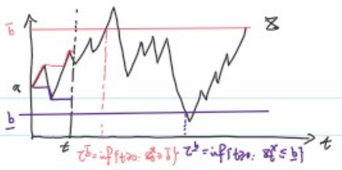


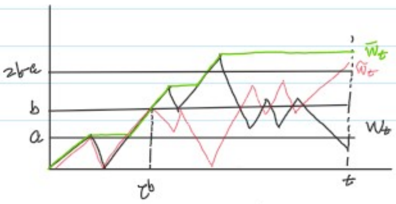
$\tau^{\bar{b}} = \inf\{t \geq 0 : X_t^x \geq \bar{b}\}$ $\tau^{\underline{b}} = \inf\{t \geq 0 : X_t^x \leq \underline{b}\}$

$\{\tau^{\bar{b}} > t\} = \{\max\limits_{0\leq s\leq t} X_s < \bar{b}\} \Leftrightarrow \{\tau^{\bar{b}} \leq t\} = \{\max\limits_{0\leq s\leq t} X_s \geq \bar{b}\}$

$\{\tau^{\underline{b}} > t\} = \{\min\limits_{0\leq s\leq t} X_s > \underline{b}\} \Leftrightarrow \{\tau^{\underline{b}} \leq t\} = \{\min\limits_{0\leq s\leq t} X_s \leq \underline{b}\}$

Reflection principle for BM. W.

Define $\overline{W}_t = \max\limits_{0\leq s\leq t} W_s$



$\tau^b = \inf\{t \geq 0 : W_t \geq b\}$ first-passage-time of b by W is an $\mathcal{F}_t$-stopping time (debut time)

$\{\overline{W}_t \geq b\} = \{\tau^b \leq t\} = \{\tilde{\tau}^b \leq t\}$ where

$\tilde{\tau}^b = \inf\{t \geq 0 : \widetilde{W}_t \geq b\}$, $\widetilde{W}_t = W_t^{\tau^b} - (W_t - W_t^{\tau^b})$

$$= \begin{cases} W_t - (W_t - W_t) = W_t, & t \leq \tau^b \\ W_{\tau^b} - (W_t - W_{\tau^b}), & t \geq \tau^b \end{cases}$$

reflection of W by b from τ^b.

Claim: $\widetilde{W}$ is also BM. (by Strong Markov property + symmetric property of BM)

Hence. $\mathbb{P}(\overline{W}_t \geq b, W_t \leq a)$

$= \mathbb{P}(\tilde{\tau}^b \leq t, \widetilde{W}_t \geq 2b-a)$ $\{\overline{W}_t \geq b\} = \{\tau^b \leq t\} = \{\tilde{\tau}^b \leq t\}$, $\{W_t \leq a\} = \{\widetilde{W}_t \geq 2b-a\}$

$= \mathbb{P}(\widetilde{W}_t \geq 2b-a)$ $\{\widetilde{W}_t \geq 2b-a\} \subseteq \{\tilde{\tau}^b \leq t\}$

$= \mathbb{P}(W_t \geq 2b-a)$

Reflection principle: $\mathbb{P}(\overline{W}_t \geq b, W_t \leq a) = \mathbb{P}(W_t \geq 2b-a)$ for $b \geq a > 0$.

$\Rightarrow$ joint distribution of $(\overline{W}, W)$.

$\mathbb{P}(\overline{W}_t \leq b, W_t \leq a)$

$= \mathbb{P}(W_t \leq a) - \mathbb{P}(\overline{W}_t \geq b, W_t \leq a)$

$= \mathbb{P}(W_t \leq a) - \mathbb{P}(W_t \geq 2b-a)$ $\Rightarrow$ joint density of $(\overline{W}, W)$: $f_{W,\overline{W}}(x, y)$

$\Rightarrow$ distribution of $\overline{W}$:

$\mathbb{P}(\overline{W}_t \geq b)$

$= \mathbb{P}(\overline{W}_t \geq b, W_t \leq b) + \mathbb{P}(\overline{W}_t \geq b, W_t \geq b)$

Reflection principle $\{\overline{W}_t \geq b\} \supseteq \{W_t \geq b\}$

$= \mathbb{P}(W_t \geq b) + \mathbb{P}(W_t \geq b)$

$= \mathbb{P}(W_t \geq b) + \mathbb{P}(\underbrace{-W_t}_{BM} \leq -b) = \mathbb{P}(|W_t| \geq b)$

Hence. $\overline{W}_t \overset{d}{=} |W_t|$ for fixed t.

non-decreasing in t

no monotone property of t

Reflection principle for BM with drift $Z_t^\mu = W_t + \mu t$.

