Interest Rate Models Enhanced with Local Volatility

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Lingling Cao
École des Ponts ParisTech

INTEREST RATE MODELS ENHANCED WITH LOCAL VOLATILITY

LINGLING CAO AND PIERRE HENRY-LABORDÈRE

Abstract. In this paper, we complement generic interest rate models with a local volatility. We derive an *exact* Dupire-like formula for the local volatility. An efficient calibration scheme is then achieved with the particle method as introduced in [5].

1. Introduction

Dupire's local volatility model, widely used in equity markets, has the property to be perfectly calibrated to Vanillas. In fixed income markets, models with a similar property are not available.

Recently, the authors in [4] have considered a one-dimensional Cheyette model enhanced with a local volatility and have derived an (approximate) Dupire-like local volatility formula for swaptions (see also [3] where a quadratic parametrisation of the local volatility is provided). A similar approach is performed in [9] for the case of a Libor market model enhanced with a local volatility. Markov functional models [8], which allows to calibrate exactly a strip of swaptions, are difficult to generalize with many factors.

In this paper, we pursue the approach started in [4] and explain how to enhance generic multi-factor interest rate models with a local volatility. Without relying on some (model-dependent) approximations as in [4], we derive a general equation which implies that a generic interest rate model is calibrated to a strip of rolling maturity swaptions. As an example, we specify this equation to the one-dimensional Cheyette model. Extension to a multi-dimensional Cheyette model or a Libor market model is straightforward, and is briefly sketched. The calibration equation is then solved numerically with a McKean particle method as introduced in [5] (see e.g. [6] for a lengthly exposition). We illustrate the efficiency of our algorithm on various numerical examples.

2. Matching a rolling maturity swaption

We consider a swap $s_t^{\alpha,\beta} \equiv \frac{P_{t,T_\alpha} - P_{t,T_\beta}}{N_t^{\alpha,\beta}}$ between maturities T_α and T_β. We denote $P_{t,T} \equiv \mathbb{E}_t^{\mathbb{Q}}[e^{-\int_t^T r_s ds}]$ the t-value of a bond with maturity T and $N_t^{\alpha,\beta} \equiv \int_{T_\alpha}^{T_\beta} P_{t,u} du$. $\mathbb{Q}$ denotes the spot measure, associated to the bank account numéraire $B_t \equiv e^{\int_0^t r_s ds}$. Note that we have used a continuous definition for the coupon-bearing bond $N_t^{\alpha,\beta}$ for the sake of simplicity. This can be easily replaced by a discrete definition $N_t^{\alpha,\beta} = \sum_{T_\alpha}^{T_\beta} P_{t,u}\delta$ in our computations below.

Under the swap measure $\mathbb{Q}^{\alpha,\beta}$, associated to the numéraire $N_t^{\alpha,\beta}$, $s_t^{\alpha,\beta}$ is driftless by construction and is given by

$$ds_t^{\alpha,\beta} = \sigma_t^{\alpha,\beta} \cdot dW_t^{\alpha\beta}$$

$W_t^{\alpha\beta}$ is a $\mathbb{Q}^{\alpha,\beta}$-Brownian motion (eventually multi-dimensional) and the volatility $\sigma_t^{\alpha,\beta}$ is a general process, left unspecified for the moment. For use below, we denote $f_{t,s}$ the forward curve defined as $P_{t,T} \equiv e^{-\int_t^T f_{t,u} du}$.

We consider a (payer) swaption with strike K and maturity t written on a rolling maturity swap $s_t^{t,t+\theta}$ with a fixed tenor θ. A payer swaption gives the owner of the swaption the right to enter into a swap where they pay the fixed leg and receive the floating leg. The present value is given by

$$\begin{aligned} C(t,K) &\equiv \mathbb{E}^{\mathbb{Q}}[\frac{1}{B_t}\left(1-P_{t,t+\theta}-KN_t^{t,t+\theta}\right)^+] \\ &= \mathbb{E}^{\mathbb{Q}}[\frac{N_t^{t,t+\theta}}{B_t}\left(s_t^{t,t+\theta}-K\right)^+] \end{aligned} \tag{1}$$

$C^{\mathrm{mkt}}(t,K)$ denotes below the market price of this swaption. Here, as θ is fixed, we have skipped the dependence of θ in C. In practise, $C^{\mathrm{mkt}}(t,K)$ is only quoted for discrete maturities, i.e., $t=(T_i)_{1\leq i\leq n}$, and we need to inter/extrapolate C^{mkt} in an arbitrage-free way (see next section). Our interest rate model is calibrated to $C^{\mathrm{mkt}}(t,K)$ if and only if the volatility $\sigma_t^{t,t+\theta}$ satisfies the following condition:

