Caplet Pricing with Backward-Looking Rates

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Abstract

We consider the Hull-White short rate model and extend the known closed-form pricing kernel to include the integrated short rate as a separate independent variable, applying it to cap/floor pricing.

1 Overview

It is by now well-known that, in the wake of a number of scandals involving illicit manipulation of LIBOR interest rates, the financial world is in the throes of a major transition away from such term rates, set in advance of the period for which they are applicable, in favour of backward-looking rates which are set daily with the rates compounded over the period for which interest is to be paid. As explained by Lyashenko and Mercurio [2019]:

> In 2013-2014, the Financial Stability Board (FSB) conducted fundamental reviews of major interest rate benchmarks and recommended developing alternative nearly risk-free rates (RFRs) that are better suited as the reference rates for certain financial transactions. By now, RFRs have been selected in all major economies: The US selected a new Treasuries repo financing rate called SOFR (Secured Overnight Funding Rate); the UK selected the reformed SONIA (Sterling Overnight Index Average); Switzerland selected SARON (Swiss Average Rate Overnight); Japan selected TONA (Tokyo Overnight Average Rate); and the Euro zone selected a new unsecured overnight rate called ESTER (Euro Short-Term Rate).

However the changes proposed have resulted in great disruption across the financial industry as institutions of all types have scrambled to make the necessary changes to allow them to trade and manage the risk on financial instruments citing the new compounded rates rather than LIBOR. Henrard [2019] presents a critical perspective, pointing out that there have been problems with a lack of precision in the proposals being put forward resulting in difficulty in progressing discussions; and a lack of measurability resulting in impediments to implementation in practice. A particular headache has been what to do about legacy LIBOR-based trades, some of which have many decades to run. Another issue has been the need to build discounting curves and forward curves based on the new rates. One advantage of the proposed change is that it dispenses (at least in principle) with the need for multiple, mutually incompatible, forward curves incorporating different tenor-dependent LIBOR spreads over the risk-free rate. Whether the advantages will outweigh the costs probably remains to be seen.

Lyashenko and Mercurio [2019] went on in their seminal paper to set out how the LIBOR Market Model, which is arguably the most important interest rate model in current use, can be extended to allow it to be used for the pricing of interest rate derivative products where the referenced underlyings may be compounded rates instead of or in addition to term rates. They apply their extended model to the pricing of interest rate futures contracts, caps and swaptions.

We look here to perform a similar task for the simpler short rate model of Hull and White [1990]. This was proposed as an extension to the model of Vasicek [1977] and is indeed often referred to as the extended Vasicek model. The model postulates that the instantaneous interest rate, or short rate, can be modelled as a mean-reverting Ornstein-Uhlenbeck process. As is well-known [Brigo and Mercurio, 2006], the Hull-White model is an affine model, in the sense that the coefficients in both the drift and diffusion terms in the governing PDE are affine functions of the short rate. The availability of analytic solutions for zero coupon bond prices in such circumstances, and indeed for cap/floor and swaption prices, has resulted in this model remaining popular with practitioners for nearly thirty years. A pricing kernel for this model was first reported by Van Steenkiste and Foresi [1999] and formally published more recently by Turfus [2019], using a method of derivation which we shall exploit further here. Option pricing formulae based on compounded rates (both discrete and continuous) were previously reported by Henrard [2004] and Henrard [2007].

2 Model Description

Rather than the short rate itself, we shall find it convenient to work with an auxiliary variable x_t satisfying the following canonical Ornstein–Uhlenbeck process:

$$dx_t = -\alpha_r(t)x_t dt + \sigma_r(t)dW_t, \tag{2.1}$$

where $\alpha_r, \sigma_r : \mathbb{R}^+ \to \mathbb{R}^+$ are piecewise continuous functions, the former L_1- and the latter L_2-integrable, and W_t is a Brownian motion under the risk-neutral measure for $t \geq 0$. Under the Hull-White model, this auxiliary variable x_t is related to the instantaneous short rate r_t by

$$r_t = \tilde{r}(t) + x_t, \tag{2.2}$$

with $\tilde{r} : \mathbb{R}^+ \to \mathbb{R}$ an (L_1-integrable) function to be determined by calibration to fit the instantaneous forward curve given by $\overline{r} : \mathbb{R}^+ \to \mathbb{R}$.