Define $\bar{Z}_t^\mu = \max_{0 \le s \le t} Z_s^\mu$.

$\{\tau^b \le t\} = \{\bar{Z}_t^\mu \ge b\}$.

$\tau^b = \inf\{t \ge 0 : Z_t^\mu \ge b\}$

$\mathbb{P}(\bar{Z}_t^\mu \le b, \ Z_t^\mu \le a)$ for $b \ge a > 0$.

Define $Q \sim P$ by R-N density

$$\frac{dQ}{dP}\Big|_{\mathcal{F}_t} = \mathcal{E}(-\mu W)_t = e^{-\mu W_t - \frac{1}{2}\mu^2 t}$$

By Girsanov, $Z_t^\mu = W_t + \mu t$ is BM under Q.

$$= \mathbb{E}\left[\mathbb{1}_{\bar{Z}_t^\mu \le b, \ Z_t^\mu \le a}\right]$$

$$= \mathbb{E}^Q\left[\frac{dP}{dQ}\Big|_{\mathcal{F}_t} \mathbb{1}_{\bar{Z}_t^\mu \le b, \ Z_t^\mu \le a}\right]$$

$$= \mathbb{E}^Q\left[e^{\mu W_t + \frac{1}{2}\mu^2 t} \mathbb{1}_{\bar{Z}_t^\mu \le b, \ Z_t^\mu \le a}\right]$$

$$= \mathbb{E}^Q\left[e^{\mu Z_t^\mu - \frac{1}{2}\mu^2 t} \mathbb{1}_{\bar{Z}_t^\mu \le b, \ Z_t^\mu \le a}\right]$$

$$= \int_{x \le a} dx \int_{y \le b} dy \ e^{\mu x - \frac{1}{2}\mu^2 t} f_{W, \bar{W}}(x, y)$$

$$\mathbb{P}(\bar{Z}_t^\mu \le b) = \mathbb{E}\left[\mathbb{1}_{\bar{Z}_t^\mu \le b}\right]$$

$$= \mathbb{E}^Q\left[\frac{dP}{dQ}\Big|_{\mathcal{F}_t} \mathbb{1}_{\bar{Z}_t^\mu \le b}\right]$$

$$= \mathbb{E}^Q\left[e^{\mu Z_t^\mu - \frac{1}{2}\mu^2 t} \mathbb{1}_{\bar{Z}_t^\mu \le b}\right]$$

$$= \int_{\mathbb{R}} dx \int_{y \le b} dy \ e^{\mu x - \frac{1}{2}\mu^2 t} f_{W, \bar{W}}(x, y)$$

Black-Cox structural model

Payoff of Corporate bond $\quad \underset{\text{no default}}{K \mathbb{1}_{\tau > T}} + \underset{\text{default.}}{V_\tau \mathbb{1}_{\tau \le T}}$

Corporate bond price $\quad E^Q\left[\mathbb{1}_{\tau > T} K e^{-rT} + \mathbb{1}_{\tau \le T} V_\tau e^{-r\tau}\right]$

For $E^Q\left[\mathbb{1}_{\tau > T} K e^{-rT}\right] = K e^{-rT} \ Q(\tau > T)$

$$= K e^{-rT} Q\left(\inf_{0 \le t \le T} \left(\frac{V_t}{D_t}\right) > 1\right)$$

$$= K e^{-rT} Q\left(\inf_{0 \le t \le T} \frac{V_0}{K e^{-dT}} e^{\sigma_V W_t + (r - \frac{1}{2}\sigma_V^2 - d)t} > 1\right)$$

$$= K e^{-rT} Q\left(\inf \sigma_V W_t + (r - \tfrac{1}{2}\sigma_V^2 - d)t > \log \frac{K e^{-dT}}{\cdot\cdot}\right)$$

$0\le t\le T$

$$= Ke^{-rT} Q\Big(\inf_{0\le t\le T} \sigma_V W_t + (r-\tfrac12\sigma_V^2-d)t > \log \frac{Ke^{-dT}}{V_0}\Big)$$

$$= Ke^{-rT} Q\Big(\sup_{0\le t\le T} \underbrace{-W_t}_{BM} \underbrace{- \frac{(r-\frac12\sigma_V^2-d)}{\sigma_V} t}_{\mu} < \frac{1}{\sigma_V}\log\frac{V_0}{Ke^{-dT}}\Big)$$

