Analytic Pricing of Options on Compounded Rates

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1 author:

Colin Turfus
Quantitative Finance

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Colin Turfus

Quantitative Analyst,
Model Risk Management,
Deutsche Bank, London

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Overview of the presentation

Definition of Hull-White model

Under the Hull-White model the short rate r_t can be expressed as

$$\begin{aligned} r_t &= \bar{r}(t) + r^*(t) + x_t \\ &=: r(x_t, t), \end{aligned} \tag{1}$$

where the auxiliary variable x_t satisfies the following canonical Ornstein–Uhlenbeck process:

$$dx_t = -\alpha_r(t) x_t dt + \sigma_r(t) dW_t \tag{2}$$

and $r^*(t)$ is a convexity adjustment function to be determined by calibration to fit the instantaneous forward curve given by $\bar{r}(t)$.

Hull-White pricing equation

Let the time-t price of a derivative paying $P(x_T)$ at time $T > t$ be denoted

$$f(x_t, t) = E\left[e^{-\int_t^T r_u du} P(x_T) \middle| \mathcal{F}_t \right] \tag{3}$$

From the Feynman-Kac theorem, $f(x, t)$ emerges as the solution to the following Kolmogorov backward diffusion equation:

$$\frac{\partial f}{\partial t} - \alpha_r(t) x \frac{\partial f}{\partial x} + \frac{1}{2}\sigma_r^2(t) \frac{\partial^2 f}{\partial x^2} - r(x, t) f = 0, \tag{4}$$

for $t \geq 0$, with $f(x, T) = P(x)$.

Notation for Hull-White pricing

We can write the zero coupon bond price under this model

$$F^T(x,t) = D(t,T)e^{-\mu^*(x,t,T)} \tag{5}$$

where we define

$$D(t_1,t_2) := e^{-\int_{t_1}^{t_2} \bar{r}(t)dt} \tag{6}$$

$$\phi_r(t,T) = e^{-\int_t^T \alpha_r(u)du}, \tag{7}$$

$$B^*(t,T) = \int_t^T \phi_r(t,u)du, \tag{8}$$

$$\Sigma_r(t,T) = \int_t^T \phi_r^2(u,T)\sigma_r^2(u)du, \tag{9}$$

$$I^*(t,T) = \int_t^T \phi_r(u,T)\Sigma_r(t,u)du, \tag{10}$$

$$\mu^*(x,t,T) = B^*(t,T)(x+r^*(t)) + \tfrac{1}{2}B^{*2}(t,T)\Sigma_r(0,t) \tag{11}$$

Hull-White pricing kernel

From [Turfus (2019)], the Hull-White pricing kernel can be written:

$$G(x,t;\xi,T) = F^T(x,t)N(\xi + I^*(t,T) - x\phi_r(t,T); \Sigma_r(t,T)), \tag{12}$$

where

$$N(x;\Sigma) := \frac{e^{-\frac{x^2}{2\Sigma}}}{\sqrt{2\pi\Sigma}}.$$

Derivative prices are obtained from

$$f(x,t) = \int_{\mathbb{R}} P(\xi)G(x,t;\xi,T)d\xi,$$

with PV resulting from setting $x = t = 0$.

Model extension for compounded rates

The payoff at T_2 for compounded interest over $[T_1, T_2]$ can conveniently be modelled as

$$P = e^{\int_{T_1}^{T_2} r(x_t,t)dt} - 1.$$

Problem: P is now path-dependent. So we introduce a new integrated variable z_t defined by

$$z_t = \int_0^t (r^*(s) + x_s)\, ds, \tag{13}$$

in terms of which we can write the payoff at T_2 as

$$P(z_{T_1}, z_{T_2}) = D(T_1, T_2)^{-1} e^{z_{T_2} - z_{T_1}} - 1. \tag{14}$$

We are thus led to consider derivative contracts whose payoff at time T has the general form $P(x_T, z_T)$.

