# PFE Engine Documentation

September 3, 2020

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**TODO list:**

- (+)XCCY swaption: calculation algorithm
- risk metrics names passed into CompatibL?
- !!!IRS and CIRS: amortisation etc.
- integration: list of variables send to Zeus
- ()FX: USD always domestic? from .docx
- Watermark
- !!!IR/FX errors absolute as default?
- Asians: geometric mean?

# 1 Introduction

This document contains detailed description of pricing and simulation models using in PFE Engine system, integration with other systems and main business processes linked with the system. One can briefly describe architecture of the system as follow:

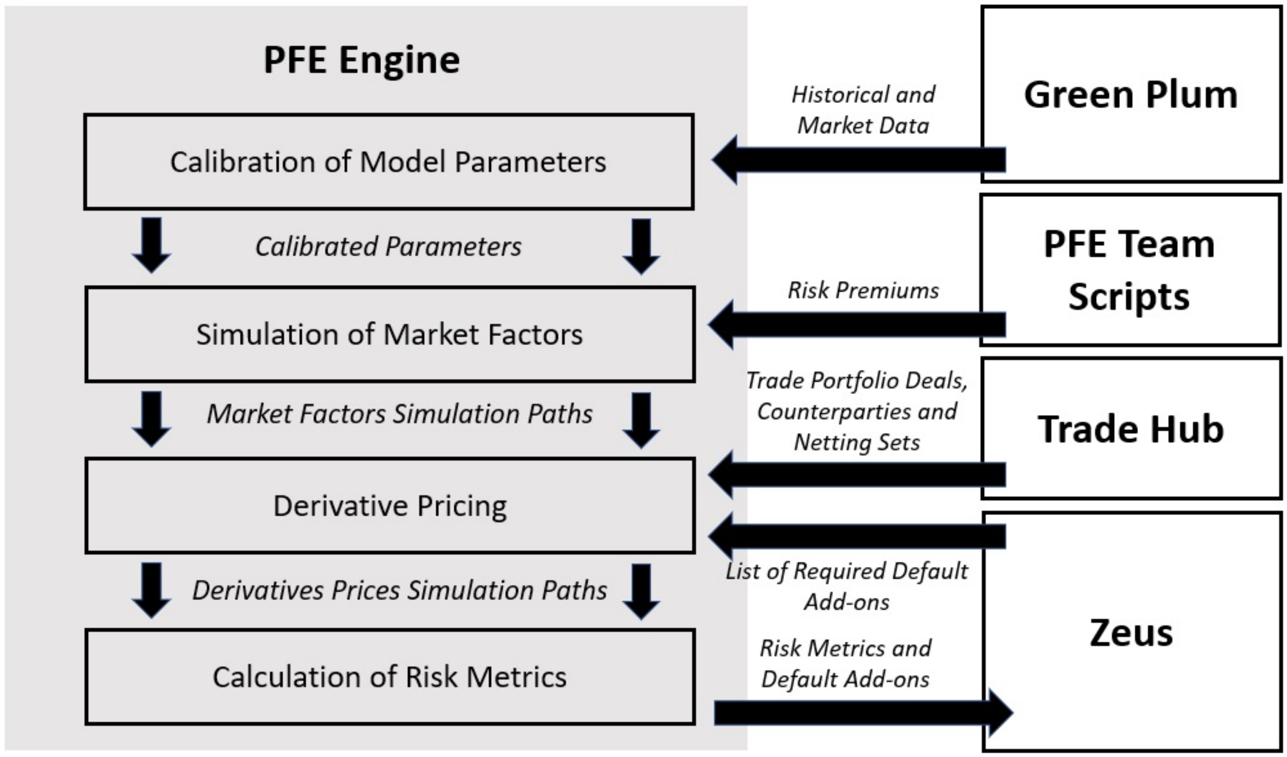


Figure 1: Simplified PFE Engine architecture

The main purpose of PFE Engine is calculation of counterparty risk metrics on portfolio/counterparty level subject to given netting sets. That calculus are made by Monte-Carlo simulations of risk factors in real world measure and analytical repricing of derivatives according to appropriate historical and market data. In general terms one can allocate all market factors into 5 groups: interest rates (IR), foreign exchange rates (FX), equity prices (EQ), commodity prices (CMDTY) and credit spreads (CR). For each of them there are related sections in "Simulation Models" and "Derivative Pricing" sections.

All interactions of PFE Engine with external system are described in Section 2. Simulation models in both real world and risk neutral measures, as well as calibration algorithms, are presented in Section 3. Section 5 contains pricing approaches and formulae for all derivatives supported by the system. In Section 6 one can find definitions of risk metrics used in PFE Engine and a brief description of usage of these metrics in business processes of the bank.

## 2 Data Sources and Integration

All interactions of PFE Engine with other bank's systems can be briefly described as follows:

- The system loads market and historical data for parameters estimates and calibrations from **GreenPlum** system.
- Parameters of derivatives to be priced as well as counterparty information are downloaded from **TradeHub**.
- After all deals are priced, the system loads information about netting sets from **Zeus** storage, calculates necessary risk metrics on counterparty/netting sets/portfolio levels and uploads them into Zeus.

Here is detailed and accurate IT integration scheme PFE Engine with above mentioned systems:

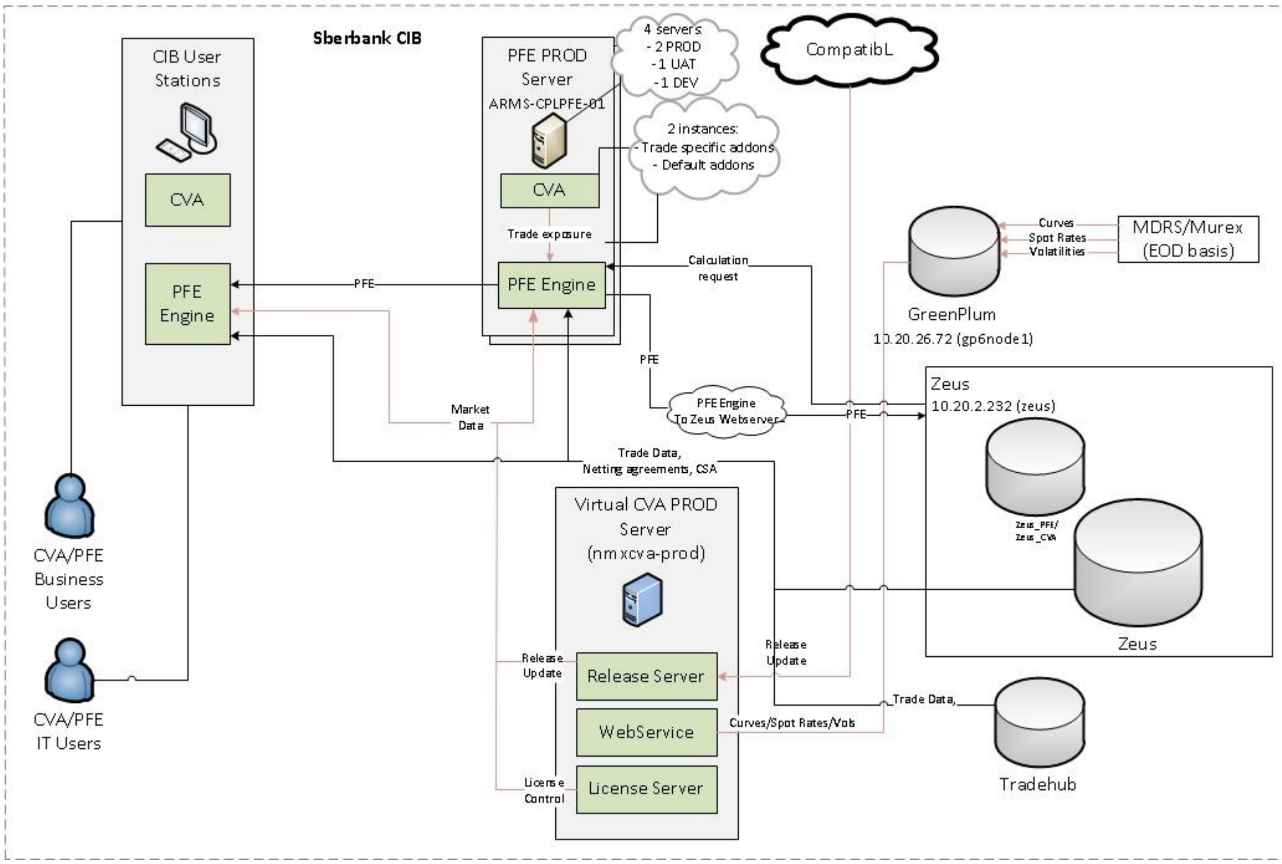


Figure 2: Scheme of IT integration of PFE Engine system

# 3 Simulation Models

## 3.1 Interest Rate Model: HW1F

### 3.1.1 Theoretical Background

Under the one factor Hull-White model (HW1F) the instantaneous short rate modelling as a combination of stochastic and deterministic components:

$$\begin{cases} r(u) = x(u) + \phi(u), \\ dx(t) = -\alpha x(t)dt + \sigma(t)d\widetilde{W}(t) \\ \qquad\;\; = \lambda_{IR}(t)\sigma(t)dt - \alpha x(t)dt + \sigma(t)dW(t), \end{cases}$$

where

- $\phi(t)$ is the deterministic component (**implicitly calibrated to the discount curve** at the martingale adjustment step; see calibration algorithm in Section 3.1.4).
- $\alpha$ is the **constant** mean-reversion speed parameter (**set manually**; see list of input parameters in Section 3.1.2).
- $\sigma(t)$ is the **piecewise** instantaneous IR volatility (**calibrated to swaption and/or cap/floor prices**; see calibration algorithm in Section 3.1.3).
- $\widetilde{W}(t)$ and $W(t)$ are standard Brownian motions under **risk neutral (RN)** and **real world (RW) measures** respectively.
- $\lambda_{IR}(t)$ is the **piecewise** risk premium (**estimated on historical data**; see Section 4 for details), that determines transition from RN to RW measure: $\widetilde{W}(t) = W(t) + \int_0^t \lambda_{IR}(s)ds$.

Note that it is possible to reformulate these formulae for the total short rate $r(t)$:

$$\begin{aligned} dr(t) &= (\theta(t) - \alpha r(t))\, dt + \sigma(t)d\widetilde{W}(t) \\ &= (\theta(t) + \lambda_{IR}(t)\sigma(t) - \alpha r(t))\, dt + \sigma(t)dW(t) \end{aligned}$$

with drift $\theta(t)$ defined as

$$\theta(t) = \partial_t\phi(t) + \alpha\phi(t).$$

**Shadow rate** is one more modification of HW1F model that used when one needs to truncate distribution of the short rate from the bottom:

$$\begin{aligned} ds(t) &= (\theta(t) - \alpha s(t))\, dt + \sigma(t)d\widetilde{W}(t), \\ r(t) &= \max(s(t),\ \overline{r}). \end{aligned}$$

Here $\overline{r}$ is the bottom fringe of $r(t)$ distribution and $s(t)$ is so-called shadow rate – that is just unbounded analogue of $r(t)$. Modification of such type can be used in case of negative interest rates for the purpose of preventing $r(t)$ to be extremely low. The need of usage of the modification and value of $\overline{r}$ for every currency **are determined expertly** and set by the user in *Cl.Engine* (see Section 3.1.2 for details).

Below we will consider a single case of piecewise constant parametrization for volatility $\sigma(t)$:

$$\sigma(\tau) = \begin{cases} 0 & \tau < 0 \\ \sigma_0 & \tau \geq 0 \wedge \tau \leq t_1 \\ \sigma_1 & \tau > t_1 \wedge \tau \leq t_2 \\ \sigma_2 & \tau > t_2 \wedge \tau \leq t_3 \\ \ldots & \\ \sigma_n & \tau > t_n \wedge \tau \leq t_{n+1} \\ \sigma_{n+1} & \tau > t_{n+1} \end{cases}$$

Introduce the following mean reversion speed function, as in [2], which will be useful to simplify many formulas below:

$$E(t) = e^{\alpha t}$$

Introduce the E-functions ratio as

$$R(t_j, t_i) = \frac{E(t_j)}{E(t_i)} = \prod_{k=j}^{i-1} \frac{E_k(t_k)}{E_k(t_{k+1})}, j < i$$

$$R_k(u, v) = \frac{E_k(u)}{E_k(v)}$$

$$R(t_j, t_i) = R_j(t_j, t_{j+1}) R_{j+1}(t_{j+1}, t_{j+2}) \dots R_{i-1}(t_{i-1}, t_i)$$

Solution of the SDE in the original spot measure $\mathbb{Q}$ is:

$$x(u) = x(v)e^{-\alpha(u-v)} + \int_v^u e^{-\alpha(u-\tau)}\sigma(\tau)dW(\tau), \quad v \leq u$$

Solution of the SDE in the $T$-forward measure $\mathbb{Q}_T$ associated with unit zero bond price $P(t,T)$, $P(T,T) = 1$, is:

$$x^{\mathbb{Q}^T}(u) = x(v)e^{-\alpha(u-v)} - \int_v^u e^{-\alpha(u-\tau)}\sigma^2(\tau)B(\tau,T)d\tau + \int_v^u e^{-\alpha(u-\tau)}\sigma(\tau)dW^{\mathbb{Q}^T}(\tau) =$$

$$= x(v)e^{-\alpha(u-v)} - M_x^T(v,u) + \int_v^u e^{-\alpha(u-\tau)}\sigma(\tau)dW^{\mathbb{Q}^T}(\tau), \quad v \leq u$$

with mean reversion function

$$B(t_1, t_2 \geq t_1) = \frac{1 - e^{-\alpha(t_2 - t_1)}}{\alpha} = E(t_1)\int_{t_1}^{t_2} \frac{1}{E(\tau)} d\tau$$

Drift term in $\mathbb{Q}_T$ is

$$M_x^T(v,u) = M_x^T(v,t_i) + M_x^T(t_i,t_{i+1}) + ... + M_x^T(t_{i+k},u),$$

$$\text{for } t_{i-1} < v \leq t_i < t_{i+1} < ... < t_{i+k} \leq u < t_{i+k+1}$$

$$M_x^T(t_i,t_{i+1}) = \frac{\sigma_i^2}{\alpha^2}\left[1 - e^{-\alpha(t_{i+1}-t_i)} + \frac{e^{-\alpha(t_{i+1}+T-2t_i)}}{2} - \frac{e^{-\alpha(T-t_{i+1})}}{2}\right]$$

**Non-Conditional Moments**

Non-conditional expectation in $\mathbb{Q}$:

$$E^{\mathbb{Q}}[r(t)] = E^{\mathbb{Q}}[r(t)|\mathcal{F}_0] = \phi(t)$$

Non-conditional expectation in $\mathbb{Q}^T$:

$$E^{\mathbb{Q}^T}[r(t)] = E^{\mathbb{Q}^T}[r(t)|\mathcal{F}_0] = -M_x^T(0,t) + \phi(t)$$

Non-conditional variances in $\mathbb{Q}$ and $\mathbb{Q}^T$:

$$
\begin{aligned}
Var^{\mathbb{Q}}\left[r(t)\right] = Var^{\mathbb{Q}}\left[x(t)\right] &= Var^{\mathbb{Q}^T}\left[x(t)\right] \\
&= \int_0^t e^{-\alpha(t-u)}\sigma(u) \int_0^t e^{-\alpha(t-v)}\sigma(v) Var^{\mathbb{Q}}\left[dW^{\mathbb{Q}}(u)dW^{\mathbb{Q}}(v)\right] \\
&= \int_0^t e^{-\alpha(t-u)}\sigma(u) \int_0^t e^{-\alpha(t-v)}\sigma(v)\delta(u-v)\mathbf{1}_{v\le u} du dv \\
&= \int_0^t e^{-2\alpha(t-u)}\sigma^2(u) du \\
&= V_x(0,t)
\end{aligned}
$$

where

$$
\begin{aligned}
V_x(T,U) &= \int_T^U e^{-2\alpha(U-s)}\sigma^2(s) ds \\
&= \sum_{i=j,t_j<T\le t_{j+1}}^{k,t_{k+1}<U\le t_{k+2}} \frac{\sigma_i^2 e^{-2\alpha U}}{2\alpha}\left[e^{2\alpha t_{i+1}} - e^{2\alpha \max(t_i,T)}\right] \\
&+ \frac{\sigma_{k+1}^2 e^{-2\alpha U}}{2\alpha}\left[e^{2\alpha U} - e^{2\alpha t_{k+1}}\right]
\end{aligned}
$$

Denote $v_i^x$ as:

$$
v_i^x(t_i,t_{i+1}) = \frac{\sigma_i^2}{2\alpha}\left[1 - e^{-2\alpha(t_{i+1}-t_i)}\right]
$$

**Conditional Moments**

Conditional expectations in $\mathbb{Q}$:

$$
\begin{aligned}
E^{\mathbb{Q}}\left[r(t)|\mathcal{F}_t\right] &= E^{\mathbb{Q}}\left[x(t)|\mathcal{F}_t\right] + \phi(t) \\
&= x(t) + \phi(t)
\end{aligned}
$$

$$
\begin{aligned}
E^{\mathbb{Q}}\left[r(T)|\mathcal{F}_t,\, t\le T\right] &= E^{\mathbb{Q}}\left[x(T)|\mathcal{F}_t,\, t\le T\right] + \phi(T) \\
&= x(t)e^{-\alpha(T-t)} + \phi(T) \\
&= E_r^{\mathbb{Q}}(t,T)
\end{aligned}
$$

Conditional expectations in $\mathbb{Q}^T$:

$$
\begin{aligned}
E^{\mathbb{Q}^T}\left[r(t)|\mathcal{F}_t\right] &= E^{\mathbb{Q}^T}\left[x(t)|\mathcal{F}_t\right] + \phi(t) \\
&= x(t) + \phi(t)
\end{aligned}
$$

$$
\begin{aligned}
E^{\mathbb{Q}^T}\left[r(U)|\mathcal{F}_t,\, t\le U\right] &= E^{\mathbb{Q}^T}\left[x(U)|\mathcal{F}_t,\, t\le U\right] + \phi(U) \\
&= x(t)e^{-\alpha(U-t)} - M_x^T(t,U) + \phi(U) \\
&= E_r^{\mathbb{Q}^T}(t,U)
\end{aligned}
$$

Conditional variances in $\mathbb{Q}$ and $\mathbb{Q}^T$:

$$\begin{aligned}
&Var^{\mathbb{Q}}\left[r(U)|\mathcal{F}_t,\, t \leq U\right] = \\
&\quad = Var^{\mathbb{Q}}\left[x(U)|\mathcal{F}_t,\, t \leq U\right] = Var^{\mathbb{Q}^T}\left[x(U)|\mathcal{F}_t,\, t \leq U\right] \\
&\quad = \int\limits_t^U e^{-\alpha(U-u)}\sigma(u) \int\limits_t^U e^{-\alpha(U-v)}\sigma(v) Var^{\mathbb{Q}}\left[dW^{\mathbb{Q}}(u)dW^{\mathbb{Q}}(v)\right] = \\
&\quad = \int\limits_t^U e^{-\alpha(U-u)}\sigma(u) \int\limits_t^U e^{-\alpha(U-v)}\sigma(v)\delta(u-v)\mathbf{1}_{v \leq u} du dv \\
&\quad = \int\limits_t^U e^{-2\alpha(U-u)}\sigma^2(u) du \\
&\quad = V_x\left(t, U\right).
\end{aligned}$$

Therefore, $r(U)$ conditional on $\mathcal{F}_t, t \leq U$ is normally distributed.

In $\mathbb{Q}$ measure:

$$r(U) \sim \mathcal{N}\left(E_r^{\mathbb{Q}}(t, U), V_x\left(t, U\right)\right).$$

In $\mathbb{Q}^T$ measure:

$$r(U) \sim \mathcal{N}\left(E_r^{\mathbb{Q}^T}(t, U), V_x\left(t, U\right)\right).$$

Integrated instantaneous short rate is needed to calculated numeraire:

$$\begin{aligned}
Y(t, T) &= \int_t^T r(s) ds \\
&= \int_t^T x(s) ds + \int_t^T \phi(s) ds \\
&= X(t, T) + \Phi(t, T),
\end{aligned}$$

with

$$\begin{aligned}
X(t, T) &= \int_t^T x(s) ds, \\
\Phi(t, T) &= \int_t^T \phi(s) ds.
\end{aligned}$$

To calculate stochastic path discount states $Y(0, t)$ one could numerically discretize integral over simulated short-rates $r(t)$

$$Y(0, t_i) = \sum_{j=0}^{i} f\left(r(t_i), r(t_{i+1}), t_i, t_{i+1}\right),$$

however, it is also possible to simulate both entities $(r(t), Y(0, t))$ as bi-variate normal distribution, using expressions for covariances and correlations (see Section 3.1.4 for details).

Non-conditional expectation in $\mathbb{Q}$:

$$E^{\mathbb{Q}}\left[Y(0, t)\right] = \int_0^t \phi(s) ds$$

Conditional expectation in $\mathbb{Q}$:

$$E^{\mathbb{Q}}\left[Y(0,t)|\,\mathcal{F}_t\right] = \int_0^t x(s)ds + \int_0^t \phi(s)ds$$
$$E^{\mathbb{Q}}\left[Y(t,T)|\,\mathcal{F}_t,\, t \le T\right] = B(t,T)x(t) + \int_t^T \phi(s)ds$$

$$\begin{aligned}
E^{\mathbb{Q}}\left[Y(T,U)|\,\mathcal{F}_t,\, t \le T \le U\right] &= \int_T^U E^{\mathbb{Q}}\left[x(s)|\,\mathcal{F}_t\right]\, ds + \int_T^U \phi(s)ds \\
&= \int_T^U e^{-\alpha(s-t)} x(t)\, ds + \int_T^U \phi(s)ds \\
&= \frac{e^{-\alpha(T-t)} - e^{-\alpha(U-t)}}{\alpha} x(t) + \int_T^U \phi(s)ds
\end{aligned}$$

Non-conditional variance in $\mathbb{Q}$:

$$\begin{aligned}
Var^{\mathbb{Q}}\left[Y(0,t)\right] = Var^{\mathbb{Q}}\left[X(0,t)\right] &= \int_0^t \int_0^t Cov^{\mathbb{Q}}\left[r(u), r(v)\right] dv\, du \\
&= 2\int_0^t \int_0^u Cov^{\mathbb{Q}}\left[r(u), r(v)\right] dv\, du \\
&= 2\int_0^t \int_0^u \int_0^v \sigma^2(s) e^{-\alpha(u-s)} e^{-\alpha(v-s)}\, ds\, dv\, du \\
&= V(0,t)
\end{aligned}$$

Conditional variance in $\mathbb{Q}$:

$$\begin{aligned}
Var^{\mathbb{Q}}\left[Y(t,T)|\,\mathcal{F}_t,\, t \le T\right] = Var^{\mathbb{Q}}\left[X(t,T)|\,\mathcal{F}_t,\, t \le T\right] &= \int_t^T \int_t^T Cov^{\mathbb{Q}}\left[r(u), r(v)\right] dv\, du \\
&= 2\int_t^T \int_t^u Cov^{\mathbb{Q}}\left[r(u), r(v)\right] dv\, du \\
&= 2\int_t^T \int_t^u \int_t^v \sigma^2(s) e^{-\alpha(u-s)} e^{-\alpha(v-s)}\, ds\, dv\, du \\
&= V(t,T).
\end{aligned}$$

Denote variance functions

$$\begin{aligned}
V(0,T) &= \sum_{i=0}^{k,t_{k+1}<T\le t_{k+2}} \left[v_i(0, T-t_i) - v_i(0, T-t_{i+1})\right] \\
&+ v_{k+1}(0, T-t_{k+1}) \\
V(S,T) &= v_j(0, T-S) \\
&+ v_{j+1}(0, T-t_{j+1}) - v_{j+1}(0, T-t_{j+2}) \\
&+ ... \\
&+ v_{k+1}(0, T-t_{k+1}) \\
&= \sum_{i=j,t_j<S\le t_{j+1}}^{k,t_{k+1}<T\le t_{k+2}} \left[v_i(0, T-\max(t_i, S)) - v_i(0, T-t_{i+1})\right] \\
&+ v_{k+1}(0, T-\max(t_{k+1}, S))
\end{aligned}$$

with

$$v_j(u,v) = \frac{\sigma_j^2}{\alpha^2}\left[(v-u) + \frac{1}{2\alpha}\left(e^{-2\alpha u} - e^{-2\alpha v}\right) + \frac{2}{\alpha}\left(e^{-\alpha v} - e^{-\alpha u}\right)\right]$$

$$\begin{aligned} Var^{\mathbb{Q}}\left[Y(t,T)\right|\mathcal{F}_t,\, t\le T] = Var^{\mathbb{Q}}\left[X(t,T)\right|\mathcal{F}_t,\, t\le T] &= \int_t^T\int_t^T Cov^{\mathbb{Q}}\left[r(u),r(v)\right]dv\,du \\ &= 2\int_t^T\int_t^u Cov^{\mathbb{Q}}\left[r(u),r(v)\right]dv\,du \\ &= 2\int_t^T\int_t^u\int_t^v \sigma^2(s)\frac{E^2(s)}{E(u)E(v)}\,ds\,dv\,du \\ &= V(t,T). \end{aligned}$$

Denote variance function

$$\begin{aligned} V(t,T) &= 2\int_t^T\int_t^u\int_t^v \sigma^2(s)\frac{E^2(s)}{E(u)E(v)}\,ds\,dv\,du \\ &= 2\int_t^T V_x(t,v)B(v,T)\,dv \\ &= \sum_{j=i,t_i<t\le t_{i+1}}^{n,t_n<T\le t_{n+1}} \frac{\sigma_j^2}{\alpha_j^2}\left(\min(t_{j+1},T) - \max(t_j,t)\right) + \\ &+ \sum_{j=i,t_i<t\le t_{i+1}}^{n,t_n<T\le t_{n+1}} \frac{\sigma_j^2}{\alpha_j^2}\left(-2B_j(\max(t_j,t),\min(t_{j+1},T)) + B_j^{2\alpha}(\max(t_j,t),\min(t_{j+1},T))\right) + \\ &+ 2\sum_{j=i,t_i<t\le t_{i+1}}^{n,t_n<T\le t_{n+1}} A_jB_j(\max(t_j,t),\min(t_{j+1},T))\frac{E_j(t_j)}{E_j(\min(t_{j+1},T))}\left(C_j - \frac{1}{\alpha_j}\right) + \\ &+ \sum_{j=i,t_i<t\le t_{i+1}}^{n,t_n<T\le t_{n+1}} C_j\sigma_j^2B_j(\max(t_j,t),\min(t_{j+1},T))B_j(\max(t_j,t),t_{j+1}) + \\ &+ 2\sum_{j=i,t_i<t\le t_{i+1}}^{n,t_n<T\le t_{n+1}} \frac{A_j}{\alpha_j}B_j^{2\alpha}(\max(t_j,t),\min(t_{j+1},T)), \end{aligned}$$

where

$$A_j = \begin{cases} \sum_{m=i,t_i<t\le t_{i+1}}^{j-1} \sigma_m^2 B_m^{2\alpha}(\max(t_m,t),t_{m+1})R^2(t_{m+1},t_j) & j>i \\ 0 & j\le i \end{cases},$$

$$C_j = \begin{cases} \sum_{m=j+1}^{n,t_n<T\le t_{n+1}} B_m(t_m,\min(t_{m+1},T))R(t_{j+1},t_m) & j<n \\ 0 & j\ge n \end{cases}$$

Due to the Gaussian distribution of $r(T)$ conditional on $\mathcal{F}_t, t\le T$, $\int_t^T r(s)ds$ is itself normally distributed. Precisely we can show that in $\mathbb{Q}$

$$\begin{aligned} \int_t^T r(s)ds &\sim \mathcal{N}\left(B(t,T)x(t) + \int_t^T \phi(s)ds, V\left(t,T\right)\right) \sim \\ &\sim \mathcal{N}\left(B(t,T)x(t) - \ln\frac{P^M\left(0,T\right)}{P^M\left(0,t\right)} + \frac{1}{2}\left[V\left(0,T\right) - V\left(0,t\right)\right], V\left(t,T\right)\right) \end{aligned}$$

Non-conditional covariances in $\mathbb{Q}$ and $\mathbb{Q}^T$ between $r(t)$ and $Y(0,t)$:

$$\begin{aligned}
Cov^{\mathbb{Q},\mathbb{Q}^T}\left[r(t), Y(0,t)\right] &= \int_0^t Cov^{\mathbb{Q},\mathbb{Q}^T}\left[r(t), r(s)\right]\, ds \\
&= \int_0^t \int_0^t \int_0^s e^{-\alpha(s-v)}\sigma(v)e^{-\alpha(t-u)}\sigma(u)\times \\
&\times Cov^{\mathbb{Q},\mathbb{Q}^T}\left[dW(v)dW(u)\right]\, dv\, du\, ds \\
&= \int_0^t \int_0^s e^{-\alpha(s+t-2v)}\sigma^2(v)\, dv\, ds \\
&= V_c(0,t)
\end{aligned}$$

Conditional covariances in $\mathbb{Q}$ and $\mathbb{Q}^T$ between $r(T)$ and $Y(t,T)$:

$$\begin{aligned}
Cov^{\mathbb{Q},\mathbb{Q}^T}\left[r(T), Y(t,T)\middle|\, \mathcal{F}_t,\, t \le T\right] &= \int_t^T Cov^{\mathbb{Q},\mathbb{Q}^T}\left[r(T), r(s)\middle|\, \mathcal{F}_t\right]\, ds \\
&= \int_t^T \int_t^T \int_t^s e^{-\alpha(s-v)}\sigma(v)e^{-\alpha(T-u)}\sigma(u)\times \\
&\times Cov^{\mathbb{Q},\mathbb{Q}^T}\left[dW(v)dW(u)\right]\, dv\, du\, ds \\
&= \int_t^T \int_t^s e^{-\alpha(s+T-2v)}\sigma^2(v)\, dv\, ds \\
&= V_c(t,T)
\end{aligned}$$

where

$$\begin{aligned}
v_j^c(u,v) &= \frac{\sigma_j^2}{2\alpha^2}\left(1 + e^{-2\alpha(v-u)} - 2e^{-\alpha(v-u)}\right) \\
V_c(0,T) &= \sum_{i=0}^{k,\, t_{k+1}<T\le t_{k+2}} \left[v_i^c(t_i,T) - v_i^c(t_{i+1},T)\right] \\
&+ v_{k+1}^c(t_{k+1},T) \\
V_c(S,T) &= v_j^c(S,T) - v_j^c(t_{j+1},T) \\
&+ v_{j+1}^c(t_{j+1},T) - v_{j+1}^c(t_{j+2},T) \\
&+ \ldots \\
&+ v_{k+1}^c(t_{k+1},T) \\
&= \sum_{i=j,\, t_j<S\le t_{j+1}}^{k,\, t_{k+1}<T\le t_{k+2}} \left[v_i^c(\max(t_i,S),T) - v_i^c(t_{i+1},T)\right] \\
&+ v_{k+1}^c(\max(t_{k+1},S),T)
\end{aligned}$$

**Zero-coupon Bond Pricing**

The price at time $t$ of a unit-notional zero-coupon bond with maturity $T$ is:

$$\begin{aligned}
P\left(t,T\right) = \mathbf{ZB}(t,T) &= E^{\mathbb{Q}}\left[e^{-\int_t^T r(s)ds}\middle|\, \mathcal{F}_t,\, t \le T\right] \\
&= E^{\mathbb{Q}}\left[e^{-\int_t^T (x(s)+\phi(s))ds}\middle|\, \mathcal{F}_t,\, t \le T\right] \\
&= E^{\mathbb{Q}}\left[e^{-\int_t^T x(s)ds}\middle|\, \mathcal{F}_t,\, t \le T\right] e^{-\int_t^T \phi(s)ds} \\
&= P^x\left(t,T\right) P^{\phi}\left(t,T\right)
\end{aligned}$$

$$
\begin{aligned}
P^{x}(t,T) &= e^{-B(t,T)x(t)+\frac{1}{2}[V(t,T)]} \\
P^{\phi}(t,T) &= \frac{P^{M}(0,T)}{P^{M}(0,t)} e^{-\frac{1}{2}[V(0,T)-V(0,t)]} \\
\Phi(t,T) &= \int_{t}^{T} \phi(s)ds \\
&= -\ln \frac{P^{M}(0,T)}{P^{M}(0,t)} + \frac{1}{2}\left[V(0,T)-V(0,t)\right] \\
\phi(t) &= -\frac{\partial \ln P^{M}(0,t)}{\partial t} + \frac{1}{2}\frac{\partial V(0,t)}{\partial t} \\
&= f^{M}(0,t) + \frac{1}{2}\frac{\partial V(0,t)}{\partial t}
\end{aligned}
$$

where

- $V(t,T)$ is the variance of the integrated stochastic component $x(t)$, defined above,
- $\phi(t)$ is the deterministic component of the short rate,
- $\mathbb{F} = \{\mathcal{F}_t, 0 \le t \le T\}$ is a right-continuous filtration,
- $P^{M}(v,u)$ is the current market zero-coupon bond prices,
- $f^{M}(0,t)$ is the market instantaneous forward rate at time 0 for maturity $t$.

