Linear systems in $\mathbb{R}^n$

$\dot{x} = Ax \qquad x \in \mathbb{R}^n \qquad A$ - $n \times n$ matrix

Linear combination of contributions

① Real eigenvalue λ_1

$$c_1 e^{\lambda_1 t}$$

② Real eigenvalue λ of multiplicity r

$$c_1 e^{\lambda t} + c_2 t e^{\lambda t} + \ldots + c_r t^{r-1} e^{\lambda t}$$

③ Complex eigenvalue $\lambda = \rho \pm i\omega$

$$e^{\rho t}(B\cos\omega t + C\sin\omega t)$$

④ Complex eigenvalue $\lambda = \rho \pm i\omega$ of multiplicity r

$$e^{\rho t}\big(B_1\cos\omega t + C_1\sin\omega t +$$
$$+ \quad B_2 t\cos\omega t + C_2 t\sin\omega t + \ldots$$
$$+ \quad B_r t^{r-1}\cos\omega t + C_r t^{r-1}\sin\omega t\big)$$

$\dot{x} = Ax \qquad x \in \mathbb{R}^n \qquad A$ - $n \times n$ matrix

$x(0) = x_0$

<u>Def.</u> Let A be a linear operator defined on $\mathbb{R}^n$. The exponential of A is the linear operator defined on $\mathbb{R}^n$ by $\boxed{e^A = \sum_{k=0}^{\infty} \frac{A^k}{k!}}$

Solution of $\dot{x} = Ax$, $x(0) = x_0$ $\qquad \boxed{x(t) = e^{tA} x_0}$

$$e^{tA} = \sum_{k=0}^{\infty} \frac{t^k}{k!} A^k$$

Can solve inhomogeneous eqns.

$\dot{x} = Ax + g(t) \qquad x(0) = x_0$

$$x(t) = e^{tA} x_0 + e^{tA} \int_0^t e^{-t'A} g(t')\, dt'$$

Normal Forms

$\dot{x} = Ax$ $\qquad$ solution $x = e^{tA}x_0$

$x(0) = x_0$

Consider linear change : $x = Py$

P - $n \times n$ invertable $\qquad \Downarrow$

$(\det P \neq 0) \qquad y = P^{-1}x$

$$\dot{y} = P^{-1}\dot{x} = P^{-1}Ax = \underbrace{P^{-1}AP}_{\Lambda}y$$

$$\Rightarrow \quad \dot{y} = \Lambda y, \quad \Lambda = P^{-1}AP$$

Strategy: choose P such that Λ takes a form for which we can calculate $e^{t\Lambda}$ (the case when Λ - diag matrix)

$y(t) = e^{t\Lambda}y_0 \qquad\qquad y_0 = P^{-1}x_0$

$$\Rightarrow \quad x(t) = Py = Pe^{t\Lambda}y_0 = \underbrace{Pe^{t\Lambda}P^{-1}}_{e^{tA}}x_0$$

① All eigenvalues of A are distinct

$\lambda_i \in \mathbb{R}$ with eigenvectors u_i , $i=1,..k$

$\lambda_j = \alpha_j \pm i\beta_j \in \mathbb{C}$ with eigenvectors

$$W_j = u_j \pm i V_j$$

$$j = k+1, k+2, \dots, e$$

Construct P as (will be invertable):

$$P = [u_1, u_2, \dots, u_k, \underset{\substack{\uparrow \\ Im(W_{k+1})}}{V_{k+1}}, \underset{\substack{\uparrow \\ Re(W_{k+1})}}{u_{k+1}}, \dots, V_e, u_e]$$

$$\Lambda = P^{-1} A P = diag[\lambda_1, .., \lambda_k, B_{k+1}, \dots, B_e]$$

$$B_{j\,(2\times2)} = \begin{bmatrix} \alpha_j & -\beta_j \\ \beta_j & \alpha_j \end{bmatrix} \quad j = k+1, \dots, e$$

$$\Lambda = \begin{bmatrix} \boxed{\lambda_1} & & & & 0 \\ & \ddots & & & \\ & & \boxed{\lambda_k} & & \\ & 0 & & \boxed{B_{k+1}} & \\ & & & & \ddots \; \boxed{B_e} \end{bmatrix}$$

$$\Rightarrow e^{tA} = P \,\mathrm{diag}\left[e^{\lambda_1 t}, \dots, e^{\lambda_k t}, E_{k+1}, \dots, E_e\right] P^{-1}$$

Here $E_{j\,(2\times 2)} = e^{\alpha_j t} \begin{bmatrix} \cos\beta_j t & -\sin\beta_j t \\ \sin\beta_j t & \cos\beta_j t \end{bmatrix}$

$j = k+1, \dots, \ell$

② A has real multiple eigenvalues

λ - multiplicity k $\longrightarrow$ $u_1, u_2, \ldots, u_k$
generalised eigenvectors

$\boxed{\text{Any nonzero solution of } (A-\lambda I)^m \cdot u = 0 \text{ for } m = 1, \ldots, k \text{ is called a generalised eigenvector}}$

$$P = [u_1, \ldots, u_k]$$

$$e^{tA} = P \operatorname{diag}[e^{\lambda t}, e^{\lambda t}, \ldots, e^{\lambda t}] \cdot \left(I + Nt + \ldots + N^{k-1} \frac{t^{k-1}}{(k-1)!} \right) \cdot P^{-1}$$

Here a linear operator N is <u>nilpotent</u> of order k

$(N^{k-1} \neq 0$ and $N^k = 0)$

Remark: • λ of multiplicity 2

$$\Lambda = \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix} \to u_1, u_2 \text{ - generalised eigenvectors}$$

$$P = [u_1 \; u_2]$$

$$(I + Nt) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + t \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & t \\ 0 & 1 \end{bmatrix}$$

$$\Rightarrow e^{At} = P \begin{bmatrix} e^{\lambda t} & te^{\lambda t} \\ 0 & e^{\lambda t} \end{bmatrix} P^{-1}$$

• λ of multiplicity 3

$$\Lambda = \begin{bmatrix} \lambda & & 0 \\ & \lambda & \\ 0 & & \lambda \end{bmatrix} \to u_1, u_2, u_3 \text{ - generalised eigenvectors}$$

$$I + Nt + \frac{N^2t^2}{2!} = \begin{bmatrix} 1 & t & \frac{t^2}{2} \\ 0 & 1 & t \\ 0 & 0 & 1 \end{bmatrix} \Rightarrow$$

$$\Rightarrow e^{At} = P \begin{bmatrix} e^{\lambda t} & te^{\lambda t} & \frac{t^2}{2}e^{\lambda t} \\ 0 & e^{\lambda t} & te^{\lambda t} \\ 0 & 0 & e^{\lambda t} \end{bmatrix} P^{-1}$$

③ A has complex multiple eigenvalues

$\lambda = \alpha \pm i\beta$ of multiplicity k

(matrix $A_{(2k \times 2k)}$) and generalised eigenvectors

$w_j = u_j \pm i v_j, \quad j = 1, .., k$

$P = [\, v_1, u_1, v_2, u_2, \dots, v_k, u_k \,]$ - invertable

$$e^{tA} = P \, \mathrm{diag}\left[e^{\alpha t} \begin{pmatrix} \cos\beta & -\sin\beta \\ \sin\beta & \cos\beta \end{pmatrix}, e^{\alpha t} \begin{pmatrix} & \end{pmatrix}, \dots \right]$$

$$\left(I + Nt + \dots + N^{k-1} \frac{t^{k-1}}{(k-1)!} \right) \cdot P^{-1}$$