: X_t^μ with $\mu = -\dfrac{r-\frac12\sigma_V^2-d}{\sigma_V}$

For $E^Q\left[\mathbb{1}_{\tau\le T}\dfrac{V_\tau}{B_\tau}\right]$

$= V_0 E^Q\left[\dfrac{V_\tau/B_\tau}{V_0}\mathbb{1}_{\tau\le T}\right]$

$= V_0 E^{Q^V}\left[\mathbb{1}_{\tau\le T}\right]$

$= V_0\, Q^V(\tau\le T)$

$= V_0\, Q^V\Big(\inf_{s\in[0,T]}\left(\dfrac{V_s}{D_s}\right)\le 1\Big)$

Define $Q^V \sim Q$ by R-N density

$\left.\dfrac{dQ^V}{dQ}\right|_{\mathcal{F}_t} = \dfrac{V_t/B_t}{V_0} = \mathcal{E}(\sigma_V W)_t$

By Girsanov. $W_t^{Q^V} = W_t - \sigma_V t$ is BM under Q^V.

Chapter 8. Stochastic Calculus for single jump processes

8.1 Functions with one-sided limits.

Def. $f: [0,T]\to\mathbb{R}$ has right limit at $t\in[0,T)$ if $f(t+) := \lim_{s\downarrow t} f(s)$ exists.

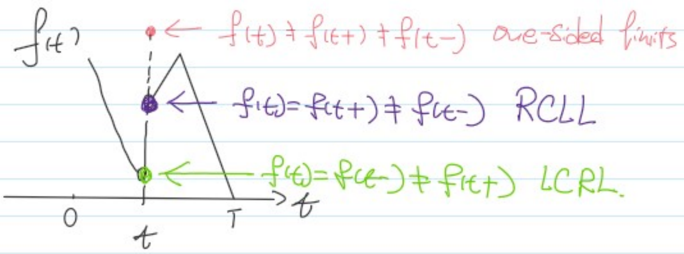
left limit at $t\in(0,T]$ if $f(t-) := \lim_{s\uparrow t} f(s)$ exists

f has one-sided limits at $t\in(0,T)$ if $f(t+)$, $f(t-)$ exist

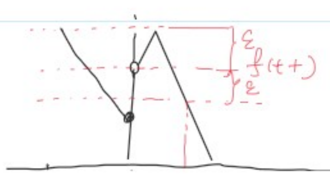
is right continuous at $t\in[0,T)$ if $f(t) = f(t+)$

left continuous at $t\in(0,T]$ if $f(t) = f(t-)$

f is continuous at $t\in(0,T)$ if $f(t) = f(t+) = f(t-)$

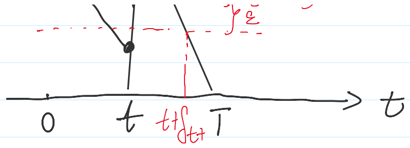


Theorem 1. $f: [0,T]\to\mathbb{R}$ has one-sided limits. Then, f is bdd on $[0,T]$



Proof: $f(t+)$ exists. i.e. $\lim_{s\downarrow t} f(s)$ exists.

$\forall\varepsilon>0$, $\exists\,\delta_{t+}>0$ s.th.



$\forall \varepsilon>0,\ \exists\, \delta_{t+}>0$ s.t. $s\downarrow t$

$$|f(s)-f(t+)|<\varepsilon \quad \text{for } s\in(t,\ t+\delta_{t+})$$

$$\Rightarrow |f(s)| \le |f(s)-f(t+)| + |f(t+)|$$
$$\le \varepsilon + |f(t+)| \quad \text{for } s\in(t,\ t+\delta_{t+})$$

Similarly, $\exists\, \delta_{t-}>0$, s.t.

$$|f(s)| \le |f(s)-f(t-)| + |f(t-)|$$
$$\le \varepsilon + |f(t-)| \quad \text{for } s\in(t-\delta_{t-},\ t)$$

Hence,
$$|f(s)| \le \max\{\varepsilon+|f(t+)|,\ \varepsilon+|f(t-)|,\ f(t)\} =: \Gamma_t$$
$$\text{for } s\in(t-\delta_{t-},\ t+\delta_{t+}) =: O_t$$

Since $\{O_t : t\in[0,T]\}$ is an open cover of $[0,T]$,
there exists $\{t_1,\cdots,t_n\}\subset[0,T]$ s.t. $[0,T]\subset \bigcup_{i=1}^{n} O_{t_i}$

Hence, $\sup_{s\in[0,T]}|f(s)| \le \max_{1\le i\le n} \Gamma_{t_i}$ #