numerator has standard term $\partial_T C(T,K)$ and extra term A

$$A = \frac{C^{T+dT}(T,K) - C^T(T,K)}{dT} \tag{23}$$

Let's calculate A

$$A = \frac{Ee^{-\int_0^{T+dT} r_t\,dt}\left(f_T(T+dT) - K\right)_+ - Ee^{-\int_0^T r_t\,dt}\left(f_T(T) - K\right)_+}{dT} \tag{24}$$

$$Ee^{-\int_0^{T+dT} r_t\,dt}\left(f_T(T+dT) - K\right)_+ - Ee^{-\int_0^T r_t\,dt}\left(f_T(T) - K\right)_+ \tag{25}$$
$$= Ee^{-\int_0^{T+dT} r_t\,dt}\left(f_T(T) - K\right)_+ - Ee^{-\int_0^T r_t\,dt}\left(f_T(T) - K\right)_+ \tag{26}$$
$$+ Ee^{-\int_0^T r_t\,dt}\left(f_T(T+dT) - K\right)_+ - Ee^{-\int_0^T r_t\,dt}\left(f_T(T) - K\right)_+ + o(dT) \tag{27}$$
$$= E^T P(T,T+dT)\left(f_T(T) - K\right)_+ - E^T\left(f_T(T) - K\right)_+ \tag{28}$$
$$+ E^T P(T,T)\left(f_T(T+dT) - K\right)_+ - E^T\left(f_T(T) - K\right)_+ + o(dT) \tag{29}$$
$$= E^T\left(P(T,T+dT) - 1\right)\left(f_T(T) - K\right)_+ \tag{30}$$
$$+ E^T\left(f_T(T+dT) - K\right)_+ - E^T\left(f_T(T) - K\right)_+ + o(dT) \tag{31}$$

to estimate the term

$$E^T\left(f_T(T+dT) - K\right)_+ - E^T\left(f_T(T) - K\right)_+ \tag{32}$$

recall that

$$G(t,T) = \int_t^T e^{-(u-t)\mu}\,du = \frac{1 - e^{-(T-t)\mu}}{\mu} \tag{33}$$

$$f(t,T) = f(0,T) + e^{-\mu(T-t)}\left(x_t + G(t,T)y_t\right) \tag{34}$$

Hence

$$
\begin{aligned}
f_T(T+dT) &= e^{-\mu dT}\left(x_T + dTy_T + o(dT)\right) && (35)\\
&= f_T(T) + dTy_T - \mu dTx_T + o(dT) && (36)\\
&= (1-\mu dT)f_T(T) + dTy_T + o(dT) && (37)
\end{aligned}
$$

We use notation $f_T = f_T(T)$ thus we get (all expressions with accuracy $o(dT)$) for the term

$$
\begin{aligned}
&E^T\left(f_T(T+dT)-K\right)_+ - E^T\left(f_T-K\right)_+ && (38)\\
&\quad= E^T\left((1-\mu dT)f_T + dTy_T - K\right)_+ - E^T\left(f_T-K\right)_+ && (39)\\
&\quad= (1-\mu dT)E^T\left(f_T - \frac{K-dTy_T}{1-\mu dT}\right)_+ - E^T\left(f_T-K\right)_+ && (40)\\
&\quad= (1-\mu dT)E^T\left(f_T - \left(K - dT\left(y_T-\mu K\right)\right)\right)_+ - E^T\left(f_T-K\right)_+ && (41)\\
&\quad= -dTE^T\left(y_T-\mu K\right)\partial_K\left(f_T-K\right)_+ - \mu dTE^T\left(f_T-K\right)_+ && (42)\\
&\quad= dTE^T\left(y_T-\mu K\right)\theta\left(f_T-K\right) - \mu dTE^T\left(f_T-K\right)_+ && (43)
\end{aligned}
$$

where $\theta(x)$ is step function.