For our calculations, we make essentially the same modelling assumptions as in the Hull-White calculations of Turfus [2019, 2021], defining

$$r^*(t) := \tilde{r}(t) - \overline{r}(t), \tag{2.3}$$

$$r(x,t) := \tilde{r}(t) + x, \tag{2.4}$$

$$D(t_1, t_2) := e^{-\int_{t_1}^{t_2} \overline{r}(t)dt}. \tag{2.5}$$

The novel feature here is that, rather than targeting options citing *forward-looking* LIBOR rates, we shall look to price options with *backward-looking* payoffs citing daily-compounded rates. The payoff at T_2 for such rates for a payment period $[T_1, T_2]$ can conveniently be modelled as

$$P = e^{\int_{T_1}^{T_2} r(x_t,t)dt} - 1.$$

Our problem here is that P is not a function of the stochastic variable x_t but of its integral over the payment period. To address this we introduce a new integrated variable z_t defined by

$$z_t = \int_0^t \left(r^*(s) + x_s\right) ds, \tag{2.6}$$

in terms of which we can write the payoff at T_2 as

$$P(z_{T_1}, z_{T_2}) = D(T_1, T_2)^{-1} e^{z_{T_2} - z_{T_1}} - 1. \tag{2.7}$$

We are thus led to consider derivative contracts whose payoff at time T has the general form $P(x_T, z_T)$, i.e. may depend on the variable z as well as x. We write the associated derivative price on this basis as $f(x, z, t)$. Of course, to address a payoff of the particular form of (2.7), we will need to calculate the derivative price in two stages, in the first instance pricing the payoff as of time T_1, then treating the result as the payoff at time T_1 for a derivative contract priced as of time $t < T_1$, as is standard practice with payoffs involving term rates.

We seek, in a natural extension of the work of Turfus [2019], to obtain an exact Green's function solution (or pricing kernel) for European-type securities whose contingent payoff at some future time T depends upon the short rate and/or its integral at that time. By this is meant a function $G(x, z, t; \xi, \zeta, T)$ such that the price at time $t \geq 0$ contingent upon $x_t = x$, $z_t = z$ of a security which pays $P(x_T, z_T)$ at time T is given by the convolution expression:

$$f(x, z, t) = \iint_{\mathbb{R}^2} G(x, z, t; \xi, \zeta, T) P(\xi, \zeta) d\xi d\zeta. \tag{2.8}$$

We will use this pricing kernel to calculate analytic expressions for the prices of caplets on backward-looking compounded rates.

3 Pricing Kernel

Noting (2.1) and the fact that (2.6) gives rise to $dz_t = (r^*(t) + x_t)\, dt$, we infer from the Feynman-Kac theorem that the function $f(x, z, t)$ emerges as the solution to the following Kolmogorov backward diffusion equation:

$$\frac{\partial f}{\partial t} - \alpha_r(t) x \frac{\partial f}{\partial x} + (r^*(t) + x) \frac{\partial f}{\partial z} + \frac{1}{2}\sigma_r^2(t) \frac{\partial^2 f}{\partial x^2} - r(x, t) f = 0, \quad t \geq 0, \tag{3.1}$$

with $r(x, t)$ given by (2.4), subject to the final condition $\lim_{t \to T^-} f(x, z, t) = P(x, z)$. As can be seen this equation is a special case of the Hull-White rates-equity pricing equation considered by Turfus [2018], with the equity volatility $\sigma_z(t) \equiv 0$. It follows straightforwardly that the pricing kernel is given by the following theorem, making use of equation (3.2) of that work.

Theorem 3.1. *The pricing kernel for the extended Hull-White pricing equation* (3.1) *is:*

$$G(x, z, t; \xi, \zeta, T) = F^T(x, t) N_2(\xi + \Sigma_{rz}(t, T) - x\phi_r(t, T), \zeta + \tfrac{1}{2}\Sigma_{zz}(t, T) - \mu^*(x, t, T) - z; \mathbf{\Sigma}^+(t, T)), \tag{3.2}$$

where $N_2(\cdot, \cdot; \mathbf{\Sigma})$ is a bivariate Gaussian distribution function with variance $\mathbf{\Sigma}$,

$$F^T(x, t) = D(t, T) e^{-\mu^*(x, t, T)} \tag{3.3}$$

is the T-maturity Hull-White zero coupon bond price and we further define:

