HypHyp model perturbation expansion

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1 Result

$$dX_t = f\left(X_t\right)\sigma\left(y_t\right)dZ_t \tag{1}$$
$$dy_t = -ky_t dt + \eta dY_t \tag{2}$$
$$\rho = \mathrm{Corr}\left(Z_t, Y_t\right) \tag{3}$$

If we consider expansion over volatility η then for HypHyp model the following approximation holds for strike z and initial state x_0

$$V = V_0 + \eta G_{\mathrm{xy}} P_{0,T} + \eta^2\left(G_{\mathrm{yy}} + G_{\mathrm{xy},2}\right)P_{0,T} \tag{4}$$

Implied volatility is defined as

$$\sigma\left(x_0, z\right)^2 = {\sigma_1}^2 \text{ if } z < x_0 \tag{5}$$
$$\sigma\left(x_0, z\right)^2 = \frac{1}{2}{\sigma_1}^2 + \frac{1}{2}{\sigma_2}^2 \text{if} z \geq x_0 \tag{6}$$

where

$$T{\sigma_1}^2 = w_{\mathrm{LV}} + 2\eta G_{\mathrm{xy}} + \eta^2\left(2\left(G_{\mathrm{yy}} + G_{\mathrm{xy},2}\right) - \ {G_{\mathrm{xy}}}^2 H_2\right) \tag{7}$$

$$T{\sigma_2}^2 = w_{\mathrm{LV}} + 2\frac{\sqrt{1 + 2\left(\eta G_{\mathrm{xy}} + \eta^2\left(G_{\mathrm{yy}} + G_{\mathrm{xy},2}\right)\right)H_2} - 1}{H_2} \tag{8}$$

and

$$P_{0,T}\left(x_0, z\right) = \phi\left(z - x_0, \sigma^2 T\right) \tag{9}$$
$$G_{\mathrm{xy}} = B_{\mathrm{xy},1}H_1 \tag{10}$$
$$B_{\mathrm{xy},1} = \rho\sigma^3\frac{e^{-kT} + kT - 1}{k^2} \tag{11}$$

$$G_{\mathrm{yy}} = B_{\mathrm{yy},0} + B_{\mathrm{yy},2}H_2 \tag{12}$$
$$B_{\mathrm{yy},0} = \frac{1}{4}\sigma^2\frac{e^{-2kT}\left(1 + e^{2kT}(-1 + 2kT)\right)}{k^2} \tag{13}$$
$$B_{\mathrm{yy},2} = \frac{1}{8}\sigma^4\frac{2e^{-2kT}\left(-1 + 4e^{kT} + e^{2kT}(-3 + 2kT)\right)}{k^3} \tag{14}$$

$$G_{\mathrm{xy},2} = B_{\mathrm{xy}2,2} H_2 + B_{\mathrm{xy}2,4} H_4 \tag{15}$$

$$B_{\mathrm{xy}2,2} = -\rho^2 \frac{1}{2} \frac{\left(7 + e^{-2kT} - 4kT - 2e^{-kT}(4+kT)\right)\sigma^4}{k^3}; \tag{16}$$

$$B_{\mathrm{xy}2,4} = \rho^2 \frac{1}{8} \frac{4e^{-2kT}\left(1 + e^{kT}(-1+kT)\right)^2 \sigma^6}{k^4} \tag{17}$$

$$H_i = H_i\left(z - x_0, {\sigma_0}^2 T\right) \tag{18}$$

$$H_i(z-x,w) = \frac{{\partial_x}^i \phi(z-x,w)}{\phi(z-x,w)} \tag{19}$$

$${\sigma_{\mathrm{LV}}}^2 = \sigma^2 T F\left(x_0, z\right) + \sigma^4 \frac{T^2}{2(z-\mathrm{x0})^2}\left(\frac{f^2\left(x_0,z\right) + f^2\left(x_0,z\right)}{2} - F\left(x_0,z\right)\right) \tag{20}$$

$$F\left(x_0,z\right) = \frac{1}{z-\mathrm{x0}} \int_{x_0}^{z} f[x]^2\, dt \tag{21}$$

and can be valuated explicitly

$$F\left(x_0,z\right) = z - x_0 + \frac{1}{2}\left(z-x_0\right)^2 \beta + \tag{22}$$

