Week3 (1) Chapter 0.1-0.6

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Chapter 0

Definition 6. (semimartingale)

An adapted process X is called a semimartingale if $X_t = M_t + A_t$ where M is a local martingale, and A is a finite variation process, i.e.

$$\sup_{D_t} \sum_l |A_{t_l} - A_{t_{l-1}}| < \infty$$

where D_t is any finite partition of $[0,t]$.

One way to distinguish between the local martingale and the finite variation process in the semimartingale is to use quadratic variation.

Definition 7. (quadratic variation)

A stochastic process M has quadratic variation if for any $t \geq 0$, when the mesh of the partition of $||D_t|| \to 0$,

$$\lim_{||D_t|| \to 0} \sum_l |M_{t_l} - M_{t_{l-1}}|^2 < \infty.$$

Proposition 4. *For a local martingale M, its quadratic variation $\langle M \rangle$ can be defined equivalently as follows:*
(1) An increasing continuous adapted process $\langle M \rangle$ such that $M_t^2 - \langle M \rangle_t, t \geq 0$, is a local martingale;
(2) From Itô's formula, $\langle M \rangle_t = M_t^2 - 2\int_0^t M_s dM_s$.

Note that a finite variation process always has zero quadratic variation.

Proposition 5. *(Criteria for local martingales being martingales)*

(1) Let M be a local martingale such that $E[\sup_{s\in[0,t]} |M_s|] < \infty$ for any $t \geq 0$, then M is a martingale; up to time t.

(2) Let M be a local martingale such that $E\langle M \rangle_t < \infty$ for any $t \geq 0$, then M is a martingale; up to time t.

(3) A bounded local martingale M is a martingale.

2 Itô's stochastic integration

predictable => prog mlb => adapted (in continuous time)

Itô's stochastic integration gives a meaning for the integrals like $\int_0^\cdot h_s dW_s$. First we define the spaces of integrands. Fix $T > 0$ in the following.

Let $\mathscr{L}^2(\mathbb{R}^d)$ be the space of $\mathbb{R}^d$-valued progressively measurable processes $h = (h^1, \ldots, h^d)$ with $\mathbf{E}[\int_0^T |h_s|^2 ds] < \infty$

Let $\mathscr{L}(\mathbb{R}^d)$ be the space of $\mathbb{R}^d$-valued progressively measurable processes $h = (h^1, \ldots, h^d)$ with $\int_0^T |h_s|^2 ds < \infty$, a.s.

Obviously, we have that $\mathscr{L}^2(\mathbb{R}^d) \subset \mathscr{L}(\mathbb{R}^d)$.

Theorem 1. *For any $h \in \mathscr{L}(\mathbb{R}^d)$, one can define the stochastic integral $\int_0^t h_s dW_s = \int_0^t \sum_{j=1}^d h_s^j dW_s^j$, $t \in [0,T]$, with the following properties:*

$h = (h^1 \cdots h^d)$

$W = \begin{pmatrix} W^1 \\ \vdots \\ W^d \end{pmatrix}$

(1) $\int_0^{\cdot} h_s dW_s$ is a local martingale;
(2) (linearity) for any $\lambda_1, \lambda_2 \in \mathbb{R}$, and $h(1), h(2) \in \mathscr{L}(\mathbb{R}^d)$,

$$\int_0^t (\lambda_1 h_s(1) + \lambda_2 h_s(2))dW_s = \lambda_1 \int_0^t h_s(1)dW_s + \lambda_2 \int_0^t h_s(2)dW_s;$$

(3) for any stopping time τ,

$$\int_0^{t\wedge\tau} h_s dW_s = \int_0^t \mathbf{1}_{\{s\le\tau\}} h_s dW_s;$$

(4) (dominated convergence) If $h(n) \in \mathscr{L}(\mathbb{R}^d)$ is a sequence with $\lim_{n\to\infty} h_t(n) = 0$ for each (ω, t), and such that $|h_t(n)| \le K$ for some $K \ge 0$, then

$$\lim_{n\to\infty} \sup_{s\in[0,t]} |\int_0^s h_u(n)dW_u| = 0$$

in probability.
(5) (Itô's isometry) If $h \in \mathscr{L}^2(\mathbb{R}^d)$,, then $\int_0^{\cdot} h_s dW_s$ is a martingale, and moreover,

$$\mathbf{E}\left[(\int_0^t h_s dW_s)^2\right] = \mathbf{E}\left[\int_0^t |h_s|^2 ds\right].$$

(6) (quadratic variation)

$$\left\langle \int_0^{\cdot} h_s dW_s \right\rangle_t = \int_0^t |h_s|^2 ds.$$

3 Itô's formula

Itô's formula says that semimartingales are invariant under C^2 transformation. Since we are interested in the multi-dimensional Itô's formula, we first define the quadratic covariation $\langle M^i, M^j \rangle_t$ as

if $i=j \Rightarrow \langle M^i \rangle_t = (M^i)^2 - 2\int_0^t M^i_s dM^i_s$ ①

$$\langle M^i, M^j \rangle_t = M^i_t M^j_t - \int_0^t M^i_s dM^j_s - \int_0^t M^j_s dM^i_s, \quad ②$$

or, equivalently, as an increasing continuous adapted process $\langle M^i, M^j \rangle$ such that $M^i_t M^j_t - \langle M^i, M^j \rangle_t, t \in [0,T]$, is a local martingale.

