Price expansion for LSV model

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Fill missing steps of price calculation from "Modelling and Simulation of Stochastic Volatility in Finance" by Christian Kahl

Model

$$dX_t = \sigma_0 h(y_t) f(X_t) dW_t \tag{1}$$
$$dy_t = -k y_t dt + \alpha\sqrt{2k} dZ_t \tag{2}$$

Expansion

$$dX_t = \epsilon\sigma_0 h(\epsilon y_t) f(X_t) dW_t \tag{3}$$

$$C(T,K) = C_0(T,K) + C_1(T,K) + C_2(T,K) \tag{4}$$
$$C_0(T,K) = \epsilon X_0 \sigma_0 \sqrt{T}(\phi(z) - z\Phi(-z)) \tag{5}$$
$$C_1(T,K) = \epsilon^2 X_0 \sigma_0 f_0 k_1 z\phi(z) \tag{6}$$
$$C_2(T,K) = \epsilon^3 X_0 \sigma_0 f_0 \sqrt{T}\times \tag{7}$$

$$\left(\frac{1}{24k^3T^2} e^{-2kT}\left(3\left(2 + kT - 2z^2 + 8e^{kT}\left(-1+z^2\right) + e^{2kT}\left(6 + 2k^2T^2 - 6z^2 + kT\left(-5+4z^2\right)\right)\right) h_2\right.$$

$$+2e^{kT}kT\sigma_0\left(12\left(1 + e^{kT}(-1+kT)\right)z^2 f_1 h_1 + 2e^{kT}k^2T^2\left(-1+z^2\right)f_1^2\sigma_0 + e^{kT}k^2T^2\left(1+2z^2\right)f_0 f_2\sigma_0\right)\Big)\phi(z)$$

$$\left.+\frac{1}{2}\frac{1}{T}\left({k_1}^2\left(z^4 - 2z^2 + 1\right) + {h_1}^2\eta^2{\rho_0}^2 n_{2,2}\left(z\sqrt{T}\right)\right)\right) \tag{8}$$

$$\eta = \alpha\sqrt{2k} \tag{9}$$
$$\rho_0 = \sqrt{1-\rho^2} \tag{10}$$
$$k_1 = h_1\eta\rho n_1 + \frac{1}{2}\sigma_0 f_1 T \tag{11}$$
$$n_1 = \frac{kT - 1 + e^{-kT}}{Tk^2} \tag{12}$$
$$n_{2,2}(b) = \frac{e^{-2Tk}}{4T^2k^3}\left(\left(e^{2Tk}(4Tk-6) + 8e^{Tk} - 2\right)b^2 + T\left(Tk - 8e^{Tk} + e^{2Tk}(Tk(2Tk-5)+6)+2\right)\right) \tag{13}$$

$$h\left(\epsilon y_t\right) = h(0) + h^{(1)}(0)\epsilon y_t + \frac{1}{2}h^{(2)}(0)\left(\epsilon y_t\right)^2 \tag{14}$$

$$h(0) = 1 \tag{15}$$
$$h_i = h^{(i)} \tag{16}$$
$$f_i = f^{(i)} \tag{17}$$

$$X_t = X_0 + \epsilon g_1 + \epsilon^2 g_2 + \epsilon^3 g_3 \tag{18}$$
$$= X_0 + \epsilon\sigma_0 f_0 X_0\sqrt{t}\left(\gamma_1 + \epsilon\gamma_2 + \epsilon^2\gamma_3\right) \tag{19}$$

$$g_1 = \sigma_0 f_0 \int_0^t dW_s \tag{20}$$
$$g_2 = \sigma_0 f_0\left(h_1\int_0^t y_s\, dW_s + \sigma_0 f_1\int_0^t\int_0^s dW_u\, dW_s\right) \tag{21}$$
$$g_3 = \sigma_0 f_0\left(\frac{h_2}{2}\int_0^t {y_s}^2\, dW_s\right. \tag{22}$$
$$+\,\sigma_0 h_1 f_1\left(\int_0^t y_s\int_0^s dW_u\, dW_s + \int_0^t\int_0^s y_u dW_u dW_s\right) \tag{23}$$
$$\left.+{\sigma_0}^2{f_1}^2\int_0^t\int_0^s\int_0^u dW_v dW_u dW_s + {\sigma_0}^2 f_0\frac{f_2}{2}\int_0^t {W_s}^2\, dW_s\right) \tag{24}$$

