Local Stochastic volatility model

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Plan

Our goal was to was to compare hedging in Black-Scholes model and in the Local Volatility model, according to the article "Historical backtesting...", with prices generated by Heston model.
We have found some discrepancies in this article:
1) The formula (11) for local volatility in terms of K and T on page 3 is incorrect: θ^2 is forgotten in the numerator; the correct formula (1.26) is on page 23 in the paper "Local Stochastic Volatility Models. Solving the Smile Problem with a Nonlinear Partial Integro-Differential Equation".
2) As we discussed last time, in the article they interpolate between different times using cubic splines - but the better way is to interpolate linearly; moreover, this interpolation should be done in the coordinates not (K,T), but (y,w) or (y,T) instead. We tried both variants. Then we also tried to extrapolate across strikes with cubic splines and linear function according to the Lee formulas (with interpolation cubic in both cases).
3) They use Cranck-Nicolson scheme to solve the PDE to find V in Local Volatility model - but it unstable when the payoff is not smooth - and the payoff of a European option is not smooth. We used fully implicit scheme.

The formula for local volatility in terms of IV

where the *local volatility* $\sigma(K,T)$ is given by

$$\sigma(K,T) = \sqrt{\frac{2\theta T \frac{\partial \theta}{\partial T} + 2(r(T) - q(T))K\theta T \frac{\partial \theta}{\partial K}}{\left[1 + d_1 K\sqrt{T}\frac{\partial \theta}{\partial K}\right]^2 + K^2\theta T\left[\frac{\partial^2 \theta}{\partial K^2} - d_1\left(\frac{\partial \theta}{\partial K}\right)^2\sqrt{T}\right]}}, \tag{11}$$

$$d_1 = \frac{\ln\left(\frac{S_0}{K}\right) + \int_0^T [r(t) - q(t)]dt + \frac{1}{2}\theta^2 T}{\theta\sqrt{T}}. \tag{12}$$

Figure: The incorrect formula from [Shevchenko, Ling]

$$\sigma_{LV}^2(K,T) = \frac{\sigma_I^2 + 2T\sigma_I\left(\frac{\partial \sigma_I}{\partial T} + (r-d)K\frac{\partial \sigma_I}{\partial K}\right)}{\left(1 + d_1 K\sqrt{T}\frac{\partial \sigma_I}{\partial K}\right)^2 + K^2\sigma_I T\left(\frac{\partial^2 \sigma_I}{\partial K^2} - d_1\sqrt{T}\left(\frac{\partial \sigma_I}{\partial K}\right)^2\right)}. \tag{1.26}$$

Where

$$d_1 = \frac{\log\left(\frac{S_0}{K}\right) + (r - d + \frac{1}{2}\sigma_I^2)T}{\sigma_I\sqrt{T}}.$$

Figure: The formula from [Barucci]

Description of the model

$$dS_t = S_t(rdt + \sigma(t, S_t)dW_t), \quad S_0 > 0. \tag{1}$$

Definition

The function $\sigma(t, s)$ such that the call option prices given by the model (1) coincide with the market prices $\widehat{C}(T, K)$ is called **local volatility**, i.e.

$$\widehat{C}(T, K) = C(T, K) := e^{-rT}E(S_T - K)^+, \quad T \in [0, T_{max}], \ K \in (0, \infty). \tag{2}$$

The aim is to calibrate the model to get the option prices in each defined time and strike price.

LV in terms of IV

$$C = C_{BS}(S_t, t, K, T, \sigma_I(K, T))$$

Applying the chain rule of differentiation and Black-Scholes formula it is possible to prove that:

$$\sigma_{LV}^2(K, T) = \frac{\sigma_I^2 + 2T\sigma_I(\frac{\partial \sigma_I}{\partial T} + (r-d)K\frac{\partial \sigma_I}{\partial K})}{(1 + d_1 K\sqrt{T}\frac{\partial \sigma_I}{\partial K})^2 + K^2\sigma_I T(\frac{\partial^2 \sigma_I}{\partial K^2} - d_1\sqrt{T}(\frac{\partial \sigma_I}{\partial K})^2)}, \qquad (3)$$

where

$$d_1 = \frac{\log(\frac{S_0}{K}) + (r - d + \frac{1}{2}\sigma_I^2)T}{\sigma_I\sqrt{T}}$$

But it is better to calculate local volatility as the function of log-moneyness $y = \log\frac{K}{S_0}$ and time T, using total variance $w = \sigma_I^2 \cdot T$.