Note that from the polarization identity

$$xy = \frac{1}{4}(|x+y|^2 - |x-y|^2)$$

and Itô's isometry in part (5) of Theorem 1, we can obtain

Proof: $E[\int h(1)dW \int h(2)dW]$
$= \frac{1}{4} E[(\int h(1)+h(2)dW)^2 - (\int h(1)-h(2)dW)^2]$ by polarisation
$= \frac{1}{4} E[\int (h(1)+h(2))^2 ds - \int (h(1)-h(2))^2 ds]$ by ⑤
$= E[\int h(1)\cdot h(2)\, ds]$

$$\mathbf{E}\left[(\int_0^t h_s(1)dW_s)(\int_0^t h_s(2)dW_s)\right] = \mathbf{E}\left[\int_0^t (h_s(1))^T h_s(2)ds\right].$$

In terms of quadratic covariation, from the polarization identity

$$\langle M^i, M^j \rangle_t = \frac{1}{4}\left(\langle M^i + M^j \rangle_t - \langle M^i - M^j \rangle_t\right) \text{ by using ① + ②}$$

and the quadratic variation result in part (6) of Theorem 1, we can obtain

$$\langle \int_0^{\cdot} h_s(1)dW_s, \int_0^{\cdot} h_s(2)dW_s \rangle_t = \int_0^t (h_s(1))^T h_s(2)ds.$$

Proof: $\langle \int h(1)dW, \int h(2)dW \rangle$
$= \frac{1}{4}(\langle \int h(1)+h(2)dW \rangle - \langle \int h(1)-h(2)dW \rangle)$ by polarisation
$= \frac{1}{4}(\int (h(1)+h(2))^2 ds - \int (h(1)-h(2))^2 ds)$ by ⑥
$= \int h(1)\cdot h(2)\, ds = \int \sum_k h^k(1)\, h^k(2)\, ds$

Theorem 2. *(Itô's formula) Let $X = (X^1, \ldots, X^n)^T$ be an n-dimensional semimartingale, and let $f \in C^2(\mathbb{R}^n)$. Then $f(X_t)$, $t \ge 0$, is also an n-dimensional semimartingale, and moreover,*

$$df(X_t) = \sum_{i=1}^n \partial_{x_i} f(X_t) \overset{\left(dM_t + dA_t\right)}{\overset{\shortparallel}{dX^i_t}} + \frac{1}{2}\sum_{i,j=1}^n \partial_{x_i x_j} f(X_t) d\langle X^i, X^j \rangle_t.$$

As an example, suppose that X solves the following time homogenous diffusion

$$dX^i_t = \mu^i(X_t)dt + \sum_{j=1}^d \sigma^{ij}(X_t)dW^j_t, \quad 1 \le i \le n,$$

or in a matrix form

$$dX_t = \mu(X_t)dt + \sigma(X_t)dW_t. \Leftrightarrow \begin{pmatrix} dX^1_t \\ \vdots \\ dX^n_t \end{pmatrix} = \begin{pmatrix} \mu^1 \\ \vdots \\ \mu^n \end{pmatrix} dt + \begin{pmatrix} \sigma^{11} & \cdots & \sigma^{1n} \\ & \ddots & \\ \sigma^{n1} & \cdots & \sigma^{nn} \end{pmatrix} \begin{pmatrix} dW^1 \\ \vdots \\ dW^n \end{pmatrix}$$

Then from Itô's formula, we have that

$$df(X_t) = \mathscr{L} f(X_t)dt + \sum_{i=1}^n \partial_{x_i} f(X_t) \sum_{j=1}^d \sigma^{ij}(X_t)dW^j_t$$

Note that $d\langle X^i, X^j \rangle$

or in a matrix form

$$dX_t = \mu(X_t)dt + \sigma(X_t)dW_t. \Leftrightarrow$$

Then from Itô's formula, we have that

$$df(X_t) = \mathscr{L}f(X_t)dt + \sum_{i=1}^{n} \partial_{x_i} f(X_t) \sum_{j=1}^{d} \sigma^{ij}(X_t) dW_t^j,$$

Note that $d\langle X^i, X^j\rangle$

$= \langle \sum_k \sigma^{ik} dW_t^k, \sum_k \sigma^{jk} dW_t^k \rangle$

$= \langle \sigma^i dW_t, \sigma^j dW_t \rangle$ where $\sigma^i = (\sigma^{i1} \dots \sigma^{in})$

$= \sum_k \sigma^{ik} \sigma^{jk} dt.$

where $\mathscr{L}$ is called the infinitesimal generator of X, given as

$$\mathscr{L}f(x) = \frac{1}{2} \sum_{i,j=1}^{n} \partial_{x_i x_j} f(x) \sum_{k=1}^{d} \sigma^{ik}(x)\sigma^{kj}(x) + \sum_{i=1}^{n} \partial_{x_i} f(x) \mu^i(x).$$

