ST403X / MA4F70
UNIVERSITY OF WARWICK
FINAL YEAR EXAMINATIONS: APRIL 2017

**BROWNIAN MOTION**
Time Allowed: 2 Hours

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Full marks may be gained by correctly answering 3 complete questions. Candidates may attempt all questions. Marks will be awarded for the best 3 answers only.

Read carefully the instructions on the answer book and make sure that the particulars required are entered on each answer book. No calculators allowed.

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1. (a) State Kolmogorov's continuity theorem. [4 marks]

   (b) Show that standard, one dimensional Brownian motion is *a.s.* $\alpha$-Hölder continuous for any $\alpha < 1/2$. [8 marks]

   (c) Let $\beta(\cdot)$ be standard, one dimensional Brownian motion. Show that

$$P\left(\sup_{s,t\in[0,1]} \frac{|\beta(t)-\beta(s)|}{|t-s|^{1/2}} = \infty\right) = 1.$$

[8 marks]

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2. (a) Let $P_x$ with $x \in \mathbb{R}^d$ denote the Wiener measure corresponding to standard, $d$-dimensional Brownian motion $\beta(\cdot)$ starting from $x$. Consider an open, bounded domain $\Omega \subset \mathbb{R}^d$ with boundary $\partial\Omega$ and $D \subset \partial\Omega$. Define

$$\tau_\Omega := \inf\{t > 0 \colon \beta(t) \notin \Omega\}$$

   (i) Show that $\tau_\Omega$ is finite *a.s.* with respect to $P_x$ for any $x \in \Omega$. [5 marks]

   (ii) For $x \in \Omega$, set up a boundary value problem to compute the probability $P_x(\beta(\tau_\Omega) \in D)$. Show that $u(x) := P_x(\beta(\tau_\Omega) \in D)$ is the *unique* $C^2(\Omega) \cap C(\overline{\Omega})$ solution of this boundary value problem (you do not need to show that $P_x(\beta(\tau_\Omega) \in D)$ is a solution to this boundary value problem). [5 marks]

   (b) Let $\beta(\cdot) := (\beta_1(\cdot), \beta_2(\cdot))$ be a two-dimensional Brownian motion and $P_x$ the corresponding Wiener measure when $\beta(0) = x, x \in \mathbb{R}^2$. Let also $a > 0$ be a positive real number. Define also

$$\tau_a := \inf\{t \colon \beta_1(t) = a\}.$$

   (i) Show that $\tau_a$ is $P_0 - a.s.$ finite. [5 marks]

   (ii) Compute the distribution of $\beta_2(\tau_a)$ under $P_x$. [5 marks]

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Continued ...

3. (a) Define the notion of *stopping time.* [2 marks]

   (b) State and prove the strong Markov property for standard, one dimensional Brownian motion. [9 marks]

   (c) Let $X$ be a random variable which takes values 1 and $-1$ with probabilities $p$ and $(1-p)$, respecitvely, where $p \in (0,1)$. Let also $\beta(\cdot)$ be a standard, one dimensional Brownian motion (you have the freedom to choose the starting point of the Brownian motion). Find a stopping time $\tau$, with respect to the filtration of $\beta(\cdot)$, so that $\beta(\tau)$ has the same distribution as the random variable $X$. You should provide full proofs that your construction of stopping time $\tau$ has the desired property.

   [9 marks]

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4. (a) Let $b > 0$. Show that $P_0\big(\max_{0\le s\le t}\beta(s) \geq b\big) = P_0\big(|\beta(t)| > b\big)$, where $\beta(\cdot)$ is one dimensional Brownian motion, starting at zero and $P_0$ is its distribution.

   [4 marks]

   (b) (i) Provide a rigorous definition of the *local time* for standard, one dimensional Brownian motion. Describe informally what local time measures. [4 marks]

   (ii) Describe precisely the monotonicity properties of local time. [4 marks]

   (iii) Let $f, g$ be arbitrary continuous functions with $f(t) = g(t) + A(t)$ with

   - $f(0) = g(0) = A(0) = 0$,
   - $A(\cdot)$ is nondecreasing and increasing only when $f(t) = 0$.

