Applications of Stochastic Calculus in Finance Chapter 6: Credit risk and credit derivatives

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1 Credit Risk

So far, the zero coupon bond price $P(t,T)$ is assumed to have the property

$$P(T,T) = 1.$$

That is, the payoff at maturity T is certain, and there is no risk of default by the issuer. This is the case for *treasury bonds*. On the other hand, *corporate bonds*, may be subject to a substantial risk of default.

Credit risk is the potential default risk that an obligor fails to honor its obligation, resulting in the losses of its counterparty. For the question of assessment of credit risk, we have not only to consider default probability, but also the probability of transitions between credit ratings. There are mainly three rating agencies: *Moody's Investors Service (Moody's)*, *Standard & Poor's (S&P)* and *Fitch Ratings*. They assign a credit rating that reflect the creditworthiness of an obligor, and adjust its ratings regularly.

In this course, we mainly assess credit risk in terms of default probability. Transitions of credit ratings can be studied by using Markov chain theory (see Bielecki and Rutkowski [1] Chapter 12). We mainly consider two types of credit derivatives in this course: *CDS* and *basket CDS*. Notorious *Collateralized Debt Obligation (CDO)*, which played a major role in the recent financial crisis, will be discussed in exercise.

2 Credit default swap (CDS)

Credit default swap (CDS) is a swap contract that the seller will compensate the buyer in a credit event, such as bankruptcy or failure to pay a debt. The buyer makes

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a series of fixed payments (CDS spread, CDS premium) to the seller, and in exchange, receives a payoff if the credit event occurs.

A CDS may refer to a specified loan or bond obligation of a reference entity, which may have nothing to do with the buyer. So the buyer does not need to own the underlying debt, and does not even have to suffer a loss from the default. In such a case, the CDS is called the naked CDS. *It is like buying fire insurance on your neighbor's house*.

The CDS buyer pays its seller a fixed rate $\delta\kappa$ with $\delta = T_i - T_{i-1}$ at a sequence of settlements dates $T_1 < T_2 < \cdots < T_n = T$ until the default time τ of the reference entity. Usually, the premium is paid every three months, so $\delta = 0.25$. In return, if the default time $\tau \in (0, T]$, then the seller pays the buyer a deterministic cash amount LGD (*loss given default*).

Conditional on $\{T_{i-1} < \tau \leq T_i\}$, the discounted payoff of the buyer's cash flow at initial time 0 is

$$\underbrace{e^{-r\tau}LGD}_{default\ leg} - \underbrace{\left(\sum_{j=1}^{i-1} e^{-rT_j}\delta\kappa + e^{-r\tau}(\tau - T_{i-1})\kappa\right)}_{premium\ leg}$$

with the convention $\sum_{j=1}^{0} = 0$, and $r \geq 0$ being the constant interest rate.

See Fig 1 for the cash flow of a CDS buyer conditional on $\{T_{i-1} < \tau \leq T_i\}$.

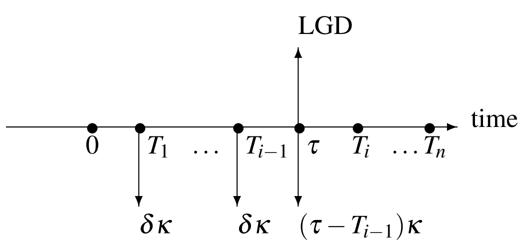

Fig 1: cash flow of a CDS buyer conditional on $\{T_{i-1} < \tau \leq T_i\}$.

Summing up from $i = 1$ to n, we obtain the discounted payoff (conditional on we knowing the default time τ):

$$\begin{aligned}
\Pi_b(0) = &\sum_{i=1}^{n} \mathbf{1}_{\{T_{i-1}<\tau\leq T_i\}} \left(e^{-r\tau}LGD - \sum_{j=1}^{i-1} e^{-rT_j}\delta\kappa - e^{-r\tau}(\tau - T_{i-1})\kappa\right) \\
&- \mathbf{1}_{\{\tau>T_n\}} \sum_{j=1}^{n} e^{-rT_j}\delta\kappa \\
= &\mathbf{1}_{\{\tau\leq T\}} e^{-r\tau}LGD \\
&- \sum_{i=1}^{n} \left(\mathbf{1}_{\{\tau>T_i\}} e^{-rT_i}\delta\kappa + \mathbf{1}_{\{T_{i-1}<\tau\leq T_i\}} e^{-r\tau}(\tau - T_{i-1})\kappa\right)
\end{aligned}$$