The corresponding matrix form is

$$df(X_t) = \left(\frac{1}{2} Trace\{\sigma(X_t)\sigma(X_t)^T \nabla_x^2 f(X_t)\} + \mu(X_t)^T \nabla_x f(X_t) \right) dt + \left(\sigma(X_t)^T \nabla_x f(X_t)\right)^T dW_t.$$

Hence,

$$df(X_t) = \sum_i \partial_{x_i} f \left(\mu^i dt + \sum_j \sigma^{ij} dW_t^j \right) + \frac{1}{2} \sum_{ij} \partial_{x_i x_j} f \sum_k \sigma^{ik}\sigma^{jk} dt.$$

$$= \Big[\underbrace{\frac{1}{2} \sum_{ij} \partial_{x_i x_j} f \sum_k \sigma^{ik}\sigma^{jk}} + \sum_i \underbrace{\partial_{x_i} f \, \mu^i}_{\nabla_x f \cdot \mu} \Big] dt + \underbrace{\sum_i \partial_{x_i} f \sum_j \sigma^{ij} dW_t^j}_{\nabla_x f^T \sigma \cdot dW = (\sigma^T \nabla_x f)^T dW}$$

$$\frac{1}{2} \sum_{ij} \partial_{x_i x_j} f (\sigma\sigma^T)_{ij}$$

$$= \frac{1}{2} \sum_i \Big(\underbrace{\sum_j \partial_{x_i x_j} f (\sigma\sigma^T)_{ij}}_{(\nabla_x^2 f \sigma\sigma^T)_{ii}} \Big)$$

$$= \frac{1}{2} Trace(\nabla_x^2 f \sigma\sigma^T)$$

Proposition 6. *(Lévy's characterization) Let $M = (M^1, \cdots, M^d)^T$ be a $\mathbb{R}^d$-valued local martingale. Then it is also a d-dimensional Brownian motion if*

$$\langle M^i, M^j \rangle_t = \delta_{ij} t$$

$\delta_{ij} = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases}$

Let $\nabla_x f = \begin{pmatrix} \partial_{x_1} f \\ \vdots \\ \partial_{x_n} f \end{pmatrix}$ $\quad \nabla_x^2 f = \begin{pmatrix} \partial_{x_1 x_1} f & \cdots & \partial_{x_1 x_n} f \\ & \ddots & \\ \partial_{x_n x_1} f & \cdots & \partial_{x_n x_n} f \end{pmatrix}$

for $1 \le i, j \le d$.

4 Stochastic differential equation (SDE)

Definition 8. Let $\mu(\cdot, x) : \Omega \times [0, \infty) \to \mathbb{R}^n$ and $\sigma(\cdot, x) : \Omega \times [0, \infty) \to \mathbb{R}^{n \times d}$ be both progressively measurable. Given a SDE

$$dX_t = \mu(t, X_t)dt + \sigma(t, X_t)dW_t, \quad X_0 = x \in \mathbb{R}^n,$$

its solution is defined as a semimartingale X such that the corresponding integral equation is satisfied

$$X_t = x + \int_0^t \mu(s, X_s)ds + \int_0^t \sigma(s, X_s)dW_s. \checkmark$$

We say X is unique if any other solution $\tilde{X}$ is a modification of X.

Examples of SDEs include
(1) the prices of the stocks:

$$dS_t^i = S_t^i \left(\mu_t^i dt + \sum_{j=1}^{d} \sigma_t^{ij} dW_t^j \right), \quad 1 \le i \le n$$

for progressively measurable processes μ^i and σ^{ij}.
(2) Time homogenous diffusion:

$$dX_t^i = \mu^i(X_t)dt + \sum_{j=1}^{d} \sigma^{ij}(X_t)dW_t^j, \quad 1 \le i \le n,$$

Note: do NOT write as $\mathcal{E}(M_t)$

for deterministic functions $\mu^i(\cdot)$ and $\sigma^{ij}(\cdot)$.
(3) Stochastic exponential: Given a local martingale M, define its stochastic exponential as $\mathscr{E}(M)_t = e^{M_t - \frac{1}{2}\langle M \rangle_t}$, which satisfies

$$d\mathscr{E}(M)_t = \mathscr{E}(M)_t dM_t.$$

Apply Itô's to $\ln \mathcal{E}(M)_t$:

$$d \ln \mathcal{E}(M)_t = \frac{1}{\mathcal{E}(M)_t} d\mathcal{E}(M)_t - \frac{1}{2\mathcal{E}(M)_t^2} d\langle \mathcal{E}(M) \rangle_t$$

$$= \frac{1}{\cancel{\mathcal{E}(M)_t}} \cancel{\mathcal{E}(M)_t} dM_t - \frac{1}{2\cancel{\mathcal{E}(M)_t^2}} \cancel{\mathcal{E}(M)_t^2} d\langle M \rangle_t$$

$$\Rightarrow d \ln \mathcal{E}(M)_t = dM_t - \frac{1}{2} d\langle M \rangle_t$$

Theorem 3. *If both $\mu(\cdot, x)$ and $\sigma(\cdot, x)$ are Lipschitz continuous in x and have at most linear growth in x, then the above SDE admits a unique solution.*

Proposition 7. *(Novikov's condition)*
Let M be a local martingale with its stochastic exponential $\mathscr{E}(M)$. If

$$\mathbf{E}[e^{\frac{1}{2}\langle M \rangle_T}] < \infty,$$

then $\mathscr{E}(M)$ is a martingale up to time T.