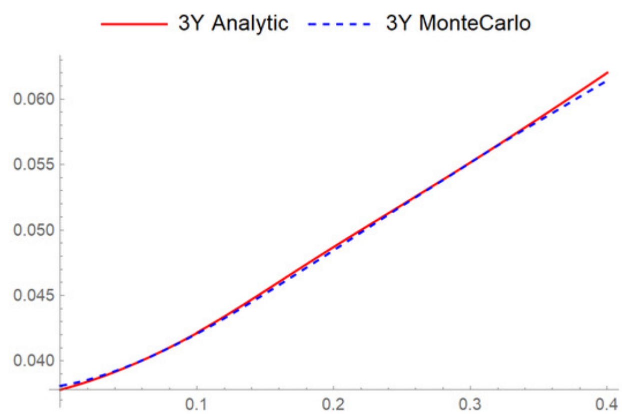
$$+\left(\frac{1}{6}\left(z-x_0\right)^3 + \frac{1}{2}\left(z-x_0\right)\right)\beta^2 - \left(z - x_0 + \left(\frac{1}{6} + \frac{1}{4}\left(z-x_0\right)^2\right)\beta\right)\mathrm{Abs}[\beta] \tag{23}$$

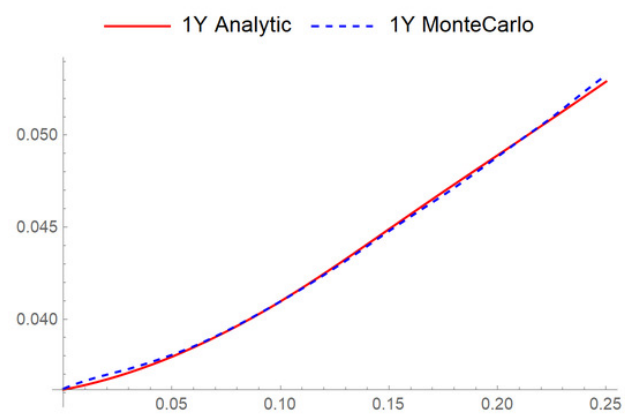
$$+\sqrt{1+\left(x_0-z\right)^2}\left(\frac{1}{4}\left(x_0-z\right)\beta^2 + \left(\frac{z-x_0}{2} + \frac{1}{6}\left(1+\left(z-x_0\right)^2\right)\beta\right)\mathrm{Abs}[\beta]\right) \tag{24}$$

$$+\left(\frac{\beta^2}{4} - \frac{\mathrm{Abs}[\beta]}{2}\right)\mathrm{ArcSinh}\left[x_0 - z\right] \tag{25}$$

2 Test with MonteCarlo simulation

$$x_0 = 10\%,\ \ \alpha = 0.4,\ \ \beta = 0.2, \sigma_0 = 0.04, \rho = 0.5, k = 0.3, \eta = \alpha\sqrt{2k} \tag{26}$$





3 Perturbation approach

Perturbation method is convenient tool which has origin from mathematical physics and widely applied for this type of models (see references for few examples).

If we have solution for operator A of equation

$$-\partial_t P = AP \tag{27}$$

then solution for operator $A + \epsilon B$ can be represented as series

$$-\partial_t P = (A+\epsilon B)PP_\epsilon = P_0 + \epsilon \int_0^T P_{0,t} B P_{t,T} dt + \epsilon^2 \int_0^T \int_0^{t_1} P_{0,t_1} B P_{t_1,t_2} B P_{t_2,T} dt ... \tag{28}$$

For example for HypHyp model we could put

$$A = -\frac{1}{2}\sigma(0)^2 {\partial_x}^2 \tag{29}$$

And perturbation operator B

$$B_t(y) = \frac{1}{2}\sigma^2\left(f(x)^2 - 1\right){\partial_x}^2 + \rho\eta f(x)\sigma(y)\partial_x\partial_y - ky\partial_y + \frac{1}{2}{\eta_t}^2{\partial_y}^2 \tag{30}$$

To simplify valuation we consider additive components of local and stochastic volatility. So we missing cross effect of factor $\rho\times$LocalVolSkew $\rho\times$LocalVolSkew. This can be fixed if necessary.

