06.02.2024. MA999. HW1 (Yulia Timofeeva)

1. Model of fishery: $\frac{dx}{dt} = x(1-x) - h\frac{x}{a+x}$; $h, a > 0$.

a) Show that the system can have 1, 2 or 3 fixed points, depending on the values of a, h. Classify their stability.

$f(x) = x - x^2 - \frac{hx}{a+x}$

Fixed point: $f(x) = 0 \iff x(1-x) - \frac{hx}{a+x} = 0$

$x\left(1 - x - \frac{h}{a+x}\right) = 0$

$x\left(\frac{(1-x)(a+x) - h}{a+x}\right) = 0 \rightarrow \left[\begin{array}{l} x_0 = 0 \text{ – the first root} \\ a + x - ax - x^2 - h = 0. \end{array}\right.$

$x^2 + x(a-1) + h - a = 0.$

$\Rightarrow D = (a-1)^2 - 4(h-a) = a^2 - 2a + 1 - 4h + 4a = (a+1)^2 - 4h$

(3) • if $D > 0$, i.e. $|a+1| > 2\sqrt{h}$, i.e. $\left[\begin{array}{l} a+1 > 2\sqrt{h} \text{ – yes, can be} \\ a+1 < -2\sqrt{h} \text{ – never can be, since } a > 0 \end{array}\right.$

we have 3 roots: $x_{1,2} = \frac{1 - a \pm \sqrt{(a+1)^2 - 4h}}{2}$ and $x_0 = 0$.

(2) • if $D = 0$, i.e. $|a+1| = 2\sqrt{h} \iff a = 2\sqrt{h} - 1$,

we have 2 roots: $x_{1,2} = \frac{1-a}{2}$ and $x_0 = 0$

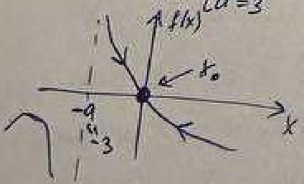
(1) • if $D < 0$, i.e. $|a+1| < 2\sqrt{h}$,

we have 1 root: $x_0 = 0$.

Let's analyse stability of x_0, x_1, x_2:

(1) $h > \frac{(a+1)^2}{4} \Rightarrow$ 1 root $(x_0 = 0)$ – stable

for example, $\left\{\begin{array}{l} h = 5 \\ a = 3 \end{array}\right.$

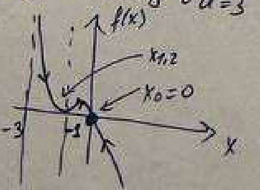


in this case x_0 – stable (from visual stability analysis, based on the sign of $f(x)$ for every x)

(2) $h = \frac{(a+1)^2}{4} \Rightarrow$ 2 roots

$\left[\begin{array}{l} x_0 = 0 \text{ – stable} \\ x_{1,2} = \frac{1-a}{2} \text{ – half-stable} \end{array}\right.$

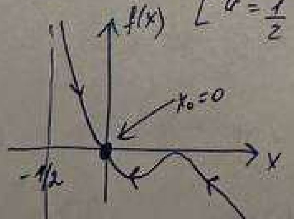
2.1) $a > 1$, e.g. $\left\{\begin{array}{l} h = 4 \\ a = 3 \end{array}\right.$



$\Rightarrow x_0 = 0$ is stable

$x_{1,2}$ – is half stable

2.2) $a < 1$, e.g. $\left\{\begin{array}{l} h = \frac{9}{16} \\ a = \frac{1}{2} \end{array}\right.$



$\Rightarrow x_0 = 0$ is stable, $x_{1,2}$ is half-stable

(3) $h < \frac{(a+1)^2}{4} \Rightarrow$ 3 roots

$\left\{\begin{array}{ll} x_0 = 0 & \longleftarrow \text{stable when } a < h, \text{ unstable when } a > h \\ x_1 = \frac{1 - a - \sqrt{(a+1)^2 - 4h}}{2} & \leftarrow \text{unstable when } a < h, \text{ stable when } a > h \\ x_2 = \frac{1 - a + \sqrt{(a+1)^2 - 4h}}{2} & \leftarrow \text{stable } \forall a, h > 0. \end{array}\right.$

How we analyse stability?