Consequently

$$
\begin{aligned}
P(t,T) &= \frac{P^{M}(0,T)}{P^{M}(0,t)} e^{-\frac{1}{2}[V(0,T)-V(t,T)-V(0,t)]-B(t,T)x(t)} \\
&= \frac{P^{M}(0,T)}{P^{M}(0,t)} e^{-\frac{1}{2}[V(0,T)-V(t,T)-V(0,t)]+B(t,T)\phi(t)} e^{-B(t,T)r(t)} \\
&= A(t,T)e^{-B(t,T)r(t)}
\end{aligned}
$$

where

$$
A(t,T) = \frac{P^{M}(0,T)}{P^{M}(0,t)} e^{-\frac{1}{2}[V(0,T)-V(t,T)-V(0,t)]+B(t,T)\phi(t)}
$$

Expression for $A(t,T)$ can be simplified:

$$
A(t,T) = \frac{P^{M}(0,T)}{P^{M}(0,t)} e^{B(t,T)f^{M}(0,t)-\frac{1}{2}B(t,T)^{2}V_{x}(0,t)}
$$

which is due to the identity:

$$
B(t,T)^{2}V_{x}(0,t) \equiv V(0,T) - V(t,T) - V(0,t) - B(t,T)\frac{\partial V(0,t)}{\partial t}
$$

If zero-coupon bond price is calculated with some rate $R$ (not equal to $r(t)$) we can rewrite above expression:

$$
P(t,T;R) = A(t,T)e^{-B(t,T)R}.
$$

Conditional on the logarithm of zero-coupon price:

$$
\begin{aligned}
\ln P(t,T)|\,\mathcal{F}_t,\ t \le T &= \ln E^{\mathbb{Q}}\left[ e^{-\int_t^T r(s)ds} \middle| \mathcal{F}_t,\ t \le T \right] \\
&= -B(t,T)\,x(t) + \frac{1}{2}V(t,T) + \ln P^{\phi}(t,T) \\
\ln P(T,U)|\,\mathcal{F}_t,\ t \le T \le U &= \ln E^{\mathbb{Q}}\left[ e^{-\int_T^U r(s)ds} \middle| \mathcal{F}_t,\ t \le T \le U \right] \\
&= -B(T,U)\ x(T)|\,\mathcal{F}_t + \frac{1}{2}V(T,U) + \ln P^{\phi}(T,U).
\end{aligned}
$$

Conditional expectation in $\mathbb{Q}$:

$$\begin{aligned}
E^{\mathbb{Q}}\left[\ln P\left(T,U\right)|\mathcal{F}_t,\, t\leq T\leq U\right] &= -B(T,U)E^{\mathbb{Q}}\left[x(T)|\mathcal{F}_t,\, t\leq T\leq U\right] \\
&+\frac{1}{2}V\left(T,U\right)-\int_T^U \phi(s)ds \\
&= -B(T,U)x(t)\frac{E(t)}{E(T)} \\
&+\frac{1}{2}V\left(T,U\right)-\int_T^U \phi(s)ds \\
&= M_P(t,T,U).
\end{aligned}$$

Conditional expectation in $\mathbb{Q}^T$:

$$\begin{aligned}
E^{\mathbb{Q}^T}\left[\ln P\left(T,U\right)|\mathcal{F}_t,\, t\leq T\leq U\right] &= \\
&= -B(T,U)E^{\mathbb{Q}^T}\left[x(T)|\mathcal{F}_t,\, t\leq T\leq U\right] \\
&+\frac{1}{2}V\left(T,U\right)-\int_T^U \phi(s)ds \\
&= -B(T,U)\left[\frac{E(t)}{E(T)}x(t)-M_x^T\left(t,T\right)\right] \\
&+\frac{1}{2}V\left(T,U\right)-\int_T^U \phi(s)ds \\
&= M_P^T(t,T,U).
\end{aligned}$$

Conditional variances in $\mathbb{Q}$ and $\mathbb{Q}^T$:

$$\begin{aligned}
Var^{\mathbb{Q}}\left[\ln P^x\left(t,T\right)|\mathcal{F}_t,\, t\leq T\right] &= Var^{\mathbb{Q}^T}\left[\ln P^x\left(t,T\right)|\mathcal{F}_t,\, t\leq T\right] \\
&= B(t,T)^2V_x\left(t,t\right)=0 \\
Var^{\mathbb{Q}}\left[\ln P^x\left(t,T\right)\right] &= Var^{\mathbb{Q}^T}\left[\ln P^x\left(t,T\right)\right] \\
&= B(t,T)^2V_x\left(0,t\right) \\
Var^{\mathbb{Q}}\left[\ln P^x\left(T,U\right)|\mathcal{F}_t,\, t\leq T\leq U\right] &= Var^{\mathbb{Q}^T}\left[\ln P^x\left(T,U\right)|\mathcal{F}_t,\, t\leq T\leq U\right] \\
&= B(T,U)^2V_x\left(t,T\right) \\
&= V_P(t,T,U).
\end{aligned}$$

It is then easy to realize that the distribution of the zero-coupon bond price $P(T,U)$ conditional on $\mathcal{F}_t, t\leq T\leq U$ is log-normal.

In $\mathbb{Q}$:

$$P(T,U)\sim\mathcal{LN}\left(M_P(t,T,U),V_P(t,T,U)\right)$$

In $\mathbb{Q}^T$:

$$P(T,U)\sim\mathcal{LN}\left(M_P^T(t,T,U),V_P(t,T,U)\right)$$

It is possible to formulate diffusion SDE for the bond price itself $P(t,T)$ in $\mathbb{Q}$:

$$dP(t,T)=P(t,T)\left(r(t)dt-\sigma_P(t,T)dW_d(t)\right),$$

where zero bond volatilities are

$$\sigma_P(t,T)=\sigma(t)\int_t^T e^{-\int_t^u \alpha(v)dv}du=B(t,T)\sigma(t). \tag{1}$$

This formula can be used in bond option pricing, cross-currency models with HW1F components or other forward-related calculations.

**Stochastic discount factor** The associated stochastic discount factor is given by the following expression:

$$SDF(t) = e^{-\int_0^t r(s)ds}. \tag{2}$$

**Expected deflator** Expected deflator (expected discount factor) $EDF(t,T)$ is a discount factor associated with $T$-forward measure. Its value coincides with bond price $P(t,T)$ and is given by:

$$EDF(t,T) = \mathbf{E}^{\mathbb{Q}}\left[e^{-\int_t^T r(s)ds}|\mathcal{F}_t\right] = A(t,T)e^{-B(t,T)r(t)}, \tag{3}$$

where $\mathbb{Q}$ is the spot risk-neutral measure, $A(t,T)$ and $B(t,T)$ are defined by:

$$A(t,T) = \frac{P^M(0,T)}{P^M(0,t)} e^{B(t,T)f^M(0,t) - \frac{1}{2}B(t,T)^2 V_x(0,t)}, \tag{4}$$

$$B(t,T) = \int_t^T e^{-\alpha(v-t)}dv, \tag{5}$$

$$V_x(t,U) = \int_t^U e^{-2\alpha(U-u)}\sigma^2(u)du. \tag{6}$$

Integrals in the expressions for $B(T,U)$ and $V_x(t,U)$ are calculated numerically using *PiecewiseConstantIntegration.CalculateFlatBackward*, $P^M(0,T)$ is market bond price with maturity $T$ (from market discount curve), forward rate $f^M(0,t)$ is approximated by:

$$f^M(0,t) = -\frac{1}{\Delta t}\log\frac{P^M(0,t+\Delta t)}{P^M(0,t)} \tag{7}$$

with $\Delta t = 0.0001$.

#### 3.1.2 Model Parameters and Market Data

1. **Discount Curve** – ClCurvePoints (defined according to ClCcyPrimaryCurve specification) – **market discount curve** $P^M(0,t)$ **used as a discount factor** in future payment discounting.

2. **Forecast Curve** – ClCurvePoints (defined according to ClCcyPrimaryCurve specification) – **used in future payment calculations** (e.g., future values of forward rates etc)

3. **Primary Curve** – ClCurvePoints (defined according to ClCcyPrimaryCurve specification) – **used in bond price simulation.** In this case, primary-secondary adjustment is applied to the bond prices (simulated using the primary curve):

$$P(t,T) := \delta(t,T)P(t,T), \tag{8}$$

   where

$$\delta(t,T) = \frac{P_{discount}(0,T)/P_{discount}(0,t)}{P_{primary}(0,T)/P_{primary}(0,t)}, \tag{9}$$

   $P_{discount}(0,t)$ is discount factor given by the discount curve, $P_{primary}(0,t)$ is a discount factor given by the primary curve.

4. **Volatility Curve – instantaneous short rate volatility** $\sigma(t)$ is defined in one of the following ways:

   (a) SbPslRwModelSettings.IRVols (Key: ClCcyKey, Value: ClFactor-Series (manually defined, no restrictions on Factor and ValueType))

   (b) SbSharedMarketSimulationModelSettings(Default Market Simulation Model Settings).IrModelDefaults (Key: ClCcyKey, Value: ClDouble (manually defined); in this case the curve is flat)

(c) ClFactorSeries (Factor: Model.CCY_IR_MODEL, ValueType: Vol) − historical implied volatilities, estimated using exponentially-weighted moving average approach (EWMA) using the volatilities calculated in the CVA system, calculated for a set of historical dates:

$$S_t = \alpha Y_t + (1-\alpha) S_{t-1}, \quad 1 < t \leq T, \tag{10}$$

where $Y_t$ is implied volatility at time $t$, $S_t$ is EWMA value at time $t$, initial value is $S_1 = \alpha Y_1$, parameter $\alpha$ $(0 < \alpha \leq 1)$ is defined by user, default value is 0.95. Final value is adjusted as follows: $EWMA = S_T \times \frac{1}{1-(1-\alpha)^T}$ (to make sum of weights equal to 1). If $\alpha = 0$, EWMA is arithmetic mean of $Y_t$.

5. **Reversion Skew** – Default Market Simulation Model Settings.IrModelDefaults (Key: ClCcyKey, Value: ClDouble (manually defined)) − **mean reversion skew (short rate mean reversion $\alpha$)** is defined manually.

6. **Risk Premium Curve** – ClCurvePoints (Curve: Risk Premium: CCY) – $DF_{RP}(t)$ **deflator being used to calculate bond prices in the real-world measure.** If the instantaneous market price of risk, $\lambda_{IR}(t)$, is provided instead of the discount curve $DF_{RP}(t)$, the corresponding discount curve can be calculated as follows:

$$DF_{RP}(T) = \exp\left[\int_0^T \left(\int_0^t e^{-\alpha(t-u)} \lambda_{IR}(u)\sigma(u)du\right) dt\right].$$

7. **Shadow Rate** – SbPslShadowRates (Key: ClCcyKey, Value: ShadowRate) – $\overline{r}$ **used as lower threshold** for simulation of instantaneous short rate $r(t)$.

### 3.1.3 Calibration

**Volatility calibration**

If BOTH *Sigma* and *TimeDependentSigmas* are set → NO Calibration, exception "Both sigma and time dependent sigma are set."

If *Sigma* is set its value will be used as the model volatility $\sigma(t)$. If at the same time *Instruments* are also set → exception "Both sigma and instruments are set."

If *TimeDependentSigmas* are set → their value will be used as the model volatility $\sigma(t)$. If at the same time *Instruments* are also set → exception "Both time dependent sigma and instruments are set."

If *Instruments* are set → they will be used for volatility calibration.

If *IterativeCalibration* is left blank → piecewise model parameter $\sigma(t)$ is calibrated with a set of European swaptions and cap/floors (see pricing methodology below and [2]). Calibration is performed numerically (method *LevenbergMarquardt.Minimize*) by minimizing weighted sum of squared errors (difference between market and model price):

$$\{\sigma_i\} = \arg\min_{\{\sigma_i\}} \sqrt{\sum_j w_j e_j^2(\{\sigma_i\})}, \tag{11}$$

where

- $\sigma_i$ is volatility at the time interval $t_{i-1} \leq t < t_i$ $(i = \overline{1,n})$;
- $w_j$ is weight of instrument $j$, all **set equal to** 1 **as default**;
- $e_j(\{\sigma_i\})$ is difference between market and model price of instrument $j$ (defined in the field *Instruments*). Can be *absolute*, *relative* or expressed *in terms of implied volatility* (defined in the *CalibrationErrorType* field):

  (i) $e_j(\{\sigma_i\}) = \pi_j^{market} - \pi_j^{model}(\{\sigma_i\})$,

  (ii) $e_j(\{\sigma_i\}) = \dfrac{\pi_j^{market} - \pi_j^{model}(\{\sigma_i\})}{\pi_j^{market}}$,

(iii) $e_j(\{\sigma_i\}) = \dfrac{\pi_j^{market} - \pi_j^{model}(\{\sigma_i\})}{\sigma_j^{imp}}$,

respectively. **Relative differences are used on default**.

If *IterativeCalibration* is set to ✓ → each $\sigma_i$ is independently calibrated using (11) and the corresponding group of instrument $i$. Groups are formed through instruments' settings (e.g., *SwaptionInstrumentsSettings* in *SwaptionInstrumentsSettings*). Otherwise calibration is performed simultaneously, when all $\sigma_i$ are all calibrated using the same group of calibration instruments.

If calibration is unsuccessful, $\sigma_i$ takes the value from *DefaultSigma*.

**Calibration details – tab description**

Common fields:

– Total Price Error (TPE) = $\sqrt{\sum_i PE_i^2 \times IW_i^2}$

– Calibration status

– Calibration Error Type

– Total Relative Price Error (TRPE) = $\sqrt{\sum_i PRE_i^2 \times IW_i^2}$

– Total Implied Vol Error (TIVE) = $\sqrt{\sum_i IVE_i^2 \times IW_i^2}$

– Total RMSE (coincides with the error defined in *Calibration Error Type*)

Individual fields:

– Instrument

– Instrument Weight (IW)

– Calibration Status

– Market Price

– Model Price

– Price Error (PE) = $marketValue - modelValue$

– Relative Price Error (RPE) = $\frac{marketValue - modelValue}{marketValue}$ if $marketValue \neq 0$ and 0 otherwise

– Market Volatility (MV)

– Implied Model Volatility (IMV)

– Implied Vol Error (IVE) = $IMV - MV$

Remark. The application might display values with lower precision compared to computations (i.e., the displayed results might have been rounded).

**Swaption Instruments Settings – tab description** Swaption Instruments Settings contains the following fields:

– StartTenor

– EndAbsoluteTenor

– EndRelativeTenor (used if EndAbsoluteTenor is not set)

– Frequency (not used)

– NoticePeriod

where **StartTenor** defines option maturity, **EndAbsoluteTenor** defines swap duration (with respect to "as-of" date), **EndRelativeTenor** defines the swap duration (with respect to option maturity date).

These tenors define conditions of the swap payments (expiryDate, swapStartDate, swapPeriod, default frequency = "3M").

Start and end dates are defined as follows:

$$startDate = StartTenor + asOfDate$$

$$endDate = EndAbsoluteTenor + asOfDate$$

or

$$endDate = EndRelativeTenor + startDate.$$

Then tenors are give by:

$$startTenor = (startDate.Value - asOfDate.Value).Days + "D"$$

$$endTenor = (endDate.Value - startDate.Value).Days + "D".$$

Maturity and expiry dates are calculated as follows:

$$Data.MaturityTenor = newClTenorKey\{TenorID = endTenor\}$$

$$Data.ExpiryTenor = newClTenorKey\{TenorID = startTenor\}$$

$$var \quad optionPeriod = new \quad Period(Data.ExpiryTenor.TenorID)$$
$$var \quad swapPeriod = new \quad Period(Data.MaturityTenor.TenorID)$$

$$effectiveDate = asOfDate$$
$$expiryDate = effectiveDate + optionPeriod$$
$$swapStartDate = expiryDate + noticeLag$$

*noticeLag* is taken from the convention ("0D" by default).

If *effectiveDate*, *expiryDate* or *swapStartDat* is not a weekday, it can be moved to the following weekday according to the convention.

**Drift calibration**

*This alternative method of calibration of the mean is not used in current version of the system.*

1. From market bond prices $P^M(0, T_i)$ interpolate bond prices $P^M(0, t_j)$ at each simulation steps $t_j$
2. Calculate $P^\phi(t_j, t_{j+1})$, then $\phi(t_j)$ and $\phi(t_{j+1})$:

$$\int_{t_j}^{t_{j+1}} \phi(s)ds = -\ln P^\phi(t_j, t_{j+1})$$
$$\phi(0) = r^M(0)$$
$$\dots$$
$$\begin{aligned}\int_{t_j}^{t_{j+1}} \phi(t_j, t_{j+1}, t)ds &= -\ln P^\phi(t_j, t_{j+1}) \\ &= -\ln P^M(t_j, t_{j+1}) + \frac{1}{2}\left[V(0, t_{j+1}) - V(0, t_j)\right] \\ \phi(t_j \le t \le t_{j+1}) &= -\left.\frac{\partial \ln P^\phi(t_j, t)}{\partial t}\right|_{t_j \le t \le t_{j+1}}\end{aligned}$$

- Loglinear DF:

$$\begin{aligned}
\ln P^M(t_j,t)\big|_{t_j\le t\le t_{j+1}} &= \ln\left[\frac{P^M(0,t_{j+1})}{P^M(0,t_j)}\right]\frac{t-t_j}{t_{j+1}-t_j}\\
-\left.\frac{\partial \ln P^M(t_j,t)}{\partial t}\right|_{t_j\le t\le t_{j+1}} &= \ln\left[\frac{P^M(0,t_j)}{P^M(0,t_{j+1})}\right]\frac{1}{t_{j+1}-t_j}\\
&\Downarrow\\
\phi(t_j\le t\le t_{j+1}) &= \ln\left[\frac{1}{P^M(t_j,t_{j+1})}\right]\frac{1}{t_{j+1}-t_j}+\frac{1}{2}\frac{\partial V(0,t)}{\partial t}
\end{aligned}$$

- Linear Continuous Rate:

$$\begin{aligned}
\ln P^M(t_j,t)\big|_{t_j\le t\le t_{j+1}} &= r_jt_j - rt\\
&= r_jt_j - \left[r_j + (r_{j+1}-r_j)\,\frac{t-t_j}{t_{j+1}-t_j}\right]t\\
r_j &= -\frac{\ln P^M(0,t_j)}{t_j}\\
-\left.\frac{\partial \ln P^M(t_j,t)}{\partial t}\right|_{t_j\le t\le t_{j+1}} &= r_j + (r_{j+1}-r_j)\,\frac{2t-t_j}{t_{j+1}-t_j}\\
&\Downarrow\\
\phi(t_j\le t\le t_{j+1}) &= -\frac{\ln P^M(0,t_j)}{t_j}\\
&+\left(\frac{\ln P^M(0,t_j)}{t_j}-\frac{\ln P^M(0,t_{j+1})}{t_{j+1}}\right)\frac{2t-t_j}{t_{j+1}-t_j}\\
&+\frac{1}{2}\frac{\partial V(0,t)}{\partial t}
\end{aligned}$$

#### 3.1.4 Simulation Algorithm

**Short rate and SDF simulation.** Values of short rate $r(t)$ and $SDF(t)$ processes are simulated at the discreet time grid $t_0 < t_1 < ... < t_n$ with the following algorithm:

Initialization:

1. Upload the input data: *Discount curve, Volatility curve, Reversion skew, Risk premium curve, Shadow Rate.*

2. Set initial values: $r_{unadj}(0) = 0$ ($r_{unadj}(t)$ is an auxiliary variable used for short rate and $SDF$ simulation) and $SDF(0) = 1$.

Iterations $i = 1, 2...n$:

3. Calculate non-adjusted $SDF(t_i)$ using the following expression:

$$SDF(t_i) = SDF(t_{i-1})e^{-r_{unadj}(t_{i-1})(t_i-t_{i-1})}. \tag{12}$$

   To make sure that the simulated SDF is consistent with discount curve $P(0,t_i) = DiscountCurve.Df(t_i)$:

$$P(0,t_i) = E^{\mathbb{P}}\left[SDF(t_i)|\mathcal{F}_0\right] \tag{13}$$

   $SDF(t_i)$ needs to be adjusted by factor $\beta(t_i)$:

$$SDF(t_i) := \beta(t_i)SDF(t_i). \tag{14}$$

For simulations in **the risk-neutral measure** ($\mathbb{P} = \mathbb{Q}$) $\beta(t_i)$ is given by

$$\beta(t_i) = \frac{P(0, t_i)}{\frac{1}{N_\omega}\sum_\omega SDF(t_i, \omega)}, \tag{15}$$

$\omega$ is path order number.

For simulations in **the real word measure** ($\mathbb{P} = \mathbb{Q}_{RW}$) $\beta(t_i)$ is adjusted by the risk premium $DF_{RP}(t_i)$ (stored in the risk premium curve):

$$\beta(t_i) = \frac{P(0, t_i)}{DF_{RP}(t_i)\frac{1}{N_\omega}\sum_\omega SDF(t_i, \omega)} \tag{16}$$

4. If shadow rate $\overline{r}$ is set, SDF is adjusted accordingly:

$$SDF(t_i) := \min\{SDF(t_i), SDF(t_{i-1})e^{-\overline{r}(t_i - t_{i-1})}\} \tag{17}$$

5. Calculate short rate $r(t_i)$ using expression:

$$r(t_i) = \frac{1}{t_i - t_{i-1}} \ln \frac{SDF(t_{i-1})}{SDF(t_i)}. \tag{18}$$

6. Calculate next value of $r_{unadj}(t_i)$ using the formula:

$$r_{unadj}(t_i) = r(t_i)\left(1 - \alpha(t_i - t_{i-1})\right) + \sigma(t_i)(W(t_i) - W(t_{i-1})). \tag{19}$$

Trajectories of the standard Brownian motion $W(t_i)$ are generated by an object of *IClOneFactorModel* ($W(t_i) - W(t_{i-1})$ is a realization of the Gaussian random variable with zero mean and variance $t_i - t_{i-1}$, $\mathcal{N}(0, t_i - t_{i-1})$), $\alpha$ and $\sigma(t_i)$ are the model parameters (used as input for *Cl.Engine*).

**Remark.** This algorithm does not use the deterministic component $\phi(t)$ of the short rate as it has been taken into account at the martingale adjustment step.

**EDF simulation.** Expected deflator $EDF$ at each element of time grid $t_i$ is calculated using expression (3):

$$EDF(t_i, T) = A(t_i, T)e^{-B(t_i, T)r(t_i)}. \tag{20}$$

To keep $EDF$ consistent with the initial discount curve, it is also adjusted by a factor $\beta_E(t)$

$$EDF(t_i, T, \omega) := \beta_E(t_i)EDF(t_i, T, \omega), \tag{21}$$

where

$$\beta_E(t_i) = \frac{P(0, T)}{\frac{1}{N_\omega}\sum_\omega SDF(t_i, \omega)} \frac{1}{\frac{1}{N_\omega}\sum_\omega EDF(t_i, T, \omega)}. \tag{22}$$

**Common bi-variate normal distribution approach**

*This alternative method of simulation is not used in current version of the system.*

It is also possible to simulate both entities $(r(t), Y(0, t))$ as bi-variate normal distribution, using expressions for covariances and correlations. This removes discretization error due to integral approximation. Expressions for bi-variate normal distribution (conditional on $r_i$ and $Y_i$) are given by:

$$\begin{pmatrix} r_{i+1} \\ Y_{i+1} \end{pmatrix} \sim \mathcal{N}(\boldsymbol{\mu}, \boldsymbol{\Sigma})$$

where

$$\begin{aligned} \boldsymbol{\mu} &= E^{\mathbb{Q}}\left[\begin{array}{c} r_{i+1} \\ Y_{i+1} \end{array} \middle| r_i,\, Y_i\right] \\ &= \begin{pmatrix} (r(t_i) - \phi(t_i))\frac{E(t_i)}{E(t_{i+1})} + \phi(t_{i+1}) \\ Y_i + B(t_i, t_{i+1})\left[r(t_i) - \phi(t_i)\right] - \ln P^M\left(t_i, t_{i+1}\right) \\ + \dfrac{1}{2}\left[V\left(0, t_{i+1}\right) - V\left(0, t_i\right)\right] \end{pmatrix} \end{aligned}$$

$$
\begin{aligned}
\boldsymbol{\Sigma} &= \begin{pmatrix} Var^{\mathbb{Q}}\left[ r_{i+1} | \, r_i, \, Y_i \right] & Cov^{\mathbb{Q}}\left[ r_{i+1}, Y_{i+1} | \, r_i, \, Y_i \right] \\ Cov^{\mathbb{Q}}\left[ r_{i+1}, Y_{i+1} | \, r_i, \, Y_i \right] & Var^{\mathbb{Q}}\left[ Y_{i+1} | \, r_i, \, Y_i \right] \end{pmatrix} \\
&= \begin{pmatrix} V_x\left(t_i, t_{i+1}\right) & V_c\left(t_i, t_{i+1}\right) \\ V_c\left(t_i, t_{i+1}\right) & V\left(t_i, t_{i+1}\right) \end{pmatrix}
\end{aligned}
$$

In case of constant $\alpha$ we have:

$$
\begin{aligned}
\boldsymbol{\Sigma} &= \begin{pmatrix} v_i^x(t_i, t_{i+1}) & v_i^c\left(t_i, t_{i+1}\right) \\ v_i^c\left(t_i, t_{i+1}\right) & v_i(0, t_{i+1} - t_i) \end{pmatrix} \\
\begin{pmatrix} r_{i+1} \\ Y_{i+1} \end{pmatrix} &= \boldsymbol{\mu} + \boldsymbol{AZ} \\
\boldsymbol{A} &= \begin{pmatrix} \sqrt{v_i^x(t_i, t_{i+1})} & 0 \\ \rho_{rY}\sqrt{v_i(0, t_{i+1} - t_i)} & \sqrt{1 - \rho_{rY}^2}\sqrt{v_i(0, t_{i+1} - t_i)} \end{pmatrix} \\
\rho_{rY} &= \frac{v_i^c\left(t_i, t_{i+1}\right)}{\sqrt{v_i(0, t_{i+1} - t_i)\, v_i^x(t_i, t_{i+1})}},
\end{aligned}
$$

$$
v_i^x(u, v) = \frac{\sigma_i^2}{2\alpha}\left[1 - e^{-2\alpha(v-u)}\right],
$$

$$
v_j^c(u, v) = \frac{\sigma_j^2}{2\alpha^2}\left(1 + e^{-2\alpha(v-u)} - 2e^{-\alpha(v-u)}\right).
$$

For simulate only $r(t)$ at times $t_j$, $j = 0 \dots n$, we may set

$$
r_{i+1} = \left(r(t_i) - \phi(t_i)\right)\frac{E(t_i)}{E(t_{i+1})} + \phi(t_{i+1}) + V_x\left(t_i, t_{i+1}\right) Z_{i+1}
$$

with $Z_1, ..., Z_n$ independent draws from $\mathcal{N}(0, 1)$.

For more details see [4].

## 3.2 Foreign Exchange Model: HW1F-BS-HW1F

### 3.2.1 Theoretical Background

The Black-Scholes with HW1F short rates assumes the following FX dynamics:

$$\begin{cases} \dfrac{dFX(t)}{FX(t)} = (r_d(t) - r_f(t))dt + \sigma(t)d\widetilde{W}^d_{FX}(t) \\ \qquad\qquad = (r_d(t) - r_f(t) + \lambda_{FX}(t)\sigma(t))dt + \sigma(t)dW_{FX}(t), \\ \\ r_d(u) = x_d(u) + \phi_d(u), \\ dx_d(t) = -\alpha_d x_d(t)dt + \sigma_d(t)d\widetilde{W}^d_d(t) \\ \qquad\qquad = \lambda^d_{IR}(t)\sigma_d(t)dt - \alpha_d x_d(t)dt + \sigma_d(t)dW_d(t), \\ \\ r_f(u) = x_f(u) + \phi_f(u), \\ dx_f(t) = -\alpha_f x_f(t)dt - \rho_{x_f,FX_f}\sigma_f(t)\sigma(t)dt + \sigma_f(t)d\widetilde{W}^d_f(t) \\ \qquad\qquad = \lambda^f_{IR}(t)\sigma_f(t)dt - \alpha_f x_d(t)dt + \sigma_f(t)dW_f(t), \end{cases}$$

where

- $FX(t)$ is the FX rate at time $t$ (units of the domestic $(d)$ currency per unit of the foreign $(f)$ currency).
- $\sigma(t)$ is the **piecewise** instantaneous volatility of FX (**calibrated to European FX options prices**; see calibration algorithm in Section 3.2.3).
- $r_d(t)$ and $r_f(t)$ are short rates associated with domestic and foreign currencies respectively (described in Section 3.1.1).
- $\alpha_d$ and $\alpha_f$ are **constant** mean-reversion speed parameters ((**set manually**; see list of input parameters in Section 3.2.2).
- $\phi_d(t)$ and $\phi_f(t)$ are deterministic components (see Section 3.1.4).
- $\sigma_d(t)$ and $\sigma_f(t)$ are **piecewise** short rate volatilities (see Section 3.1.3 for calibration algorithm).
- $\{\widetilde{W}^d_{FX}(t),\ \widetilde{W}^d_d(t),\ \widetilde{W}^d_f(t)\}$ are standard Brownian motions under **domestic risk neutral (dRN)** and $\{W_{FX}(t),\ W_d(t),\ W_f(t)\}$ are standard Brownian motions under **real world (RW) measure**.
- $\lambda_{FX}(t)$ is the **piecewise** idiosyncratic FX risk premium (**estimated on historical data**; see Section 4 for details), that determines transition from dRN to RW measure: $\widetilde{W}^d_{FX}(t) = W(t) + \int_0^t \lambda_{FX}(s)ds$.
- $\lambda^d_{IR}(t)$ and $\lambda^f_{IR}(t)$ are domestic and foreign **piecewise** risk premiums (**estimated on historical data**; see Section 4 for details), that determines transition from RN to RW measure: $\widetilde{W}^*(t) = W^*(t) + \int_0^t \lambda^*_{IR}(s)ds$.