   Show that the functions $f, A$ are uniquely determined by $g$ and that for every $t$ it holds that

$$A(t) = \sup_{0\le s\le t}\big(-g(s)\big).$$

   [4 marks]

   (iv) Denote by $P_0$ the distribution of one dimensional, standard Brownian motion starting at zero and denote by $(L_t(0))_{t>0}$ its local time process at zero. Compute $E_0[\,(L_t(0))^2\,]$. State clearly any result you might want to quote. [4 marks]

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**END**

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1. (a) *State Kolmogorov's continuity theorem.* [4 marks]

**Answer [Bookwork]:** Let $\mathcal{X} \subset C([0,1])$ and $\mathcal{B}$ be the Borel $\sigma$-algebra and assume we have a collection of consistent finite dimensional distributions $\{\mu_F \colon F \subset [0,1] \text{ finite}\}$. If for every $s,t \in [0,1]$ there exist constants $\alpha, \beta, C > 0$ such that

$$\int |x-y|^\beta \mu_{\{t,s\}}(dx,dy) \leq C|t-s|^{1+\alpha},$$

then there exists a unique probability measure $Q$ on $(\mathcal{X}, \mathcal{B})$ such that

- $Q\Big|_F = \mu_F$, for every $F \subset [0,1]$, finite.
- and
$$Q\Big(\sup_{0<s,t<1} \frac{|x(t)-x(s)|}{|t-s|^{\alpha/\beta}} < \infty\Big) = 1$$

(b) *Show that standard, one dimensional Brownian motion is a.s. $\alpha$-Hölder continuous for any $\alpha < 1/2$.* [8 marks]

**Answer [seen example]:** Use Kolmogorov's continuity theorem. Let $n$ arbitrary integer and compute

$$E\big[|\beta(t)-\beta(s)|^{2n}\big] = C_n E\big[|\beta(t)-\beta(s)|^2\big]^n = C_n|t-s|^n$$

by the fact that Brownian increments are gaussian and the higher moments of Gaussian variables are bounded by a power of the second moment. Apply Kolmogorov's continuity theorem with $\beta = 2n$ and $\alpha = n-1$. Then it follows that *a.s.* Brownian paths are Hölder $(n-1)/2n$. Since $n$ is arbitrary, we let $n$ tend to infinity and we see that the exponent converges to $1/2$. So Brownian motion is $\alpha$-Hölder for any $\alpha < 1/2$.

(c) *Let $\beta(\cdot)$ be standard, one dimensional Brownian motion. Show that*

$$P\left(\sup_{s,t\in[0,1]} \frac{|\beta(t)-\beta(s)|}{|t-s|^{1/2}} = \infty\right) = 1.$$

[8 marks]

**Answer [example given as an exercise but not covered in lecture]:** Consider the sequence of times $t_n$ converging monotonically to zero. Compute

$$P\left(\sup_{s,t\in[0,1]}\frac{|\beta(t)-\beta(s)|}{|t-s|^{1/2}}<\infty\right)=\lim_{M\to\infty}P\left(\sup_{s,t\in[0,1]}\frac{|\beta(t)-\beta(s)|}{|t-s|^{1/2}}<M\right)$$
$$\leq\lim_{M\to\infty}P\left(\cap_{n>N_0(M)}\left\{\frac{|\beta(t_n)-\beta(t_{n+1})|}{|t_n-t_{n+1}|^{1/2}}<\sqrt{\log n}\right\}\right)$$

where $N_0(M)$ is large enough depending on $M$. The right hand side probability is bounded by

$$\lim_{N\to\infty}P\left(\cap_{N>n>N_0(M)}\left\{\frac{|\beta(t_n)-\beta(t_{n+1})|}{|t_n-t_{n+1}|^{1/2}}<\sqrt{\log n}\right\}\right)$$
$$=\lim_{N\to\infty}\prod_{n=N_0(M)}^{N}P\left(\frac{|\beta(t_n)-\beta(t_{n+1})|}{|t_n-t_{n+1}|^{1/2}}<\sqrt{\log n}\right)$$

and using the gaussian scaling of Brownian motion this equals

$$\lim_{N\to\infty}\prod_{n=N_0(M)}^{N}P\left(|\beta(1)|<\sqrt{\log n}\right)=\lim_{N\to\infty}\prod_{n=N_0(M)}^{N}\left(1-P(|\beta(1)|\geq\sqrt{\log n})\right)$$
$$\leq C\lim_{N\to\infty}\exp\left(-\sum_{n=N_0(M)}^{N}\frac{(\sqrt{\log n})^2}{2}\right)$$
$$\leq C\lim_{N\to\infty}\exp\left(-\sum_{n=N_0(M)}^{N}\frac{1}{\sqrt{n}}\right)=0$$