$$\gamma_i = \frac{1}{\sigma_0 f_0 X_0\sqrt{t}}g_i \tag{25}$$

We assume $X_0 = 1$

$$\gamma_1 = \frac{1}{\sqrt{t}} \int_0^t dW_s = \frac{W_t}{\sqrt{t}} \tag{26}$$

$$\gamma_2 = \frac{1}{\sqrt{t}} \left(h_1 \int_0^t y_s \, dW_s + \sigma_0 f_1 \int_0^t \int_0^s dW_u \, dW_s \right) \tag{27}$$

$$\gamma_3 = \frac{1}{\sqrt{t}} \Bigg(\tag{28}$$

$$\frac{h_2}{2} \int_0^t {y_s}^2 \, dW_s \tag{29}$$

$$+ \sigma_0 h_1 f_1 \left(\int_0^t y_s \int_0^s dW_u \, dW_s + \int_0^t \int_0^s y_u dW_u dW_s \right) \tag{30}$$

$$+{\sigma_0}^2 {f_1}^2 \int_0^t \int_0^s \int_0^u dW_v dW_u dW_s \Bigg) \tag{31}$$

$$+ \frac{1}{2} \frac{1}{\sqrt{t}} {\sigma_0}^2 f_0 f_2 \int_0^t {W_s}^2 \, dW_s \tag{32}$$

for γ_3 difference with pure stochastic volatility case is the last extra term

$$\frac{1}{2} \frac{1}{\sqrt{t}} {\sigma_0}^2 f_0 f_2 \int_0^t {W_s}^2 \, dW_s \tag{33}$$

Price formula

$$C(T, K) = E \left(X_0 + \epsilon \sigma_0 f_0 X_0 \sqrt{t} \left(\gamma_1 + \epsilon \gamma_2 + \epsilon^2 \gamma_3 \right) - K \right)_+ \tag{34}$$

$$= \epsilon \sigma_0 f_0 X_0 \sqrt{t} E \left(\gamma_1 + \epsilon \gamma_2 + \epsilon^2 \gamma_3 - z \right)_+ \tag{35}$$

$$= \epsilon \sigma_0 f_0 X_0 \sqrt{t} E H \left(\gamma_1 + \epsilon \gamma_2 + \epsilon^2 \gamma_3 - z \right) \tag{36}$$

$$z = \frac{k - X_0}{X_0 \epsilon \sigma \sqrt{t}} \tag{37}$$

$$C(T, K) = \epsilon \sigma_0 f_0 X_0 \sqrt{t} E H \left(\gamma_1 - z + \epsilon \gamma_2 + \epsilon^2 \gamma_3 \right) \tag{38}$$

$$= \epsilon \sigma_0 f_0 X_0 \sqrt{t} E \left(H_0 + \epsilon \gamma_2 H_1 + \epsilon^2 \left(H_1 \gamma_3 + \frac{1}{2} H_2 {\gamma_2}^2 \right) \right) \tag{39}$$

$$H_i = H^{(i)}(\gamma_1 - z) \tag{40}$$
$$H_0(x) = (x-z)_+ \tag{41}$$
$$H_1(x) = 1[x-z>0] \tag{42}$$
$$H_2(x) = \delta[x-z] \tag{43}$$

$$C(T,K) = \epsilon\sigma_0 f_0 X_0 \sqrt{t}\left(EH_0 + \epsilon E\left(\mathrm{E}\gamma_2 H_1|\gamma_1\right) + \epsilon^2 E\left(E\left(H_1\gamma_3 + \frac{1}{2}H_2{\gamma_2}^2|\gamma_1\right)\right)\right) \tag{44}$$

Conditional Expectations

Conditional expectation of γ_2

for SV (Stochastic Volatility) case formula 4.91

$$\gamma_2 = \frac{1}{\sqrt{t}}\left(h_1\int_0^t y_s\,dW_s + \sigma_0\int_0^t\int_0^s dW_u\,dW_s\right) \tag{45}$$

$$\eta = \alpha\sqrt{2k} \tag{46}$$

$$E\left(\gamma_2(T)|\gamma_1\right) = \frac{1}{\sqrt{T}}k_1\left({\gamma_1}^2 - 1\right) \tag{47}$$

$$k_1 = h_1\eta\rho n_1 + \frac{1}{2}\sigma_0 T \tag{48}$$
$$n_1 = \frac{kT - 1 + e^{-kT}}{Tk^2} \tag{49}$$
$$\gamma_1(T) = \frac{W_T}{\sqrt{T}} \tag{50}$$

