Game of Vols

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1 An Arbitrage-Free Alternative to an Implied Volatility Smile

Implied volatilities are used in practice as an alternative to quoting option prices. At a fixed maturity date and a fixed measure of moneyness such as strike price or delta, the absence of arbitrage imposes constraints on option prices. For example, a call price is bounded below by its intrinsic value and above by a discounted spot price. A put price is bounded below by its intrinsic value and above by a discounted strike price. Furthermore, a co-terminal call and put of the same moneyness are bound by put call parity [*]. The use of implied volatility is a convenient way to capture these no arbitrage constraints. So long as a market maker quotes an implied variance which is non-negative, the option price calculated by the Black Scholes formula respects the above no arbitrage bounds and put call parity. While the use of implied variance for a single strike-maturity pair is arbitrage-free the use of implied variance at two or more strike-maturity pairs need not be. For example, at some fixed strike, no arbitrage and a positive implied variance quote at a near maturity imposes a positive lower bound on the implied variance quote at the next maturity. If a market maker violates this lower bound by quoting a low but positive implied variance, a calendar spread arbitrage arises [†]. This problem is easily fixed by quoting positive implied forward variances instead.

However the use of forward implied variances does not eliminate cross strike arbitrages involving two or more co-terminal options. No arbitrage imposes both lower and upper bounds on the slope in strike of forward implied variances. No arbitrage further imposes a lower bound on the convexity in strike of forward implied variances. One can solve these problems by instead quoting a risk-neutral density at each maturity [‡] , but it is easier to instead quote a positive surface of local variance rates. When option prices are obtained by numerically solving a PDE, the resulting prices are free of all arbitrages.

The objective of this paper is to propose an alternative arbitrage-free quoting mechanism which automatically incorporates other constraints one may wish to impose. For example, the use of a positive local variance rate surface does not guarantee that the price of the underlying asset will remain strictly positive [§]. If one is valuing stock index or currency options, strict positivity of the underlying's price is often considered desirable. Second, one may wish to generate option pricing function which is linearly homogeneous in spot price and strike price. The use of a positive local variance rate surface does not guarantee that doubling both the spot price and strike price will also double the option premium [¶]. Third, one may wish to implicitly work with a risk-neutral PDF which is log concave in the log exchange rate. The use of a positive local variance rate surface does not guarantee log concavity of this PDF. In contrast, our alterna-

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tive arbitrage-free quoting mechanism leads to strictly positive price processes [‖] and to option pricing functions that are linearly homogeneous in spot price and strike price [**]. A sufficient condition for generating these properties is that the risk-neutral PDF be log concave in the log exchange rate [††] .

Another drawback of local variance rate surface is that it cannot be determined when the input data is a single complete arbitrage-free implied volatility slice, as opposed to a complete arbitrage-free implied volatility surface. In contrast, we develop a concept which is determined by a complete arbitrage-free implied volatility slice. For some underlyings, eg. for options on commodity futures, market quotes are available for only one maturity. A change in the option's maturity results in a change in the maturity of the underlying commodity futures contract. When implied volatility slices are quoted at two or more maturities, our approach treats each maturity separately and yet remains free of calendar arbitrage. In contrast, a local variance rate approach would require the supplied pair of implied volatility slices to be at maturities which are infinitesimally close [*]

An overview of this paper is as follows. The next section explains our setup. In particular, we focus on a pair of distribution functions, which capture the probability of an option finishing in-the-money under two equivalent martingale measures. The following section shows that our use of the pair of distribution functions is consistent with no calendar arbitrage. In the following section, we relate the two distribution functions to each other. We show that cross strike arbitrage is eliminated via the use of a function. We show how to generate a function merely by specifying a positive function of a real-valued variable. One can interpret this positive function as a substitute for the implied variance rate at the given maturity.

2 Probabilities of Finishing In-The-Money

Let T be a fixed future date. Let C_+ denote a contract whose payoff at T is a fixed number of units of one currency. The reason for the $+$ subscript will become clear shortly. Let C_- be a second contract whose payoff at T is a fixed number of units of any other currency. For both contracts, the number of units that will be delivered is assumed to be strictly positive. C_+ and C_- are not values per se, but merely notations for describing a contract maturing at T.

A forward contract is the obligation to exchange a fixed amount of one currency for a fixed amount of another currency at the expiration date T. The long side of the forward contract is required to exchange the C_- contract for the C_+ contract at T. If the contract C_+ is worth more than the contract C_- when both are valued using one pricing currency, then C_+ is also worth more than C_- when they are both valued using a different pricing currency. As a

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result, we can express the forward contract $\mathcal{F}$ in a numeraire-free manner as [†]:

$$\mathcal{F} = C_+ - C_-. \tag{2.1}$$

The reason for the subscripts $+$ and $-$ in C_+ and C_- is now clear: C_+ is the contract received by the long side at the expiry of the forward contract, while C_- is the contract delivered.

To value the forward contract, let $N_+ > 0$ and $N_- > 0$ be the present values of the C_+ and C_- contracts in some chosen numeraire:

$$\langle C_- \rangle = N_-, \quad \langle C_+ \rangle = N_+. \tag{2.2}$$

N_+ and N_- each be prices expressed in the same units. The common numeraire in which these prices are expressed can be either of the two currencies involved in the required exchange or it can be some third currency. We refer to the common currency used to value each leg as the pricing currency. When the $\langle \cdot \rangle$ operator acts on a contract C, it returns the present value of the contract in the pricing currency. For any pricing currency, the pricing operator is linear and hence:

$$\langle \mathcal{F} \rangle = N_+ - N_-. \tag{2.3}$$

Thus, the arbitrage-free value of the forward contract in our pricing currency is just the difference in the two positive values of each leg.

A European-style vanilla option is an optional forward contract giving its holder the right to exchange a fixed amount of one currency for a fixed amount of the other currency at the expiration date T. Like the corresponding forward contract $\mathcal{F}$, we can express the vanilla option contract $\mathcal{O}$ in a numeraire-free manner as:

$$\mathcal{O} = (C_+ - C_-)^+. \tag{2.4}$$

Like C_+, C_-, and $\mathcal{F}$, $\mathcal{O}$ is not a value, but merely a way of describing a particular contract called an option.

The option can be thought of more traditionally as a C_+ denominated put written on C_- with the relative strike $\frac{N_+}{N_-}$ and notional N_-. The option also has an alternative traditional interpretation as a C_- denominated call written on C_+ with the relative strike $\frac{N_-}{N_+}$ and notional N_+ [*].

Instead of describing the option as either a call or a put, we have chosen a symmetric formulation which allows both interpretations simultaneously .

In our symmetric formulation, the value of the option is a linearly homogeneous function $P : \mathbb{R}^+ \times \mathbb{R}^+ \mapsto \mathbb{R}^+$ of the two notionals N_+ and N_-:

$$\langle \mathcal{O} \rangle = P(N_+, N_-), \quad \forall \lambda > 0 : \; P(\lambda N_+, \lambda N_-) = \lambda P(N_+, N_-). \tag{2.5}$$

Differentiating w.r.t. λ and setting $\lambda = 1$:

$$\left(N_+ \partial_{N_+} + N_- \partial_{N_-} \right) P(N_+, N_-) = P(N_+, N_-). \tag{2.6}$$

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Each partial derivative is closely related to the probability of the event $C_+ > C_-$ under the appropriate martingale measure:

$$\partial_{N_+} P \equiv \Delta_+ = Pr_+(C_+ > C_-), \quad \partial_{N_-} P \equiv \Delta_- = -Pr_-(C_+ > C_-), \tag{2.7}$$

Thus, the option value can be represented as:

$$P = N_+ \Delta_+ - N_- \Delta_-. \tag{2.8}$$

This result can be easily interpreted as follows. Δ_+ is the value of a digital option paying one unit of C_+ if $C_+ > C_-$ at T measured in C_+ units. Similarly, Δ_- is the value of a digital option paying one unit of C_- if $C_+ > C_-$ at T measured in C_- units. A static portfolio which holds the first digital option and is short the second digital option will match the payoff of the vanilla option if held to expiry. Hence, (2.8) follows from no arbitrage.

Since each probability is a partial derivative of a first-order homogenous function, both of the probabilities are a order homogeneous function, i.e., each probability depends only on the ratio $R = \frac{N_+}{N_-}$ of the two notionals:

$$\Delta_\pm(N_+, N_-) = \Delta_\pm\left(\frac{N_+}{N_-}, 1\right) = \Delta_\pm(R, 1) \equiv \Delta_\pm(R). \tag{2.9}$$

Since N_+ and N_- are both prices expressed in in the same pricing currency, the ratio R is dimensionless. In differential form, this condition is equivalent to:

$$N_+ \tfrac{\partial \Delta_\pm}{\partial N_+} = -N_- \tfrac{\partial \Delta_\pm}{\partial N_-} = \tfrac{\partial \Delta_\pm}{\partial x}, \tag{2.10}$$

where:

$$x = \ln(R) \tag{2.11}$$

denotes the natural logarithm of the ratio. As the cross partial of the option price does not depend on the order in which we differentiate, we have that:

$$\partial_{N_+} \Delta_- + \partial_{N_-} \Delta_+ = 0. \tag{2.12}$$

Multiplying (2.12) by N_+ and combining with (2.10), we also have that:

$$\tfrac{\partial \Delta_-}{\partial x} = R \tfrac{\partial \Delta_+}{\partial x}, \tag{2.13}$$

and, therefore:

$$R = \tfrac{\partial \Delta_-}{\partial \Delta_+}. \tag{2.14}$$

Once again, this relation has a simple probabilistic interpretation. As we have mentioned $\Delta_\pm$ are probabilities that the option finishes in-the-money in the appropriate numeraires. This happens if and only the appreciation rate of C_- relative to C_+ does not exceed R. Thus $\Delta_\pm$ can be thought of as cumulative distribution functions of relative change of the exchange rate, or logarithmic return if expressed in terms of x *. The partial derivatives of $\Delta_\pm$ w.r.t. x are probability density functions:

$$\rho_\pm = \frac{\partial \Delta_\pm}{\partial x}.$$

Equation (2.14) says that the ratio of the two notionals has to match the ratio of the two PDF's:

$$R = \frac{\rho_-}{\rho_+}. \tag{2.15}$$

Note that the cumulative distribution functions in both numeraires should increase with R, which, in particular, requires Δ_- to be a convex function of Δ_+. As a consequence:

$$\Delta_- \leq \Delta_+. \tag{2.16}$$

As an example, suppose a call option gives its owner the right to exchange K dollars for one pound. In this example C_+ is a contract paying one pound at T and $N_+ = 1$ pound, or equivalently S dollars, where S is the spot FX rate in dollars per pound . C_- is a contract paying K dollars at T and $= K$ dollars. The ratio $R = \frac{N_+}{N_-} = \frac{S}{K}$ is dimensionless and $x = \ln(S/K)$. Since the currency received upon exercise is pounds, Δ_+ is the probability that the call finishes in-the-money under pound measure. Since the currency delivered upon exercise is dollars, Δ_- is the probability that the call finishes in-the-money under dollar measure. Consider the formula for these two deltas in the Black Scholes model. $\Delta_+ = N(d_1)$ is the probability that the call finishes in-the-money under pound measure. $\Delta_- = N(d_2)$ is the probability that the call finishes in-the-money under dollar measure *. Both of these probabilities are increasing in $R = \frac{S}{K}$ and hence in $x = ln(S/K)$. Their derivatives w.r.t. x are $\rho_+ = N'(d_1)$ and $\rho_- = N'(d_2)$. As is well known:

$$\frac{S}{K} = \frac{N'(d_2)}{N'(d_1)}, \tag{2.17}$$

which is consistent with (2.15). Also, $N(d_2) \leq N(d_1)$ which is consistent with (2.16).

As a second example, suppose a put option gives its owner the right to exchange one pound for K dollars. In this example C_+ is a contract paying K dollars at T and $N_+ = K$ dollars, or equivalently $\frac{K}{S}$ pounds, where S is still the spot FX rate in dollars per pound . C_- is a contract paying one pound at T and $N_- = 1$ pound. The ratio $R = \frac{N_+}{N_-} = \frac{K}{S}$ and $x = \ln(K/S)$. Since the currency received upon exercise is dollars, Δ_+ is the probability that the put finishes in-the-money under dollar measure. Since the currency delivered upon exercise is pounds, Δ_- is the probability that the put finishes in-the-money under pound measure. Consider the formula for these two deltas in the Black Scholes model. $\Delta_+ = N(-d_2)$ is the probability that the put finishes in-the-money under dollar measure. $\Delta_- = N(-d_1)$ is the probability that the put finishes in-the-money under pound measure. Both of these probabilities are increasing in $R = \frac{K}{S}$

* If there is no ambiguity we shall use the same notation R, x in reference to relative return.

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and hence in $x = ln(K/S)$. Their derivatives w.r.t. x are $\rho_+ = N'(d_2)$ and $\rho_- = N'(d_1)$. As is well known:

$$\frac{K}{S} = \frac{N'(d_1)}{N'(d_2)}, \tag{2.18}$$

which is consistent with both (2.15) and (2.17). Also, $N(-d_1) \leq N(-d_2)$ which is consistent with (2.16).

3 Consistency with No Calendar Arbitrage

As the function $\Delta_+ \to \Delta_-$ is convex, for any positive pair of $N_\pm > 0$, the linear combination $N_+\Delta_+ - N_-\Delta_-$ is a concave function of Δ_+, and, therefore, has a unique maximum.

This maximum is achieved at the ratio (2.14) and therefore is equal to the price of the option with notionals $N_\pm$:

$$P(N_+, N_-) = \sup_{\Delta_\pm} \left(N_+\Delta_+ - N_-\Delta_-\right). \tag{3.1}$$

When we find the supremum on the RHS subject to the constraint that Δ_- is a particular convex function of Δ_+, we obtain the function P linking the arbitrage-free option price to the two notionals N_+ and N_- describing the option's contractual structure.

Let us now consider developing an alternative to an implied volatility surface. At each expiration date T, there is a functional relationship between Δ_- and Δ_+:

$$\Delta_- = \Delta_- \left(\Delta_+, T\right). \tag{3.2}$$

Let us demonstrate, that for any fixed value of Δ_+, the value of Δ_- monotonically decreases with expiration T.

