Applications of Stochastic Calculus in Finance
Chapter 1: Interest rates and related contracts

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1 Zero-coupon bonds

A bond is a securitized form of a loan, which is a financial instrument in the market where the time value of money is traded.

Zero-coupon bond of maturity T: A financial instrument paying to its holder 1 unit of cash at maturity T. It is also called a T-bond. The fixed final payoff is called face value/notional value/principal value/nominal value. Essentially, T-bond represents the time value of 1 unit of cash with maturity T.

Fig 1: cash flow of a T-bond holder.

Assumption 1 *The term structure of zero-coupon bond as a mapping $T \mapsto P(t,T)$ for any fixed $t \in [0,T)$ is differentiable.*

Statistical estimation of term structure can be found in Filipovic [1] Chapter 3.

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2 Related interest rates

Forward rate agreement (FRA): A financial contract in which two parties agree to exchange, at some future date(s), interest rate payments on the nominal value of a contract.

See Fig 2 for the cash flow of an FRA payer with $S > T \geq t$. The cash flow of an FRA receiver is then obtained by reversing the sign of the cash flow of the FRA payer.

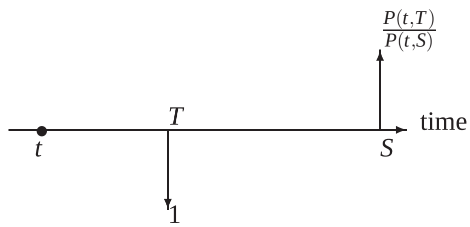

Fig 2: cash flow of an FRA payer.

Note that we can replicate the FRA by selling one T-bond and buying $\frac{P(t,T)}{P(t,S)}$ units of S-bonds at time t. Hence, a payer FRA is essentially a forward investment of 1 at T yielding $\frac{P(t,T)}{P(t,S)}$ at time S.

Next, we define six different types of interest rates based on calculating yields of such an FRA.

(1) Simple forward rate $F(t;T,S)$ for $[T,S]$ at t:

$$1+(S-T)F(t;T,S) = \frac{P(t,T)}{P(t,S)} \Leftrightarrow F(t;T,S) = \frac{1}{S-T}\left(\frac{P(t,T)}{P(t,S)} - 1\right).$$

(2) Simple short/spot rate $F(t,T)$ for $[t,T]$:

$$F(t,T) = F(t;t,T),$$

so

$$F(t,T) = \frac{1}{T-t}\left(\frac{1}{P(t,T)} - 1\right).$$

(3) Continuously compounded forward rate $R(t;T,S)$ for $[T,S]$ at t:

$$e^{(S-T)R(t;T,S)} = \frac{P(t,T)}{P(t,S)} \Leftrightarrow R(t;T,S) = -\frac{\ln P(t,S) - \ln P(t,T)}{S-T}.$$

Note that if $S-T$ is infinitesimal,

$$e^{(S-T)R(t;T,S)} = 1+(S-T)R(t;T,S) + o(S-T).$$

Hence, simple forward rate $F(t;T,S)$ and continuously compounded forward rate $R(t;T,S)$ coincide if the time period $S-T$ is infinitesimal.

On the other hand,

$$e^{(S-T)R(t;T,S)} = \lim_{m\uparrow\infty}\left(1+\frac{(S-T)R(t;T,S)}{m}\right)^m$$

where m represents the number of reinvestments in $[T,S]$.

(4) Continuously compounded short/spot rate $R(t,T)$ for $[t,T]$:

$$R(t,T) = R(t;t,T),$$

so

$$R(t,T) = -\frac{\ln P(t,T)}{T-t}.$$

The mapping $T \mapsto R(t,T)$ is called yield curve, so $R(t,T)$ is also called yield-to-maturity.

(5) Forward rate $f(t,T)$ for $[t,T]$:

$$f(t,T) = \lim_{S\downarrow T} R(t;T,S) = -\partial_T \ln P(t,T).$$

The mapping $T \mapsto f(t,T)$ is called forward curve, and $f(t,T)$ represents the interest rate over an infinitesimal time interval $[T,T+\Delta]$ as seen from time t.