Extended Hull-White pricing equation

The relevant pricing equation for derivative prices $f(x, z, t)$ is in this case:

$$\frac{\partial f}{\partial t} - \alpha_r(t) x \frac{\partial f}{\partial x} + (r^*(t) + x) \frac{\partial f}{\partial z} + \frac{1}{2}\sigma_r^2(t) \frac{\partial^2 f}{\partial x^2} - r(x, t) f = 0, \quad (15)$$

for $t \geq 0$, with $f(x, z, T) = P(x, z)$.

The two-factor pricing kernel for this problem is implicitly addressed through the derivation by [Turfus (2018b)] of a rates-equity kernel (setting the equity volatility to zero).

This result was originally derived by [Van Steenkiste and Foresi(1999)].

Hull-White pricing kernel for RFR

We infer the Hull-White pricing kernel for compounded backward-looking rates can be written:

$$G(x,z,t;\xi,\zeta,T) = F^T(x,t)N_2(\xi + I^*(t,T) - x\phi_r(t,T), \\ \zeta + \tfrac{1}{2}K^*(t,T) - \mu^*(x,t,T) - z; \mathbf{\Sigma}^+(t,T)), \quad (16)$$

with $N_2(\cdot,\cdot;\mathbf{\Sigma})$ a bivariate Gaussian with covariance $\mathbf{\Sigma}$ and

$$K^*(t,T) = 2\int_t^T I^*(t,u)du, \quad (17)$$

$$\mathbf{\Sigma}^+(t,T) = \begin{pmatrix} \Sigma_r(t,T) & I^*(t,T) \\ I^*(t,T) & K^*(t,T) \end{pmatrix}. \quad (18)$$

In this case, derivatives prices are obtained from

$$f(x,z,t) = \iint_{\mathbb{R}^2} P(\xi,\zeta)G(x,z,t;\xi,\zeta,T)d\xi d\zeta.$$

Caplet payoff for compounded rates - I

The caplet payoff at time T_2 with strike K for rates compounded over $[T_1, T_2]$ can be written

$$P_{\text{caplet}}(z_1, z_2) = \left[D(T_1, T_2)^{-1} e^{z_{T_2} - z_{T_1}} - \kappa^{-1}\right]^+, \qquad (19)$$

where we define $\kappa = (1 + K\delta(T_1, T_2))^{-1}$. Applying (16), we can price $P_{\text{caplet}}(\cdot)$ as of time T_1:

$$\begin{aligned} f(x, z, T_1) &= \iint_{\mathbb{R}^2} P_{\text{caplet}}(z, \zeta) G(x, z, T_1; \xi, \zeta, T_2) d\xi d\zeta \\ &= \Phi\left(-\hat{d}_2(x)\right) - \kappa^{-1} F^{T_2}(x, T_1) \Phi\left(-\hat{d}_1(x)\right), \end{aligned}$$

with $\Phi(\cdot)$ a Gaussian cumulative distribution function. [Note how the z-dependence drops out.]

Caplet payoff for compounded rates - II

Here

$$\hat{d}_2(x) := \frac{\Delta z^* - \mu^*(x, T_1, T_2) - \frac{1}{2}K^*(T_1, T_2)}{\sqrt{K^*(T_1, T_2)}}, \tag{20}$$

$$\hat{d}_1(x) := \hat{d}_2(x, t) + \sqrt{K^*(T_1, T_2)}. \tag{21}$$

with Δz^* the critical value of $z_{T_2} - z_{T_1}$, namely

$$\Delta z^* := \ln\left(\kappa^{-1} D(T_1, T_2)\right). \tag{22}$$

Caplet price for compounded rates

Taking the T_1-price as a payoff at time T_1, we can price this as of $t = 0$ by applying $G(0,0,0;\xi,\zeta,T_1)$, to obtain

$$PV_{\text{caplet}} = D(0,T_1)\Phi(-d_2) - \kappa^{-1}D(0,T_2)\Phi(-d_1), \tag{23}$$

where

$$d_2 = \frac{\Delta z^* - \frac{1}{2}\left(B^{*2}(T_1,T_2)\Sigma_r(0,T_1) + K^*(T_1,T_2)\right)}{\sqrt{B^{*2}(T_1,T_2)\Sigma_r(0,T_1) + K^*(T_1,T_2)}}, \tag{24}$$

$$d_1 = d_2 + \sqrt{B^{*2}(T_1,T_2)\Sigma_r(0,T_1) + K^*(T_1,T_2)}. \tag{25}$$

We see that setting the convexity adjustment $K^*(T_1,T_2) \equiv 0$ recovers the well-known Hull-White caplet formula.

