Simplified commodity Cheyette model for XVA

The general commodity Cheyette model evolves commodity futures according to

$$\frac{dF(t,T)}{F(t,T)} = A_t \left(\frac{g_1(T)}{g_1(t)} \sigma^1(t)\, dW_t^1 + \frac{g_2(T)}{g_2(t)} \sigma^2(t)\, dW_t^2 \right)$$

where
$g_i(t) = e^{-\int_0^t a_i(s) ds}$, $< dW_t^1, dW_t^2 >= \rho_t\, dt$ and A_t is a multiplicative local and possibly stochastic volatilty.

The local volatility in the model is non-parameteric (Dupire) and is calibrated to successive commodity futures implied volatility surface segments and the necessary early-expiry implied volatilites for each succesive future contract are generated concistently within the model.

The full-blown model is overkill and too computationally heavy for XVA computations
so we suggest the following simplifications/approximations:

- Use only one Brownian driver for the curve dynamics (with a signle mean-reversion)
- Use a Dupire spot local volatility calibrated on "commodity spot" implied volatilities : since the commodity underlyings currently under scope for XVA feature options on futures that expire at most a few days before the maturity of the corresponding future, we can build a single approximated commodity spot implied volatility surface from market quotes of options of futures implied volatilities. The Dupire spot local voltility is then calibrated to the spot implied volatility surface.

The simplified model SDE reads

$$\frac{dF(t,T)}{F(t,T)} = A(S_t)\; g(t,T)\; \sigma(t)\; dW_t \quad \text{where } S_t = F(t,t) \text{ and } g(t,T) = \frac{g(T)}{g(t)}$$

And the commodity futures are Markovian function of two state variables x and $y

$$F(t,T) = F(0,T)\; \exp\left[g(t,T)\, x_t - \frac{1}{2} g^2(t,T)\, y_t \right]$$

where the state variables are solutions to the SDEs

$$\begin{aligned} dx_t &= -a(t)\, x_t\, dt + A(S_t)\sigma(t)\; dW_t \\ dy_t &= \left(-2\, a(t)\, y_t + \left(A(S_t)\sigma(t)\right)^2 \right)\; dt \end{aligned}$$

The commodity spot S_t is simply given by

$$S_t = F(0,T)\; \exp\left(x_t - \frac{1}{2}\, y_t \right)$$

and hence has the following mean-reverting dynamics

$\frac{dS_t}{S_t} = (\theta_t - a(t)\ \log(S_t))\,dt + A(S_t)\sigma(t)\ dW_t$

where $\theta_t = a(t)\left(F(0,t) + \frac{1}{2}y_t\right) + \frac{d}{dt}\log(F(0,t))$

Spot local volatility calibration

Given a marked mean-reversion $a(t)$, the deterministic volatilities $\sigma(t)$ are boot-strapped to implied futures option volatilities at a reference strike (usually ATMF).
We denote δ the time lag between a future option expiry and the maturity of the underlying future.
Within the deterministic volatility model, we have
$\Sigma^2(t, t+\delta) = e^{-2\int_t^{t+\delta} a(s)ds}\ \Sigma^2(t,t)$ where $\Sigma(t,T)$ is the implied volatility of an option with expiry t on the commodity future maturing at T.

We assume that the same relation holds, at all strikes, in the smile-on case, leading to the approximate commodity spot implied volatilities:
$\Sigma^2(K,t) \approx e^{2\int_t^{t+\delta} a(s)ds}\ \Sigma^2(K,t,t+\delta)$

We can now calibrate the spot local volatility $A(S_t)$ to spot implied volatilities $\Sigma(K,t)$.

In the risk-neutral measure (i.e. where the numéraire is the continously compounded money market account), recall that for a process S_t following
$\frac{dS_t}{S_t} = \mu_t\ dt + A(S_t)\sigma(t)\ dW_t$
the local volatility that matches the underlying marginal densities is given by
$A^2(K)\ \sigma^2(t) = \frac{\partial_T C_t(K,T) - \mathbb{E}_t[\mu_T\ S_T\ 1_{S_T>K}]}{\frac{1}{2}K^2\partial_K{}^2 C_t(K,T)}$
where $C_t(K,T)$ is the undiscounted price at time t of a call option struck at K and expiring at time $T > t$.
In order to keep things tractable and have a deterministic mean-reversion level θ_t for the commodity spot in the local volatility computation, we will approximate y_t with its deterministic volatility counterpart so that