Denote FX rate in above form as $FX_d(t)$ and reverse rate as $FX_f(t)$ (units of the foreign $(f)$ currency per unit of the domestic $(d)$ currency). To compute FX variance $Var^{\mathbb{Q}_d}\left[\ln FX_f(t_i)\right]$ and terminal correlations between different risk factors, one needs the following covariances:

$$Cov^{\mathbb{Q}_d}\left[x_d(t_i), FX_d(t_i)\right] = \int\limits_0^{t_i} \frac{E_d(u)}{E_d(t_i)} B_d(u,t_i)\sigma_d^2(u)du = \sum_{j=0}^{i-1} R_d(t_{j+1},t_i) M_{x_d}^{t_i}(t_j,t_{j+1})$$

$$Cov^{\mathbb{Q}_d}\left[x_f(t_i), FX_f(t_i)\right] = \int\limits_0^{t_i} \frac{E_f(u)}{E_f(t_i)} B_f(u,t_i)\sigma_f^2(u)du = \sum_{j=0}^{i-1} \frac{E_f(t_{j+1})}{E_f(t_i)} M_{x_f}^{t_i}(t_j,t_{j+1})$$

$$\begin{aligned}
Cov^{\mathbb{Q}_d}\left[x_d(t_i), x_f(t_i)\right] &= \rho_{x_d,x_f} \int\limits_0^{t_i} \frac{E_d(u)E_f(u)}{E_d(t_i)E_f(t_i)} \sigma_d(u)\sigma_f(u)du = \\
&= \rho_{x_d,x_f} \sum_{j=0}^{i-1} \sigma_{d,j}\sigma_{f,j} B_j^{d+f}(t_j,t_{j+1}) \frac{E_{d+f}(t_{j+1})}{E_{d+f}(t_i)}
\end{aligned}$$

$$\begin{aligned}
Cov^{\mathbb{Q}_d}\left[x_d(t_i), FX_f(t_i)\right] &= \rho_{x_d,x_f} \int\limits_0^{t_i} \frac{E_d(u)}{E_d(t_i)} B_f(u,t_i)\sigma_d(u)\sigma_f(u)du \\
&= \rho_{x_d,x_f} \sum_{j=0}^{i-1} \sigma_{d,j}\sigma_{f,j} \frac{E_d(t_{j+1})}{E_d(t_i)} \Bigg[ \frac{B_j^d(t_j,t_{j+1}) - B_j^{d+f}(t_j,t_{j+1})}{\alpha_{f,j}} + \\
&+ \widetilde{B}_{j+1}^f(t_{j+1},t_i) B_j^{d+f}(t_j,t_{j+1}) \Bigg]
\end{aligned}$$

$$\begin{aligned}
Cov^{\mathbb{Q}_d}\left[x_f(t_i), FX_d(t_i)\right] &= \rho_{x_d,x_f} \int\limits_0^{t_i} \frac{E_f(u)}{E_f(t_i)} B_d(u,t_i)\sigma_d(u)\sigma_f(u)du \\
&= \rho_{x_d,x_f} \sum_{j=0}^{i-1} \sigma_{d,j}\sigma_{f,j} \frac{E_f(t_{j+1})}{E_f(t_i)} \Bigg[ \frac{B_j^f(t_j,t_{j+1}) - B_j^{d+f}(t_j,t_{j+1})}{\alpha_{d,j}} + \\
&+ \widetilde{B}_{j+1}^d(t_{j+1},t_i) B_j^{d+f}(t_j,t_{j+1}) \Bigg]
\end{aligned}$$

$$
\begin{aligned}
Cov^{\mathbb{Q}_d}\left[FX_d(t_i), FX_f(t_i)\right] &= \rho_{x_d,x_f} \int_0^{t_i} B_d(u,t_i) B_f(u,t_i) \sigma_d(u) \sigma_f(u) du \\
&= \rho_{x_d,x_f} \sum_{j=0}^{i-1} \int_{t_j}^{t_{j+1}} B_j^d(u,t_{j+1}) B_j^f(u,t_{j+1}) \sigma_{d,j} \sigma_{f,j} du \\
&+ \rho_{x_d,x_f} \sum_{j=0}^{i-1} \int_{t_j}^{t_{j+1}} B_j^d(u,t_{j+1}) \frac{E_j^f(u)}{E_j^f(t_{j+1})} \widetilde{B}_{j+1}^f(t_{j+1},t_i) \sigma_{d,j} \sigma_{f,j} du \\
&+ \rho_{x_d,x_f} \sum_{j=0}^{i-1} \int_{t_j}^{t_{j+1}} \frac{E_j^d(u)}{E_j^d(t_{j+1})} \widetilde{B}_{j+1}^d(t_{j+1},t_i) B_j^f(u,t_{j+1}) \sigma_{d,j} \sigma_{f,j} du \\
&+ \rho_{x_d,x_f} \sum_{j=0}^{i-1} \int_{t_j}^{t_{j+1}} \frac{E_j^d(u)}{E_j^d(t_{j+1})} \widetilde{B}_{j+1}^d(t_{j+1},t_i) \frac{E_j^f(u)}{E_j^f(t_{j+1})} \widetilde{B}_{j+1}^f(t_{j+1},t_i) \sigma_{d,j} \sigma_{f,j} du \\
&= \rho_{x_d,x_f} \sum_{j=0}^{i-1} \frac{\sigma_{d,j}\sigma_{f,j}}{\alpha_{d,j}\alpha_{f,j}} \Big( t_{j+1} - t_j - B_j^d(t_j,t_{j+1}) - B_j^f(t_j,t_{j+1}) + B_j^{d+f}(t_j,t_{j+1}) \Big) \\
&+ \rho_{x_d,x_f} \sum_{j=0}^{i-1} \sigma_{d,j}\sigma_{f,j} \widetilde{B}_{j+1}^d(t_{j+1},t_i) \frac{B_j^d(t_j,t_{j+1}) - B_j^{d+f}(t_j,t_{j+1})}{\alpha_{f,j}} \\
&+ \rho_{x_d,x_f} \sum_{j=0}^{i-1} \sigma_{d,j}\sigma_{f,j} \widetilde{B}_{j+1}^f(t_{j+1},t_i) \frac{B_j^f(t_j,t_{j+1}) - B_j^{d+f}(t_j,t_{j+1})}{\alpha_{d,j}} \\
&+ \rho_{x_d,x_f} \sum_{j=0}^{i-1} \sigma_{d,j}\sigma_{f,j} \widetilde{B}_{j+1}^d(t_{j+1},t_i) \widetilde{B}_{j+1}^f(t_{j+1},t_i) B_j^{d+f}(t_j,t_{j+1})
\end{aligned}
$$

$$
\begin{aligned}
Cov^{\mathbb{Q}_d}\left[x_d(t_i), \ln FX_f(t_i)\right] &= \rho_{x_d,FX_f} \int\limits_0^{t_i} \frac{E_d(u)}{E_d(t_i)} \sigma_d(u)\sigma_{FX_f}(u)du \\
&+ Cov^{\mathbb{Q}_d}\left[x_d(t_i), FX_d(t_i)\right] \\
&- Cov^{\mathbb{Q}_d}\left[x_d(t_i), FX_f(t_i)\right] \\
&= \rho_{x_d,FX_f} \sum_{j=0}^{i-1} \sigma_{d,j}\sigma_{FX_f,j} B_j^d(t_j, t_{j+1}) \frac{E_d(t_{j+1})}{E_d(t_i)} \\
&+ Cov^{\mathbb{Q}_d}\left[x_d(t_i), FX_d(t_i)\right] \\
&- Cov^{\mathbb{Q}_d}\left[x_d(t_i), FX_f(t_i)\right] \\
Cov^{\mathbb{Q}_d}\left[x_f(t_i), \ln FX_d(t_i)\right] &= \rho_{x_f,FX_f} \int\limits_0^{t_i} \frac{E_f(u)}{E_f(t_i)} \sigma_f(u)\sigma_{FX_f}(u)du \\
&+ Cov^{\mathbb{Q}_d}\left[x_f(t_i), FX_d(t_i)\right] \\
&- Cov^{\mathbb{Q}_d}\left[x_f(t_i), FX_f(t_i)\right] \\
&= \rho_{x_f,FX_f} \sum_{j=0}^{i-1} \sigma_{f,j}\sigma_{FX_f,j} B_j^f(t_j, t_{j+1}) \frac{E_f(t_{j+1})}{E_f(t_i)} \\
&+ Cov^{\mathbb{Q}_d}\left[x_f(t_i), FX_d(t_i)\right] \\
&- Cov^{\mathbb{Q}_d}\left[x_f(t_i), FX_f(t_i)\right] \\
Cov^{\mathbb{Q}_d}\left[FX_d(t_i), \ln FX_f(t_i)\right] &= \rho_{x_d,FX_f} \int\limits_0^{t_i} B_d(u, t_i)\sigma_d(u)\sigma_{FX_f}(u)du \\
&+ Var^{\mathbb{Q}_d}\left[FX_d(t_i)\right] \\
&- Cov^{\mathbb{Q}_d}\left[FX_d(t_i), FX_f(t_i)\right] \\
&= \rho_{x_d,FX_f} \sum_{j=0}^{i-1} \frac{\sigma_{d,j}\sigma_{FX_f,j}}{\alpha_{d,j}} \left[t_{j+1} - t_j - B_j^d(t_j, t_{j+1})\right] \\
&+ \rho_{x_d,FX_f} \sum_{j=0}^{i-1} \sigma_{d,j}\sigma_{FX_f,j} \widetilde{B}_{j+1}^d(t_{j+1}, t_i) B_j^d(t_j, t_{j+1}) \\
&+ Var^{\mathbb{Q}_d}\left[FX_d(t_i)\right] \\
&- Cov^{\mathbb{Q}_d}\left[FX_d(t_i), FX_f(t_i)\right] \\
Cov^{\mathbb{Q}_d}\left[FX_f(t_i), \ln FX_f(t_i)\right] &= \rho_{x_f,FX_f} \int\limits_0^{t_i} B_f(u, t_i)\sigma_f(u)\sigma_{FX_f}(u)du \\
&+ Cov^{\mathbb{Q}_d}\left[FX_d(t_i), FX_f(t_i)\right] \\
&- Var^{\mathbb{Q}_d}\left[FX_f(t_i)\right] \\
&= \rho_{x_f,FX_f} \sum_{j=0}^{i-1} \frac{\sigma_{f,j}\sigma_{FX_f,j}}{\alpha_{f,j}} \left[t_{j+1} - t_j - B_j^f(t_j, t_{j+1})\right] \\
&+ \rho_{x_f,FX_f} \sum_{j=0}^{i-1} \sigma_{f,j}\sigma_{FX_f,j} \widetilde{B}_{j+1}^f(t_{j+1}, t_i) B_j^f(t_j, t_{j+1}) \\
&+ Cov^{\mathbb{Q}_d}\left[FX_d(t_i), FX_f(t_i)\right] \\
&- Var^{\mathbb{Q}_d}\left[FX_f(t_i)\right]
\end{aligned}
$$

$$
\begin{aligned}
Cov^{\mathbb{Q}_d}\left[x_{f_1}(t_i), \ln FX_{f_2}(t_i)\right] &= Cov^{\mathbb{Q}_d}\left[x_{f_1}(t_i), FX_d(t_i)\right] - Cov^{\mathbb{Q}_d}\left[x_{f_1}(t_i), FX_{f_2}(t_i)\right] \\
&+ \rho_{x_{f_1}, FX_{f_2}} \int\limits_0^{t_i} \sigma_{f_1}(u)\sigma_{FX_{f_2}}(u)du \\
&= Cov^{\mathbb{Q}_d}\left[x_{f_1}(t_i), FX_d(t_i)\right] - Cov^{\mathbb{Q}_d}\left[x_{f_1}(t_i), FX_{f_2}(t_i)\right] \\
&+ \rho_{x_{f_1}, FX_{f_2}} \sum_{j=0}^{i-1} \sigma_{f_1,j}\sigma_{FX_{f_2},j}(t_{j+1} - t_j) \\
Cov^{\mathbb{Q}_d}\left[FX_{f_1}(t_i), \ln FX_{f_2}(t_i)\right] &= Cov^{\mathbb{Q}_d}\left[FX_{f_1}(t_i), FX_d(t_i)\right] - Cov^{\mathbb{Q}_d}\left[FX_{f_1}(t_i), FX_{f_2}(t_i)\right] \\
&+ \rho_{x_{f_1}, FX_{f_2}} \int\limits_0^{t_i} B_{f_1}(u,t_i)\sigma_{f_1}(u)\sigma_{FX_{f_2}}(u)du \\
Cov^{\mathbb{Q}_d}\left[\ln FX_{f_1}(t_i), \ln FX_{f_2}(t_i)\right] &= Var^{\mathbb{Q}_d}\left[FX_d(t_i)\right] - Cov^{\mathbb{Q}_d}\left[FX_d(t_i), FX_{f_1}(t_i)\right] \\
&- Cov^{\mathbb{Q}_d}\left[FX_d(t_i), FX_{f_2}(t_i)\right] + Cov^{\mathbb{Q}_d}\left[FX_{f_1}(t_i), FX_{f_2}(t_i)\right] \\
&+ \rho_{FX_{f_1}, FX_{f_2}} \int\limits_0^{t_i} \sigma_{FX_{f_1}}(u)\sigma_{FX_{f_2}}(u)du \\
&+ \rho_{x_d, FX_{f_1}} \int\limits_0^{t_i} B_d(u,t_i)\sigma_d(u)\sigma_{FX_{f_1}}(u)du \\
&+ \rho_{x_d, FX_{f_2}} \int\limits_0^{t_i} B_d(u,t_i)\sigma_d(u)\sigma_{FX_{f_2}}(u)du \\
&- \rho_{x_{f_1}, FX_{f_2}} \int\limits_0^{t_i} B_{f_1}(u,t_i)\sigma_{f_1}(u)\sigma_{FX_{f_2}}(u)du \\
&- \rho_{x_{f_2}, FX_{f_1}} \int\limits_0^{t_i} B_{f_2}(u,t_i)\sigma_{f_2}(u)\sigma_{FX_{f_1}}(u)du
\end{aligned}
$$

where $\alpha_{d+f} = \alpha_d + \alpha_f$,

$$E(t) = e^{\alpha t},$$

$$B(t_1, t_2 \geq t_1) = \frac{1 - e^{-\alpha(t_2 - t_1)}}{\alpha} = E(t_1)\int_{t_1}^{t_2} \frac{1}{E(\tau)} d\tau,$$

From covariance formulas one can get terminal correlation between listed risk factors, using the standard formula:

$$Corr\left[X, Y\right] = \frac{Cov\left[X, Y\right]}{\sqrt{Var\left[X\right] Var\left[Y\right]}}$$

### 3.2.2 Model Parameters and Market Data

1. **Domestic Curve** – ClCurvePoints (defined according to ClCcyPrimaryCurve specification) – **used in bond price simulation in domestic currency.** (see Section 3.1.2)

2. **Foreign Curve** – ClCurvePoints (defined according to ClCcyPrimaryCurve specification) – **used in bond price simulation in foreign currency.** (see Section 3.1.2)

3. **FX Spot** – ClFactorValue (Factor: FxPair.CCY1/CCY2, ValueType: FxSpot) – $FX(0)$ **used as initial point for simulations.**

4. **Volatility Curve** – **instantaneous FX volatility** $\sigma(t)$ is defined in one of the following ways:

(a) SbPslRwModelSettings.FXVols (Key: ClFxPairKey, Value: ClFactorSeries (manually defined, no restrictions on Factor and ValueType))

(b) SbSharedMarketSimulationModelSettings(Default Market Simulation Model Settings).FxModelDefaults (Key: ClCcyKey1,ClCcyKey2 Value: ClDouble (manually defined); in this case the curve is flat)

(c) ClFactorSeries (Factor: Model.CCY_FX_MODEL, ValueType: Vol) – historical implied volatilities (normally used for real world simulations) estimated using exponentially-weighted moving average (EWMA) of the volatility time series passed from the CVA system. EWMA approach works as follows:

$$S_t = \alpha Y_t + (1-\alpha)S_{t-1}, \quad 1 < t \le T, \tag{23}$$

where $Y_t$ is volatility at time $t$, $S_t$ is EWMA value at time $t$, initial value is $S_1 = \alpha Y_1$, parameter $\alpha$ ($0 < \alpha \le 1$) is defined by user, default value is 0.95. Final value is adjusted to make sum of weights equal to 1: $EWMA = S_T \times \frac{1}{1-(1-\alpha)^T}$. If $\alpha = 0$, EWMA is arithmetic mean of $Y_t$.

5. **Idiosyncratic Risk Premium Curve** – ClCurvePoints (Curve: Idiosyncratic Risk Premium: CCY1/CCY2) – $DF_{RP}(t)$ **deflator being used to calculate FX spot in the real-world measure.** If the instantaneous market price of risk, $\lambda_{FX}(t)$, is provided instead of the discount curve $DF_{RP}(t)$, the corresponding discount curve can be calculated as follows:

$$DF_{RP}(t) = \exp\left[-\int_0^t \lambda_{FX}(u)\sigma_{FX}(u)du\right].$$

#### 3.2.3 Calibration

To calibrate $\sigma(t)$ (given interest rate models) a European option set is used. Calibration is performed numerically (using *LevenbergMarquardt.Minimize*) by minimization difference between model and market price (based on either absolute error, relative error or implied volatility error):

$$\{\sigma_i\} = \arg\min_{\{\sigma_i\}} \sqrt{\sum_j w_j e_j^2(k, \{\alpha_i, \sigma_i\}}, \tag{24}$$

where

- $\sigma_i$ is volatility at the time interval $t_{i-1} \le t < t_i$ $(i = \overline{1,n})$;
- $w_j$ is weight of instrument $j$, all **set equal to** 1 **as default**;
- $e_j(\{\sigma_i\})$ is difference between market and model price of instrument $j$ (defined in the field *Instruments*). Can be *absolute*, *relative* or expressed *in terms of implied volatility* (defined in the *CalibrationErrorType* field):

  (i) $e_j(\{\sigma_i\}) = \pi_j^{market} - \pi_j^{model}(\{\sigma_i\})$,

  (ii) $e_j(\{\sigma_i\}) = \dfrac{\pi_j^{market} - \pi_j^{model}(\{\sigma_i\})}{\pi_j^{market}}$,

  (iii) $e_j(\{\sigma_i\}) = \dfrac{\pi_j^{market} - \pi_j^{model}(\{\sigma_i\})}{\sigma_j^{imp}}$,

  respectively. **Relative differences are used on default**.

If *IterativeCalibration* is set to ✓ → each $\sigma_i$ is independently calibrated using (24) and the corresponding group of instrument $i$. Groups are formed through instruments' settings (e.g., *SwaptionInstrumentsSettings* in *SwaptionInstrumentsSettings*). Otherwise calibration is performed simultaneously, when all $\sigma_i$ are all calibrated using the same group of calibration instruments.

If calibration is unsuccessful, $\sigma_i$ takes the value from *DefaultSigma*.

**Calibration details – tab description**

Common fields:

- Calibration status
- Calibration Error Type
- Total Price Error (TPE) = $\sqrt{\sum_i PE_i^2 \times IW_i^2}$
- Total Relative Price Error (TRPE) = $\sqrt{\sum_i PRE_i^2 \times IW_i^2}$
- Total Implied Vol Error (TIVE) = $\sqrt{\sum_i IVE_i^2 \times IW_i^2}$
- Total RMSE (coincides with the error defined in *Calibration Error Type*)

Individual fields:

- Instrument
- Instrument Weight (IW)
- Calibration Status
- Market Price
- Model Price
- Price Error (PE) = $marketValue - modelValue$
- Relative Price Error (RPE) = $\frac{marketValue - modelValue}{marketValue}$ if $marketValue \neq 0$ and 0 otherwise
- Market Volatility (MV)
- Implied Model Volatility (IMV)
- Implied Vol Error (IVE) = $IMV - MV$

Remark. The application might display values with lower precision compared to computations (i.e., the displayed results might have been rounded).

**Calibration Instruments Settings** Calibration instrument price is given by the volatility surface (FxVolSurface). Volatility surface contains market European option volatility for various levels of delta (by column: at-the-money volatility as well as butterfly and risk reversal volatilities) and option maturity (by row).

### 3.2.4 Simulation Algorithm

FX process values at a discrete grid $0 = t_0 < t_1 < ... < t_n$ can be performed using the following algorithm

**Initialization**

1. Upload the input parameters: Domestic curve, Foreign curve, Volatility curve, Idiosyncratic Risk Premium curve, FX Spot (see Section 3.2.2 for the description of these parameters).
2. Initialize the initial state $FX(0)$ with the initial FX Spot value.

**Iterations** $i = 1, \ldots, n$

3. Value $X(t_i)$ in the BS model can be computed using the formula:

$$FX(t_i) = FX(t_{i-1}) \exp\left[ \left(r_d(t_{i-1}) - r_f(t_{i-1})\right) \Delta t_i - \frac{1}{2}\sigma_i^2 \Delta t_i + \sigma_i \sqrt{\Delta t_i} \epsilon_i \right],$$

$$\Delta t_i = t_i - t_{i-1},$$

$$\sigma_i = \sqrt{\frac{1}{\Delta t_i} \int_{t_{i-1}}^{t_i} \sigma^2(t) dt},$$

integral of piecewise volatility can be calculated using *ClIntegrationUtils.IntegratePiecewiseRegular*, $\epsilon_i$ is a random draw from $N(0,1)$ (generated by *IClOneFactorModel*).

Short rate values $r_d(t)$ and $r_f(t)$ can be taken from either the short rate model simulations or the discount curves (using method *DiscountCurve.ShortRate*)

4. The obtained values $FX(t_i)$ are adjusted by a factor $\beta(t)$:

$$FX(t_i, \omega) := \beta(t_i) FX(t_i, \omega), \tag{25}$$

in order to satisfy martingale property:

$$P_f(t,T)FX(t) = E^{\mathbb{Q}^d}\left[FX(T) \times SDF_d(t,T)|\mathcal{F}_t\right]. \tag{26}$$

This can be proved as follows:

$$\begin{aligned} E^{\mathbb{Q}^d}\left[FX(T) \times SDF_d(t,T)|\mathcal{F}_t\right] = P_d(t,T)E^{\mathbb{Q}^d_T}\left[FX(T)|\mathcal{F}_t\right] = \\ = P_d(t,T)F(t,T) = P_d(t,T)FX(t)\frac{P_f(t,T)}{P_d(t,T)} = P_f(t,T)FX(t), \end{aligned}$$

where $P_{d/f}(0,t_i)$ are domestic/foreign bond prices (taken from the corresponding discount curve), $SDF_{d/f}(t_i,\omega)$ are corresponding stochastic discount factors, $F(t,T)$ is an FX forward between $t$ and $T$.

Hence, average over all paths $\omega$ should coincide with the corresponding values of the forward curve: where

$$\beta(t_i) = \frac{FX(0)P_f(0,t_i)}{\frac{1}{N_\omega}\sum_\omega SDF_d(t_i,\omega)FX(t_i)}. \tag{27}$$

**Real-world simulation.** To simulate trajectories in the real-world measure each FX trajectory $FX(t_i)$ is multiplied by a discounted multiplier associated with idiosyncratic risk premium curve (*idiosyncraticRiskPremiumCurve_.Df*$(t_i)$):

$$X_{RW}(t_i) = \frac{X(t_i)}{DF_{RP}(t_i)}. \tag{28}$$

### 3.3 Commodity Price Model: Schwartz1F

#### 3.3.1 Theoretical Background

Schwartz model assumes the following risk-neutral dynamics for the spot price $S(t)$ (see [8] for details):

$$\begin{aligned}\frac{dS(t)}{S(t)} &= \left(\theta(t) + \frac{\sigma(t)^2}{2} - k \ln S(t)\right) dt + \sigma(t) d\widetilde{W}(t) \\ &= \left(\theta(t) + \lambda_{CMDTY}(t)\sigma(t) + \frac{\sigma(t)^2}{2} - k \ln S(t)\right) dt + \sigma(t) dW(t),\end{aligned}$$

or, applying Ito's lemma:

$$\begin{aligned}d \ln S(t) &= (\theta(t) - k \ln S(t))\, dt + \sigma(t) d\widetilde{W}(t) \\ &= (\theta(t) + \lambda_{CMDTY}(t)\sigma(t) - k \ln S(t))\, dt + \sigma(t) dW(t),\end{aligned}$$

where

- $\theta(t)$ is the risk-neutral drift (**implicitly calibrated to futures prices** at the martingale adjustment step; see calibration algorithm in Section 3.3.4). It can be related to the instantaneous short-rate and the convenience yield of the commodity.
- $k$ is **constant** mean-reversion speed parameter (**set manually**; see list of input parameters in Section 3.3.2).
- $\sigma(t)$ is the **piecewise** instantaneous volatility of the log of the spot (**calibrated to European commodity options prices**; see calibration algorithm in Section 3.3.3).
- $\widetilde{W}(t)$ and $W(t)$ are standard Brownian motions under **risk neutral (RN)** and **real world (RW) measures** respectively.
- $\lambda_{CMDTY}(t)$ is the **piecewise** risk premium (**estimated on historical data**; see Section 4 for details), that determines transition from RN to RW measure: $\widetilde{W}(t) = W(t) + \int_0^t \lambda_{CMDTY}(s) ds$.

It is more convenient to work with the log of the spot price, $X(t)$:

$$S(t) = e^{X(t)}, \tag{29}$$

$$dX(t) = (\theta(t) - kX(t))\, dt + \sigma(t) dW(t), \tag{30}$$

where $W(t)$ is the standard Brownian motion, $\sigma(t)$ is volatility, $\theta(t)$ is risk-neutral drift which price can be calculated using the forward curve, $k$ is the mean reversion speed, $\theta(t)$ is the mean reversion target.

The process $X(t)$ is given by:

$$X(t) = X(s)e^{-k(t-s)} + \int_s^t e^{-k(t-u)}\theta(u) du + \int_s^t e^{-k(t-u)}\sigma(u) dW(u)$$

It can be split into deterministic and stochastic components:

$$\begin{aligned}X(t) &= \alpha(t) + x(t) \\ \alpha(t) &= X(0)e^{-kt} + \int_0^t e^{-k(t-u)}\theta(u) du \\ x(t) &= \int_0^t e^{-k(t-u)}\sigma(u) dW(u)\end{aligned}$$

whose dynamics is:

$$\begin{aligned}\dot{\alpha}(t) &= \theta(t) - k\alpha(t) \\ dx(t) &= -kx(t)dt + \sigma(t)dW(t)\end{aligned}$$

We indicate the price of a Futures contract expiring in $T$ and observed in $t$ with with $F(t,T)$. It is given by:

$$\begin{aligned} F(t,T) &= \mathbb{E}_t\big[S(T)\big] = \mathbb{E}_t\big[e^{\alpha(T)+x(T)}\big] \\ &= e^{\alpha(T)}\mathbb{E}_t\big[e^{x(t)e^{-k(T-t)}+\int_t^T e^{-k(T-u)}\sigma(u)dW(u)}\big] \Rightarrow \\ \ln F(t,T) &= \alpha(T) + x(t)e^{-k(T-t)} + \frac{1}{2}V_x(t,T) \end{aligned}$$

where:

$$V_x(t,T) = \int_t^T e^{-2k(T-u)}\sigma^2(u)du$$

is the state factor variance. The model's drift is then calibrated to the spot Futures curve $F(0,T)$:

$$\alpha(T) = \ln F(0,T) - \frac{1}{2}V_x(t,T)$$

so that the Futures price is given by the reconstruction formula:

$$\begin{aligned} \ln F(t,T) &= \ln F(0,T) + x(t)e^{-k(T-t)} - \frac{1}{2}(V_x(0,T) - V_x(t,T)) \\ &= \ln F(0,T) + x(t)e^{-k(T-t)} - \frac{1}{2}e^{-2k(T-t)}V_x(0,t) \end{aligned}$$

Futures Prices are martingales in the risk-neutral measure:

$$\frac{dF(t,T)}{F(t,T)} = \sigma(t)e^{-k(T-t)}dW(t)$$

**Futures contract prices in the risk-neutral measure**

Let $F(t,T) = \mathbf{E}\left[S(T)|\mathcal{F}_t\right]$ denote price of the futures contract with maturity $T$ at time $t$. Its value is given by:

$$F(t,T) = F(0,T)\exp\left(x(t)e^{-k(T-t)} - \frac{1}{2}\int_0^t e^{-2k(T-u)}\sigma^2(u)du\right), \tag{31}$$

where $F(0,T)$ is the initial price (given in the forward curve), $x(t)$ is the stochastic process which given by:

$$x(t) = x(s)e^{-k(t-s)} + \int_s^t e^{-k(t-u)}\sigma(u)dW(u), \quad x(0) = 0. \tag{32}$$

**Futures contract prices in the real-world measure**

An analogous formula is valid for the real-world measure if we add a risk-premium adjustment:

$$\widetilde{F}(t,T) = \frac{F(0,T)}{D_R(T)}\exp\left((\widetilde{x}(t) + \ln D_R(t))e^{-k(T-t)} - \frac{1}{2}\int_0^t e^{-2k(T-u)}\sigma^2(u)du\right), \tag{33}$$

$$\widetilde{x}(t) = \widetilde{x}(s)e^{-k(t-s)} + \Big(-\ln D_R(t) + e^{-k(t-s)}\ln D_R(s)\Big) + \int_s^t e^{-k(t-u)}\sigma(u)d\widetilde{W}(u), \tag{34}$$

where $\widetilde{x}(0) = 0$, $\widetilde{W}(t)$ is a standard Brownian motion in the real-world measure, $D_R(t)$ is the discount curve:

$$D_R(t) = \exp\left(-\int_0^t e^{-k(t-u)}\lambda(u)\sigma(u)du\right), \tag{35}$$

$\lambda(t)$ is the market price of risk.

#### 3.3.2 Model Parameters and Market Data

1. **Discount Curve** – defined according to ClRateCurveField specification – **market discount curve** $P^M(0,t)$ **used as a discount factor** in future payment discounting.

2. **Spot Value** – double – $S(0)$ **used as initial point for simulations.**

3. **Convenience Yield Curve** – defined according to ClRateCurveField specification – containing discount factors **allowing** $Df_{CY}(t)$ **forward prices reconstruction:** $F(0,t) = S(0)\frac{Df_{CY}(t)}{P^M(0,t)}$.

4. **Volatility Curve** – defined according to ClCurveField specification – **instantaneous volatility of logarithm of spot** $\sigma(t)$.

5. **Mean Reversion Value** – double – **mean reversion skew (log-spot mean reversion** $k$**)** is defined manually.

6. **Commodity Risk Premium Curve** $D_R(t)$ – defined according to ClRateCurveField specification – $D_R(t)$ **deflator being used to calculate commodity spot prices in the real-world measure.** If the instantaneous market price of risk, $\lambda_{CMDTY}(t)$, is provided instead of the discount curve $D_R(T)$, the corresponding discount curve can be calculated as follows:

$$D_R(t) = \exp\left(-\int_0^t e^{-k(t-u)}\lambda_{CMDTY}(u)\sigma(u)du\right).$$

#### 3.3.3 Calibration

To calibrate model parameter $\sigma(t)$ (piecewise constant function) we use a set of European options. Calibration is performed numerically (using method *LevenbergMarquardt.Minimize*) through minimisation of error between model and market price (which can be based on either absolute error, relative error or implied volatility error):

$$\{\sigma_i\} = \arg\min_{\{\sigma_i\}} \sqrt{\sum_j w_j e_j^2(k, \{\sigma_i\})}, \tag{36}$$

where $\sigma_i$ is volatility at the interval $t_{i-1} \le t < t_i$ $(i = \overline{1,n})$, $w_j$ is weight of the instrument $j$, $e_j(k,\{\sigma_i\})$ is calibration error (between model and market price). Model price is calculated using (146) for the given values of $k$ and $\{\sigma_i\}$.

**Calibration details – tab description**
Common fields:

- Calibration status
- Calibration Error Type
- Total Price Error (TPE) = $\sqrt{\sum_i PE_i^2 \times IW_i^2}$
- Total Relative Price Error (TRPE) = $\sqrt{\sum_i PRE_i^2 \times IW_i^2}$
- Total Implied Vol Error (TIVE) = $\sqrt{\sum_i IVE_i^2 \times IW_i^2}$
- Total RMSE (coincides with the error defined in *Calibration Error Type*)

Individual fields:

- Instrument
- Instrument Weight (IW)
- Calibration Status
- Market Price

– Model Price

– Price Error (PE) = $marketValue - modelValue$

– Relative Price Error (RPE) = $\frac{marketValue - modelValue}{marketValue}$ if $marketValue \neq 0$ and 0 otherwise

– Market Volatility (MV)

– Implied Model Volatility (IMV)

– Implied Vol Error (IVE) = $IMV - MV$

Remark. The application might display values with lower precision compared to computations (i.e., the displayed results might have been rounded).

**Calibration Instruments Settings** Calibration instrument price is given by the volatility surface (CommodityVolSurface). Volatility surface contains market European option volatility for various levels of delta (by column) and option maturity (by row).

#### 3.3.4 Simulation Algorithm

**Risk-neutral simulation.** For the time grid $0 = t_0 < t_1 < ... < t_n$ price dynamics can be simulated using (31) and (32) as follows:

1. Calculate trajectory of the process (32) using the expression (method *ClSchwartz1FSimulation.CalculateTo*):

$$x(t_i) = x(t_{i-1})e^{-k(t_i - t_{i-1})} + e^{-kt_i}\epsilon_i\sqrt{\int_{t_{i-1}}^{t_i} e^{2ku}\sigma^2(u)du}, \tag{37}$$

   where $x(t_0) = 0$, $\epsilon_i \sim N(0,1)$ is a standard normal random variable. The integral can be calculated numerically (using *DoubleExponentialTransformation.Integrate*).

2. Formula (31) is used to calculate trajectory of $F(t,T)$ $(F(t_0,T), F(t_1,T), ...F(t_{,n},T))$. $F(0,T)$ is given in the forward curve

**Real-world simulation.** Simulation of the process (34) is performed with an adjustment:

$$\widetilde{x}(t_i) = x(t_i) - \ln D_R(t_i) + e^{-k(t_i - t_{i-1})}\ln D_R(t_{i-1}). \tag{38}$$

Value $D_R(t)$ is give by the market price of risk discount curve.

Futures prices are simulated using the same principle according to (33).

**Martingale adjustment.** Martingale adjustment is a consequence of the fact that $F(t,T)$ is a martingale in the risk neutral measure and means that each path $F(t_i,T,\omega)$ ($\omega$ is a path index) is normalised by a factor $\beta(t_i,T)$ which makes sample mean of $F(t_i,T,\omega)$ equal to the initial value $F(0,T)$, i.e.

$$\frac{1}{N_\omega}\sum_\omega F(t_i,T,\omega) = F(0,T), \tag{39}$$

which corresponds to the definition of a martingale process:

$$E\left(F(t_i,T)|\mathcal{F}_0\right) = F(0,T). \tag{40}$$

This normalisation is performed as follows:

$$F(t_i,T,\omega) := \beta(t_i,T)F(t_i,T,\omega), \tag{41}$$

where $\beta(t,T)$ is a sample mean of all paths $F(t,T)$:

$$\beta(t_i,T) = \frac{F(0,T)}{\frac{1}{N_\omega}\sum_\omega F(t_i,T,\omega)}, \tag{42}$$

$\omega$ is a path index.

## 3.4 Credit Spread Model: BK1F

### 3.4.1 Theoretical Background

In this section we consider derivative instruments on credit events (such as bond obligor's default). We assume that the probability of no default occurring within time frame from $t$ to $T$ given that default has not happened before (survival probability $ESP(t,T)$) is given by:

$$ESP(t,T) = \mathbf{E}^{\mathbb{Q}}\left[e^{-\int_t^T \lambda(s)ds}|\mathcal{F}_t\right], \tag{43}$$

where $\mathbb{Q}$ is the risk-neutral measure, $\lambda(s)$ is a stochastic hazard rate:

$$\begin{cases} \lambda(t) = \exp\left(\alpha(t) + x(t)\right), \\ dx(t) = -kx(t)dt + \sigma d\widetilde{W}(t), \end{cases}$$

where

- $\alpha(t)$ is an adjustment term (**implicitly calibrated** by equating $e^{-\int_0^T \lambda(s)ds}$ to the corresponding values of the credit discount curve *MarketCreditCurve.Df*; see calibration algorithm in Section 3.4.4).
- $k$ is **constant** mean-reversion speed parameter (**set manually**; see list of input parameters in Section 3.4.2).
- $\sigma$ is **constant** volatility parameter (**set manually**; see list of input parameters in Section 3.4.2).
- $\widetilde{W}(t)$ is standard Brownian motion under **risk neutral (RN) measure.**

At any time theoretical value instantaneous hazard rate can be calculated as follows:

$$\lambda(t) = -\frac{d}{dT}\log ESP(t,T)|_{T=t}. \tag{44}$$

In practice, we use the following expression:

$$\lambda(t) \approx -\frac{1}{\Delta t}\log ESP(t, t+\Delta t). \tag{45}$$

**Market credit curve reconstruction** Market credit curve $MarketCreditCurve.Df(t) = ESP(0,t)$ is reconstructed from the set of credit default swaps market quotes $CDS_{market}(0,T_1), CDS_{market}(0,T_2), \dots CDS_{market}(0,T_i)$ by iteratively solving of the equations

$$P(0,T_i)ESP(0,T_i)C - P(0,T_i)N\nu\left(ESP(0,T_{i-1}) - ESP(0,T_i)\right) = CDS_{market}(0,T_i) - CDS_{market}(0,T_{i-1})$$

with respect to $ESP(0,T_i)$ (given all the previous values $ESP(0,0) = 1, ESP(0,T_1)\dots ESP(t,T_{i-1})$). Here $P(t,T)$ is the zero-coupon bond price and $\nu = (1-R)$. Equation is solved numerically using *Brent.FindRoot*.