Adjustment for daily compounding

If instead the compounding is daily with fixing on dates $t_0, t_1, \ldots, t_{n-1}$, following [Henrard(2007)], we must replace

$$K^*(T_1, T_2) \longrightarrow \sum_{i=0}^{n-1} (B^{*2}(t_i, t_n) - B^{*2}(t_i, t_{n-1}))\Sigma_r(t_{i-1}, t_i),$$

which is effectively a quadrature formula for, so very close in value to, the continuously compounded value.

Average rate caplet price

If instead of compounding the daily rates an average is used, we obtain a Bachelier formula:

$$PV_{\text{caplet}} = D(0,T_2)\sqrt{B^{*2}(T_1,T_2)\Sigma_r(t,T_1) + K^*(T_1,T_2)} \left(N(-d_0) - d_0\Phi(-d_0)\right), \quad (26)$$

where

$$d_0 := \frac{\Delta z_B^* - K^*(T_1,T_2)}{\sqrt{B^{*2}(T_1,T_2)\Sigma_r(0,T_1) + K^*(T_1,T_2)}}$$

and the critical value of $z_{T_2} - z_{T_1}$ is now redefined to be

$$\Delta z_B^* = K\delta(T_1,T_2) - \int_{T_1}^{T_2} \overline{r}(t_1)dt_1. \quad (27)$$

Definition of Black-Karasinski model

Under the Black-Karasinski model the short rate r_t can be expressed as

$$r_t = \tilde{r}(t) e^{x_t - \frac{1}{2}\Sigma_r(0,t)} \\ =: r(x_t, t), \tag{28}$$

where the auxiliary variable x_t satisfies the Ornstein–Uhlenbeck process (2) and $\tilde{r}(t)$ is to be determined by calibration to fit the instantaneous forward curve given by $\bar{r}(t)$.

The pricing equation is given by (4) as for the Hull-White case, subject to the redefinition of $r(x, t)$.

We offer a sketch proof of the B-K caplet price derivation. For more details, see [Turfus (2020)].

Black-Karasinski pricing kernel

We follow [Turfus (2018a)] and [Turfus (2020)] in defining $\epsilon = \|\tilde{r}(\cdot)\|$ for some suitable norm and writing the corresponding pricing kernel in the limit as $\epsilon \to 0$ as

$$G(x, z, t; \xi, \zeta, T) = D(t, T)e^{-F_1(x,t,T)(1-\partial_z)}G_0(x, z, t; \xi, \zeta, T) \\ + D(t, T)\sum_{n=1}^{\infty}(-1)^n (1-\partial_z)^n G_n(x, z, t; \xi, \zeta, T), \quad (29)$$

with $G_n(\cdot) = \mathcal{O}(\epsilon^n)$,

$$G_0(x, z, t; \xi, \zeta, T) := \frac{1}{\sqrt{\Sigma_r(t, T)}} N\left(\frac{\xi - \phi_r(t, T)x}{\sqrt{\Sigma_r(t, T)}}\right)\delta(\zeta - z)$$

$$F_1(x, T_1, T_2) := \int_{T_1}^{T_2}(R(x, T_1, t_1) - \bar{r}(t_1))dt_1,$$

and $R(x, t_1, t_2) = \mathcal{O}(\epsilon)$ defined as in [Turfus (2018a)].

Black-Karasinski caplet payoff

As in the Hull-White case we have

$$V_{\mathrm{caplet}}(x, T_1) = \int_{z+\Delta z^*}^{\infty} \int_{\infty}^{\infty} P_{\mathrm{caplet}}(z, \zeta) \\ G(x, z, T_1; \xi, \zeta, T_2) d\xi d\zeta. \qquad (30)$$

Problem: the leading order z-dependence is singular (a step function)!