The proof will be by contradiction. Indeed, let us assume that for a pair of maturity dates T_1 and T_2 with $T_2 > T_1$, there exists a Δ_+ such that :

$$\Delta_-(\Delta_+, T_2) \geq \Delta_-(\Delta_+, T_1). \tag{3.3}$$

Let $N_\pm$ be a pair of notionals for the option with expiration T_2 and the chosen value of Δ_+. Applying here the no calendar arbitrage constraint in combination with (2.8) and (3.1), we have:

$$\begin{aligned} &P(N_+, N_-, T_2) = N_+\Delta_+ - N_-\Delta_-(\Delta_+, T_2) > P(N_+, N_-, T_1) \\ &= \sup_\delta N_+\delta - N_-\Delta_-(\delta, T_1) > N_+\Delta_+ - N_-\Delta_-(\Delta_+, T_1). \end{aligned}$$

and, therefore, $\Delta_-(\Delta_+, T_2) < \Delta_-(\Delta_+, T_2)$, which contradicts our original assumption. Thus we have proved that for any fixed value of Δ_+, the value of Δ_- monotonically decreases with expiration T. Analogously, it can be shown that for any fixed value of Δ_-, the value of Δ_+ would monotonically increase with expiration. As monotonic functions are invertible, the expiration T can be represented as a positive function $T(\Delta_+, \Delta_-) : [0,1] \times [0,1] \mapsto \mathbb{R}^+$:

$$T = T(\Delta_+, \Delta_-) \tag{3.4}$$

which is increasing with Δ_+ and is decreasing with Δ_-.

4 Constant Quanto Shift Parametrization

At each fixed T, increasing the ratio R from 0 to ∞ causes both of the probabilities Δ_- and Δ_+ to move from 0 to 1. In this section, we find a pair of restrictions on the probability measures C_+ and C_- which allows us to think of R as an increasing function of a real-valued moneyness measure z, which correspondingly moves from $-\infty$ to ∞. In particular, let us guess that at each fixed T, there exists a real-valued moneyness measure $z \in (-\infty, +\infty)$ such that:

$$\Delta_- = \Omega(z). \tag{4.1}$$

Here, Ω is an increasing function of z taking values between 0 and 1. It can be thought of as a cumulative distribution function giving the probability under measure that a random variable $Z_T \leq z$. Our first restriction on the two probability measures is that the effect of changing probability measures from to can be captured by a positive quantity τ independent of z such that:

$$\Delta_+ = \Omega(z + \tau). \tag{4.2}$$

The function of z described by $\Omega(z + \tau)$ is the distribution function of Z_T in the measure. *

The quanto correction τ does not depend on z, but must be an increasing function of T with $\tau = 0$ when $T = 0$. Since T is fixed in this section, our notation does not reflect this dependence. Note, that under this parametrization, the condition (2.16) is automatically satisfied for all positive $\tau > 0$. To complete our interpretation of the parametrization, we note that the ratio R of the two notionals can be expressed as a function of z and τ:

$$R(z, \tau) = \frac{\partial \Delta_-}{\partial \Delta_+} = \frac{\frac{\partial \Delta_-}{\partial z}}{\frac{\partial \Delta_+}{\partial z}} = \frac{\Omega'(z)}{\Omega'(z + \tau)} = \frac{\omega(z)}{\omega(z + \tau)}, \tag{4.3}$$

where $\omega(z) \equiv \Omega'(z) \geq 0$ is defined as the probability density function (PDF) of Z_T under measure. Since $\omega(z) \geq 0$, the RHS of (4.3) guarantees that R is a positive function of $z \in \mathbb{R}$ and $\tau > 0$. We now seek a second restriction on the probability measures C_+ and C_- which guarantees that the ratio R of the two notionals is an increasing function of z for each $\tau > 0$. Towards this end, note that the logarithm of the ratio of the two notionals can be calculated as:

$$\ln(R(z, \tau)) = \ln(\omega(z)) - \ln(\omega(z + \tau)). \tag{4.4}$$

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Let us represent the PDF of Z_T under measure as:

$$\omega(z) = e^{-h(z)}, \tag{4.5}$$

where $h(z)$ is twice differentiable Then the logarithm of the ratio of the two notionals can be calculated as:

$$\ln(R(z,\tau)) = h(z+\tau) - h(z) = \int_z^{z+\tau} h'(y)\,dy. \tag{4.6}$$

Differentiating (4.6) w.r.t. z implies:

$$= h'(z+\tau) - h'(z) = \int_z^{z+\tau} h''(y)\,dy. \tag{4.7}$$

Suppose that $h''(z) \geq 0$ for all $z \in \mathbb{R}$. Since $R(z,\tau) > 0$, we must also have > 0 * so that the ratio R is increasing in z at each $\tau > 0$.

Thus, if one picks any positive function $p(z)$ to describe the second derivative of $h(z)$, one can double integrate, negate, and exponentiate to get the PDF of Z_T under C_+ measure. When this PDF is used to get the two probabilities Δ_- and Δ_+, the ratio R, and the option price P, the resulting option pricing formula is arbitrage-free. Thus, this positive function $p(z)$ can be regarded as a substitute for the function relating eg. Black implied variance to log of the strike over the forward. Our positive function $p(z)$ produces arbitrage-free pricing formulas, while it is well known that an arbitrary direct positive specification of Black implied variance can produce cross-strike or cross-maturity arbitrage. One can of course also use $p(z)$ to produce an arbitrage-free implied volatility slice.

5 Dupire Local Variance Rate in Delta Form

The Dupire formula for a local variance rate σ^2 can be written as:

$$\sigma^2 = \frac{2\left(\frac{\partial P}{\partial t}\right)_x}{N_+^2 \frac{\partial^2 P}{(\partial N_+)^2}}. \tag{5.1}$$

Note that both the numerator and the denominator of the formula are first-order homogeneous functions of $N_\pm$, which makes the expression in the right part depend on the strike (the ratio) only. If we assume that $N_- = 1$, then the formula reduces to the traditional Dupire formula written in term of puts. By the symmetry argument between numeraire choice, the denominator of the formula can be also written as:

$$N_+^2 \frac{\partial^2 P}{\partial {N_+}^2} = N_-^2 \frac{\partial^2 P}{\partial {N_-}^2} = N_+ \frac{\partial \Delta_+}{\partial x} = N_- \frac{\partial \Delta_-}{\partial x}. \tag{5.2}$$

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Substituting this along with (2.8) into (5.1), we have:

$$\frac{\sigma^2}{2} = \frac{\left(\frac{\partial\Delta_+}{\partial t}\right)_x}{\left(\frac{\partial\Delta_+}{\partial x}\right)_t} - \frac{\left(\frac{\partial\Delta_-}{\partial t}\right)_x}{\left(\frac{\partial\Delta_-}{\partial x}\right)_t} = \left(\frac{\partial x}{\partial t}\right)_{\Delta_-} - \left(\frac{\partial x}{\partial t}\right)_{\Delta_+}. \tag{5.3}$$

Thus, we conclude that Dupire's local variance rate can be calculated as:

$$\sigma^2 = 2\left(\left(\frac{\partial x}{\partial t}\right)_{\Delta_-} - \left(\frac{\partial x}{\partial t}\right)_{\Delta_+}\right). \tag{5.4}$$

Let us apply this result to compute the local variance rate in the parametrization (3.4). According to (4.3), the ratio of the two notionals can be calculated as:

$$R = \left(\frac{\partial\Delta_-}{\partial\Delta_+}\right)_T = -\frac{\partial_+ T}{\partial_- T} \tag{5.5}$$

Here and afterwards, we use the notations:

$$\partial_\pm = \frac{\partial}{\partial\Delta_\pm}. \tag{5.6}$$

The formula (5.4) in this parametrization becomes:

$$\tfrac{1}{2}\sigma^2 = \left(\frac{\partial x}{\partial T}\right)_{\Delta_-} - \left(\frac{\partial x}{\partial T}\right)_{\Delta_+} = \frac{\partial_+ x}{\partial_+ T} - \frac{\partial_- x}{\partial_- T}. \tag{5.7}$$

Substituting (5.5) in (5.7) and simplifying the result, we have:

$$\tfrac{1}{2}\sigma^2 = \frac{\partial_+^2 T}{\left(\partial_+ T\right)^2} - 2\frac{\partial_+\partial_- T}{\partial_+ T\partial_- T} + \frac{\partial_-^2 T}{\left(\partial_- T\right)^2}. \tag{5.8}$$

Changing the time parametrization to $T = T(\tau)$, (5.8) becomes:

$$\sigma^2 = \frac{\sigma_\tau^2}{\frac{dT}{d\tau}}, \tag{5.9}$$

where:

$$\sigma_\tau^2 = 2\frac{\partial_+^2\tau}{2\left(\partial_+\tau\right)^2} - 4\frac{\partial_+\partial_-\tau}{\partial_+\tau\partial_-\tau} + 2\frac{\partial_-^2\tau}{2\left(\partial_-\tau\right)^2}. \tag{5.10}$$

In the spirit of our assumption (4.1), let us also change the parametrization for the $\Delta_\pm$ variables to:

$$\Delta_+ = \Omega_+(z_+), \quad \Delta_- = \Omega_-(z_-). \tag{5.11}$$

Then, the first term in (5.10) can be represented as:

$$\begin{aligned}\frac{\partial_+^2\tau}{\left(\partial_+\tau\right)^2} &= -\partial_+\left(\partial_+\tau\right)^{-1} = -\omega_+(z_+)^{-1}\partial_+\left(\frac{\omega_+(z_+)}{\frac{\partial\tau}{\partial z_+}}\right) \\ &= \left(\frac{\partial\tau}{\partial z_+}\right)^{-1} h_+'(z_+) + \left(\frac{\partial\tau}{\partial z_+}\right)^{-2}\frac{\partial^2\tau}{(\partial z_+)^2},\end{aligned} \tag{5.12}$$

where, as per the representation in (4.5):

$$\omega_+(z) = \Omega'_+(z) = e^{-h_+(z)}. \tag{5.13}$$

Analogously, the last term in (5.10) can be represented as:

$$\frac{\partial_-^2\tau}{(\partial_-\tau)^2} = \left(\frac{\partial\tau}{\partial z_-}\right)^{-1} h'_-(z_-) + \left(\frac{\partial\tau}{\partial z_-}\right)^{-2} \frac{\partial^2\tau}{(\partial z_-)^2}. \tag{5.14}$$

Also, it is easy to see than the middle term in (5.10) can be represented in terms of the new variables as:

$$\frac{\partial_+\partial_-\tau}{\partial_+\tau\partial_-\tau} = \left(\frac{\partial\tau}{\partial z_+}\frac{\partial\tau}{\partial z_-}\right)^{-1} \frac{\partial^2\tau}{\partial z_+\partial z_+}. \tag{5.15}$$

Substituting (5.12) to (5.15) in (5.10), we arrive at the following local variance formula:

$$\begin{aligned}\tfrac{1}{2}\sigma_\tau^2 &= \left(\frac{\partial\tau}{\partial z_+}\right)^{-1} h'_+(z_+) + \left(\frac{\partial\tau}{\partial z_-}\right)^{-1} h'_-(z_-)\\ &+ \left(\frac{\partial\tau}{\partial z_+}\right)^{-2} \frac{\partial^2\tau}{(\partial z_+)^2} - 2\left(\frac{\partial\tau}{\partial z_+}\frac{\partial\tau}{\partial z_-}\right)^{-1} \frac{\partial^2\tau}{\partial z_+\partial z_+} + \left(\frac{\partial\tau}{\partial z_-}\right)^{-2} \frac{\partial^2\tau}{(\partial z_-)^2}.\end{aligned} \tag{5.16}$$

Our quanto parametrization assumption can be formulated as:

$$\Omega_+(z_+) = \Omega_-(z_-) \equiv \Omega(z), \quad \tau = z_+ - z_-. \tag{5.17}$$

Equation (5.17) can be interpreted as follows: z_- and z_+ are two random variables deterministically related to each other via the formula $z_+ = z_- + \tau$, where τ is some constant. The distribution of z_+ under the C_+ probability measure follows the same law as the distribution of z_- under the C_- measure.

Plugging this ansatz into the local variance formula (5.16), we finally arrive at the following result:

$$\tfrac{1}{2}\sigma_\tau^2 = h'(z_+) - h'(z_-) = \int_{z_-}^{z_+} h''(z)dz. \tag{5.18}$$

As per (4.7), this result can be also interpreted as:

$$\sigma_\tau^2 = 2\left(\frac{\partial x}{\partial z_\pm}\right)_\tau. \tag{5.19}$$

Note, that due to (5.17), the partial derivatives with respect to the variables $z_\pm$ are the same.

6 Power Options and Characteristic Functions

A power option is an European claim with the payoff equal to an exchange rate $\frac{C_-}{C_+}$ of a pair of currencies or assets to the power λ delivered in the C_+ units. It can be thought of as a geometric average of two assets

$$\left(\frac{C_-}{C_+}\right)^\lambda C_+ = C_-^\lambda C_+^{1-\lambda}.$$

Note that the formula above specifies the contract in the numeraire independent fashion similarly to (2.4)*. We shall call a characteristic function a present value of the power options in units of the spot power claim†

$$\chi(\lambda, T) = \frac{\left\langle C_-^{\lambda} C_+^{1-\lambda} \right\rangle_T}{\left\langle C_-^{\lambda} C_+^{1-\lambda} \right\rangle_t}. \tag{6.1}$$

In the special cases $\lambda \in \{0, 1\}$ the power option is reduced to one of the underlying contracts and, therefore,

$$\chi(0, T) = \chi(1, T) = 1. \tag{6.2}$$

The power options should not necessarily be of finite present value for any λ. An infinite present value would indicate that the seller of the option will incur unlimited unhedgeable risk. Let us call a Laplace support the area of λ associated with finite contracts. To analyse its properties let us divide the claim into left and right tail contracts paying in case of positive and negative returns, respectively. The value of the left tail contract monotonically decreases with λ and, therefore, the area of λ for which it is finite must be of the form $[-\lambda_-, +\infty)$, $(-\lambda_-, +\infty)$, or, $(-\infty, +\infty)$. Due to (6.2) the contract is always finite for $\lambda = 0$ and, therefore, $\lambda_- \geq 0$. Analogously, it can be shown that the area of λ for which the right tail contract is finite can be represented as $(-\infty, 1 + \lambda_+]$, $(-\infty, 1 + \lambda_+)$, or, $(-\infty, +\infty)$, where $\lambda_+ \geq 0$. The power option is finite if and only if its both tail contracts are also finite. Therefore, the Laplace support must be a closed or semi closed interval between $-\lambda_-$, and $1 + \lambda_+$, with each $\lambda_\pm$ being either nonnegative real number or positive infinity. The bounds $\lambda_\pm$ can be related to asymptotical behavior of the implied distribution and, therefore, option prices:

$$P\left(N_+, N_-\right)_{N_+ \to 0} \sim N_+^{1+\lambda_-}, \quad P\left(N_+, N_-\right) - N_+ + N_-{}_{N_- \to 0} \sim N_-^{1+\lambda_+} \tag{6.3}$$

or, equivalently,

$$P\left(N_+, N_-\right)_{N_- \to \infty} \sim N_-^{-\lambda_-}, \quad P\left(N_+, N_-\right) - N_+ + N_-{}_{N_+ \to \infty} \sim N_+^{-\lambda_+}. \tag{6.4}$$

In terms of implied vol this behavior translates into asymptotically linear implies variance w.r.t. logarithmic strike

$$\lim_{x \to \pm\infty} \frac{\sigma^2_{\text{impl}}(x, T)}{|x|} = 2 \frac{\sqrt{1 + \frac{1}{\lambda_\pm}} - 1}{\sqrt{1 + \frac{1}{\lambda_\pm}} + 1}.$$

The maximum slope 2 is reached for $\lambda_\pm = 0$. The slower than linear grows of implied variance indicates an unbounded Laplace support in the corresponding tail.