It can be shown that first order approximation for local volatility is given as averaging variance along Brownian bridge see [2].

$$T\sigma_{\text{imp}}(z)^2 \approx \int_{x_0}^{z} E\left(\sigma_{\text{loc}}\left(X_t\right)^2 | X_T = z\right) dt \tag{31}$$

where X_t precess with constant volatility σ and hence can be represented through standard Brownian bridge B_t

$$X_t = x_0 + (z - x_0)\frac{t}{T} + \sigma B_t(T) \tag{32}$$

$$\tag{33}$$

$$T\sigma_{\text{imp}}(z)^2 \approx \int_{x_0}^{z} \mathrm{E}\sigma_{\text{loc}}\left(x_0 + (z - x_0)\frac{t}{T} + \sigma B_t(T)\right)^2 dt \tag{34}$$

Expected value of squared local volatility we can estimate through value from mean point and convexity adjustment

$$\mathrm{E}\sigma_{\text{loc}}\left(x_0 + (z - x_0)\frac{t}{T} + \sigma B_t(T)\right)^2 \tag{35}$$

$$\approx \mathrm{E}\sigma_{\text{loc}}\left(x_0 + (z - x_0)\frac{t}{T}\right)^2 + \frac{1}{2}\sigma^2\left(\sigma_{\text{loc}}\left(x_0 + (z - x_0)\frac{t}{T}\right)^2\right)'' \mathrm{Var}\left(B_t(T)\right) \tag{36}$$

$$\mathrm{Var}\left(B_t(T)\right) = \frac{t(T - t)}{T} \tag{37}$$

this can be rewritten in form

$$\mathrm{E}\sigma_{\text{loc}}\left(x_0 + (z - x_0)\frac{t}{T} + \sigma B_t(T)\right)^2 \tag{38}$$

$$\approx \sigma^2 T F\left(x_0, z\right) \tag{39}$$

$$+ \sigma^4\frac{T^2}{2(z - \mathrm{x0})^2}\left(\frac{f^2\left(x_0, z\right) + f^2\left(x_0, z\right)}{2} - F\left(x_0, z\right)\right) \tag{40}$$

$$F\left(x_0, z\right) = \frac{1}{z - \mathrm{x0}}\int_{x_0}^{z} f^2[x]^2\, dt \tag{41}$$

4 Stochastic volatility component

It is convenient to remove drift by transformation. Derivation is close to [1], [3]. [3] gives generic algorithm for any LSV models.

$$\tilde{y}_t = e^{kt} y_t \tag{42}$$
$$d\tilde{y}_t = k\tilde{y}_t dt + -k\tilde{y}_t dt + \eta \mathrm{e}^{kt} dY_t \tag{43}$$

Then

$$dX_t = \sigma\left(e^{-kt}\tilde{y}_t\right) dZ_t \tag{44}$$

Using the same notation for y_t

$$dX_t = \sigma\left(e^{-kt} y_t\right) dZ_t \tag{45}$$
$$dy_t = \eta_t dY_t \tag{46}$$
$$\eta_t = \eta e^{kt} \tag{47}$$

We will consider this model as perturbation of flat volatility model

$$dX_t = \sigma\left(e^{-kt} y_t\right) dZ_t \tag{48}$$
$$y_t = y \tag{49}$$

Let's write the solution in terms of total variance $W_{t,T}(y)$ with initial $y_0 = y$

$$y_t = y \tag{50}$$
$$V_{t,T}(x, y) = V\left(x, W_{t,T}(y)\right) \tag{51}$$

$$V(x, w) = \sqrt{w}\phi\left[\frac{x - x_0}{\sqrt{w}}\right] - (x - x_0)\ \Phi\left[-\frac{x - x_0}{\sqrt{w}}\right] \tag{52}$$

where ϕ, Φ are density and distribution functions for normal distribution.

$$W_{t,T}(y) = \int_t^T \sigma\left(ye^{-ku}\right)^2 du = \int_t^T b\left(ye^{-ku}\right) du \tag{53}$$

$$b(y) = \sigma(y)^2 = \sigma^2 g(y)^2 = \sigma^2\left(1 + 2y + 2y^2 + ...\right) \tag{54}$$
$$\sigma(x) = \sigma_0 g(x) \tag{55}$$
$$g(y) = y + \sqrt{1 + y^2} = 1 + y + \frac{1}{2}y^2 + ... \tag{56}$$

transition density from time t to time T is

$$p_{t,T}(x,z,y) = \phi\left(z - x, W_{t,T}(y)\right) \tag{57}$$

Option value V and density p solves PDE
perturbation operator B

$$\partial_t p = -\frac{1}{2}\sigma\left(ye^{-kt}\right)^2 {\partial_x}^2 p \tag{58}$$

