Applications of Stochastic Calculus in Finance
Chapter 5: LIBOR market model

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1 Motivation and Heuristic Derivation

Let us first make a comparison between the pricing of equity options and interested rate related contracts implemented in industry.

option pricing theory	interest rate theory
equity log normal under $\mathbf{Q}$;	forward rate log normal under $\mathbf{Q}$
equity option;	interest rate related contract
Black-Scholes formula;	Black's formula

Therefore, in order to adapt the Black-Schoels formula for equity options to interest rate related contracts, it would be desirable to build up a model in which forward rate is log normal under the spot measure $\mathbf{Q}$. However, *the log normal assumption of the forward rate $f(t,T)$ will cause it to explode*. To illustrate this issue, we take on dimensional case, and assume that volatility $\sigma(t,T) = \sigma f(t,T)$ for $\sigma > 0$. Then the HJM condition yields that the forward rate $f(t,T)$ must follow

$$\begin{aligned} df(t,T) &= \sigma(t,T)\int_t^T \sigma(t,s)dsdt + \sigma(t,T)dW_t^{\mathbf{Q}} \\ &= \sigma^2 f(t,T)\int_t^T f(t,s)dsdt + \sigma f(t,T)dW_t^{\mathbf{Q}} \end{aligned}$$

Suppose further the initial forward curve is flat, i.e. $f(0,T) = 1$. Then, Itô's formula yields

$$f(t,T) = \exp\left\{\int_0^t (\sigma^2 \int_s^T f(s,u)du)ds\right\}\exp\left\{\sigma W_t^{\mathbf{Q}} - \frac{1}{2}\sigma^2 t\right\}$$

Differentiating against T on both sides yields

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$$\partial_T f(t,T) = f(t,T)\int_0^t \sigma^2 f(s,T)ds$$
$$= \frac{\sigma^2}{2}\partial_t(\int_0^t f(s,T)ds)^2.$$

Integrating against t from 0 to 1 on both sides yields

$$\int_0^1 \partial_T f(t,T)dt = \frac{\sigma^2}{2}(\int_0^1 f(s,T)ds)^2$$

Let $X_T = \int_0^1 f(s,T)ds$. Then the above equation reduces to an ODE

$$\frac{dX_T}{dT} = \frac{\sigma^2}{2}X_T^2$$

whose solution is given by

$$X_T = \frac{X_1}{1-\frac{\sigma^2}{2}X_1(T-1)}.$$

It is clear that when $T \to 1+\frac{2}{\sigma^2 X_1}$, $X_T \to \infty$. Since X_T is nondecreasing, we also have $X_T = \infty$ for $T > 1+\frac{2}{\sigma^2 X_1}$ Hence, $f(s,T)$ becomes ∞ for time $s \leq 1$ and $T \geq 1+\frac{2}{\sigma^2 X_1}$. It will further imply that the zero-coupon bond prices become zero so the market admits an arbitrage opportunity. Alteratively stated, *log normal forward rates are inconsistent with a market satisfying no-arbitrage condition*.

To overcome this issue, we need to use LIBOR instead. Anyway, most of interest rate related contracts in practice are written on LIBOR rather than the forward rate $f(t,T)$. Recall the forward δ-period LIBOR for $[T,T+\delta]$ at time t is given by $L(t,T) = F(t;T,T+\delta)$. When $t = T$, we call $L(T,T)$ the spot LIBOR, or simply LIBOR. The positive δ is called the tenor of the LIBOR.

Recall that

$$L(t,T) = F(t;T,T+\delta) = \frac{1}{\delta}(\frac{P(t,T)}{P(t,T+\delta)}-1).$$

Moreover, $\frac{P(t,T)}{P(t,T+\delta)}$ is a martingale under the $(T+\delta)$-forward measure $\tilde{\mathbf{Q}}^{T+\delta}$:

$$d\frac{P(t,T)}{P(t,T+\delta)} = \frac{P(t,T)}{P(t,T+\delta)}(\sigma^*(t,T)-\sigma^*(t,T+\delta))dW_t^{\tilde{\mathbf{Q}}^{T+\delta}}.$$