In LSV case almost the same

$$\gamma_2 = \frac{1}{\sqrt{t}}\left(h_1\int_0^t y_s\,dW_s + \sigma_0 f_1\int_0^t\int_0^s dW_u\,dW_s\right) \tag{51}$$

$$E\left(\gamma_2(T)|\gamma_1\right) = \frac{1}{\sqrt{T}}k_1\left({\gamma_1}^2 - 1\right) \tag{52}$$

$$k_1 = h_1\eta\rho n_1 + \frac{1}{2}\sigma_0 f_1 T \tag{53}$$

Conditional expectation of $\gamma_2{}^2$

for SV case formula 4.99

$$\rho_0 = \sqrt{1-\rho^2} \tag{54}$$
$$\tag{55}$$
$$E\left(\gamma_2{}^2(T)|\gamma_1\right) = \frac{1}{T}\left(k_1{}^2\left(\gamma_1{}^4 - 2\gamma_1{}^2 + 1\right) + h_1{}^2\eta^2\rho_0{}^2 n_{2,2}\left(\gamma_1\sqrt{T}\right)\right) \tag{56}$$

$$n_{2,2}(b) = \frac{e^{-2Tk}}{4T^2k^3}\left(\left(e^{2Tk}(4Tk-6) + 8e^{Tk} - 2\right)b^2 + T\left(Tk - 8e^{Tk} + e^{2Tk}(Tk(2Tk-5)+6)+2\right)\right) \tag{57}$$

In generic case is the same just f_1 enters in k_1 expression for k_2 is the same

$$E\left(\gamma_2{}^2(T)|\gamma_1\right) = \frac{1}{T}k_2 \tag{58}$$
$$\tag{59}$$
$$k_2 = k_1{}^2\left(\gamma_1{}^4 - 2\gamma_1{}^2 + 1\right) + h_1{}^2\eta^2\rho_0{}^2 n_{2,2}\left(\gamma_1\sqrt{T}\right) \tag{60}$$

Conditional expectation of γ_3

$$\gamma_3 = \frac{1}{\sqrt{t}}\left(\frac{h_2}{2}\int_0^t y_s{}^2\, dW_s \right. \tag{61}$$
$$+ \sigma_0 h_1 f_1\left(\int_0^t y_s \int_0^s dW_u\, dW_s + \int_0^t \int_0^s y_u dW_u dW_s\right) \tag{62}$$
$$\left. +\sigma_0{}^2 f_1{}^2 \int_0^t \int_0^s \int_0^u dW_v dW_u dW_s\right) + \frac{1}{\sqrt{t}}\sigma_0{}^2 f_0 \frac{f_2}{2}\int_0^t W_s{}^2\, dW_s \tag{63}$$

the first part is almost the same as in pure SV case

$$M_1 = \int_0^t y_s{}^2\, dW_s \tag{64}$$
$$M_2 = \int_0^t y_s \int_0^s dW_u\, dW_s \tag{65}$$
$$M_3 = \int_0^t \int_0^s y_u dW_u dW_s \tag{66}$$
$$M_4 = I_{1,1,1} \tag{67}$$
$$M_5 = \int_0^t W_s{}^2\, dW_s \tag{68}$$

$$\gamma_3 = \frac{1}{\sqrt{t}}\left(\frac{h_2}{2}M_1 + \sigma_0 h_1 f_1\left(M_2 + M_3\right) + {\sigma_0}^2 {f_1}^2 M_4\right) + \frac{1}{\sqrt{t}}{\sigma_0}^2 f_0 \frac{f_2}{2}M_5 \tag{69}$$

$$m_i(b) = E\left(M_i | W_T = b\right) \tag{70}$$

Lemma 4.21

$$m_1(b) = \frac{1}{4T^3k^3}\left(be^{-2Tk}\left(-2b^2 + 8e^{Tk}\left(b^2 - 3T\right) + 6T + T^2k + e^{2Tk}\left((4Tk-6)b^2 + T(Tk-2)(2Tk-9)\right)\right)\right) \tag{71}$$

Lemma 4.22

$$m_2(b) = \frac{1}{2T^3k^3}\left(be^{-Tk}\left(b^2\left(2Tk + e^{Tk}\left(T^2k^2 - 2\right) + 2\right) - T\left(4Tk + e^{Tk}(Tk(Tk+2) - 6) + 6\right)\right)\right) \tag{72}$$

Lemma 4.23

$$m_3(b) = \frac{1}{2T^3k^3}b\left(b^2 - 3T\right)\left(Tk(Tk-2) - 2e^{-Tk} + 2\right) \tag{73}$$