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2. (a) *Let $P_x$ with $x\in\mathbb{R}^d$ denote the Wiener measure corresponding to standard, d-dimensional Brownian motion $\beta(\cdot)$ starting from $x$. Consider an open, bounded domain $\Omega\subset\mathbb{R}^d$ with boundary $\partial\Omega$ and $D\subset\partial\Omega$. Define*

$$\tau_\Omega:=\inf\{t>0\colon\beta(t)\notin\Omega\}$$

(i) *Show that $\tau_\Omega$ is finite a.s. with respect to $P_x$ for any $x\in\Omega$.* [5 marks]

**Answer [seen example]:** Let $R$ be the maximum distance from $x$ to the boundary $\partial\Omega$ and let $\tau_d$ be the exit time from the ball with center $x$ and radius $R$ and compute

$$P_x(\tau_\Omega>n)=E_x[P_x(\tau_\Omega>n\,|\,\mathcal{F}_1)\,;\tau_\Omega>1]$$

and by the strong Markov property this equals

$$E_x[P_{\beta(1)}(\tau_\Omega>n-1)\,;\tau_\Omega>1]\leq P_x(\tau_\Omega>1)\sup_{z\in\Omega}P_z(\tau_\Omega>n-1)$$

But

$$P_x(\tau_\Omega>1)\leq P_x(\tau_R>1)$$

and by translation invariance this is bounded by $\rho:=P_0(\tau_{B(0,R)}>1)<1$, where $B(0,R)$ is the ball of the radius $R$ and center 0. So we get that

$$\sup_{z\in\Omega}P_z(\tau_\Omega>n)\leq\rho\sup_{z\in\Omega}P_z(\tau_\Omega>n-1)$$

and iterating

$$\sup_{z\in\Omega} P_z(\tau_\Omega > n) \leq \rho^n,$$

which proves the result by taking the limit.

(ii) *For $x \in \Omega$. Set up a boundary value problem to compute the probability $P_x(\beta(\tau_\Omega) \in D)$. Show that $u(x) := P_x(\beta(\tau_\Omega) \in D)$ is the unique solution of this boundary value problem (you do not need to show that $P_x(\beta(\tau_\Omega) \in D)$ is a solution to this boundary value problem).* [5 marks]

**Answer [similar examples seen]:** $u(x)$ solves the boundary value problem

$$\begin{cases} \partial_t u = \frac{1}{2}\Delta u, & \text{in } \Omega \\ u\big|_{\partial\Omega} = 1_D \end{cases}$$

To prove uniqueness, we use that

$$u(\beta(t)) - \frac{1}{2}\int_0^t \Delta u(\beta(s))\, ds,$$

is a Martingale. Since $\Delta u = 0$, it follows that $u(\beta(t))$ is a Martingale and so

$$u(x) = E_x[u(\beta(\tau_\Omega \wedge t)].$$

Using the previous question and passing to the limit using dominated convergence since $u \in C^2(\Omega) \cap C(\bar{\Omega})$, we have that

$$u(x) = E_x[u(\beta(\tau_\Omega)] = E_x[1_D(\beta(\tau_\Omega))].$$

(b) *Let $\beta(\cdot) := (\beta_1(\cdot), \beta_2(\cdot))$ be a two-dimensional Brownian motion and $P_x$ the corresponding Wiener measure when $\beta(0) = x, x \in \mathbb{R}^2$. Let also $a > 0$ be a positive real number. Define also*

$$\tau_a := \inf\{t \colon \beta_1(t) = a\}.$$

**[unseen example]:**

(i) *Show that $\tau_a$ is $P_0 -$ a.s. finite.* [5 marks]

**Answer:** The student needs to notice that it suffices to show that one dimensional Brownian motion (the first coordinate of the $d$ Brownian motion) hits a level $a$, a.s. But one dimensional Brownian motion Brownian motion is recurrent.

(ii) *Compute the distribution of $\beta_2(\tau_a)$ under $P_x$.* [5 marks]