*In general, any first order homogeneous function of n-argument defines an n asset derivative claim.

†Here we assume that $C_\pm$ are self-financing (e.i., carry/discount free. Otherwise, they should be replace by zero coupon bonds expiring together with the option).

6.1 Term structure and local vol

It easy to see that to satisfy the calendar consistency condition in the tails the Laplace support must not increase with expiration, i.e., $\lambda_\pm$ must be weakly decreasing w.r.t T. In addition, for a given calendar expiration T the process $\lambda_\pm(T,t)$ must not decrease with the current time t. Indeed, if the Laplace support shrunk over time, a long position in a power option disappearing from the market would acquire an infinite value, which should not be possible. First of all, the market would not be able to fulfil an infinite claim. Besides, with the price rising the option holder would eventually close his position selling the option. This ensures that the option once traded will stay on the market and, therefore, will maintain a finite present value. We like to call it the No Al Dorado principle. As a consequence of weak monotonicity, under any continues process the bounds $\lambda_\pm(T,t)$ must not be volatile.

Application of the Dupire formula (5.1) to the asymptotic behavior (6.4) in the put tail yields:

$$2\sigma^2 \approx \frac{\partial_T R^{\lambda_-+1}}{\partial_R^2 R^{\lambda_-+1}} = \frac{x\partial_T \lambda_-}{\lambda_-\left(\lambda_-+1\right)} = -x\partial_T \ln\left(1+\tfrac{1}{\lambda_-}\right).$$

Together with an analogous relation in the call tail it can be written as

$$\lim_{x->\pm\infty} \sigma^2 = \tfrac{1}{2}\left|x\right|\partial_T \varphi_\pm, \quad \text{where} \quad \varphi_\pm = \ln\left(1+\tfrac{1}{\lambda_\pm}\right). \tag{6.5}$$

Thus we established that the local variance will grow asymptotically linearly with large logarithmic strikes as long as the Laplace support bound associated with the tail remains strictly positive. The assumption of time homogeneous local vol implies piecewise linear interpolation of $\varphi_\pm$ over maturity T. In particular, this would lead to Laplace support exponentially shrinking towards $[0,1]$ for long expirations

$$\lambda_\pm = \left(e^{\varphi_\pm}-1\right)^{-1} \sim e^{-\varphi'_\pm T}. \tag{6.6}$$

On the contrary, the assumption of time-homogeneous, stationary (mean-reverting) stochastic vol/jump diffusion leads to asymptotically independent consecutive long term returns. This results in cumulants being asymptotically linear in T and, therefore, the Laplace support for large maturities stabilizing in the area wider than $[0,1]$ *, i.e.,

$$\lim_{T\to\infty} \lambda_\pm > 0 \Leftrightarrow \lim_{T\to\infty} \varphi_\pm < \infty \tag{6.7}$$

causing asymptotic variance slopes (6.5) to flatten for long expiration. This result is inline with empirically observation of Dupire local vol flattening for long expirations. Consequently, under reasonable vol extrapolation in expiration dimension, it is plausible to assume the slope potentials $\varphi_\pm$ growing slower than linearly.

*This can be demonstrated for known stochastic vol models, e.g., Heston

6.2 Characteristic function in delta representation

The characteristic function (6.1) can be computed as an expectation of R^λ in the C_+ measure

$$\chi(\lambda, T) = \mathrm{E}_+\left[R^\lambda\right] = \mathrm{E}_+\left[e^{\lambda x}\right].$$

Applying here (4.1), (4.5), (4.7) to compute the characteristic function in the delta form we shall have

$$\chi(\lambda, T) = \int_{-\infty}^{+\infty} e^{\lambda(h(z+\tau)-h(z))-h(z+\tau)}dz = \int_{-\infty}^{+\infty} e^{(\lambda-1)h(z+\tau)-\lambda h(z)}dz. \quad (6.8)$$

The integral will not converge unless the expression under the exponential will grow sufficiently negative in the tails $z \to \pm\infty$. In particular, testing the condition in the call tail for $\lambda > 1$ we shall have

$$(\lambda - 1)h(z+\tau) - \lambda h(z) < 0 \Leftrightarrow h(z+\tau) < \frac{\lambda}{\lambda-1}h(z).$$

Therefore, the integral will diverge in the call tail unless

$$\lim_{z\to+\infty} \frac{\ln\left(h(z)\right)}{z} < \tfrac{1}{\tau}\ln\left(1 + \tfrac{1}{\lambda-1}\right).$$

Substituting, here the critical value $\lambda = 1 + \lambda_+$ while applying (6.5) and combining it with the analogous result for the put tail yields the following relation between the Laplace support bounds and asymptotic behavior of $h(z)$

$$\varphi_\pm = a_\pm \tau, \quad a_\pm = \lim_{z\to\pm\infty} \frac{\ln\left(h(z)\right)}{|z|}. \quad (6.9)$$

Comparing this formula, with (6.5), we observe, that to be consistent with time homogeneous local vol extrapolation τ should be assumed asymptotically leaner with time to expiration, while under the stochastic vol jump assumptions it should increase slower than leaner, or, should be bounded:

$$\lim_{T\to\infty} \varphi_\pm = a_\pm \lim_{T\to\infty} \tau < \infty. \quad (6.10)$$

6.3 Power asymptotical behavior of $h(z)$

Let us assume that the leading term of $h(z)$ in one of the tails behaves is of the form

$$h(z)_{z\to\pm\infty} \sim |z|^{1+\alpha} \quad (\alpha > 0).$$

Then the asymptotic of the logarithmic return (4.7) will become

$$x(z)_{z\to\pm\infty} \sim |z|^\alpha$$

and, as a consequence, the asymptotic of logarithmic density in terms of the logarithmic return will become

$$-\log(\rho(x))_{x\to\pm\infty} \sim |x|^{1+\frac{1}{\alpha}}.$$

In particular, in the Gaussian-like case ($\alpha = 1$), as expected:

$$-\log\left(\rho(x)\right)_{x\to\pm\infty} \sim x^2.$$

Let as now consider a case of linear asymptotic ($\alpha = 0$), when the slop h' reaches it maximum values

$$h(z)_{z\to\pm\infty} = x_\pm\,|z| + o\left(|z|\right).$$

Then the distribution of logarithmic return will be confined to the interval

$$x \in \left(-x_-, x_+\right).$$

Finally for $\alpha \to \infty$ the asymptotic of logarithmic density will approach its theoretical limit corresponding to the bounded Laplace support:

$$-\log(\rho(x))_{x\to\pm\infty} \sim |x|\,.$$

7 Asset Demise in Delta Parametrization

Let us now consider the case when the distribution $\Omega(z)$ is bounded in one of the tails or both of them

$$z_{\min} < z < z_{\max}. \tag{7.1}$$

In this case the logarithmic density $h(z)$ is defined only within the interval $(z_{\min}, z_{\max})$. The convexity requirement leads to the following asymptotical behavior of h' near the distribution bounds

$$\lim h'(z)_{z\to z_{\min}} = \lim h'(z)_{z\to z_{\max}} = \pm\infty. \tag{7.2}$$

In addition, it must satisfy the following normalization condition

$$\int_{z_{\min}}^{z_{\max}} e^{-h(z)}dz = 1. \tag{7.3}$$

The limits of $h(z)$ itself may either remain finite if

$$h(z)_{z\to z_{\min\,/\,\max}} \approx \alpha - \beta\left|z - z_{\min\,/\,\max}\right|^{\gamma} \quad (\beta > 0,\ 0 < \gamma < 1)\,, \tag{7.4}$$

or, otherwise, approach positive infinity

$$\lim h(z)_{z\to z_{\min}} = \lim h(z)_{z\to z_{\max}} = +\infty. \tag{7.5}$$

Note that $z_{\min}$, $z_{\max}$ are bounds of Z_- distribution in the C_- measure, while, in the same time being bounds of Z_+ distribution in the C_+ measure. In particular, the event $z_{\min} < Z_+ < z_{\min} + \tau$ is within the distribution bound in C_+ measure. However, the corresponding $Z_- = Z_+ - \tau$ would be outside the bounds making this even impossible under the C_- measure. This may occur only if the asset

C_- loses all its values, which can be interpreted as an event of default. The probability of C_- demise in the C_+ measure can be calculated as

$$\Pr_+\left[C_- = 0\right] = \int_{z_{\min}}^{z_{\min}+\tau} e^{-h(z)} dz. \tag{7.6}$$

Analogously, for the probability of C_+ demise in the C_- measure becomes

$$\Pr_-\left[C_+ = 0\right] = \int_{z_{\max}-\tau}^{z_{\max}} e^{-h(z)} dz. \tag{7.7}$$

As expected, both default probability increase with τ *. If z is bounded from both tails, there is a critical value of τ equal to the size of distribution

$$\tau_{\max} = z_{\max} - z_{\min}, \tag{7.8}$$

such as one of the asset must default before the corresponding expiration is reached. At this point, the corresponding implied volatility becomes infinite. For example, such situation may occur if $C_\pm$ are opposing bets in the game which must be completed before the critical expiration (e.g, boxing match). An other, more conventional, way to interpret (7.8) would be to assume that the function $\tau(T)$ is bounded at the $\tau_{\max}$ level

$$\lim_{T\to\infty} \tau(T) = \tau_{\max}, \tag{7.9}$$

This is consistent with the τ extrapolation assumption (6.10) derived based on analysis of Laplace support term structure.

8 Interpolation and extrapolation of τ

So far we have established that the volatility surface is consistent in both strike and time dimensions as long as $h(z)$ is convex and τ is increasing with T starting at $\tau(t) = 0$. In the Black-Scholes case

$$\tau = \sigma\sqrt{T-t}. \tag{8.1}$$

In the short term limit $T \to t$ this is the common behavior for all continuous processes. Indeed, in the limit $\tau \to 0$ the relation between z and logarithmic return can be approximated as

$$x \approx \tau h'(z)$$

leading to the following approximation for the variance of short term logarithmic return

$$\frac{\sigma^2(T-t)}{\tau^2} \approx \mathrm{E}[x^2] \approx \int \left(h'(z)\right)^2 dz = \int h'(z) de^{-h(z)} = \int \left(h''(z)\right) e^{-h(z)} dz.$$

*To be precise, in the circumstances described here neither C_- nor C_+ could be used as a conventional numeraire hence a numeraire asset must remain strictly positive. Such defective numeraires can only be applied to a distribution conditioned on absence of default.

This allows to generalise the equation (8.1) as follows

$$\tau_{T\to t} \approx \sigma\sqrt{(T-t)\mathrm{E}[h'']}. \tag{8.2}$$

Thus we established, that in the short term τ should behave as a square root of the time to expirations. In the previous sections we have also analyzed the relations between asymptotical behavior of tail vols and default probabilities to extrapolation of τ. Here we are going to propose a generic technique for interpolation and extrapolation of τ, which seems plausible in most of the those cases. As h is a convex function of z it must have the minimum value $h_{\min}$, and for any $H > 0$, there should be exactly two solutions of the equation

$$h(z) = h_{\min} + H.$$

Let us defined a relation between H and τ as follows

$$\tau = z_2(H) - z_1(H),$$

where $z_1(H) < z_2(H)$ is the ordered pair of such solutions. This relation is obviously monotonic and, therefore, a bijection. It can be also defined as

$$H = \inf_z \max\left(h(z), h(z+\tau)\right) - h_{\min}. \tag{8.3}$$

The idea is to use H as a proxy for interpolation and extrapolation of τ w.r.t. expiration. For instance, we can interpolate and extrapolate H piecewise linearly w.r,t. T. As a function of τ, H is asymptotically quadratic for small τ, which is consistent with (8.2). In the mutual demise case (7.5) it ensures, the behavior (7.9). In the bounded Laplace support case (6.9), H is asymptotically exponential with large τ leading to τ being logarithmic w.r.t. the expiration proxy variable H, which is consistent with stochastic vol/jump diffusion assumptions.