$dZ_t = Z_t dB_t.$

5 Girsanov's theorem

Proposition 8. (Bayes Rule)
Let $\mathscr{E}(M)$ be a martingale up to $T > 0$. Define a new probability measure $\mathbf{Q}$ by the Radon-Nikodym density

$$\left.\frac{d\mathbf{Q}}{d\mathbf{P}}\right|_{\mathscr{F}_t} := \mathbf{E}^{\mathbf{P}}[\mathscr{E}(M)_T|\mathscr{F}_t] = \mathscr{E}(M)_t.$$

define on $\mathcal{F}_T$: $\frac{dQ}{dP} = \mathcal{E}(M)_T$.

then, on $\mathcal{F}_t$, for $t \leq T$, $\frac{dQ}{dP} = E[\mathcal{E}(M)_T | \mathcal{F}_t] = \mathcal{E}(M)_t$

Then for any r.v. $Y_t \in \mathscr{F}_t$, we have that

$$\mathbf{E}^{\mathbf{Q}}[Y_t] = \mathbf{E}^{\mathbf{P}}[Y_t\mathscr{E}(M)_t];$$
$$\mathbf{E}^{\mathbf{Q}}[Y_t|\mathscr{F}_s] = \mathbf{E}^{\mathbf{P}}\left[\left.\frac{Y_t\mathscr{E}(M)_t}{\mathscr{E}(M)_s}\right|\mathscr{F}_s\right], \quad \text{for } 0 \leq s \leq t \leq T.$$

Proof. For the first equality, it is sufficient to prove for $Y_t = \mathbf{1}_A$ where $A \in \mathscr{F}_t$. Indeed,

$$\begin{aligned}\mathbf{E}^{\mathbf{Q}}[\mathbf{1}_A] &= \mathbf{Q}(A)\\ &= \int_A \mathscr{E}(M)_t d\mathbf{P}\\ &= \int_\Omega \mathbf{1}_A\mathscr{E}(M)_t d\mathbf{P} = \mathbf{E}^{\mathbf{P}}[\mathbf{1}_A\mathscr{E}(M)_t].\end{aligned}$$

For the second equality, we need to show its RHS is the conditional expectation of Y_t given $\mathscr{F}_s$ under $\mathbf{Q}$. Indeed, for any $A \in \mathscr{A}_s$,

$$\begin{aligned}\mathbf{E}^{\mathbf{Q}}\left[\mathbf{1}_A\mathbf{E}^{\mathbf{P}}\left[\left.\frac{Y_t\mathscr{E}(M)_t}{\mathscr{E}(M)_s}\right|\mathscr{F}_s\right]\right] &= \mathbf{E}^{\mathbf{P}}\left[\mathbf{1}_A\mathbf{E}^{\mathbf{P}}\left[\left.\frac{Y_t\mathscr{E}(M)_t}{\mathscr{E}(M)_s}\right|\mathscr{F}_s\right]\mathscr{E}(M)_s\right]\\ &= \mathbf{E}^{\mathbf{P}}[\mathbf{1}_A Y_t\mathscr{E}(M)_t]\\ &= \mathbf{E}^{\mathbf{Q}}[\mathbf{1}_A Y_t] \qquad \square\end{aligned}$$

Theorem 4. (Girsanov's theorem)
Let $\mathscr{E}(M)$ be a martingale up to $T > 0$. Define a new probability measure $\mathbf{Q}$ by the Radon-Nikodym density

$$\left.\frac{d\mathbf{Q}}{d\mathbf{P}}\right|_{\mathscr{F}_t} = \mathscr{E}(M)_t.$$

Let X be a semimartingale under $\mathbf{P}$ with the decomposition: $X_t = N_t + A_t$ where N is a $\mathbf{P}$-local martingale and A is a finite variation process. Then X is also a semimartingale under $\mathbf{Q}$ with the decomposition: $X_t = \tilde{N}_t + \tilde{A}_t$, where $\tilde{N}$ is a $\mathbf{Q}$-local martingale:

$$\tilde{N}_t = N_t - \int_0^t \frac{1}{\mathscr{E}(M)_s} d\langle\mathscr{E}(M), N\rangle_s,$$

and $\tilde{A}$ is a finite variation process:

is FV by assumption

$$\tilde{A}_t = A_t + \int_0^t \frac{1}{\mathscr{E}(M)_s} d\langle \mathscr{E}(M), N\rangle_s.$$ is FV since $\langle \mathscr{E}(M), N\rangle_t \uparrow t$, so FV.