**Answer:** By translation invariance we have

$$\begin{aligned} P_{(x_1,x_2)}(\beta_2(\tau_a) = y) &= P_0(\beta_2(\tau_{a-x_1}) = y - x_2) \\ &= \int_0^\infty P_0(\beta_2(\tau_{a-x_1}) = y - x_2\, ;\, \tau_{a-x_1} = t)dt \\ &= \int_0^\infty P_0(\beta_2(t) = y - x_2\, ;\, \tau_{a-x_1} = t)dt \\ &= \int_0^\infty P_0(\beta_2(t) = y - x_2)P(\,\tau_{a-x_1} = t)dt, \end{aligned}$$

where in the last step we used the independence of $\beta_1, \beta_2$. By the reflection principle we know that

$$P(\tau_{a-x_1} = t) = \frac{|a - x_1|}{\sqrt{2\pi t^3}} e^{-\frac{(x_1-a)^2}{2t}}$$

and inserting this above we get that it equals

$$\int_0^\infty \frac{1}{\sqrt{2\pi t}} e^{-\frac{(x_2-y)^2}{2t}} \frac{|a - x_1|}{\sqrt{2\pi t^3}} e^{-\frac{(x_1-a)^2}{2t}} dt = \frac{1}{\pi} \frac{|x_1 - a|}{(x_1 - a)^2 + (x_2 - y)^2}$$

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Continued ...

3. (a) *Define the notion of stopping time.* [2 marks]

**Answer [bookwork]:** Given a filtration $\mathcal{F}_t$, a stopping time $\tau$ is a random variable, such that any event $\{\tau \le t\}$ is measurable with respect to $\mathcal{F}_t$.

(b) *State and prove the strong Markov property for standard, one dimensional Brownian motion.* [9 marks]

**Answer [bookwork]:** The strong Markov states that for any stopping time $\tau$ the process $\{\beta(t+\tau) - \beta(\tau) : t > 0\}$ is independent of $\mathcal{F}^+_\tau$ (the germ $\sigma$-algebra) and is distributed as a standard one dimensional Brownian motion starting from 0.
To prove this we start by defining

$$\tau_n = \sum_{k \ge 1} \frac{k}{2^n} 1_{\tau \in \left(\frac{k-1}{2^n}, \frac{k}{2^n}\right]}.$$

We now check that $\beta(\cdot + \tau_n) - \beta(\tau_n)$ is independent of $\mathcal{F}^+_{\tau_n}$. This is done as follows. Let $E \in \mathcal{F}^+_{\tau_n}$. Then

$$\begin{aligned} P(\beta(\cdot + \tau_n) - \beta(\tau_n) \in A\,;\, E) &= \sum_{k=0}^{\infty} P(\beta(\cdot + \tau_n) - \beta(\tau_n) \in A\,;\, E\,;\, \tau_n = k/2^n) \\ &= \sum_{k=0}^{\infty} P(\beta(\cdot + k2^{-n}) - \beta(k2^{-n}) \in A\,;\, E\,;\, \tau_n = k/2^n) \end{aligned}$$

and by the Markov property this equals

$$\sum_{k=0}^{\infty} P(\beta(\cdot) \in A)\, P(E\,;\, \tau_n = k/2^n) = P(\beta(\cdot) \in A)\, P(E),$$

which implies that $\{\beta(\cdot + \tau_n) - \beta(\tau_n)\}$ is independent of $\mathcal{F}^+_{\tau_n}$, which contains $\mathcal{F}^+_\tau$. The proof is completed by taking the limit $n \to \infty$ and using the continuity of Brownian motion.

(c) *Let $X$ be a random variable which takes values 1 and $-1$ with probabilities $p$ and $(1-p)$, respecitvely, where $p \in (0,1)$. Let also $\beta(\cdot)$ be a standard, one dimensional Brownian motion. Find a stopping time $\tau$, with respect to the filtration of $\beta(\cdot)$, so that $\beta(\tau)$ has the same distribution as the random variable $X$. You should provide full proofs that your construction of stopping time $\tau$ has the desired property.* [9 marks]

**Answer [similar example seen - this is a particular case of the Skorokhod embedding theorem]:** Consider $\beta(\cdot)$ starting from $x \in (-1,1)$, which will be chosen appropriately, and $\tau$ to be the exit time from the interval $(-1,1)$. Then

$$P_x(\beta(\tau)=1) = \frac{x+1}{2} \quad \text{and} \quad P_x(\beta(\tau)=-1) = \frac{1-x}{2}$$

and then choose $x$ so that $(x+1)/2 = p$.

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4. (a) *Let $b > 0$. Show that $P_0\big(\max_{0\le s\le t}\beta(s) \ge b\big) = P_0\big(|\beta(t)| > b\big)$, where $\beta(\cdot)$ is one dimensional Brownian motion, starting at zero and $P_0$ is its distribution.* [4 marks]

**Answer [Bookwork]:** This is the reflection principle and the students will have to reproduce the standard proof. This is as follows: Let $\tau_a$ be the hitting time of level $a$. Then

$$\begin{aligned} P_0(\beta(t) > a) &= P_0(\beta(t) > a,\, \tau_a \le t) \\ &= E_0\big[P_0(\beta(t) > a \,|\, \mathcal{F}_{\tau_a})\,;\, \tau_a \le t\big] \\ &= E_0\big[P_a(\beta(t-\tau_a) > a)\,;\, \tau_a \le t\big] \\ &= \frac{1}{2}P_0(\tau \le t) \end{aligned}$$

where in the third equality we used the strong Markov property. The result follows by taking the 2 to the left hand side.