Proposition 2.1. *$C(t,K)=C^{\mathrm{mkt}}(t,K)$ for all (t,K) if and only if*

$$\begin{aligned} \mathbb{E}^{\mathbb{Q}^{t,t+\theta}}[(\sigma_t^{t,t+\theta})^2|s_t^{t,t+\theta}=K] &= 2\frac{\partial_t C^{\mathrm{mkt}}(t,K)-KC^{\mathrm{mkt}}(t,K)+K^2\partial_K C^{\mathrm{mkt}}(t,K)}{\partial_K^2 C^{\mathrm{mkt}}(t,K)} \\ &+ 2\frac{\mathbb{E}^{\mathbb{Q}}[\frac{1_{s_t^{t,t+\theta}>K}}{B_t}\left(f_{t,t}-f_{t,t+\theta}P_{t,t+\theta}\right)]}{\partial_K^2 C^{\mathrm{mkt}}(t,K)} \end{aligned} \tag{2}$$

The proof is reported in the appendix. Note that

$$\begin{aligned} \mathbb{E}^{\mathbb{Q}^{t,t+\theta}}[(\sigma_t^{t,t+\theta})^2|s_t^{t,t+\theta}=K] &= \frac{\mathbb{E}^{\mathbb{Q}^{t,t+\theta}}[(\sigma_t^{t,t+\theta})^2\delta(s_t^{t,t+\theta}-K)]}{\mathbb{E}^{\mathbb{Q}^{t,t+\theta}}[\delta(s_t^{t,t+\theta}-K)]} \\ &= \frac{\mathbb{E}^{\mathbb{Q}}[\frac{N_t^{t,t+\theta}}{B_t}(\sigma_t^{t,t+\theta})^2\delta(s_t^{t,t+\theta}-K)]}{\mathbb{E}^{\mathbb{Q}}[\frac{N_t^{t,t+\theta}}{B_t}\delta(s_t^{t,t+\theta}-K)]}\frac{\mathbb{E}^{\mathbb{Q}}[\delta(s_t^{t,t+\theta}-K)]}{\mathbb{E}^{\mathbb{Q}}[\delta(s_t^{t,t+\theta}-K)]} \\ &= \frac{\mathbb{E}^{\mathbb{Q}}[\frac{N_t^{t,t+\theta}}{B_t}(\sigma_t^{t,t+\theta})^2|s_t^{t,t+\theta}=K]}{\mathbb{E}^{\mathbb{Q}}[\frac{N_t^{t,t+\theta}}{B_t}|s_t^{t,t+\theta}=K]} \end{aligned}$$

Remark 2.2. In the case where $\theta=\theta(t)$ is a deterministic function of time, we should add on the right-hand side of Equation (2) the term

$$-2\theta'(t)\frac{\mathbb{E}^{\mathbb{Q}}[\frac{1_{s_t^{t,t+\theta(t)}>K}}{B_t}\left(f_{t,t+\theta(t)}-K\right)P_{t,t+\theta(t)}]}{\partial_K^2 C^{\mathrm{mkt}}(t,K)}$$

For example, for T-coterminal swaptions, we take $\theta(t)=T-t$.

Remark 2.3. Additionally, if we impose the model to be calibrated to the volatility cube, that is $C(t,K,\theta)=C^{\mathrm{mkt}}(t,K,\theta)$, $\forall\ (t,K,\theta)$, Equation 2 is replaced by:

$$\mathbb{E}^{\mathbb{Q}^{t,t+\theta}}[(\sigma_t^{t,t+\theta})^2|s_t^{t,t+\theta}=K]=2\frac{\partial_t C^{\mathrm{mkt}}(t,K,\theta)-\partial_\theta C^{\mathrm{mkt}}(t,K,\theta)}{\partial_K^2 C^{\mathrm{mkt}}(t,K,\theta)}+2\frac{\mathbb{E}^{\mathbb{Q}}[\frac{1_{s_t^{t,t+\theta}>K}}{B_t}\left(f_{t,t}-K\right)]}{\partial_K^2 C^{\mathrm{mkt}}(t,K,\theta)}$$

Note that it is not obvious how to construct arbitrage-free 3-dimensional surface of $C^{\mathrm{mkt}}(t,K,\theta)$ from liquid quoted prices. In particular, naive interpolation could cause arbitrage.

Alternative representation. By using the approximation

$$\left(f_{t,t}-f_{t,t+\theta(t)}P_{t,t+\theta(t)}\right)\approx s_t^{t,t+\theta}\left(1-P_{t,t+\theta(t)}\right)$$

which consists in assuming that the rate curve is flat at t with a constant rate equal to $s_t^{t,t+\theta}$, the last term (model-dependent) in (2) can be approximated by

$$\begin{aligned}\mathbb{E}^{\mathbb{Q}}[\frac{1_{s_t^{t,t+\theta}>K}}{B_t}\left(f_{t,t}-f_{t,t+\theta(t)}P_{t,t+\theta(t)}\right)] &\approx \mathbb{E}^{\mathbb{Q}}[\frac{N_t^{t,t+\theta}}{B_t}1_{s_t^{t,t+\theta}>K}(s_t^{t,t+\theta(t)})^2]\\ &= N_0^{t,t+\theta}\mathbb{E}^{\mathbb{Q}^{t,t+\theta}}[1_{s_t^{t,t+\theta}>K}(s_t^{t,t+\theta(t)})^2]\end{aligned}$$

