HypHyp asymptotic

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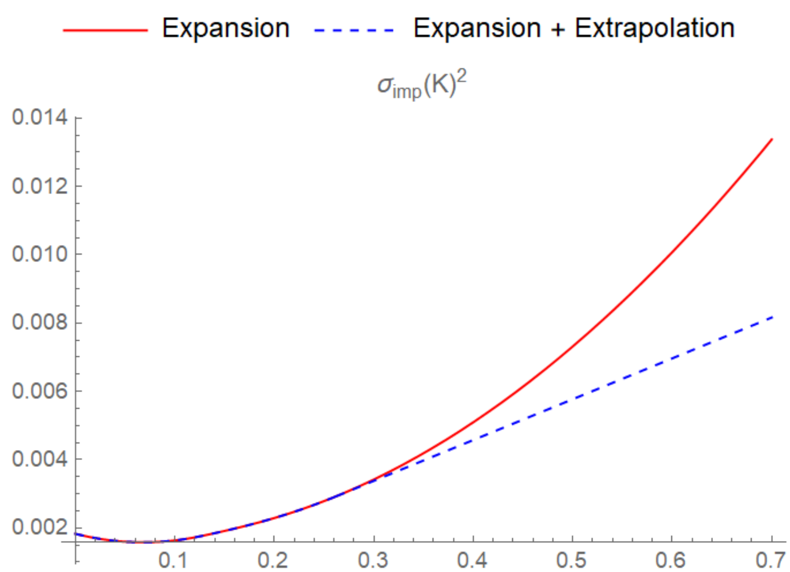
1 Problem

Typical issue for any expansion method is that implied volatility σ_{imp} or ${\sigma_{\text{imp}}}^2$ is expressed in terms power series. Hence the smile wings can be significantly overestimated or unstable for model parameters.
In original papers by P. Jackel and C. Kahl this problem is mitigated by replacing original expansion by SVI parametrization by matching derivatives at ATM. Actually it is not properly justified because tail behaviour is not defined by dynamics around ATM.
It seems that more natural approach is to find asymptotic of wings (it is approximately linear in terms of ${\sigma_{\text{imp}}}^2$) and extrapolate the expansion as linear from the point where skew given by expansion becomes larger than asymptotic skew. By practical reasons we will concentrate on positive correlation $\rho > 0$ and right wing asymptotic.



2 Asymptotic skew

Taking into account that for large y $g(y)$ becomes linear for large y $g(y) \sim \gamma y, \gamma = 2$ SDE for the process has the following form

$$dX_t = \gamma\sigma y_t dZ_t \tag{1}$$
$$dy_t = -k y_t dt + \eta dY_t \tag{2}$$
$$\rho = \text{Corr}\left(Z_t, Y_t\right) \tag{3}$$

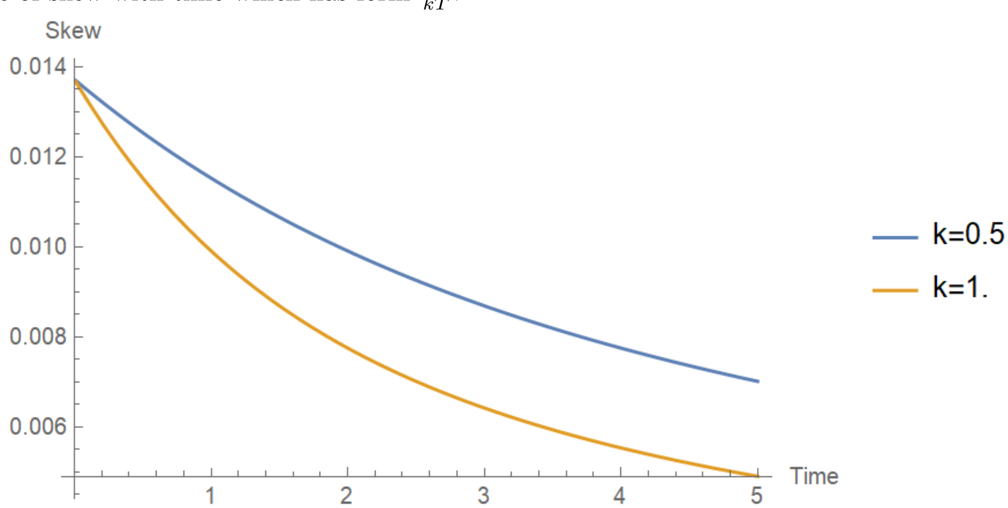
Result

Asymptotically square of implied volatility is approximately linear with slope value

