ST403X / MA4F70

UNIVERSITY OF WARWICK

FINAL YEAR EXAMINATIONS: SUMMER 2015

BROWNIAN MOTION

Time Allowed: 2 Hours

Full marks may be gained by correctly answering 3 complete questions. Candidates may attempt all questions. Marks will be awarded for the best 3 answers only.

Read carefully the instructions on the answer book and make sure that the particulars required are entered on each answer book

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1. Let $\beta(\cdot)$ be standard, one dimensional Brownian motion starting from zero and P_0 denote its distribution.

   (a) State and prove Blumenthal's 0-1 law. [5 marks]

   (b) Let $\tau_0 := \inf\{t > 0 \colon \beta(t) < 0\}$. Show that $P_0(\tau_0 > 0) = 0$. [5 marks]

   (c) Show that, for any $t \in [0, 1]$, $P_0(t \text{ is a local maximum }) = 0$. [5 marks]

   (d) Show that the times in $(0, 1)$ where Brownian motion has local maxima is, almost surely, a dense set. [5 marks]

Continued ...

2. Let $P_x, x \in \mathbb{R}$, denote the Wiener measure corresponding to standard, one dimensional Brownian motion starting from x.

   (a) State the strong Markov property for Brownian motion. [5 marks]

   (b) Let $b > 0$. Show that $P_0\big(\max_{0\le s\le t}\beta(s) \ge b\big) = P_0\big(|\beta(t)| > b\big)$. [5 marks]

   (c) Denote $u(t,x) := P_x\big(\max_{0\le s\le t}\beta(s) < b\big)$. Set up the boundary value problem that $u(t,x)$ satisfies. Justify your derivation. [5 marks]

   (d) Let $\beta_1(\cdot)$ and $\beta_2(\cdot)$ be two independent, standard, one dimensional Brownian motions starting from x_1, x_2, respectively. Compute the probability that they do not meet in the time interval $[0,t]$, $t > 0$. [5 marks]

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3. (a) State the Feynman-Kac formula related to parabolic PDEs. [5 marks]

   (b) State the Feynman-Kac formula related to the Dirichlet problem of a bounded domain in $\mathbb{R}^d$. State any further assumptions needed. [5 marks]

   (c) Assume that $d = 1$, i.e. that the Brownian motion $\beta(\cdot)$ is one dimensional. Let $\gamma, \alpha > 0$ and $\tau := \inf\{t\colon |\beta(t)| \ge \alpha\}$ Compute $E_0[e^{\gamma\tau}]$. Does this exist for all $\gamma > 0$? Justify you answer. [10 marks]

Continued ...

4. (a) State Donsker's Theorem. [4 marks]

   (b) Let $\beta(\cdot)$ be standard Brownian motion starting from zero and $L := \sup\{t \in [0,1] \colon \beta(t) = 0\}$. Let $s \in (0,1)$ and show that

$$P_0(L \leq s) = \frac{1}{\pi} \int_{1-s}^{\infty} \frac{1}{(sr^3)^{1/2}} \frac{rs}{r+s} dr.$$

[8 marks]

   (c) Let $(S_n)_{n \geq 0}$ be a symmetric, simple random walk, starting at the origin. Let $L_n := \sup\{i \leq n \colon S_i = 0\}$. Compute

$$\lim_{n \to \infty} \mathbb{P}_0 \left(\frac{L_n}{n} \leq s \right), \qquad s < 1.$$

   State any assumptions you make. [8 marks]

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1. Let $\beta(\cdot)$ be standard, one dimensional Brownian motion starting from zero and P_0 denote its distribution.