For the first term we have

$$
\begin{aligned}
&E^T(P(T,T+dT)-1)\left(f_T-K\right)_+ P(T,T+dT) && (44)\\
&\quad= e^{-G(T,T+dT)x_T - \frac{1}{2}G(T,T+dT)^2 y_T} && (45)\\
&\quad= e^{-dTx_T - \frac{1}{2}dT^2 y_T} = 1 - dTx_T + o(dT) && (46)
\end{aligned}
$$

So

$$
E^T(P(T,T+dT)-1)\left(f_T-K\right)_+ = -dTE^T f_T\left(f_T-K\right)_+ + o(dT) \qquad (47)
$$

So we get expression for A

$$
\begin{aligned}
A &= E^T\left(y_t-\mu K\right)\theta\left(f_T-K\right) - \mu E^T\left(f_T-K\right)_+ - E^T f_T\left(f_T-K\right)_+ && (48)\\
&= E^T y_t\theta\left(f_T-K\right) - E^T f_T\left(f_T-K\right)_+ && (49)\\
&\quad - \mu KE^T\theta\left(f_T-K\right) - \mu E^T\left(f_T-K\right)_+ && (50)\\
&= E^T y_t\theta\left(f_T-K\right) - E^T f_T\left(f_T-K\right)_+ + \mu K\partial_K C - \mu C && (51)
\end{aligned}
$$

Generic local volatility formula

Finally we have expression for local volatility

$$
\begin{aligned}
&\sigma_{\mathrm{loc}}(T,K)^2 = 2\frac{\partial_t C^T(t,K)|_{t=T}}{\partial_{K,K}C^T} = 2\frac{\partial_T C(T,K) - A}{\partial_{K,K}C} && (52)\\
&\quad= 2\frac{\partial_T C(T,K) + E^T f_T\left(f_T-K\right)_+ - E^T y_T\theta\left(f_T-K\right) + \mu\left(C - K\partial_K C\right)}{\partial_{K,K}C} && (53)
\end{aligned}
$$

Alternative derivation can be done using results from [2] or [4].

Note that for case of instantaneous forward rate

$$f_T = x_T \tag{54}$$

and

$$\sigma_r(x, y, T) = \sigma_{\text{loc}}(T, x) \tag{55}$$

Note that only the term

$$E^T y_T \theta\left(f_T - K\right) = E^T y_T \theta\left(x_T - K\right) \tag{56}$$

can not be calculated from implied distribution and depends on dynamic of process defined by local volatility.

Hence this expression is not explicit. One approach to valuate right side is based on Monte-Carlo approach (see [2], [4]).

Total variance approximation

To avoid Monte-Carlo valuation we will try to estimate the term

$$E^T y_T \theta\left(x_T - K\right) \tag{57}$$

analytically.

$$y_T = \int_0^T e^{-2\mu(T-t)} \sigma_{\text{loc}}\left(t, x_t\right)^2 dt \tag{58}$$

$$E^T\left(y_T | x_T = x\right) = E^T\left(\int_0^T e^{-2\mu(T-t)} \sigma_{\text{loc}}\left(t, x_t\right)^2 dt | x_T = x\right) \tag{59}$$

Note that from perturbation approach this term can be used as approximation of implied volatility

$$T\sigma_{\text{imp}}(T, K)^2 \approx E^T\left(\int_0^T e^{-2\mu(T-t)} \sigma_{\text{loc}}\left(t, x_t\right)^2 dt | x_T = x\right) \tag{60}$$

Hence we use approximation

$$E^T y_T \theta\left(x_T - K\right) = E^T T \sigma_{\text{imp}}\left(T, f_T\right)^2 \theta\left(f_T - K\right) \tag{61}$$

Now we have the following formula for local volatility

$$\sigma_{\text{loc}}(T, K)^2 = 2\frac{\partial_T C(T, K)}{\partial_{K,K} C} \tag{62}$$

$$+ 2\frac{E^T f_T\left(f_T - K\right)_+ - T E^T \sigma_{\text{imp}}\left(T, f_T\right)^2 \theta\left(f_T - K\right)}{\partial_{K,K} C} \tag{63}$$

$$+ 2\mu\frac{C - K\partial_K C}{\partial_{K,K} C} \tag{64}$$

Now all terms on the right side can be fully evaluated from implied distribution.