Hazard rate values are reconstructed as follows:

$$\lambda(T_i) \approx -\frac{1}{T_i - T_{i-1}}\log\frac{ESP(0,T_i)}{ESP(0,T_{i-1})}. \tag{46}$$

### 3.4.2 Model Parameters and Market Data

1. **Discount Curve** – defined according to ClRateCurveField specification – **market discount curve** $P^M(0,t)$ **used as a discount factor** in future payment discounting.
2. **Credit Curve** – defined according to ClRateCurveField specification – **market credit curve** contains default probabilities.
3. **Volatility Parameter** – double – **instantaneous volatility of logarithm of instantaneous hazard rate** $\sigma$ is defined manually.

4. **Mean Reversion Value** – double – **mean reversion skew (log of instantaneous hazard rate mean reversion** $k$**)** is defined manually.

5. **Hazard Rate Real World Adjustment Curve** – defined according to ClRateCurveField specification – $A(t)$ adjustment curve **used for transferring instantaneous hazard rate,** $\lambda(t)$**, dynamics** from RN to RW measure. Currently **not used.**

6. **Survival Probability Real-World Adjustment Parameter** – double – $c$ constant **used for transferring default time,** $\tau_{\omega i}$**, dynamics** from RN to RW measure. Currently **not used.**

#### 3.4.3 Calibration

Current version of PFE Engine does not support any type of calibration. Therefore, model parameters $\{k, \sigma\}$ are set manually.

#### 3.4.4 Simulation Algorithm

**State variable simulation.** First of all, we generate traectories of the process $x(t)$ (*unadjHazardRate_*). They are generated on the discrete grid $0 = t_0 < t_1 < ... < t_n$ as follows:

$$x(t_i) = x(t_{i-1})e^{-k\Delta t_i} + \epsilon_i\sqrt{V(t_{i-1}, t_i)}, \quad x(t_0) = 0, \tag{47}$$

where

$$V(t_1, t_2) = \sigma\frac{1 - e^{-2k(t_2 - t_1)}}{2k}, \tag{48}$$

$x(t_0) = 0$, $\Delta t_i = t_i - t_{i-1}$, $\epsilon_i$ is the ormal random variable $\epsilon_i \sim (0, 1)$.

**Stochastic survival probability.** Stochastic survival probability (unconditional), $SSP(t)$, is given by (in logarithmic terms, *lnSspDict_*):

$$\ln SSP(t_i) = \ln SSP(t_{i-1}) - \Delta t_i e^{\alpha(t_i) + x(t_i)}, \tag{49}$$

where $SSP(t_0) = 1$, $\Delta t_i = t_i - t_{i-1}$, $\alpha(t_i)$ is calculated as root of:

$$\frac{1}{N_\omega}\sum_\omega \exp\left[\ln SSP(t_{i-1}, \omega) - \Delta t_i e^{\alpha(t_i) + x(t_i, \omega)}\right] =$$

$$= MarketCreditCurve.Df(t_i), \tag{50}$$

where $\omega$ is a trajectory number, $N_\omega$ is the number of trajectories,
$MarketCreditCurve.Df(t_i)$ is the value of credit discount curve at point $t_i$. The equation is solved numerically using *Brent.FindRoot*. Initial value of $\alpha$ is given by:

$$\alpha(t_0) = \log\left(-\frac{\log MarketCreditCurve.Df(t_1)}{t_1 - t_0}\right). \tag{51}$$

**Hazard rate.** Hazard rate trajectories $\lambda(t_i)$ are then given by:

$$\lambda(t_i, \omega) = \exp\left(\alpha(t_i) + x(t_i, \omega)\right). \tag{52}$$

Initial value of hazard rate is given by:

$$\lambda(t_0) = \exp\left(\alpha(t_0)\right) = -\frac{\log MarketCreditCurve.Df(t_1)}{t_1 - t_0}. \tag{53}$$

**Expected survival probability.** Expected survival probability (conditional) is calculated using Semi-Analytical Lattice Integration approach (SALI). For each time step $t_i$, the discrete lattice $x_{i\alpha}$ is the uniform grid for the interval from $-[w\sqrt{V(t_0, t_i)} + \max(0, \log A(t))]$ to $[w\sqrt{V(t_0, t_i)} + \max(0, \log A(t))]$ with $N(t_i)$ vertices:

$$N(t_i) = \left\lceil \frac{2P[w\sqrt{V(t_0, t_i)} + \max(0, \log A(t))]}{\sqrt{V(t_{i-1}, t_i)}} \right\rceil \tag{54}$$

where $P$ is the precision parameter, $w$ is the width parameter (in terms of standard deviation), $A(t)$ is given by the corresponding value of the risk premium discount curve (*hazardRateRealWorldAdjustmentCurve_.Df*).

To calculate the survival probability we introduce a set of auxiliary variables. We consider matrix $G_{\alpha\beta}(t_i, t_j)$ with elements

$$G_{\alpha\beta}(t_i, t_j) = P\left[x(t_j) = x_{j\beta} | x(t_i) = x_{i\alpha}\right], \tag{55}$$

($\alpha$ is row number, $\beta$ is column number). It is the probability of $x(t)$ being close to grid elemenent $x_{j\beta}$ at time $t_j$ given $x(t_i) = x_{i\alpha}$. One step transition probabilities are given by:

$$G_{\alpha\beta}(t_{i-1}, t_{i+1}) = \mathbf{GLQ}(\mu_\alpha, \sqrt{V(t_{i-1}, t_{i+1})}, m^l_\alpha, q^l_\alpha, x_{\alpha-1}, x_\alpha) + \mathbf{GLQ}(\mu_\alpha, \sqrt{V(t_{i-1}, t_{i+1})}, m^r_\alpha, q^r_\alpha, x_\alpha, x_{\alpha+1}), \tag{56}$$

where

$$\mathbf{GLQ}(\mu, \sigma, m, q, x_l, x_u) = (m\mu + q)\left(\Phi\left(\frac{x_u - \mu}{\sigma}\right) - \Phi\left(\frac{x_l - \mu}{\sigma}\right)\right) + m\sigma\left(\phi\left(\frac{x_u - \mu}{\sigma}\right) - \phi\left(\frac{x_l - \mu}{\sigma}\right)\right),$$

$$m^l_\alpha = \frac{1}{x_\alpha - x_{\alpha-1}}, \quad m^u_\alpha = -\frac{1}{x_{\alpha+1} - x_\alpha},$$

$$q^l_\alpha = \frac{1}{2} - m^l_\alpha \frac{x_{\alpha-1} + x_\alpha}{2}, \quad q^u_\alpha = \frac{1}{2} - m^u_\alpha \frac{x_\alpha + x_{\alpha+1}}{2},$$

$\Phi(.)$ is the standard normal c.d.f., $\phi(.)$ is the standard normal p.d.f. Multi-step transition probability matrices $G(t_i, t_j)$ are calculated recursively (method *ClOUGrid.CalcCondProbs*):

$$G(t_i, t_j) = \prod_{s=i}^{j-1} Q(t_s, t_{s+1}) \tag{57}$$

We also introduce matrix $Q(t_i, t_{i+1})$ with elements $Q_{\alpha\beta}(t_i, t_{i+1})$ which are known as Arrow-Debreu prices ($\alpha$ is row number, $\beta$ is column number):

$$Q_{\alpha\beta}(t_i, t_{i+1}) = \exp\left(-(t_{i+1} - t_i)e^{\alpha(t_{i+1}) + x(t_{i+1})}\right) G_{\alpha\beta}(t_i, t_{i+1}). \tag{58}$$

In this case, expected survival probability $ESP(t_i, T)$ for each element of the discrete lattice value $x_{i\alpha}$ of slice $x(t_i)$ can be expressed in terms of its future values $ESP_\beta(t_j, T)$, $(t_i < t_j \le T)$ as follows:

$$ESP_\alpha(t_i, T) = \sum_\beta \left[\prod_{s=i}^{j-1} Q(t_s, t_{s+1})\right]_{\alpha\beta} ESP_\beta(t_j, T). \tag{59}$$

The summation is performed by all elements $\alpha$ of the matrix $\prod_{s=i}^{j-1} Q(t_s, t_{s+1})$ (product by all intermediate subintervals $(t_s, t_{s+1})$ of the interval $(t_i, t_j)$).

Value of $ESP(t, T)$ at $t$ (which do not belong to the discrete time grid) can be calculated via linear interpolation. First, interpolation is performed by the second argument, between slices $ESP(t_{j+1}, t_{j+1}) = 1$ and $ESP(t_j, t_{j+1})$ $(t_j < T \le t_{j+1})$.

Afterwards, interpolation is performed by the state variable $x(t)$:

$$ESP(t, T; x(t)) = ESP_\alpha(t, T) + \frac{ESP_\alpha(t_{i+1}, T) - ESP_\alpha(t_i, T)}{x_{t_{i+1},\alpha} - x_{t_i,\alpha}}(x(t) - x_{t_i,\alpha}) \tag{60}$$

where $t_i \le t < t_{i+1}$, $\alpha$ is the nearest lattice element.

**Calibration to the credit discount curve** Values $\alpha(t_i)$ are calibrated through iterative solution of the following equation (with respect to $\alpha(t_i)$):

$$\sum_\beta Q_\beta(t_i) = MarketCreditCurve.Df(t_i). \tag{61}$$

where $Q_\beta(t_i)$ is defined by the recursive rule:

$$Q_\beta(t_i) = \sum_\omega Q_\omega(t_{i-1}) \exp\left(-(t_i - t_{i-1})e^{\alpha(t_i)+x(t_i)}\right) G_{\alpha\beta}(t_{i-1}, t_i), \tag{62}$$

where $Q_\omega(t_0) = 1$. Equation (61) is solved numerically, using
*Brent.FindRoot* method.

**Real-world simulation.** In the real-world measure, hazard rate simulation is performed according to the same principle. The only distinction is that risk-neutral hazard rate is simulated in the real-world measure $\mathbb{P}$:

$$\lambda_{RW}(t) = \lambda(t) \times A(t). \tag{63}$$

Real-world adjustment $A(t)$ is given by the corresponding value of the risk premium discount curve (*hazardRateRealWorldAdjustmentCurve_.Df*) which is provided externally. This curve can be interpreted in terms of historical and risk neutral drifts of $\log\lambda(t)$:

$$\log A(t) = E^{\mathbb{P}}\left[\log\lambda(t)|\mathcal{F}_0\right] - E^{\mathbb{Q}}\left[\log\lambda(t)|\mathcal{F}_0\right]. \tag{64}$$

The stochastic survival probabilities in the real-world measure are defined as follows

$$SSP^{RW}(t,\omega) = e^{-\int_0^t h(s)ds} \tag{65}$$

where $h(s)$ is the real-world default intensity expressed in the real-world measure which is assumed to be proportional to the risk-neutral intensity expressed in the real world measure:

$$h(t) = c\lambda_{RW}(t). \tag{66}$$

Note that $SSP^{RW}(t,\omega)$ differs from the risk-neutral probability expressed in the real-world measure: the latter one represents the dynamics of risk-neutral market expectations regarding future defaults while actual default probability might be different. Hence, we obtain

$$SSP^{RW}(t,\omega) = e^{c\ln SSP(t,\omega)}. \tag{67}$$

**Default event simulation.** To simulate default events, we create $M$ copies of each trajectory. For each copy $i$ we generate a uniform random number $U_{\omega i}$:

$$U_{\omega i} = \Phi^{-1}(Z_{\omega i}) \tag{68}$$

where $Z_{\omega i}$ are random draws from the standard normal distribution and $\Phi^{-1}(z)$ is the inverse c.d.f. of the standard normal distribution. Values $Z_{\omega i}$ are correlated with the analogous values for the other obligors and the corresponding correlations are defined in the *Cpties Correlation* tab.

The default times $\tau_{\omega i}$ on each path are defined as follows:

$$\tau_{\omega i} = \inf\{t : U_{\omega i} \geq SSP^{RW}(t,\omega)\}. \tag{69}$$

The prices of the derivative instruments at the defaulted paths are adjusted and set to their value in case of the default.

## 3.5 Equity Price Model: BS1F with Buehler Parametrisation

### 3.5.1 Theoretical Background

Here we discuss the model parametrization first introduced by Buehler at al. [7]. Using the following notation:

- $S(t)$: Equity price at time $t$
- $\{\tau_k, k = 1 \dots n\}$: Equity ex-dividend dates
- $\{\alpha_k, k = 1 \dots n\}$: Absolute dividend amounts
- $\{\beta_k, k = 1 \dots n\}$: Proportional dividend factors
- $d(t)$: Dividend paid at $t$.
- $F(t, T)$: Equity forward for delivery at time $T$, observed at time $t$
- $P(t, T)$: Risk-free discount factor between times $t$ and $T$
- $N(t)$: Risk-neutral numeraire at time $t$
- $\mathbb{Q}$: Risk-neutral pricing measure

we first define how dividends affect the equity price on the ex-div dates $\{\tau_k\ k \in 1 \dots n\}$, when first a proportional dividend and then an absolute one is paid:

$$d(\tau_k) = \alpha_k + \beta_k S(\tau_k^-)$$

By no-arbitrage then the stock price jumps as follows:

$$S(\tau_k) = S(\tau_k^-)(1 - \beta_k) - \alpha_k$$

By investing the proportional dividends (both discrete and continuous) in the stock itself and forward selling the absolute ones, we can write the equity forward for delivery at time $T$ as:

$$F(t, T) = \frac{S(t) - D(t, T)}{\bar{P}(t, T)} \tag{70}$$

where:

$$\begin{aligned}
\bar{P}(t, T) &= \frac{P(t, T)}{Q(t, T)} \\
Q(t, T) &= \prod_{k: t < \tau_k \le T} (1 - \beta_k) \\
D(t, T) &= \sum_{k: t < \tau_k \le T} \alpha_k \bar{P}(t, \tau_k)
\end{aligned}$$

From this we can derive an expression for the risk-neutral dynamics of $S(t)$. As $F(t, T)P(t, T)$ is a tradeable there must exist a $\mathbb{Q}$-martingale $Y_T(t)$ such that, $\forall t \le T$:

$$F(t, T)P(t, T) = Q(t, T)(S(t) - D(t, T)) = N(t)Y_T(t) \tag{71}$$

Notice that $Y_T(t)$ and $F(t, T)$ are continuous processes, even if the stock price jumps across an ex-dividend date. We can show this by looking at how $Q(t, T)(S(t) - D(t, T))$ behaves when crossing such a date, say $\tau_1$. Using the following results:

$$\begin{aligned}
S(\tau_1) &= S(\tau_1^-)(1 - \beta_1) - \alpha_1 \\
Q(\tau_1, T) &= \frac{Q(\tau_1^-, T)}{1 - \beta_1} \\
D(\tau_1, T) &= \Big(D(\tau_1^-, T) - \frac{\alpha_1}{1 - \beta_1}\Big)(1 - \beta_1)
\end{aligned}$$

we can in fact prove that:

$$Q(\tau_1,T)(S(\tau_1)-D(\tau_1,T)) = Q(\tau_1^-,T)(S(\tau_1^-)-D(\tau_1^-,T))$$

Also, if we choose $T > \tau_n$ and introduce the discounted value of all future absolute dividends $D(t)$:

$$D(t) = \sum_{k:\tau_k>t} \alpha_k \bar{P}(t,\tau_k)$$

we can write:

$$\begin{aligned} Q(t,T)(S(t)-D(t)) &= N(t)Y_T(t) \\ Q(0,T)(S(0)-D(0)) &= Y_T(0) \end{aligned}$$

Hence [1]:

$$S(t) = Q(0,t)(S(0)-D(0))N(t)\frac{Y_T(t)}{Y_T(0)} + D(t)$$

The last equation can be written in terms of a martingale $X(t)$ that does not depend on the maturity $T$:

$$S(t) = Q(0,t)(S(0)-D(0))N(t)X(t) + D(t) \tag{72}$$

$$\mathbb{E}^{\mathbb{Q}}\Big[X(t)\Big] = 1$$

As observed above, the process $X(t)$ is free from any jumps due to dividend payments: for such a reason it is often referred to as "Pure Equity Process".

Please also notice that the dynamics of the Spot price in Eq. 72 is consistent with the expression for the Forward given by Eq. 71. That is:

$$F(t,T) = \mathbb{E}^{\mathbb{Q}^T}\Big[S(T)\Big|t\Big] = \frac{N(t)}{P(t,T)}\mathbb{E}^{\mathbb{Q}}\Big[\frac{S(T)}{N(T)}\Big|t\Big]$$

We can also express the equity drift in terms of the spot Forward:

$$S(t) = P(0,t)\Big(F(0,t) - \sum_{k:\tau_k>t}\alpha_k\bar{P}(0;t,\tau_k)\Big)N(t)X(t)$$

where we have introduced the forward discount factors:

$$\bar{P}(0;t,\tau_k) = \frac{\bar{P}(0,\tau_k)}{\bar{P}(0,t)}$$

In particular, when rates are deterministic, we have $P(0,t) = \frac{1}{N(t)}$ as well as $\bar{P}(0;t,\tau_k) = \bar{P}(t,\tau_k)$ so the stock process reduces to:

$$S(t) = (F(0,t)-D(t))X(t) + D(t) \tag{73}$$

which is the parametrization introduced by Buehler at al. in [7].

Since the equity forwards are positive because of no-arbitrage, we also have the additional positivity constraint.

$$X(t) > 0$$

Below we discuss the two different dynamics that we have implemented for the Pure Equity process.

The simplest way to enforce the positivity constraint on the Pure Equity Process is to postulate a lognormal dynamics:

$$\begin{aligned} X(0) &= 1 \\ \frac{dX(t)}{X(t)} &= \sigma_X(t)dW^{\mathbb{Q}}(t) \end{aligned}$$

where:

[1] $Q(t,T) = \frac{Q(0,T)}{Q(0,t)}$

- $\sigma_X(t)$ is a lognormal instantaneous volatility curve
- $W^{\mathbb{Q}}(t)$ is a Wiener process in the $\mathbb{Q}$ measure.

The Buehler model [7] (here we assume zero continuous dividend and hazard rate) assumes dividends $\alpha_k + \beta_k S(\tau_k^-)$ are paid at times $\tau_1, ...\tau_n$ (where $\alpha_k$ is absolute dividend, $\beta_k$ is proportional dividend rate, $S(\tau_k^-)$ is the stock price at time $\tau_k$).

In this case at time $t$ the stock price $S(t)$ can is give by:

$$S(t) = Q(0,t)(S(0) - D(0))X(t)N(t) + D(t), \tag{74}$$

where $N(t)$ is risk neutral numeraire at time $t$, $P(t,T) = \mathbf{E}\left[N(t)/N(T)|\mathcal{F}_t\right]$ is risk neutral discount factor between $t$ and $T$ (e.g., bond with maturity $T$, if $N(t) = e^{\int_0^t r(s)ds}$; price $P(t,T)$ can be calculated using forward curve or using the Hull-White model),

$$D(t) = \sum_{k:\tau_k > t} \alpha_k \frac{P(t,\tau_k)}{Q(t,\tau_k)}, \tag{75}$$

$$Q(t,T) = \prod_{k:t<\tau_k \le T} (1 - \beta_k), \tag{76}$$

$X(t)$ is driftless geometric Brownian motion (with respect to the risk neutral measure):

$$\frac{dX(t)}{X(t)} = \sigma_X(t)dW(t), \quad X(0) = 1, \tag{77}$$

$S(0)$ is price at time $t = 0$.

In this case price of the forward contract with maturity $T$ at time $t$ is given by:

$$F(t,T) = \frac{Q(t,T)(S(t) - D(t,T))}{P(t,T)}, \tag{78}$$

$$D(t,T) = \sum_{k:t<\tau_k \le T} \alpha_k \frac{P(t,\tau_k)}{Q(t,\tau_k)}. \tag{79}$$

#### 3.5.2 Model Parameters and Market Data

1. **Discount Curve** – ClCurvePoints (defined according to ClCcyPrimaryCurve specification) – **market discount curve $P^M(0,t)$ used as a discount factor** in future payment discounting.
2. Volatility Curve (containing short rate volatility $\sigma(t)$) is defined according to ClCurveField specification
3. **Fixed Dividend Curve** – is defined according to ClFixedDividendData specification – **fixed dividend rates** $\alpha_k$.
4. **Proportional Dividend Curve** – is defined according to ClFloatDividendData specification – **proportional dividend rates** $\beta_k$.
5. **Equity Risk Premium Curve** – ClCurvePoints (Curve: Risk Premium: equity) – $R_F(0,t)$ **deflator is used to calculated equity prices in the real-world measure.** If the instantaneous market price of risk, $\lambda_{EQ}(t)$, is provided instead of the discount curve $R_F(0,t)$, the corresponding discount curve can be calculated as follows:

$$R_F(0,T) = \frac{1}{F(0,T)}\left[\frac{R_X(T)}{R_P(0,T)}\left(F(0,T) - \frac{1}{P(0,T)}\sum_{k:\tau_k>T}\alpha_k\frac{P(0,\tau_k)}{Q(0,\tau_k)} + \frac{1}{P_{RW}(0,T)}\sum_{k:\tau_k>T}\alpha_k\frac{P_{RW}(0,\tau_k)}{Q(0,\tau_k)}\right)\right],$$

where

$$R_X(t) = \exp\left[\int_0^t \lambda_{EQ}(u)\sigma_X(u)du\right].$$

.

### 3.5.3 Calibration

To calibrate the model parameter $\sigma_X(t)$ (piecewise function), a set of European options can be used. Calibration ca be done numerically (using method *LevenbergMarquardt.Minimize*) with minimizing difference between model and market price (based on either absolute error, relative error or implied volatility error):

$$\{\sigma_i\} = \arg\min_{\{\sigma_i\}} \sqrt{\sum_j w_j^2 e_j^2(k, \{\sigma_i\})}, \tag{80}$$

where $\sigma_i$ is value of $\sigma_X(t)$ for $t$ from the interval $t_{i-1} \le t < t_i$ ($i = \overline{1,n}$), $w_j$ is weight of the instrument $j$, $e_j(\{\sigma_i\})$ is calibration error, calculated with (210) (with $t = 0$) given the set of parameters $\{\sigma_i\}$ (values $P(0,t_i)$, $F(0,t_i)$ are picked from the market data), and market price of this instrument.

Market price of a calibration option with expiry $t_i$ is given by:

$$V_{EU}^{market}(t_i, K, \nu) = Bl(\nu, K, F(0,t_i), \sigma_i^{market}\sqrt{t_i}, P(0,t_i), 0), \tag{81}$$

where $Bl(...)$ is the Black formula (method *BlackFormula.BlackFormulaMain*), $\sigma_i^{market}$ is market volatility for the corresponding period (picked from the market data).

**Calibration details – tab description**
Common fields:

- Calibration status
- Calibration Error Type
- Total Price Error (TPE) = $\sqrt{\sum_i PE_i^2 \times IW_i^2}$
- Total Relative Price Error (TRPE) = $\sqrt{\sum_i PRE_i^2 \times IW_i^2}$
- Total Implied Vol Error (TIVE) = $\sqrt{\sum_i IVE_i^2 \times IW_i^2}$
- Total RMSE (coincides with the error defined in *Calibration Error Type*)

Individual fields:

- Instrument
- Instrument Weight (IW)
- Calibration Status
- Market Price
- Model Price
- Price Error (PE) = $marketValue - modelValue$
- Relative Price Error (RPE) = $\frac{marketValue - modelValue}{marketValue}$ if $marketValue \neq 0$ and 0 otherwise
- Market Volatility (MV)
- Implied Model Volatility (IMV)
- Implied Vol Error (IVE) = $IMV - MV$

Remark. The application might display values with lower precision compared to computations (i.e., the displayed results might have been rounded).

**Calibration Instruments Settings** Calibration instrument price is given by the volatility surface (EquityVolSurface). Volatility surface contains market European option volatility for various levels of strike (by column) and option maturity (by row).

#### 3.5.4 Simulation Algorithm

**Risk neutral simulation.** To simulate trajectory $S(t)$ on the discrete grid $t_0 < t_1 < t_2 < ... < t_n$ we first simulate process $X(t)$. Process $X(t)$ is generated by the formula:

$$X(t_i) = X(t_{i-1})\times \tag{82}$$

$$\times \exp\left(-\frac{1}{2}(t_i - t_{i-1})\sigma^2(t_{i-1}, t_i) + (W(t_i) - W(t_{i-1}))\sigma(t_{i-1}, t_i)\right), \tag{83}$$

where $X(t_0) = 1$, trajectories of the Wiener process $W(t_i)$ is generated by *IClOneFactorModel*, values $\sigma(t_{i-1}, t_i)$ is stored in the curve *IClVolCurve*.

Then, $X(t)$ are plug into expression (74). Numeraire $N(t)$ is calculated using $1/SDF$, corresponding to the rolling-spot risk neutral measure, calculated using short-rate process (method *IClTermStructureSimulation.Sdf*).

Forward-process $F(t, T)$ can be simulated using expression (78).

**Real world simulations.** To generate a trajectory for the equity process in the real world measure we need to recalculate $X(t)$ trajectories in the real world measure using the following expression:

$$\widetilde{X}(t_i) = \widetilde{X}(t_{i-1})\times \tag{84}$$

$$\times \exp\left(-\frac{1}{2}(t_i - t_{i-1})\sigma^2(t_{i-1}, t_i) + (W(t_i) - W(t_{i-1}))\sigma(t_{i-1}, t_i)\right)\times \tag{85}$$

$$\times R_X(T_i)/R_X(T_{i-1}), \tag{86}$$

where

$$R_X(T) = R_P(0,T)\frac{F(0,T)R_F(0,T) - \frac{1}{P_{RW}(0,T)}\sum_{k:\tau_k > T}\alpha_k\frac{P_{RW}(0,\tau_k)}{Q(0,\tau_k)}}{F(0,T) - \frac{1}{P(0,T)}\sum_{k:\tau_k > T}\alpha_k\frac{P(0,\tau_k)}{Q(0,\tau_k)}}, \tag{87}$$

$R_F(0, T)$ is a discount factor (calculated using equity risk premium curve), $R_P(0, T)$ is risk premium discount factor associated with the interest rate risk, $P_{RW}(0, T)$ is real world bond price. If interest rate is deterministic, $R_P(0, T) = 1$, $P_{RW}(0, T) = P(0, T)$.

At the next step, trajectories $\widetilde{X}(t_i)$ can be used in the expressions (74) and (78).

### 3.6 Generation of Correlated Shocks

PFE Engine system first calculates correlation matrix and then simulates vectors of correlated shocks that used in all simulation models.

**Common Correlation Matrix** Object Hierarchy Correlations Provider (*ClCorrelationDataProviderHierarchyData*) contains three matrixes:

- Pre-defined correlations;
- Implied correlations;
- Historical correlations.

The list is presented in order of decreasing priority. When constructing the correlation matrix, user-defined **pre-defined correlations** matrix (*ClPredefinedCorrelationDataProviderData*) is taken into account first.

Next in priority are **implied correlations** (*SbPslImpliedVolCorrelationDataProviderData*). Implied Correlations are only available for FX and are downloaded for Murex service. Implied correlation between logarithms of $\frac{CUR_X}{CUR_Y}$ and $\frac{CUR_X}{CUR_Z}$ is calculating as:

$$\widehat{\rho}^{\,imp}_{X/Y,X/Z} = \frac{\widehat{\sigma}^2_{X/Y} + \widehat{\sigma}^2_{X/Z} - \widehat{\sigma}^2_{Z/Y}}{2\ \widehat{\sigma}_{X/Y}\ \widehat{\sigma}_{X/Z}}.$$

Last of all **historical correlations** (*SbPslHistoricalCorrelationDataProviderData*) are taken into account. Historical correlation of two factors is calculated as follows:

1) load historical values of these factors;
2) truncate time series so that only the time intervals with both variables observed remain;
3) filtering the series with given frequency (weekly by default);
4) make series stationary:
   (a) choose appropriate proxy variable (e.g. 3M rates) and take simple differences for IR factors;
   (b) take log-differences for FX, EQ and CMDTY factors;
   (c) CR factors correlations do not supported in current version of PFE Engine;
5) calculate **Pearson correlation coefficient** by following sample formula:

$$\rho_{x,y} = \frac{\sum(x_i - \overline{x})(y_i - \overline{y})}{\sqrt{\sum(x_i - \overline{x})^2}\sqrt{\sum(y_i - \overline{y})^2}}.$$

As a result, final matrix may account for all three types of correlations.

**Correlated Shocks Generation** Correlated shocks simulation algorithm is based on common correlation matrix and consists of the following steps:

1) loading of $n$ necessary risk factors for given counterparty/portfolio;
2) choice of appropriate $n \times n$ submatrix $S$ of common correlation matrix;
3) test $S$ for positive semi-definition, if not then use correction algorithm form [1];
4) make Cholesky decomposition $S = L \times L^T$, where $L$ is lower rectangular matrix;
5) generate $n \cdot T$ number of $M$-dimensional standard normal vectors $V$, i.e. $V_m \sim \mathcal{N}(0, 1)$;
6) create correlated shocks vectors: $V^* = L \times V$.

As a result, we have $n \cdot T$ $M$-dimensional normal vectors with correlation matrix $S$, where $n$ is **a number of risk factors** used in calculus, $T$ is **a number of time steps** and $M$ ia **a number of simulation paths.** After that, correlated shocks are used as inputs in relevant simulation models.

**Setting of Simulation Parameters** Parameters $T$ and $n$ are defined by characteristics of derivative portfolio to be simulated, i.e. **date of longest payment** and **number on market factors** involved in the portfolio, respectively.

**The default value of** $M$ (*PathCount* in *Cl.Analyst.Finance.ClTimelineSettingsData*) is set as 2000 for CRC's (see Section 6) and 8000 for trade specific simulations. Market factors are simulating on daily frequency. Pricing grid (*intervals* in *Cl.Analyst.Finance.ClTimelineSettingsData*) is defined with different frequencies for each time interval. Here are these frequencies:

- 1D for [0, 2W),
- 1W for [2W, 6M),
- 2W for [6M, $T_{max}$],
- maturities longer then $T_{max}$ are not supported.

$T_{max}$ is user defined parameter, with current value 20Y. Note that all frequencies are calculating for given year base (*DaysInYear* in *Cl.Analyst.Finance.ClTimelineSettingsData*), 365 as a default.

# 4 Market Prices of Risk

This section contains a methodology of estimating of *the Market Price of Risk* (MPR) for all typologies of underlyings used in PFE Engine. Relevant literary references are also provided.