LV in terms of IV Part 1

Denote σ_I by θ for convenience.
Then we take formula (1) and change the variables from (K,T) to (y,w).

$$y = ln(\frac{K}{S_0}) = lnK - lnS_0$$

$$w = \theta^2 T$$

$$\frac{\partial\theta}{\partial K} = \frac{\partial\theta}{\partial y} \cdot \frac{\partial y}{\partial K} = \boxed{\frac{\partial\theta}{\partial y} \cdot \frac{1}{K}}$$

$$\frac{\partial^2\theta}{\partial K^2} = \frac{\partial}{\partial K}\left(\frac{\partial\theta}{\partial K}\right) = \frac{\partial}{\partial K}\left(\frac{\partial\theta}{\partial y} \cdot \frac{1}{K}\right) = \boxed{\frac{\partial^2\theta}{\partial y^2} \cdot \frac{1}{K^2} - \frac{\partial\theta}{\partial y} \cdot \frac{1}{K^2}}$$

LV in terms of IV Part 2

$$\frac{\partial \theta}{\partial T} = \frac{\partial \theta}{\partial w} \cdot \frac{\partial w}{\partial T} = \frac{\partial \theta}{\partial w} \left(\theta^2 + 2\theta T \frac{\partial \theta}{\partial T} \right)$$

$$\Longrightarrow \frac{\partial \theta}{\partial T} \left(2\theta T \frac{\partial \theta}{\partial w} - 1 \right) = -\theta^2 \frac{\partial \theta}{\partial w}$$

$$\Longrightarrow \boxed{\frac{\partial \theta}{\partial T} = \frac{\theta^2 \frac{\partial \theta}{\partial w}}{1 - 2\theta T \frac{\partial \theta}{\partial w}}}$$

We put these expressions into formula (1).

$$\sigma_{LV}^2 =$$

$$\frac{\theta^2 + 2\theta T\left(\frac{\theta^2 \frac{\partial\theta}{\partial w}}{1-2\theta T\frac{\partial\theta}{\partial w}} + rK\frac{\partial\theta}{\partial y}\frac{1}{K}\right)}{\left(1 + d_1 K\sqrt{T}\frac{1}{K}\frac{\partial\theta}{\partial y}\right)^2 + K^2\theta T\left(\frac{\partial^2\theta}{\partial y^2}\cdot\frac{1}{K^2} - \frac{\partial\theta}{\partial y}\cdot\frac{1}{K^2} - d_1\sqrt{T}\left(\frac{\partial\theta}{\partial y}\right)^2\frac{1}{K^2}\right)} \tag{4}$$

$$= \frac{\theta^2 + 2\theta T\left(\frac{\theta^2 \frac{\partial\theta}{\partial w}}{1-2\theta T\frac{\partial\theta}{\partial w}} + r\frac{\partial\theta}{\partial y}\right)}{\left(1 + d_1\sqrt{T}\frac{\partial\theta}{\partial y}\right)^2 + \theta T\left(\frac{\partial^2\theta}{\partial y^2} - \frac{\partial\theta}{\partial y} - d_1\sqrt{T}\left(\frac{\partial\theta}{\partial y}\right)^2\right)} \tag{5}$$

LV in terms of IV Part 4

$$\sigma_{LV}^2 = \frac{\frac{\theta^3}{\theta - 2w\frac{\partial\theta}{\partial w}} + 2r\frac{\partial\theta}{\partial y}\frac{w}{\theta}}{\left(1 + d_1\sqrt{T}\frac{\partial\theta}{\partial y}\right)^2 + \frac{w}{\theta}\left(\frac{\partial^2\theta}{\partial y^2} - \frac{\partial\theta}{\partial y} - d_1\sqrt{T}\left(\frac{\partial\theta}{\partial y}\right)^2\right)} \tag{6}$$

LV in terms of IV Part 5

Thus,

• to compute the local volatility function (4), we need partial derivatives of the implied volatility surface θ;

• in practice we only have a finite number of market data points, typically 5 values for a given maturity and about 10 maturities; We need some interpolating procedure for θ;

• This is an ill-posed problem, and there are a number of ways to interpolate these data points. We use natural cubic splines to interpolate across log moneyness, linear ones to interpolate across maturities.

Interpolation across log moneyness

Suppose we have market data for N different maturities and that for each maturity M options are available. Let $y_j^{(i)}$ and $\theta_j^{(i)}$ denote the log moneyness and implied volatility of the j-th vanilla option with maturity $T_i(w_i)$.