Lightweight form

To avoid numerical integration in previous formula we can simplify by taking expansion on skew of implied variance.

Assume that implied variance $w(T,K) = Tv(T,K)^2, v(T,K) = \sigma_{\text{imp}}(T,K)$ has a linear form.

$$w(T,x) = T\,(bx + w_0) \tag{65}$$

We will take first order terms in terms of skew parameter b. Then we will get that

$$\begin{aligned} B &= E^T f_T\,(f_T - K)_+ - T E^T \sigma_{\text{imp}}\,(T, f_T)\,^2 \theta\,(f_T - K) &(66)\\ &= \frac{1}{2} b t^2 w_0 p_0(x,t) + o(b) &(67)\end{aligned}$$

$$p_0(x,t) := \frac{e^{-\frac{x^2}{2tw_0}}}{\sqrt{2\pi}\sqrt{tw_0}} \tag{68}$$

or in terms of implied variance

$$B = \frac{1}{2}\; w(t,K)(w(t,K))' p_0(x,t) \tag{69}$$

Rewriting in terms of implied variance $w(T,K)$ and implied volatility $v(T,K)$ we get

$$\text{Dupire} = \frac{w_t}{\left(1 - \frac{(K - \text{S0}) v_k}{v}\right)^2 + tvv_{\text{kk}}} \tag{70}$$

$$\begin{aligned} \sigma_{\text{loc}}(T,K)^2 &= \text{Dupire} + \frac{ww_k + \mu\,(2w - xw_k)}{\left(1 - \frac{(K - \text{S0}) v_k}{v}\right)^2 + tvv_{\text{kk}}} &(71)\\ &= \text{Dupire} + \frac{2t^2 v^3 v_k + 2\mu\left(tv^2 - (K - \text{S0}) v v_k\right)}{\left(1 - \frac{(K - \text{S0}) v_k}{v}\right)^2 + tvv_{\text{kk}}} &(72)\end{aligned}$$

Our additional non trivial term to standard Dupire formula is

$$2v\frac{t^2 v^2 \partial_K v + \mu\,(tv - (K - \text{S0})\partial_K v)}{\left(1 - \frac{(K - \text{S0}) v_k}{v}\right)^2 + tvv_{\text{kk}}} \tag{73}$$

Perturbation

Ii is convenient to work in T forward measure.

$$dx_t = \left(y_t - (T-t)\sigma\left(x_t, t\right)^2\right) dt + \sigma\left(x_t, t\right) dW_t dy_t = \sigma_r\left(x_t, t\right)^2 dt \tag{74}$$

$$f(t,T) = f(0,T) + e^{-\mu(T-t)}\left(x_t + G(t,T)y_t\right) \tag{75}$$
$$f_t = f(t,T) - f(0,T) \tag{76}$$
$$x_t = f_t e^{\mu(T-t)} - G(t,T)y_t \tag{77}$$

in terms of forward rate

$$df_t = e^{-\mu(T-t)}\sigma\left(x_t, t\right) dW_t \tag{78}$$
$$dy_t = \left(\sigma\left(x_t, t\right)^2 - 2\mu y_t\right) dt \tag{79}$$