8.1 Example: Explosive tails

Assume

$$h(z) = \tfrac{A}{2}\cosh(z) + \ln(\mathcal{N}),$$

where,

$$\mathcal{N} = \int_{-\infty}^{+\infty} \exp\left(-\tfrac{A}{2}\cosh(z)\right) dz.$$

Then

$$x = h(z_+) - h(z_-) = A\sinh\left(\tfrac{\tau}{2}\right)\sinh(\bar{z}),$$

where

$$\bar{z} = \frac{z_- + z_+}{2}.$$

Differentiating both side w.r.t z we shall have

$$\frac{dx}{dz_\pm} = A\sinh\left(\tfrac{\tau}{2}\right)\cosh(\bar{z}).$$

Therefore, the densities $\rho_\pm$ of the logarithmic return x in the $C_\pm$ measures can be calculate as

$$\rho_\pm = \frac{\exp\left(-\frac{A}{2}\cosh(z_\pm)\right)}{\mathcal{N} A \sinh\left(\frac{\tau}{2}\right)\cosh(\bar{z})} = \frac{\exp\left(-\frac{A}{2}\left(\cosh(\bar{z})\cosh\left(\frac{\tau}{2}\right) \pm \sinh(\bar{z})\sinh\left(\frac{\tau}{2}\right)\right)\right)}{\mathcal{N} A \sinh\left(\frac{\tau}{2}\right)\cosh(\bar{z})}$$

Substituting here

$$\sinh(\bar{z}) = \frac{x}{A\sinh(\tau)}, \quad \cosh(\bar{z}) = \sqrt{1 + \left(\sinh(\bar{z})\right)^2}$$

we can express this density in terms of logarithmic return

$$\rho_\pm = \frac{\exp\left(-\frac{1}{2}\left(\sqrt{\left(A\cosh\left(\frac{\tau}{2}\right)\right)^2 + \left(\coth\left(\frac{\tau}{2}\right)x\right)^2} \pm x\right)\right)}{\mathcal{N}\sqrt{\left(A\sinh\left(\frac{\tau}{2}\right)\right)^2 + x^2}}$$

Computing λ bounds from the asymptotic of the logarithmic density we shall have

$$\lambda_\pm = -\lim_{x\to\pm\infty} \frac{\ln\left(\rho_\mp(x)\right)}{|x|} = \frac{l}{2}\left(1 - \coth\left(\frac{\tau}{2}\right)\right) = \left(e^\tau - 1\right)^{-1}$$

In the notations (6.5) this formula can be written in the form $\varphi_\pm = \tau$, which is in agreement with the general result (6.9), where

$$a_\pm = \lim_{z\to\pm\infty} \frac{\ln\left(\cosh(z)\right)}{|z|} = 1. \tag{8.4}$$

Finally, to simplify interpolation/extrapolatin in the maturity dimension we can express τ it terms of the time proxy variable (8.3)

$$H = \frac{A}{2}\left(\cosh\left(\frac{\tau}{2}\right) - 1\right).$$

Then

$$A\cosh\left(\frac{\tau}{2}\right) = 2H + A,$$
$$A\sinh\left(\frac{\tau}{2}\right) = 2\sqrt{H(H+A)},$$

and, therefore,

$$\rho_\pm = \frac{\exp\left(-\frac{1}{2}\left(\sqrt{(2H+A)^2 + \frac{((2H+A)x)^2}{4H(H+A)}} \pm x\right)\right)}{\mathcal{N}\sqrt{4H(H+A) + x^2}}.$$

9 Inverse Quanto-Shift Parametrization and Interpolation in Time Dimension

For the purpose of this section, we shall scale z parameters making $\tau = 1$ but allow the h-function to depend on expiration so that implied distribution at each expiration can be obtained from the following implicit relations:

$$\Omega(z,T) = \int e^{-h(z,T)} dz, \tag{9.1a}$$

$$\Delta_\pm = \Omega(z_\pm, T), \tag{9.1b}$$

$$x = h(z_+, T) - h(z_-, T) \tag{9.1c}$$

$$z_+ = z_- + 1. \tag{9.1d}$$

Let us go one step further and suggest to parametrize option market in terms of the the inverse function

$$z_\pm = z(\Omega_\pm, T). \tag{9.2}$$

Then, the equation (9.1d) can be used in parameterizing an option market in the implicit version of the form (3.4). Namely,

$$z(\Delta_+, T) - z(\Delta_-, T) = 1. \tag{9.3}$$

Let us first investigate properties of the inverse quato shift parametrization in strike dimension, and omit the expiration parameter T. In term of the parametrization (9.2) the logarithmic z-density becomes

$$h = \ln\left(\frac{dz(\Omega)}{d\Omega}\right) = \ln\left(z'(\Omega)\right) \tag{9.4}$$

Therefore, according to (9.1c), pair of deltas $\Delta_\pm$ associated with a given relative strike $R = \exp(x)$ can be obtain by solving the following system of equations

$$z'(\Delta_+) = Rz'(\Delta_-), \tag{9.5a}$$

$$z(\Delta_+) = z(\Delta_-) + 1. \tag{9.5b}$$

Knowing $\Delta_\pm$ vanilla options can be evaluated via (2.8). The specification is consistent if h is convex w.r.t. z. This means that the derivative

$$\frac{dh}{dz} = \frac{dh}{d\Omega}\frac{d\Omega}{dz} = \frac{\frac{d\ln\left(z'(\Omega)\right)}{dz}}{z'(\Omega)} = \frac{z''(\Omega)}{\left(z'(\Omega)\right)^2} = -\frac{d}{d\Omega}\left(z'(\Omega)\right)^{-1}$$

is monotonically increasing w.r.t. z and therefore w.r.t. Ω . Thus we demonstrated that the convexity condition for the function $h(z)$ is equivalent to concavity condition for $\left(z'(\Omega)\right)^{-1}$:

$$\frac{d^2}{\left(d\Omega\right)^2}\left(z'(\Omega)\right)^{-1} \leq 0 \tag{9.6}$$

Note that the function $z(\Omega)$ is defined modulo constant. In particular, adding constant to $z(\Omega)$ would not affect the equations (9.5). Let $\{z_i(\Omega)\}_{i=1}^2$ be a pair of function satisfying the concavity condition (9.6) and therefore corresponding to consistent co-terminal option markets. Then any linear combination of these functions with positive coefficients

$$z(\Omega) = \alpha_1 z_1(\Omega) + \alpha_2 z_2(\Omega). \tag{9.7}$$

would also satisfy the inequality (9.6). This is the consequence of the following lemma

Lemma 1. *Harmonic average of positive concave functions is also concave.*

Proof. See Appendix. □

Note, that constant shifts in functions z_i, in the component function $z_i(\Omega)$ will result in a constant shift in their linear combination (9.7). Therefore, the linear combo operations is not affected by modulo constant uncertainty. The linear mixing (9.7) allows to build consistent co-terminal option marked out of other co-terminal option market. As we are going to demonstrate here it can be also used as a tool for consistent interpolation in time dimension. In order to do it let us first formulate the calendar consistency condition in terms of inverse quanto shift parametrization.

Lemma 2. *The inverse quoto-shift option market is calendar consistent if*

$$\forall T_1 < T_2 : \quad z(\Omega, T_1) - z(\Omega, T_2) \textit{ not decreasing in } \Omega. \tag{9.8}$$

Proof. Let $\Delta_\pm$ be a pair of deltas satisfying (9.5b) for the shorter expiration:

$$z(\Delta_+, T_1) = z(\Delta_-, T_1) + 1.$$

We can alway adjust constant shifts in z so that $z(\Delta_-, T_1) = z(\Delta_-, T_2) = 0$. Then as a consequence of $\Delta_+ > \Delta_-$ under the lemma's assumptions

$$z(\Delta_+, T_2) \le z(\Delta_+, T_1) = z(\Delta_-, T_1) + 1 = 1 = z(\Delta'_+, T_2) - z(\Delta_-, T_2) = z(\Delta'_+, T_2),$$

where Δ'_+ is the counterpart of Δ_- at the longer expirations T_2. As $z(\Omega, T)$ is non-decreasing in Ω, this is also implies that $\Delta'_+ > \Delta_+$. Thus, we established that under the lemma assumption Δ_+ associated with the same Δ_- is nondecreasing in time, which (as it has been demonstrated in Sec 3) is an no calendar arbitrage condition in terms of relation between deltas. □

It easy to see that, if two co-terminal markets satisfy the Lemma 2, so will all the intermediated expirations $T_1 < T < T_2$ linearly interpolated in-terms of $z(\Omega)$ via any monotonic function of expiration $\eta(T)$ (e.g., $\eta(T) = T^{-\frac{1}{2}}$):

$$z(\Omega, T) = \frac{(\eta(T_2) - \eta(T))\, z(\Omega, T_1) + (\eta(T) - \eta(T_1))\, z(\Omega, T_2)}{\eta(T_2) - \eta(T_1)}. \tag{9.9}$$

Moreover, if the $z(\Omega, T_{1|2})$ can be linearly decomposed

$$z(\Omega, T_{1|2}) = \sum_{i=1}^{N} z_i(\Omega, T_{1|2})$$

so that component z_i satisfies (9.8) separately, one can chose different interpolation function $\eta_i(T)$ per each component:

$$z_i(\Omega, T) = \frac{(\eta_i(T_2) - \eta_i(T))\, z(\Omega, T_1) + (\eta_i(T) - \eta_i(T_1))\, z(\Omega, T_2)}{\eta_i(T_2) - \eta_i(T_1)}$$
$$z(\Omega, T) = \sum_{i=1}^{N} z_i(\Omega, T)$$

Finally, Let us compute local vol in the inverse quanto shift parametrization. We shall do it base on the Dupire formula in the delta form (5.4). After introduce expiration dependency the equation (9.5) becomes

$$x = \log\left(z_{,\Omega}(\Delta_+, T)\right) - \log\left(z_{,\Omega}(\Delta_-, T)\right), \tag{9.10a}$$
$$z(\Delta_+, T) = z(\Delta_-, T) + 1. \tag{9.10b}$$

Hereafter, subscript indices following a comma denote differentiation. Differentiating the above equations we shall have

$$\begin{aligned} dx =& \frac{z_{,\Omega\Omega}(\Delta_+, T)}{z_{,\Omega}(\Delta_+, T)} d\Delta_+ - \frac{z_{,\Omega\Omega}(\Delta_-, T)}{z_{,\Omega}(\Delta_-, T)} d\Delta_- \\ &+ \left(\frac{z_{,\Omega T}(\Delta_+, T)}{z_{,\Omega}(\Delta_+, T)} - \frac{z_{,\Omega T}}{z_{,\Omega}}(\Delta_-, T) \right) dT \end{aligned} \tag{9.11a}$$
$$\begin{aligned} 0 =& z_{,\Omega}(\Delta_+, T) d\Delta_+ - z_{,\Omega}(\Delta_-, T) d\Delta_- \\ &+ \left(z_{,T}(\Delta_+, T) - z_{,T}(\Delta_-, T)\right) dT = 0 \end{aligned} \tag{9.11b}$$

Solving (9.11b) we shall obtain

$$\begin{aligned} \left(\frac{\partial \Delta_+}{\partial T}\right)_{\Delta_-} &= \frac{z_{,T}(\Delta_-, T) - z_{,T}(\Delta_+, T)}{z_{,\Omega}(\Delta_+, T)}. \\ \left(\frac{\partial \Delta_-}{\partial T}\right)_{\Delta_+} &= \frac{z_{,T}(\Delta_-, T) - z_{,T}(\Delta_+, T)}{z_{,\Omega}(\Delta_-, T)} \end{aligned} \tag{9.12}$$

Substituting (9.11a) to (5.4) and applying (9.12) we should finally have

$$\sigma^2_{\text{loc}} = 2\left(z_{,T}(\Delta_-, T) - z_{,T}(\Delta_+, T)\right) \left(\frac{z_{,\Omega\Omega}}{z^2_{,\Omega}}(\Delta_+, T) - \frac{z_{,\Omega\Omega}}{z^2_{,\Omega}}(\Delta_-, T) \right). \tag{9.13}$$

9.1 Default and Tail vol explosion

In term if inverse quato-shift parametrization, the condition (7.1) for one of the underlying be able to default translates into finite limit of the functions $z(\Omega)$ in the corresponding tail:

$$\lim_{\Omega \to 0} = z_{\min}, \quad \lim_{\Omega \to 1} = z_{\max}. \tag{9.14}$$

To express tail explosion in terms of $z(\Omega)$ let us apply (6.9) with $\tau = 1$, substitute (9.4) and express the limit in terms of Ω:

$$\lim_{\Omega \to 0} \frac{\ln\left(\ln\left(z'(\Omega)\right)\right)}{z(\Omega)} = -\varphi_-, \quad \lim_{\Omega \to 1} \frac{\ln\left(\ln\left(z'(\Omega)\right)\right)}{z(\Omega)} = \varphi_+. \tag{9.15}$$

It easy to see that this translates into the following asymptotic behavior

$$\Omega \to 0 : z \approx -\frac{1}{\varphi_-} \ln\left(-\ln(\Omega)\right), \quad \Omega \to 1 : z \approx \frac{1}{\varphi_+} \ln\left(-\ln(1-\Omega)\right). \tag{9.16}$$

9.2 Examples

Obviously, the Black-Scholes case in the inverse quanto shift parametrization becomes

$$z = \frac{1}{\delta} W, \quad W \equiv \mathcal{N}^{-1}(\Omega). \tag{9.17}$$

Here δ is Black-Scholes implied standard deviation and $\mathcal{N}^{-1}$ inverse cumulative normal distribution. This formula can be generalized to the case with different flattening vols tails as follows

$$\begin{aligned} z = & \frac{1}{2\delta_-} \left((W - C_-) - \sqrt{(W - C_-)^2 + q_-} \right) \\ & + \frac{1}{2\delta_+} \left((W + C_+) + \sqrt{(W + C_+)^2 + q_+} \right), \end{aligned} \tag{9.18}$$

where $\delta_\pm$ are asymptotic put and call implied standard deviation, respectively, and $C_\pm$ are large enough. * This function can be shown to satisfy the consistency condition (9.6) for any positive $\delta_\pm$ and $q_\pm$. In the limit $\delta_\pm \to$ one of the term become trivial, e.g.,

$$z = \frac{1}{2\delta_-} \left((W - C_-) - \sqrt{(W - C_-)^2 + q_-} \right), \tag{9.19}$$

and the limit of z in the corresponding tail becomes finite

$$\lim_{\Omega \to 0} z(\Omega) = 0 > -\infty$$

indicating possibility of default. In terms of the $h(z)$ function the parametrization (9.18) can be shown to be †

$$h = \tfrac{1}{2} \left(\frac{q}{4z\delta} - \delta z + C \right)^2 - \ln\left(\frac{q}{4z^2\delta} + \delta \right) + \tfrac{1}{2} \ln(2\pi). \tag{9.20}$$

In the case of $\delta_- = \delta_+$, $q_- = q_+$, $C_+ = -C_-$ the formula (9.18) turns back int to the Black Scholes one.