$$\sigma_{\text{imp}}(T,z)^2 \sim z \times \sigma\eta\gamma\sqrt{\text{f1}[kT]}\frac{\rho + \sqrt{\rho^2 + \frac{4}{6^{1/2}}\left(1-\rho^2\right)\frac{\sqrt{\text{f2}[kT]}}{\text{f1}[kT]}}}{4} \tag{4}$$

$$f2(T) = 3\frac{e^{-4T} + (4T-5) + e^{-2T}(4+8T)}{8T^4} \tag{5}$$
$$f1(T) = \frac{e^{-2T} - 1 + 2T}{2T^2} \tag{6}$$
$$f1(0) = f2(0) = 1 \tag{7}$$

From this expression we see that as usual k is responsible for the decreasing rate of skew with time which has form $\frac{1}{kT}$.



Remark

The option price for above SDE align can be calculate by characteristic function in the same way as for Heston model. Indeed, let

$$v_t = {y_t}^2 \tag{8}$$

then by Ito lemma

$$dX_t = \gamma\sigma\sqrt{v_t}dZ_t \tag{9}$$
$$dv_t = 2y_t\left(-ky_tdt + \eta dY_t\right) + \eta^2 dt = \left(\eta^2 - 2kv_t\right)dt + 2\eta\sqrt{v_t}dY_t \tag{10}$$

We can follow standard derivation for corresponding results as in classical Heston case. The only difference is that we have here normal case instead of lognormal.

3 Derivation

Case $k = 0$

First lets consider the case when $k = 0$

$$dX_t = \sigma g\left(y_t\right)dZ_t \tag{11}$$
$$dy_t = \eta dY_t \tag{12}$$

Hence

$$y_t = \eta Y_t \tag{13}$$

Then total variance for process X_t from 0 to T is

$$W_{0,T} = \int_0^T \sigma^2 g\left(\eta Y_t\right)^2 dt \tag{14}$$

for large y, $g(y) \approx \gamma y$

$$W_{0,T} = \gamma^2\sigma^2\eta^2 \int_0^T {Y_t}^2 dt \tag{15}$$

For zero correlation $W_{0,T}$ and Z_t are independent and price of option can be calculated as expected value of Bachelier price over realization of total variance. In general case we can rewrite align as

$$dX_t = \sigma dZ_t + \gamma\sigma\eta\left(\rho Z_t + \rho_2 Y_t\right)dZ_t \tag{16}$$
$$\rho_2 = \sqrt{1-\rho^2} \tag{17}$$
$$dX_t = \sigma dZ_t + \gamma\sigma\eta\rho Z_t dZ_t + \gamma\sigma\eta\rho_2 Y_t dZ_t \tag{18}$$
$$X_t = \sigma Z_t + \frac{1}{2}\gamma\sigma\eta\rho\left({Z_t}^2 - t\right) + \gamma\sigma\eta\rho_2\int_0^t Y_u\, dZ_u \tag{19}$$

the component

$$\int_0^t Y_u \, dZ_u \tag{20}$$

is distributed as normal variable with stochastic variance

$$W_{0,T} = \int_0^T {Y_t}^2 dt \tag{21}$$

then we will take approximation

$$X_T \approx \sigma Z_T + \frac{1}{2}\gamma\sigma\eta\rho\left({Z_T}^2 - T\right) + \gamma\sigma\eta\rho_2\sqrt{\frac{1}{T}\int_0^T {Y_t}^2 dt Z_T} \tag{22}$$

$$\text{Distribution}\,(X_T) \sim \sigma\sqrt{T}Z_1 + \frac{1}{2}\gamma\sigma\eta\rho T\left({Z_1}^2 - 1\right) + \gamma\sigma\eta\rho_2 T\sqrt{\int_0^1 {Y_t}^2 dt Z_1} \tag{23}$$

Distribution of

$$\int_0^1 {Y_t}^2 dt \tag{24}$$

is known in form of Laplace transform

$$Ee^{-\lambda \int_0^T {Y_t}^2 \, dt} = \sqrt{\text{Sech}\left[t\sqrt{2\lambda}\right]} \tag{25}$$

R.H.Cameron and W.T.Martin,The Wiener measure of Hilbert neighborhoods in the space of real continuous functions, Jour.Math.Phys.Massachusetts Inst.Technology, 23 (1944), 195−−209.
M.Kac,On distributions of certain Wiener functionals,Transactions of the American Mathematical Society, vol .65, issue .1, pp .65-66, 1949.