Proof. We only need to show that $\tilde{N}$ is a **Q**-local martingale. The following is the key observation to prove this.

*If $\tilde{N}_t\mathscr{E}(M)_t$, $t \in [0,T]$, is a **P**-local martingale, then $\tilde{N}_t$, $t \in [0,T]$, is a **Q**-local martingale.*

Hence, we first prove that $\tilde{N}_t\mathscr{E}(M)_t$ is a **P**-local martingale. Note that

$$\begin{aligned} d\tilde{N}_t &= dN_t - \frac{1}{\mathscr{E}(M)_t} d\langle \mathscr{E}(M), N\rangle_t, \\ d\mathscr{E}(M)_t &= \mathscr{E}(M)_t dM_t, \end{aligned}$$

Applying Itô's formula to $\tilde{N}_t\mathscr{E}(M)_t$ yields that

$$\begin{aligned} d\tilde{N}_t\mathscr{E}(M)_t &= \mathscr{E}(M)_t d\tilde{N}_t + \boxed{\tilde{N}_t d\mathscr{E}(M)_t} + d\langle \tilde{N}, \mathscr{E}(M)\rangle_t \\ &= \mathscr{E}(M)_t dN_t - d\langle \mathscr{E}(M), N\rangle_t + \boxed{\tilde{N}_t d\mathscr{E}(M)_t} + d\langle N, \mathscr{E}(M)\rangle_t \\ &= \mathscr{E}(M)_t dN_t + \tilde{N}_t d\mathscr{E}(M)_t \end{aligned}$$

which is obviously a **P**-local martingale. That is, there exits an increasing sequence of stopping times $T_n \uparrow \infty$ such that $(\tilde{N}\mathscr{E}(M))^{T_n}$ is a martingale.

Next we prove that $\tilde{N}$ is a **Q**-local martingale. Consider the above stopping time sequence T_n such that $\{s \leq T_n\}$, by Bayes rule, since $\tilde{N}_t^{T_n} \in \mathscr{F}_t$,

$$\mathbf{E}^{\mathbf{Q}}\left[\tilde{N}_t^{T_n} | \mathscr{F}_s\right] = \mathbf{E}^{\mathbf{P}}\left[\frac{\tilde{N}_t^{T_n}\mathscr{E}(M)_t}{\mathscr{E}(M)_s} | \mathscr{F}_s\right].$$

On the event $\{t > T_n\} \in \mathscr{F}_{T_n}$ (since $T_n \in \mathscr{F}_{T_n}$), the optional stopping theorem implies that

$$\mathbf{E}^{\mathbf{P}}[\mathscr{E}(M)_t | \mathscr{F}_{T_n}] = \mathscr{E}(M)_{T_n}.$$

On the other hand, $\tilde{N}_t^{T_n} = \tilde{N}_{T_n} \in \mathscr{F}_{T_n}$ and the tower property yields

$$\mathbf{E}^{\mathbf{P}}\left[\tilde{N}_t^{T_n}\mathscr{E}(M)_t | \mathscr{F}_s\right] = \mathbf{E}^{\mathbf{P}}\left[\tilde{N}_t^{T_n}\mathbf{E}^{\mathbf{P}}[\mathscr{E}(M)_t | \mathscr{F}_{T_n}] | \mathscr{F}_s\right] = \mathbf{E}^{\mathbf{P}}\left[\tilde{N}_{T_n}\mathscr{E}(M)_{T_n} | \mathscr{F}_s\right].$$

Hence,

$$\begin{aligned} \mathbf{E}^{\mathbf{Q}}\left[\tilde{N}_t^{T_n} | \mathscr{F}_s\right] &= \frac{1}{\mathscr{E}(M)_s}\mathbf{E}^{\mathbf{P}}\left[\tilde{N}_t^{T_n}\mathscr{E}(M)_t\mathbf{1}_{\{t \leq T_n\}} + \tilde{N}_t^{T_n}\mathscr{E}(M)_t\mathbf{1}_{\{t > T_n\}} | \mathscr{F}_s\right] \\ &= \frac{1}{\mathscr{E}(M)_s}\mathbf{E}^{\mathbf{P}}\left[\tilde{N}_t\mathscr{E}(M)_t\mathbf{1}_{\{t \leq T_n\}} + \tilde{N}_{T_n}\mathscr{E}(M)_{T_n}\mathbf{1}_{\{t > T_n\}} | \mathscr{F}_s\right] \\ &= \frac{1}{\mathscr{E}(M)_s}\mathbf{E}^{\mathbf{P}}[(\tilde{N}\mathscr{E}(M))_t^{T_n} | \mathscr{F}_s] = \frac{1}{\mathscr{E}(M)_s}(\tilde{N}\mathscr{E}(M))_s^{T_n} = \tilde{N}_s \end{aligned}$$

on the event $\{s \leq T_n\}$.