Corollary 4.16

$$m_4(b) = E\left(M_4 | W_T\right) \qquad = \frac{1}{6}b\left(b^2 - 3T\right) \tag{74}$$

$$m_5(b) = \frac{1}{3}b^3 - \frac{1}{2}bT \tag{75}$$

Price valuation

$$C(T,K) = C_0(T,K) + C_1(T,K) + C_2(T,K) \tag{76}$$

$$C_0(T,K) = \epsilon\sigma_0 f_0 X_0 \sqrt{t} E H_0 \tag{77}$$

$$C_1(T,K) = \epsilon\sigma_0 f_0 X_0 \sqrt{t}\epsilon E\left(\mathrm{E}\gamma_2 H_1 | \gamma_1\right) \tag{78}$$

$$C_2(T,K) = \epsilon\sigma_0 f_0 X_0 \sqrt{t}\epsilon^2 E\left(E\left(H_1\gamma_3 + \frac{1}{2}H_2{\gamma_2}^2 | \gamma_1\right)\right) \tag{79}$$

$$C_0(T,K) = \epsilon X_0 \sigma_0 \sqrt{T}(\phi(z) - z\Phi(-z)) \tag{80}$$

$$C_1(T,K) = \epsilon^2 X_0 \sigma_0 f_0 k_1 z\phi(z) \tag{81}$$

Valuation C_1

$$\gamma_1 = \frac{W_T}{\sqrt{T}} \tag{82}$$

$$E\left(\gamma_2(T)|\gamma_1\right) = \frac{1}{\sqrt{T}} k_1 \left({\gamma_1}^2 - 1\right) \tag{83}$$

$$\mathrm{E}1\left[\frac{W_T}{\sqrt{T}} - z > 0\right]\gamma_2(T) = \frac{1}{\sqrt{T}} k_1 E1\left[\frac{W_T}{\sqrt{T}} - z > 0\right]\left(\left(\frac{W_T}{\sqrt{T}}\right)^2 - 1\right) \tag{84}$$

$$= \frac{1}{\sqrt{T}} k_1 E1\left[W_1 - z > 0\right]\left({W_1}^2 - 1\right) \tag{85}$$

this type can be calculated using properties of Hermit polynomials

$$\frac{1}{2}\left(W^2 - 1\right) = I_{(1,1)} \quad = \int_0^1 \int_0^{t_1} dW_{t_2}\, dW_{t_1} = \frac{1}{2}\mathcal{H}_2(W) \quad = \frac{1}{2}\frac{d_{x,x}\phi(W)}{\phi(W)} \tag{86}$$

$$E1\left[W_1 - z > 0\right]\left({W_1}^2 - 1\right) = \int_z^\infty \frac{\phi_{x,x}(x)}{\phi(x)}\phi(x)dx \quad = -\phi'(z) = z\phi(z) \tag{87}$$

Valuation C_2

$$1\left[\frac{W_T}{\sqrt{T}} - z > 0\right]\gamma_3(T) + \frac{1}{2}\delta\left(\frac{W_T}{\sqrt{T}} - z\right){\gamma_2}^2(T) = \tag{88}$$

$$\frac{e^{-2kT}\left(2 + kT - 2z^2 + 4e^{kT}\left(-2 + (2 + kT)z^2\right) + e^{2kT}\left(6 - 5kT - 6z^2 + 2k^3T^3z^2 + 2k^2T^2\left(1 + 2z^2\right)\right)\right)}{4k^3\ T^{3/2}}\phi(z) \tag{89}$$

$$+ \frac{1}{2}\frac{1}{T}\left({k_1}^2\left(z^4 - 2z^2 + 1\right) + {h_1}^2\eta^2{\rho_0}^2 n_{2,2}\left(z\sqrt{T}\right)\right) \tag{90}$$

$$\tag{91}$$

$$\begin{aligned}
\eta &= \alpha\sqrt{2k} \\
\rho_0 &= \sqrt{1 - \rho^2} \\
k_1 &= h_1\eta\rho n_1 + \frac{1}{2}\sigma_0 f_1 T \\
k_2 &= {k_1}^2\left({\gamma_1}^4 - 2{\gamma_1}^2 + 1\right) + {h_1}^2\eta^2{\rho_0}^2 n_{2,2}\left(\gamma_1\sqrt{T}\right) \\
n_{2,2}(b) &= \frac{e^{-2Tk}}{4T^2k^3}\left(\left(e^{2Tk}(4Tk - 6) + 8e^{Tk} - 2\right)b^2 + T\left(Tk - 8e^{Tk} + e^{2Tk}(Tk(2Tk - 5) + 6) + 2\right)\right)
\end{aligned}$$