This approximation is common when pricing CMS by replication on swaptions. For a model such that $C(t,\cdot)=C^{\mathrm{mkt}}(t,\cdot)$, the payoff $1_{s_t^{t,t+\theta}>K}(s_t^{t,t+\theta(t)})^2$ can then be replicated over $C^{\mathrm{mkt}}(t,\cdot)$ by

$$N_0^{t,t+\theta}\mathbb{E}^{\mathbb{Q}^{t,t+\theta}}[1_{s_t^{t,t+\theta}>K}(s_t^{t,t+\theta(t)})^2]=-K^2\partial_K C^{\mathrm{mkt}}(t,K)+2KC^{\mathrm{mkt}}(t,K)+2\int_K^\infty C^{\mathrm{mkt}}(t,x)dx$$

By using $\mathbb{E}^{\mathbb{Q}}[\frac{N_t^{t,t+\theta}}{B_t}1_{s_t^{t,t+\theta}>K}(s_t^{t,t+\theta(t)})^2]$ as a control variate term, Equation (2) can then be (exactly) written as

Corollary 2.4. *$C(t,K)=C^{\mathrm{mkt}}(t,K)$ for all (t,K) if and only if*

$$\mathbb{E}^{\mathbb{Q}^{t,t+\theta}}[(\sigma_t^{t,t+\theta})^2|s_t^{t,t+\theta}=K] = \sigma_{\mathrm{loc}}(t,K)^2+2\frac{\mathbb{E}^{\mathbb{Q}}[\frac{1_{s_t^{t,t+\theta}>K}}{B_t}\left(f_{t,t}-f_{t,t+\theta}P_{t,t+\theta}-s_t^{t,t+\theta}(1-P_{t,t+\theta})\right)]}{\partial_K^2C^{\mathrm{mkt}}(t,K)} \tag{3}$$

with

$$\sigma_{\mathrm{loc}}(t,K)^2\equiv 2\frac{\partial_t C^{\mathrm{mkt}}(t,K)+KC^{\mathrm{mkt}}(t,K)+2\int_K^\infty C^{\mathrm{mkt}}(t,x)dx}{\partial_K^2C^{\mathrm{mkt}}(t,K)} \tag{4}$$

Generic interest rate models under consideration. Formula (2) is valid for a generic interest rate model. However, for the simulation and for the calibration algorithm described below, this model needs to admit a (finite-dimensional) Markov representation, meaning that a t-bond value $P_{t,T}=f(t,T,X_t,(\sigma_s^X)_{s\in[0,t]})$ can be written as a functional f of a finite-dimensional stochastic processes X_t where f can depend only on the volatility σ^X of X between $[0,t]$. Moreover, we should have $f(0,T,X_0,\sigma_0^X)=P_{0T}^{\mathrm{mkt}}$. This includes Cheyette's model and Libor market models but excludes short-rate models enhanced with a local volatility.

3. Cheyette's model: an example

As an example, we consider the Cheyette model [2] which is defined by a 2-dimensional Markov representation of $P_{t,T}$:

$$\begin{aligned}P_{t,T} &= \frac{P_{0,T}}{P_{0,t}}e^{\left(\frac{e^{-\Lambda(T-t)}-1}{\Lambda}\right)x_t-\frac{1}{2}\left(\frac{e^{-\Lambda(T-t)}-1}{\Lambda}\right)^2y_t}\\ dx_t &= (y_t-\Lambda x_t)dt+\sigma_t dW_t\\ dy_t &= (\sigma_t^2-2\Lambda y_t)dt\end{aligned} \tag{5}$$

which depends on 2 processes (x_t,y_t). We should emphasize that σ_t is an arbitrary stochastic process. Under the spot measure $\mathbb{Q}$, associated to the numéraire B_t, the dynamics of $P_{t,T}$ is

$$\frac{dP_{t,T}}{P_{t,T}}=r_tdt+\sigma_P(t,T)dW_t$$

where $\sigma_P(t,T) \equiv \sigma_t \frac{(e^{-\Lambda(T-t)}-1)}{\Lambda}$. The short-rate is

$$r_t = -\partial_T \ln P_{0,T}|_{T=t} + x_t$$

In the case of the Cheyette model, the instantaneous volatility of a forward swap $s_t^{\alpha\beta}$ is

$$\sigma_t^{\alpha,\beta} = \partial_x s^{\alpha,\beta}(t, x_t, y_t)\sigma_t$$

4. Calibration and particle's algorithm

We take $\sigma_t = \sigma(t, s^{t,t+\theta})/\partial_x s^{t,t+\theta}(t, x_t, y_t)$, $\sigma(\cdot,\cdot)$ being the (local) volatility of the swap $s^{t,t+\theta}$. From Equation (3), we match the market values of rolling-maturity swaptions $C^{\mathrm{mkt}}(t,K)$ for all $(t,K) \in [0,T] \times \mathbb{R}$ if and only if $\sigma(t,K)$ is given by

$$\sigma(t,K)^2 = \sigma_{\mathrm{loc}}(t,K)^2 + 2\frac{\Xi(t,K)}{\partial_K^2 C^{\mathrm{mkt}}(t,K)} \tag{6}$$

with

$$\Xi(t,K) \equiv \mathbb{E}^{\mathbb{Q}}[e^{-\int_0^t r_s ds}\xi_t], \quad \xi_t \equiv 1_{s_t^{t,t+\theta}>K}\left(f_{t,t} - f_{t,t+\theta}P_{t,t+\theta} - s_t^{t,t+\theta}(1-P_{t,t+\theta})\right)$$