The CDS spread/CDS premium at initial time 0, denoted as $R_{CDS}(0)$, is the fixed rate κ such that $\mathbf{E}^{\mathbf{Q}}[\Pi_b(0)] = 0$ under the risk neutral probability measure $\mathbf{Q}$, which is

$$R_{CDS}(0) = \frac{\mathbf{E}^{\mathbf{Q}}[\mathbf{1}_{\{\tau\le T\}}e^{-r\tau}]LGD}{\sum_{i=1}^{n}\left(\mathbf{Q}\{\tau > T_i\}e^{-rT_i}\delta + \mathbf{E}^{\mathbf{Q}}[\mathbf{1}_{\{T_{i-1}<\tau\le T_i\}}e^{-r\tau}(\tau - T_{i-1})]\right)}$$

2.1 CDS with continuum payments and counterparty default

In an ideal situation where the CDS premium is payed continuously until the default time τ, the discounted payoff of the CDS buyer at time 0 is

$$\Pi_b(0) = \mathbf{1}_{\{\tau\le T\}}e^{-r\tau}LGD - \int_0^{\tau\wedge T} e^{-rs}\kappa ds.$$

Thus, the CDS spread at initial time 0 is

$$R_{CDS}(0) = \frac{\mathbf{E}^{\mathbf{Q}}[\mathbf{1}_{\{\tau\le T\}}e^{-r\tau}]LGD}{\mathbf{E}^{\mathbf{Q}}[\int_0^{\tau\wedge T} e^{-rs}ds]} = r\frac{\mathbf{E}^{\mathbf{Q}}[\mathbf{1}_{\{\tau\le T\}}e^{-r\tau}]LGD}{1-\mathbf{E}^{\mathbf{Q}}[e^{-r(\tau\wedge T)}]}.$$

In practice, the CDS seller may also default, so we also need to take account of the default risk of the CDS seller as the counterparty (so called *counterparty risk*). Let $\bar{\tau}$ be the default time of the CDS seller, then the discounted payoff of the CDS buyer at time 0 is modified as

$$\Pi_b(0) = \mathbf{1}_{\{\tau\le T\}}\mathbf{1}_{\{\bar{\tau}>\tau\}}e^{-r\tau}LGD - \int_0^{\tau\wedge\bar{\tau}\wedge T} e^{-rs}\kappa ds.$$

Thus, the CDS spread at initial time 0 is

$$R_{CDS}(0) = \frac{\mathbf{E}^{\mathbf{Q}}[\mathbf{1}_{\{\tau\le T\}}\mathbf{1}_{\{\bar{\tau}>\tau\}}e^{-r\tau}]LGD}{\mathbf{E}^{\mathbf{Q}}[\int_0^{\tau\wedge\bar{\tau}\wedge T} e^{-rs}ds]} = r\frac{\mathbf{E}^{\mathbf{Q}}[\mathbf{1}_{\{\tau\le T\}}\mathbf{1}_{\{\bar{\tau}>\tau\}}e^{-r\tau}]LGD}{1-\mathbf{E}^{\mathbf{Q}}[e^{-r(\tau\wedge\bar{\tau}\wedge T)}]}.$$

3 Basket CDS

Basket CDS: A CDS that refers to multiple reference entities, which may correlate with each other. The contract specifies the number of defaults after which the payoff is generated, based on which basket CDS is classified as first-to-default CDS (F2D), second-to-default CDS (S2D) or more generally mth-to-default CDS (M2D).

