Necessary and sufficient conditions for survival strategies in asset markets with endogenous prices

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Table of Contents

Motivation

- This work continues the strand of the literature emerged in the seminal paper of Blume and Easley (1992) and extends the results obtained by Amir et al. (2013).
- Agent's strategy is called *survival*, if its fraction of capital in the total market capital is separated from zero on the infinite horizon, for any strategies of other agents.
- The goal is to investigate the question of existence of survival strategies in a stochastic model of market with discrete time and endogeneous prices.
- We obtain also sufficient conditions for survival strategies, while the majority of papers in this direction either construct particular survival strategies in models of various generality or provide necessary conditions for survival.

General model

- $t = 1, 2, \ldots$
- $N \geq 2$ agents.
- $K \geq 2$ assets, each asset pays random dividend A_t^k.
- Assets are "short-lived": they exist for one period, pay dividends and should be purchased again in the beginning of the next period.
- Each agent n in each moment of time t selects a vector of proportions $\lambda_t^n = (\lambda_t^{n,1}, \ldots, \lambda_t^{n,K})$, in which he invests his capital W_t^n in each of K assets at the moment of time t.
- λ_t^n : $\mathcal{F}_t$-measurable vector from the set $\Delta^K := \{(a_1, ..., a_K) \in \mathbb{R}_+^K : a_1 + ... + a_K = 1\}$.
- The prices are determined endogenously from equality of supply and demand for each of assets.

Dynamics of capital

Let the dynamics of agent's wealth W_t^n be described by the relation

$$W_{t+1}^n = \sum_{k=1}^{K} \frac{\lambda_t^{n,k} W_t^n}{\sum_{n=1}^{N} \lambda_t^{n,k} W_t^n} A_{t+1}^k \qquad (1)$$

Economical interpretation:

- Let $\bar{P}_t = (P_t^1, \dots, P_t^K)$ be the vector of prices at time t. Agent following strategy λ_t^n allocated fraction $\lambda_t^{n,k}$ of his capital W_t^n into asset k, hence he has $X_t^n := \frac{\lambda_t^{n,k} W_t^n}{P_t^k}$ shares of asset k.
- The prices are determined from the balance of supply and demand:

$$1 = \sum_{n=1}^{N} X_t^{n,k} = \sum_{n=1}^{N} \frac{\lambda_t^{n,k} W_t^n}{P_t^k} \Rightarrow \boxed{P_t^k = \sum_{n=1}^{N} \lambda_t^{n,k} W_t^n}$$

- Hence the dynamics of agents' wealth are the following

$$W_{t+1}^n = \sum_{k=1}^{K} X_t^{n,k} A_{t+1}^k = \sum_{k=1}^{K} \frac{\lambda_t^{n,k} W_t^n}{P_t^k} A_{t+1}^k = (1)$$

Survival strategies

- We are interested in the evolution of *relative wealth* of agents, defined as follows $r_t^n := \frac{W_t^n}{W_t}$.

Definition 1

We will call the strategy λ surviving on the set of outcomes $\Gamma \in \mathcal{F}$, if for any vector of initial wealth W_0 and any profile of strategies $\Lambda = (\lambda^1, ..., \lambda^N)$, consisting of the given strategy $\lambda^n = \lambda$ and arbitrary strategies λ^j of agents $j \neq n$, holds the inequality $\liminf_{t \to 0} r_t^n > 0$ a.s. on the set Γ.

- Th strategy is called a.s.-survival (surviving with probability 1), if this inequality holds on the set of probability 1.
- In the considered model the strategy $\lambda^* := \mu_t$, where $\mu_t^k := \mathbb{E}_t R_{t+1}^k$ with $\mathbb{R}_t^k := \frac{A_t^k}{\sum_{k=1}^K A_t^k}$ is survival with probability 1.