Note that in the Cheyette model, ξ_t is an explicit function of t and the 2 processes x_t, y_t: $\xi_t = \xi(t, x_t, y_t)$. As ζ_t is of order 2 in θ, we could use as an approximation the Dupire-like formula

$$\sigma(t,K)^2 \approx \sigma_{\mathrm{loc}}(t,K)^2$$

Taking for granted that this approximation is valid (order two in θ), we deduce that the model can be calibrated to our option prices $C^{\mathrm{mkt}}(t,K)$ if and only if $\sigma_{\mathrm{loc}}(t,K)^2 > 0$, equivalent to

$$\partial_K^2 C^{\mathrm{mkt}}(t,K) > 0 \tag{7}$$

$$\partial_t C^{\mathrm{mkt}}(t,K) + K C^{\mathrm{mkt}}(t,K) + 2\int_K^\infty C^{\mathrm{mkt}}(t,x)dx \geq 0 \tag{8}$$

In particular, conditions (7, 8) are sufficient for concluding the arbitrage-freeness of C^{mkt} as they are attained by our local volatility model.

Going beyond this approximation requires the use of the particle algorithm (see Chapter 10 in [6] for mathematical complements). Indeed, the processes (x_t, y_t) with σ_t defined above is an example of a non-linear McKean SDE: The volatility σ_t depends on the t-marginals of (x_t, y_t) through the expression $\Xi(t,\cdot)$ and the resulting Fokker-Planck equation is non-linear. The particle algorithm is then the appropriate method for solving such a non-linear PDE. It can then be briefly described by the following meta-algorithm:

Algorithm. We consider N independent particles with coordinates $(x_t^i, y_t^i, r_t^i, \int_0^t r_s^i)_{i=1,\ldots,N}$ at t following

$$dx_t^i = (y_t^i - \Lambda x_t^i)dt + \sigma_t^i dW_t^i \tag{9}$$

$$dy_t^i = ((\sigma_t^i)^2 - 2\Lambda y_t^i)dt \tag{10}$$

$$r_t^i = -\partial_T \ln P_{0,T}|_{T=t} + x_t^i \tag{11}$$

with $\sigma_t^i = \sigma(t, s^{t,t+\theta}(t, x_t^i, y_t^i))/\partial_x s^{t,t+\theta}(t, x_t^i, y_t^i)$. $(W_\cdot^i)_{i=1,\ldots,N}$ are N independent Brownian motions.

Divide the interval $[0,T]$ into intervals of size Δ.

(1) $t \equiv 0$, set $\sigma(t,S) = \sigma_{\mathrm{loc}}(t,S)$ between 0 and Δ and diffuse the N particles up to Δ - say with an Euler discretization scheme of SDEs (9, 10, 11).

(2) Compute $\Xi(\Delta, K)$ using Monte Carlo:

$$\Xi(\Delta, K) = \frac{1}{N}\sum_{i=1}^{N} e^{-\int_0^t r_s^i ds}\xi(t, x_t^i, y_t^i)$$

Then, compute the local volatility $\sigma(\Delta, S)$ for all S using Equation (6), and set $\sigma(t, S) = \sigma(\Delta, S)$ for all $t \in [\Delta, 2\Delta]$.

(3) Iterate up to a maturity T.

Remark 4.1. Note that if for a strike S, $\sigma(k\Delta, S)^2 < 0$, then we set $\sigma(k\Delta, S) = 0$. This indicates that $C^{\mathrm{mkt}}(t, K)$ is not attainable by our model. This could come from our interpolation/extrapolation of market prices $(C^{\mathrm{mkt}}(T_i, K))_{i=1,\dots,n}$ which are usually quoted in terms of a normal implied volatility $(\sigma^{\mathrm{mkt}}(T_i, K))_{i=1,\dots,n}$. $\sigma^{\mathrm{mkt}}(T, K)$ is defined as the number which, when put in the Bachelier formula for a swaption with strike K and maturity T, reproduces the market price $C^{\mathrm{mkt}}(T_i, K)$:

$$C^{\mathrm{mkt}}(T_i, K) = \mathrm{B}(\sigma^{\mathrm{mkt}}(T, K)^2 T|K)$$

where

$$\mathrm{B}(v|K) \equiv N_0^{T,T+\theta}\sqrt{v}\left(N'(x_T) - x_T(1 - N(x_T))\right), \quad x_T = \frac{K - s_0^{T,T+\theta}}{\sqrt{v}}$$

N is the Gaussian cumulative function. As rates can become negative, the Bachelier formula is more appropriate than the Black-Scholes formula. In our numerical experiments, we have used a cubic spline interpolation in strike of $\sigma^{\mathrm{mkt}}(T, K)$ and a linear interpolation of $\sigma^{\mathrm{mkt}}(T, K)^2 T$ in time T.