## 4.1 Market Price of Interest Rate Risk

### 4.1.1 Theory

The system uses 1-factor Hull-White (HW1F) model for short-rate dynamics (see Section 3.1.1):

$$\begin{aligned} r(u) &= (\theta(t) - \alpha(t)x(t))dt + \sigma(t)dW_t^{RN} \\ &= (\theta(t) + \Lambda_{IR}(r_t;t)\sigma(t) - \alpha(t)x(t))dt + \sigma(t)dW_t^{RW}, \end{aligned}$$

where $\Lambda_{IR}(r_t;t)$ is a general form of MPR. Below we will use following well-known properties and notations of HW1F model:

$$\begin{aligned} P(t,T) &\triangleq \mathbb{E}_t^{RN}\left[e^{-\int_t^T r_s ds}\right] = A(t,T)e^{-B(t,T)r_t}, \\ A(t,T) &\triangleq, \\ B(t,T) &\triangleq, \\ E(t) &\triangleq e^{\int_0^t \alpha(s)ds}. \end{aligned}$$

It is common practice to assume the same type of the model under both RN and RW measures, therefore $\Lambda_{IR}(r_t;t)$ can only have affine form:

$$\Lambda_{IR}(r_t;t) = \lambda_0(t) + \lambda_1(t)r_t.$$

Market practice as well as academic research shows that mean-reversion speed is the same under both RN and RW measures. So we can assume $\lambda_1(t) \equiv 0$, therefore

$$\Lambda_{IR}(r_t;t) = \lambda(t).$$

Hereinafter we will follow the methodology proposed in ???, where authors are working with historical instantaneous forward curves $f(t,T) = -\frac{\partial \ln P(t,T)}{\partial T}$. For these purposes, one should rephrase above short-rate model into the class of forward market HJM models. Under T-forward measure, $\mathfrak{F}^{Q^T}$, this model takes the form:

$$df(t,T) = \sigma_f(t,T)dW_t^{Q^T}.$$

According to Theorem ??? from ???, under risk-neutral measure we have:

$$\begin{aligned} df(t,T) = \alpha_f(t,T)dt + \sigma_f(t,T)dW_t^{RN} &\triangleq \left(\sigma_f(t,T)\int_t^T \sigma_f(t,s)ds\right)dt + \sigma_f(t,T)dW_t^{RN} \\ &= \left(\sigma_f(t,T)\int_t^T \sigma_f(t,s)ds + \sigma_f(t,T)\lambda(t)\right)dt + \sigma_f(t,T)dW_t^{RW}. \end{aligned}$$

Instantaneous volatility of (???) forward rate in HW1F model is $\sigma_f(t,T) = B'_T(t,T)\sigma(t)$. Therefore HJM model takes the form

$$\begin{aligned} df(t,T) &= \left(B'_T(t,T)\sigma(t)\int_t^T B'_T(t,T)\sigma(t)ds + B'_T(t,T)\sigma(t)\lambda(t)\right)dt + B'_T(t,T)\sigma(t)dW_t^{RW} \\ &= \frac{E(t)}{E(T)}\sigma(t)\left[\left(E(t)\int_t^T \frac{ds}{E(s)} - 1\right)\sigma(t) + \lambda(t)\right]dt + \frac{E(t)}{E(T)}\sigma(t)dW_t^{RW}. \end{aligned}$$

For econometric estimation procedure one need to discretize the model. The simplest way for that is Euler-Maruyama scheme with descretization step $\tau$. Note that from this point on we move on to other definition of $T$ – from the absolute time to the relative one $T \to t+T$. Therefore we obtain the general econometric model

$$f(t+\tau, t+T) = f(t, t+T) + \frac{E(t)}{E(T)}\sigma(t)\left[\left(E(t)\int_t^T \frac{ds}{E(s)} - 1\right)\sigma(t) + \lambda(t)\right]\tau + \frac{E(t)}{E(T)}\sigma(t)\sqrt{\tau}\varepsilon_t,\ \ \varepsilon_t \sim \mathcal{N}(0,1).$$

Since we use historical calibration for above model (see Section 4.1.3), mean-reversion parameter, $\alpha$, and instantaneous volatility, $\sigma(t)$, are constants across this procedure: $\alpha(t) \equiv \alpha,\ \sigma(t) \equiv \sigma$. Moreover, for simplicity sake, we will consider more straightforward particular form of affine MPR:

$$\lambda(t) \equiv \lambda.$$

The final econometric model takes the form

$$f(t+\tau, t+T) = f(t, t+T) + \left[e^{-\alpha T}\sigma^2\left(\frac{1-\alpha-e^{-\alpha T}}{\alpha}\right) + e^{-\alpha T}\sigma\lambda\right]\tau + e^{-\alpha T}\sigma\sqrt{\tau}\varepsilon_t. \qquad (88)$$

??? assumptions ($f(t,T)$ as market expectations)

### 4.1.2 Data

Main currencies (table: start date, tenors, frequency, max tenor)
LIBOR vs. treasure
Visualisation
Crisis period exclusion
sampling + tau + delta t

### 4.1.3 Estimation

??? Econometrics (lags as optimal TS instruments?)
OLS/MLE properties
Historical calibrations
Results

### 4.1.4 Mapping and Re-estimating

Principles of mapping (article)
Table of mapping
Tests of re-estimating
Results: 1Y frequency

## 4.2 Market Price of Foreign Exchange Risk

TBD

## 4.3 Market Price of Commodity Risk

TBD

### 4.4 Market Price of Credit Spread Risk

TBD

## 4.5 Market Price of Equity Risk

TBD

# 5 Derivative Pricing

## 5.1 Interest Rate Derivatives

The system contains the following typologies:

- Cap/floor,
- Digital cap/floor,
- Cap/floor with European barrier,
- Forward rate agreement,
- Asian forward rate agreement,
- Interest rate swap,
- European swaption,
- Interest rate swap with range accrual.

Aside from these derivative instruments, there are a few auxiliary instruments that are used to price more complicated typologies:

- Zero-coupon bond option,
- Digital zero coupon bond option,
- Interest rate range accrual.

In this section, we describe pricing methodologies for each of the above instruments.

### 5.1.1 Zero-coupon Bond Option

Consider European option with fixing time $T_F$, payment time $T_P$ and strike $K$ on a zero-coupon bond with maturity $T_M$. We assume that bond is quoted is associated with currency $f$ (currency of the forecast curve) but currency of payment is $d$ (currency of the discount curve). Price at time $t$ of the call option is given by:

$$\mathbf{ZBC}\left(t, T_F, T_M, T_P, K\right) = P_d(t, T_P)\left(F(t, T_F, T_M, T_P)\Phi\left(d_1\right) - K\Phi\left(d_2\right)\right) FX_f(t),$$

price of the put option is:

$$\mathbf{ZBP}\left(t, T_F, T_M, T_P, K\right) = P_d(t, T_P)\left(-F(t, T_F, T_M, T_P)\Phi\left(-d_1\right) + K\Phi\left(-d_2\right)\right) FX_f(t),$$

where $\Phi(.)$ - is cumulative distribution function of the standard normal distribution, $FX_f(t)$ is exchange rate (units of $d$ per unit of $f$)

$$F(t, T_F, T_M, T_P) = q\left(t, T_F, T_M, T_P\right)\frac{P_f(t, T_M)}{P_f(t, T_F)},$$

$$d_1 = \frac{1}{\sigma_p(t, T_F, T_M)}\ln\frac{F(t, T_F, T_M, T_P)}{K} + \frac{1}{2}\sigma_p(t, T_F, T_M),$$

$$d_2 = d_1 - \sigma_p(t, T_F, T_M),$$

$$\sigma_p(t, T, U) = B(T, U)\sqrt{V_x\left(t, T\right)}, \tag{89}$$

expression $B(T,U)$ is calculated using (141), expression $V_x(t,T)$ is given by (6). Integrals in these expressions $B(T,U)$ and $V_x(t,U)$ are calculated numerically using method *PiecewiseConstantIntegration.CalculateFlatBackward*,

$$q(t,T_F,T_M,T_P) = TA(t,T_F,T_M,T_P)\,QA(t,T_F,T_M,T_P),$$

$TA(t,T_F,T_M,T_P)$ is timing adjustment (equals 1 if $T_P = T_M$):

$$TA(t,T_F,T_M,T_P) = \exp\left(\int_t^{T_F} \sigma_f^2(u)\left(B_f(u,T_F) - B_f(u,T_M)\right)\left(B_f(u,T_P) - B_f(u,T_M)\right)du\right),$$

$QA_f(t,T_F,T_M,T_P)$ is quanto adjustment corresponding to transition from forward measure associated with currency $f$ (currency of the instrument) to domestic (currency of payment $d$) forward measure (equals 1 if $f$ is the same currency as payment currency):

$$\begin{aligned} QA(t,T_F,T_M,T_P) = \exp\Bigg(\int_t^{T_F} &\sigma_f(u)\left(B_f(u,T_M) - B_f(u,T_F)\right)\big(B_f(u,T_P)\sigma_f(u) + \\ &+ \sigma_{FX_{f,d}}(u)\rho_{FX} - \sigma(u)B(u,T_P)\rho_{f,d}\big)\,du\Bigg), \end{aligned}$$

$\sigma_{FX_{f,d}}(t)$ is FX rate volatility (we express FX rate in units of domestic currency per unit of $f$), $\rho_{FX}$ is correlation between increments of interest rate and FX rate, $\sigma_f(t)$, $B_f(t,T)$ are analogous for $\sigma(t)$ and $B(t,T)$ but correspond to the interest rate model for currency $f$, $\rho_{f,d}$ is correlation between increments of the short rates (corresponding to the cap/floor base currency and the payment currency).

### 5.1.2 Digital Zero-coupon Bond Option

Consider digital option with notional $Q$, fixing time $T_F$, payment time $T_P$ and strike $K$ on a zero-coupon bond with maturity $T_M$. We assume that bond is quoted is associated with currency $f$ but currency of payment is $d$. Price at time $t$ of the call option is given by:

$$\mathbf{CoN}_{ZBC}(t,T_F,T_M,T_P,K,Q) = Qq(t,T_F,T_M,T_P)\,P_d(t,T_P)\Phi(d_2),$$

the put option price is given by:

$$\mathbf{CoN}_{ZBP}(t,T_F,T_M,T_P,K,Q) = Qq(t,T_F,T_M,T_P)\,P_d(t,T_P)\Phi(-d_2),$$

where $\Phi(.)$ - is cumulative distribution function of the standard normal distribution,

$$d_1 = \frac{1}{\sigma_p(t,T_F,T_M)} \ln\frac{F(t,T_F,T_M,T_P)}{K} + \frac{1}{2}\sigma_p(t,T_F,T_M),$$

$$d_2 = d_1 - \frac{1}{2}\sigma_p(t,T_F,T_M),$$

$$F(t,T_F,T_M,T_P) = q(t,T_F,T_M,T_P)\frac{P_f(t,T_F)}{P_f(t,T_M)},$$

$$\sigma_p(t,T,U) = B(T,U)\sqrt{V_x(t,T)}, \tag{90}$$

$$q(t,T_F,T_M,T_P) = TA(t,T_F,T_M,T_P)\,QA(t,T_F,T_M,T_P),$$

expression $B(T,U)$ is calculated using (141), expression $V_x(t,T)$ is given by (6). Integrals in these expressions $B(T,U)$ and $V_x(t,U)$ are calculated numerically using method *PiecewiseConstantIntegration.CalculateFlatBackward*, $TA(t,T_F,T_M,T_P)$ is timing adjustment, $QA(t,T_F,T_M,T_P)$ is quanto adjustment (both defined in Section 5.1.1).

### 5.1.3 Cap/Floor

Consider cap/floor with strike $X$, notional $N$. Denote fixing times with $\mathcal{T} = \{T_0, T_1, ..., T_n\}$ and payment times with $\mathcal{T}^P = \{T_0^P, ..., T_{n-1}^P\}$. Besides, let $\tau_i$ denote interval between $T_{i-1}$ and $T_i$. Cap/floor price can be calculated as a sum of the constituting caplets/floorlets:

$$\mathbf{Cap}(t, \mathcal{T}, \mathcal{T}^P, N, X) = \sum_{i:T_i^P > t} \mathbf{Caplet}(t, T_{i-1}, T_i, T_{i-1}^P, N, X) \tag{91}$$

$$\mathbf{Floor}(t, \mathcal{T}, \mathcal{T}^P, N, X) = \sum_{i:T_i^P > t} \mathbf{Floorlet}(t, T_{i-1}, T_i, T_{i-1}^P, N, X) \tag{92}$$

1. **Model price** At time $t$ price of the caplet with fixing time $T_{i-1}$ and maturity time $T_i$ can be expressed via zero-coupon bond:

$$\mathbf{Caplet}(t, T_{i-1}, T_i, N, X) = N(1 + X\tau_i)\mathbf{ZBP}\left(t, T_{i-1}, T_i, \frac{1}{1 + X\tau_i}\right), \tag{93}$$

$$\mathbf{Floorlet}(t, T_{i-1}, T_i, T_{i-1}^P, N, X) = N(1 + X\tau_i)\mathbf{ZBC}\left(t, T_{i-1}, T_i, T_{i-1}^P, \frac{1}{1 + X\tau_i}\right). \tag{94}$$

   Expressions **ZBC** and **ZBP** are defined in 5.1.1.

2. **Market price** Market price of the caplet/floorlet (with maturity $T_{i-1}$ and $T_i$) at moment $t = 0$ is given by:

$$\begin{aligned}
\mathbf{Caplet}^M(0, T_{i-1}, T_i, N, X) &= N\tau_i P^M(0, T_i)\mathbf{Bl}(X, F(0, T_{i-1}, T_i), \sigma(X, T_{i-1})\sqrt{T_{i-1}}, 1) \\
\mathbf{Floorlet}^M(0, T_{i-1}, T_i, N, X) &= N\tau_i P^M(0, T_i)\mathbf{Bl}(X, F(0, T_{i-1}, T_i), \sigma(X, T_{i-1})\sqrt{T_{i-1}}, -1)
\end{aligned}$$

   where $\sigma(X, T_{i-1})$ is market volatility (defined in a special market volatility curve), $F(t, T, S)$ is simply-compounded forward interest rate:

$$F(t, T, S) = \frac{1}{S - T}\left(\frac{P^M(t, T)}{P^M(t, S)} - 1\right),$$

   $P^M(0, T)$ is market zero-coupon bond price at time $t = 0$ and maturing at $T$ (stored in market discount curve), $\mathbf{Bl}(K, F, v, w)$ is the Black formula (method *BlackFormula.BlackFormulaMain*):

$$\begin{aligned}
\mathbf{Bl}(K, F, v, w) &= Fw\Phi(wd_1(K, F, v)) - Kw\Phi(wd_2(K, F, v)), \\
d_1(K, F, v) &= \frac{ln(\frac{F}{K}) + \frac{v^2}{2}}{v}, \\
d_2(K, F, v) &= \frac{ln(\frac{F}{K}) - \frac{v^2}{2}}{v}.
\end{aligned}$$

### 5.1.4 Digital Cap/Floor

Consider digital cap/floor with strike $X$ and notional $N$. We denote by $\mathcal{T} = \{T_0, T_1, ..., T_n\}$ set of maturity times of the caplets/floorlets included in this instrument ($T_0$ is the first reset time). Besides, let $\tau_i$ denote length of the time interval between $T_{i-1}$ and $T_i$ (expressed in years) and $h_i$ - is time from fixing date $T_i$ and payment date. Digital cap/floor price can be computed as a sum of corresponding components (caplets/floorlets):

$$\mathbf{Cap}_D(t, \mathcal{T}, N, X) = \sum_{i:T_i + h_i > t} \mathbf{Caplet}_D(t, T_{i-1}, T_i, T_i + h_i, X, N)$$

$$\mathbf{Floor}_D(t,\mathcal{T},N,X) = \sum_{i:T_i+h_i>t} \mathbf{Floorlet}_D(t,T_{i-1},T_i,T_i+h_i,X,N)$$

At time $t$ digital caplet/floorlet with strike $X$, covering period between $T_{i-1}$ and $T_i$ (if $T_i > t$) is given by:

$$\mathbf{Caplet}_D(t,T_{i-1},T_i,T_i+h_i,X,N) = NP(t,T_i+h_i)\Phi\left(-d_2\right),$$

$$\mathbf{Floorlet}_D(t,T_{i-1},T_i,T_i+h_i,X,N) = NP(t,T_i+h_i)\Phi\left(d_2\right),$$

if $T_i \le t < T_i + h_i$:

$$\mathbf{Caplet}_D(t,T_{i-1},T_i,T_i+h_i,X,N) = \begin{cases} N(1+X\tau_i)q\left(t,T_{i-1},T_i,T_i+h_i\right)P(t,T_i+h_i), \text{ if } L(T_{i-1},T_i) > X, \\ 0, \text{ else} \end{cases}$$

$$\mathbf{Floorlet}_D(t,T_{i-1},T_i,T_i+h_i,X,N) = \begin{cases} N(1+X\tau_i)q\left(t,T_{i-1},T_i,T_i+h_i\right)P(t,T_i+h_i), \text{ if } L(T_{i-1},T_i) < X, \\ 0, \text{ else} \end{cases}$$

where $L(T_{i-1},T_i)$ is simply compounded spot interest rate prevailing at time $T_{i-1}$ for the maturity $T_i$,

$$d_2 = \frac{1}{\sigma_p(t,T_{i-1},T_i)} \ln \frac{q\left(t,T_{i-1},T_i,T_i+h_i\right)P(t,T_{i-1})(1+X\tau_i)}{P(t,T_i)} - \frac{1}{2}\sigma_p(t,T_{i-1},T_i),$$

$$q\left(t,T,U,T+h\right) = TA\left(t,T,U,T+h\right)QA\left(t,T,U,T+h\right),$$

$TA\left(t,T_F,T_M,T_P\right)$ is timing adjustment, $QA\left(t,T_F,T_M,T_P\right)$ is quanto adjustment (both are defined in section 5.1.1).

### 5.1.5 Cap/Floor with European Barrier

Consider barrier cap/floor of the type $x$ (where $x$ is either down-and-in (DI), up-and-out (UO), up-and-in (UI), or down-and-out (DO)) with strike $X$, barrier $H$, notional $N$. Denote fixing times with $\mathcal{T} = \{T_0, T_1, ..., T_n\}$ and payment times with $\mathcal{T}^P = \{T_0^P, ..., T_{n-1}^P\}$. Besides, let $\tau_i$ denote length of the interval from $T_{i-1}$ to $T_i$. Price of a cap/floor is a sum of corresponding caplets/floorlets:

$$\mathbf{Cap}_x(t,\mathcal{T},\mathcal{T}^P,N,X,H) = \sum_{i=1}^{n} \mathbf{Caplet}(t,T_{i-1},T_i,T_{i-1}^P,N,X,H) \tag{95}$$

$$\mathbf{Floor}_x(t,\mathcal{T},\mathcal{T}^P,N,X,H) = \sum_{i=1}^{n} \mathbf{Floorlet}(t,T_{i-1},T_i,T_{i-1}^P,N,X,H) \tag{96}$$

At time $t$ price of a barrier caplet with strike $X$, barrier $H$, fixing time $T_{i-1}$, maturity $T_i$ can be calculated using the formulas:

$$\mathbf{Caplet}_{DI}(t,T_{i-1},T_i,T_{i-1}^P,N,X,H) = \begin{cases} V_{EU}(t,T_{i-1},T_i,X,1) - V_{EU}(t,T_{i-1},T_i,H,1) - \\ -V_{DIGITAL}(t,T_{i-1},T_i,H,H-X,1), \text{ if } X < H, \\ 0, \text{ else,} \end{cases}$$

$$\mathbf{Caplet}_{UO}(t,T_{i-1},T_i,T_{i-1}^P,N,X,H) = \begin{cases} V_{EU}(t,T_{i-1},T_i,T_{i-1}^P,X,1) - V_{EU}(t,T_{i-1},T_i,T_{i-1}^P,H,1) - \\ -V_{DIGITAL}(t,T_{i-1},T_i,T_{i-1}^P,H,H-X,1), \text{ if } X < H, \\ 0, \text{ else,} \end{cases}$$

$$\mathbf{Caplet}_{UI}(t,T_{i-1},T_i,T_{i-1}^P,N,X,H)=\begin{cases} V_{EU}(t,T_{i-1},T_i,T_{i-1}^P,H,1)+ \\ +V_{DIGITAL}(t,T_{i-1},T_i,T_{i-1}^P,H,H-X,1), \text{ if } X<H, \\ V_{EU}(t,T_{i-1},T_i,T_{i-1}^P,X,1), \text{ else,} \end{cases}$$

$$\mathbf{Caplet}_{DU}(t,T_{i-1},T_i,T_{i-1}^P,N,X,H)=\begin{cases} V_{EU}(t,T_{i-1},T_i,T_{i-1}^P,H,1)+ \\ +V_{DIGITAL}(t,T_{i-1},T_i,T_{i-1}^P,H,H-X,1), \text{ if } X<H, \\ V_{EU}(t,T_{i-1},T_i,T_{i-1}^P,X,1), \text{ else.} \end{cases}$$

Analogous floorlet prices are given by:

$$\mathbf{Floorlet}_{DI}(t,T_{i-1},T_i,T_{i-1}^P,N,X,H)=\begin{cases} V_{EU}(t,T_{i-1},T_i,T_{i-1}^P,H,-1)+ \\ +V_{DIGITAL}(t,T_{i-1},T_i,T_{i-1}^P,H,X-H,-1), \text{ if } X<H, \\ V_{EU}(t,T_{i-1},T_i,T_{i-1}^P,X,-1), \text{ else,} \end{cases}$$

$$\mathbf{Floorlet}_{UO}(t,T_{i-1},T_i,T_{i-1}^P,N,X,H)=\begin{cases} V_{EU}(t,T_{i-1},T_i,H,-1)+ \\ +V_{DIGITAL}(t,T_{i-1},T_i,T_{i-1}^P,H,X-H,-1), \text{ if } X<H, \\ V_{EU}(t,T_{i-1},T_i,T_{i-1}^P,X,-1), \text{ else,} \end{cases}$$

$$\mathbf{Floorlet}_{UI}(t,T_{i-1},T_i,T_{i-1}^P,N,X,H)=\begin{cases} V_{EU}(t,T_{i-1},T_i,T_{i-1}^P,X,-1)-V_{EU}(t,T_{i-1},T_i,T_{i-1}^P,H,-1)- \\ -V_{DIGITAL}(t,T_{i-1},T_i,T_{i-1}^P,H,X-H,-1), \text{ if } X<H, \\ 0, \text{ else,} \end{cases}$$

$$\mathbf{Floorlet}_{DU}(t,T_{i-1},T_i,T_{i-1}^P,N,X,H)=\begin{cases} V_{EU}(t,T_{i-1},T_i,T_{i-1}^P,X,-1)-V_{EU}(t,T_{i-1},T_i,T_{i-1}^P,H,-1)- \\ -V_{DIGITAL}(t,T_{i-1},T_i,T_{i-1}^P,H,X-H,-1), \text{ if } X<H, \\ 0, \text{ else.} \end{cases}$$

Expressions $V_{EU}(t,T_{i-1},T_i,T_{i-1}^P,X,\nu)$, $V_{DIGITAL}(t,T_{i-1},T_i,T_{i-1}^P,X,\nu)$ are defined by:

$$V_{EU}(t,T_{i-1},T_i,T_{i-1}^P,X,1)=N(1+X\tau_i)\mathbf{ZBC}\left(t,T_{i-1},T_i,T_{i-1}^P,\frac{1}{1+X\tau_i}\right),$$

$$V_{EU}(t,T_{i-1},T_i,T_{i-1}^P,X,-1)=N(1+X\tau_i)\mathbf{ZBP}\left(t,T_{i-1},T_i,T_{i-1}^P,\frac{1}{1+X\tau_i}\right),$$

$$V_{DIGITAL}(t,T_{i-1},T_i,T_{i-1}^P,X,Q,1)=N(1+X\tau_i)\mathbf{CoN}_{ZBC}\left(t,T_{i-1},T_i,T_{i-1}^P,\frac{1}{1+X\tau_i},Q\right),$$

$$V_{DIGITAL}(t,T_{i-1},T_i,T_{i-1}^P,X,Q,-1)=N(1+X\tau_i)\mathbf{CoN}_{ZBP}\left(t,T_{i-1},T_i,T_{i-1}^P,\frac{1}{1+X\tau_i},Q\right).$$

Expressions **ZBC** and **ZBP** are defined in section 5.1.1. Expressions $\mathbf{CoN}_{ZBC}$ and $\mathbf{CoN}_{ZBP}$ are defined in section 5.1.2.

#### 5.1.6 Forward Rate Agreement

FRA is a contract with notional $N$, forward start time $T$, forward end time $S>T$ and maturity (pay) time $U$. Denote by $\tau$ the year fraction from $T$ to $S$. Then the FRA price at time $t$ is:

The total value of the contact at time $t<T$ is

$$\mathbf{FRA}(t,T,S,N,K)=NP(t,S)\tau\left(K-F(t,T,S)\right)$$

where $F(t,T,S)$ is simply-compounded forward interest rate at time $t$ for the expiry $T > t$ and maturity $S > T$

$$F(t,T,S) = \frac{1}{\tau}\left(\frac{P(t,T)}{P(t,S)}TA(t,T,S,U) - 1\right),$$

$TA\left(t,T_F,T_M,T_P\right)$ is timing adjustment defined in section 5.1.1.

If $T \le t < S$, FRA price is approximately defined by:

$$\mathbf{FRA}(t,T,S,N,K) = NP(t,S)\tau\left(K - \frac{1}{\tau}\left(\frac{1}{P(T,S)} - 1\right)\right)$$

### 5.1.7 Asian forward rate agreement

Consider an Asian FRA with average notional $\overline{N}$, weights $w_i$ (given in the input data), fixing times $T_i^F$, fixing end times $T_i^M = T_{i+1}^F - 1D$, fixing period length $s$ (libor length), averaging period end $T^M$ and payment time $T^P$. Then the FRA price at time $t < T_F$ is

$$\begin{aligned}\mathbf{FRA}(t,K,\overline{N}) = {} & \overline{N}P\left(t,T^P\right)\times \\ & \times\left(\sum_{i=1}^{n}\left(F(t,T_i^F,T_i^F+\tau,T^P)w_i^{float}\tau^{float} - Kw_i^{fixed}\tau^{fixed}\right)\right),\end{aligned}$$

where

$$\tau^{float} = \frac{f(CalcEndDate - CalcStartDate + 1; AccrualBasisFloat)}{f(\sum_{i=1}^{n}\delta_i; AccrualBasisFloat)},$$

$$\tau^{fixed} = \frac{f(CalcEndDate - CalcStartDate + 1; AccrualBasisFixed)}{f(\sum_{i=1}^{n}\delta_i; AccrualBasisFixed)},$$

where $f(x;y)$ is a function converting $x$ days into year fraction in accordance with convention $y$, $\delta_i = T_{i+1}^F - T_i^F$ for $i = 1,...n-1$ and $\delta_n = T^M - T_n^F$, $w_i^{float}$ and $w_i^{fixed}$ are values of $w_i$ converted into year fractions in accordance with *AccrualBasisFloat* and *AccrualBasisFixed* conventions respectively, $F(t,T_F,T_M,T_P)$ is a simply-compounded forward interest rate at time $t$ with fixing start $T_F > t$, fixing end $T_M > T_F$ and payment time:

$$F(t,T_F,T_M,T_P) = \frac{1}{T_M - T_F}\left(\frac{P(t,T_F)}{P(t,T_M)}TA(t,T_F,T_M,T_P) - 1\right).$$

In case of $T_F \le t < T_P$, approximate value of $F(t,T_F,T_M,T_P)$ is:

$$F(t,T_F,T_M,T_P) = \frac{1}{T_M - T_F}\left(\frac{1}{P(T_F,T_M)} - 1\right).$$

$TA$ is a timing adjustment:

$$TA\left(t,T_F,T_M,T_P\right) = \exp\left(\int_t^{T_F}\sigma^2(u)\left(B(u,T_F) - B(u,T_M)\right)\left(B(u,T_P) - B(u,T_M)\right)du\right),$$

$\sigma(u)$ is the interest rate volatility,

$$B(t,T) = \int_t^T e^{-\alpha(v-t)}dv$$

$\alpha$ is the mean-reversion speed.

$N_i$ is notional at the $i$-th fixing date, $n$ is number of fixing dates, Weights $w_i$ are used in the input data but they have been calculated according to the formula:

$$w_i = \frac{N_i \delta_i}{\overline{N}},$$

where $N_i$ is $i$-th notional.

Average notional have been calculated according to the principle:

$$\overline{N} = \frac{\sum_{i=1}^{n} N_i \delta_i}{\sum_{i=1}^{n} \delta_i}.$$

#### 5.1.8 Interest Rate Swap

Consider a cross-currency interest rate swap with strike with two legs. Each leg has fixing times $T_1^F, ..., T_n^F$, payment times $T_1^P, ..., T_n^P$ and maturity times $T_1^M, ..., T_n^M$.

Its price at time $t$ is given by:

$$\mathbf{IRS} = FX_1(t) L_1(t) - FX_2(t) L_2(t), \tag{97}$$

where $FX_k(t)$ is an exchange rate of the currency $ccy_k$ of leg $k$ (expressed in units of the domestic currency per unit of the foreign currency), $L_k(t)$ is present value of leg $k$ (expressed in currency $ccy_k$):

$$L_k(t) = \sum_{i:T_i^P > t} CF_k(t, T_i^F, T_i^M, T_i^P), \tag{98}$$

where cash flows $CF_k$ can be of the following types: notional payment, fixed payment, floating payment. If payment is notional:

$$CF_k(t, T_i^F, T_i^M, T_i^P) = N_i P_k(t, T_i^P),$$

where $N_i$ is notional, $P_k(t, T_i^P)$ is price of the bond associated with the currency $k$. If payment if fixed:

$$CF_k(t, T_i^F, T_i^M, T_i^P) = N_i K \tau_i P_k(t, T_i^P),$$

where $K$ is the fixed rate, $\tau_i = T_i^M - T_i^F$ is accrual period. If payment is floating:

$$CF_k(t, T_i^F, T_i^M, T_i^P) = N_i F(t, T_i^F, T_i^M, T_i^P) \tau_i P_k(t, T_i^P),$$

where $F(t, T_i^F, T_i^M, T_i^P)$ for $t < T_i^F$ is defined as follows:

$$\begin{aligned} F(t, T_i^F, T_i^M, T_i^P) = \left( F(t, T_i^F, T_i^M, T_i^M) + \frac{1}{\tau_i} \right) \times \\ \times TA_k(t, T_i^F, T_i^M, T_i^P) QA_k(t, T_i^F, T_i^M, T_i^P) - \frac{1}{\tau_i}, \end{aligned}$$

$$F(t, T_i^F, T_i^M, T_i^M) = \frac{1}{\tau_i} \left( \frac{P_k(t, T_i^F) - P_k(t, T_i^M)}{P_k(t, T_i^M)} \right),$$

$TA_k$, $QA_k$ are timing and quanto adjustments calculated for transition from currency $ccy_k$ to the domestic one (expressions are defined in section 5.1.1).

If $T_i^F < t < T_i^P$:

$$F(t, T_i^F, T_i^M, T_i^M) = \frac{1}{\tau_i} \left( \frac{1}{P_k(t, T_i^M)} - 1 \right).$$

For rates with overnight accrual (OIS):

$$F(t, T_i^F, T_i^M, T_i^M) = \frac{1}{\tau_i} \left( \frac{\prod_{T_i^F < t_j < t} \left(1 + R_{t_j}(t_{j+1} - t_j)\right)}{P_k(t, T_i^M)} - 1 \right),$$

where $R_{t_j}$ values of OIS rate, $t_j$ are corresponding accrual dates.

#### 5.1.9 European Swaption

Consider a European swaption with strike $X$, expiration $T$, notional $N$, fixing times $\mathcal{T} = \{T_1, ..., T_n\}, T_1 > T$. Let $\tau_i$ denote interval between $T_{i-1}$ and $T_i$.

1. **Model price** Model price is calculated using the Jamshidian decomposition, as described in [3]. Introduce $c_i := X\tau_i$ for $i = 1, ..., n-1$ and $c_n := 1 + X\delta_i$. Let $r^*$ denote the root of equation:

$$\sum_{i=1}^{n} c_i A(T, T_i) e^{-B(T,T_i)r^*} = 1, \tag{99}$$

which can be found numerically (method *Brent.FindRoot*).

Now setting $X_i := A(T, T_i)e^{-B(T,T_i)r^*}$, we can obtain swaption price (payer and receiver swaptions) at time $t < T$ using expressions:

$$\mathbf{PS}(t, T, \mathcal{T}, N, X) = N \sum_{i=1}^{n} c_i \mathbf{ZBP}(t, T, T_i, T_i, X_i), \tag{100}$$

$$\mathbf{RS}(t, T, \mathcal{T}, N, X) = N \sum_{i=1}^{n} c_i \mathbf{ZBC}(t, T, T_i, T_i, X_i). \tag{101}$$

Expressions **ZBC** and **ZBP** are defined in section 5.1.1.

2. **Market price** Market price of a European swaption at time $t = 0$ can be calculated using Black formula:

$$\mathbf{PS}^M(0, T, \mathcal{T}, N, X, \sigma) = N\mathbf{Bl}(X, S, \sigma\sqrt{T}, 1) \sum_{i=1}^{n} \tau_i P^M(0, T_i),$$
$$\mathbf{RS}^M(0, T, \mathcal{T}, N, X, \sigma) = N\mathbf{Bl}(X, S, \sigma\sqrt{T}, -1) \sum_{i=1}^{n} \tau_i P^M(0, T_i)$$

where $\sigma$ is market implied volatility (stored in market volatility curve), $S(t)$ is a swap rate:

$$S(t) = \frac{\sum_{i=1}^{\tilde{n}} P^M(t, T_i) \left( \frac{P^M(t, T_{i-1})}{P^M(t, T_i)} - 1 \right)}{\sum_{i=1}^{n} \tau_i P^M(t, T_i)},$$

$P^M(0, T)$ is market zero-coupon bond price at time $t = 0$ and maturing at $T$ (stored in market discount curve), $\mathbf{Bl}(K, F, v, w)$ is the Black formula (method *BlackFormula.BlackFormulaMain*):

$$\mathbf{Bl}(K, F, v, w) = Fw\Phi(wd_1(K, F, v)) - Kw\Phi(wd_2(K, F, v)),$$
$$d_1(K, F, v) = \frac{ln(\frac{F}{K}) + \frac{v^2}{2}}{v},$$
$$d_2(K, F, v) = \frac{ln(\frac{F}{K}) - \frac{v^2}{2}}{v}.$$

#### 5.1.10 IR Range Accrual

For this instrument we have a series of dates $\{t_i\}$ with corresponding weights $\{w_i\}$ (we assume, that $\sum_i w_i = 1$, otherwise scale them accordingly), notional $N$ and payment date $T$. Calculation depends on payment type. If accrual on given date happens when index is inside of given strip $[K_l, K_u]$ (one of the bounds can be infinite), the fair price at time $t$ is calculated as

$$RA_{inside}(t, \{t_i\}, \{w_i\}, T, K_l, K_u, N) = N \sum_i w_i (Bin^c(t, t_i, T, K_l) - Bin^c(t, t_i, T, K_u)), \tag{102}$$

where $Bin^c(t, t_i, T, K)$ is the price at time $t$ of a digital call option with fixing date $t_i$, payment date $T$ and strike $K$ (and notional 1).

If accrual on given date happens when index is outside of given strip $[K_l, K_u]$ (one of the bounds can be infinite), the fair price at time $t$ is calculated as

$$RA_{outside}(t, \{t_i\}, \{w_i\}, T, K_l, K_u, N) = N \sum_i w_i(Bin^c(t, t_i, T, K_u) + Bin^p(t, t_i, T, K_l)) =$$

$$= N - RA_{inside}(t, \{t_i\}, \{w_i\}, T, K_l, K_u, N), \tag{103}$$

$Bin^p(t, t_i, T, K)$ is the price at time $t$ of a digital put option with fixing date $t_i$, payment date $T$ and strike $K$ (and notional 1).