   (a) State and prove Blumenthal's 0-1 law. [5 marks]

   Answer. [BOOKWORK] Let $\mathcal{F}^+(t) := \cap_{s>t}\mathcal{F}_s$ be the germ σ algebra. Then any event $A \in \mathcal{F}^+(0)$ has probability zero or one **(1 mark)**. For the proof we show first that $\beta(\cdot + t) - \beta(t)$ is Brownian motion independent of $\mathcal{F}^+(t)$. **(2 marks)**Then letting $t \to 0$ and using continuity of Brownian motion, shows that any $A \in \mathcal{F}^+(0)$ is independent of itself **(2 marks)**.

   (b) Let $\tau_0 := \inf\{t > 0 \colon \beta(t) < 0\}$. Show that $P_0(\tau_0 > 0) = 0$. [5 marks]

   Answer. [SEEN EXAMPLE] $\{\tau_0 = 0\}$ is in the germ σ algebra, so the event will have probability zero or one **(2 marks)**. But $P_0(\tau_0 = 0) \geq \lim_{t\to 0} P_0(\tau_0 \leq t) \geq \lim_{t\to 0} P_0(\beta(t) < 0) = 1/2$, so $P_0(\tau_0 = 0) = 1$ **(3 marks)**.

   (c) Show that, for any $t \in [0, 1]$, $P_0(t \text{ is a local maximum }) = 0$. [5 marks]

   Answer [SIMILAR EXAMPLE SEEN] By the use of Markov property at time t, this reduces to the previous question.

   (d) Show that the times in $(0, 1)$ where Brownian motion has local maxima is a dense set. [5 marks]

   Answer. [UNSEEN EXAMPLE]. Show first that Brownian motion is nowhere monotone: take any interval $[a, b]$ and break it into n subintervals. By independent increments the probability that BM takes monotone values at these intervals will tend to zero as n tends to infinity :

$$
\begin{aligned}
P(\text{BM is monotone in } [a,b]) = P\big(\cap_{i=1}^{n} \{\beta(t_{i+1}) \geq \beta(t_i)\}\big) &= \prod_{i=1}^{n} P(\beta(t_{i+1}) \geq \beta(t_i)) \\
&= P(\beta(1) \geq 0)^n
\end{aligned}
$$

   (2 marks).
   Now, take the collection of dyadic intervals to see that there is is no point of monotonicity **(1 mark)**.
   Having this, if the set of local maxima was not dense, the would be an interval of monotonicity and this is a contradiction **(2 marks)**.

Continued ...

2. Let $P_x, x \in \mathbb{R}$, denote the Wiener measure corresponding to standard, one dimensional Brownian motion starting from x.

   (a) State the strong Markov property for Brownian motion. [5 marks]

   Answer. [BOOKWORK]. Let $\mathcal{F}_t^+$ be the germ filtrations at t. Let also τ be a stopping time and $\mathcal{F}_\tau^+$ the σ algebra associated to this stopping time. Then the process $\beta(\cdot+\tau)-\beta(\tau)$ is Brownian motion, independent of $\mathcal{F}_\tau^+$.

   (b) Let $b>0$. Show that $P_0\big(\max_{0\le s\le t}\beta(s)\ge b\big)=P_0\big(|\beta(t)|>b\big)$. [5 marks]

   Answer [BOOKWORK] Start from $P(\beta(t)\ge b)=P(\beta(t)\ge b,\ \tau_b\le t)$ where τ_b is the hitting time of b **(2 marks)**. Use the strong Markov property at $\mathcal{F}^+(\tau_b)$**(2marks)**. Finally, use the fact that for any t, $P_0(\beta(t)\ge 0)=1/2$ **(2 marks)**.

$$
\begin{aligned}
P(\beta(t)\ge b) &= P(\beta(t)\ge b,\ \tau_b\le t)=E\big[P(\beta(t)\ge b,\ \tau_b\le t)\,|\,\mathcal{F}_{\tau_b}\big]\\
&= E[P_b(\beta(t-\tau_b)\ge b)\,1_{\tau_b\le t}]=\frac{1}{2}P(\tau_b\le t)
\end{aligned}
$$