Note that the first term in (2), depending on C^{mkt}, can be written in terms of the implied volatility:

$$\frac{\left(\partial_t C^{\mathrm{mkt}} - KC^{\mathrm{mkt}} + K^2\partial_K C^{\mathrm{mkt}}\right)}{\partial_K^2 C^{\mathrm{mkt}}} = \frac{\partial_t\sigma^{\mathrm{mkt}}\sqrt{t} + K^2\partial_K\sigma^{\mathrm{mkt}}\sqrt{t} - (S_0^{t,t+\theta}+K)\sigma^{\mathrm{mkt}}\sqrt{t}}{B_t} + \frac{N(x_t)-1}{N'(x_t)}\cdot\frac{A_t}{B_t}$$

where

$$A_t = \frac{f_{0,t}P_{0,t} - f_{0,t+\theta}P_{0,t+\theta}}{N_0^{t,t+\theta}}, \quad B_t = \frac{\partial^2(\sigma^{\mathrm{mkt}}\sqrt{t})}{\partial K^2} + \frac{\left(1 - x_t\frac{\partial(\sigma^{\mathrm{mkt}}\sqrt{t})}{\partial K}\right)^2}{\sigma\sqrt{t}}$$

Acceleration techniques. It is not necessary to compute $\sigma(t, S)$ for all S using Equation (6). One can save much time by computing $\sigma(t, S)$ for a grid $G_{s,t}$ of values of S, of size much smaller than N, say $N_{f,t}$ and then inter- and extrapolating as explained in [5, 6]. We use cubic splines, with a flat extrapolation, and $N_{f,t} = \max(N_f\sqrt{t}, N_f')$; typical values are $N_f = 30$ and $N_f' = 15$ but could vary when the numerical conditions are harsh. The range of the grid can be inferred from the prices of undiscounted digital options: $-\partial_K C^{\mathrm{mkt}}(t, \max(G_{f,t}))/N_0^{t,t+\theta} = 1 + \partial_K C^{\mathrm{mkt}}(t, \min(G_{f,t}))/N_0^{t,t+\theta} = \alpha$. In practice, we take $\alpha = 1 - N(2.5)$ where N is the cumulative function of the Gaussian law.

Extension 1: Multi-dimensional Cheyette or Libor market models. Our algorithm can be easily extended to a multi-dimensional Cheyette model for which

$$dx_t = (\cdots)dt + \Sigma(t, x_t)\cdot dW_t$$

$\Sigma(t, x_t)$ is an $N \times N$ volatility and W_t an N-dimensional Brownian motion. Here the drift term has not been written explicitly. This implies that under $\mathbb{Q}^{\alpha\beta}$

$$ds_t^{\alpha,\beta} = \left(\nabla_x s_t^{\alpha,\beta}\right)\Sigma(t, x_t)\cdot dW_t^{\alpha,\beta}$$

Take $\Sigma(t,x_t) = (\nabla_x s_t^{t,t+\theta})^{-1}\sigma(t,s_t^{t,t+\theta})\Phi(t)$ where $\sigma(t,s_t^{t,t+\theta})$ is a scaling function and $\Phi(t)$ a deterministic $N\times N$ matrix. The calibration condition reads then

$$\sigma(t,K)^2 \mathrm{Tr}\left(\Phi(t)^\dagger\cdot\Phi(t)\right) = \sigma_{\mathrm{loc}}(t,K)^2 + 2\frac{\Xi(t,K)}{\partial_K^2 C^{\mathrm{mkt}}(t,K)}$$

from which we can apply our particle algorithm.

Similarly, for a Libor market model [1], we have

$$ds_t^{\alpha,\beta} = \sum_{i=\alpha+1}^{\beta}\frac{\partial s_t^{\alpha,\beta}}{\partial L_t^i}\Sigma_t^i\cdot dW_t^{\alpha,\beta}$$

where Σ_t^i is the instantaneous volatility of Libor $L_t^i \equiv L(t,T_i,T_{i+1})$. We could take

$$\Sigma_t^i = \left(\frac{\partial s_t^{t,t+\theta}}{\partial L_t^i}\right)^{-1}\sigma(t,s_t^{t,t+\theta})\Phi_i(t)$$

Extension 2: Matching N tenors. By introducing a multi-factor interest rate models depending on N local volatilities $(\sigma_i(\cdot,\cdot))_{i=1,\ldots,N}$, we have enough flexibility to match rolling maturity swaptions $C^{\mathrm{mkt}}(t,K,\theta_i)$ for N tenors $(\theta_i)_{i=1,\ldots,N}$. For example, in the case of a N-dimensional Cheyette model, we can parameterize Σ_t as: For all $p=1,\ldots,N$,

$$(\nabla_x s_t^{t,t+\theta_p}.\Sigma(t,x_t))^\dagger.\nabla_x s_t^{t,t+\theta_p}.\Sigma(t,x_t) = \sum_{i,j=1}^{N}\partial_{x^i}s_t^{t,t+\theta_p}\sigma_i(t,s_t^{t,t+\theta_i})\sigma_j(t,s_t^{t,t+\theta_j})\partial_{x^j}s_t^{t,t+\theta_p} \tag{12}$$