The last equation can be written in integral form

$$y_t = \int_0^t e^{-2\mu(t-u)}\sigma\left(x_u, u\right)^2 du \tag{80}$$

If we have solution for operator A of equation

$$-\partial_t P = AP \tag{81}$$

then solution for operator $A + \epsilon B$ can be represented as series

$$-\partial_t P = (A + \epsilon B)P \tag{82}$$

$$P_\epsilon = P_0 + \epsilon \int_0^T P_{0,t} B P_{t,T} dt + \epsilon^2 \int_0^T \int_0^{t_1} P_{0,t_1} B P_{t_1,t_2} B P_{t_2,T} dt ... \tag{83}$$

Not perturbed case

We take σ which defines total variance as implied variance i.e.

$$\sigma_{\text{imp}}(T,K)^2 T = \sigma^2 \int_0^T e^{-2\mu(T-t)}\, dt \tag{84}$$

$$df_t = e^{-\mu(T-t)}\sigma dW_t \tag{85}$$
$$dy_t = \left(\sigma^2 - 2\mu y_t\right) dt \tag{86}$$

$$-\partial_t C = \frac{1}{2}\sigma^2 \partial_{f,f} C + \left(\sigma^2 - 2\mu y\right)\partial_y C \tag{87}$$

$$y_T = y_t + \sigma^2 \int_0^{T-t} e^{-2\mu(T-t-u)}\, du = y_t + w_{t,T} \tag{88}$$

$$w_{t,T} = \sigma^2 \frac{1 - e^{-2(T-t)\mu}}{2\mu} \tag{89}$$

Value of call option is defined by Bachelier formula

$$C_{t,T} = \mathrm{BH}_{t,T}\left(f, K, w_{t,T}\right) \tag{90}$$

where $\mathrm{BH}_{t,T}(K, w)$ Bachelier price of call option at moment t with maturity T with variance $w_{t,T}$ with forward rate f.

Perturbation

Let' s consider local volatility case as perturbation of constant volatility case with $\sigma = \sigma_{\mathrm{imp}}(T, K)$

$$df_t = e^{-\mu(T-t)} \sigma\left(x_t, t\right) dW_t \tag{91}$$

$$dy_t = \left(\sigma\left(x_t, t\right)^2 - 2\mu y_t\right) dt \tag{92}$$

$$-\partial_t C = \frac{1}{2} e^{-2\mu(T-t)} \sigma(x,t)^2 \partial_{f,f} C + \left(\sigma(x,t)^2 - 2\mu y\right) \partial_y C \tag{93}$$

perturbation operator is

$$B_t = \frac{1}{2} e^{-2\mu(T-t)} \left(\sigma(x,t)^2 - \sigma^2\right) \partial_{f,f} + e^{-2\mu(T-t)} \left(\sigma(x,t)^2 - \sigma^2\right) \partial_y \tag{94}$$

Then for option value with maturity T $V_{0,T}$ defined by process with local volatility we have

$$V_{0,T} = C_{0,T} + \int_0^T P_{0,t} B C_{t,T} dt + o(B) \tag{95}$$

$$\left(\partial_{f,f} + \partial_y\right) C_{t,T} = p_{t,T} \tag{96}$$

$$p_{t,T}\left(f_t, f_T, y_t, y_T\right) = p\left(f_T - f_t, w_{t,T}\right) p(x,w) = \frac{1}{\sqrt{2\pi w}} e^{-\frac{x^2}{2w}} \tag{97}$$

Taking into account that $\sigma = \sigma_{\mathrm{imp}}(T, K)$ is implied volatility

$$V_{0,T} = C_{0,T} \tag{98}$$

and hence

$$\int_0^T P_{0,t}e^{-2\mu(T-t)}\left(\sigma(x,t)^2-\sigma^2\right)P_{t,T}dt=0 \tag{99}$$