In operator form corresponding semi-group is

$$P_{t,T} = e^{-\frac{1}{2}\int_t^T \sigma\left(ye^{-ku}\right)^2 {\partial_x}^2 du} \tag{59}$$

$$\tag{60}$$

$$\begin{aligned} B_t(y) &= \frac{1}{2}\left(b_t(y) - 1\right){\partial_x}^2 + \rho\eta_t\sigma_t(y)\partial_x\partial_y + \frac{1}{2}{\eta_t}^2{\partial_y}^2 && (61)\\ &= B_{\mathrm{xx}} + B_{\mathrm{xy}} + B_{\mathrm{yy}} && (62)\end{aligned}$$

$$\begin{aligned} \sigma_t(y) &= \sigma\left(e^{-kt}y\right) && (63)\\ b_t(y) &= \sigma\left(e^{-kt}y\right)^2 && (64)\end{aligned}$$

Cross term $B_{\mathbf{xy}}$

$$B_{\mathrm{xy}} = \rho\eta_t\sigma_t(y)\partial_x\partial_y \tag{65}$$

$$\partial_y V_{t,T} = \partial_y W_{t,T}\partial_W V_{t,T} \tag{66}$$

using that

$$\partial_W V_{t,T} = \frac{1}{2}\partial_{x,x}V_{t,T} \tag{67}$$

we can continue

$$\partial_y V_{t,T} = \frac{1}{2}\partial_y W_{t,T}\partial_{x,x}V_{t,T} = \frac{1}{2}\partial_y W_{t,T}P_{t,T} \tag{68}$$

$$\partial_x \partial_y V_{t,T} = \partial_x \frac{1}{2} \partial_y W_{t,T} P_{t,T} = \frac{1}{2} \partial_y W_{t,T} \partial_x P_{t,T} \tag{69}$$

Using transitivity of ∂_x and $P_{0,t}$

$$P_{0,t} B_{\mathrm{xy}} V_{t,T} = \rho \eta_t \sigma_t(y) \frac{1}{2} \partial_y W_{t,T} P_{0,t} \partial_x P_{t,T} = \frac{1}{2} \rho \eta_t \sigma_t(y) \partial_y W_{t,T} \partial_x P_{0,T} \tag{70}$$

$$\int_0^T P_{0,t} B_{\mathrm{xy}} V_{t,T} dt = \frac{1}{2} \rho \ \partial_x P_{0,T} \int_0^T \sigma_t(y) \eta_t \partial_y W_{t,T} dt \tag{71}$$

for $y = y_0 = 0$ and HypHyp case

$$\int_0^T P_{0,t} B_{\mathrm{xy}} V_{t,T} dt = B_{\mathrm{xy},1} \partial_x P_{0,T} \tag{72}$$

$$B_{\mathrm{xy},1} = \frac{1}{2} \rho \int_0^T \sigma(0) \eta_t \partial_y W_{t,T} dt = \eta \rho \sigma^3 \frac{e^{-kT} \left(1 + e^{kT}(-1 + kT)\right)}{k^2} \tag{73}$$

In terms of implied variance

$$\Delta \mathrm{W} = 2 \sigma T \Delta \sigma = 2 B_{\mathrm{xy},1} \partial_x P_{0,T} \tag{74}$$

$$\partial_x P_{0,T} = \frac{z - x_0}{W_{0,T}} \tag{75}$$

In terms of implied vol

$$\Delta \sigma = \frac{1}{\sigma T} B_{\mathrm{xy},1} \partial_x P_{0,T} = \eta \rho \frac{\left(kT + e^{-kT} - 1\right)}{k^2 T^2} (z - x_0) \tag{76}$$

for short time $T \to 0$

$$\frac{1}{2} \eta \rho \sigma (z - x_0) \tag{77}$$

So for short time skew is $\frac{1}{2} \eta \rho \sigma$ and decrease as $\frac{1}{\mathrm{kT}}$ with expiration.
for long time $T \to \infty$ asymptotic is

$$\Delta \sigma = \eta \rho \frac{z - x_0}{kT} \tag{78}$$

That is why we need local volatility component to have non zero skew for long maturities.