Hence, we obtain that

$$dL(t,T) = \frac{1}{\delta}d\frac{P(t,T)}{P(t,T+\delta)} = (L(t,T)+\frac{1}{\delta})(\sigma^*(t,T)-\sigma^*(t,T+\delta))dW_t^{\tilde{\mathbf{Q}}^{T+\delta}}.$$

Suppose that there exist an $\mathbb{R}^d$-valued deterministic and bounded function $\lambda(t,T)$ such that

$$L(t,T)\lambda(t,T) = (L(t,T) + \frac{1}{\delta})(\sigma^*(t,T) - \sigma^*(t,T+\delta)). \tag{1}$$

Then we would have a log normal model for $L(t,T)$:

$$dL(t,T) = L(t,T)\lambda(t,T)dW_t^{\tilde{\mathbf{Q}}^{T+\delta}},$$

i.e.

$$L(t,T) = L(0,T)\mathscr{E}(\int_0^{\cdot} \lambda(s,T)dW_s^{\tilde{\mathbf{Q}}^{T+\delta}})_t.$$

The objective is then to construct a family of log normal models for $L(t,T)$ with different settlement dates T under their respective $(T+\delta)$-forward measures.

2 LIBOR Market Model

Fix a finite time horizon $T_M = M\delta$ for $M \in \mathbb{N}$, and a filtered probability space $(\Omega, \mathscr{F}, \{\mathscr{F}_t\}_{t\geq 0}, \tilde{\mathbf{Q}}^{T_M})$, which satisfies the usual conditions, and supports a d-dimensional Brownian motion $W^{\tilde{\mathbf{Q}}_{T_M}}$.

Assumption 1 *We are given the following data.*

(1) For $1 \leq i \leq M$, the settlement date $T_i = i\delta$;

(2) For $1 \leq i \leq M$, the initial term structure of T_i-bond $P(0,T_i)$. Hence, we obtain the initial forward LIBOR as

$$L(0,T_i) = \frac{1}{\delta}(\frac{P(0,T_i)}{P(0,T_{i+1})} - 1)$$

for $1 \leq i \leq M-1$;

(3) For $1 \leq i \leq M-1$, the volatility $\lambda(t,T_i)$ of the forward LIBOR $L(t,T_i)$, where $\lambda(t,T_i)$ is a deterministic and bounded function for $t \in [0,T_i]$.

We construct a <u>LIBOR market model</u> as follows.

<u>Step 1:</u> For $t \in [0,T_{M-1}]$, we are given the Brownian motion $W^{\tilde{\mathbf{Q}}_{T_M}}$ under the $\tilde{\mathbf{Q}}^{T_M}$, the volatility $\lambda(t,T_{M-1})$, and the initial zero-coupon bond prices $P(0,T_{M-1})$, $P(0,T_M)$. We construct $L(t,T_{M-1})$ as

$$\begin{cases} dL(t,T_{M-1}) = L(t,T_{M-1})\lambda(t,T_{M-1})dW_t^{\tilde{\mathbf{Q}}^{T_M}}, \\ L(0,T_{M-1}) = \frac{1}{\delta}(\frac{P(0,T_{M-1})}{P(0,T_M)} - 1) \end{cases}$$

which gives us

$$L(t,T_{M-1})=\frac{1}{\delta}(\frac{P(0,T_{M-1})}{P(0,T_M)}-1)\mathscr{E}(\int_0^{\cdot}\lambda(s,T_{M-1})dW_s^{\tilde{\mathbf{Q}}^{T_M}})_t.$$

Now based on (1), we define

$$\sigma^*(t,T_{M-1})-\sigma^*(t,T_M)=\frac{\delta L(t,T_{M-1})}{\delta L(t,T_{M-1})+1}\lambda(t,T_{M-1}).$$

This gives us the SDE of $\frac{P(t,T_{M-1})}{P(t,T_M)}$ under $\tilde{\mathbf{Q}}^{T_M}$:

$$d\frac{P(t,T_{M-1})}{P(t,T_M)}=\frac{P(t,T_{M-1})}{P(t,T_M)}(\sigma^*(t,T_{M-1})-\sigma^*(t,T_M))dW_t^{\tilde{\mathbf{Q}}^{T_M}}.$$