$$\phi_r(t, T) = e^{-\int_t^T \alpha_r(u) du}, \tag{3.4}$$

$$B^*(t, T) = \int_t^T \phi_r(t, u) du, \tag{3.5}$$

$$\Sigma_{rr}(t, T) = \int_t^T \phi_r^2(u, T) \sigma_r^2(u) du, \tag{3.6}$$

$$\Sigma_{rz}(t, T) = \int_t^T \phi_r(u, T) \Sigma_{rr}(t, u) du, \tag{3.7}$$

$$\Sigma_{zz}(t, T) = 2\int_t^T \Sigma_{rz}(t, u) du, \tag{3.8}$$

$$\mu^*(x, t, T) = B^*(t, T)(x + r^*(t)) + \tfrac{1}{2} B^{*2}(t, T) \Sigma_{rr}(0, t), \tag{3.9}$$

$$\mathbf{\Sigma}^+(t, T) = \begin{pmatrix} \Sigma_{rr}(t, T) & \Sigma_{rz}(t, T) \\ \Sigma_{rz}(t, T) & \Sigma_{zz}(t, T) \end{pmatrix}. \tag{3.10}$$

We further note, following Turfus [2019], that to fit the instantaneous forward curve (the condition of being arbitrage-free), we must choose

$$r^*(t) = \Sigma_{rz}(0, t) \tag{3.11}$$

4 Applications

4.1 Compounded Rates Payment

Let us now consider the valuation of the payoff (2.7) as of some time $t \in [0, T_1]$. As indicated above, we must first carry out the valuation as of T_1. Applying (3.2), we obtain

$$\begin{aligned} V(x, T_1) &= \iint_{\mathbb{R}^2} P(z, \zeta) G(x, z, T_1; \xi, \zeta, T_2) d\xi d\zeta \\ &= \frac{e^{-\mu^*(x,T_1,T_2)}}{\sqrt{\Sigma_{zz}(T_1, T_2)}} \int_{\mathbb{R}} \left(e^{\zeta - z} - D(T_1, T_2)\right) N\left(\frac{\zeta + \frac{1}{2}\Sigma_{zz}(T_1, T_2) - \mu^*(x, T_1, T_2) - z}{\sqrt{\Sigma_{zz}(T_1, T_2)}}\right) d\zeta \\ &= 1 - F^{T_2}(x, T_1), \end{aligned} \tag{4.1}$$

with $N(\cdot)$ a normal distribution function. Applying (3.2) again to price (4.1) as of some prior time $t \geq 0$, we obtain straightforwardly

$$V(x, t) = F^{T_1}(x, t) - F^{T_2}(x, t). \tag{4.2}$$

In other words the value of a backward-looking compounded rate payment is identical to that of a forward-looking LIBOR-type payment, as must be the case from the constraint of absence of arbitrage.

4.2 Delayed Compounded Rates Payment

We consider next how the valuation of the payoff (2.7) is affected if the payoff is delayed until some later time $T > T_2$. This not uncommon in contracts involving compounded rates. In this case we obtain

$$\begin{aligned} V(x, T_1) &= \iint_{\mathbb{R}^2} F^T(\xi, T_2) P(z, \zeta) G(x, z, T_1; \xi, \zeta, T_2) d\xi d\zeta \\ &= \frac{F^T(x, T_1)}{\sqrt{\Sigma_{zz}(T_1, T_2)}} \int_{\mathbb{R}} \left(\frac{1}{F^{T_2}(x, T_1)} e^{-\mu^*(x,T_1,T_2)+\zeta-z} - 1\right) \\ &\qquad\qquad N\left(\frac{\zeta + \frac{1}{2}\Sigma_{zz}(T_1, T_2) - \mu^*(x, T_1, T_2) - z}{\sqrt{\Sigma_{zz}(T_1, T_2)}}\right) d\zeta \\ &= F^T(x, T_1) \left(\frac{1}{F^{T_2}(x, T_1)} - 1\right), \end{aligned} \tag{4.3}$$

with $N(\cdot)$ a normal distribution function. We see the result is at this point identical to the corresponding LIBOR calculation considered in §4.3.2 of Turfus [2021], to which the reader is referred for details. We note in particular that the PV of the delayed compounded rates payment can on this basis be written

$$PV_{\text{delayed}} = D(0, T) \left(\frac{F_{\text{convexity}}(T_1, T_2, T)}{D(T_1, T_2)} - 1\right). \tag{4.4}$$

where

$$F_{\text{convexity}}(T_1, T_2, T) := e^{-\phi_r(T_1,T_2)B^*(T_2,T)B^*(T_1,T_2)\Sigma_{rr}(0,T_1)}. \tag{4.5}$$

As can be seen, since $F_{\text{convexity}}(T_1, T_2, T_2) = 1$, the standard LIBOR result is obtained in the limit as $T \to T_2$.