Pure stochastic term $B_{\mathbf{yy}}$

$$B_{\text{yy}} = \frac{1}{2}{\eta_t}^2{\partial_y}^2 \tag{79}$$

$$\partial_W V_{t,T} = \frac{1}{2}\partial_{x,x}V_{t,T} \tag{80}$$

$$\partial_y V_{t,T} = \partial_y W_{t,T}\partial_W V_{t,T} = \frac{1}{2}\partial_y W_{t,T}\partial_{x,x}V_{t,T} = \frac{1}{2}\partial_y W_{t,T}P_{t,T} \tag{81}$$

$$\partial_{y,y}V_{t,T} = \frac{1}{2}\partial_y\left(\partial_y W_{t,T}P_{t,T}\right) = \frac{1}{2}\partial_{y,y}W_{t,T}P_{t,T} + \frac{1}{2}\partial_y W_{t,T}\partial_y P_{t,T} \tag{82}$$

$$\partial_y P_{t,T} = \partial_y W_{t,T}\partial_W P_{t,T} = \frac{1}{2}\partial_y W_{t,T}\partial_{x,x}P_{t,T} \tag{83}$$

$$\partial_{y,y}V_{t,T} = \frac{1}{2}\left({\partial_y}^2 W_{t,T}\right)P_{t,T} + \frac{1}{4}\left(\partial_y W_{t,T}\right)^2{\partial_x}^2 P_{t,T} \tag{84}$$

$$P_{0,t}{\eta_t}^2\partial_{y,y}V_{t,T} = \frac{1}{2}{\eta_t}^2\left(\partial_{y,y}W_{t,T}\right)P_{0,T} + \frac{1}{4}{\eta_t}^2\left(\partial_y W_{t,T}\right)^2{\partial_x}^2 P_{0,T} \tag{85}$$

For any variance function $W_{t,T}(y)$ we get expression for pure SV term of derivatives

$$\int_0^T P_{0,t}B_{\text{yy}}V_{t,T}dt = \frac{1}{4}P_{0,T}\int_0^T {\eta_t}^2\partial_{y,y}W_{t,T}dt + \frac{1}{8}{\partial_x}^2 P_{0,T}\int_0^T {\eta_t}^2\left(\partial_y W_{t,T}\right)^2 dt \tag{86}$$

In particular for HypHyp model we get

$$\partial_y W_{t,T} = \int_t^T e^{-ku}b'\left(ye^{-ku}\right)du \tag{87}$$

$$\tag{88}$$

$${\partial_y}^2 W_{t,T} = \int_t^T e^{-2ku}b''\left(ye^{-ku}\right)du \tag{89}$$

For $y = y_0 = 0$

$$\partial_y W_{t,T} = b'(0)\frac{e^{-kt} - e^{-kT}}{k} \tag{90}$$

$$\partial_y{}^2 W_{t,T} = b''(0)\frac{e^{-2kt} - e^{-2kT}}{2k} \tag{91}$$

$$B_{\text{yy},0} = \frac{1}{4}\int_0^T {\eta_t}^2 \partial_{y,y} W_{t,T} = \frac{1}{4}\eta^2\sigma^2 \frac{e^{-2kT}\left(1 + e^{2kT}(-1+2kT)\right)}{k^2} \tag{92}$$

$$B_{\text{yy},2} = \frac{1}{8}\int_0^T {\eta_t}^2 \left(\partial_y W_{t,T}\right)^2 dt = \frac{1}{8}\eta^2\sigma^4 \frac{2e^{-2kT}\left(-1 + 4e^{kT} + e^{2kT}(-3+2kT)\right)}{k^3} \tag{93}$$

And finally

$$\int_0^T P_{0,t} B_{\text{yy}} V_{t,T} dt = B_{\text{yy},0} P_{0,T} + B_{\text{yy},2} \partial_x{}^2 P_{0,T} \tag{94}$$