On the other hand, on the event $\{s > T_n\}$, since $\tilde{N}_t^{T_n} = \tilde{N}_{T_n}$, it follows that

$$\mathbf{E}^{\mathbf{Q}}\left[\tilde{N}_t^{T_n} | \mathscr{F}_s\right] = \tilde{N}_{T_n},$$

which shows that $\tilde{N}$ is indeed a $\mathbf{Q}$-local martingale. □

Theorem 5. *(Girsanov's theorem in Brownian motion case)*
Let $M_t = \int_0^t h_s dW_s$ with

$$= E^P\left[e^{\frac{1}{2}\langle M\rangle_T}\right] = E^P\left[e^{\frac{1}{2}\langle\int_0^{\cdot} h_s dW_s\rangle_T}\right] \text{ by } 6$$

$$\mathbf{E}^{\mathbf{P}}[\exp(\frac{1}{2}\int_0^T |h_s|^2 ds)] < \infty,$$

so that $\mathscr{E}(M)$ is a martingale. Define a new probability measure $\mathbf{Q}$ by the Radon-Nikodym density

$$\left.\frac{d\mathbf{Q}}{d\mathbf{P}}\right|_{\mathscr{F}_t} = \mathscr{E}(M)_t.$$

Then $W_t^{\mathbf{Q}} := W_t - \int_0^t h_s ds$ is a d-dimensional Brownian motion under $\mathbf{Q}$.

$$d\mathcal{E}(h)_t = \mathcal{E}(h)_t h_t dW_t$$

$$dW_t = dW_t$$

Proof. From Theorem 4, the Brownian motion W, as a $\mathbf{P}$-local martingale, is modified as

$$W_t - \int_0^t \frac{1}{\mathscr{E}(M)_s} d\langle \mathscr{E}(M), W\rangle_s = W_t - \int_0^t h_s ds,$$

$$\int_0^t \frac{1}{\mathcal{E}(h)_s} d\langle \mathcal{E}(h), W\rangle_s$$

$$= \int_0^t \frac{1}{\mathcal{E}(h)_s} \mathcal{E}(h)_s h_s ds = \int_0^t h_s ds$$

which is a $\mathbf{Q}$-local martingale. Since

$$\langle W^{\mathbf{Q},i}, W^{\mathbf{Q},j}\rangle_t = \langle W^i, W^j\rangle_t = \delta_{ij} t,$$

we conclude that $W^{\mathbf{Q}}$ is a Brownian motion under $\mathbf{Q}$ by using Lévy's characterization. □

6 Martingale representation

Theorem 6. *(Martingale representation)*
Assume that the filtration $\{\mathscr{F}_t\}$ is generated by the d-dimensional Brownian motion $W = (W^1, \ldots, W^d)^T$. For any local martingale M, there exists a density process $h = (h^1, \ldots, d^d)$ in $\mathscr{L}(\mathbb{R}^d)$ such that

$$M_t = M_0 + \int_0^t h_s dW_s.$$

Moreover, if M is a martingale, then $h \in \mathscr{L}^2(\mathbb{R}^d)$.

7 Feynman-Kac formula

Proposition 9. *(Markov property)*
Suppose that X solves the following SDE:

$$dX_t = \mu(t,X_t)dt + \sigma(t,X_t)dW_t,$$

where $\mu(\cdot,\cdot):[0,T]\times\mathbb{R}^n \to \mathbb{R}^n$ and $\sigma(\cdot,\cdot):[0,T]\times\mathbb{R}^n \to \mathbb{R}^{n\times d}$ are both deterministic functions. Then X admits the Markov property: For any measurable function $f(\cdot)$ defined on $\mathbb{R}^n$,

$$\mathbf{E}[f(X_t)|\mathscr{F}_s] = \mathbf{E}[f(X_t)|X_s] = F(s,X_s)$$

for some measurable function $F(\cdot,\cdot)$.

Proof. We first show the case $X_t = W_t$, i.e. a BM admits the Markov property. We claim that

$$\mathbf{E}[f(W_t)|\mathscr{F}_s] = g(W_s) \tag{1}$$

with $g(x) = \mathbf{E}[f(W_t - W_s + x)]$.

If (1) holds, then the tower property yields

$$\begin{aligned}\mathbf{E}[f(W_t)|W_s] &= \mathbf{E}[\mathbf{E}[f(W_t)|\mathscr{F}_s]|W_s]\\ &= \mathbf{E}[g(W_s)|W_s] = g(W_s).\end{aligned}$$

To show (1), we aim to show that for any $A \in \mathscr{F}_s$,

$$\mathbf{E}[f(W_t)\mathbf{1}_A] = \mathbf{E}[g(W_s)\mathbf{1}_A].$$

We first approximate $f(W_t)$ pointwisely by the functions of the form

$$f(W_t - W_s + W_s) = \lim_{m\to\infty}\sum_{n=1}^{m}\varphi_n^1(W_s)\varphi_n^2(W_t - W_s).$$