Proposition 1. $P(t,T) = e^{-\int_t^T f(t,u)du}$.

Proof. Since $f(t,T) = \frac{-1}{P(t,T)}\partial_T P(t,T)$, that is,

$$\partial_T P(t,T) = -P(t,T)f(t,T), \tag{1}$$

which has a general solution $P(t,T) = Ce^{-\int_t^T f(t,u)du}$ for some constant C. Since $P(t,t) = 1$, we obtain that $C = 1$.

(6) Short/spot rate r_t at time t:

$$r_t = \lim_{T\downarrow t} R(t,T) = \lim_{T\downarrow t} -\frac{\ln P(t,T)}{T-t}.$$

Using the L'Hopital's rule and ODE (1), we further obtain

$$\lim_{T\downarrow t} -\frac{\ln P(t,T)}{T-t} = \lim_{T\downarrow t} -\partial_T P(t,T) = \lim_{T\downarrow t} f(t,T)P(t,T) = \lim_{T\downarrow t} f(t,T)$$

which is denoted as $f(t,t) := \lim_{T\downarrow t} f(t,T)$.

Hence. r_t is the interest rate at time t over an infinitesimal time interval $[t,t+\Delta]$. Based on r_t, we can define money-market account/bank account/money account as $B_t = e^{\int_0^t r_s ds}$.

Example: LIBOR (London interbank offered rate)[1] is the simple rate at which high-credit financial institutions in London can borrow in interbank market. Its maturities range from overnight to 12 months. It is quoted on a simple basis. For instance, the three months forward LIBOR for $[T, T+\frac{1}{4}]$ at time t is given by

$$L(t,T) = F(t;T,T+\frac{1}{4}).$$

The three months spot LIBOR for $[T, T+\frac{1}{4}]$ is given by

$$L(T,T) = F(T,T+\frac{1}{4}).$$

Other interbank offered rates include MIBOR, SIBOR and HIBOR, Euribor and Shibor.

3 Interest rates related contracts

(1) Fixed coupon bond (FCB): A bond that has a sequence of fixed coupons at a sequence of settlement dates: $T_1 < T_2 < \cdots < T_n$ with maturity T_n. Let $\delta = T_i - T_{i-1}$, as the tenor structure of such an FCB.

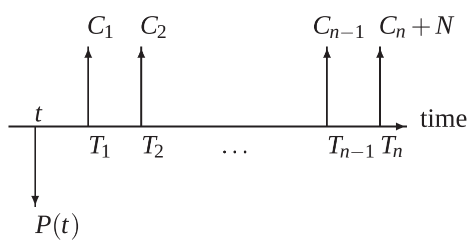

Fig 3: cash flow of an FCB holder.

The holder of FCB receives fixed cash flow $C_i = \kappa\delta N$ at settlement date T_i for $1 \le i \le n$, and the capital N at maturity T_n, where κ is some fixed rate. Since the time t value of C_i is $P(t,T_i)C_i = P(t,T_i)\kappa\delta N$, and the time t value of N is $P(t,T_n)N$, the fair value of FCB is

$$P(t) = (\kappa\delta\sum_{i=1}^{n} P(t,T_i) + P(t,T_n))N.$$

(2) Floating rate note (FRN): A bond that has a sequence of variable coupons, which are usually determined by some reference rates (e.g. LIBOR) at settlement dates: $T_1 < T_2 < \cdots < T_n$ with maturity T_n.

[1] LIBOR has been replaced by risk-free rate (RFR) since 2022 due to its role worsening the 2008 financial crisis and manipulation scandal.

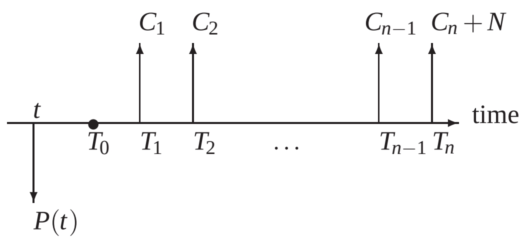

Fig 4: cash flow of an FRN holder.