Suppose there are M reference entities in a basket CDS, and we consider the discounted payoff of the mth-to-default CDS. Let $\tau(m)$ represent the mth default time and $N(t)$ be the number of defaults by time t. Then

$$\{N(t)=m\}=\{\tau(m)\le t<\tau(m+1)\},\quad \{\tau(m)>t\}=\cup_{j=0}^{m-1}\{N(t)=j\},$$

and conditional on $\tau(m)$ being known, the discounted payoff of the mth-to-default CDS at time 0 is

$$\begin{aligned}&\mathbf{1}_{\{\tau(m)\le T_n\}}P(0,\tau(m))LGD(m)\\ &-\sum_{i=1}^{n}\left(\mathbf{1}_{\{\tau(m)>T_i\}}P(0,T_i)\delta\kappa+\mathbf{1}_{\{T_{i-1}<\tau(m)\le T_i\}}P(0,\tau(m))(\tau(m)-T_{i-1})\kappa\right),\end{aligned}$$

where $LGD(m)$ is the loss given default for the mth default. Note that if the spot rate is a constant, then $P(0,T)=e^{-rT}$.

Similar to the determination of the spread/premium of CDS, we define the basket CDS spread at time $t=0$ as the fixed rate κ such that

$$R_{BCDS}(0)=\frac{\mathbf{E}^{\mathbf{Q}}[\mathbf{1}_{\{\tau(m)\le T_n\}}P(0,\tau(m))]LGD(m)}{\sum_{i=1}^{n}\left(\mathbf{Q}\{\tau(m)>T_i\}P(0,T_i)\delta+\mathbf{E}^{\mathbf{Q}}[\mathbf{1}_{\{T_{i-1}<\tau(m)\le T_i\}}P(0,\tau(m))(\tau(m)-T_{i-1})]\right)}$$

The rest of this credit risk section is to develop stochastic models for the default times τ, $\bar\tau$ and $\tau(m)$, and calculate their distributions. This will in turn provide us with CDS pricing formulas.

Exercise 1. (Collateralized Debt Obligation (CDO))

A CDO is a structured finance product that is backed by a pool of loans and other assets and sold to institutional investors. A CDO is a particular type of derivative because, as its name implies, its value is derived from another underlying asset. If the underlying is bond/loan, it is called *cash CDO*. If the underlying is CDS, it is called *synthetic CDO*.

The CDO is sliced into sections known as *tranches*. If some underlying defaults and the cash collected by the CDO is insufficient to pay all of its investors, those in the lowest, most junior tranches suffer losses first. The last to lose payment from default are the safest, most senior tranches. Consequently, coupon payments vary by tranche with the safest/most senior tranches receiving the lowest rates and the lowest tranches receiving the highest rates to compensate for higher default risk.

Consider a synthetic CDO written on M CDSs with maturity T and with 3 tranches (*junior, mezzanine and senior*). Let τ_m be the default time of the mth CDS (note that it is NOT $\tau(m)$). The total loss is

$$L(t)=\sum_{m=1}^{M}L_m(t)=\sum_{m=1}^{M}\mathbf{1}_{\{\tau_m\le t\}}LGD_m$$

where LGD_m is the loss given default for the mth CDS. Then, the loss of the jth tranche (for $j=1,2,3$) at time t is

$$Z_j(t)=\begin{cases}0, & L(t)\le Z_{Lj};\\ L(t)-Z_{Lj}, & Z_{Lj}<L(t)\le Z_{Uj};\\ Z_{Uj}-Z_{Lj}, & L(t)>Z_{Uj},\end{cases}$$

where Z_{Lj} and Z_{Uj} are the *attachment* and *detachment* points of the jth tranche. It is clear that we must have $Z_{Lj} = Z_{U(j-1)}$.

The discounted payoff of the premium leg that the jth tranche CDO holder will receive is

$$\sum_{i=1}^{n} \delta \kappa_j [Z_{Uj} - Z_{Lj} - Z_j(t_i)] e^{-rt_i}.$$

The corresponding discounted payoff of the default leg is

$$\sum_{i=1}^{n} [Z_j(t_i) - Z_j(t_{i-1})] e^{-rt_i}.$$

Compute the CDO premium at initial time 0 for the jth tranche.

References

1. Bielecki, Tomasz, and Marek Rutkowski. *Credit risk: modeling, valuation and hedging*. Springer, 2013.