Then, the DCT/MCT yields

$$\begin{aligned}g(x) &= \mathbf{E}[\sum_n \varphi_n^1(x)\varphi_n^2(W_t - W_s)]\\ &= \sum_n \varphi_n^1(x)\mathbf{E}[\varphi_n^2(W_t - W_s)],\end{aligned}$$

and using the fact that $W_t - W_s$ is independent of $\mathscr{F}_s$,

$$\begin{aligned}\mathbf{E}[f(W_t)\mathbf{1}_A] &= \mathbf{E}[\sum_n \varphi_n^1(W_s)\varphi_n^2(W_t - W_s)\mathbf{1}_A] \\ &= \sum_n \mathbf{E}[\varphi_n^1(W_s)\mathbf{1}_A \mathbf{E}[\varphi_n^2(W_t - W_s)|\mathscr{F}_s]] \\ &= \sum_n \mathbf{E}[\varphi_n^1(W_s)\mathbf{1}_A \mathbf{E}[\varphi_n^2(W_t - W_s)]] \\ &= \mathbf{E}\left[\left(\sum_n \varphi_n^1(x)\mathbf{E}[\varphi_n^2(W_t - W_s)]\right)\Big|_{x=W_s} \mathbf{1}_A\right] = \mathbf{E}[g(W_s)\mathbf{1}_A].\end{aligned}$$

The general case uses the fact that X_t is $\sigma(X_s, W_r - W_s, r \in [s,t])$-measurable, so there exists a measurable function F such that

$$X_t = F(X_s, W_r - W_s, r \in [s,t]).$$

See Oksendal [2] Chapter 7.1 for a detailed proof. □

Theorem 7. *(Feynman-Kac Formula)*

Suppose that X solves the SDE as in Proposition 9, so that it admits the Markov property: For any measurable functions $f(\cdot)$ and $q(\cdot)$ on $\mathbb{R}^n$,

$$\mathbf{E}[e^{-\int_t^T q(X_s)ds} f(X_T)|\mathscr{F}_t] = \mathbf{E}[e^{-\int_t^T q(X_s)ds} f(X_T)|X_t] = F(t, X_t)$$

for some measurable function $F(\cdot,\cdot)$.

If $F \in C^{1,2}([0,T) \times \mathbb{R}^n)$ then $F(t,x)$ solves the following PDE on $[0,T) \times \mathbb{R}^n$:

$$\begin{cases} \partial_t F(t,x) + \mathscr{L}F(t,x) - q(x)F(t,x) = 0, \\ F(T,x) = f(x) \end{cases} \tag{2}$$

where $\mathscr{L}$ is the infinitesimal generator of X.

Proof. We first show the Markov property. The general idea is to replace the underlying Markov process X with a new Markov process which corresponds to the "killing" of the sample paths of X at the rate $q(\cdot)$. However, we give a more direct proof by introducing an auxiliary process

$$dY_t = -q(X_t)Y_t dt$$

Then, it is clear that $Y_T = Y_t e^{-\int_t^T q(X_s)ds}$. We may view (X,Y) as a pair of Markov processes, so $e^{-\int_t^T q(X_s)ds} f(X_T) = Y_T f(X_T)/Y_t$. By Proposition 9,

$$\mathbf{E}[\frac{Y_T f(X_T)}{Y_t}|\mathscr{F}_t] = \frac{\mathbf{E}[Y_T f(X_T)|(X_t, Y_t)]}{Y_t} = \mathbf{E}[e^{-\int_t^T q(X_s)ds} f(X_T)|(X_t, Y_t)].$$

However, the term $e^{-\int_t^T q(X_s)ds} f(X_T)$ does not depend on Y_t, from which we conclude.

To derive the PDE, the key observation is that

$$e^{-\int_0^t q(X_s)ds}F(t,X_t) = \mathbf{E}[e^{-\int_0^T q(X_s)ds}F(T,X_T)|\mathscr{F}_t]$$

so $e^{-\int_0^t q(X_s)ds}F(t,X_t)$, $t\in[0,T]$, is a martingale. Applying Itô's formula to $e^{-\int_0^t q(X_s)ds}F(t,X_t)$ gives us

$$\begin{aligned} de^{-\int_0^t q(X_s)ds}F(t,X_t) &= e^{-\int_0^t q(X_s)ds}dF(t,X_t) - e^{-\int_0^t q(X_s)ds}q(X_t)F(t,X_t)dt \\ &= e^{-\int_0^t q(X_s)ds}(\partial_t F(t,X_t) + \mathscr{L}F(t,X_t) - q(X_t)F(t,X_t))dt \\ &\quad + e^{-\int_0^t q(X_s)ds}(\sigma(X_t)^T\nabla_x F(t,X_t))^T dW_t \end{aligned}$$