#### 5.1.11 IRS with Range Accrual

Consider interest rate fixed/fixed swap with fixed rates $K_1$ and $K_2$, fixing times $T_1^{fix}, ..., T_{n_{fix}}^{fix}$ and payment times $T_1, T_2, ..., T_n$ (let $\tau_i$ denote length of time interval from $T_{i-1}$ to $T_i$ expressed in years).

Depending on the value of *RALegNum*, the swap price at time $t$ is given either by:

$$\mathbf{IRS}_{RA}^1(t, \{T_i\}, N, K) = N \sum_{i:T_i>t} \tau_i \mathbf{RA}_i K_1 - N \sum_{i:T_i>t} \tau_i P(t, T_i) K_2,$$

if $RALegNum = 1$; or by:

$$\mathbf{IRS}_{RA}^2(t, \{T_i\}, N, K) = N \sum_{i:T_i>t} \tau_i P(t, T_i) K_1 - N \sum_{i:T_i>t} \tau_i \mathbf{RA}_i K_2,$$

if $RALegNum = 2$. Depending on the type (defined in *RAType* and can have one of the following values: *IN*, *OUT*, *BELOW*, *ABOVE*) $\mathbf{RA}_i$ can be calculated using one of the following expressions:

$$\mathbf{RA}_i^{IN} = \frac{1}{n_i} \sum_{j:T_{i-1}<T_j^{fix}\le T_i} \left( Bin^c(t, T_j^{fix}, T_i, K_l) - Bin^c(t, T_j^{fix}, T_i, K_u) \right),$$

$$\mathbf{RA}_i^{OUT} = 1 - \mathbf{RA}_i^{IN},$$

$$\mathbf{RA}_i^{ABOVE} = \frac{1}{n_i} \sum_{j:T_{i-1}<T_j^{fix}\le T_i} Bin^c(t, T_j^{fix}, T_i, K_l),$$

$$\mathbf{RA}_i^{BELOW} = 1 - \frac{1}{n_i} \sum_{j:T_{i-1}<T_j^{fix}\le T_i} Bin^c(t, T_j^{fix}, T_i, K_u),$$

where $K_l$, $K_u$ are lower and upper barriers (defined in *RAUpBarrier*, *RADownBarrier*), $n_i$ is the number of fixing dates in the interval $(T_{i-1}, T_i]$, $Bin^c(t, T_j^{fix}, T_i, K)$ is the price of the digital call option corresponding to a particular index (defined in *RAIndex*) at time $t$ with fixing time $T_j^{fix}$, payment time $T_i$ and strike $K$ (and notional 1). If $T_j^{fix} < t < T_i$ is computed as the discounted realized payoff (according to the methodology for the corresponding instrument).

## 5.2 Foreign Exchange Derivatives

The system contains the following typologies:

- FX forward,
- Asian FX forward,
- FX swap,
- European FX option,
- Digital FX option,
- FX range accrual,
- European FX option with European barrier,
- Asian FX option (approximate calculation using lognormal approximation),
- Asian digital FX option (approximate calculation using lognormal approximation),
- Asian FX option with European barrier (approximate calculation using lognormal approximation),
- Asian Digital FX Option with European Barrier (approximate calculation using lognormal approximation),
- European FX option with American barrier,
- Cross-currency interest rate swaption (approximate calculation using lognormal approximation).

In this section, we describe pricing methodologies for each of the above instruments.

### 5.2.1 FX Forward

Consider an FX forward contract with notional $N_d$ in the domestic currency and $N_f$ in the foreign currency and expiration time $T$. Its value at time $t$ is:

$$\begin{aligned}\mathbf{Fwd}(t,T,N_f,N_d) &= \mathbb{E}_t^{\mathbb{Q}}\left[D(t,T)\left(N_f FX(T)-N_d\right)\right]\\ &= N_f P_d(t,T)\left(FX(t)\frac{P_f(t,T)}{P_d(t,T)}-\frac{N_d}{N_f}\right),\end{aligned}$$

where $P_d(t,T)$ is the domestic bond price, $P_f(t,T)$ is the foreign bond price, $D(t,T)=e^{-\int_t^T r(s)ds}$ is a stochastic discount factor and $\mathbb{Q}$ is the RN measure.

### 5.2.2 Asian FX Forward

Consider an Asian FX forwardwith notional $N_d$ in the domestic currency and $N_f$ in the foreign currency, averaging times $T_1, T_2, ...T_n$ and expiration time $T$ ($t \leq T_1 < T_2 \leq T$). At time $t$ its value is given by:

$$\begin{aligned}\mathbf{Fwd}_{Asian}(t,\{T_i\},T,N_f,N_d) &= \mathbb{E}_t^{\mathbb{Q}}\left[D(t,T)\left(\frac{N_f}{n}\sum_{i=1}^{n}FX(T_i)-N_d\right)\right]\\ &= N_f P_d(t,T)\left(\frac{1}{n}\sum_{i=1}^{n}FX(t)\frac{P_f(t,T_i)}{P_d(t,T_i)}-\frac{N_d}{N_f}\right),\end{aligned}$$

where $P_d(t,T)$ is the domestic bond price, $P_f(t,T)$ is the foreign bond price, $D(t,T)=e^{-\int_t^T r(s)ds}$ is a stochastic discount factor and $\mathbb{Q}$ is the RN measure.

### 5.2.3 FX Swap

Consider an FX swap with fixed rate $K$, fixing dates $T_1, ..., T_n$ (let $\tau_i$ denote interval length from $T_{i-1}$ to $T_i$ expressed in years) and notional in foreign currency $N_1^f, ...N_n^f$.

Its price at time $t$ is given by:

$$\mathbf{Swap}(t, \{T_i\}, \{N_i^f\}) = \sum_{i=1}^{n} N_i^f P_d(t, T_i) \left( FX(t) \frac{P_f(t, T_i)}{P_d(t, T_i)} - K \right), \tag{104}$$

where $P^d(t, T)$ is a domestic bond, $P^f(t, T)$ is a foreign bond.

### 5.2.4 European FX Option

1. **Model price** Price of a Eurpoean FX option with expiration date $T$, strike $K$ at time $t$ is given by:

$$V_{EU}(t, T, K, \nu) = \nu P_d(t, T) \left(F(t, T)\Phi(\nu d_1) - K\Phi(\nu d_2)\right) \tag{105}$$

where $\Phi(.)$ is c.d.f. of the stadard normal distribution $N(0, 1)$, $\nu = 1$ is for a call option, $\nu = -1$ is for a put option,

$$F(t, T) = FX(t)\frac{P_f(t, T)}{P_d(t, T)}, \tag{106}$$

$$d_1 = \frac{\ln \frac{F(t,T)}{K} + \frac{1}{2}\Sigma(t, T)}{\sqrt{\Sigma(t, T)}}, \tag{107}$$

$$d_2 = d_1 - \sqrt{\Sigma(t, T)}, \tag{108}$$

$$\Sigma(t, T) = a(t, T) + b(t, T) + \int_t^T \sigma^2(u) du, \tag{109}$$

$$a(t, T) = \Sigma_d(t, T) + \Sigma_f(t, T) - 2\Sigma_{df}(t, T), \tag{110}$$

$$b(t, T) = 2 \left( \int_t^T \rho_{d,FX}\sigma_d(s)B_d(s, T)ds - \int_t^T \rho_{x_f,FX_f}\sigma_f(s)B_f(s, T)ds \right). \tag{111}$$

where integral can be calculated using the method *PiecewiseConstantIntegration.CalculateFlatBackward*. Values $\Sigma_d(t, T)$, $\Sigma_f(t, T)$ (variances of the domestic and foreign bonds), $\Sigma_{df}(t, T)$ (their covariance) are calculated with *SbPslIrHw1fModel.ZeroBondVariance* and *SbPslIrHw1fModel.ZeroBondCovariance*. Integrals in the expression for $b(t, T)$ can be approximated with method *SbPslFxBsModel.Covariance* ($\sigma_d(s)$, $B_d(s, T)$, $\sigma_f(s)$, $B_f(s, T)$ are expressions from the corresponding interest rate models: short rate volatility and factor $B(t, T)$ calculated in method *SbPslIrHw1fModel.B*).

2. **Market price** The analogous market price (at time $t = 0$) is given by:

$$V_{EU}^M(0, T, K, \nu) = P_d^M(0, T)\mathbf{Bl}(K, F, \sigma\sqrt{T}, \nu), \tag{112}$$

where $F = F(0, T) = FX(0)\frac{P_f^M(0,T)}{P_d^M(0,T)}$ is an FX forward with maturity $T$, $P_d^M(0, T)$ is domestic discount factor, $P_f^M(0, T)$ is a foreign discount factor, $\sigma = \sigma(T, K)$ is market volatility (that corresponds to maturity $T$ and strike $K$ (from the volatility curve), $\nu = 1$ for a call option, $\nu = -1$ for a put option, $\mathbf{Bl}(K, F, v, w)$ is the Black formula (method *BlackFormula.BlackFormulaMain*):

$$\mathbf{Bl}(K, F, v, w) = Fw\Phi(wd_1(K, F, v)) - Kw\Phi(wd_2(K, F, v)),$$

$$d_1(K, F, v) = \frac{ln(\frac{F}{K}) + \frac{v^2}{2}}{v},$$

$$d_2(K, F, v) = \frac{ln(\frac{F}{K}) - \frac{v^2}{2}}{v}.$$

### 5.2.5 European Digital FX Option

Consider a digital option with strike $K$, notional $N_f$, expiration date $T$ and payment date $T_p$. Its price at time $t < T_p$ is given by:

$$V_{DIGITAL}(t, T, T_p, K, N, \nu) = N_f P_d(t, T_p)\Phi(\nu d_2) \tag{113}$$

where $\Phi(.)$ is the c.d.f. of the standard normal distribution $N(0, 1)$, $\nu = 1$ is for a call option, $\nu = -1$ is for a put option, $K$ is the strike,

$$d_2 = \frac{\ln \frac{F(t,T)}{K} - \frac{1}{2}\Sigma(t, T)}{\sqrt{\Sigma(t, T)}}, \tag{114}$$

$$F(t, T) = FX(t)\frac{P_f(t, T)}{P_d(t, T)}, \tag{115}$$

$$\Sigma(t, T) = a(t, T) + b(t, T) + \int_t^T \sigma^2(u)du, \tag{116}$$

$$a(t, T) = \Sigma_d(t, T) + \Sigma_f(t, T) - 2\Sigma_{df}(t, T), \tag{117}$$

$$b(t, T) = 2\left(\int_t^T \rho_{d,FX}\sigma_d(s)B_d(s, T)ds - \int_t^T \rho_{x_f,FX_f}\sigma_f(s)B_f(s, T)ds\right). \tag{118}$$

where integral can be calculated using the method *PiecewiseConstantIntegration.CalculateFlatBackward.* Values $\Sigma_d(t, T)$, $\Sigma_f(t, T)$ (variances of the domestic and foreign bonds), $\Sigma_{df}(t, T)$ (their covariance) are calculated with *SbPslIrHw1fModel.ZeroBondVariance* and *SbPslIrHw1fModel.ZeroBondCovariance.* Integrals in the expression for $b(t, T)$ can be approximated with method *SbPslFxBsModel.Covariance* ($\sigma_d(s)$, $B_d(s, T)$, $\sigma_f(s)$, $B_f(s, T)$ are expressions from the corresponding interest rate models: short rate volatility and factor $B(t, T)$ calculated in method *SbPslIrHw1fModel.B*).

### 5.2.6 FX Range Accrual

For this instrument we have a series of dates $\{t_i\}$ with corresponding weights $\{w_i\}$ (we assume, that $\sum_i w_i = 1$, otherwise scale them accordingly), notional $N$ and payment date $T$. Calculation depends on payment type. If accrual on given date happens when index is inside of given strip $[K_l, K_u]$ (one of the bounds can be infinite), the fair price at time $t$ is calculated as

$$\begin{aligned} RA_{inside}(t, \{t_i\}, \{w_i\}, T, K_l, K_u, N) = N\sum_i w_i(V_{DIGITAL}(t, t_i, T, K_l, 1, 1)- \\ - V_{DIGITAL}(t, t_i, T, K_u, 1, 1)), \end{aligned}$$

where $V_{DIGITAL}(t, t_i, T, K, 1, 1)$ is the price at time $t$ of a digital call option with fixing date $t_i$, payment date $T$ and strike $K$ (and notional 1).

If accrual on given date happens when index is outside of given strip $[K_l, K_u]$ (one of the bounds can be infinite), the fair price at time $t$ is calculated as

$$\begin{aligned} RA_{outside}(t, \{t_i\}, \{w_i\}, T, K_l, K_u, N) = N\sum_i w_i(V_{DIGITAL}(t, t_i, T, K_u, 1, 1)+ \\ + V_{DIGITAL}(t, t_i, K_l, T, 1, -1) = \\ = N - RA_{inside}(t, \{t_i\}, \{w_i\}, T, K_l, K_u, N), \end{aligned}$$

$V_{DIGITAL}(t, t_i, T, K, 1, -1)$ is the price at time $t$ of a digital put option with fixing date $t_i$, payment date $T$ and strike $K$ (and notional 1).

#### 5.2.7 European FX Option with European Barrier

Consider a barrier call options of each of the following types down-and-in (DI), up-and-out (UO), up-and-in (UI), down-and-out (DO) with strike $K$, barrier $H$, expiration payment date $T_p$. Their prices at time $t$ are given by:

$$\mathbf{C}_{DI}(t,\{T_i\},T_p,K,H)=\begin{cases} V_{EU}(t,\{T_i\},T_p,K,1)-V_{EU}(t,\{T_i\},T_p,H,1)- \\ -V_{DIGITAL}(t,\{T_i\},T_p,H,H-K,1), \text{ if } K<H, \\ 0, \text{ else}, \end{cases}$$

$$\mathbf{C}_{UO}(t,\{T_i\},T_p,K,H)=\begin{cases} V_{EU}(t,\{T_i\},T_p,K,1)-V_{EU}(t,\{T_i\},T_p,H,1)- \\ -V_{DIGITAL}(t,\{T_i\},T_p,H,H-K,1), \text{ if } K<H, \\ 0, \text{ else}, \end{cases}$$

$$\mathbf{C}_{UI}(t,\{T_i\},T_p,K,H)=\begin{cases} V_{EU}(t,\{T_i\},T_p,H,1)+ \\ +V_{DIGITAL}(t,T_e,T,H,H-K,1), \text{ if } K<H, \\ V_{EU}(t,\{T_i\},T_p,K,1), \text{ else}, \end{cases}$$

$$\mathbf{C}_{DU}(t,\{T_i\},T_p,K,H)=\begin{cases} V_{EU}(t,\{T_i\},T_p,H,1)+ \\ +V_{DIGITAL}(t,\{T_i\},T_p,H,H-K,1), \text{ if } K<H, \\ V_{EU}(t,\{T_i\},T_p,K,1), \text{ else}. \end{cases}$$

For put options the expressions are as follows:

$$\mathbf{P}_{DI}(t,\{T_i\},T_p,K,H)=\begin{cases} V_{EU}(t,\{T_i\},T_p,H,-1)+ \\ +V_{DIGITAL}(t,\{T_i\},T_p,H,K-H,-1), \text{ if } K<H, \\ V_{EU}(t,\{T_i\},T_p,K,-1), \text{ else}, \end{cases}$$

$$\mathbf{P}_{UO}(t,\{T_i\},T_p,K,H)=\begin{cases} V_{EU}(t,\{T_i\},T_p,H,-1)+ \\ +V_{DIGITAL}(t,\{T_i\},T_p,H,K-H,-1), \text{ if } K<H, \\ V_{EU}(t,\{T_i\},T_p,K,-1), \text{ else}, \end{cases}$$

$$\mathbf{P}_{UI}(t,\{T_i\},T_p,K,H)=\begin{cases} V_{EU}(t,\{T_i\},T_p,K,-1)-V_{EU}(t,\{T_i\},T_p,H,-1)- \\ -V_{DIGITAL}(t,\{T_i\},T_p,H,K-H,-1), \text{ if } K<H, \\ 0, \text{ else}, \end{cases}$$

$$\mathbf{P}_{DU}(t,\{T_i\},T_p,K,H)=\begin{cases} V_{EU}(t,\{T_i\},T_p,K,-1)-V_{EU}(t,\{T_i\},T_p,H,-1)- \\ -V_{DIGITAL}(t,\{T_i\},T_p,H,K-H,-1), \text{ if } K<H, \\ 0, \text{ else}. \end{cases}$$

Here $V_{EU}(t,T_e,T,K,\nu)$, $V_{DIGITAL}(t,T_e,T,K,\nu)$ are expressions for European and European digital option that can be calculated using (150) and (159) respectively.

#### 5.2.8 European FX Option with American Barrier

Consider a call options with American-style barrier of each of the following types down-and-in (DI), up-and-out (UO), up-and-in (UI), down-and-out (DO) with strike $K$, barrier $H$, expiration date $T_p$. Their prices at time $t$ are given by [5, 6]

$$\mathbf{C}_{UI}(t,T_p,N,K,H)=\begin{cases}A(1,-1)+E(1,-1) & \text{if } K>H,\\ B(1,-1)-C(1,-1)+D(1,-1)+E(1,-1) & \text{if } K<H;\end{cases}$$

$$\mathbf{C}_{UO}(t,T_p,N,K,H)=\begin{cases}F(1,-1) & \text{if } K>H,\\ A(1,-1)-B(1,-1)+C(1,-1)-D(1,-1)+F(1,-1) & \text{if } K<H;\end{cases}$$

$$\mathbf{C}_{DI}(t,T_p,N,K,H)=\begin{cases}C(1,1)+E(1,1) & \text{if } K>H,\\ A(1,1)-B(1,1)+D(1,1)+E(1,1) & \text{if } K<H;\end{cases}$$

$$\mathbf{C}_{DO}(t,T_p,N,K,H)=\begin{cases}A(1,1)-C(1,1)+F(1,1) & \text{if } K>H,\\ B(1,1)-D(1,1)+F(1,1) & \text{if } K<H;\end{cases}$$

$$\mathbf{P}_{UI}(t,T_p,N,K,H)=\begin{cases}A(-1,-1)-B(-1,-1)+D(-1,-1)+E(-1,-1) & \text{if } K>H,\\ C(-1,-1)+E(-1,-1) & \text{if } K<H;\end{cases}$$

$$\mathbf{P}_{UO}(t,T_p,N,K,H)=\begin{cases}B(-1,-1)-D(-1,-1)+F(-1,-1) & \text{if } K>H,\\ A(-1,-1)-C(-1,-1)+F(-1,-1) & \text{if } K<H;\end{cases}$$

$$\mathbf{P}_{DI}(t,T_p,N,K,H)=\begin{cases}B(-1,1)-C(-1,1)+D(-1,1)+E(-1,1) & \text{if } K>H,\\ A(-1,1)+E(-1,1) & \text{if } K<H;\end{cases}$$

$$\mathbf{P}_{DO}(t,T_p,N,K,H)=\begin{cases}A(-1,1)-B(-1,1)+C(-1,1)-D(-1,1)+F(-1,1) & \text{if } K>H,\\ F(-1,1) & \text{if } K<H.\end{cases}$$

The expressions for $A - F$ are as follows:

$$A(\phi,\eta) = \phi P(t,T_p)N(\phi x_1(t)) - \phi K P(t,T_{i-1})N(\phi x_1(t) - \phi\sqrt{\Sigma(t,T)}),$$

$$B(\phi,\eta) = \phi P(t,T_p)N(\phi x_2(t)) - \phi K P(t,T_{i-1})N(\phi x_2(t) - \phi\sqrt{\Sigma(t,T)}),$$

$$\begin{aligned} C(\phi,\eta) &= \phi P(t,T_p)(H/S)^{2(\mu(t)+1)}N(\eta y_1(t)) \\ &\quad - \phi K P(t,T_{i-1})(H/S)^{2(\mu(t)+1)}N(\eta y_1(t) - \eta\sqrt{\Sigma(t,T)}), \end{aligned}$$

$$\begin{aligned} D(\phi,\eta) &= \phi P(t,T_p)(H/S)^{2(\mu(t)+1)}N(\eta y_2(t)) \\ &\quad - \phi K P(t,T_{i-1})(H/P(t,T_p))^{2(\mu+1)}N(\eta y_2(t) - \eta\sqrt{\Sigma(t,T)}), \end{aligned}$$

$$E(\phi,\eta) = XP(t,T_p)(N(\eta x_2(t) - \eta\sqrt{\Sigma(t,T)}) - (H/S)^{2\mu}(\eta y_2(t) - \eta\sqrt{\Sigma(t,T)})),$$

$$F(\phi,\eta) = X[(H/S)^{\mu(t)+\lambda(t)}N(\eta z(t)) + (H/S)^{\mu(t)-\lambda(t)}N(\eta z(t) - 2\eta\lambda\sqrt{\Sigma(t,T)})],$$

where

$$x_1(t) = \frac{\log(S/K)}{\sqrt{\Sigma(t,T)}} + (1+\mu(t))\sqrt{\Sigma(t,T)},$$

$$x_2(t) = \frac{\log(S/H)}{\sqrt{\Sigma(t,T)}} + (1+\mu(t))\sqrt{\Sigma(t,T)},$$

$$y_1(t) = \frac{\log(H^2/SK)}{\sqrt{\Sigma(t,T)}} + (1+\mu(t))\sqrt{\Sigma(t,T)},$$

$$y_2(t) = \frac{\log(H/S)}{\sqrt{\Sigma(t,T)}} + (1+\mu(t))\sqrt{\Sigma(t,T)},$$

$$z(t) = \frac{\log(H/S)}{\sqrt{\Sigma(t,T)}} + \lambda\sqrt{\Sigma(t,T)},$$

$$\mu(t) = \frac{r(t) - \frac{\Sigma(t,T)}{2(T-t)}}{\frac{\Sigma(t,T)}{T-t}},$$

$$\lambda(t) = \sqrt{\mu(t)^2 + \frac{2r(t)(T-t)}{\Sigma(t,T)}},$$

$$r(t) = r_d(t) - r_f(t),$$

$$\Sigma(t,T) = a(t,T) + b(t,T) + \int_t^T \sigma^2(u)du, \tag{119}$$

$$a(t,T) = \Sigma_d(t,T) + \Sigma_f(t,T) - 2\Sigma_{df}(t,T), \tag{120}$$

$$b(t,T) = 2\left(\int_t^T \rho_{d,FX}\sigma_d(s)B_d(s,T)ds - \int_t^T \rho_{x_f,FX_f}\sigma_f(s)B_f(s,T)ds\right). \tag{121}$$

where integral can be calculated using the method *PiecewiseConstantIntegration.CalculateFlatBackward*. Values $\Sigma_d(t,T)$, $\Sigma_f(t,T)$ (variances of the domestic and foreign bonds), $\Sigma_{df}(t,T)$ (their covariance) are calculated with *SbPslIrHw1fModel.ZeroBondVariance* and *SbPslIrHw1fModel.ZeroBondCovariance*. Integrals in the expression for $b(t,T)$ can be approximated with method *SbPslFxBsModel.Covariance* ($\sigma_d(s)$, $B_d(s,T)$, $\sigma_f(s)$, $B_f(s,T)$ are expressions from the corresponding interest rate models: short rate volatility and factor $B(t,T)$ calculated in method *SbPslIrHw1fModel.B*).

#### 5.2.9 Asian FX Option

Consider an Asian option with strike $K$, averaging times $T_1 < T_2 < ... < T_n$, payment dates $T_p > T_n$. Its price at time $t < T_1$ is given by:

$$V_{ASIAN}(t,\{T_i\},T_p,K,\nu) = \nu P(t,T)\left(F_A(t,\{T_i\})\Phi(\nu d_1) - K\Phi(\nu d_2)\right) \tag{122}$$

where $\Phi(.)$ is the c.d.f. of the standard normal distribution $N(0,1)$, $\nu = 1$ is for a call option, $\nu = -1$ is for a put option, $K$ is the strike,

$$d_1 = \frac{\ln\frac{F_A(t,\{T_i\})}{K} + \frac{1}{2}\Sigma_A(t,\{T_i\},T_p)}{\sqrt{\Sigma_A(t,\{T_i\},T_p)}}, \tag{123}$$

$$d_2 = \frac{\ln\frac{F_A(t,\{T_i\})}{K} - \frac{1}{2}\Sigma_A(t,\{T_i\},T_p)}{\sqrt{\Sigma_A(t,\{T_i\},T_p)}}, \tag{124}$$

$$F_A(t,\{T_i\}) = \frac{1}{n}\sum_{i=1}^{n} F(t,T_i), \tag{125}$$

$$F(t,T) = FX(t)\frac{P_f(t,T)}{P_d(t,T)}, \tag{126}$$

$$\Sigma_A(t,\{T_i\},T_p) = \ln\left(\frac{\frac{1}{n^2}\sum_{k=1}^{n}\sum_{l=1}^{n} F(t,T_k)F(t,T_l)e^{H(t,T_k,T_l,T_p)}}{F_A^2(t,\{T_i\})}\right), \tag{127}$$

$$H(t,T_k,T_l,T_p) = \frac{\min\{T_k - t, T_l - t\}}{T_p - t}\Sigma(t,T_p). \tag{128}$$

$$\Sigma(t,T) = \int_t^T \sigma^2(u)du. \tag{129}$$

The integral can be calculated using the method *PiecewiseConstantIntegration.CalculateFlatBackward*.

#### 5.2.10 Asian Digital FX Option

Consider an Asian digital option with strike $K$, notional $N_f$, averaging times $T_1 < T_2 < ... < T_n$, payment date $T_p > T_n$. Its price at time $t < T_1$ is given by:

$$V_{ASIAN-DIGITAL}(t,\{T_i\},T_p,K,N,\nu) = N_f P_d(t,T)\Phi(\nu d_2) \tag{130}$$

where $\Phi(.)$ is the c.d.f. of the standard normal distribution $N(0,1)$, $\nu = 1$ is for a call option, $\nu = -1$ is for a put option, $K$ is the strike,

$$d_2 = \frac{\ln\frac{F_A(t,\{T_i\})}{K} - \frac{1}{2}\Sigma_A(t,\{T_i\},T_p)}{\sqrt{\Sigma_A(t,\{T_i\},T_p)}}, \tag{131}$$

$$F_A(t,\{T_i\}) = \frac{1}{n}\sum_{i=1}^{n} F(t,T_i), \tag{132}$$

$$F(t,T) = X(t)\frac{P_f(t,T)}{P_d(t,T)}, \tag{133}$$

$$\Sigma_A(t,\{T_i\},T_p) = \ln\left(\frac{\frac{1}{n^2}\sum_{k=1}^{n}\sum_{l=1}^{n}F(t,T_k)F(t,T_l)e^{H(t,T_k,T_l,T_p)}}{F_A^2(t,\{T_i\})}\right), \tag{134}$$

$$H(t,T_k,T_l,T_p) = \frac{\min\{T_k - t, T_l - t\}}{T_p - t}\Sigma(t,T_p). \tag{135}$$

$$\Sigma(t,T) = \int_t^T \sigma^2(u)du. \tag{136}$$

The integral can be calculated using the method *PiecewiseConstantIntegration.CalculateFlatBackward*.

If $t \geq T_1$ falls into $[T_{i-1}, T_i)$ (i.e., by the moment $t$ $X(T_1), X(T_2), ...X(T_{i-1})$ are already known), expressions $F_A(t,\{T_i\})$ and $\Sigma_A(t,\{T_i\},T_p)$ should use known spot values $X(T_i)$ instead of forward values $F(t,T_i)$, and $H(t,T_k,T_l,T_p) = 0$ if $T_k < t$ or $T_l < t$.

### 5.2.11 Asian FX Option with European Barrier

Consider Asian barrier call options of each of the following types down-and-in (DI), up-and-out (UO), up-and-in (UI), down-and-out (DO) with strike $K$, barrier $H$, averaging times $T_1 < T_2 < ... < T_n$, payment date $T_p > T_n$. Their prices at time $t$ are given by:

$$\mathbf{C}_{DI}(t,\{T_i\},T_p,K,H) = \begin{cases} V_{ASIAN}(t,\{T_i\},T_p,K,1) - V_{ASIAN}(t,\{T_i\},T_p,H,1)- \\ -V_{ASIAN-DIGITAL}(t,\{T_i\},T_p,H,H-K,1), \text{ if } K < H, \\ 0, \text{ else}, \end{cases}$$

$$\mathbf{C}_{UO}(t,\{T_i\},T_p,K,H) = \begin{cases} V_{ASIAN}(t,\{T_i\},T_p,K,1) - V_{ASIAN}(t,\{T_i\},T_p,H,1)- \\ -V_{ASIAN-DIGITAL}(t,\{T_i\},T_p,H,H-K,1), \text{ if } K < H, \\ 0, \text{ else}, \end{cases}$$

$$\mathbf{C}_{UI}(t,\{T_i\},T_p,K,H) = \begin{cases} V_{ASIAN}(t,\{T_i\},T_p,H,1)+ \\ +V_{ASIAN-DIGITAL}(t,T_e,T,H,H-K,1), \text{ if } K < H, \\ V_{ASIAN}(t,\{T_i\},T_p,K,1), \text{ else}, \end{cases}$$

$$\mathbf{C}_{DU}(t,\{T_i\},T_p,K,H) = \begin{cases} V_{ASIAN}(t,\{T_i\},T_p,H,1)+ \\ +V_{ASIAN-DIGITAL}(t,\{T_i\},T_p,H,H-K,1), \text{ if } K < H, \\ V_{ASIAN}(t,\{T_i\},T_p,K,1), \text{ else}. \end{cases}$$

For put options the expressions are as follows:

$$\mathbf{P}_{DI}(t,\{T_i\},T_p,K,H) = \begin{cases} V_{ASIAN}(t,\{T_i\},T_p,H,-1)+ \\ +V_{ASIAN-DIGITAL}(t,\{T_i\},T_p,H,K-H,-1), \text{ if } K < H, \\ V_{ASIAN}(t,\{T_i\},T_p,K,-1), \text{ else}, \end{cases}$$

$$\mathbf{P}_{UO}(t,\{T_i\},T_p,K,H) = \begin{cases} V_{ASIAN}(t,\{T_i\},T_p,H,-1)+ \\ +V_{ASIAN-DIGITAL}(t,\{T_i\},T_p,H,K-H,-1), \text{ if } K < H, \\ V_{ASIAN}(t,\{T_i\},T_p,K,-1), \text{ else}, \end{cases}$$

$$\mathbf{P}_{UI}(t,\{T_i\},T_p,K,H)=\begin{cases} V_{ASIAN}(t,\{T_i\},T_p,K,-1)-V_{ASIAN}(t,\{T_i\},T_p,H,-1)- \\ -V_{ASIAN-DIGITAL}(t,\{T_i\},T_p,H,K-H,-1), \text{ if } K<H, \\ 0, \text{ else,} \end{cases}$$

$$\mathbf{P}_{DU}(t,\{T_i\},T_p,K,H)=\begin{cases} V_{ASIAN}(t,\{T_i\},T_p,K,-1)-V_{ASIAN}(t,\{T_i\},T_p,H,-1)- \\ -V_{ASIAN-DIGITAL}(t,\{T_i\},T_p,H,K-H,-1), \text{ if } K<H, \\ 0, \text{ else.} \end{cases}$$

Here $V_{ASIAN}(t,T_e,T,K,\nu)$, $V_{ASIAN-DIGITAL}(t,T_e,T,K,\nu)$ are expressions for Asian and Asian digital option that can be calculated using (150) and (159) respectively.

#### 5.2.12 Asian Digital FX Option with European Barrier

Consider Asian barrier digital call options of each of the following types down-and-in (DI), up-and-out (UO), up-and-in (UI), down-and-out (DO) with strike $K$, barrier $H$, averaging times $T_1<T_2<...<T_n$, payment date $T_p>T_n$. Their prices at time $t$ are given by:

$$\mathbf{C}_{DIGITAL-DI}(t,\{T_i\},T_p,K,H)=\begin{cases} 0, \text{ if } K\geq H, \\ V_{ASIAN-DIGITAL}(t,\{T_i\},T_p,K,N,1)- \\ -V_{ASIAN-DIGITAL}(t,\{T_i\},T_p,H,N,1), \text{ else,} \end{cases}$$

$$\mathbf{C}_{DIGITAL-UO}(t,\{T_i\},T_p,K,H)=\begin{cases} 0, \text{ if } K\geq H, \\ V_{ASIAN-DIGITAL}(t,\{T_i\},T_p,K,N,1)- \\ -V_{ASIAN-DIGITAL}(t,\{T_i\},T_p,H,N,1), \text{ else,} \end{cases}$$

$$\mathbf{C}_{DIGITAL-UI}(t,\{T_i\},T_p,K,H)=V_{ASIAN-DIGITAL}(t,\{T_i\},T_p,\max\{K,H\},N,1),$$

$$\mathbf{C}_{DIGITAL-DU}(t,\{T_i\},T_p,K,H)=V_{ASIAN-DIGITAL}(t,\{T_i\},T_p,\max\{K,H\},N,1).$$

Put option prices are given by:

$$\mathbf{P}_{DIGITAL-DI}(t,\{T_i\},T_p,K,H)=V_{ASIAN-DIGITAL}(t,\{T_i\},T_p,\min\{K,H\},N,-1)$$

$$\mathbf{P}_{DIGITAL-UO}(t,\{T_i\},T_p,K,H)=V_{ASIAN-DIGITAL}(t,\{T_i\},T_p,\min\{K,H\},N,-1)$$

$$\mathbf{P}_{DIGITAL-UI}(t,\{T_i\},T_p,K,H)=\begin{cases} V_{ASIAN-DIGITAL}(t,\{T_i\},T_p,K,N,-1)- \\ -V_{ASIAN-DIGITAL}(t,\{T_i\},T_p,H,N,-1), \text{ if } K<H, \\ 0, \text{ else,} \end{cases}$$

$$\mathbf{P}_{DIGITAL-DU}(t,\{T_i\},T_p,K,H)=\begin{cases} V_{ASIAN-DIGITAL}(t,\{T_i\},T_p,K,N,-1)- \\ -V_{ASIAN-DIGITAL}(t,\{T_i\},T_p,H,N,-1), \text{ if } K<H, \\ 0, \text{ else,} \end{cases}$$

Here $V_{ASIAN-DIGITAL}(t,T_e,T,K,\nu)$ is the expressions for Asian digital option that can be calculated using (159).