Example of a non-trivial survival set

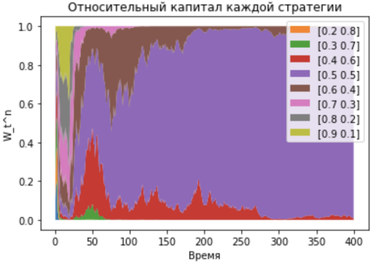
Suppose the asset payoffs are given by the sequence of random vectors

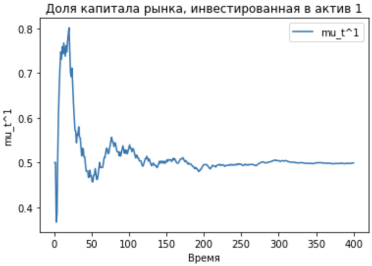
$$A_t = A_t^{(1)}\,\mathrm{I}(t < \theta) + A_t^{(2)}\,\mathrm{I}(t \geq \theta),$$

where $A_t^{(1)}$ and $A_t^{(2)}$ are sequences of i.i.d. random vectors and θ is a random moment of time with values in $\mathbb{Z}_+ \cup \{\infty\}$ representing a *changepoint* in the sequence of payoffs. The random event $\{\omega : \theta = \infty\}$ corresponds to the absence of a changepoint. If the mean vectors $\mu^{(1)} = \mathrm{E}\,A_t^{(1)}$ and $\mu^{(2)} = \mathrm{E}\,A_t^{(2)}$ are different, then the strategy $\lambda_t = \mu^{(1)}$ survives on the set $\Gamma = \{\omega : \theta = \infty\}$, while it does not survive on its complement if at least one agent in the strategy profile uses the strategy $\lambda'_t = \mu^{(2)}$.

Example of evolution of strategies

- Only two assets.
- Denote by $\xi_t^k = \frac{1}{W_t}\sum_{n=1}^{N}\lambda_t^{n,k}W_t^n$ the fraction of W_t invested in k-th asset.
- Payoff of each of two assets is equal either $1+\xi_t^k$ with probability p, or zero with probability $1-p$, $p = 2/3$.
- Survival strategy is $\Lambda^* = (1/2, 1/2)$.
- There are 9 agents with strategies $\Lambda^n = (n/10, 1-n/10)$, where $n = 1, 2, \dots, 9$.





Main theorem (part 1)

Theorem 1

1) Assume that the strategy λ has positive components ($\lambda_t^k > 0$ for all $k = 1, \ldots, K$, $t \geq 0$). Then λ survives on the set

$$\Gamma = \left\{ \omega : \sum_{t=0}^{\infty} \sum_{k=1}^{K} \mu_t^k \ln \frac{\mu_t^k}{\lambda_t^k} < \infty \right\}. \tag{2}$$

Moreover, if there exists a constant $\varepsilon > 0$ such that $\mu_t^k \geq \varepsilon$ for all $t \geq 0$, $k = 1, \ldots, K$, then λ survives on the set

$$\Gamma' = \{\omega : \sum_{t=0}^{\infty} \|\mu_t - \lambda_t\|^2 < \infty\}$$

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Main theorem (part 2)

Main theorem (part 2)

2) Fix a strategy profile and the vector of initial capitals.
Let $\bar{\lambda}_t$ denote the *weighted strategy* of all agents, given by formula

$$\bar{\lambda}_t^k := \sum_{n=1}^{N} \lambda_t^{n,k} r_t^n$$

Assume that the profile contains a strategy λ with positive components.
Then

$$\sum_{t=0}^{\infty} \|\mu_t - \bar{\lambda}_t\|^2 < \infty \text{ on } \Gamma,$$

where Γ is the set from equation (2).
In particular, $\lim_{t\to\infty}(\mu_t - \bar{\lambda}_t) = 0$ on Γ.