Let's understand how these x_0, x_1, x_2 are located with respect to zero.

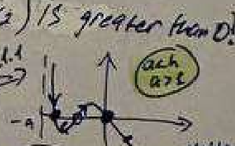
3.1) When x_1 (and hence x_2) is greater than 0!

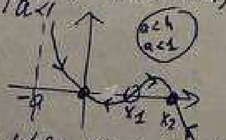
This happens when

$\frac{1 - a - \sqrt{(a+1)^2 - 4h}}{2} \overset{?}{>} 0$

3.1.2 $\left\{\begin{array}{l} (a+1)^2 - 4h < (1-a)^2 \\ a < 1 \end{array}\right. \iff \left\{\begin{array}{l} a < h \\ a < 1 \end{array}\right.$

3.1.1





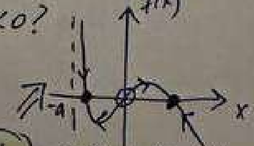
$\Rightarrow x_0$ and x_2 are stable

x_1 is unstable

3.2) When $x_2 > 0$, and $x_1 < 0$?

This happens when

$\frac{1 - a + \sqrt{(a+1)^2 - 4h}}{2} \overset{?}{>} 0$

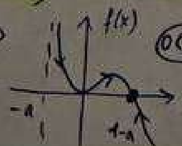


$(a+1)^2 - 4h > (1-a)^2 \iff a > h$ $\Rightarrow x_1$ and x_2 – stable

x_0 is unstable

3.3) When $a = h$, then $\left\{\begin{array}{l} x_{1,2} = 0 \\ x_3 = 1 - a \end{array}\right.$

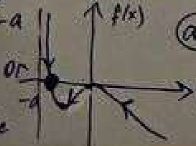
$\Rightarrow$



x_3 – stable

x_0 – half-stable

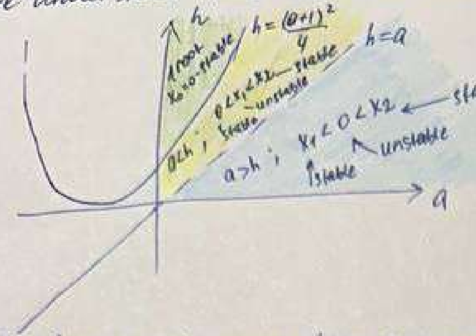
or



x_3 – stable

x_0 – half stable

So we understood that behaviour changes when we reach curves $h=a$ and $h=\frac{(a+1)^2}{4}$



(*)

b) Analyse the dynamics near $x=0$ and show that a bifurcation occurs when $h=a$. What type of bifurcation is it?

To analyse stability at $x=0$, let's look at $f'(0)$.

$$f(x) = x - x^2 - \frac{hx}{a+x}$$

$$f'(x) = 1 - 2x - \frac{h\cdot(1\cdot(a+x) - 1\cdot x)}{(a+x)^2} = 1 - 2x - \frac{ha}{(a+x)^2}$$

$$\Rightarrow f'(0) = 1 - \frac{ha}{a^2} = 1 - \frac{h}{a} = \frac{a-h}{a}$$
$\to >0$, when $a>h \Rightarrow$ unstable
$\to <0$, when $a<h \Rightarrow$ stable

When $h=a$, linear analysis is insufficient and we need larger order derivative:

$$f''(x) = -2 + \frac{2ha}{(a+x)^3} \underset{\substack{h=a\\x=0}}{=} -2 + \frac{2a^2}{a^3} = -2\left(1 - \frac{1}{a}\right)$$ - may be <0 or >0, depending on a.