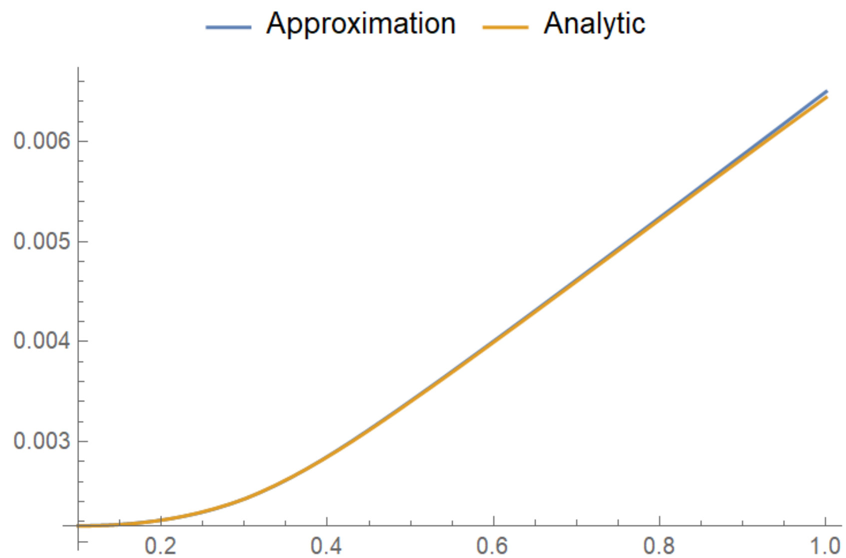
However for our goal it is convenient to use approximation as square of normal variable i.e.

$$\int_0^1 {Y_t}^2 dt \approx a{Y_1}^2 + b \tag{26}$$

where coefficients a, b are calculated by matching first two moments. After calculation we will get

$$\int_0^1 {Y_t}^2 dt \approx \frac{1}{\sqrt{6}}{Y_1}^2 + \frac{3-\sqrt{6}}{6} \tag{27}$$

Actually this approximation is quite good if we compare square of implied volatility for zero correlation case calculated with using analytic density from inverse Laplace transform and approximation



Now we have the following approximation

$$\text{Distribution}\left(X_T\right) \sim \sigma\sqrt{T}Z_1 + \frac{1}{2}\gamma\sigma\eta\rho T\left({Z_1}^2 - 1\right) + \gamma\sigma\eta\rho_2 T\sqrt{\frac{1}{\sqrt{6}}{Y_1}^2 + \frac{3-\sqrt{6}}{6}}Z_1 \tag{28}$$

asymptotically main term will be (we take only the second order term)

$$X_T \approx \frac{1}{2}\gamma\sigma\eta\rho T{Z_1}^2 + \frac{1}{6^{1/4}}\gamma\sigma\eta\rho_2 T Y_1 Z_1 \tag{29}$$

Asymptotically option value for normal model with variance w has order

$$e^{-\frac{z^2}{2w}} \tag{30}$$

Option value can be calculated from integral

$$\int\int e^{-\frac{1}{2}\left(x^2+y^2\right)}\left(bx^2 + cyx - z\right)_+ dxdy \tag{31}$$

$$b = \frac{1}{2}\gamma\sigma\eta\rho T \tag{32}$$

$$c = \frac{1}{6^{1/4}}\gamma\sigma\eta\rho_2 T \tag{33}$$

As $z \to \infty$ the leading term will corresponds to maximum value

$$e^{-\frac{1}{2}\left(x^2+y^2\right)} \tag{34}$$

under constrain

$$bx^2 + cyx \geq z \tag{35}$$

Solving this maximization problem (see Appendix) we will get

$$x^2 + y^2 = 2\frac{\sqrt{c^2+b^2}-b}{c^2}z \tag{36}$$

$$\mathrm{Exp}\left(-\frac{1}{2}\left(x^2+y^2\right)\right) = \mathrm{Exp}\left(-\frac{\sqrt{c^2+b^2}-b}{c^2}z\right) \tag{37}$$

For flat volatility case asymptotic in terms of variance w is

$$\mathrm{Exp}\left(-\frac{z^2}{2w}\right) \tag{38}$$

hence implied variance will be

$$w = \frac{\sqrt{c^2+b^2}+b}{2}z \tag{39}$$

For correlation 1, $c = 0$

$$w = bz = \frac{1}{2}\gamma\sigma\eta T \tag{40}$$

For correlation zero, $b = 0$

$$w = \frac{c}{2}z = \frac{1}{2}\frac{1}{6^{1/4}}\gamma\sigma\eta\rho_2 T z \tag{41}$$