   (c) Denote $u(t,x):=P_x\big(\max_{0\le s\le t}\beta(s)<b\big)$. Set up the boundary value problem that $u(t,x)$ satisfies. Justify your derivation. [5 marks]

   Answer [UNSEEN EXAMPLE]

$$
\begin{aligned}
u_t &= \frac{1}{2}u_{xx}, && s\in(0,t), x<b,\\
u(0,x) &= 1\\
u(s,b) &= 0, && s\in(0,t).
\end{aligned}
$$

   (1 mark) To obtain this one needs to consider that

$$
u(t-s,\beta(s))-\int_0^t(-u_t+\frac{1}{2}u_{xx})(t-s,\beta(s))\,ds
$$

   is a martingale **(2 marks)**. With u being the solution of the boundary value problem, we get that $u(t-s,\beta(s))$ is a martingale. Make use of the optimal stopping theorem **(2 marks)**.

   (d) Let $\beta_1(\cdot)$ and $\beta_2(\cdot)$ two standard, one dimensional Brownian motions starting from x_1, x_2, respectively. Compute the probability that they do not meet in the time interval $[0,t]$, $t>0$. [5 marks]

   Answer [UNSEEN EXAMPLE] The difference $W(s):=(\beta_1(s)-\beta_2(s))/\sqrt{2}$ is a Brownian motion **(3 marks)**. Therefore, they will not meet by time t, if $W(s)$ does not hit zero by time $2t$. This now reduces to question (b) **(2 marks)**.

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3. (a) State the Feynman-Kac formula related to parabolic PDEs. [5 marks]

   Answer. [BOOKWORK] Let $V(t,x)\in C_b^{1,2}$, a bounded function with bounded derivatives and $f\in C_b(\mathbb{R})$. The solution to the equation

$$
\begin{aligned}
u_t &= \frac{1}{2}\Delta u+Vu, && t>0, x\in\mathbb{R}^d,\\
u(0,x) &= f(x), && x\in\mathbb{R}.
\end{aligned}
$$

Then the solution of this problem is given by

$$u(t,x) = E_x\left[f(\beta(t))e^{\int_0^t V(t-s,\beta(s))ds}\right].$$

(b) State the Feynman-Kac formula related to the Dirichlet problem of a bounded domain in $\mathbb{R}^d$. State any further assumptions needed. [5 marks]

Answer [BOOKWORK] Assume that the boundary ∂D is smooth. Let $V(x) \in C_b^2 D$, a bounded function with bounded derivatives and $f \in C_b(\partial D)$. The solution to the equation

$$\begin{aligned} \frac{1}{2}\Delta u + Vu &= 0, \qquad x \in D, \\ u(x) &= f(x), \qquad x \in \partial D. \end{aligned}$$

Then the solution of this problem is given by

$$u(x) = E_x\left[f(\beta(\tau))e^{\int_0^\tau V(\beta(s))ds}\right], \qquad x \in D$$

and τ is the hitting time of the boundary.

(c) Assume that $d = 1$, i.e. that the Brownian motion $\beta(\cdot)$ is one dimensional. Let $\gamma, \alpha > 0$ and $\tau := \inf\{t\colon |\beta(t)| \geq a\}$ Compute $E_0[e^{\gamma\tau}]$. Does this exist for all $\gamma > 0$? Justify you answer. [10 marks]

Answer. [UNSEEN EXAMPLE] Apply Feynman-Kac formula to see that $u(x) := E_x[e^{\gamma\tau}], x \in (-a,a)$, solves the problem

$$\begin{aligned} \frac{1}{2}u_{xx} + \gamma u &= 0, \qquad x \in (-a,a), \\ u(a) = u(-a) &= 1, \end{aligned}$$