#### 5.2.13 Cross-currency Interest Rate Swaption

Consider a cross-currency interest rate swaption with maturity $T$ and notional $N$ on a swap consisting of two legs: leg 1 denominated in currency $f_1$ and leg 2 denominated in $f_2$. Each leg can be either fixed or floating. Denote the sequence of fixing dates of the leg $k$ with $T^f_{k0} < T^f_{k1} < ... < T^f_{kN_k}$ and the sequence of payment dates with $T^p_{k0} < T^p_{k1} < ... < T^p_{kN_k}$. In this document, we assume that there is a mutually-exclusive correspondence between fixing dates and payment dates (if we have a few fixing dates with the same payment date or vice versa, we denote these pairs with different indices: there can be $T_i$ with different indices but the same value). Let $\tau_i$ denote interval length between $T_{i-1}$ and $T_i$. We also denote currency in which the instrument is denominated with $d$ ("domestic") and FX rate of currency $f_k$ (with respect to $d$) with $FX_{f_k}(t)$ (expressed in units of domestic currency per unit of currency $f_k$).

Terminal payoff of the cross-currency swaption at maturity $t = T$ is given by (expressed in domestic currency):

$$N\left(wL_1(T) - wL_2(T)\right)^+, \tag{137}$$

where $L_k(T)$ is time $T$ value of the leg $k$ (expressed in domestic currency), $w$ is either 1 or $-1$ (depending on whether buy or sell is applied to the first leg). Thus, at time $t < T$ price of the instrument is given by:

$$\mathbf{QES}(t) = NP^d(t,T)E^{\mathbb{Q}^d_T}\left[(wL_1(T) - wL_2(T))^+\middle|\mathcal{F}_t\right],$$

where $P^d(t,T)$ is domestic bond, $\mathbb{Q}^d_T$ is domestic $T$-forward measure. Assuming that $L_k(T)$ can be approximated by lognormal random variables with parameters $\mu_k(t,T)$ and $\sigma^2_k(t,T)$ (calculations for each type of legs are presented below), we obtain:

$$\begin{aligned}\mathbf{QES}(t) &= NP^d(t,T)\int_{-\infty}^{+\infty} E^{\mathbb{Q}^d_T}\left[(wL_1(T) - wL_2(T))^+\middle|\mathcal{F}_t, \frac{\ln L_2(T) - \mu_2(t,T)}{\sigma_2(t,T)} = x\right]\phi(x)dx = \\ &= NP^d(t,T)\int_{-\infty}^{+\infty} E^{\mathbb{Q}^d_T}\left[\left(wL_1(T) - we^{\mu_2(t,T)+x\sigma_2(t,T)}\right)^+\middle|\mathcal{F}_t, \frac{\ln L_2(T) - \mu_2(t,T)}{\sigma_2(t,T)} = x\right]\phi(x)dx = \\ &= NP^d(t,T)\int_{-\infty}^{+\infty} f(x)\phi(x)dx,\end{aligned}$$

where $\phi(x)$ is p.d.f. of the standard normal distribution, integral can be calculated numerically, $f(x)$ is conditional expectation of the payoff given $L_2(T) = e^{\mu_2(t,T)+x\sigma_2(t,T)}$. It can be calculated using the Black formula (without discount factor):

$$f(x) = w\left[e^{\mu_{1|2}+\frac{1}{2}\sigma^2_{1|2}}\Phi\left(w\frac{\mu_{1|2} + \sigma^2_{1|2} - (\mu_2 + \sigma_2 x)}{\sigma_{1|2}}\right) - e^{\mu_2+\sigma_2 x}\Phi\left(w\frac{\mu_{1|2} - (\mu_2 + \sigma_2 x)}{\sigma_{1|2}}\right)\right],$$

where $\rho_{12}$ is correlation between $\ln L_1(T)$ and $\ln L_2(T)$, $\mu_{1|2}$ and $\sigma_{1|2}$ are conditional expectation and variance of $\ln L_1(T)$ given $L_2(T) = e^{\mu_2(t,T)+\sigma_2(t,T)x}$:

$$\mu_{1|2} := \mu_{1|2}(t,T) = \mu_1(t,T) + \rho_{12}\sigma_1(t,T)x,$$

$$\sigma^2_{1|2} := \sigma^2_{1|2}(t,T) = (1 - \rho^2_{12})\sigma^2_1(t,T).$$

Expressions $\mu_k(t,T)$ and $\sigma_k(t,T)$ can be calculated as follows:

$$\sigma_k(t,T)^2 = \ln\left(\frac{E^{\mathbb{Q}^d_T}\left[L^2_k(T)|\mathcal{F}_t\right]}{E^{\mathbb{Q}^d_T}\left[L_k(T)|\mathcal{F}_t\right]^2}\right), \tag{138}$$

$$\mu_k(t,T) = \ln E^{\mathbb{Q}^d_T}\left[L_k(T)|\mathcal{F}_t\right] - \frac{1}{2}\sigma^2_k(t,T). \tag{139}$$

However, values of the first and the second moments of $L_k(T)$ depend on the type of leg $k$. The formulas corresponding to each type of legs are presented below.

**Negative interest rates.** If one of the legs has negative value, it is reasonable to approximate both legs with displaced lognormal distribution (where difference $L_k(T) + D$ has lognormal distribution, where $D$ is large enough to compensate all possible negative values of legs). Hence, we choose the following expression for both $\mu_k$ and $\sigma_k$ for both legs:

$$\sigma_k(t,T)^2 = \ln\left(\frac{E^{\mathbb{Q}^d_T}\left[L_k^2(T)|\mathcal{F}_t\right] + 2DE^{\mathbb{Q}^d_T}\left[L_k(T)|\mathcal{F}_t\right] + D^2}{\left(E^{\mathbb{Q}^d_T}\left[L_k(T)|\mathcal{F}_t\right] + D\right)^2}\right),$$

$$\mu_k(t,T) = \ln\left(E^{\mathbb{Q}^d_T}\left[L_k(T)|\mathcal{F}_t\right] + D\right) - \frac{1}{2}\sigma_k^2(t,T),$$

where

$$D = -2\min\{0, E^{\mathbb{Q}^d_T}\left[L_1(T)|\mathcal{F}_t\right], E^{\mathbb{Q}^d_T}\left[L_2(T)|\mathcal{F}_t\right]\}.$$

The rest of the formulas will remain unchanged.

**Fixed leg.** If leg $k$ is a fixed leg with fixed rate $K$, its expectation can be calculated as follows:

$$E^{\mathbb{Q}^d_T}\left[L_k(T)|\mathcal{F}_t\right] = K \sum_{i:T_i^f>t} \tau_i E^{\mathbb{Q}^d_T}\left[P^{f_k}(T,T_i^p)FX_{f_k}(T_i^p)|\mathcal{F}_t\right].$$

Here (and onwards) for the sake of simplicity, we use $T_i$ istead of $T_{ki}^f$. Taking into account that all factors are lognormal, the expectations can be factorized:

$$\begin{aligned}
&E^{\mathbb{Q}^d_T}\left[P^{f_k}(T,T_i^p)FX_{f_k}(T_i^p)|\mathcal{F}_t\right] = E^{\mathbb{Q}^d_T}\left[P^{f_k}(T,T_i^p)|\mathcal{F}_t\right] E^{\mathbb{Q}^d_T}\left[FX_{f_k}(T_i^p)|\mathcal{F}_t\right] \times \\
&\times e^{\mathbf{cov}(\ln P^{f_k}(T,T_i^p), \ln FX_{f_k}(T_i^p)|\mathcal{F}_t)} = \\
&= M_P^{f_k}(t,T,T_i^p) Fwd_{f_k}^{FX}(t,T,T_i^p) e^{C_{f_k}(t,T,T_i^p)}.
\end{aligned}$$

Here we use the property of the lognormal distribution: $E(XY) = E(X)E(Y)e^{cov(\ln X, \ln Y)}$. In the previous expression we have used the following notation:

$$\begin{aligned}
M_P^{f_k}(t,T,T_i^p) &= E^{\mathbb{Q}^d_T}\left[P(T,T_i^p)|\mathcal{F}_t\right] = E^{\mathbb{Q}^{f_k}_T}\left[P^{f_k}(T,T_i^p)|\mathcal{F}_t\right] QA_{f_k}(t,T,T_i^p,T) = \\
&= \frac{P^{f_k}(t,T_i^p)}{P^{f_k}(t,T)} QA_{f_k}(t,T,T_i^p,T)
\end{aligned}$$

(please note that quanto-adjustment has been added because of conversion from foreign forward measure to domestic spot measure);

$$Fwd_{f_k}^{FX}(t,T,T_i^p) = FX_{f_k}(t)\frac{P^{f_k}(t,T_i^p)P^d(t,T)}{P^{f_k}(t,T)P^d(t,T_i^p)},$$

$$C_{f_k}(t,T,T_i) \approx \rho_{f_k,FX_k} B_{f_k}(T,T_i)\sqrt{V_{f_k}(t,T)\Sigma_{FX_{f_k}}(t,T_i)}, \tag{140}$$

$$B_{f_k}(t,T) = \int_t^T e^{-\int_t^v \alpha_{f_k}(u)du} dv, \tag{141}$$

$$V_{f_k}(t,T) = \int_t^T e^{-2\alpha_{f_k}(T-u)}\sigma_{IR_{f_k}}^2(u)du, \tag{142}$$

$$\Sigma_{FX_{f_k}}(t,T) = \int_t^T \sigma_{FX_{f_k}}^2(s)ds, \tag{143}$$

$$QA_{f_k}(t,T_F,T_M,T_P) = \exp(\int_t^{T_F} \sigma_{IR_{f_k}}(u)(B_{f_k}(u,T_M) - B_{f_k}(u,T_F))(B_{f_k}(u,T_P)\sigma_{IR_{f_k}}(u) + $$
$$+ \sigma_{FX_{f_k,d}}(u)\rho_{FX_{f_k},d} - \sigma_d(u)B_d(u,T_P)\rho_{f_k,d})du),$$

The second moment can be calculated as follows:

$$E^{\mathbb{Q}_T^d}\left[L_k^2(T)|\mathcal{F}_t\right] = K^2 \sum_{i,j:T_i^f,T_j^f>t} \tau_i\tau_j E^{\mathbb{Q}_T^d}\left[P^{f_k}(T,T_i^p)FX_{f_k}(T_i^p)P^{f_k}(T,T_j^p)FX_{f_k}(T_j^p)|\mathcal{F}_t\right],$$

$$E^{\mathbb{Q}_T^d}\left[P^{f_k}(T,T_i^p)FX_{f_k}(T_i^p)P^{f_k}(T,T_j^p)FX_{f_k}(T_j^p)|\mathcal{F}_t\right] = M_P^{f_k}(t,T,T_i^p)M_P^{f_k}(t,T,T_j^p)\times$$
$$\times Fwd_{f_k}^{FX}(t,T,T_i^p)Fwd_{f_k}^{FX}(t,T,T_j^p)e^{H_{f_k}(t,T,T_i^p,T_j^p)},$$

where

$$H_{f_k}(t,T,T_i,T_j) = C_{f_k}(t,T,T_i,T_i) + C_{f_k}(t,T,T_j,T_j) + C_{f_k}(t,T,T_i,T_j) + C_{f_k}(t,T,T_j,T_i) +$$
$$+ \Sigma_{FX_{f_k}}(t,\min(T_i,T_j)) + B_{f_k}(T,T_i)B_{f_k}(T,T_j)V_{f_k}(t,T),$$

$$C_{f_k}(t,T,T_j,T_i) \approx \rho_{f_k,FX_k}B_{f_k}(T,T_j)\sqrt{V_{f_k}(t,T)\Sigma_{FX_{f_k}}(t,T_i)}. \tag{144}$$

**Floating leg.** The first moment of the floating leg can be calculated as follows:

$$E^{\mathbb{Q}_T^d}\left[L_k(T)|\mathcal{F}_t\right] = \sum_{i:T_i^f>t} \tau_i E^{\mathbb{Q}_T^d}\left[P^{f_k}(T,T_i^p)F(T;T_{i-1},T_i)FX_{f_k}(T_i^P)|\mathcal{F}_t\right] =$$
$$= \sum_{i:T_i^f>t} E^{\mathbb{Q}_T^d}\left[P^{f_k}(T,T_{i-1})FX_{f_k}(T_i^p)\frac{P^{f_k}(T,T_i^p)}{P^{f_k}(T,T_i)}|\mathcal{F}_t\right] -$$
$$- \sum_{i:T_i^f>t} E^{\mathbb{Q}_T^d}\left[P^{f_k}(T,T_i)FX_{f_k}(T_i^p)\frac{P^{f_k}(T,T_i^p)}{P^{f_k}(T,T_i)}|\mathcal{F}_t\right] \approx$$
$$\approx \sum_{i:T_i^f>t} \frac{P^{f_k}(T,T_i^p)}{P^{f_k}(T,T_i)} E^{\mathbb{Q}_T^d}\left[P^{f_k}(T,T_{i-1})FX_{f_k}(T_i^p)|\mathcal{F}_t\right] -$$
$$- \sum_{i:T_i^f>t} \frac{P^{f_k}(T,T_i^p)}{P^{f_k}(T,T_i)} E^{\mathbb{Q}_T^d}\left[P^{f_k}(T,T_i)FX_{f_k}(T_i^p)|\mathcal{F}_t\right].$$

Terms $E^{\mathbb{Q}_T^d}\left[P^{f_k}(T,T_i)FX_{f_k}(T_j)|\mathcal{F}_t\right]$ can be calculated according to the same principle:

$$E^{\mathbb{Q}_T^d}\left[P^{f_k}(T,T_{i-1})FX_{f_k}(T_j)|\mathcal{F}_t\right] =$$
$$= M_P^{f_k}(t,T,T_{i-1})Fwd_{f_k}^{FX}(t,T,T_j)e^{C_{f_k}(t,T,T_{i-1},T_i)},$$

where

$$C_{f_k}(t,T,T_j,T_i) \approx \rho_{f_k,FX_k}B_{f_k}(T,T_j)\sqrt{V_{f_k}(t,T)\Sigma_{FX_{f_k}}(t,T_i)}. \tag{145}$$

The second moment of the floating leg is calculated as follows:

$$E^{\mathbb{Q}_T^d}[L_k^2(T)|\mathcal{F}_t] = \sum_{i,j:T_i^f,T_j^f>t} E^{\mathbb{Q}_T^d}[(P^{f_k}(T,T_{i-1}) - P^{f_k}(T,T_i))(P^{f_k}(T,T_{j-1}) - P^{f_k}(T,T_j))\times$$
$$\times FX_{f_k}(T_i)FX_{f_k}(T_j)|\mathcal{F}_t] =$$
$$= \sum_{i,j:T_i^f,T_j^f>t} \frac{P^{f_k}(T,T_i^p)}{P^{f_k}(T,T_i)}\frac{P^{f_k}(T,T_j^p)}{P^{f_k}(T,T_j)} E^{\mathbb{Q}_T^d}[P^{f_k}(T,T_{i-1})FX_{f_k}(T_i)P^{f_k}(T,T_{j-1})FX_{f_k}(T_j)|\mathcal{F}_t]-$$
$$- \sum_{i,j:T_i^f,T_j^f>t} \frac{P^{f_k}(T,T_i^p)}{P^{f_k}(T,T_i)}\frac{P^{f_k}(T,T_j^p)}{P^{f_k}(T,T_j)} E^{\mathbb{Q}_T^d}[P^{f_k}(T,T_i)FX_{f_k}(T_i)P^{f_k}(T,T_{j-1})FX_{f_k}(T_j)|\mathcal{F}_t]-$$
$$- \sum_{i,j:T_i^f,T_j^f>t} \frac{P^{f_k}(T,T_i^p)}{P^{f_k}(T,T_i)}\frac{P^{f_k}(T,T_j^p)}{P^{f_k}(T,T_j)} E^{\mathbb{Q}_T^d}[P^{f_k}(T,T_{i-1})FX_{f_k}(T_i)P^{f_k}(T,T_j)FX_{f_k}(T_j)|\mathcal{F}_t]+$$
$$+ \sum_{i,j:T_i^f,T_j^f>t} \frac{P^{f_k}(T,T_i^p)}{P^{f_k}(T,T_i)}\frac{P^{f_k}(T,T_j^p)}{P^{f_k}(T,T_j)} E^{\mathbb{Q}_T^d}[P^{f_k}(T,T_i)FX_{f_k}(T_i)P^{f_k}(T,T_j)FX_{f_k}(T_j)|\mathcal{F}_t].$$

Each of the above terms can be calculated according to the following principle:

$$E^{\mathbb{Q}_T^d}\left[P^{f_k}(T,T_{i_1})FX_{f_k}(T_{i_2})P^{f_k}(T,T_{j_1})FX_{f_k}(T_{j_2})|\mathcal{F}_t\right] = M_P^{f_k}(t,T,T_{i_1})M_P^{f_k}(t,T,T_{j_1})\times$$
$$\times Fwd_{f_k}^{FX}(t,T,T_{i_2}^p)Fwd_{f_k}^{FX}(t,T,T_{j_2}^p)e^{H_{f_k}(t,T,T_{i_1},T_{i_2},T_{j_1},T_{j_2})},$$

where

$$H_{f_k}(t,T,T_{i_1},T_{i_2},T_{j_1},T_{j_2}) = C_{f_k}(t,T,T_{i_1},T_{i_2}) + C_{f_k}(t,T,T_{j_1},T_{j_2}) + C_{f_k}(t,T,T_{i_1},T_{j_2}) + C_{f_k}(t,T,T_{j_1},T_{i_2})+$$
$$+ \Sigma_{FX_{f_k}}(t,\min(T_{i_2},T_{j_2})) + B_{f_k}(T,T_{i_1})B_{f_k}(T,T_{j_1})V_{f_k}(t,T).$$

**Notional exchange.** If the swaption contains notional exchange at time $T_{NE}$, its leg $k$ takes the form:

$$L_k^{NE}(T) = L_k(T) + P^{f_k}(T,T_{NE})FX_{f_k}(T_{NE}).$$

Thus, the first moment contains an additional term:

$$E^{\mathbb{Q}_T^d}\left[L_k^{NE}(T)|\mathcal{F}_t\right] = E^{\mathbb{Q}_T^d}\left[L_k(T)|\mathcal{F}_t\right] + E^{\mathbb{Q}_T^d}\left[P^{f_k}(T,T_{NE})FX_{f_k}(T_{NE})|\mathcal{F}_t\right],$$

which can be calculated according to the formula for $E^{\mathbb{Q}_T^d}\left[P^{f_k}(T,T_i)FX_{f_k}(T_i)|\mathcal{F}_t\right]$ above.

The second moment contains a few additional terms:

$$E^{\mathbb{Q}_T^d}\left[\left(L_k^{NE}(T)\right)^2|\mathcal{F}_t\right] = E^{\mathbb{Q}_T^d}\left[L_k^2(T)|\mathcal{F}_t\right] + 2E^{\mathbb{Q}_T^d}\left[L_k(T)P^{f_k}(T,T_{NE})FX_{f_k}(T_{NE})|\mathcal{F}_t\right]+$$
$$+ E^{\mathbb{Q}_T^d}\left[\left(P^{f_k}(T,T_{NE})FX_{f_k}(T_{NE})\right)^2|\mathcal{F}_t\right].$$

The last term can be calculated with according to the formulas above while the second one can be approximated by the product:

$$E^{\mathbb{Q}_T^d}\left[L_k(T)P^{f_k}(T,T_{NE})FX_{f_k}(T_{NE})|\mathcal{F}_t\right] \approx$$
$$\approx E^{\mathbb{Q}_T^d}\left[L_k(T)|\mathcal{F}_t\right]E^{\mathbb{Q}_T^d}\left[P^{f_k}(T,T_{NE})FX_{f_k}(T_{NE})|\mathcal{F}_t\right],$$

whose factors are calculated above.

**Physical settlement.** Unlike the case of cash settlement (where $\mathbf{QES}(t) = 0$ for $t > T$), physically settled swaption can be prices even if $t > T$. In this case its price is defined by:

$$\mathbf{QES}(t) = V(t)I_{V(T)>0},$$

where $V(t)$ is a price of the underlying swap.

**Notional exchange.** If the swaption contains notional exchange at time $T_{NE}$, $L_k^i$ are replaced by $\bar{L}_k^i$ which are defined by:

$$\bar{L}_k^1 = L_k^1 + M_P^{f_k}(t, T, T_{NE}) Fwd_{f_k}^{FX}(t, T, T_{NE}) e^{C_{f_k}(t,T,T_{NE})},$$

$$\bar{L}_k^2 = L_k^2 + 2L_k^1 M_P^{f_k}(t, T, T_{NE}) Fwd_{f_k}^{FX}(t, T, T_{NE}) e^{C_{f_k}(t,T,T_{NE})} + \\ + Y(T_{NE}, T_{NE}, T_{NE}, T_{NE}).$$

**Additional fixed payments.** If leg $k$ contains additional fixed payment with notional $N_m$ (in domestic currency) at time $T_m$, $L_k^i$ can be adjusted by additional terms:

$$\bar{L}_k^1 = L_k^1 + \frac{N_m}{N} M_P^{f_k}(t, T, T_m) Fwd_{f_k}^{FX}(t, T, T_m) e^{C_{f_k}(t,T,T_m)},$$

$$\bar{L}_k^2 = L_k^2 + 2\frac{N_m}{N} L_k^1 M_P^{f_k}(t, T, T_m) Fwd_{f_k}^{FX}(t, T, T_m) e^{C_{f_k}(t,T,T_m)} + \\ + \frac{N_m^2}{N^2} Y(T_m, T_m, T_m, T_m).$$

**Amortization.** If leg $k$ contains payments with different notional values, $N$ can be chosen arbitrarily but the $i$-th notional can be taken into account by coefficient $\alpha_i = N_i/N$ applied to each corresponding cash flow.

### 5.3 Commodity Derivatives

#### 5.3.1 Single currency derivatives

#### 5.3.2 European option

European option price with expiration date $T_e$, strike $K$ on a futures with maturity $T$, at time $t$ is given by:

$$V_{EU}(t, T_e, T, K, \nu) = \nu \left( F(t,T)\Phi(\nu d_1) - K\Phi(\nu d_2) \right) \tag{146}$$

where $\Phi(.)$ is a c.d.f. of the standard normal distribution $N(0,1)$, $\nu = 1$ for a call option, $\nu = -1$ for a put option,

$$d_1 = \frac{\ln \frac{F(t,T)}{K} + \frac{1}{2}\Sigma(t, T_e, T)}{\sqrt{\Sigma(t, T_e, T)}}, \tag{147}$$

$$d_2 = d_1 - \sqrt{\Sigma(t, T_e, T)}, \tag{148}$$

$$\Sigma(t, T_e, T) = \mathbf{Var}\left[\ln F(T_e, T)|\mathcal{F}_t\right] = e^{-2kT} \int_t^{T_e} e^{2ku} \sigma^2(u) du. \tag{149}$$

Integral $\Sigma(t, T_e, T)$ is calculated numerically (using method *DoubleExponentialTransformation.Integrate*).

#### 5.3.3 Asian option

Asian option with strike $K$, averaging times $T_1 < T_2 < ... < T_n$, payment date $T_p > T_n$ at time $t < T_1$ has value of

$$V_{ASIAN}(t, \{T_i\}, T_p, K, \nu) = \nu \left( F_A(t, \{T_i\})\Phi(\nu d_1) - K\Phi(\nu d_2) \right) \tag{150}$$

where $\Phi(.)$ is the c.d.f. of the standard normal distribution $N(0,1)$, $\nu = 1$ for a call option, $\nu = -1$ for a put option,

$$d_1 = \frac{\ln \frac{F_A(t,\{T_i\})}{K} + \frac{1}{2}\Sigma_A(t, \{T_i\}, T_p)}{\sqrt{\Sigma_A(t, \{T_i\}, T_p)}}, \tag{151}$$

$$d_2 = d_1 - \sqrt{\Sigma_A(t, \{T_i\}, T_p)}, \tag{152}$$

$$F_A(t, \{T_i\}) = \frac{1}{n} \sum_{i=1}^{n} F(t, T_i), \tag{153}$$

$$\Sigma_A(t, \{T_i\}, T_p) = \ln \left( \frac{\frac{1}{n^2} \sum_{k=1}^{n} \sum_{l=1}^{n} F(t, T_k) F(t, T_l) e^{H(t, T_k, T_l, T_p)}}{F_A^2(t, \{T_i\})} \right), \tag{154}$$

$$H(t, T_k, T_l, T_p) = e^{k(T_k + T_l)} \int_t^{T_p} e^{2ku} \sigma(u) du. \tag{155}$$

Integral $\Sigma(t, T_e, T)$ and $H(t, T_k, T_l, T_p)$ is calculated numerically (using method *DoubleExponentialTransformation.Integrate*).

If $t \geq T_1$ and falls into $[T_{i-1}, T_i)$ (i.e. by the time $t$ $S(T_1), S(T_2), ...S(T_{i-1})$ are already known), futures values $F(t, T_i)$ in the formulas for $F_A(t, \{T_i\})$ and $\Sigma_A(t, \{T_i\}, T_p)$ should be replaced by the spot values $S(T_i)$. Besides, $H(t, T_k, T_l, T_p) = 0$ if $T_k < t$ or $T_l < t$.

#### 5.3.4 Digital option

Digital option with notional $N$, expiration date $T_e$ and strike $K$ at time $t$ has a value of

$$V_{DIGITAL}(t, T_e, T, K, N, \nu) = N\Phi(\nu d_2) \tag{156}$$

where $\Phi(.)$ is the c.d.f. of the standard normal distribution $N(0,1)$, $\nu = 1$ for a call option, $\nu = -1$ for a put option,

$$d_2 = \frac{\ln \frac{F(t,T_e)}{K} - \frac{1}{2}\Sigma(t, T_e, T)}{\sqrt{\Sigma(t, T_e, T)}}, \tag{157}$$

$$\Sigma(t, T_e, T) = e^{-2kT} \int_t^{T_e} e^{2ku}\sigma^2(u)du, \tag{158}$$

$T$ is the maturity of the nearest futures contract to the expiration date $T_e$ (method *GetFutureMaturityTime*). Integral $\Sigma(t, T_e, T)$ is calculated numerically (using method *DoubleExponentialTransformation.Integrate*).

#### 5.3.5 Asian digital option

Asian digital option with notional $N$, strike $K$, averaging times $T_1 < T_2 < ... < T_n$, payment date $T_p > T_n$ at time $t < T_1$ has value of

$$V_{ASIAN-DIGITAL}(t, \{T_i\}, T_p, K, N, \nu) = N\Phi(\nu d_2) \tag{159}$$

where $\Phi(.)$ is the c.d.f. of the standard normal distribution $N(0,1)$, $\nu = 1$ for a call option, $\nu = -1$ for a put option,

$$d_2 = \frac{\ln \frac{F_A(t,\{T_i\})}{K} - \frac{1}{2}\Sigma_A(t, \{T_i\}, T_p)}{\sqrt{\Sigma_A(t, \{T_i\}, T_p)}}, \tag{160}$$

$$F_A(t, \{T_i\}) = \frac{1}{n}\sum_{i=1}^{n} F(t, T_i), \tag{161}$$

$$\Sigma_A(t, \{T_i\}, T_p) = \ln\left(\frac{\frac{1}{n^2}\sum_{k=1}^{n}\sum_{l=1}^{n} F(t, T_k)F(t, T_l)e^{H(t,T_k,T_l,T_p)}}{F_A^2(t, \{T_i\})}\right), \tag{162}$$

$$H(t, T_k, T_l, T_p) = e^{k(T_k+T_l)} \int_t^{T_p} e^{2ku}\sigma(u)du. \tag{163}$$

Integral $\Sigma(t, T_e, T)$ and $H(t, T_k, T_l, T_p)$ is calculated numerically (using method *DoubleExponentialTransformation.Integrate*).

If $t \geq T_1$ and falls into $[T_{i-1}, T_i)$ (i.e. by the time $t$ $S(T_1), S(T_2), ...S(T_{i-1})$ are already known), futures values $F(t, T_i)$ in the formulas for $F_A(t, \{T_i\})$ and $\Sigma_A(t, \{T_i\}, T_p)$ should be replaced by the spot values $S(T_i)$. Besides, $H(t, T_k, T_l, T_p) = 0$ if $T_k < t$ or $T_l < t$.

#### 5.3.6 Barrier option

The price of barrier call down-and-in (DI), up-and-out (UO), up-and-in (UI), down-and-out (DO) options with strike $K$, barrier $H$, expiration date $T_e$ at time $t$ is given by:

$$\mathbf{C}_{DI}(t, T_e, K, H) = \begin{cases} V_{EU}(t, T_e, T_e, K, 1) & -V_{EU}(t, T_e, T_e, H, 1) - \\ & -V_{DIGITAL}(t, T_e, T, H, H - K, 1), \text{ if } K < H, \\ 0, \text{ else}, & \end{cases}$$

$$\mathbf{C}_{UO}(t, T_e, K, H) = \begin{cases} V_{EU}(t, T_e, T_e, K, 1) & -V_{EU}(t, T_e, T_e, H, 1) - \\ & -V_{DIGITAL}(t, T_e, T, H, H - K, 1), \text{ if } K < H, \\ 0, \text{ else}, & \end{cases}$$

$$\mathbf{C}_{UI}(t,T_e,K,H)=\begin{cases}V_{EU}(t,T_e,T_e,H,1)+V_{DIGITAL}(t,T_e,T,H,H-K,1), & \text{if } K<H,\\ V_{EU}(t,T_e,T_e,K,1), & \text{else,}\end{cases}$$

$$\mathbf{C}_{DU}(t,T_e,K,H)=\begin{cases}V_{EU}(t,T_e,T_e,H,1)+V_{DIGITAL}(t,T_e,T,H,H-K,1), & \text{if } K<H,\\ V_{EU}(t,T_e,T_e,K,1), & \text{else.}\end{cases}$$

For put options, the formulas are as follows:

$$\mathbf{P}_{DI}(t,T_e,K,H)=\begin{cases}V_{EU}(t,T_e,T_e,H,-1)+V_{DIGITAL}(t,T_e,T,H,K-H,-1), & \text{if } K<H,\\ V_{EU}(t,T_e,T_e,K,-1), & \text{else,}\end{cases}$$

$$\mathbf{P}_{UO}(t,T_e,K,H)=\begin{cases}V_{EU}(t,T_e,T_e,H,-1)+V_{DIGITAL}(t,T_e,T,H,K-H,-1), & \text{if } K<H,\\ V_{EU}(t,T_e,T_e,K,-1), & \text{else,}\end{cases}$$

$$\mathbf{P}_{UI}(t,T_e,K,H)=\begin{cases}V_{EU}(t,T_e,T_e,K,-1) & -V_{EU}(t,T_e,T_e,H,-1)-\\ & -V_{DIGITAL}(t,T_e,T,H,K-H,-1), \ \text{if } K<H,\\ 0, \ \text{else,}\end{cases}$$

$$\mathbf{P}_{DU}(t,T_e,K,H)=\begin{cases}V_{EU}(t,T_e,T_e,K,-1) & -V_{EU}(t,T_e,T_e,H,-1)-\\ & -V_{DIGITAL}(t,T_e,T,H,K-H,-1), \ \text{if } K<H,\\ 0, \ \text{else.}\end{cases}$$

Here we use expressions $V_{EU}(t,T_e,T,K,\nu)$, $V_{DIGITAL}(t,T_e,T,K,\nu)$ for the European and digital options, that are calculated according to (210) and (156).