Mixed case

$$w = \frac{\sqrt{\left(\frac{1}{6^{1/4}}\gamma\sigma\eta\rho_2 T\right)^2 + \left(\frac{1}{2}\gamma\sigma\eta\rho T\right)^2} + \frac{1}{2}\gamma\sigma\eta\rho T}{2}z = \frac{1}{2}\gamma\sigma\eta T\frac{\sqrt{\frac{4}{6^{1/2}}{\rho_2}^2+\rho^2}+\rho}{2}z \tag{42}$$

Case $k > 0$

We follow the same path as for $k = 0$

$$dX_t = \gamma\sigma y_t dZ_t \tag{43}$$
$$dy_t = -ky_t dt + \eta dY_t \tag{44}$$

$$y_t = \int_0^t \eta e^{-k(t-u)} dY_u = \rho \int_0^t \eta e^{-k(t-u)} dZ_u + \rho_2 \int_0^t \eta e^{-k(t-u)} dU_u \tag{45}$$

$$dX_t = \gamma\sigma \left(\rho \int_0^t \eta e^{-k(t-u)} dZ_u + \rho_2 \int_0^t \eta e^{-k(t-u)} dU_u \right) dZ_t \tag{46}$$

$$dX_t = \gamma\sigma\rho\eta \int_0^t e^{-k(t-u)} dZ_u dZ_t + \rho_2\gamma\sigma\eta \int_0^t e^{-k(t-u)} dU_u dZ_t \tag{47}$$

$$X_T = \gamma\sigma\rho\eta \int_0^T \int_0^t e^{-k(t-u)} dZ_u dZ_t + \rho_2\gamma\sigma\eta \int_0^T \int_0^t e^{-k(t-u)} dU_u dZ_t \tag{48}$$

approximations of terms by matching the moments (see Appendix. VaR Z.)

$$\int_0^T \int_0^t e^{-k(t-u)} dZ_u dZ_t \approx \frac{1}{2} f_1(kT) \left({Z_T}^2 - T \right) = T\beta_T \left({Z_1}^2 - 1 \right) \tag{49}$$

$$\beta_T = \frac{1}{2}\sqrt{f_1(kT)} \tag{50}$$

$$f_1(T) = \frac{e^{-2T} - 1 + 2T}{2T^2} \tag{51}$$

for the second term (see Appendix. VaR W.).

$$\int_0^T \int_0^t e^{-k(t-u)} dU_u dZ_t \approx \sqrt{\alpha_T {Y_T}^2 + \text{const}} Z_T \approx \sqrt{\alpha_T} \text{Abs}\left(Y_T\right) Z_T \approx \sqrt{\alpha_T} T \text{Abs}\left(Y_1\right) Z_1 \tag{52}$$

Remark: We could match moment of integral itself, but more accurate result will be in case when we match moment of total variance.

$$\int_0^T \left(\int_0^t e^{-k(t-u)} dU_u \right)^2 dt = \alpha_T {Y_T}^2 T + \text{Mean} \tag{53}$$

$$\alpha_T = \sqrt{\frac{f_2[kT]}{6}} \tag{54}$$

$$f_2(T) = 3\frac{e^{-4T}\left(1 + e^{4T}(-5+4T) + e^{2T}(4+8T)\right)}{8(T)^4} \tag{55}$$

$$X_T \approx \sigma \mathrm{Z}_T + \gamma\sigma\rho\eta\beta_T T \left({Z_1}^2 - 1 \right) + \rho_2\gamma\sigma\eta\sqrt{\alpha_T} T \text{Abs}\left(Y_1\right) Z_1 \tag{56}$$

$$b = \gamma\sigma\rho\eta\beta_T T \tag{57}$$

$$c = \rho_2\gamma\sigma\eta\sqrt{\alpha_T}T \tag{58}$$

$$w = \frac{\sqrt{c^2+b^2}+b}{2}z \tag{59}$$

$$\frac{\sqrt{c^2+b^2}+b}{2} = \frac{\gamma\sigma\eta T}{2}\left(\rho\beta_T + \sqrt{\rho^2{\beta_T}^2 + {\rho_2}^2\alpha_T}\right) \tag{60}$$

$$= \gamma\sigma\eta T\beta_T\frac{\rho + \sqrt{\rho^2 + {\rho_2}^2\alpha_T/{\beta_T}^2}}{2} \tag{61}$$

$$\sigma(z)^2 \sim \gamma\sigma\eta\beta_T\frac{\rho + \sqrt{\rho^2 + {\rho_2}^2\alpha_T/{\beta_T}^2}}{2}z \tag{62}$$

$$\sigma(z)^2 \sim \gamma\sigma\eta\sqrt{f_1(kT)}\frac{\rho + \sqrt{\rho^2 + \frac{4}{6^{1/2}}{\rho_2}^2\frac{\sqrt{f_2[kT]}}{f_1(kT)}}}{4}z \tag{63}$$