Equation (2) reads

$$\mathbb{E}^{\mathbb{Q}^{t,t+\theta_p}}[\sum_{i,j=1}^{N}\partial_{x^i}s_t^{t,t+\theta_p}\sigma_i(t,s_t^{t,t+\theta_i})\sigma_j(t,s_t^{t,t+\theta_j})\partial_{x^j}s_t^{t,t+\theta_p}|s_t^{t,t+\theta_p}=K] = \mathcal{P}_p(t,K)$$

where $\mathcal{P}_p(t,K)$ is the right hand-side of Equation (2) for $\theta=\theta_p$. Once $\mathcal{P}_\cdot$ and $s_t^{t,t+\theta_\cdot}$ are computed and simulated using the particle algorithm, the above nonlinear equations can be solved using a fixed-point algorithm (see [7] where a similar method is used). Note that the equation (12) can be however constraining. There might be other parameterizations that can offer greater flexibility.

5. Numerical experiments

We have checked the accuracy of our algorithm on the EUR market smile (15-April-2016) using a one-dimensional Cheyette model as described previously (see Equation (5)). In particular, we have chosen the mean-reversion $\Lambda = 1\%$. The time-discretization Δ has been set to $\Delta = 1/50$ and we have used $N = 2^{12}$ particles. After calibrating the model using the particle algorithm, we have computed rolling-tenor swaption smile using a (quasi) Monte Carlo pricer with $N = 2^{15}$ paths and a timestep $\Delta t = 1/250$ (see Figures 1, 2). Swaption fair values are quoted in (normal) implied volatility and the strikes are quoted in standard deviation with respect to the ATM volatility σ_{ATM}: $K = s_0^{t,t,+\theta} + \sigma_{\mathrm{ATM}}\sqrt{t}$ stdev with stdev $\in [-2,2]$. The computation time is around 8 seconds for maturities up to 10 years. As shown in Figures (1-2), the absolute error in implied volatility is of few basis points even in the tails.

[thp]

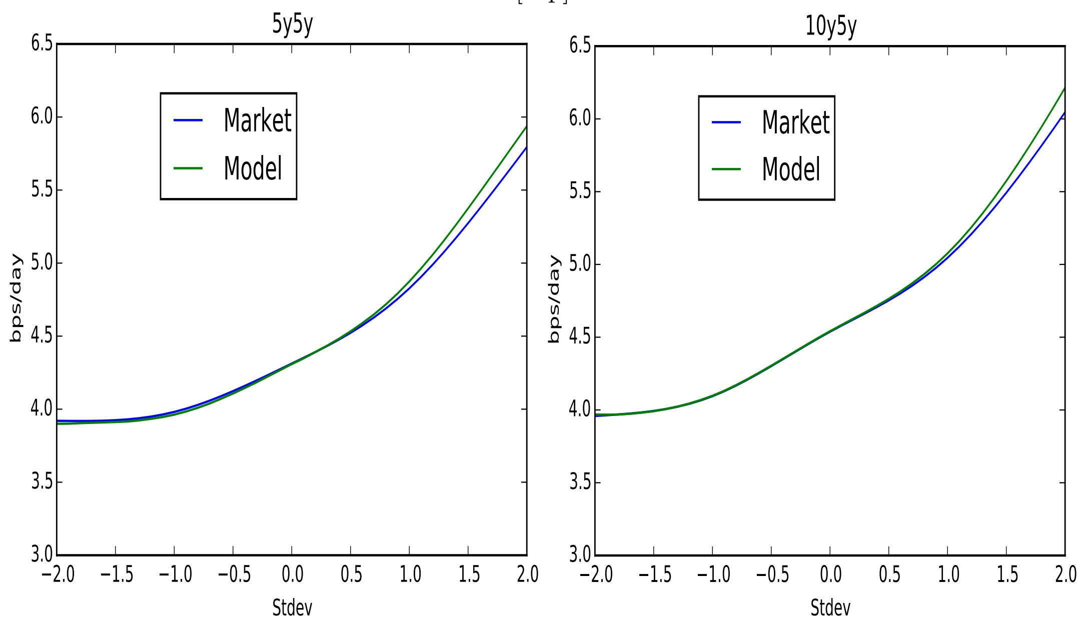

FIGURE 1. Swaption smile with maturities 5Y/10Y and tenor of 5Y compared to implied volatilities (EUR, 15-April-2016). We used $N = 2^{12}$ particles, 2^{15} simulations.