* Minimum values of $C_\pm$ depend on $q_\pm$ and can be negative for small $q_\pm$. See Appendix for detail

† See Appendix

Finally. in order to introduce tail explosions let us modify the formula (9.18) as follows:

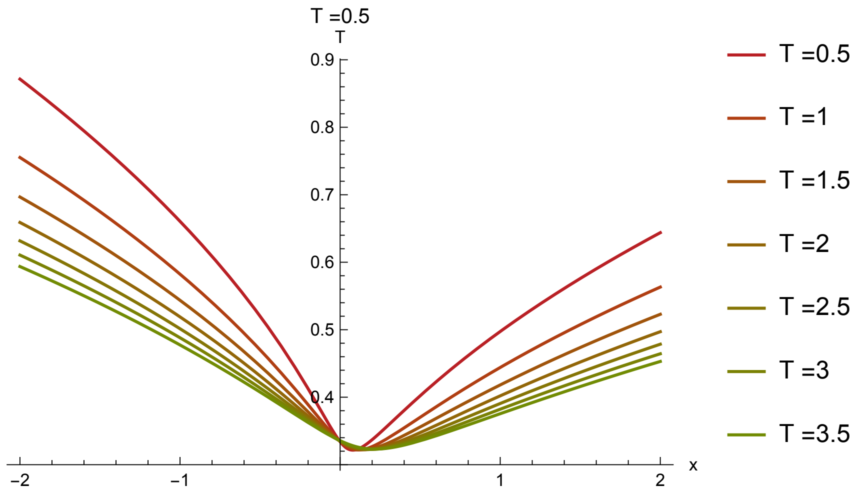
$$z = -\frac{2}{\varphi_-}\ln\left(1-\frac{\varphi_-}{4\delta_-}\left((W-C_-)-\sqrt{(W-C_-)^2+q_-}\right)\right) + \frac{2}{\varphi_+}\ln\left(1+\frac{\varphi_+}{4\delta_+}\left((W+C_+)+\sqrt{(W+C_+)^2+q_+}\right)\right). \tag{9.21}$$

The modified function can be still shown to satisfy the consistency condition (9.6) for any positive $q_\pm$, $\delta_\pm$ and $\varphi_\pm$. In addition it is monotonically decreasing w.r.t $\delta_\pm$ and $\varphi_\pm$. In the limit $\varphi_\pm \to 0$, it turns to the explosion free case (9.18).

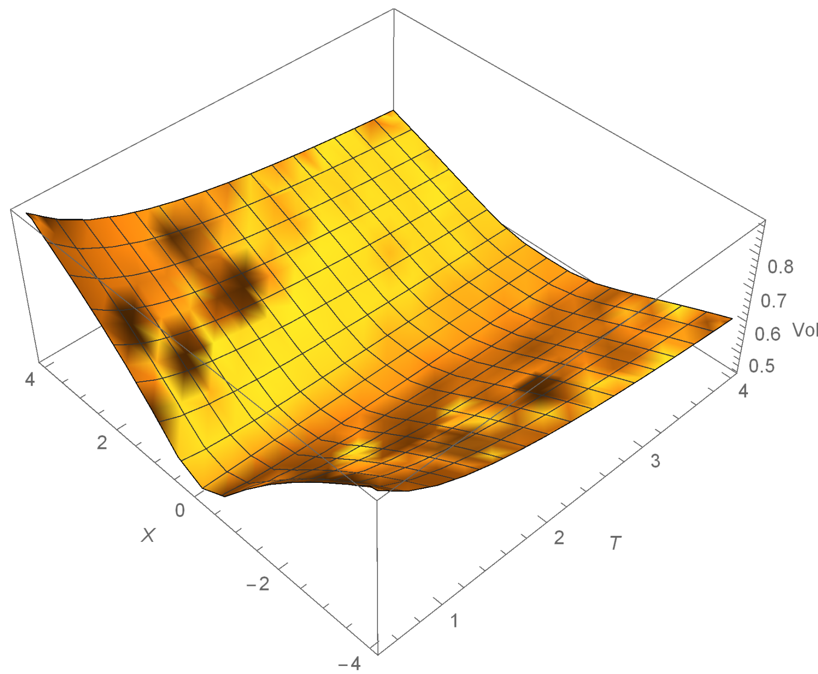
9.3 Illustrations

The example a base on extrapolating a single co-terminal vol for short and long expirations

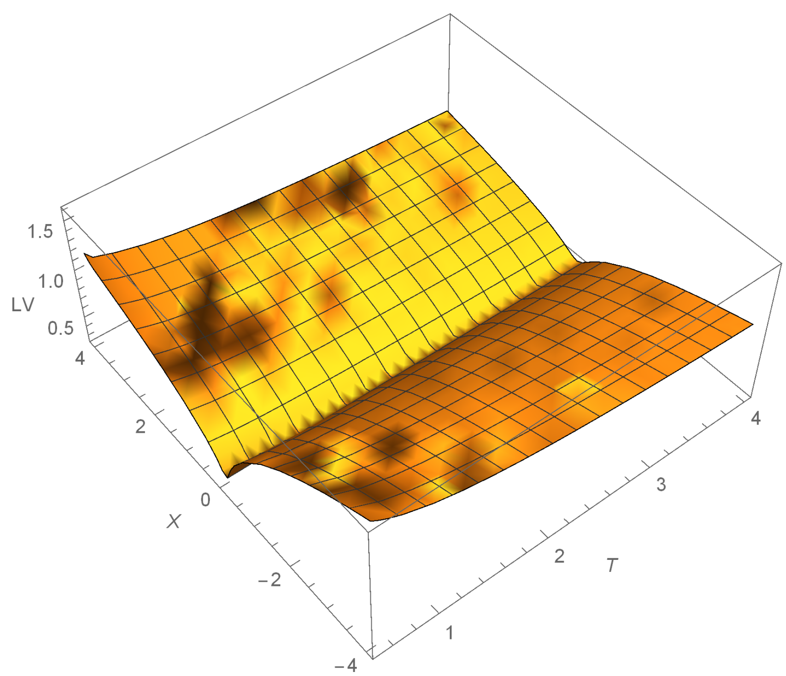
- Implied vol term structure:



- Implied vol 3D plot:

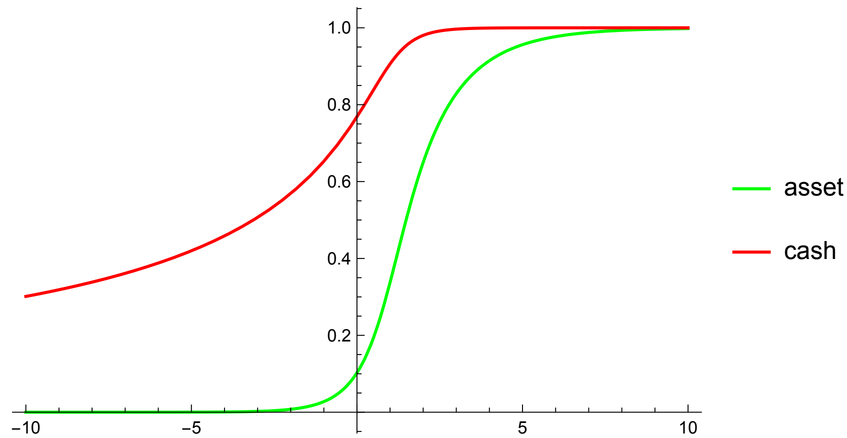


- Local vol 3D plot:

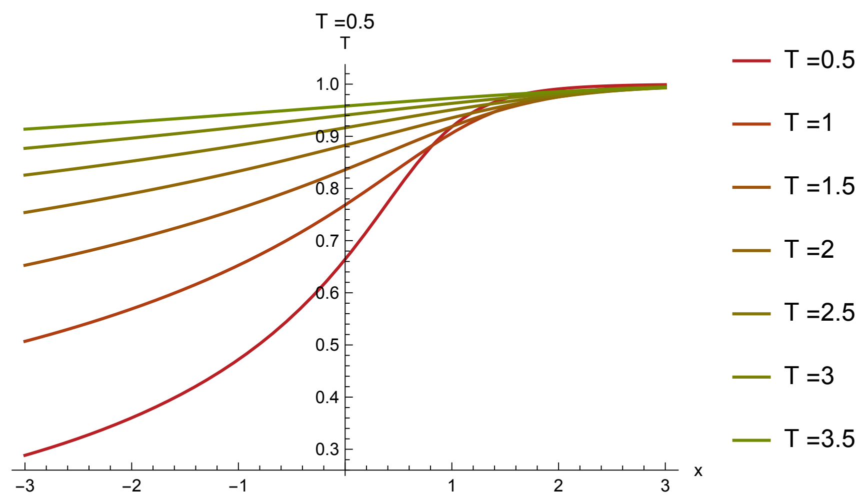


This is a similar example for volatility surface with embedded default:

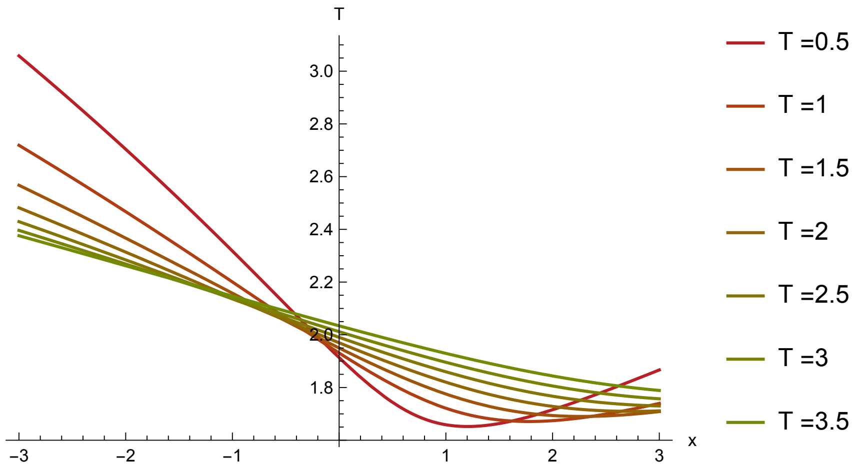
- Implied CDF in cash an asset measure. Not that cash CDF does not asymptotes to zero in the put tail indicating default:



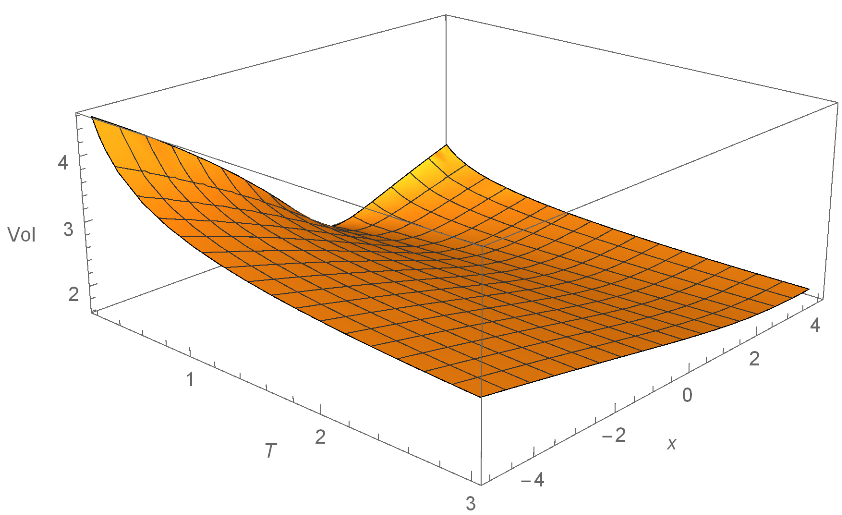
- Term-structure of the cash CDF (the chosen example is rather extreme) .



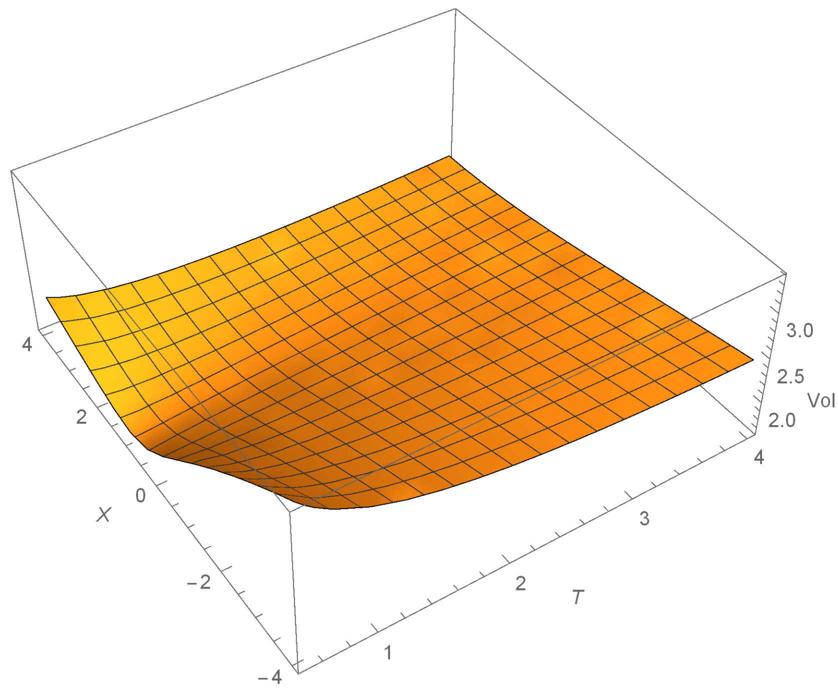
- Implied vol term structure:



- Implied vol 3D plot:



- Implied local vol 3D plot:



10 Boom and Crash or Empires Must Fall

So far our only concerned was building a volatility surface free of static arbitrage while remaining agnostic about its process. With such models one can consistently price any european claim but the scope of the covered products ends here. It is not sufficient for pricing american, compound or path dependent contracts.

Here we shall demonstrate that a family of fully blown arbitrage free stochastic vol models can be constructed within the class of constant quato-shift vol surfaces. Under the proposed process both underlying might be completely annihilated in finite time. If both of them are currencies one may interpret the result as a justification of the Empires Must Fall philosophy thought is more feasible for emerging markets. Alternatively, we can relate it to an equity process. The fact that a share may completely lose its value is not at all surprising. The suggestion that the currency can follow the same predicament is a bit more remote. However, we can suggest a less extreme interpretation under which rather than linking such event to demise of domestic currency we regard it as an approximation to a boom behavior when the share propelled to qualitatively higher level. In more general note, it seems plausible that the boom and crash behavior is natural under the deterministic rates/proportional dividend approximation. In reality, stochastic rates guard against the boom and crash phenomena by introducing mean-reversion to the spot process potentially eliminating such events completely in finite and even infinite time.