Then

$$\int_0^T P_{0,t}e^{-2\mu(T-t)}\sigma\left(x_t,t\right)^2 P_{t,T}dt=\int_0^T P_{0,t}e^{-2\mu(T-t)}\sigma^2 P_{t,T}dt \tag{100}$$

$$=\sigma^2\int_0^T e^{-2\mu(T-t)}P_{0,t}P_{t,T}dt=\sigma^2 P_{0,T}\int_0^T e^{-2\mu(T-t)}\,dt \tag{101}$$

$$\sigma^2\int_0^T e^{-2\mu(T-t)}\,dt=\frac{1}{P_{0,T}}\int_0^T P_{0,t}e^{-2\mu(T-t)}\sigma\left(x_t,t\right)^2 P_{t,T}dt \tag{102}$$

in other words in first order approximation we have

$$\sigma^2\int_0^T e^{-2\mu(T-t)}\,dt=E^T\left(\int_0^T e^{-2\mu(T-t)}\sigma\left(x_t,t\right)^2 dt|x_T=K\right) \tag{103}$$

$$=E^T\left(\int_0^T e^{-2\mu(T-t)}\sigma\left(x_t,t\right)^2 dt|x_T=K\right) \tag{104}$$

$$=E^T\left(y_T|x_T=K\right) \tag{105}$$

We get as claimed

$$\sigma_{\text{imp}}(T,K)^2T=E^T\left(y_T|x_T=K\right) \tag{106}$$

Integration over implied distribution

We will use notations $v(x)$ - implied volatility for strike x, and $w(x)=tv(x)^2$ - corresponding implied variance.
Integration over implied distribution can be presented in form of implied volatility

$$\frac{E^T f_T\left(f_T-K\right)_+-E^T T\sigma_{\text{imp}}\left(T,f_T\right)^2\theta\left(f_T-K\right)}{p_0(K,T)}=\int_K^\infty p(x,K)F(x)dx \tag{107}$$

$$p_0(K,t)=\frac{1}{v(K)\sqrt{(2\pi t)}}e^{-\frac{K^2}{2tv(K)^2}} \tag{108}$$

$$p(x,K)=e^{\frac{\frac{K^2}{v(K)^2}-\frac{x^2}{v(x)^2}}{2t}}\frac{v(K)}{v(x)}\left(\left(1-x\left(v'(x)/\,v(x)\right)\right)^2+tv(x)v''(x)\right) \tag{109}$$

$$F(x) = x(x-K) - w(t,x) = x(x-K) - tv(t,x)^2 \tag{110}$$

To simplify notations assume that S0=0, in other words K is moneyness $K-\mathrm{S0}$, for case μ=0

$$\sigma_{\mathrm{loc}}{}^2 = \frac{\partial_t \left(tv^2\right) + 2\int_K^\infty p(x,K)F(x)dx}{\left(1-\frac{Kv_k}{v}\right)^2 + tvv_{\mathrm{kk}}} \tag{111}$$

$$= \frac{\partial_t \left(tv^2\right) + 2\int_K^\infty e^{\frac{K^2}{2tv(K)^2}-\frac{x^2}{2tv(x)^2}} \frac{v(K)}{v(x)} \mathrm{Denom}\left(x,v,v_x,v_{\mathrm{xx}}\right)F(x)dx}{\mathrm{Denom}\left(K,v,v_k,v_{\mathrm{kk}}\right)} \tag{112}$$

Or in other notations

$$p(x,t) = e^{-\frac{x^2}{2tv(x)^2}} \frac{1}{v(x)} \mathrm{Denom}\left(x,v,v_x,v_{\mathrm{xx}}\right) \tag{113}$$

$$\mathrm{Denom}\left(x,v,v_x,v_{\mathrm{xx}}\right) = \left(1-\frac{Kv_x}{v}\right)^2 + tvv_{\mathrm{xx}} \tag{114}$$

$$\sigma_{\mathrm{loc}}{}^2 = \frac{\partial_t \left(tv^2\right)}{\mathrm{Denom}\left(K,v,v_k,v_{\mathrm{kk}}\right)} + \frac{2}{p(K,t)} \int_K^\infty p(x,t)F(x)dx \tag{115}$$

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