4 Appendix

Minimization

Obviously minimal value should hold on boundary

$$bx^2 + cyx = z \tag{64}$$

$$y \to \frac{-bx^2+z}{cx} \tag{65}$$

$$f(x) = x^2 + y^2 = x^2 + \frac{\left(-bx^2+z\right)^2}{c^2x^2} \tag{66}$$

$$f' = \left(2 + \frac{2b^2}{c^2}\right)x - \frac{2z^2}{c^2x^3} = 0 \tag{67}$$

$$x \to \frac{\sqrt{z}}{\left(b^2+c^2\right)^{1/4}} \tag{68}$$

$$x^2 + y^2 = \frac{2\left(\sqrt{b^2+c^2}-b\right)z}{c^2} \tag{69}$$

Var Z

We will approximate integral as

$$\int_0^T \int_0^t e^{-k(t-u)} dZ_u dZ_t = \sqrt{\frac{1}{2}\frac{e^{-2kT} - 1 + 2kT}{4k^2T^2}} \left(Z_T{}^2 - T\right) \tag{70}$$

$$\mathrm{VaR}\left(\int_0^T \int_0^t e^{-k(t-u)} dZ_u dZ_t\right) = \frac{e^{-2kT} - 1 + 2kT}{4k^2} \tag{71}$$

$$\tag{72}$$

$$E\left(\int_0^T \int_0^t e^{-k(t-u)} dZ_u dZ_t\right)^2 \tag{73}$$

$$= E\int_0^T \int_0^{t1} \int_0^T \int_0^{t2} e^{-k(t1-u1)} e^{-k(t2-u2)} dZ_{u1} dZ_{t1} dZ_{u2} dZ_{t2} \tag{74}$$

$$= E\int_0^T \int_0^T \int_0^T \int_0^T 1[u1 < t1 < T]1[u2 < t2 < T] e^{-k(t1-u1)} e^{-k(t2-u2)} dZ_{u1} dZ_{t1} dZ_{u2} dZ_{t2} \tag{75}$$

$$= I_{u1=u2,t1=t2} + I_{u1=t2,t1=u2} \tag{76}$$

$$I_{u1=t2,t1=u2} \tag{77}$$

$$= \int_0^T \int_0^T \int_0^T \int_0^T 1[u1 < t1 < T]1[t1 < u1 < T] e^{-k(t1-u1)} e^{-k(u1-t1)} du1dt1 \tag{78}$$

$$= 0 \tag{79}$$

$$I_{u1=u2,t1=t2} = \tag{80}$$

$$= \int_0^T \int_0^T 1[u1 < t1 < T] e^{-2k(t1-u1)} du1dt1 \tag{81}$$

$$= \int_0^T \int_0^{t1} e^{-2k(t1-u1)} du1dt1 \tag{82}$$

$$= \frac{e^{-2kT} - 1 + 2kT}{4k^2} \tag{83}$$

taking into account expression for variance of $Z_T{}^2$ we will get declared expression.

$$\mathrm{VaR}\left(Z_T{}^2\right) = \mathrm{E}Z_T{}^4 - \left(\mathrm{E}Z_T{}^2\right)^2 = 3T^2 - T^2 = 2T^2 \tag{84}$$

VaR W

$$\text{VaR}W = E\left(\int_0^T \left(\int_0^t e^{-k(t-u)} dY_u\right)^2 dt\right)^2 = E\left(\int_0^T \int_0^t \int_0^t e^{-k(t-u)} e^{-k(t-v)} dY_u dY_v dt\right)^2 \tag{85}$$

$$=E \int_0^T \int_0^{t1} \int_0^{t1} \int_0^T \int_0^{t2} \int_0^{t2} e^{-k(t1-u1)} e^{-k(t1-v1)} e^{-k(t2-u2)} e^{-k(t2-v2)} dY_{u1} dY_{v1} dt1 dY_{u2} dY_{v2} dt2$$

$$= E \int_0^T \int_0^T \int_0^T \int_0^T \int_0^T \int_0^T 1[\max(u1, v1) < t1 < T] 1[\max(u2, v2) < t2 < T]$$
$$e^{-k(t1-u1)} e^{-k(t1-v1)} e^{-k(t2-u2)} e^{-k(t2-v2)} dY_{u1} dY_{v1} dt1$$
$$= I_{u1=v1,u2=v2} + I_{u1=u2,v1=v2} + I_{u1=v2,u2=v1}$$