Under the proposed process we shall assume that the $h(z)$ function does not change. However, we shall allow the $\tau(T)$ curve process to become stochastic and volatile. Then for any given option the corresponding values of the quanto shift parameters would follow some stochastic processes, such that

$$d\tau = dz_+ - dz_-, \tag{10.1}$$

with the common value of $d\tau$ for all strikes. For the disambiguation, in this section x will refer to logarithmic relative strikes only, while for the logarithmic forward adjusted return we shall use the variable y. The change or logarithmic at-the-moneyness of the options

$$dx = d\left(h(z_+) - h(z_-)\right) \tag{10.2}$$

is opposite to the logarithmic return

$$dx = -dy \tag{10.3}$$

and, therefore, must me the same for all strikes and maturities with the convexity drift equal to

$$\langle dx\rangle_\pm = -\langle dy\rangle_\pm = \pm\tfrac{1}{2}dx^2. \tag{10.4}$$

The first step is to express $dz_\pm$ quadratic covariation rates via the strike independent dx, $d\tau$ covariation rates. From (10.1) and (10.2) we can express the dx $d\tau$ covariations via the $dz_\pm$ covariations, as follows

$$\begin{aligned}
(d\tau)^2 &= (dz_+)^2 + (dz_-)^2 - dz_+dz_-,\\
(dx)^2 &= \left(h'(z_+)\right)^2(dz_+)^2 + \left(h'(z_-)\right)^2(dz_-)^2 - 2h'(z_+)h'(z_-)dz_+dz_-,\\
dxd\tau &= h'(z_+)(dz_+)^2 + h'(z_-)(dz_-)^2 - \left(h'(z_+) + h'(z_-)\right)dz_+dz_-.
\end{aligned}$$

Solving above for $dz_\pm$ covariation rates we shall have

$$\begin{aligned}
(dz_+)^2 &= \frac{(dx)^2 + \left(h'(z_-)\right)^2(d\tau)^2 - 2h'(z_-)dxd\tau}{\left(h'(z_+) - h'(z_-)\right)^2}\\
(dz_-)^2 &= \frac{(dx)^2 + \left(h'(z_+)\right)^2(d\tau)^2 - 2h'(z_+)dxd\tau}{\left(h'(z_+) - h'(z_-)\right)^2}\\
dz_+dz_- &= \frac{(dx)^2 + h'(z_+)h'(z_-)(d\tau)^2 - 2\left(h'(z_+) + h'(z_-)\right)dxd\tau}{\left(h'(z_+) - h'(z_-)\right)^2}
\end{aligned} \tag{10.5}$$

Now we should compute risk neutral τ-drift $\langle d\tau\rangle_\pm$ in one of the numeraires and require it to be constant across strikes. First recall that $\Delta_\pm$ must be martingales in the $C_\pm$ measures, respectively:

$$\begin{aligned}
0 = \langle d\Delta_\pm\rangle_\pm = \langle d\Omega(z_\pm)\rangle_\pm &= \Omega''(z_\pm)\langle dz_\pm\rangle + \tfrac{1}{2}\Omega'(z_\pm)\left(dz_\pm\right)^2\\
&= e^{-h(z_\pm)}\left(\langle dz_\pm\rangle_\pm - \tfrac{1}{2}h'(z_\pm)\left(dz_\pm\right)^2\right)
\end{aligned}$$

and, therefore,

$$\langle dz_\pm \rangle_\pm = \tfrac{1}{2} h'(z_\pm)\,(dz_\pm)^2 .$$

As the drifts $z_\pm$ are computed in deferent measure we should convert one of them by applying the appropriate quanto correction (Girsanov Theorem)

$$\forall \psi: \quad \langle d\psi \rangle_+ - \langle d\psi \rangle_- = d\psi dx. \tag{10.6}$$

Here ψ is an arbitrary continuous process of finite quadratic variation. In particular, it is consistent with (10.4). Now, we are ready to compute the τ-drift:

$$\begin{aligned} \langle d\tau \rangle_+ &= \langle d\,(z_+ - z_-) \rangle_+ = \langle dz_+ \rangle - \langle dz_- \rangle - dz_+ dx \\ &= \tfrac{1}{2} h'(z_+)\,(dz_+)^2 + \tfrac{1}{2} h'(z_-)\,(dz_-)^2 - h'(z_+) dz_+ dz_- . \end{aligned}$$

In combination with a similar result for $\langle d\tau \rangle_-$ it can be written as

$$\pm \langle d\tau \rangle_\pm = \tfrac{1}{2} h'(z_+)\,(dz_+)^2 + \tfrac{1}{2} h'(z_-)\,(dz_-)^2 - h'(z_\pm) dz_+ dz_- . \tag{10.7}$$

Finally substituting here (10.5) we shall conclude

$$-\langle d\tau \rangle_\pm = \frac{(dx)^2 + h'(z_+) h'(z_-)\,(d\tau)^2 - 2h(z_\pm) dx d\tau}{2\,(h'(z_+) - h'(z_-))} \tag{10.8}$$

Note, that at the time when expiration T is reached, $\tau(T)$ must turn 0. In order for this to happen the drift of τ should be predominately negative. As the τ-drift must be constant across all strikes, the right part of (10.8) must depend only on the difference $\tau = z_+ - z_-$ but not on each of $z_\pm$ separately. It easy to see that the drifts (10.8) are consistent with the Girsanov theorem (10.6)

$$\langle d\tau \rangle_+ - \langle d\tau \rangle_- = d\tau dx. \tag{10.9}$$

Consequently, as long as the τ drift in one of the numeraires is constant across strikes, the other one will automatically follow. To represent the problem in more symmetric fashion we shell use the average measure, such that

$$\langle d\psi \rangle_0 \equiv \tfrac{1}{2} \left(\langle d\psi \rangle_+ + \langle d\psi \rangle_- \right). \tag{10.10}$$

In particular, under this measure logarithmic return is martingale

$$\langle dx \rangle_0 = \langle dy \rangle_0 = 0.$$

Then the equations (10.8) becomes

$$-\langle d\tau \rangle_0 = \frac{(dx)^2 + h'(z_+) h'(z_-)\,(d\tau)^2 - (h(z_+) + h(z_-))\, dx d\tau}{2\,(h'(z_+) - h'(z_-))}. \tag{10.11}$$

Solving this equations for $h'(z_+)$ and substituting $z_- = z$, $z_+ = z + \tau$ we shall have

$$h'(z+\tau) = \frac{(2\,\langle d\tau \rangle_0 + dx d\tau)\, h'(z) - dx^2}{d\tau^2 h'(z) + 2\,\langle d\tau \rangle_0 - dx d\tau}. \tag{10.12}$$

Thus to make consistent volatility dynamics possible under the constant h-function assumption its derivative h' must be invariant under the τ-shift of the argument accompanied by a fractional linear transformation. It can be shown that to solve this condition for all τ the function $h(z)$ must be of the form

$$h(z) = -\frac{1}{\omega^2}\log\left(\cos(\omega z+\theta)\right) - \frac{1}{\omega}\tan(\theta)z + \ln(\mathcal{N}). \tag{10.13}$$

Here $\mathcal{N}$ is the normalization factor. Note that, to make this function well defined and convex we should restrict its support to the interval $\left(-\frac{\pi-2\theta}{2\omega}, \frac{\pi-2\theta}{2\omega}\right)$ with the normalization constant, consequently, defined as

$$\mathcal{N} = \int_{-\frac{\pi-2\theta}{2\omega}}^{\frac{\pi-2\theta}{2\omega}} \cos\left(\omega z\right)^{\frac{1}{\omega^2}} e^{\frac{\tan(\theta)}{\omega}z} dz = \frac{1}{\omega}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos(u)^{\frac{1}{\omega^2}} e^{\frac{\tan(\theta)}{\omega^2}(u-\theta)} du$$

In addition, the value of τ must stay below its infinite volatility bound (7.1)

$$\tau < \tau_{\max} = z_{\max} - z_{\min} = \frac{\pi}{\omega} \tag{10.14}$$

The parameters $\omega > 0$, $\theta \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$ specify the entire class of acceptable $h(z)$ up to the choice of z scale and shift. We selected the shift to satisfy the gauge constrain

$$h'(0) = 0$$

and the scale to be consistent with the Black Scholes limit

$$\omega \to 0: \quad h(z) \approx \frac{z^2}{2}.$$

To satisfy the equations (10.12) the covariations and drifts must be as follows

$$\begin{aligned} &(d\tau)^2 = \omega^2 ds, \quad (dx)^2 = \sec(\theta)^2 ds, \quad dxd\tau = -\omega\tan(\theta)ds, \\ &\langle dx\rangle_0 = 0, \quad \langle d\tau\rangle_0 = -\frac{\omega}{2}\cot\left(\tau\omega\right)ds, \end{aligned} \tag{10.15}$$

in some volatility stopping time s. The dynamics is well defined as long as τ is within the bounds (10.14). Finally, expressing increments of the option specific processes x in terms of the global return process y we shall have

$$\begin{aligned} &(d\tau)^2 = \omega^2 ds, \quad (dy)^2 = \sec(\theta)^2 ds, \quad dyd\tau = \omega\tan(\theta)ds, \\ &\langle dy\rangle_0 = 0, \quad \langle d\tau\rangle_0 = -\frac{\omega}{2}\cot\left(\tau\omega\right)ds, \end{aligned} \tag{10.16}$$

The curve process $\tau(T)$ represents the volatility term-structure. Its correlation with the spot level can be calculated as

$$\rho = \frac{dyd\tau}{\sqrt{(dy)^2(d\tau)^2}} = \sin(\theta). \tag{10.17}$$

Assuming a single factor vol model * we can specify the processes (10.16) under average measure in the conventional SDE form

$$dy = \sec(\theta)dW_1, \quad d\tau(T) = -\frac{\omega}{2}\cot\left(\tau(T)\omega\right)ds + \omega dW_2, \tag{10.18}$$

where W_1, W_2 are standard Wiener processes in stopping time s withe correlation (10.17). The relation between the stoping time s and the calendar time t is as follows. At the moment when $\tau(T)$ turns 0 the calendar times becomes $t = T$. The case when $\tau(T)$ never reaches zero in the stopping time s indicates demise of one of the underlying before T. To keep this relation self consistent for any pair of expirations $T_1 < T_2$ the event $\tau(T_1) = 0$ must occur before the event $\tau(T_2) = 0$. This property is enforced as long as as an arbitrage free volatility surface is maintained, which means that the curve $\tau(T)$ remains monotonically increasing and stays below the bound (10.14). Let us demonstrate that the process (10.18) satisfies this requirement. The random component of the τ-curve process is a parallel shift which preserves monotonicity. Analysis of the drift term is easier to perform in terms the following η curve

$$\eta(T) \equiv \cos\left(\omega\tau(T)\right). \tag{10.19}$$

Within the interval $\left[0, \frac{\pi}{\omega}\right]$ the relation between η and τ is strictly monotonically decreasing and, therefore, is a bijection. The condition for $\tau(T)$ to be a monotonically increasing function taking value in the interval $\left[0, \frac{\pi}{\omega}\right]$ is equivalent to the condition for $\eta(T)$ to be a monotonically decreasing function taking values in the interval $[1, -1]$. In terms of the η representation the drift contribution results in const scaling

$$d\eta = -\omega\sin(\omega\tau)\left\langle d\tau\right\rangle_0 = \tfrac{1}{2}\omega^2\eta ds, \tag{10.20}$$

which preserves monotonicity. However, the process does not preclude $\tau(T)$ from reaching its maximum value $\frac{\pi}{\omega}$, making demise of one of the asset imminent before the prescribed future time T (like in the boxing match example). It can be related to a penny stock which should soon either disappear from the market of rebound.

11 Multi-cross Options and Distributions

11.1 Basic options

Let us generalize the framework to handle european derivatives of $n+1$ $(n > 1)$ underlying currencies. Let $\mathcal{C} = \{C_i\}_{i=0}^{n}$ be a set of contracts denominated in each of the currencies with present values $N_i = \langle C_i \rangle$. To each pair of contracts we shall define logarithmic strike as

$$x_{ij} = \ln\left(\tfrac{N_j}{N_i}\right). \tag{11.1}$$

*A multi-factor $\tau(T)$ process is also possible

They obviously satisfy the following relations

$$\forall i : x_{ii} = 0, \quad \forall i, k, j : \ x_{ij} = x_{ik} + x_{kj}. \tag{11.2}$$

To any pair of non-overlapping subsets

$$\mathcal{C}_\pm \subseteq \mathcal{C}, \quad \mathcal{C}_+ \cap \mathcal{C}_- = \varnothing,$$

we can assign an european claim

$$\mathcal{O}_{\mathcal{C}_+,\mathcal{C}_-} = \left(\inf_{C \in \mathcal{C}_+} C - \sup_{C \in \mathcal{C}_-} C \right)^+ . \tag{11.3}$$

Note, that the payoff of the option is nontrivial if and only if all the underlying from the subset $\mathcal{C}_+$ finish strictly below all the underlying from the subset $\mathcal{C}_-$. If each of the set consist of one contract only $\dim(\mathcal{C}_\pm) = 1$ the option is reduced to a standard single cross vanilla one. If $\dim(\mathcal{C}_+) = 1$ and $\mathcal{C}_0 = \varnothing$ the option degenerate to one of the underlying contracts. We shall disallow an empty $\mathcal{C}_+ = \varnothing$ as its payoff is always trivial. The options based on the full set of currencies form a linear basis for the rest of the options (11.3). Namely,

$$\mathcal{O}_{\mathcal{C}_+,\mathcal{C}_-} = \sum_{\substack{\mathcal{C}'_- \supseteq \mathcal{C}_-,\ \mathcal{C}'_+ \supseteq \mathcal{C}_+ \\ \mathcal{C}'_+ = \mathcal{C} \setminus \mathcal{C}'_-}} \mathcal{O}_{\mathcal{C}'_+,\mathcal{C}'_-}. \tag{11.4}$$

In particular,

$$\forall C \in \mathcal{C} : \quad C = \mathcal{O}_{\{C\},\varnothing} = \sum_{\mathcal{C}'_+ \ni C} \mathcal{O}_{\mathcal{C}'_+,\mathcal{C} \setminus \mathcal{C}'_-}. \tag{11.5}$$

Similarly to the single cross case values of the options (11.3) can be represented by the first order homogeneous function of notionals N_i.

$$\left\langle \mathcal{O}_{\mathcal{C}_+,\mathcal{C}_-} \right\rangle = P_{\mathcal{C}_+,\mathcal{C}_-}(N_*).$$