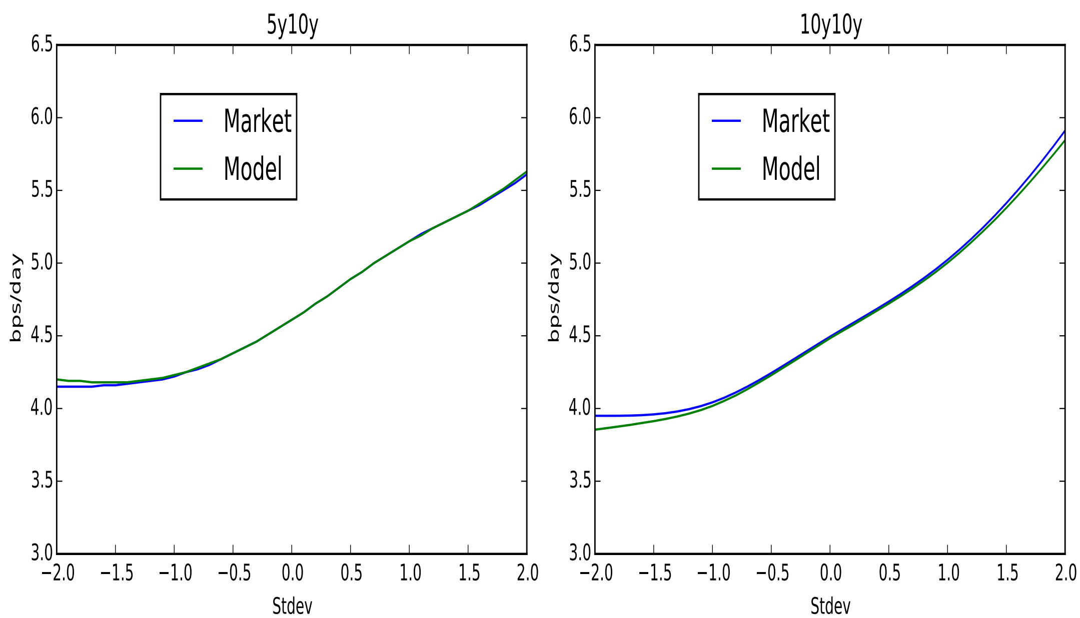

FIGURE 2. Swaption smile with maturities 5Y/10Y and tenor of 10Y compared to implied volatilities (EUR, 15-April-2016). We used $N = 2^{12}$ particles, 2^{15} simulations.

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Appendix

Proof. Under the spot measure $\mathbb{Q}$, associated to the numéraire B_t, we have

$$\frac{dP_{t,T}}{P_{t,T}} = r_t dt + \sigma_P(t,T) dW_t^{\mathbb{Q}}$$

For the sake of simplicity of the proof, we have assumed that $W_t^{\mathbb{Q}}$ is a one-dimensional $\mathbb{Q}$-Brownian motion. This can be easily generalized by replacing for example expression $\sigma_P(t,T)dW_t^{\mathbb{Q}}$ by $\sigma_P(t,T) \cdot dW_t^{\mathbb{Q}}$ where $\cdot$ denotes a matrix multiplication. Formula (2) remains valid in this multi-dimensional setup.

We have

$$\frac{dN_t^{\alpha,\beta}}{N_t^{\alpha\beta}} = r_t dt + \frac{\int_{T_\alpha}^{T_\beta} \sigma_P(t,u) P_{t,u} du}{\int_{T_\alpha}^{T_\beta} P_{t,u} du} dW_t^{\mathbb{Q}}$$

Under the spot measure $\mathbb{Q}$, we obtain

$$ds_t^{\alpha,\beta} = \sigma_t^{\alpha,\beta} dW_t^{\mathbb{Q}} - \sigma_t^{\alpha,\beta} \frac{\int_{T_\alpha}^{T_\beta} \sigma_P(t,u) P_{t,u} du}{\int_{T_\alpha}^{T_\beta} P_{t,u} du} dt$$

In particular,

$$ds_t^{t,t+\theta(t)} = \sigma_t^{t,t+\theta(t)} dW_t^{\mathbb{Q}} - \sigma_t^{t,t+\theta(t)} \frac{\int_t^{t+\theta(t)} \sigma_P(t,u) P_{t,u} du}{\int_t^{t+\theta(t)} P_{t,u} du} dt + \frac{\partial s_t^{u,u+\theta(u)}}{\partial u}|_{u=t} dt \tag{13}$$

Under $\mathbb{Q}$, the ratio $\frac{N_t^{\alpha,\beta}}{B_t}$ is driftless

$$d\frac{N_t^{\alpha,\beta}}{B_t} = \frac{N_t^{\alpha,\beta}}{B_t} \left(\frac{\int_{T_\alpha}^{T_\beta} \sigma_P(t,u) P_{t,u} du}{\int_{T_\alpha}^{T_\beta} P_{t,u} du} dW_t^{\mathbb{Q}} \right)$$

This implies that

$$d\frac{N_t^{t,t+\theta(t)}}{B_t} = \frac{N_t^{t,t+\theta(t)}}{B_t}\left(\frac{\int_t^{t+\theta(t)}\sigma_P(t,u)P_{t,u}du}{\int_t^{t+\theta(t)}P_{t,u}du}dW_t^{\mathbb{Q}}\right) + \frac{\partial_u N_t^{u,u+\theta(u)}|_{u=t}}{B_t}dt \tag{14}$$