The holder of FRN receives floating cash flow

$$C_i = L(T_{i-1}, T_{i-1})\delta N = F(T_{i-1}, T_i)\delta N = \frac{N}{P(T_{i-1}, T_i)} - N$$

at settlement date T_i for $1 \le i \le n$, where T_0 is called the reset date. The time t value of $-N$ is $-P(t, T_i)N$. The time t value of $\frac{N}{P(T_{i-1},T_i)}$ is determined by the following replication argument:

(a) At time t, pay $P(t, T_{i-1})N$ to buy N units of T_{i-1} bonds;

(b) At time T_{i-1}, receive N from T_{i-1} bonds. Meanwhile, pay N to buy $\frac{N}{P(T_{i-1},T_i)}$ units of T_i bonds. This is a net-zero investment;

(c) At time T_i, receive $\frac{N}{P(T_{i-1},T_i)}$ from T_i bonds.

Hence, we construct a strategy replicating the payoff $\frac{N}{P(T_{i-1},T_i)}$, and its time t value is $P(t, T_{i-1})N$. In total, the time t value of C_i is $(P(t, T_{i-1}) - P(t, T_i))N$. The fair value of FRN is

$$P(t) = \sum_{i=1}^{n} (P(t, T_{i-1}) - P(t, T_i))N + P(t, T_n)N = P(t, T_0)N.$$

In the US, the main issuers of FRNs are Fannie Mae (Federal Natioanl Mortage Association) and Freddie Mac (Federal Home Loan Mortage Corporation). In Europe, the main issuers are investment banks.

(3) Interest rate swap (IRS): A financial contract in which two parties agree to exchange interest rate cash flows, based on a nominal amount from a fixed rate to a floating rate (e.g. LIBOR), or vice versa. Hence, IRS is a sequence of exchanges between fixed and floating coupon payments.

See Fig 5 for the cash flow of an IRS payer.

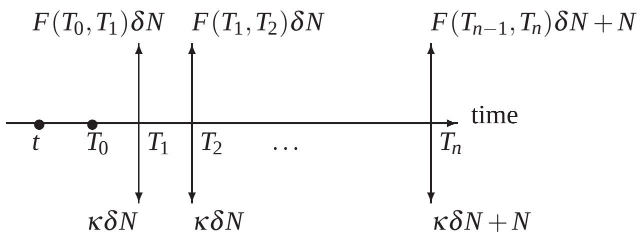

Fig 5: cash flow of an IRS payer.

The IRS payer pays fixed $\kappa\delta N$ for floating $F(T_{i-1},T_i)\delta N$ at settlement date T_i for $1 \le i \le n$. Based on time t values of FCB and FRN, the time t value of payer IRS is

$$\begin{aligned}\Pi_p(t) &= P(t,T_0)N - (\kappa\delta\sum_{i=1}^{n} P(t,T_i) + P(t,T_n))N \\ &= (P(t,T_0) - P(t,T_n) - \kappa\delta\sum_{i=1}^{n} P(t,T_i))N.\end{aligned}$$

The cash flow of an IRS receiver is then obtained by reversing the sign of the cash flow of the IRS payer. Hence, the time t value of receiver IRS is $\Pi_c(t) = -\Pi_p(t)$.

Forward swap rate (par swap rate) $R_{Swap}(t)$ at time t is the fixed rate κ such that $\Pi_p(t) = \Pi_c(t) = 0$, which is

$$R_{Swap}(t) = \frac{P(t,T_0) - P(t,T_n)}{\delta\sum_{i=1}^{n} P(t,T_i)}.$$

(4) Interest rate cap/floor: A financial instrument where the issuer has the obligation to pay cash to the holder if a particular interest rate exceeds/falls below a mutually agreed level at some future date(s).