Main theorem (part 3)

Theorem 1 (part 3)

3) For any strategy λ, surviving on some set $\Gamma \in \mathcal{F}$, holds the inequality

$$\sum_{t=0}^{\infty} \|\mu_t - \lambda_t\|^2 < \infty \quad \text{on } \Gamma. \tag{3}$$

Moreover, if $\mu_t^k \geq \varepsilon > 0$ on Γ for all $t \geq 0$ and $k = 1, \ldots, K$, then condition (3) is necessary and sufficient for the strategy λ with positive components to survive on Γ.

Proof of main theorem (part 1)

The idea of the proof is the following:
assume agent $i = 1$ uses a strategy λ with strictly positive components.
Define the following random sequences U_t and Z_t, $t \geq 0$:

$$U_t = \sum_{s=0}^{t} \sum_{k=1}^{K} \mu_s^k \ln \frac{\mu_s^k}{\lambda_s^k},$$

$$Z_0 = r_0^1, \qquad Z_t = \ln r_t^1 + U_{t-1},\ t \geq 1.$$

Gibb's inequality implies that $U_t \geq 0$. Observe that the set Γ in formula (2) is nothing but the set of convergence of U_t.
Let us prove that Z_t is a local submartingale. It is enough to show that $\mathrm{E}(Z_{t+1}^+ \mid \mathcal{F}_t) < \infty$ and $\mathrm{E}(Z_{t+1} - Z_t \mid \mathcal{F}_t) \geq 0$. The first inequality is clear, since $Z_{t+1} \leq U_t$.

Proof of main theorem (part 1)

To show that $\mathrm{E}(Z_{t+1} - Z_t \mid \mathcal{F}_t) \geq 0$, we observe that

$$\ln r_{t+1}^1 - \ln r_t^1 = \ln\left(\sum_{k=1}^{K} \frac{\lambda_t^k}{\bar{\lambda}_t^k} R_{t+1}^k\right) \geq \sum_{k=1}^{K} R_{t+1}^k \ln \frac{\lambda_t^k}{t,k},$$

where we used the convexity of the logarithm and treated R_{t+1}^k as coefficients of the convex combination of the numbers $\lambda_t^k/\bar{\lambda}_t^k$. This implies the bound

$$\mathrm{E}(\ln r_{t+1}^1 - \ln r_t^1 \mid \mathcal{F}_t) \geq \sum_{k=1}^{K} \mu_t^k \ln \frac{\lambda_t^k}{\bar{\lambda}_{t,k}}.$$

From the definition of Z_t, we find

$$\mathrm{E}(Z_{t+1} - Z_t \mid \mathcal{F}_t) \geq \sum_{k=1}^{K} \mu_t^k \ln \frac{\mu_t^k}{\bar{\lambda}_t^k} \geq 0. \qquad (4)$$

Consequently, Z is a local submartingale.

Proof of main theorem (part 1)

Since $Z_t \leq U_{t-1}$ and U_t converges on the set Γ, $\lim_{t\to\infty} Z_t$ exists on Γ. As a result, $\rho = \lim_{t\to\infty} \ln r_t^1$ also exists on Γ, which implies that $\lim_{t\to\infty} r_t^1 = e^\rho > 0$, hence λ survives on Γ.
To prove that λ survives on Γ' provided that the condition $\mu_{t,k} \geq \varepsilon > 0$ holds true, we can use the following bounds which follow from reverse Pinsker's inequality :

$$\sum_{k=1}^{K} \mu_t^k \ln \frac{\mu_t^k}{\lambda_t^k} \leq \frac{|\mu_t - \lambda_t|^2}{2 \min_k \lambda_t^k} = O(\|\mu_t - \lambda_t\|^2) \text{ as } t \to \infty \text{ on } \Gamma',$$

where we used that $\|\mu_t - \lambda_t\| \to 0$ on Γ', so $\min_k \lambda_t^k$ is asymptotically bounded away from zero. Then $\Gamma' \subseteq \Gamma$, which implies the survival property on Γ'.

Bibliography

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