Second order cross term

$$\text{CrossTerm}_2 \tag{95}$$

$$= P_{0,t_1} B_{\text{xy}} P_{t_1,t_2} B_{\text{xy}} V_{t_2,T} \quad = \rho^2 P_{0,t_1} \sigma_{t_1}(y)\eta_{t_1} \partial_x \partial_y P_{t_1,t_2} \sigma_{t_2}(y)\eta_{t_2} \partial_x \partial_y V_{t_2,T} \tag{96}$$

from the first order cross term we have

$$A = P_{t_1,t_2} \sigma_{t_2}(y)\eta_{t_2} \partial_x \partial_y V_{t_2,T} = \frac{1}{2}\sigma_{t_2}(y)\eta_{t_2} \partial_y W_{t_2,T} \partial_x P_{t_1,T} \tag{97}$$

$$\partial_y A = \frac{1}{2}\partial_y \left(\sigma_{t_2}(y)\eta_{t_2} \partial_y W_{t_2,T} \partial_x P_{t_1,T}\right) \tag{98}$$

$$= \frac{1}{2}\partial_y \left(\sigma_{t_2}(y)\eta_{t_2} \partial_y W_{t_2,T}\right) \partial_x P_{t_1,T} + \frac{1}{2}\sigma_{t_2}(y)\eta_{t_2} \partial_y W_{t_2,T} \partial_y \partial_x P_{t_1,T} \tag{99}$$

$$= \frac{1}{2}\partial_y \left(\sigma_{t_2}(y)\eta_{t_2} \partial_y W_{t_2,T}\right) \partial_x P_{t_1,T} + \frac{1}{4}\sigma_{t_2}(y)\eta_{t_2} \partial_y W_{t_2,T} \partial_y W_{t_1,T} \partial_x{}^3 P_{t_1,T} \tag{100}$$

$$P_{0,t_1}\sigma_{t_1}(y)\eta_{t_1}\partial_x\partial_y A \tag{101}$$
$$= \frac{1}{2}\sigma_{t_1}(y)\eta_{t_1}\partial_y\left(\sigma_{t_2}(y)\eta_{t_2}\partial_y W_{t_2,T}\right)\partial_x{}^2 P_{0,T} \tag{102}$$
$$+ \frac{1}{4}\sigma_{t_1}(y)\eta_{t_1}\sigma_{t_2}(y)\eta_{t_2}\partial_y W_{t_2,T}\partial_y W_{t_1,T}\partial_x{}^4 P_{0,T} \tag{103}$$

$$\mathrm{CrossTerm}_2 \tag{104}$$
$$= \frac{1}{2}\rho^2\partial_x{}^2 P_{0,T}\int_0^T\int_{t_1}^T \sigma_{t_1}(y)\eta_{t_1}\eta_{t_2}\partial_y\left(\sigma_{t_2}(y)\partial_y W_{t_2,T}\right)dt_2dt_1 \tag{105}$$
$$+ \frac{1}{4}\rho^2\partial_x{}^4 P_{0,T}\int_0^T\int_{t_1}^T \sigma_{t_1}(y)\eta_{t_1}\sigma_{t_2}(y)\eta_{t_2}\partial_y W_{t_2,T}\partial_y W_{t_1,T}dt_2dt_1 \tag{106}$$

$$\mathrm{CrossTerm}_2 = \tag{107}$$
$$=\frac{1}{2}\rho^2\partial_x{}^2 P_{0,T}\int_0^T\int_{t_1}^T \sigma_{t_1}(y)\eta_{t_1}\eta_{t_2}\partial_y\left(\sigma_{t_2}(y)\partial_y W_{t_2,T}\right)dt_2dt_1 \tag{108}$$
$$+\frac{1}{4}\rho^2\partial_x{}^4 P_{0,T}\int_0^T\int_{t_1}^T \sigma_{t_1}(y)\eta_{t_1}\sigma_{t_2}(y)\eta_{t_2}\partial_y W_{t_2,T}\partial_y W_{t_1,T}dt_2dt_1 \tag{109}$$