1. Interpolation across log moneyness with fixed t (w): For each market maturity $T_i, i \in 1, ..., N$ fit a natural cubic spline $z_i(y)$ through

$$(y_1^{(i)}, \theta_1^{(i)}), ..., (y_M^{(i)}, \theta_M^{(i)})$$

Note that $z_i^{'}(y) = \frac{\partial \theta}{\partial y}$ in (y, T_i) and $z_i^{''}(y) = \frac{\partial^2 \theta}{\partial y^2}$ in (y, T_i).

2. Extrapolation across log moneyness with fixed t (w): For each maturity $T_i, i \in 1, ..., N$ fit a linear function with coefficient determined by Lee formula.

3. To find $\frac{\partial \theta}{\partial y}$ at any given (y, w) fit another linear spline $f(w)$ through

$$(w_1, z_1^{'}(y)), ..., (w_N, z_N^{'}(y))$$

Then $\frac{\partial \theta}{\partial y} = f(w)$.

4. Similarly to find $\frac{\partial^2 \theta}{\partial y^2}$ at any given (y, w) fit another linear spline $g(w)$ through

$$(w_1, z_1^{''}(y)), ..., (w_N, z_N^{''}(y))$$

Then $\frac{\partial^2 \theta}{\partial y^2} = g(w)$.

5. To find θ and $\frac{\partial \theta}{\partial T}$ at (y, w), fit a linear spline $u(w)$ through

$$(w_1, z_1(y)), ..., (w_N, z_N(y))$$

Then $\theta(y, w) = u(w)$ and $\frac{\partial \theta}{\partial T} = u'(w)$.

LV in terms of IV

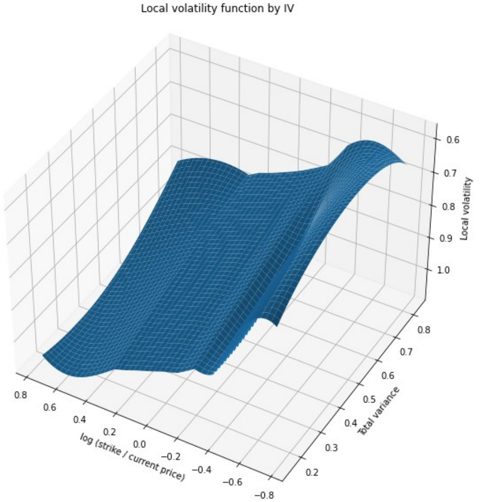

Figure: The local volatility surface

Implicit scheme

First we need a mesh of (price,time) pairs.

1. $\{t_i\}_{i=0,\dots,N},\ t_i = \Delta t * i,\ \Delta t = T/N.$
2. $\{s_j\}_{j=0,\dots,M},\ s_j = S_0 e^{-D+j\Delta x},$
   $D = \gamma\bar{\theta}\sqrt{T}$, $\bar{\theta}$ is the average of the at the money implied volatilities,
   $\Delta x = 2D/M.$
   (The time interval is $[0, T]$ and the price interval is $[S_0 e^{-D}, S_0 e^{D}]$.)

Implicit scheme

$$\frac{\partial V}{\partial t} + (r - q - \frac{1}{2}[\sigma(e^{X_t}, t)]^2)\frac{\partial V}{\partial X} + \frac{1}{2}[\sigma(e^{X_t}, t)]^2 \frac{\partial^2 V}{\partial X^2} - rV = 0. \quad (7)$$

Lets denote by $\nu(t, x) = r - q - \frac{1}{2}[\sigma(e^x, t)]^2$ and $V_j^i = V(t_i, s_j)$.
The implicit scheme is given by

$$c_{j-1}^{i-1} V_{j-1}^{i-1} + d_j^{i-1} V_j^{i-1} + e_j^{i-1} V_{j+1}^{i-1} = V_j^i, \quad (8)$$

where $i \in \{1, \dots, N\}$ and

$$c_{j-1}^{i-1} = \frac{\tau \nu_j^{i-1}}{2h} - \frac{\tau}{2h^2}(\sigma^2)_j^{i-1},$$

$$d_j^{i-1} = 1 + \frac{\tau}{h^2}(\sigma^2)_j^{i-1} + r\tau,$$

$$e_j^{i-1} = -\frac{\tau \nu_j^{i-1}}{2h} - \frac{\tau}{2h^2}(\sigma^2)_j^{i-1}.$$

Implicit scheme

For boundary conditions we use the fact that $\lim_{S \to 0} \frac{\partial V}{\partial S} = 0$ and $\lim_{S \to \infty} \frac{\partial V}{\partial S} = 1$. This leads to the following equations

$$V_0^i - V_1^i = 0, \tag{9}$$

$$V_M^i - V_{M-1}^i = s_M - s_{M-1}. \tag{10}$$

To initiate the scheme, we set for all $j = 0, \ldots, M$

$$V_j^N = (s_j - K)^+. \tag{11}$$

We then repeatedly solve the system until we obtain $(V_1^0, \ldots, V_M^0)^T$.