Note that

$$I_{u1=u2,v1=v2} =$$
$$= \int_0^T \int_0^T \int_0^T \int_0^T 1[\max(u1, v1) < t1 < T] 1[\max(u1, v1) < t2 < T]$$
$$e^{-k(t1-u1)} e^{-k(t1-v1)} e^{-k(t2-u1)} e^{-k(t2-v1)} du1 dv2 dt1 dt2$$
$$= \left(E \int_0^T \left(\int_0^t e^{-k(t-u)} dY_u\right)^2 dt\right)^2$$

Hence variance

$$VaR(W) = I_{u1=v1,u2=v2} + I_{u1=v2,u2=v1} \tag{86}$$

$$I_{u1=v1,u2=v2} =$$
$$= 2 \int_0^T \int_{t1}^T \int_0^{t1} \int_0^{t1} 1[\max(u1, v1) < t1 < T] 1[\max(u1, v1) < t2 < T]$$
$$e^{-k(t1-u1)} e^{-k(t1-v1)} e^{-k(t2-u1)} e^{-k(t2-v1)} dv1 du1 dt2 dt1$$
$$= \frac{e^{-4kT} \left(1 + e^{4kT}(-5 + 4kT) + e^{2kT}(4 + 8kT)\right)}{16k^4}$$

In similar way

$$
\begin{aligned}
&I_{u1=v2,u2=v1} = \\
&\int_0^T \int_0^T \int_0^T \int_0^T 1[\max(u1,v1) < t1 < T] 1[\max(v1,u1) < t2 < T] \\
&\quad e^{-k(t1-u1)} e^{-k(t1-v1)} e^{-k(t2-v1)} e^{-k(t2-u1)} du1dv1dt1dt2 \\
&= \frac{e^{-4kT}\left(1 + e^{4kT}(-5+4kT) + e^{2kT}(4+8kT)\right)}{16k^4}
\end{aligned}
$$

Integral approximation

Short remark on approximation

$$\int_0^T y_t \, dZ_t \approx \sqrt{\frac{1}{T}\int_0^T {y_t}^2 dt} Z_T \tag{87}$$

This can be justified using Brownian Bridge representation

$$Z_t = \frac{t}{T} Z_T + B_t \tag{88}$$

$$dZ_t = \frac{Z_T}{T} dt + dB_t \tag{89}$$

$$\mathrm{Cov}\,(dB_s, dB_t) = -\frac{dsdt}{T} \tag{90}$$

$$\int_0^T y_t \, dZ_t = \frac{Z_T}{T} \int_0^T y_t \, dt + \int_0^T y_t \, dB_t \tag{91}$$

First term has second order in terms of Z, y scale and the second is first order. And they are independent for fixed realization of y_t.

$$
\begin{aligned}
\mathrm{VaR}\left(\int_0^T y_t \, dB_t\right) &= E\left(\int_0^T y_t \, dB_t\right)^2 &(92)\\
&= E\int_0^T \int_0^T y_s y_t dB_s dB_t &(93)\\
&= \int_0^T {y_t}^2 \, dB_s - \frac{1}{T}\int_0^T \int_0^T y_s y_t ds dt &(94)\\
&= \int_0^T {y_t}^2 \, dt - \frac{1}{T}\left(\int_0^T y_t \, dt\right)^2 &(95)
\end{aligned}
$$

$$\mathrm{VaR}\left(\frac{Z_T}{T}\int_0^T y_t\,dt\right)=\frac{1}{T}\left(\int_0^T y_t\,dt\right)^2 \tag{96}$$

It is consistent with

$$\mathrm{VaR}\left(\int_0^T y_t\,dZ_t\right)=\int_0^T {y_t}^2\,dt=\mathrm{VaR}\left(\frac{Z_T}{T}\int_0^T y_t\,dt\right)+\mathrm{VaR}\left(\int_0^T y_t\,dB_t\right) \tag{97}$$