(b) (i) *Provide a rigorous definition of the local time for standard, one dimensional Brownian motion. Describe informally what local time measures.* [4 marks]

**Answer [bookwork]:** The local time $L_t(x)$ at $x$ measures how much time the one dimensional brownian motion has spent at $x$ by time $t$.
Assume $x = 0$ and let $a < 0 < b$. Denote by $D(a,b;t)$ the number of *downcrossings* by time t. Students will have to define this notion properly: First define the stopping times:

$$\begin{aligned} \sigma_1 &= \inf\{t > 0 \colon \beta(t) = b\} \\ \tau_1 &= \inf\{t > \sigma_1 \colon \beta(t) = a\} \\ \sigma_2 &= \inf\{t > \tau_1 \colon \beta(t) = b\} \\ \tau_2 &= \inf\{t > \sigma_2 \colon \beta(t) = a\} \end{aligned}$$

and so on. Then $D(a,b;t) : \max\{j \colon \tau_j \le t\}$.
The local time at 0 by time $t$ can be defined as

$$L_t(0) = \lim_{a\uparrow 0 \downarrow b} 2(b-a)D(a,b;t).$$

(ii) *Describe precisely the monotonicity properties of local time.* [4 marks]

**Answer [bookwork]:** The local time at zero is a non decreasing processes, whose points of increase coincide with the zero set of Brownian motion.

(iii) *Let $f, g$ be arbitrary continuous functions with $f(t) = g(t) + A(t)$ with*

- $f(0) = g(0) = A(0) = 0$,
- *$A(\cdot)$ is nondecreasing and increasing only when $f(t) = 0$.*

*Show that the functions $f, A$ are uniquely determined by $g$ and that for every $t$ it holds that*

$$A(t) = \sup_{0 \le s \le t} \big( - g(s) \big).$$

[4 marks]

**Answer [seen example]:** Students will first have to check that

$$A(t) = \sup_{0 \le s \le t} \big( - g(s) \big).$$

has the requested properties. This is done as follows: Because of the sup the process $A(t)$ is non decreasing. To see that it increases when $f$ is zero, we write

$$f(t) = g(t) + A(t) = \sup_{s \le t} (g(t) - g(s)).$$

So $A$ increases only when the process $-g$ reaches a new maximum, which means that the process $g$ reaches a new minimum, but then $\sup_{s \le t}(-g(s)) = -g(t)$ and so $f(t) = 0$. Uniqueness is more tricky. Suppose there given $g$ are two pairs of solution $(f_1, A_1)$ and $(f_2, A_2)$ with the requested properties. Consider

$$\begin{aligned}
\big(A_1(t) - A_2(t)\big)^2 &= 2 \int_0^t \big(f_1(s) - f_2(s)\big) \big(dA_1(s) - dA_2(s)\big) \\
&= -2 \int_0^t f_1(s)\, dA_2(s) - 2 \int_0^t f_2(s)\, dA_1(s) \\
&\le 0,
\end{aligned}$$

where the second equality follows from the fact that $A_i$ increases only at the zero set of $f_i$ and the inequality from the fact that $A_i$ is non decreasing.

(iv) *Denote by $P_0$ the distribution of one dimensional, standard Brownian motion starting at zero and denote by $(L_t(0))_{t>0}$ its local time process at zero. Compute $E_0[\,(L_t(0))^2\,]$. State clearly any result you might want to quote.* [4 marks]

**Answer [unseen example]:** We will use the fact that $L_t(0)$ has the same distribution as $\max_{0 \le s \le t} \beta(s)$ (this also follows from the previous question but a proof is not requested). So we have,

$$E_0[\,(L_t(0))^2\,] = 2 \int_0^\infty x P_0(L_t(0) > x)\, dx = 2 \int_0^\infty x P_0(\max_{0 \le s \le t} \beta(s) > x)\, dx,$$

and by the first question this equals

$$2 \int_0^\infty x P_0(|\beta(s)| > x)\, dx = E_0((\beta(t))^2) = t.$$

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**END**