4.3 Caplet Pricing

Cap and caplet prices are obtained from the following theorem.

Theorem 4.1. *Consider a caplet based on the compounded risk-free rate over a payment period $[T_1, T_2]$ and a payoff with strike K at time T_2 of*

$$
\begin{aligned}
P_{caplet}(z_1, z_2) &= \left[e^{\int_{T_1}^{T_2} r(x_t,t)dt} - 1 - K\delta(T_1, T_2)\right]^+ \\
&= \left[D(T_1, T_2)^{-1} e^{z_2 - z_1} - \kappa^{-1}\right]^+
\end{aligned}
\tag{4.6}
$$

where $z_{T_1} = z_1$, $z_{T_2} = z_2$ and $\kappa = (1 + K\delta(T_1, T_2))^{-1}$. Defining the critical value of $z_2 - z_1$ as

$$
\Delta z^* = \ln\left(\kappa^{-1} D(T_1, T_2)\right), \tag{4.7}
$$

the caplet price as of time $t \geq 0$ will be given by

$$
V_{caplet}(x, t) = F^{T_1}(x, t)\Phi(-\hat{d}_1(x, t)) - \kappa^{-1} F^{T_2}(x, t)\Phi(-\hat{d}_2(x, t)), \tag{4.8}
$$

with $\Phi(\cdot)$ a cumulative normal distribution function and, for $t \leq T_1$,

$$
\hat{d}_1(x, t) := \frac{\Delta z^* - \mu^*(x, t, T_2) + \mu^*(x, t, T_1) - \frac{1}{2}\left(B^{*2}(T_1, T_2)\Sigma_{rr}(t, T_1) + \Sigma_{zz}(T_1, T_2)\right)}{\sqrt{B^{*2}(T_1, T_2)\Sigma_{rr}(t, T_1) + \Sigma_{zz}(T_1, T_2)}}, \tag{4.9}
$$

$$
\hat{d}_2(x, t) := \hat{d}_1(x, t) + \sqrt{B^{*2}(T_1, T_2)\Sigma_{rr}(t, T_1) + \Sigma_{zz}(T_1, T_2)}. \tag{4.10}
$$

Proof. Making use of (3.2), the caplet value as of time T_1 will be

$$
\begin{aligned}
V_{\text{caplet}}(x, T_1) &= \iint_{\mathbb{R}^2} P_{\text{caplet}}(z, \zeta) G(x, z, T_1; \xi, \zeta, T_2) d\xi d\zeta \\
&= \frac{e^{-\mu^*(x, T_1, T_2)}}{\sqrt{\Sigma_{zz}(T_1, T_2)}} \int_{z + \Delta z^*}^{\infty} \left(e^{\zeta - z} - \kappa^{-1} D(T_1, T_2)\right) \\
&\qquad N\left(\frac{\zeta + \frac{1}{2}\Sigma_{zz}(T_1, T_2) - \mu^*(x, T_1, T_2) - z}{\sqrt{\Sigma_{zz}(T_1, T_2)}}\right) d\zeta \\
&= \Phi\left(-\hat{d}_1(x, T_1)\right) - \kappa^{-1} F^{T_2}(x, T_1)\Phi\left(-\hat{d}_2(x, T_1)\right),
\end{aligned}
\tag{4.11}
$$

Taking this caplet price as the payoff at T_1 and valuing as of time t, we obtain

$$
\begin{aligned}
V_{\text{caplet}}(x, t) &= \iint_{\mathbb{R}^2} V_{\text{caplet}}(\xi, T_1) G(x, z, t; \xi, \zeta, T_1) d\xi d\zeta \\
&= \frac{F^{T_1}(x, t)}{B^*(T_1, T_2)\sqrt{\Sigma_{rr}(t, T_1)}} \int_{\mathbb{R}} V_{\text{caplet}}(\xi, T_1) N\left(\frac{\xi + \Sigma_{rz}(t, T_1) - x\phi_r(t, T_1)}{B^*(T_1, T_2)\sqrt{\Sigma_{rr}(t, T_1)}}\right) d\xi \\
&= F^{T_1}(x, t)\Phi(-\hat{d}_1(x, t)) - \kappa^{-1} F^{T_2}(x, t)\Phi(-\hat{d}_2(x, t))
\end{aligned}
\tag{4.12}
$$

as required. □

Note that (4.11) can be viewed as a generalisation of Theorem 4 of Henrard [2004], which is reproduced by setting $x = 0$ and taking σ_r to be constant, whereupon we have $\frac{1}{2}\Sigma_{zz}(t, T) = \frac{1}{2}\sigma_r^2 \int_t^T B^{*2}(t, u) du$. Note that our $B^*(t, u)$ is $\nu(t, u)$ in his notation.