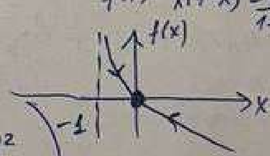
$$f'''(x) = -\frac{6\cdot ha}{(a+x)^4} \underset{\substack{h=a\\x=0}}{=} \frac{-6a^2}{a^4} = \frac{-6}{a^2} < 0,\ \forall a>0.$$

When $h=a>1$, $f''(0)<0 \Rightarrow x_0$ - stable

When $h=a<1$, $f''(0)>0 \Rightarrow x_0$ - unstable

When $h=a=1$, then

$$f(x) = x(1-x) - \frac{x}{1+x} = \frac{x(1-x^2-1)}{1+x} = \frac{-x^3}{1+x}$$



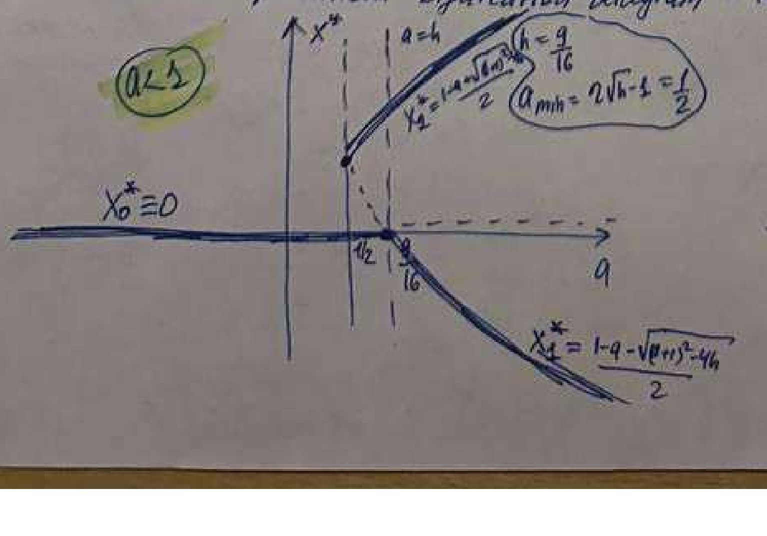
$\Rightarrow x_0$ is stable

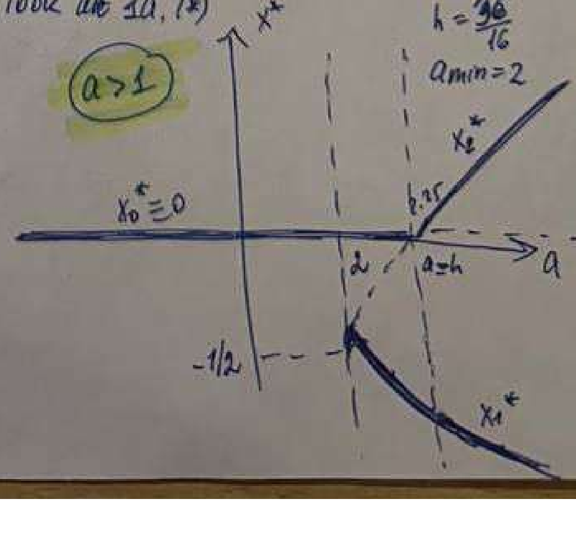
This bifurcation is transcritical - it will be seen at the graph from 1d, that there meet 4 curves.

c) Show that another bifurcation occurs when $h=\frac{(a+1)^2}{4}$. Classify this bifurcation.

We showed that at $h=\frac{(a+1)^2}{4}$ we start have 3 roots instead of 1 root or vice versa. This is a saddle node bifurcation, because if we are by one side of the point, we have one point, and if we are by the other side of the curve, we have 3.

d) Sketch the two-parameter bifurcation diagram - look at 1a. (*)





② Consider $\ddot{\theta} + 2a\dot{\theta} + \omega^2 \sin\theta = 0$. ; $a > 0$, ω^2

a) Rewrite this model as a 2-D system introducing x_1 and x_2.

b) Find fixed points and classify their stability.

c) Sketch a phase portrait with nullclines for $\omega > a$.

d) Numerically solve and provide plot.