(3 marks)
The general solution to this equation is $u(x) = A\cos(\sqrt{2\gamma}x) + B\sin(\sqrt{2\gamma}x)$. Satisfying the boundary conditions gives

$$\begin{aligned} 1 &= A\cos(\sqrt{2\gamma}a) + B\sin(\sqrt{2\gamma}a) \\ 1 &= A\cos(\sqrt{2\gamma}a) - B\sin(\sqrt{2\gamma}a) \end{aligned}$$

(3 marks)
Adding and subtracting the equations gives $A\cos(\sqrt{2\gamma}a) = 1$ and $B\sin(\sqrt{2\gamma}a) = 0$. From this we can choose $B = 0$. However, if $\cos(\sqrt{2\gamma}a) = 0$ then there is no choice of A that can satisfy the boundary condition. So, if $\sqrt{2\gamma}a < \pi/2$ then we can choose

$$A = \frac{\cos(\sqrt{2\gamma}x)}{\cos(\sqrt{2\gamma}a)}.$$

(2 marks)
If γ is such that $\sqrt{2\gamma}a \geq \pi/2$ then there is no solution, either because the boundary condition will force $A = B = 0$ or the above solution takes negative values. So for $\sqrt{2\gamma}a \geq \pi/2$ the exponential moment of τ is infinite **(2 marks)**.

Continued ...

4. (a) State Donsker's Theorem. [4 marks]

Answer. [BOOKWORK] Let $(S_n)_{n\geq 1}$ be random walk with jump increments having mean zero and variance one. Set $S_N^*(t)$ be the continuous function, which equals S_{tN}, for $t \in N^{-1}\mathbb{N}$ and linearly interpolated between these values, otherwise. Then $S_N^*(\cdot)$ converges in distribution, on the space of continuous function equipped with the *sup* topology to Brownian motion.

(b) Let $\beta(\cdot)$ be standard Brownian motion starting from zero and $L := \sup\{t \in [0,1] \colon \beta(t) = 0\}$. Compute $P_0(L \leq s)$, for $s \in (0,1)$. [8 marks]

Answer. [UNSEEN EXAMPLE] Let τ_0 be the first hitting time of zero. Condition on $\mathcal{F}_s$ and using the Markov property write **(4 marks)**

$$\begin{aligned} P_0(L \leq s) &= \int_{-\infty}^{\infty} p_s(0,x) P_x(\tau_0 > 1-s) \\ &= 2\int_0^{\infty} p_s(0,x) P_x(\tau_0 > 1-s) \end{aligned}$$

using the reflection principle $P_x(\tau_0 \in dr) = (2\pi r^3)^{-1/2} x e^{-x^2/2r} dr$ **(4 marks)** and so the above integral writes as

$$2\int_0^{\infty} dx (2\pi s)^{-1/2} e^{-x^2/2s} \int_{1-s}^{\infty} (2\pi r^3)^{-1/2} x e^{-x^2/2r} dr.$$

Interchange the integral and write it as

$$\frac{1}{\pi}\int_{1-s}^{\infty} \frac{1}{(sr^3)^{1/2}} \frac{rs}{r+s} dr.$$

(2 marks)

(c) Let $(S_n)_{n\geq 0}$ be a symmetric, simple random walk, starting at the origin. Let $L_n := \sup\{i \leq n \colon S_i = 0\}$. Compute

$$\lim_{n\to\infty} \mathbb{P}_0\left(\frac{L_n}{n} \leq s\right), \qquad s < 1.$$

State any assumptions you make. [8 marks]

Answer. [UNSEEN EXAMPLE] By Donsker's theorem this will converge to $P_0(L \leq s)$ where L is the last hitting time, in $(0,1)$, of zero for Brownian motion **(2 marks)**. Some care is needed, though since L is not a continuous functional of $C(0,1)$. We can perturb a (smooth) continuous path so that the last time it hits zero changes a lot **(2 marks)**. However, with probability one, the last hitting time of Brownian motion is not an isolated point (this follows from Blumenthal's 0-1 law) and therefore Donskers theorem can be applied **(4 marks)**.