$$\mathrm{CrossTerm}_2 = \tag{110}$$
$$=\frac{1}{2}\rho^2\partial_x{}^2 P_{0,T}\int_0^T\int_{t_1}^T \sigma\eta_{t_1}\eta_{t_2}\partial_y\left(\sigma_{t_2}(y)\partial_y W_{t_2,T}\right)dt_2dt_1 \tag{111}$$
$$+\frac{1}{4}\rho^2\partial_x{}^4 P_{0,T}\int_0^T\int_{t_1}^T \sigma_{t_1}(y)\eta_{t_1}\sigma_{t_2}(y)\eta_{t_2}\partial_y W_{t_2,T}\partial_y W_{t_1,T}dt_2dt_1 \tag{112}$$
$$=\frac{1}{2}\rho^2\partial_x{}^2 P_{0,T}\int_0^T \frac{\left(e^{kt_2}-1\right)\eta\sigma}{k}\eta_{t_2}\partial_y\left(\sigma_{t_2}(y)\partial_y W_{t_2,T}\right)dt_2 \tag{113}$$
$$+\frac{1}{8}\rho^2\partial_x{}^4 P_{0,T}\left(\int_0^T \sigma\eta_{t_1}\partial_y W_{t_1,T}dt_1\right)^2 \tag{114}$$
$$= \frac{1}{2}\eta^2\rho^2\sigma\partial_x{}^2 P_{0,T}\int_0^T \frac{\left(e^{kt_2}-1\right)}{k}e^{kt_2}\partial_y\left(\sigma_{t_2}(y)\partial_y W_{t_2,T}\right)dt_2 \tag{115}$$
$$+\eta^2\rho^2\sigma^2\frac{1}{8}\partial_x{}^4 P_{0,T}\left(\int_0^T e^{kt_1}\partial_y W_{t_1,T}dt_1\right)^2 \tag{116}$$
$$=B_{\mathrm{xy}2,2}\partial_x{}^2 P_{0,T} + B_{\mathrm{xy}2,4}\partial_x{}^4 P_{0,T} \tag{117}$$

$$B_{\mathrm{xy2,2}} = -\eta^2\rho^2\frac{1}{2}\frac{\left(7+e^{-2kT}-4kT-2e^{-kT}(4+kT)\right)\sigma^4}{k^3} \tag{118}$$

$$B_{\mathrm{xy2,4}} = \eta^2\rho^2\frac{1}{8}\frac{4e^{-2kT}\left(1+e^{kT}(-1+kT)\right)^2\sigma^6}{k^4} \tag{119}$$

$$B_{\mathrm{xy2,2}} = -\eta^2\rho^2\frac{1}{2}\frac{\left(7+e^{-2kT}-4kT-2e^{-kT}(4+kT)\right)\sigma^4}{k^3} \tag{120}$$

$$B_{\mathrm{xy2,4}} = \eta^2\rho^2\frac{1}{8}\frac{4e^{-2kT}\left(1+e^{kT}(-1+kT)\right)^2\sigma^6}{k^4} \tag{121}$$

References

[1] Bergomi, Lorenzo and Guyon, Julien (2011) The smile in stochastic volatility models, *SSRN*

[2] R. Bos, A. Gairat, A. Shepeleva (2003) Dealing with discrete dividends, *Risk.net*

[3] Lorig, Matthew and Pagliarani, Stefano and Pascucci, Andrea (2003) Explicit implied volatilities for multifactor local - stochastic volatility models, *SSRN*