$$E1\left[\frac{W_T}{\sqrt{T}}-z>0\right]\gamma_3(T)= \tag{92}$$

$$\frac{e^{-2kT}\left(2+kT-2z^2+4e^{kT}\left(-2+(2+kT)z^2\right)+e^{2kT}\left(6-5kT-6z^2+2k^3T^3z^2+2k^2T^2\left(1+2z^2\right)\right)\right)}{4k^3\ T^{3/2}}\phi(z) \tag{93}$$

$$E\frac{1}{2}\delta\left(\frac{W_T}{\sqrt{T}}-z\right)\gamma_2{}^2(T)=\frac{1}{2}E\left(\gamma_2{}^2(T)|\gamma_1=z\right)\quad=\frac{1}{2}\frac{1}{T}\left(k_1{}^2\left(z^4-2z^2+1\right)+h_1{}^2\eta^2\rho_0{}^2n_{2,2}\left(z\sqrt{T}\right)\right) \tag{94}$$

Valuation γ_3

$$E\left(\gamma_3|W_T\right)=\frac{1}{\sqrt{t}}\left(\frac{h_2}{2}M_1+\sigma_0h_1f_1\left(M_2+M_3\right)+\sigma_0{}^2f_1{}^2M_4\right)+\frac{1}{\sqrt{t}}\sigma_0{}^2f_0\frac{f_2}{2}M_5 \tag{95}$$

Can be valuated using Hermit polynomials. But easier by brute force of Mathematica

$\mathbf{m1[b\_]:=\frac{1}{4T^3k^3}\left(be^{-2Tk}\left(-2b^2+8e^{Tk}\left(b^2-3T\right)+6T+T^2k+e^{2Tk}\left((4Tk-6)b^2+T(Tk-2)(2Tk-9)\right)\right)\right);}$

$\mathbf{m2[b\_]:=\frac{1}{2T^3k^3}\left(be^{-Tk}\left(b^2\left(2Tk+e^{Tk}\left(T^2k^2-2\right)+2\right)-T\left(4Tk+e^{Tk}(Tk(Tk+2)-6)+6\right)\right)\right)}$

$\mathbf{m3[b\_]:=\frac{1}{2T^3k^3}b\left(b^2-3T\right)\left(Tk(Tk-2)-2e^{-Tk}+2\right);}$

$\mathbf{m4[b\_]:=\frac{1}{6}b\left(b^2-3T\right);}$

$\mathbf{m5[b\_]:=\frac{1}{3}b^3-\frac{1}{2}bT;}$

$\mathbf{\frac{h_2}{2}\frac{1}{\sqrt{2\pi}}\int_z^\infty m1\left[x\sqrt{T}\right]e^{-\frac{x^2}{2}}dx+\sigma_0h_1f_1\left(\frac{1}{\sqrt{2\pi}}\int_z^\infty m2\left[x\sqrt{T}\right]e^{-\frac{x^2}{2}}dx+\frac{1}{\sqrt{2\pi}}\int_z^\infty m3\left[x\sqrt{T}\right]e^{-\frac{x^2}{2}}dx\right)+\sigma_0{}^2f_1{}^2\frac{1}{\sqrt{2}}}$
$\mathbf{\sigma_0{}^2f_0\frac{f_2}{2}\frac{1}{\sqrt{2\pi}}\int_z^\infty m5\left[x\sqrt{T}\right]e^{-\frac{x^2}{2}}dx//Simplify}$

$\frac{1}{24k^3\sqrt{2\pi}T^{3/2}}e^{-2kT-\frac{z^2}{2}}\left(3\left(2+kT-2z^2+8e^{kT}\left(-1+z^2\right)+e^{2kT}\left(6+2k^2T^2-6z^2+kT\left(-5+4z^2\right)\right)\right)h_2+2e^{kT}\right.$

Valuation $\gamma_2{}^2(T)$

$$E\frac{1}{2}\delta\left(\frac{W_T}{\sqrt{T}}-z\right)\gamma_2{}^2(T)=\frac{1}{2}E\left(\gamma_2{}^2(T)|\gamma_1=z\right) \tag{96}$$

$$=\frac{1}{T}\left(k_1{}^2\left(z^4-2z^2+1\right)+h_1{}^2\eta^2\rho_0{}^2n_{2,2}\left(z\sqrt{T}\right)\right) \tag{97}$$

$$E\left(\gamma_2{}^2(T)|\gamma_1\right) = \frac{1}{T}\left(k_1{}^2\left(\gamma_1{}^4 - 2\gamma_1{}^2 + 1\right) + h_1{}^2\eta^2\rho_0{}^2 n_{2,2}\left(\gamma_1\sqrt{T}\right)\right)$$