### 5.3.7 Asian barrier option

The price of Asian barrier call down-and-in (DI), up-and-out (UO), up-and-in (UI), down-and-out (DO) options with strike $K$, barrier $H$, averaging times $T_1<T_2<...<T_n$, payment date $T_p>T_n$ at time $t$ is given by

$$\mathbf{C}_{DI}(t,\{T_i\},T_p,K,H)=\begin{cases} V_{ASIAN}(t,\{T_i\},T_p,K,1)-V_{ASIAN}(t,\{T_i\},T_p,H,1)-\\ -V_{ASIAN-DIGITAL}(t,\{T_i\},T_p,H,H-K,1), \ \text{if } K<H,\\ 0, \ \text{else,}\end{cases}$$

$$\mathbf{C}_{UO}(t,\{T_i\},T_p,K,H)=\begin{cases} V_{ASIAN}(t,\{T_i\},T_p,K,1)-V_{ASIAN}(t,\{T_i\},T_p,H,1)-\\ -V_{ASIAN-DIGITAL}(t,\{T_i\},T_p,H,H-K,1), \ \text{if } K<H,\\ 0, \ \text{else,}\end{cases}$$

$$\mathbf{C}_{UI}(t,\{T_i\},T_p,K,H)=\begin{cases} V_{ASIAN}(t,\{T_i\},T_p,H,1)+\\ +V_{ASIAN-DIGITAL}(t,T_e,T,H,H-K,1), \ \text{if } K<H,\\ V_{ASIAN}(t,\{T_i\},T_p,K,1), \ \text{else,}\end{cases}$$

$$\mathbf{C}_{DU}(t,\{T_i\},T_p,K,H)=\begin{cases} V_{ASIAN}(t,\{T_i\},T_p,H,1)+\\ +V_{ASIAN-DIGITAL}(t,\{T_i\},T_p,H,H-K,1), \ \text{if } K<H,\\ V_{ASIAN}(t,\{T_i\},T_p,K,1), \ \text{else.}\end{cases}$$

For put options, the formulas are as follows:

$$\mathbf{P}_{DI}(t,\{T_i\},T_p,K,H) = \begin{cases} V_{ASIAN}(t,\{T_i\},T_p,H,-1)+ \\ +V_{ASIAN-DIGITAL}(t,\{T_i\},T_p,H,K-H,-1), \text{ if } K<H, \\ V_{ASIAN}(t,\{T_i\},T_p,K,-1), \text{ else,} \end{cases}$$

$$\mathbf{P}_{UO}(t,\{T_i\},T_p,K,H) = \begin{cases} V_{ASIAN}(t,\{T_i\},T_p,H,-1)+ \\ +V_{ASIAN-DIGITAL}(t,\{T_i\},T_p,H,K-H,-1), \text{ if } K<H, \\ V_{ASIAN}(t,\{T_i\},T_p,K,-1), \text{ else,} \end{cases}$$

$$\mathbf{P}_{UI}(t,\{T_i\},T_p,K,H) = \begin{cases} V_{ASIAN}(t,\{T_i\},T_p,K,-1) - V_{ASIAN}(t,\{T_i\},T_p,H,-1)- \\ -V_{ASIAN-DIGITAL}(t,\{T_i\},T_p,H,K-H,-1), \text{ if } K<H, \\ 0, \text{ else,} \end{cases}$$

$$\mathbf{P}_{DU}(t,\{T_i\},T_p,K,H) = \begin{cases} V_{ASIAN}(t,\{T_i\},T_p,K,-1) - V_{ASIAN}(t,\{T_i\},T_p,H,-1)- \\ -V_{ASIAN-DIGITAL}(t,\{T_i\},T_p,H,K-H,-1), \text{ if } K<H, \\ 0, \text{ else.} \end{cases}$$

Here $V_{ASIAN}(t,T_e,T,K,\nu)$, $V_{ASIAN-DIGITAL}(t,T_e,T,K,\nu)$ are prices of the Asian and digital Asian options that can be calculated using formulas (150) and (159) correspondingly.

#### 5.3.8 Cross currency derivatives

In this section we assume that commodity prices are quoted in a foreign currency $f$, payment is fixed in another currency $p$ and both differ from the domestic currency $d$. Here we introduce the correspoding short rate and FX processes:

$$dr_d(t) = (\theta_d(t) - k_d r_d(t))dt + \sigma_d(t)dW^d_{r_d}(t), \tag{164}$$

$$dr_f(t) = (\theta_f(t) - k_f r_f(t))dt + \sigma_f(t)\left(dW^d_{r_f}(t) - \rho_{r_f,S_f}\sigma_f(t)dt\right), \tag{165}$$

$$dr_p(t) = (\theta_p(t) - k_p r_p(t))dt + \sigma_p(t)\left(dW^d_{r_p}(t) - \rho_{r_p,S_p}\sigma_p(t)dt\right), \tag{166}$$

$$\frac{dS_f(t)}{S_f(t)} = (r_d(t) - r_f(t))dt + \sigma_{S_f}(t)dW^d_{S_f}(t), \tag{167}$$

$$\frac{dS_p(t)}{S_p(t)} = (r_d(t) - r_p(t))dt + \sigma_{S_p}(t)dW^d_{S_p}(t). \tag{168}$$

where $r_i(t)$ is a short rate process associated with currency $i$ in domestic risk-neutral measure, $S_i(t)$ is an FX process converting currency $i$ into the domestic one (in domestic risk-neutral measure), $\theta_i(t)$, $k_i$, $\sigma_i(t)$ are the model parameters, $\rho_{XY}$ is the instantaneous correlation between $W_X(t)$ and $W_Y(t)$.

#### 5.3.9 Averaging swap

Averaging swap with fixed rate $K$ (in currency $f$), with payments dates $T^i_p$ (in currency $p$), volumes $V_i$, averaging dates $T_{i1} < T_{i2} < ... < T_{in_i} \le T^i_p$, at time $t$ has the value of

$$V_{SWAP}(t,\{T_{in_i}\},\{T^i_p\},K,\{V_i\}) = \sum_{i=1}^{n}(K - F_i(t))V_i P_p(t,T^i_p), \tag{169}$$

where $P_p(t,T)$ is the bond price corresponding to the currency $p$,

$$F_i(t) = \alpha_{i2}X_{i1} + \alpha_{i1}X_{i2} + X_{i1}X_{i2}, \tag{170}$$

$$X_{i1} = \frac{1}{n_i} \sum_{j:T_{ij}>t} F(t, T_{ij}), \tag{171}$$

$$\alpha_{i1} = \frac{1}{n_i} \sum_{j:T_{ij}\le t} S(T_{ij}), \tag{172}$$

$$X_{i2} = \frac{1}{n_i} \sum_{i:T_{ij}>t} F_{fp}(t, T_{ij}), \tag{173}$$

$F_{fp}(t, T_{ij})$ is an FX forward between currencies $f$ and $p$ (method *SbPslRepricingCalc.FxFwdSlice*),

$$\alpha_{i2} = \frac{1}{n_i} \sum_{i:T_{ij}\le t} \frac{S_f(T_{ij})}{S_p(T_{ij})}. \tag{174}$$

#### 5.3.10 At-Average composite option

At-Average composite with strike $K$ (in currency $f$), averaging times $T_1 < T_2 < ... < T_n$, payment time $T_p > T_n$ at time $t$ has the value of

$$V_{AT-AVERAGE}(t, \{T_i\}, T_p, K, \nu) = \nu P_p(t, T_p) \left(F(t)\Phi(\nu d_1) - K_{eff}\Phi(\nu d_2)\right) \tag{175}$$

where $\Phi(.)$ is the c.d.f. of the standard normal distribution $N(0, 1)$, $\nu = 1$ for a call option, $\nu = -1$ for a put option, $P_p(t, T)$ is the bond corresponding currency $p$,

$$d_1 = \frac{\ln \frac{F(t)}{K_{eff}} + \frac{1}{2}\Sigma(t)}{\sqrt{\Sigma(t)}}, \tag{176}$$

$$d_2 = d_1 - \sqrt{\Sigma(t)}, \tag{177}$$

$$F(t) = \alpha_2 X_1 + \alpha_1 X_2 + X_1 X_2, \tag{178}$$

$$X_1 = \frac{1}{n} \sum_{i:T_i>t} F(t, T_i), \tag{179}$$

$$\alpha_1 = \frac{1}{n} \sum_{i:T_i\le t} S(T_i), \tag{180}$$

$$X_2 = \frac{1}{n} \sum_{i:T_i>t} F_{fp}(t, T_i), \tag{181}$$

$F_{fp}(t, T_i)$ is the FX forward between currencies $f$ and $p$ (method *SbPslRepricingCalc.FxFwdSlice*),

$$\alpha_2 = \frac{1}{n} \sum_{i:T_i\le t} \frac{S_f(T_i)}{S_p(T_i)}, \tag{182}$$

$$K_{eff} = K - \alpha_1\alpha_2, \tag{183}$$

$$\Sigma(t) = \ln \frac{F_2(t)}{F^2(t)}, \tag{184}$$

$$\begin{aligned} F_2(t) = {} & \alpha_2^2 X_1^2 e^{\Sigma_1(t)} + \alpha_1^2 X_2^2 e^{\Sigma_2(t)} + X_1^2 X_2^2 e^{\Sigma_3(t)} + 2\alpha_1\alpha_2 X_1 X_2 + \\ & + 2\alpha_1 X_1 X_2^2 e^{\rho_{23}\sqrt{\Sigma_2(t)\Sigma_3(t)}} + 2\alpha_1 X_1^2 X_2 e^{\rho_{13}\sqrt{\Sigma_1(t)\Sigma_3(t)}}, \end{aligned} \tag{185}$$

$$\Sigma_3(t) = \Sigma_1(t) + \Sigma_2(t) - 2\rho_{12}\sqrt{\Sigma_1(t)\Sigma_2(t)}, \tag{186}$$

$$\rho_{12} = \frac{\sigma_{S_f}\rho_{S_f,F} - \sigma_{S_p}\rho_{S_p,F}}{\sqrt{\sigma_{S_f}^2 + \sigma_{S_p}^2 - 2\rho_{S_f,S_p}\sigma_{S_f}\sigma_{S_p}}}, \tag{187}$$

$$\rho_{23} = \frac{\sqrt{\Sigma_2(t)} + \rho_{12}\sqrt{\Sigma_1(t)}}{\Sigma_3(t)}, \tag{188}$$

$$\rho_{13} = \frac{\sqrt{\Sigma_1(t)} + \rho_{12}\sqrt{\Sigma_2(t)}}{\Sigma_3(t)}, \tag{189}$$

where $\Sigma_1(t)$ is calculated according to (154), $\Sigma_2(t)$ is calculated according to the analogous expression for the FX volatility.

## 5.4 Credit Derivatives

The system contains the following typologies:

- Single-name credit default swap,
- N-th-to-default credit default swap.

### 5.4.1 Credit Default Swap

Price of a Credit Default Swap (CDS) at time $t$ with fixed payments $C$, notional $N$, payment dates $\{T_i\}$ and recovery rate $R$ is given by:

$$CDS(t,N,C,R,\{T_i\}) = \sum_{i=1}^{n} P(t,T_i)ESP(t,T_i)C -$$

$$-\sum_{i=1}^{n} P(t,T_i)N\nu\left(ESP(t,T_{i-1}) - ESP(t,T_i)\right), \tag{190}$$

where $P(t,T)$ is the bond discount factor, $\nu = R$ for a funded CDS and $\nu = -(1-R)$ for an unfunded one.

If $t > \tau$ ($\tau$ is the default time), $CDS(t,N,C,R,\{T_i\}) = 0$.

### 5.4.2 N-th-to-Default Credit Default Swap

Price of an $N$-th-to-default swap on a set of counterparts $\{1,2,\dots M\}$ at time $t$ with fixed payments $C$, notional $H$, payment dates $\{T_i\}$ and recovery rate $R$ is given by:

$$NTD(t,N,H,C,R,\{T_i\}) = PL(t,N,H,C,R,\{T_i\}) - DL(t,N,H,C,R,\{T_i\}), \tag{191}$$

where

$$PL(t,N,H,C,R,\{T_i\}) = \sum_{i=1}^{n} P(t,T_i)\left(1 - s_N(t,T_i)\right)C, \tag{192}$$

$$DL(t,N,H,C,R,\{T_i\}) = \sum_{i=1}^{n} P(t,T_i)H\nu\left(s_N(t,T_i) - s_N(t,T_{i-1})\right) \tag{193}$$

if default payment appears on one of the payment dates $\{T_i\}$ or

$$DL(t,N,H,C,R,\{T_i\}) = -H\nu \int_t^{T_n} P(t,u)\frac{\partial s_N(t,u)}{\partial u}du \tag{194}$$

if default payment appears at the moment of the $N$-th default. The integral is approximated by the sum on a discrete grid $t = u_0 < u_1 < ... < u_{n_u} = T_n$:

$$DL(t,N,H,C,R,\{T_i\}) \approx \sum_{i=1}^{n_u} P(t,u_i)H\nu\left(s_N(t,u_i) - s_N(t,u_{i-1})\right). \tag{195}$$

Here $P(t,T)$ is the bond discount factor, $\nu = R$ for a funded swap and $\nu = -(1-R)$ for an unfunded one. Overall recovery rate, $R$, is defined as:

$$R = \frac{\sum_{i=1}^{M} R_i(1 - ESP_i(t,T))}{\sum_{i=1}^{M}(1 - ESP_i(t,T))}, \tag{196}$$

$R_i$ is recovery rate that corresponds to counterparty $i$.

Value $s_N(t,T)$ is the probability of at least $N$ defaults occurring by the time $T$ conditional on sigma-algebra at time $t$:

$$s_N(t,T) = P\left(\tau_{(N)}^{M} \le T|\mathcal{F}_t\right), \tag{197}$$

$\tau_i$ is the time of counterpart $i$ default, $\tau^M_{(N)}$ is the $N$-th order statistics of the set default times $\{\tau_1, \tau_2, \dots \tau_M\}$. To describe correlations of default times, we use one-factor copula model that assumes that:

$$\tau_i = F_i^{-1}\left(\Phi\left(\rho_i X + \sqrt{1-\rho_i^2} Z_i\right)\right) \tag{198}$$

where $\Phi(.)$ is standard normal c.d.f., $F_i(T) = 1 - ESP_i(t,T)$ is the c.d.f. of the counterpart $i$ default time, $ESP_i(t,T)$ is the corresponding expected survival probability $X$ is standard normally distributed common risk-factor, $Z_i$ are independent standard normally distributed individual risk-factors, $\rho_i$ is correlation of $\Phi^{-1}(F_i(\tau_i))$ with the common risk-factor $X$. Correlations $\rho_i$ can be inferred from default time correlations $\rho_{ij}$ (defined in *Cpties Correlation*) as (198) implies $\rho_{ij} = \rho_i \rho_j$. Hence, $\rho_i$ can be found through quadratic error minimization:

$$\{\rho_i\} = \arg\min_{\{\rho_i\}} \sum_{j<k} (\rho_j \rho_k - \rho_{jk})^2. \tag{199}$$

In particular, equation (198) implies:

$$P(\tau_i \le T) = P\left(\rho_i X + \sqrt{1-\rho_i^2} Z_i \le \Phi^{-1}\left(1 - ESP_i(t,T)\right)\right). \tag{200}$$

Conditional on $X = x$, default times $\tau_i$ are independent and the corresponding conditional probability $s_N(t,T|x) = P\left(\tau^M_{(N)} \le T | \mathcal{F}_t, X = x\right)$ can be calculated as follows:

$$s_N(t,T|x) = P\left(\tau^M_{(N)} \le T | \mathcal{F}_t, X = x\right) = P\left(Y_M(T) \ge N | \mathcal{F}_t, X = x\right) = 1 - \sum_{i=1}^{N-1} P\left(Y_M(T) = i | \mathcal{F}_t, X = x\right). \tag{201}$$

where $Y_k(T)$ is the process counting number of defaults by the time $T$ among the first $k$ counterparts $\{1, 2, \dots k\}$, $1 \le k \le M$. Probability $P\left(Y_M(T) = i | \mathcal{F}_t, X = x\right)$ can be decomposed as follows

$$\begin{aligned} P\left(Y_M(T) = i | \mathcal{F}_t, X = x\right) &= P\left(Y_{M-1}(T) = i | \mathcal{F}_t, X = x\right) P\left(\tau_M > T | \mathcal{F}_t, X = x\right) + \\ &+ P\left(Y_{M-1}(T) = i-1 | \mathcal{F}_t, X = x\right) P\left(\tau_M \le T | \mathcal{F}_t, X = x\right) \end{aligned}$$

Hence, if we denote $A(t,T;M,i,x) = P\left(Y_M(T) = i | \mathcal{F}_t, X = x\right)$ and $B(t,T;M,x) = P\left(\tau_M > T | \mathcal{F}_t, X = x\right)$, $A(t,T;M,i,x)$ satisfies the recurrence relation:

$$A(t,T;M,i,x) = A(t,T;M-1,i,x) B(t,T;M,x) + A(t,T;M-1,i-1,x)\left(1 - B(t,T;M,x)\right) \tag{202}$$

with the following initial conditions:

$$A(t,T;i,i,x) = P\left(\tau_1 \le T, \tau_2 \le T, \dots \tau_i \le T | \mathcal{F}_t, X = x\right) = \prod_{j=1}^{i} \left(1 - B(t,T;j,x)\right) \tag{203}$$

$$A(t,T;k,0,x) = P\left(\tau_1 > T, \tau_2 > T, \dots \tau_k > T | \mathcal{F}_t, X = x\right) = \prod_{j=1}^{k} B(t,T;j,x), \tag{204}$$

while values $B(t,T;j,x)$ are given by:

$$B(t,T;k,x) = P\left(\tau_k > T | \mathcal{F}_t, X = x\right) = 1 - P\left(\tau_k \le T | \mathcal{F}_t, X = x\right) = 1 - \Phi\left(\frac{\Phi^{-1}\left(1 - ESP_k(t,T)\right) - \rho_k x}{\sqrt{1-\rho_k^2}}\right). \tag{205}$$

Finally, unconditional probability $s_N(t,T)$ is given by:

$$s_N(t,T) = \int_{-\infty}^{+\infty} s_N(t,T|x) \phi(x) dx \tag{206}$$

where $\phi(x)$ is the p.d.f. of the latent factor $X$ (standard normal p.d.f.).

## 5.5 Equity Derivatives

The system contains the following typologies:

- Equity forward,
- Total return swap on equity,
- European equity option,
- European digital equity option,
- European equity option with European barrier.

In this section, we describe pricing methodologies for each of the above instruments. In all formulas for derivative instruments, it is assumed that **interest rate volatility is negligibly small**.

### 5.5.1 Equity Forward

Price of a forward contract with strike $K$ on equity (quoted in the same currency as $K$), maturity $T$ at time $t$ is given by:

$$V_{FWD}(t,T,K) = P(t,T)\left(F(t,T) - K\right), \tag{207}$$

where $P(t,T)$ is a discount bond.

### 5.5.2 Total Return Swap on Equity

The price of the Total Return Swap (TRS) on Equity, with accrual periods $\{T_i^{START}, T_i^{END}\}$, interest rate cash flow legs $\{L_j\}$ payment date $T_p$, is given by:

$$TRS(t, \{T_i^{START}, T_i^{END}\}, \{L_j\}, T_p) = \sum_{i=1}^{n} \left[V_{FWD}(t, T_i^{END}, 0) - V_{FWD}(t, T_{i-1}^{END}, 0)\right] +$$

$$+\sum_{i=1}^{n} \left[V_{DIV}\left(t, T_i^{START}, T_i^{END}\right)\right] + V_{IRS}(t)$$

where $V_{FWD}$ is the equity forward (defined in (207)), $V_{DIV}$ is the dividend component:

$$V_{DIV}\left(t, T^{START}, T^{END}\right) = P(t,T_p) \sum_{T^{START} \leq \tau_k \leq T^{END}} \left(\alpha_k + \beta_k F(t,\tau_k)\right), \tag{208}$$

$\tau_k$ are dividend payment dates, $\alpha_k$ are fixed dividends, $\beta_k$ are proportional dividends, $V_{IRS}(t)$ is the interest rate swap component:

$$V_{IRS}(t) = \sum_j FX_j(t) L_j(t), \tag{209}$$

where $L_j(t)$ is the present value of the interest rate cash flow leg $j$ (defined in (98)), $FX_j(t)$ is the exchange rate of the corresponding currency.

### 5.5.3 European Equity Option

Price of a European option with expiry $T$, strike $K$ on equity (quoted in the same currency as $K$) at time $t$ is given by:

$$V_{EU}(t,T,K,\nu) = \nu P(t,T)\left(F(t,T)\Phi(\nu d_1) - K\Phi(\nu d_2)\right) \tag{210}$$

where $\Phi(.)$ is c.d.f. of the standard normal distribution $N(0,1)$, $\nu = 1$ for a call option, $\nu = -1$ for a put option,

$$d_1 = \frac{\ln \frac{F(t,T)}{K} + \frac{1}{2}\Sigma(t,T)}{\sqrt{\Sigma(t,T)}}, \tag{211}$$

$$d_2 = d_1 - \sqrt{\Sigma(t,T)}, \tag{212}$$

$$\Sigma(t,T) = \int_t^T \sigma_X^2(u)du. \tag{213}$$

Integral of a piecewise function in the expression for $\Sigma(t,T)$ is calculated using *PiecewiseConstantIntegration.CalculateFlatBackward*.

### 5.5.4 European Digital Equity Option

Price of a European option with expiry $T$, strike $K$ on equity (quoted in the same currency as $K$) at time $t$ is given by:

$$V_{EU-DIGITAL}(t,T,K,\nu) = \nu P(t,T)\Phi(\nu d_2) \tag{214}$$

where $\Phi(.)$ is c.d.f. of the standard normal distribution $N(0,1)$, $\nu = 1$ for a call option, $\nu = -1$ for a put option,

$$d_2 = \frac{\ln\frac{F(t,T)}{K} - \frac{1}{2}\Sigma(t,T)}{\sqrt{\Sigma(t,T)}}, \tag{215}$$

$$\Sigma(t,T) = \int_t^T \sigma_X^2(u)du. \tag{216}$$

Integral of a piecewise function in the expression for $\Sigma(t,T)$ is calculated using *PiecewiseConstantIntegration.CalculateFlatBackward*.

### 5.5.5 European Equity Option with European Barrier

Price for an European barrier call options of each of the following types: down-and-in (DI), up-and-out (UO), up-and-in (UI), down-and-out (DO), with strike $K$, barrier $H$, payment date $T_p$ at time $t$ is given by:

$$\mathbf{C}_{DI}(t,T_p,K,H) = \begin{cases} V_{EU}(t,T_p,K,1) - V_{EU}(t,T_p,H,1) - \\ -V_{EU-DIGITAL}(t,T_p,H,H-K,1), \text{ if } K < H, \\ 0, \text{ else}, \end{cases}$$

$$\mathbf{C}_{UO}(t,T_p,K,H) = \begin{cases} V_{EU}(t,T_p,K,1) - V_{EU}(t,T_p,H,1) - \\ -V_{EU-DIGITAL}(t,T_p,H,H-K,1), \text{ if } K < H, \\ 0, \text{ else}, \end{cases}$$

$$\mathbf{C}_{UI}(t,T_p,K,H) = \begin{cases} V_{EU}(t,T_p,H,1) + \\ +V_{EU-DIGITAL}(t,T_p,H,H-K,1), \text{ if } K < H, \\ V_{EU}(tT_p,K,1), \text{ else}, \end{cases}$$

$$\mathbf{C}_{DU}(t,T_p,K,H) = \begin{cases} V_{EU}(t,T_p,H,1) + \\ +V_{EU-DIGITAL}(tT_p,H,H-K,1), \text{ if } K < H, \\ V_{EU}(tT_p,K,1), \text{ else}. \end{cases}$$

For corresponding put options, the expressions are as follows:

$$\mathbf{P}_{DI}(t,T_p,K,H) = \begin{cases} V_{EU}(t,T_p,H,-1) + \\ +V_{EU-DIGITAL}(t,T_p,H,K-H,-1), \text{ if } K < H, \\ V_{EU}(t,T_p,K,-1), \text{ else}, \end{cases}$$

$$\mathbf{P}_{UO}(t,T_p,K,H)=\begin{cases} V_{EU}(t,T_p,H,-1)+ \\ +V_{EU-DIGITAL}(t,T_p,H,K-H,-1), \text{ if } K<H, \\ V_{EU}(t,T_p,K,-1), \text{ else,} \end{cases}$$

$$\mathbf{P}_{UI}(t,T_p,K,H)=\begin{cases} V_{EU}(t,T_p,K,-1)-V_{EU}(t,T_p,H,-1)- \\ -V_{EU-DIGITAL}(t,T_p,H,K-H,-1), \text{ if } K<H, \\ 0, \text{ else,} \end{cases}$$

$$\mathbf{P}_{DU}(t,T_p,K,H)=\begin{cases} V_{EU}(t,T_p,K,-1)-V_{EU}(t,T_p,H,-1)- \\ -V_{EU-DIGITAL}(t,T_p,H,K-H,-1), \text{ if } K<H, \\ 0, \text{ else.} \end{cases}$$

Here $V_{EU}(t,T,K,\nu)$, $V_{EU-DIGITAL}(t,T,K,\nu)$ are the European and Digital European options defined by (210) and (214) respectively.

## 5.6 American Monte-Carlo

TBD...

# 6 Calculation of Risk Metrics

After the real-world future distribution of the derivative portfolio has evaluated, PFE ENGINE starts computing various risk metrics (RM) of it.

**Netting Sets and Portfolio Aggregation**

Any RM can be calculated for one deal or for a netting set (NS). Lets introduce a portfolio of assets (derivatives, structural products etc.), its value then: $V(t) = \sum_{i=1}^{n} V_i(t)$, where $V_i(t)$ is a value of $i$'th asset/derivative. Suppose we are willing to calculate some RM of this portfolio. In case of independent calculation we have

$$RM(V(t)) = \sum_{i=1}^{n} RM\left(V_i(t)\right),$$

but if all these assets constitute one common NS $V^{NS}(t)$ then

$$RM(V^{NS}(t)) = RM\left(\sum_{i=1}^{n} V_i(t)\right).$$

Note that $RM(V^{NS}(t)) \leq RM(V(t))$, what allows the counterparty has more deals $V_i$ with same risk limit.

Calculation of all RM are carried out at three aggregation levels:

- Deal,
- NS,
- Pre-settlement risk limit.

In general, deals related to one NS may be covered by limits on different types of financing. In this case, to calculate the RM, netting is applied at the intersection of the NS and the limit on the financing type. That is, the NS used for the RM calculation is always nested in the limit.

**Discounting and the Limit Currency**

Value of a NS (in case without netting every deal constitute its own NS) in above formula is presented in currency of the limit and discounted to day of calculation $t = 0$:

$$V_{ij}(t) = e^{-\int_0^t r_j(s)ds} FX_{ij}(t)\widehat{V}_{ij}(t),$$

where $\widehat{V}_{ij}(t)$ is value of the NS in nominal (time $t$) of currency the derivatives are priced, $FX_{ij}(t)$ is exchange rate between $\widehat{V}_i(t)$ price currency and the limit currency, $r_j(t)$ is risk-free short rate in the limit currency (RUB by default), $i \in 1..N$ denotes an index of the asset in the NS and $j \in 1..M$ denotes an index of the simulation.

In such notations value of the NS $V_j(t)$ on $j$-th simulation at time $t$ is

$$V_j(t) = \sum_{i=1}^{N} V_{ij}(t)$$

and sample distribution of value of the NS $V(t)$ at time $t$ is

$$V(t) = \{V_j(t)\}_{j=1}^{M}.$$

## 6.1 Potential Future Exposure

Define *the risk profile* $PFE_t^\alpha$ at future time $t$ on one deal or a NS with value $V(t)$ subject to counterparty default at time $\tau$ as

$$PFE_t^\alpha(V(t)) = (q^\alpha(V(t)|\tau = t))^+,$$

where $\alpha$ is the significance level.

Under the assumption of independence of $V(t)$ and probability of the counterparty default one can rewrite above formula for $PFE_t^\alpha$ as

$$PFE_t^\alpha(V(t)) = (q^\alpha(V(t)))^+.$$

This **independence assumption is accepted on default** as well as **confidence level** $\alpha = 95\%$**.**

The resulting risk metric $PFE^\alpha$ of the NS $V$ is maximum of the risk profile

$$PFE^\alpha(V) = \max_{0\le t\le T} PFE_t^\alpha V(t),$$

where $T$ is the maturity of longest derivative in $V$.

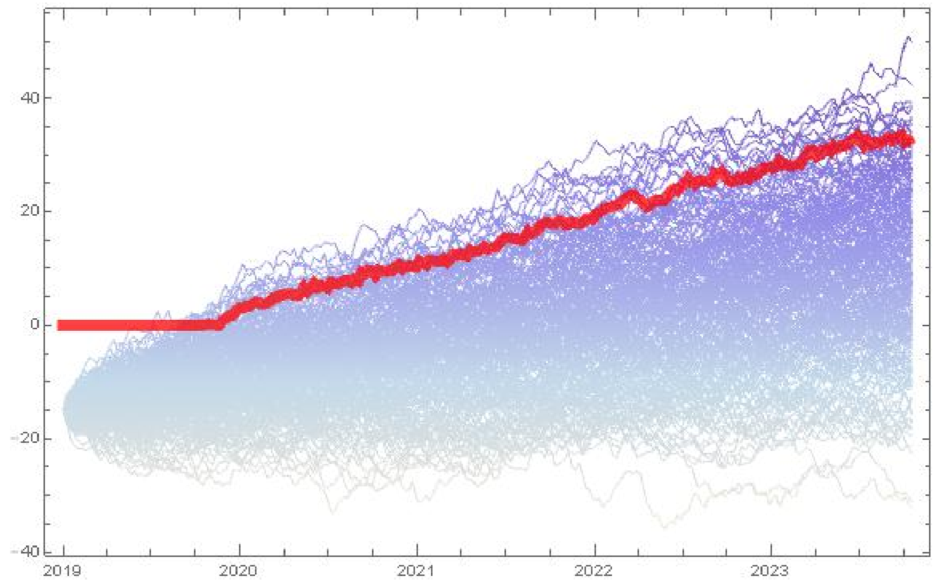


Figure 3: An example of $PFE_t^\alpha$ shape

**Credit Risk Coefficient and the Credit Add-on**

In the case when fast calculation is needed or full calculation is not possible *simplified calculation* is used. For that purpose we use next formula

$$PFE^\alpha(V) = MtM(V) + NK,$$

where $N$ is the notional of the deal $V$, $MtM(V)$ is the mark-to-market value of the deal at current date and $K$ is *the credit risk coefficient* (CRC).

CRC's are calculated by products, risk factors, maturity, direction (buy/sell), limit currency and other deal parameters: for each set of these parameters, the PFE is calculated for a synthetic deal having these parameters and a notional amount equal to the equivalent of one unit of limit currency. The default frequency of a recalculation is one working day.

*The credit add-on* refers to the added risk of a transaction or a NS and, in the case of unsecured transactions, is calculated as

$$\textit{Add-On}(V) = PFE^\alpha(V) - MtM(V).$$

This metric is used directly for utilization of the risk limits.

**Metrics under CSA**

For purposes of utilization of the PSL limit for deals made under CSA *the short add-on* $\textit{Add-On}^{(h)}$ is used. This metric reflects the maximum possible change in the fair value of a NS $V$ with a counterparty over period $h$ with given significance level $\alpha$. $\textit{Add-On}^{(h)}$ for the NS $V$ with the longest maturity $T$ is calculating as

$$\textit{Add-On}^{(h)}(V) = \max_{0\le t\le (T-h)^+} q^\alpha(V(min(t+h,T)) - V(t)),$$

where **time interval** $h$ **equals** $10D$ **by default.**

### 6.2 Expected Positive/Negative Exposure

Let us define a variable *Expected Exposure* (EE)

$$EE(V_j(t)) = (V_j(t))^+$$

and its contrary *Negative Expected Exposure* (NEE)

$$NEE(V_j(t)) = (V_j(t))^-.$$

These metrics represent the exposure expected at time $t$ on $j$-th simulation on the counterparty and on the bank, respectively.

Now let us define risk profiles for *Expected Positive Exposure* $EPE_t$

$$EPE_t(V(t)) = \mathbb{E}[EE(V(t))] \approx \frac{1}{M}\sum_{j=1}^{M}(V_j(t))^+$$

and its contrary *Expected Negative Exposure* $ENE_t$

$$ENE_t(V(t)) = \mathbb{E}[NEE(V(t))] \approx \frac{1}{M}\sum_{j=1}^{M}(V_j(t))^-$$

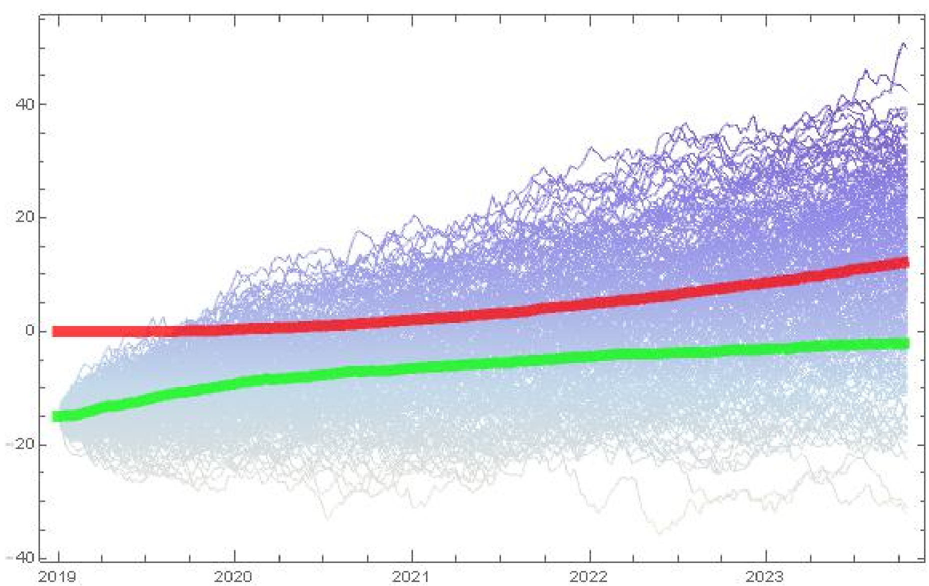


Figure 4: An example of $EPE_t$ (red) and $ENE_t$ (green) shape

Resulting risk metrics $EPE(V)$ and $ENE(V)$ of the NS $V$ are defined as follow

$$EPE(V) = \overline{EPE_t(V(t))} \approx \frac{1}{T}\sum_{k=1}^{L} EPE_t(V(t_k))(t_k - t_{k-1}),$$

$$ENE(V) = \overline{ENE_t(V(t))} \approx \frac{1}{T}\sum_{k=1}^{L} ENE_t(V(t_k))(t_k - t_{k-1}),$$

where $\overline{x(t)}$ denotes averaging by time and $\{t_k\}_{k=1}^{L}$ are simulation times with $t_0 = 0$ and $t_L = T$.

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