5 Formulas

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Φ[x_]:=(1/2)(Erf[x/(√2)] + 1);
ϕ[x_]:= e^(-x^2/2) / (√(2π));
ϕ[x_, w_]:=(1/(√(2πw)))e^(-x^2/(2w));
BH[x_, t_, K_, σ_]:=σ√tϕ[(K − x)/(σ√t)] − (K − x) Φ[−((K − x)/(σ√t))];
BH[x_, y_, w_]:=√wϕ[(y − x)/(√w)] − (y − x) Φ[−((y − x)/(√w))];
V0[x_, z_, w_]:=BH[x, z, w];
IVBH[S_, t_, K_, Price_]:=σ/. FindRoot[BH[S, t, K, σ] == Price, {σ, 0.1}];
H1[x_, z_, w_]:=(z − x)/w;
H2[x_, z_, w_]:= ((x − z)^2 − w) /w^2;
H3[x_, z_, w_]:= ((−3w + (z − x)^2) (z − x)) /w^3;
H4[x_, z_, w_]:= (3w^2 − 6w(x − z)^2 + (x − z)^4) /w^4;
```

$g[\text{y\_}]\text{:=}y+\sqrt{1+y^2};$

$f[\text{x0\_},\text{x\_},\beta\text{\_}]\text{:=}1+\frac{1}{2}\left(\beta (x-\text{x0})+\text{Abs}[\beta]\sqrt{(x-\text{x0})^2+1}-\text{Abs}[\beta]\right);$

$F[\text{x0\_},\text{z\_},\beta\text{\_}]\text{:=}z-\text{x0}+\frac{1}{2}(z-\text{x0})^2\beta +\left(\frac{1}{6}(z-\text{x0})^3+\frac{1}{2}(z-\text{x0})\right)\beta^2-\left(z-\text{x0}+\left(\frac{1}{6}+\frac{1}{4}(z-\text{x0})^2\right)\beta\right)\text{Abs}[\beta]+$

$\sqrt{1+(\text{x0}-z)^2}\left(\frac{1}{4}(\text{x0}-z)\beta^2+\left(\frac{z-\text{x0}}{2}+\frac{1}{6}\left(1+(z-\text{x0})^2\right)\beta\right)\text{Abs}[\beta]\right)+\left(\frac{\beta^2}{4}-\frac{\text{Abs}[\beta]}{2}\right)\text{ArcSinh}[\text{x0}-z];$

$\text{lvConvexity}[\text{x0\_},\text{z\_},\sigma\text{\_},\beta\text{\_},\text{T\_}]\text{:=}$

$\text{If}\left[z==\text{x0},0,\sigma^2\frac{T^2}{2(z-\text{x0})^2}\left(\frac{f[\text{x0},z,\beta]^2+f[\text{x0},\text{x0},\beta]^2}{2}-\frac{1}{(z-\text{x0})}F[\text{x0},z,\beta]\right)\right];$

$\text{IV}[\text{z\_},\text{T\_},\rho\text{\_},\text{k\_},\eta\text{\_},\beta\text{\_}]\text{:=}$

$\text{Module}[\{\text{WT},\text{dW1},\text{dW2},x,\sigma,\text{Iatm},\text{I3},\sigma 2,\text{G1},\text{G2},\text{dW},\text{V1},\text{V2},\text{V2cross},\text{Wy0},\text{Wyy0},\text{Byy0},\text{Byy2},$

$\text{Bxy1},\text{VarLV},\text{Bxy22},\text{Bxy24}\},$

$\sigma =\sigma 0;$

$x=1;$

$\text{WT}=T\sigma^2;$

$\text{Bxy1}=\rho \sigma^3\frac{e^{-kT}\left(1+e^{kT}(-1+kT)\right)}{k^2};$

$\text{V1}=\ \text{H1}[\text{x0},z,\text{WT}]\text{Bxy1};$

$\text{Byy0}=\frac{1}{4}\sigma^2\frac{e^{-2kT}\left(1+e^{2kT}(-1+2kT)\right)}{k^2};$

$\text{Byy2}=\frac{1}{8}\sigma^4\frac{2e^{-2kT}\left(-1+4e^{kT}+e^{2kT}(-3+2kT)\right)}{k^3};$

$\text{Bxy22}=-\rho^2\frac{1}{2}\frac{\left(7+e^{-2kT}-4kT-2e^{-kT}(4+kT)\right)\sigma^4}{k^3};$

$\text{Bxy24}=\rho^2\frac{1}{8}\frac{4e^{-2kT}\left(1+e^{kT}(-1+kT)\right)^2\sigma^6}{k^4};$

$\text{V2cross}=\text{Bxy22}\text{H2}[\text{x0},z,\text{WT}]+\text{Bxy24}\text{H4}[\text{x0},z,\text{WT}];$

$\text{V2} = \text{Byy0} + \text{Byy2H2}[\text{x0}, z, \text{WT}] + \text{V2cross};$

$\text{dW1} = 2\text{V1};$

$\text{dW2} = 2\text{V2} - \ \text{V1}^2\text{H2}[\text{x0}, z, \text{WT}];$

$\text{VarLV} = \sigma^2 T\text{If}\left[z == \text{x0}, 1, \frac{1}{z-\text{x0}}F[\text{x0}, z, \beta]\right] + \sigma^4\text{lvConvexity}[\text{x0}, z, \sigma, \beta, T];$

$\sqrt{\frac{\text{VarLV}+\eta\text{dW1}+\eta^2\text{dW2}}{T}}$

];