Letting $t \to 0$ in (4.8), the resultant expression for the PV is

$$
PV_{\text{caplet}} = D(0, T_1)\Phi(-d_2) - \kappa^{-1} D(0, T_2)\Phi(-d_1), \tag{4.13}
$$

where

$$
d_1 = \frac{\Delta z^* - \frac{1}{2}\left(B^{*2}(T_1, T_2)\Sigma_{rr}(0, T_1) + \Sigma_{zz}(T_1, T_2)\right)}{\sqrt{B^{*2}(T_1, T_2)\Sigma_{rr}(0, T_1) + \Sigma_{zz}(T_1, T_2)}}, \tag{4.14}
$$

$$
d_2 = d_1 + \sqrt{B^{*2}(T_1, T_2)\Sigma_{rr}(0, T_1) + \Sigma_{zz}(T_1, T_2)}. \tag{4.15}
$$

Comparing this with the well-known Hull-White result (see e.g. Turfus, 2019), we find the difference resulting from using compounded rates is only in the appearance of the variance adjustment $\Sigma_{zz}(T_1, T_2)$, and the critical condition which is required. Indeed it can be seen that, letting $\Sigma_{zz}(T_1, T_2) \to 0$ (for example by specifying $\sigma_r(t) = 0$ for $t \in [T_1, T_2]$), the standard result is recovered. Similarly, or by invoking the principle of put-call parity, we find the floorlet price to be

$$V_{\text{floorlet}}(x,t) = \kappa^{-1} F^{T_2}(x,t)\Phi(\hat{d}_2(x,t)) - F^{T_1}(x,t)\Phi(\hat{d}_1(x,t)). \tag{4.16}$$

Also, analogous to (4.4), we see the PV of a caplet whose payoff is delayed until $T > T_2$ is given by

$$PV_{\text{delayed caplet}} = D(0,T)\left(\frac{F_{\text{convexity}}(T_1, T_2, T)}{D(T_1, T_2)}\Phi(-d_1) - \kappa^{-1}\Phi(-d_2)\right), \tag{4.17}$$

where, in the definition of d_1 and d_2, Δz^* is re-interpreted as

$$\Delta z^* = \ln \frac{\kappa^{-1} D(T_1, T_2)}{F_{\text{convexity}(T_1, T_2, T)}} \tag{4.18}$$

Daily Compounding A word should also be said about the adequacy of our approximate representation of the daily compounding conventions of SONIA, SOFR, etc. by continuous compounding. Effectively, the use of daily compounding means that the rate for each day is set at the beginning of that day, rather than at the end based on a market-observed forward rate. This may sound like a trivial distinction and the difference would not be be expected to be too great. In fact it is possible to calculate by iterative application of the Hull-White pricing kernel given by equation (4) of Turfus [2019] the exact value for daily compounding rates. The problem was previously considered by Henrard [2007] from within a HJM framework, the result for the PV being presented as his Theorem 1.

4.4 European Swaption Pricing

The calculation for European swaption prices for compounded rates likewise follows closely the derivation for the LIBOR case. We reuse the notation there introduced. The exercise value of the swaption will then be the positive part of the time-t_0 value of the swap underlying, with $t_0 < T_0$ and $[T_{i-1}, T_i]$ the swap payment periods. For a payer swaption, making use of (4.2), this will be

$$P_{\text{payer}}(x) = \sum_{i=1}^{n} \left[F^{T_{i-1}}(x, t_0) - \kappa^{-1} F^{T_i}(x, t_0)\right] \mathbb{1}_{x > x^*}, \tag{4.19}$$

with x^* now denoting the value of x at which the swap underlying comes into the money. But this is identical to the expression for LIBOR-based swaptions. Consequently the results for European swaption prices with compounded rates are identical to those obtained by standard means for the LIBOR case, as was pointed out by Lyashenko and Mercurio [2019]. See for example Chapter 4 of Turfus [2021] for more details.

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