As a consequence of (11.5)

$$N_i = \sum_{\mathcal{C}'_+ \ni C_i} P_{\mathcal{C}'_+,\mathcal{C} \setminus \mathcal{C}'_-}(N_*). \tag{11.6}$$

The above relations generalize parity relations for single cross vanilla options. The total number of the nontrivial options (11.3) involving all the contracts is $2^{n+1} - 1$. As notionals (11.6) are not vega sensitive, only $2^{n+1} - n - 2$ are independent. In particular, in the single cross case $n = 1$, it leaves only one independent variable as expected. Finally, there is on more interesting contract not included in the set (11.3)

$$\mathcal{O}_{\mathcal{C}} = \sup\nolimits_{C \in \mathcal{C}} C. \tag{11.7}$$

However, it can be shown to be equivalent to the sum:

$$\mathcal{O}_{\mathcal{C}} = \sum_{\mathcal{C}'_+ \subseteq \mathcal{C}} \mathcal{O}\left(\mathcal{C}'_+, \mathcal{C} \setminus \mathcal{C}'_+\right). \tag{11.8}$$

The easiest way do understand this relations (11.4), (11.8) is to consider an extreme case when some subset of underlaying $\mathcal{C}_0 \subset \mathcal{C}$ annihilate and the rest finishes with the same value, which we shall use as a numeraire. Then the value of the multi-cross option is simply a probability of the following event

$$P_{\mathcal{C}_+,\mathcal{C}_-} = \mathbb{Q}\left\{\mathcal{C}_0 \supseteq \mathcal{C}_- \ \&\ \mathcal{C}_0 \cap \mathcal{C}_+ = \varnothing\right\} \tag{11.9}$$

and

$$P_{\mathcal{C}} = 1. \tag{11.10}$$

Derivatives of option price w.r.t. notionals from $\mathcal{C}_+$ set

$$\Delta_{\mathcal{C}_+,\mathcal{C}_-,i} = \partial_{N_i} P_{\mathcal{C}_+,\mathcal{C}_-} \quad (C_i \in \mathcal{C}_+) \tag{11.11}$$

can be interpreted as the probabilities of the underlying C_i to finich below the rest of underlying from $\mathcal{C}_+$ set and above all the underlying from $\mathcal{C}_-$ set in C_i measure. Similarly, derivatives of option price w.r.t. notionals from $\mathcal{C}_-$ set

$$\Delta_{\mathcal{C}_+,\mathcal{C}_-,i} = -\partial_{N_i} P_{\mathcal{C}_+,\mathcal{C}_-} \quad (C_i \in \mathcal{C}_-) \tag{11.12}$$

can be interpreted as the probabilities of the underlying C_i to finich above the rest of underlying from $\mathcal{C}_-$ set and below all the underlying from $\mathcal{C}_+$ set in C_i measure. These probailities are increasing and decreasing w.r.t. notionals of the contracts from the $\mathcal{C}_+$ and $\mathcal{C}_-$ subsets, respectively.

11.2 Mutli-currency constant quanto-shift framework

The state of muti-cross exchange rate for $n+1$ currencies is n dimensional. Therefore in the multi-currency setup, the constant quanto shift variable z be interpreted as an element affine space $\mathcal{Z}$ of the dimension $\dim \mathcal{Z} \geq n$ *. Accordingly, the τ-shifts themselves should be associated with the vectors from the corresponding linear space $\mathcal{T}$. We should assign $\tau_{ij} \in \mathcal{T}$ to any pair of underlying currencies, such that

$$\forall i : \tau_{ii} = 0, \quad \forall i,k,j : \ \tau_{ij} = \tau_{ik} + \tau_{kj}. \tag{11.13}$$

This implies that there only n independent τ-vectors, e.g., $\{\tau_{0j}\}_{j=1,N}$, and the rest can be expressed in terms of them:

$$\tau_{ij} = \tau_{0j} - \tau_{0i}$$

In the same spirit we should assign to each currency a variable z_i such that

$$\forall i,j : \ z_j = z_i + \tau_{ij}. \tag{11.14}$$

* We may consider $\dim \mathcal{Z} > n$ to accommodate additional stochastic vol/correlation factors.

In the 3 currency case z_i can be interpreted as vertices of the family of triangles related to each other via parallel shifts. In the 4 currency case - a family of tetrahedrons. For larger number of currencies we shall call it τ-hedron (for the luck of the better word)[†]. Analogously, to the single cross case we assume existence of the convex function

$$h: \ \mathcal{Z} \to \mathbb{R}$$

such that $\forall i$:

$$\varphi_i = e^{-h(z_i)} \tag{11.15}$$

is a density of the variable z_i in C_i measure. Then similarly to the single cross case it can be shown that

$$x_{ji} = h(z_j) - h(z_i), \tag{11.16}$$

where x_{ji} is logarithmic return of C_i. relative to C_j This map is an injection[*] , i.e., there can be no more than one set of z_i simultaneously satisfying (11.14) and (11.16) for given set of τ_{ij} and x_{ij} satisfying (11.13) and (11.2). Otherwise would violate convexity of the h-function[†].

12 Multi-Cross Term Structure and Local Covariations

12.1 Kolmogorov backward and forward equation

Let us use $\mathcal{Z}$ space to parameterize the future state of the market with the logarithmic returns defined via formula (11.16). Any function $\psi : Z \to \mathbb{R}$ representing a value of a self-financing account in units of another self-financing account must satisfy backward Kolmagorov equation of the form

$$\left(\partial_t + \left(\tfrac{1}{2}\sigma_z^{\alpha\beta}\partial_\beta + \mu_z^\alpha\right)\partial_\alpha\right)\psi = 0. \tag{12.1}$$

Here we use Einstein notations with Greek letters $\alpha, \beta = \{1 \ldots, n\}$ indexing $\mathcal{Z}$ coordinates. Accordingly, we use lower indices for covariant vectors/tensors, e.g., derivative ∂_* w.r.t., $\mathcal{Z}$ coordinates, and upper indices for contravariant vectors, e.g., covariation rates $\sigma_z^{\alpha\beta}$ and drifts μ_z^α. As usual,the same letter for a pair of upper and lower indices implies sum. For the purpose which becomes clear later, we shall apply the following equivalent modification

$$\left(\partial_t + \left(\tfrac{1}{2}\tilde{\sigma}_z^{\alpha\beta}\partial_\beta + \mu^\alpha\right)\partial_\alpha\right)\psi = 0, \tag{12.2}$$

[†] Polyhedron is taken to describe $3D$ shapes.

[*] Normally, this map is bijection, except the case of asymptotically flat h corresponding to a bounded implied distribution. Area of definition of h can be also bounded indicates possible annihilation of underlying currencies.

[†] I haven't find a neat prove yet. I hope you can help. Also, the uniqueness is not really necessary, and we can have a consistent co-terminal implied distribution with non-convex h functions

where the $\tilde{\sigma}_z$ symmetric component is equal to σ_z:

$$\tilde{\sigma}_z^{\alpha\beta} + \tilde{\sigma}_z^{\beta\alpha} = 2\sigma_z^{\alpha\beta}. \tag{12.3}$$

We shall also rewrite the equation (12.2) in the form

$$\left(\partial_t + \left(\partial_\beta \tfrac{1}{2}\tilde{\sigma}_z^{\alpha\beta} + \eta_z^\alpha\right)\partial_\alpha\right)\psi = 0, \tag{12.4}$$

where

$$\eta_z^\alpha = \mu_z^\alpha - \tfrac{1}{2}\partial_\beta \tfrac{1}{2}\tilde{\sigma}_z^{\alpha\beta}. \tag{12.5}$$

The prices of Arrow-Debreu options φ (implied density) must satisfy the forward Kolmogorov equation adjoin to the equation (12.4)

$$\left(-\partial_T + \partial_\alpha\left(\tfrac{1}{2}\tilde{\sigma}_z^{\alpha\beta} - \eta_z^\alpha\right)\right)\varphi = 0 \tag{12.6}$$

So far we have not been specific regarding which currency we use as an numeraire, and which $z_i \in \mathcal{Z}$ we use to parameterize the market state. As z_i parametrization are related to each other via parallel shifts (11.14) this choice would not influence partial derivatives w.r.t. z. However, the partial time derivative will be affected. To clarify, let $\partial_{T|i}$ denote partial time derivatives in z_i parametrization. It easy to see that they are related to each other as follows

$$\partial_{T|j} = \partial_{T|i} + \dot{\tau}_{ij}\nabla, \tag{12.7}$$

where

$$\dot{\tau}_{ij} = \frac{d\tau_{ij}}{dT}.$$

If the measure and parametrization are chosen consistently (z_i parametrization with C_i measure) the corresponding density (11.15) is time-independent, and, therefore, the equation (12.6) becomes

$$\left(\partial_\alpha\left(\tfrac{1}{2}\tilde{\sigma}^{\alpha\beta}\partial_\beta - \eta_i^\alpha\right)\right)e^{-h(z_i)} = 0. \tag{12.8}$$

Here η_i denote and η_z vectors in z_i parametrization under C_i measure. Note, that the covariation matrix σ_z is the same for all z parameterizations. We shall assume that skew-symmetric part of $\tilde{\sigma}$ is also chosen to be the same. However, the drift component is affected by both change of parametrization (12.7) and quanto adjustment:

$$\eta_j - \eta_i = \mu_j - \mu_i = \dot{\tau}_{ij} - \sigma_z\nabla\left(h(z_j) - h(z_i)\right), \tag{12.9}$$

or, in the coordinate form,

$$\eta_j^\alpha - \eta_i^\alpha = \dot{\tau}_{ij}^\alpha - \sigma_z^{\alpha\beta}\partial_\alpha\left(h(z_j) - h(z_i)\right). \tag{12.10}$$

To find an appropriate multi-cross local covariance surface corresponding to the european option market in the constant quanto shift specifications, we should solve equations (12.8), (12.9) for σ and μ. In the multi-cross case there can be multiple solutions to this problem. To resolve this ambiguity let us replace the equations (12.8) with the following stronger condition

$$\left(\left(\tfrac{1}{2}\tilde{\sigma}_z^{\alpha\beta}\partial_\beta - \eta_i^\alpha\right)\right) e^{-h(z_i)} = 0.$$

Solving it for η we shall have

$$\eta_i^\alpha = -\tfrac{1}{2}\tilde{\sigma}_z^{\alpha\beta}\partial_\beta h(z_i). \tag{12.11}$$

Substituting this into (12.10) we obtain

$$-\tfrac{1}{2}\tilde{\sigma}_z^{\alpha\beta}\partial_\beta\left(h(z_j) - h(z_i)\right) = \dot{\tau}_{ij}^\alpha - \sigma_z^{\alpha\beta}\left(\partial_\alpha h(z_j) - \partial_\alpha h(z_i)\right),$$

which is equivalent to

$$\tfrac{1}{2}\tilde{\sigma}_z^{\beta\alpha}\partial_\beta\left(h(z_j) - h(z_i)\right) = \dot{\tau}_{ij}^\alpha. \tag{12.12}$$

The above equation are linearly dependent via cyclic relations (11.13). They are equivalent to the n-dimensional independent subset

$$\tfrac{1}{2}\tilde{\sigma}_z^{\beta\alpha}\partial_\beta\left(h(z_i) - h(z_0)\right) = \dot{\tau}_{i0}^\alpha \quad (i = 1, \ldots, n)\,. \tag{12.13}$$

This equation can be easily solved for $\tilde{\sigma}_z$ by inverting the $n \times n$ matrix

$$\partial_\beta\left(h(z_i) - h(z_0)\right).$$

Quadratic variation rates of the logarithmic exchange rates can be also calculated directly from (12.12):

$$dx_{ij}^2 = \partial_\alpha\left(h(z_j) - h(z_i)\right)\tfrac{1}{2}\tilde{\sigma}_z^{\beta\alpha}\partial_\beta\left(h(z_j) - h(z_i)\right) = d\tau_{ij}^\beta\partial_\beta\left(h(z_j) - h(z_i)\right),$$

or, in the coordinate-less form

$$dx_{ij}^2 = \tfrac{1}{2}d\tau_{ij}\nabla\left(h(z_j) - h(z_i)\right). \tag{12.14}$$

This result generalises the single-cross formula (5.18) Note, that covariation matrix obtained from (12.12) is not automatically positive definite. It can be shown to be positive definite in case of similar τ-hedrons:*

$$\dot{\tau}_{ij} = u\tau_{ij} \quad (u > 0)$$

13 Constant Quanto Shift and Forward Skew

We have demonstrated how constant quanto shift approach can be applied to describe co-terminal options. Int this section we shall demonstrate how the methodology can be extended to cover bi-terminal option market. We shall define bi-terminal market, as a combination of vanilla options traded at the

*I wish ◡̈. I can proof positive cross rate variance rates(12.14). The statement seems plausible though.

consecutive expiration $T_1 < T_2$ and any path-depended claims with fixings and payments on those days. Though the scope of such problem still does not comprise a fully complete term-structure cross model, it is larger that the scope of the co-terminal problem. In particular, it would allow valuation of cliquets. For simplicity, we shall assume that both underlying assets/currencies are *value preserving*, i.e., no dividends, borrow cost, interest rates, etc. The assumptions can be trivially extended to deterministic rates and proportional dividends. Under the assumptions, without losing generality we can restrict ourself to the case when all the payments are deferred to the longer expiration T_2. Let $C_\pm$ be a pair of underlying assets/currencies. Let us chose as an numeraire one of the underlying assets, e.g., C_+. Then, under the assumptions the cross rate $X = \frac{C_-}{C_+}$ should be martingale in the corresponding measure, i.e.,

$$E_+[X(T_2)|X(T_1) = X_1] = X_1. \tag{13.1}$$

Let us now assume that both co-terminal distributions can be specified in the constant-quanto-shift form. This means that to the each expiration T_i $(i = 1, 2)$ we can assign a random variable z_{+i}, in a such a way that the its distribution in C_+ measure is the same as the distribution of the shifted variable $z_-(T_i) = z_+(T_i) - 1$ in C_- measure. Let $\rho_i(*)$ be the distribution density function for $z_{i+} \equiv z_+(T_i)$ in the C_+ measure. Under the assumption, this function can be also used for the distribution of z_{i-} in the C_- measure. Therefore, the cross rate can be calculated as

$$X_i = X_0 \frac{\rho_i\left(z_{-i}\right)}{\rho_i\left(z_{+i}\right)} = \frac{\rho_i\left(z_{+i} - 1\right)}{\rho_i\left(z_{+i}\right)}. \tag{13.2}$$

Then, as expected, the martingale conditions for the spot distributions are satisfied automatically. Indeed:

$$\begin{aligned} E_+[X(T_i)] &= \int X_i \rho_i\left(z_{+i}\right) dz_{+i} = \int X_0 \frac{\rho_i\left(z_{+i} - 1\right)}{\rho_i\left(z_{+i}\right)} \rho_i\left(z_{+i}\right) dz_{+i} \\ &= X_0 \int \rho_i\left(z_{+i} - 1\right) dz_{+i} = X_0. \end{aligned} \tag{13.3}$$

However, this is not sufficient to satisfy (13.1). Let us now make an addition assumption on the the joint distribution of z_{+i} $(i = 1, 2)$. Namely, let us assume that the z-increment $\delta z = z_{+2} - z_{+1}$ and the terminal z-value z_{+2} are statistically independent.* Then the join distribution density in the C_+ measure can be represented in the form:

$$\rho_{1,2}\left(z_{+1}, z_{+2}\right) = \rho_2\left(z_{+2}\right) \rho_{1|2}\left(z_{+1} - z_{+2}\right). \tag{13.4}$$

* Note, that this assumption is different from standard Levy processes assumption that consecutive increments are independent. Also, in the context of option pricing the Levy assumption is normally applied to the logarithmic return rather that some abstract z-variable.