By applying Itô-Tanaka's formula on $\frac{N_t^{t,t+\theta(t)}}{B_t}\left(s_t^{t,t+\theta(t)} - K\right)^+$, we get

$$d\left(\frac{N_t^{t,t+\theta(t)}}{B_t}\left(s_t^{t,t+\theta(t)} - K\right)^+\right) = \left(s_t^{t,t+\theta(t)} - K\right)^+ d\frac{N_t^{t,t+\theta(t)}}{B_t} + \frac{N_t^{t,t+\theta(t)}}{B_t}\left(1_{s_t^{t,t+\theta(t)}>K}ds_t^{t,t+\theta(t)}\right.$$

$$\left.+\frac{1}{2}\delta(s_t^{t,t+\theta(t)} - K)d\langle s_\cdot^{t,t+\theta(t)}\rangle_t\right) + 1_{s_t^{t,t+\theta(t)}>K}d\langle\frac{N_\cdot^{t,t+\theta(t)}}{B_\cdot}, s_\cdot^{t,t+\theta(t)}\rangle_t$$

By taking the expectation w.r.t. $\mathbb{Q}$ on both side, and by using Equations (13), (14), we get

$$\begin{aligned} &\frac{1}{2}\partial_K^2 C(t,K)\mathbb{E}^{\mathbb{Q}^{t,t+\theta(t)}}[(\sigma_t^{t,t+\theta(t)})^2|s_t^{t,t+\theta(t)} = K] = \partial_t C(t,K) \\ - \quad &\mathbb{E}^{\mathbb{Q}}[\frac{N_t^{t,t+\theta(t)}}{B_t}1_{s_t^{t,t+\theta(t)}>K}\frac{\partial s_t^{u,u+\theta(u)}}{\partial u}|_{u=t}] - \mathbb{E}^{\mathbb{Q}}[\frac{\partial_u N_t^{u,u+\theta(u)}|_{u=t}}{B_t}\left(s_t^{t,t+\theta(t)} - K\right)^+] \end{aligned}$$

with C defined by (1). We have also used the above identity

$$\begin{aligned} \mathbb{E}^{\mathbb{Q}}[\frac{N_t^{t,t+\theta(t)}}{B_t}\delta(s_t^{t,t+\theta(t)} - K)(\sigma_t^{t,t+\theta})^2] &= N_0^{t,t+\theta(t)}\mathbb{E}^{\mathbb{Q}^{t,t+\theta(t)}}[\delta(s_t^{t,t+\theta(t)} - K)(\sigma_t^{t,t+\theta})^2] \\ &= \mathbb{E}^{\mathbb{Q}^{t,t+\theta(t)}}[(\sigma_t^{t,t+\theta})^2|s_t^{t,t+\theta(t)} = K]\partial_K^2 C(t,K) \end{aligned}$$

Then, note that

$$\begin{aligned} N_t^{t,t+\theta(t)}\frac{\partial s_t^{u,u+\theta(u)}}{\partial u}|_{u=t} &= (-f_{t,t} + f_{t,t+\theta(t)}(1+\theta'(t))P_{t,t+\theta(t)}) - s_t^{t,t+\theta(t)}(P_{t,t+\theta(t)}(1+\theta'(t)) - 1) \\ \partial_u N_t^{u,u+\theta(u)}|_{u=t} &= P_{t,t+\theta(t)}(1+\theta'(t)) - 1 \end{aligned}$$

This implies that

$$\begin{aligned} &\frac{1}{2}\partial_K^2 C(t,K)\mathbb{E}^{\mathbb{Q}^{t,t+\theta(t)}}[(\sigma_t^{t,t+\theta(t)})^2|s_t^{t,t+\theta(t)} = K] = \partial_t C(t,K) \\ + \quad &\mathbb{E}^{\mathbb{Q}}[\frac{1_{s_t^{t,t+\theta(t)}>K}}{B_t}\left(f_{t,t} - f_{t,t+\theta(t)}(1+\theta'(t))P_{t,t+\theta(t)} + s_t^{t,t+\theta(t)}(P_{t,t+\theta(t)}(1+\theta'(t)) - 1)\right)] \\ - \quad &\mathbb{E}^{\mathbb{Q}}[\frac{P_{t,t+\theta(t)}(1+\theta'(t)) - 1}{B_t}\left(s_t^{t,t+\theta(t)} - K\right)^+] \end{aligned}$$

Then, use the identities

$$\mathbb{E}^{\mathbb{Q}}[\frac{P_{t,t+\theta(t)} - 1}{B_t}\left(s_t^{t,t+\theta(t)} - K\right)^+] = -\mathbb{E}^{\mathbb{Q}}[\frac{N_{t,t+\theta(t)}}{B_t}s_t^{t,t+\theta(t)}\left(s_t^{t,t+\theta(t)} - K\right)^+]$$

and

$$\mathbb{E}^{\mathbb{Q}}[\frac{1_{s_t^{t,t+\theta(t)}>K}}{B_t}s_t^{t,t+\theta(t)}(P_{t,t+\theta(t)} - 1)] = -\mathbb{E}^{\mathbb{Q}}[\frac{N_{t,t+\theta(t)}}{B_t}1_{s_t^{t,t+\theta(t)}>K}(s_t^{t,t+\theta(t)})^2]$$

This gives our final result. □