In order to guarantee that $e^{-\int_0^t q(X_s)ds}F(t,X_t)$ is a martingale, we must have

$$\partial_t F(t,X_t) + \mathscr{L}F(t,X_t) - q(X_t)F(t,X_t) = 0,$$

which gives us the PDE for $F(\cdot,\cdot)$. $\square$

Remark 1. Note that from the martingale property of $e^{-\int_0^t q(X_s)ds}F(t,X_t)$, we also have

$$\int_0^{\cdot} e^{-\int_0^t q(X_s)ds}(\sigma(X_t)^T\nabla_x F(t,X_t))^T dW_t \tag{3}$$

is a martingale. Hence, we can obtain a reverse statement, which also refers to the Feynman-Kac formula in the literature as it provides a formula to compute conditional expectation. Suppose $F\in C^{1,2}([0,T)\times\mathbb{R}^n)$ solves (2) and the martingale condition (3) holds, then

$$F(t,X_t) = \mathbf{E}[e^{-\int_t^T q(X_s)ds}f(X_T)|\mathscr{F}_t].$$

8 Exercises

Exercise 1. (Ornstein-Uhlenbeck Process and Vasicek Model)

1. Consider the following linear SDE system:

$$dX_t = (a(t)+b(t)X_t)dt + \sigma(t)dW_t,\quad X_0 = x,$$

where $W=(W^1,\cdots,W^d)^T$ is a d-dimensional Brownian motion. Assume that $a(t),b(t):[0,\infty)\to\mathbb{R}^d$ and $\sigma(t):[0,\infty)\to\mathbb{R}^{d\times d}$ are deterministic and bounded functions. By using integration by parts, prove that

$$X_t = \Phi(t)[x+\int_0^t \Phi^{-1}(s)a(s)ds + \int_0^t \Phi^{-1}(s)\sigma(s)dW_s],$$

where $\Phi(\cdot)$ solves the following ODE system:

$$d\Phi(t) = b(t)\Phi(t)dt,\quad \Phi(0)=1.$$

$\Rightarrow$ Apply Itô to
$\Phi(t)[x+\int_0^t\Phi^{-1}(s)a\,ds+\int_0^t\Phi^{-1}(s)\sigma\,dW_s]$
to verify the SDE.

$\Rightarrow$ Apply Itô to $e^{-bt}Z_t$.

$$d\,e^{-bt}Z_t = -b e^{-bt}Z_t dt + e^{-bt}[(a+bZ_t)dt+\sigma dW_t]$$
$$= e^{-bt}[a\,dt+\sigma dW_t]$$
$$\Rightarrow e^{-bt}Z_t - x = \int_0^t e^{-bs}(a\,ds+\sigma dW_s)$$
$$\Rightarrow Z_t = e^{bt}[x+\int_0^t e^{-bs}a\,ds+\int_0^t e^{-bs}\sigma dW_s]$$

2. Now consider the one-dimensional case, and all the coefficients are constants $a(t) = a > 0$, $b(t) = -b < 0$, and $\sigma(t) = \sigma > 0$. Write down the explicit formula for X_t.
3. Under the conditions of (2), compute $E(X_t)$ and $Var(X_t)$ and their limits when $t \to \infty$.
4. Under the conditions of (2), define $Y_t = \int_0^t \overset{\text{Gaussian}}{e^{bs}} dW_s$, and $B_t = \sqrt{2b}\, Y_{\frac{\ln(t+1)}{2b}}$. Prove that $B = (B_t)_{t \geq 0}$ is a Brownian motion, and X_t has the representation:

Recall. for BW W.

$\tilde{W}_t = \sqrt{2}\, W_{\frac{t}{2}}$ is also BM.

$$X_t = xe^{-bt} + \frac{a}{b}(1 - e^{-bt}) + \frac{\sigma e^{-bt}}{\sqrt{2b}} B_{e^{2bt}-1},$$

by using the fact that any zero mean continuous Gaussian process $B = (B_t)_{t \geq 0}$ with covariance $E[B_t B_s] = s \wedge t$ is a Brownian motion. (A stochastic process is Gaussian if its any finite dimensional distribution is Gaussian distributed.)

Exercise 2. (Stochastic Exponential)

Let $M = (M_t)_{t \geq 0}$ be a continuous local martingale with $M_0 = 0$. Its stochastic exponential is defined as $\mathscr{E}(M)_t = e^{M_t - \frac{1}{2}\langle M \rangle_t}$.

1. By using the Itô's formula, prove that $\mathscr{E}(M)$ satisfies the following SDE:

$$d\mathscr{E}(M)_t = \mathscr{E}(M)_t dM_t, \quad \mathscr{E}(M)_0 = 1.$$

Therefore $\mathscr{E}(M)$ is a nonnegative continuous local martingale.
2. Prove that $\mathscr{E}(M)$ is a supermartingale.
3. Prove that $\mathscr{E}(M)_t^{-1} = \mathscr{E}(-M)_t e^{\langle M \rangle_t}$.
4. Let N be another a continuous local martingale with $N_0 = 0$. Prove that

$$\mathscr{E}(M)_t \mathscr{E}(N)_t = \mathscr{E}(M+N)_t e^{\langle M,N \rangle_t}.$$

References

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2. Oksendal, Bernt. *Stochastic differential equations: an introduction with applications*. Springer, 2013.
3. Revuz, Daniel, and Marc Yor. *Continuous martingales and Brownian motion*. Springer, 2013.