In particular,

$$\rho_1(z_1) = \int \rho_2(z_2)\,\rho_{1|2}(z_1 - z_2)\,dz_2. \tag{13.5}$$

Note, that, like in the co-terminal case, the same distribution density function can be applied to join distribution of z_{-i} $(i = 1, 2)$ variables in the C_- measure. This additional condition is sufficient to satisfy the martingale requirement (13.1). Indeed, applying (13.2) to the second expiration we shall have:

$$\begin{aligned}
&E_+\left[X(T_2)|z_+(T_1) = z_{+1}\right]\rho_1(z_{+1}) = \int X_2\rho_2(z_{+2})\,\rho_{1|2}(z_{+1} - z_{+2})\,dz_{+2} \\
&= \int X_0\frac{\rho_2(z_{-2})}{\rho_2(z_{+2})}\rho_2(z_{+2})\,\rho_{1|2}(z_{+1} - z_{+2})\,dz_{+2} \\
&= \int X_0\rho_2(z_{-2})\,\rho_{1|2}(z_{+1} - z_{+2})\,dz_{+2} = \int X_0\rho_2(z_{-2})\,\rho_{1|2}(z_{-1} - z_{-2})\,dz_{-2}.
\end{aligned} \tag{13.6}$$

Finally, applying here (13.2) to the first expiration together with (13.5) we shall obtain the desired result

$$E_+\left[X(T_2)|z_+(T_1) = z_{+1}\right] = X_0\frac{\rho_1(z_{-1})}{\rho_1(z_{+1})} = X(T_1).$$

13.1 Constant Quanto Shift for Two-Term distribution

Here we shall extend the scope of a constant quanto shift framework to multiple expirations. Let us assume that the path of the cross values $X_i = X(T_i)$ $(i = 1, 2)$ can be parametrized by the some z-path: $z_{\pm i} = z_\pm(T_i)$ such that $z_+(T) = z_-(T) + 1$. Similarly to the co-terminal case, we shall assume that the joint distribution of z_{+i} in the C_+ measure is identical to the joint distribution of z_{-i} in the C_- measure. Here, both measures are associated with zero coupon bonds matured at T_2. Let $\rho_{1,2}(z_1, z_2)$ be a joint distribution density function for z_{+i} in the C_+ measure. Under our assumptions it can also serve as a distribution density function for z_{-i} in the C_- measure. Consequently, the cross value at the second expiration can be calculate as follows

$$X_2 = X_0\frac{\rho_{1,2}(z_{-1}, z_{-2})}{\rho_{1,2}(z_{+1}, z_{+2})}. \tag{13.7}$$

The value of the first cross X_1 should depend on the first values of z_1 only. Otherwise would violate causality. Therefore:

$$X_1 = X_0\frac{\rho_1(z_{-1})}{\rho_1(z_{+1})}, \tag{13.8}$$

where

$$\rho_1(z_1) = \int \rho_{1,2}(z_1, z_2)\,dz_2. \tag{13.9}$$

For conditional expectation of X_2 in the C_+ measure we shall have

$$\begin{aligned}
&E_+\left[X(T_2)|z_+(T_1)=z_{+1}\right]\rho_1\left(z_{+1}\right)=\int X_2\rho_{1,2}\left(z_{+1},z_{+2}\right)dz_{+2}\\
&=\int X_0\frac{\rho_{1,2}\left(z_{-1},z_{-2}\right)}{\rho_{1,2}\left(z_{+1},z_{+2}\right)}\rho_{1,2}\left(z_{+1},z_{+2}\right)dz_{+2}\\
&=\int X_0\rho_{1,2}\left(z_{-1},z_{-2}\right)dz_{+2}=\int X_0\rho_{1,2}\left(z_{-1},z_{-2}\right)dz_{-2}\\
&=X_0\rho_1\left(z_{-1}\right).
\end{aligned}\tag{13.10}$$

Applying here (13.8) to we conclude that our assumptions guaranty martingale dynamics:

$$E_+\left[X(T_2)|z_+(T_1)=z_{+1}\right]=X_0\frac{\rho_1\left(z_{-1}\right)}{\rho_1\left(z_{+1}\right)}=X(T_1).$$

The framework can be easily extended to an arbitrary number of expirations.

Note, that under these assumptions only the first co-terminal distribution is explicitly represented in the one-dimensional constant-quanto-shift form. The second term distribution is generally not. If the second co-terminal distribution is also of constant-quanto-shift form, we should be able to introduce a related z-variable. Let us denoted this variable as

$$\tilde{z}_2=\tilde{z}_2\left(z_1,z_2\right).\tag{13.11}$$

The value of this variable must fully determine the second cross value thus defining the map

$$\tilde{z}_2\to X_2=X_2\left(\tilde{z}_2\right).$$

Under the measure change the variable $\tilde{z}_2$ is shifted by 1. Therefore, the map (13.11) must satisfy

$$\tilde{z}_2\left(z_1+1,z_2+1\right)=\tilde{z}_2\left(z_1,z_2\right)+1.\tag{13.12}$$

This means that we can use the following change of bi-terminal z-variables while preserving the model specification

$$\left(z_1,z_2\right)\to\left(z_1,\tilde{z}_2\right).$$

Thus, without losing generality, we can assume that the variable $\tilde{z}_2$ is the same as z_2:

$$\tilde{z}_2\left(z_1,z_2\right)=z_2$$

and, therefore,

$$X_2=X_0\frac{\rho_2\left(z_{-2}\right)}{\rho_2\left(z_{+2}\right)},\tag{13.13}$$

where

$$\rho_2(z_2)=\int\rho_{1,2}\left(z_1,z_2\right)dz_1.\tag{13.14}$$

Comparing (13.13) with (13.7) we shall conclude

$$\rho_{1|2}\left(z_{-1}-z_{-2}, z_{-2}\right)=\rho_{1|2}\left(z_{+1}-z_{+2}, z_{+2}\right). \tag{13.15}$$

Here $\rho_{1|2}$ is a distribution of the increment $z_1 - z_2$ conditional on the terminal value z_2

$$\rho_{1|2}(\Delta z, z_2)=\frac{\rho_{1,2}\left(z_2+\Delta z, z_2\right)}{\rho_2\left(z_{+2}\right)} \tag{13.16}$$

According to (13.15) this conditional distribution must be periodic in the second variable

$$\rho_{1|2}\left(\Delta z, z+1\right)=\rho_{1|2}\left(\Delta z, z\right). \tag{13.17}$$

This result can be interpreted as follows. Terms structure of co-terminal z-distributions can be obtained by solving the reverse time stochastic process with periodic coefficients. The coefficients may include the drift, the local variance and the jump intensities. Note that z-jumps are necessary to incorporate continuous stochastic vol for forward process. These jumps result from averaging and do not necessarily indicate jumps in the actual cross process. In the same time absence of z-jumps would not mean that the cross process is genuine local vol. There is no guaranty, that all the co-terminal z-distributions obtained form solving backward z-process would be log-convex. Then the same cross value might be attributed to a set of different z-values corresponding to distinguishable discrete volatility states.

13.2 Constant Quanto Shift for Muti-Term distribution

The methodology proposed in the previous sections can be easily extended to multi-term distribution with arbitrary expiration schedule T_i $(i = 1, \ldots, N)$. The cross rate process is defined a Radon-Nikodym derivative of the pair of measures $C_\pm$. It is also assumed to be subordinated (not necessarily in Markovian fashion to some z_+ -processes in such a way that the $z+$ dynamics in C_+ measure is identical to $z_- \equiv z_+ - 1$ dynamics in C_- measure. Then the equation (13.7), (13.8) should be generalized as follow

$$X_k = X_0 \frac{\rho_{1,\ldots,k}\left(z_{-1}, \ldots, z_{-k}\right)}{\rho_{1,\ldots,k}\left(z_{+1}, \ldots, z_{+k}\right)}. \tag{13.18}$$

Here $\rho_{1,\ldots,k}$ is a joint distribution density function for $\{z_{+i}\}_{i=1}^{k}$ in the C_+ measure, or, equivalently, a joint distribution density function for $\{z_{-i}\}_{i=1}^{k}$ in the C_- measure. All this distribution obtained from $\rho_{1,\ldots,N}$ via the following chain relation:

$$\rho_{1,\ldots,k-1}(z_1, \ldots, z_{k-1}) = \int \rho_{1,\ldots,z_k}\left(z_1, \ldots, z_k\right) dz_k. \tag{13.19}$$

Similarly to (13.10), for conditional expectation of X_{k+1} in the C_+ measure we shall have

$$
\begin{aligned}
&E_+\left[X(T_k)|\left\{z_+(T_i)=z_{+i}\right\}_{i=1}^{k-1}\right]\rho_{1,\ldots,k}\left(z_{+1},\ldots,z_{+k-1}\right)\\
&=\int X_k\rho_{1,\ldots,k}\left(z_{+1},\ldots z_{+k}\right)dz_{+k}\\
&=\int X_0\frac{\rho_{1,\ldots,k}\left(z_{-1},\ldots z_{-k}\right)}{\rho_{1,\ldots,k}\left(z_{+1},\ldots,z_{+k}\right)}\rho_{1,\ldots,k}\left(z_{+1},\ldots,z_{+k}\right)dz_{+k}\\
&=\int X_0\rho_{1,\ldots,k}\left(z_{-1},\ldots,z_{-k}\right)dz_{+k}=\int X_0\rho_{1,\ldots,k}\left(z_{-1},\ldots,z_{-k}\right)dz_{-k}\\
&=X_0\rho_{1,\ldots,k-1}\left(z_{-1},\ldots,z_{-k-1}\right),
\end{aligned}
$$

and, therefore,

$$
E_+\left[X(T_k)|\left\{z_+(T_i)=z_{+i}\right\}_{i=1}^{k-1}\right]=X_0\frac{\rho_{1,\ldots,k-1}\left(z_{-1},\ldots,z_{-k-1}\right)}{\rho_{1,\ldots,k}\left(z_{+1},\ldots,z_{+k-1}\right)}=X(T_{k-1}).
$$

Thus, we demonstrated that, as expected, the cross process X is, indeed, martingale in the C_+ measure.

However, as we mentioned, the relations between, the z-process and the cross-process X is not Markovian. Namely, X_k may depend on the all prier values of z_i $(i\le N)$:

$$
\begin{aligned}
X_1&=X_1\left(z_{+1}\right),\\
X_2&=X_1\left(z_{+1},z_{+2}\right),\\
&\cdots\\
X_k&=X_1\left(z_{+1},\ldots,z_{+k}\right).
\end{aligned}
$$

To avoid path-dependency, we should require

$$
X_k=X_k\left(z_k\right),
$$

which, due to the relations (13.18), leads to the periodicity requirement for conditional z-distributions:

$$
\rho_{1,\ldots,k-1|k}\left(z_1+1,\ldots,z_k+1\right)=\rho_{1,\ldots,k-1|k}\left(z_1,\ldots,z_k\right). \tag{13.20}
$$

Here

$$
\rho_{1,\ldots,k-1|k}\left(z_1,\ldots,z_k\right)=\frac{\rho_{1,\ldots,k}\left(z_1,\ldots,z_k\right)}{\rho_k\left(z_k\right)}, \tag{13.21}
$$

In particular, integrating both part of the equation over $\{z_1\}_{i=1}^{k-2}$ we shall have

$$
\rho_{k-1|k}\left(z_{k-1}+1,z_k+1\right)=\rho_{k-1|k}\left(z_{k-1},z_k\right). \tag{13.22}
$$

It easy to see that the condition (13.20) are satisfied for z-distributions of the *reverse Markovian* form

$$
\rho_{1,\ldots,k}\left(z_1,\ldots,z_k\right)=\rho_k\left(z_k\right)\rho_{k-1|k}\left(z_{k-1},z_k\right)\cdots\rho_{1|2}\left(z_1,z_2\right), \tag{13.23}
$$

where pair-wise conditional distributions satisfy (13.22) . The class of such solutions contains z-process with reverse independent increments

$$\rho_{1,\ldots,k}\left(z_1,\ldots,z_k\right)=\rho_k\left(z_k\right)\hat{\rho}_{k-1|k}\left(z_{k-1}-z_k\right)\cdots\hat{\rho}_{1|2}\left(z_1-z_2\right). \tag{13.24}$$

In this case

$$\rho_{k-1|k}\left(z_{k-1},z_k\right)=\hat{\rho}_{k-1|k}\left(z_{k-1}-z_k\right).$$

In the Black-Scholes case such z-increment distributions can be shown to be:

$$\begin{aligned}\hat{\rho}_{k-1|k}\left(z\right)&=\sigma\sqrt{\frac{T_{k-1}T_k}{2\pi\left(T_k-T_{k-1}\right)}}\exp\left(-\frac{\sigma^2T_{k-1}T_kz^2}{2\left(T_k-T_{k-1}\right)}\right)\\ \rho_k\left(z\right)&=\sigma\sqrt{\frac{T_k}{2\pi}}\exp\left(-\frac{\sigma^2T_kz^2}{2}\right)\end{aligned} \tag{13.25}$$

Any z-distribution generated by a reverse-time Levy process will do too. The question is can we do something more generic than this?