Introduce $\begin{cases} \dot{x} = y \text{ (where } y \text{ is } \dot\theta) \\ \dot{y} = -2ay - \omega^2 \sin x \end{cases}$

fixed points: $\begin{cases} y=0 \\ -2ay - \omega^2 \sin x = 0 \end{cases} \Rightarrow \begin{cases} y=0 \\ \sin x = 0 \end{cases} \Rightarrow \begin{cases} y=0 \\ x = \pi k; k \in \mathbb{N}. \end{cases}$

Analyse stability. Use Jacobian: $J = \begin{pmatrix} \frac{\partial f}{\partial x} & \frac{\partial f}{\partial y} \\ \frac{\partial g}{\partial x} & \frac{\partial g}{\partial y} \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -\omega^2 \cos x & -2a \end{pmatrix}\Bigg|_{x^* = \pi k} = \begin{pmatrix} 0 & 1 \\ -\omega^2 & -2a \end{pmatrix}$

Find eigenvalues:

$(A - \lambda E) = \begin{pmatrix} -\lambda & 1 \\ -\omega^2 & -2a-\lambda \end{pmatrix} \Rightarrow \lambda(2a+\lambda) + \omega^2 = 0.$

$\lambda^2 + 2a\lambda + \omega^2 = 0$

$\Rightarrow \quad D = 4a^2 - 4\omega^2 = 4(a^2 - \omega^2)$

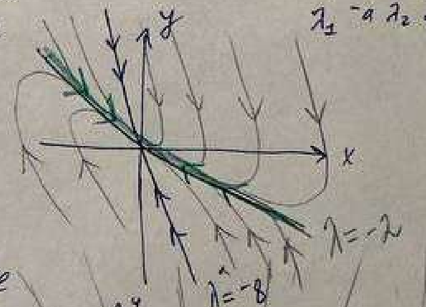
(1) • если $a > \omega \Rightarrow \lambda_{1,2} = \dfrac{-2a \pm \sqrt{a^2 - \omega^2}}{2} = -a \pm \sqrt{a^2 - \omega^2} < 0$ $\lambda_1 \quad -a \quad \lambda_2 \quad 0$

$\Rightarrow$ this is a stable node

e.g. $a = 5$, $\omega = 4$ $\lambda_{1,2} = \dfrac{-10 \pm 6}{2} = \begin{cases} -8 \\ -2 \end{cases}$

$e_1 = (-1; 8)$

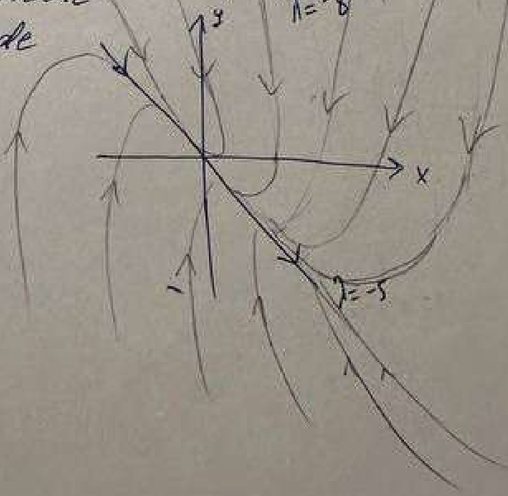
$e_2 = (-1; 2)$



(2) if $a = \omega$, then $\lambda_1 = \lambda_2 = -a$ – one eigenvalue

$\Rightarrow$ this is a stable degenerate node

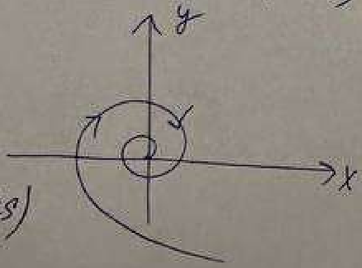
e.g. $a = 5$, $\omega = 5$ $\lambda = -5$; $e_1 = (1; -5)$



(3) if $a < \omega \Rightarrow \lambda_{1,2} = -a \pm i\sqrt{\omega^2 - a^2}$ – complex

$\Rightarrow$ this is a stable spiral (= focus)

e.g. $a = 4$, $\omega = 10$



(or $a = 4$, $\omega = 5$ also works)

! nullclines are $\begin{cases} y = 0 \\ -\omega^2 x - 2ya = 0 \Leftrightarrow y = -\dfrac{\omega^2}{2a} x \end{cases}$