Hence, we may use $P(t,T_{M-1})$ as the numeraire, and introduce the T_{M-1}-forward measure $\tilde{\mathbf{Q}}^{T_{M-1}}$ by

$$\frac{d\tilde{\mathbf{Q}}^{T_{M-1}}}{d\tilde{\mathbf{Q}}^{T_M}}\bigg|_{\mathscr{F}_t}=\mathscr{E}(\int_0^{\cdot}\sigma^*(s,T_{M-1})-\sigma^*(s,T_M))dW_s^{\tilde{\mathbf{Q}}^{T_M}})_t=\frac{P(t,T_{M-1})P(0,T_M)}{P(t,T_M)P(0,T_{M-1})}.$$

By Girsanov's theorem in BM case,

$$W_t^{\tilde{\mathbf{Q}}^{T_{M-1}}}=W_t^{\tilde{\mathbf{Q}}^{T_M}}-\int_0^t\sigma^*(s,T_{M-1})-\sigma^*(s,T_M))ds$$

is a d-dimensional Brownian motion under the T_{M-1}-forward measure $\tilde{\mathbf{Q}}^{T_{M-1}}$.

Step 2: For $t\in[0,T_{M-2}]$, we are given the Brownian motion $W^{\tilde{\mathbf{Q}}_{T_{M-1}}}$ under the $\tilde{\mathbf{Q}}^{T_{M-1}}$, the volatility $\lambda(t,T_{M-2})$, and the initial zero-coupon bond prices $P(0,T_{M-2})$, $P(0,T_{M-1})$. We construct $L(t,T_{M-2})$ as

$$\begin{cases} dL(t,T_{M-2})=L(t,T_{M-2})\lambda(t,T_{M-2})dW_t^{\tilde{\mathbf{Q}}^{T_{M-1}}}, \\ L(0,T_{M-2})=\frac{1}{\delta}(\frac{P(0,T_{M-2})}{P(0,T_{M-1})}-1) \end{cases}$$

which gives us

$$L(t,T_{M-2})=\frac{1}{\delta}(\frac{P(0,T_{M-2})}{P(0,T_{M-1})}-1)\mathscr{E}(\int_0^{\cdot}\lambda(s,T_{M-2})dW_s^{\tilde{\mathbf{Q}}^{T_{M-1}}})_t.$$

Now based on (1), we define

$$\sigma^*(t,T_{M-2})-\sigma^*(t,T_{M-1})=\frac{\delta L(t,T_{M-2})}{\delta L(t,T_{M-2})+1}\lambda(t,T_{M-2}).$$

This gives us the SDE of $\frac{P(t,T_{M-2})}{P(t,T_{M-1})}$ under $\tilde{\mathbf{Q}}^{T_{M-1}}$:

$$d\frac{P(t,T_{M-2})}{P(t,T_{M-1})}=\frac{P(t,T_{M-2})}{P(t,T_{M-1})}(\sigma^*(t,T_{M-2})-\sigma^*(t,T_{M-1}))dW_t^{\tilde{\mathbf{Q}}^{T_{M-1}}}.$$

Hence, we may use $P(t,T_{M-2})$ as the numeraire, and introduce the T_{M-2}-forward measure $\tilde{\mathbf{Q}}^{T_{M-2}}$ by

$$\left.\frac{d\tilde{\mathbf{Q}}^{T_{M-2}}}{d\tilde{\mathbf{Q}}^{T_{M-1}}}\right|_{\mathscr{F}_t} = \mathscr{E}\Big(\int_0^{\cdot} \sigma^*(s,T_{M-2}) - \sigma^*(s,T_{M-1}))dW_s^{\tilde{\mathbf{Q}}^{T_{M-1}}}\Big)_t = \frac{P(t,T_{M-2})P(0,T_{M-1})}{P(t,T_{M-1})P(0,T_{M-2})}.$$

By Girsanov's theorem in BM case,

$$W_t^{\tilde{\mathbf{Q}}^{T_{M-2}}} = W_t^{\tilde{\mathbf{Q}}^{T_{M-1}}} - \int_0^t \sigma^*(s,T_{M-2}) - \sigma^*(s,T_{M-1}))ds$$

is a d-dimensional Brownian motion under the T_{M-2}-forward measure $\tilde{\mathbf{Q}}^{T_{M-2}}$.