Hedging

We compared hedging in two setups: in Black-Scholes model and in Local Volatility model. We took a stoke with $S_0 = 100$ and simulated the evolution of the stocke with Heston model. Then we tried to hedge 3 different options: with strikes 120,100,80.

- In the Black-Scholes setup the needed delta is $\Phi(d1)$, where

$$d_1 = \frac{ln(\frac{S0}{K}) + 0.5\sigma^2(T-t)}{\sigma\sqrt{T-t}}$$

Here σ is the iv, taken at point t, and Φ is cdf of $\mathcal{N}(0,1)$.

- In the local volatility model, delta is V'_{S_t}. V we have found on the grid using the implicit scheme for PDE. Now at fixed time t we fit a cubic spline on the $V(S_t)$ - and obtain delta at any spot at this fixed time t. Here it is important that we solved the PDE on the grid of (y_extended, w_extended).

Results

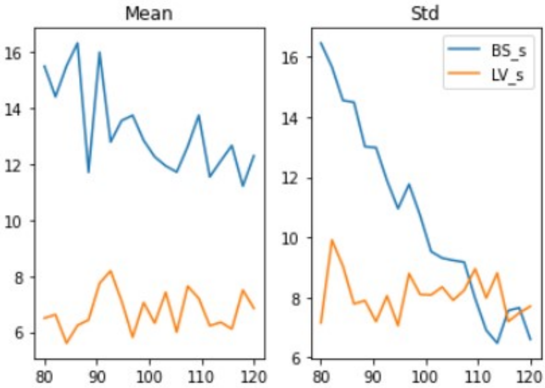

Figure: Mean and std of hedging errors for options with different strikes and $S_0 = 100$

LSV model

Considering the Heston-like LSV model, the dynamics of the spot price and the stochastic variance under the risk neutral measure are:

$$dS_t = rS_t dt + L(S_t, t)\sqrt{V_t} S_t dW_t^1 \tag{12}$$

$$dV_t = \kappa(\theta - V_t)dt + \lambda\sqrt{V_t} dW_t^2$$

$$dW_t^1 \cdot dW_t^2 = \rho dt$$

LSV model is exactly calibrated to market smiles if and only if

$$\sigma_{Dup}(t, S)^2 = L(t, S)^2 \, \mathrm{E}^{\mathrm{Q}}[V_t | S_t = S]$$

Therefore, SDE (12) of asset price can be rewritten as

$$dS_t = rS_t dt + \sqrt{\frac{\sigma_{Dup}(t, S_t)^2}{\mathrm{E}^{\mathrm{Q}}[V_t | S_t]}} \sqrt{V_t} S_t dW_t^1 \tag{13}$$

Particle method

Consider the McKean SDE for the n-process X_t:

$$dX_t = b(t, X_t, \mathrm{P}_t)dt + \sigma(t, X_t, \mathrm{P}_t)dW_t, \quad Law(X_t) = \mathrm{P}_t, \;\; X_0 \in \mathbb{R}^n, \tag{14}$$

where W_t is d-dimensional Brownian motion.
The fundamental concept to develop the particle method to simulate the stochastic McKean SDE) consists of approximate the law P_t with the empirical distribution of a fixed large number N of particles $(\xi_t^{i,N})_{i=1}^N$

$$\mathrm{P}_t \approx \mathrm{P}_t^N := \frac{1}{N}\sum_{i=1}^{N} \delta_{\xi_t^{i,N}},$$

where $(\xi_t^{i,N})_{i=1}^N$ are solutions to the $(\mathbb{R}^n)^N$-dim classical linear SDE:

$$d\xi_t^{i,N} = b(t, \xi_t^{i,N}, \mathrm{P}_t^N)dt + \sigma(t, \xi_t^{i,N}, \mathrm{P}_t^N)dW_t^i, \quad Law(\xi_0^{i,N}) = \mathrm{P}_0, \;\; 1 \leq i \leq N$$

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