Work-around: noting that

$$e^{a\frac{\partial}{\partial x}} f(x) = f(x+a),$$

we can write

$$e^{-F_1(x,T_1,T_2)(1-\partial_z)} G_0(x, z, T_1; \xi, \zeta, T_2) = \\ e^{-F_1(x,T_1,T_2)} G_0(x, z + F_1(x, T_1, T_2), T_1; \xi, \zeta, T_2).$$

Composite marginal kernel

To obtain $V_{\text{caplet}}(x, T_1)$, a similar shift is needed in the *variance* of z by

$$F_2(x, T_1, T_2) := \int_{T_1}^{T_2} R(x, T_1, t_2) \int_{T_1}^{t_2} R(x, T_1, t_1) \left(e^{\phi_r(t_1,t_2)\Sigma_r(T_1,t_1)} - 1\right) dt_1 dt_2$$

This can be incorporated into a composite second-order accurate expansion for the marginal

$$\int_{\mathbb{R}} G(x, z, T_1; \xi, \zeta, T_2) d\xi.$$

Problem: this adjustment is not deterministic, but depends on the state variable x!

Small volatility approximation

For tractability, approximate $F_2(x, T_1, T_2) \approx 2K^*_{\text{B-K}}(T_1, T_2)$ with

$$K^*_{\text{B-K}}(T_1, T_2) := 2\int_{T_1}^{T_2} \bar{r}(t_2)\int_{T_1}^{t_2} \bar{r}(t_1)\left(e^{\phi_r(t_1,t_2)\Sigma_r(T_1,t_1)} - 1\right) e^{\Delta x^*(T_1,t_1,t_2)} dt_1 dt_2, \quad (31)$$

with

$$\Delta x^*(T_1, t_1, t_2) := \phi_r(T_1, t_1)\phi_r(T_1, t_2)\Sigma_r(0, T_1). \quad (32)$$

and adjust by expanding the result in a Taylor series w.r.t. $F_2(x, T_1, T_2) - \frac{1}{2}K^*_{\text{B-K}}(T_1, T_2)$.

Caplet price

In this way the caplet price can be obtained as

$$PV_{\text{caplet}} = PV^{(0)}_{\text{caplet}} + \Delta PV_{\text{caplet}}, \tag{33}$$

where $PV^{(0)}_{\text{caplet}}$ is essentially the LIBOR caplet price (see [Turfus (2020)]) with the substitution

$$\Sigma_r(0, T_1) \to \Sigma_r(0, T_1) + K^*_{\text{B-K}}(T_1, T_2),$$

and

$$\begin{aligned}\Delta PV_{\text{caplet}} \sim{} & \\ & \frac{2D(0,T_1)}{K^*_{\text{B-K}}(T_1,T_2)} \int_{T_1}^{T_2} \bar{r}(t_2) \int_{T_1}^{t_2} \bar{r}(t_1) \left(e^{\phi_r(t_1,t_2)\Sigma_r(T_1,t_1)} - 1 \right) e^{\Delta x^*(T_1,t_1,t_2)} \\ & \qquad (N(-d_2(x^* - \Delta x^*(T_1,t_1,t_2), 0, T_1)) - N(-d_2(x^*, 0, T_1)))\, dt_1 dt_2\end{aligned}$$

with errors $= \mathcal{O}(\Sigma_r^2(T_1, T_2))$.

Conclusions

1. Backward-looking rates impact on caplet pricing through a small positive adjustment to the effective term variance: $K^*(T_1, T_2)$ in the case of Hull-White and $K^*_{\text{B-K}}(T_1, T_2)$ for Black-Karasinski.
2. The main difference between the two models is in the appearance of the Black-Karasinski adjustment term $\Delta PV_{\text{caplet}}$. The effect of this is to increase the value of OTM caplets and of ITM floorlets.
   [Intuitively, why should this be so?]

Thank you!

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