$\mu_t = \theta_t - a(t)\ \log(S_t) \approx a(t)\left(F(0,t) + \frac{1}{2}y^{\det}(t) - \log(S_t)\right) + \frac{d}{dt}\log(F(0,t))$
with
$y^{\det}(t) = g^2(t)\int_0^t \frac{\sigma^2(s)}{g^2(s)}ds$

We now have that

$\mathbb{E}_t[\mu_T\ S_T\ 1_{S_T>K}] = \theta_T\ (C_t(K,T) + K\ \partial_K C_t(K,T)) - a(T)\ \mathbb{E}_t\left[S_T\ \log(S_T)\ 1_{S_T>K}\right]$

The last expectation only depends on the terminal value of the commodity spot S_T and can therefore be computed by vanilla replication or equivalently by direct integration against the commodity spot marginal density obtained from $\Sigma(K,t)$.

Use for XVA computations in a Monte-Carlo setup

Given a marked mean-reversion $a(t)$ and boot-strapped deterministic instantaneous volatility

- Build the commodity spot implied volatility surface from market implied volatilities.
- Calibrate the commodity spot local volatility surface as outiled in the previous section.
- Diffuse the state variables x_t and y_t to the desired time horizon.
- Compute forward exposures either directly from the state variables for linear trades or via American regression on the state variables for non-linear trades.

Stochastic interest rates

Our simplified commodity spot model does not model the commodity convenience yield and accordingly does not feature the interest process in its drift. Rates stochasticity however results in a spread between the commodity futures and forwards.
That spread reflects the drift change between the risk-neutral and forward measures and is proportional to the term covariance between the stochastic discount factor and the commodity spot. In other words, commodity futures are martingales in the risk-neutral measure whereas each commodity forward is a martingale in its forward measure.
The market trades options used to calibrate our model are options on commodity futures and the resulting quoted implied volatilities correspond to prices computed in the forward measure where the expectation of the at-expiry future is no longer equal to the future price obtained from the initial futures curve $F(0,T)$.

1. smile-off drift correction.
   In the risk-neutral measure, the zero-coupon bond $P(t,T)$ has the SDE
   $\frac{dP(t,T)}{P(t,T)} = r_t\ dt - \sigma^P(t,T)\ dW_t^{r,RN}$
   where r_r is the short rate and dW^r its Brownian driver.
   The commodity futures remain martingales in the risk-neutral measure:
   $\frac{dF(t,T)}{F(t,T)} = g^C(t,T)\ \sigma_t^C\ dW_t^{C,RN}$
   The change of measure from the risk-neutral measure to the T-forward measure reads
   $dW_t^{r,T} = dW_t^{r,RN} - \sigma^P(t,T)\ dt$
   and we therefore have
   $\frac{dF(t,T)}{F(t,T)} = g^C(t,T)\ \sigma_t^C\ \left(\rho^{r,C}(t)\ \sigma^P(t,T)\ dt + dW_t^{C,T}\right)$
   where $\rho^{r,C}$ is the instantaneous correlation between the short rate and the commodity driver.
   Assuming deterministic ZC bond volatility (e.g. Hull-White) and disregarding commodity smile in the drift adjustment, we obtain
   $\mathbb{E}^T\left[F(t,T)\right] = F(0,T)\ \exp\left(\int_0^t\ g^C(u,T)\ \sigma^C(u)\ \rho^{r,C}(u)\ \sigma^P(u,T)\ du\right)$
   From market quoted implied volatilities, options prices can be recovered (with the futures curve $F(0,T)$) and then corrected IR-vols-aware commodity implied volatilites can be re-implied from those prices, with the drift-adjusted curve $\mathbb{E}^T\left[F(t_{\exp}, t_m)\right]$ where $t_{\exp}$ and t_m are the option expiry and future maturity respectively.
2. Full Dupire local volatility expression.

The above correction should be sufficiently precise but for the sake of completeness, the full commodity spot local volatility expression in presence of stochastic interest rates is given by

$$A^2(K)\ (\sigma^C(t))^2 = \frac{\partial_T C_t(K,T) + \mathbb{E}_t\left[D_{t,T}\ r_T\ \ (S_T - K)^+\right] - \mathbb{E}_t\left[D_{t,T}\ \mu_T\ \ S_T\ \ 1_{S_T > K}\right]}{\frac{1}{2} K^2 \partial_K{}^2 C_t(K,T)}$$

where $C_t(K,T)$ is the *discounted* price at time t of a call option struck at K, $D_{t,T} = e^{-\int_0^t r_u\ du}$, and all expectations are taken in the risk-neutral measure.