The cash flow of interest rate cap at settlement date T_i is

$$(F(T_{i-1},T_i)\delta N - \kappa\delta N)^+$$

for $1 \le i \le n$. For an IRS receiver holding such an interest rate cap, her cash flow is always nonnegative:

$$(\kappa\delta N - F(T_{i-1},T_i)\delta N) + (F(T_{i-1},T_i)\delta N - \kappa\delta N)^+ \ge 0.$$

The cash flow of interest rate floor at settlement date T_i is

$$(\kappa\delta N - F(T_{i-1},T_i)\delta N)^+$$

for $1 \le i \le n$. For an IRS payer holding such an interest rate floor, her cash flow is always nonnegative:

$$(F(T_{i-1},T_i)\delta N-\kappa\delta N)+(\kappa\delta N-F(T_{i-1},T_i)\delta N)^+\geq 0.$$

Proposition 2. *The time T_{i-1} value of the T_i cash flow of interest rate cap is some put option payoff. Similarly, the time T_{i-1} value of the T_i cash flow of interest rate floor is some call option payoff.*

Proof. Since

$$\begin{aligned}P(T_{i-1},T_i)(F(T_{i-1},T_i)\delta N-\kappa\delta N)^+ &= P(T_{i-1},T_i)(\frac{1}{\delta}(\frac{1}{P(T_{i-1},T_i)}-1)\delta N-\kappa\delta N)^+\\ &= N(1+\kappa\delta)(\frac{1}{1+\kappa\delta}-P(T_{i-1},T_i))^+,\end{aligned}$$

the time T_{i-1} value can then be regarded as the payoff of a put option on a T_i-bond with strike price $\frac{1}{1+\kappa\delta}$ and the option maturity T_{i-1}

Proposition 3. *Let $F_l(t)$ and $C_p(t)$ represent the time t values of interest rate floor and cap, respectively. Then $C_p(t)-F_l(t)=\Pi_P(t)$.*

Proof. We only need to observe that

$$(F(T_{i-1},T_i)\delta N-\kappa\delta N)^+-(\kappa\delta N-F(T_{i-1},T_i)\delta N)^+=F(T_{i-1},T_i)\delta N-\kappa\delta N.$$

The above three terms respectively correspond to the time T_i cash flows of interest rate cap, floor and payer swap.

4 Exercises

Exercise 1. (Forward rate predicts future short rate in a deterministic world)

1. In a deterministic world, prove that the price of the zero-coupon bond satisfies the following semigroup property:

$$P(t,S)=P(t,T)P(T,S)$$

   for any $t\leq T\leq S$. (Hint: Prove by using the replication argument as it is used for calculating the fair value of FRN on Page 5.)
2. Using the above semigroup property, prove that for any $t\leq T\leq S$,

$$\int_T^S f(t,u)du=\int_T^S f(T,u)du$$

   and

$$f(t,S)=f(T,S)=r(S).$$

3. Using the above semigroup property, recalculate the fair value of the FRN.

Exercise 2. (Interest rate swaption)

A European payer (receiver) swaption with the strike rate K is an option giving the right to enter a payer (receiver) interest rate swap with fixed rate K at a given future date, the swaption maturity. Usually, the swaption maturity is the first reset date T_0 of the underlying interest rate swap. Since the value of the payer IRS at T_0 is given by

$$N[P(T_0,T_0)-P(T_0,T_n)-K\delta\sum_{i=1}^{n}P(T_0,T_i)] = N\delta[\sum_{i=1}^{n}P(T_0,T_i)(F(T_0;T_i-1,T_i)-K)],$$

the payoff of the payer swaption is

$$N\delta[\sum_{i=1}^{n}P(T_0,T_i)(F(T_0;T_i-1,T_i)-K)]^{+}.$$

Prove the above payoff can also be written as

$$N\delta(R_{Swap}(T_0)-K)^{+}\sum_{i=1}^{n}P(T_0,T_i),$$

where $R_{Swap}(T_0)$ is the forward swap rate at T_0:

$$R_{Swap}(T_0) = \frac{P(T_0,T_0)-P(T_0,T_n)}{\delta\sum_{i=1}^{n}P(T_0,T_i)}.$$

References

1. Filipovic, Damir. *Term-Structure Models. A Graduate Course.* Springer, 2009.