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Step $M-1$: For $t \in [0,T_1]$, we are given the Brownian motion $W^{\tilde{\mathbf{Q}}^{T_2}}$ under the $\tilde{\mathbf{Q}}^{T_2}$, the volatility $\lambda(t,T_1)$, and the initial zero-coupon bond prices $P(0,T_1)$, $P(0,T_2)$. We construct $L(t,T_1)$ as

$$\begin{cases} dL(t,T_1) = L(t,T_1)\lambda(t,T_1)dW_t^{\tilde{\mathbf{Q}}^{T_2}}, \\ L(0,T_1) = \frac{1}{\delta}\big(\frac{P(0,T_1)}{P(0,T_2)} - 1\big) \end{cases}$$

which gives us

$$L(t,T_1) = \frac{1}{\delta}\Big(\frac{P(0,T_1)}{P(0,T_2)} - 1\Big)\mathscr{E}\Big(\int_0^{\cdot} \lambda(s,T_1)dW_s^{\tilde{\mathbf{Q}}^{T_2}}\Big)_t.$$

Based on (1), we may define $\sigma^*(t,T_1) - \sigma^*(t,T_2)$, and therefore the T_1-forward measure $\tilde{\mathbf{Q}}^{T_1}$ and the corresponding Brownian motion $W^{\tilde{\mathbf{Q}}^{T_1}}$. However, they will not be used.

Step M: Obviously, we have that

$$L(0,T_0) = \delta\Big(\frac{1}{P(0,T_1)} - 1\Big).$$

In summary, we have constructed a family of log normal models for $L(t,T_i)$ under their respective T_{i+1}-forward measure $\tilde{\mathbf{Q}}^{T_{i+1}}$, for $1 \le i \le M-1$, and have defined an arbitrage free market for the zero coupon bonds with maturities $T_1, T_2, \ldots, T_M$.

Theorem 1. *Given the Brownian motion $W^{\tilde{\mathbf{Q}}^{T_M}}$ under $\tilde{\mathbf{Q}}^{T_M}$, the dynamics of the forward LIBOR $L(t,T_i)$ for $i = 1,\cdots,M-1$ is given by*

$$dL(t,T_i) = L(t,T_i)\lambda(t,T_i)\left(dW_t^{\tilde{\mathbf{Q}}^{T_M}} - \sum_{j=i+1}^{M-1} \frac{\delta L(t,T_j)}{\delta L(t,T_j)+1}\lambda(t,T_j)dt\right).$$

Proof. We proceed backwards. It is obvious for $i = M-1$, since $\sum_M^{M-1} = 0$ by convention. Now for $i = M-2$, by our construction, we have that

$$dL(t,T_{M-2}) = L(t,T_{M-2})\lambda(t,T_{M-2})dW_t^{\tilde{\mathbf{Q}}^{T_{M-1}}},$$

where

$$W_t^{\tilde{\mathbf{Q}}^{T_{M-1}}} = W_t^{\tilde{\mathbf{Q}}^{T_M}} - \int_0^t \sigma^*(s,T_{M-1}) - \sigma^*(s,T_M))ds$$

The conclusion then follows from (1). □

3 Black's Formula for Interest Rate Caps

Recall that, for $1 \le i \le M$, the holder of the interest rate cap at the settlement date T_i (with the reset date T_{i-1}) receives

$$\delta N(F(T_{i-1},T_i) - K)^+ = \delta N(L(T_{i-1},T_{i-1}) - K)^+$$

The time t value of the above payoff is

$$C_p(t;T_{i-1},T_i) = \mathbf{E}^{\mathbf{Q}}[\frac{B_t}{B_{T_i}}\delta N(L(T_{i-1},T_{i-1}) - K)^+|\mathscr{F}_t].$$

From Bayes formula, the above conditional expectation can also be calculated under the T_i-forward measure $\tilde{\mathbf{Q}}^{T_i}$:

$$C_p(t;T_{i-1},T_i) = \delta NP(t,T_i)\mathbf{E}^{\tilde{\mathbf{Q}}^{T_i}}[(L(T_{i-1},T_{i-1}) - K)^+|\mathscr{F}_t].$$

But under the LIBOR market model,

$$L(T_{i-1},T_{i-1}) = L(t,T_{i-1})e^{\int_t^{T_{i-1}}\lambda(s,T_{i-1})dW^{\tilde{\mathbf{Q}}^{T_i}} - \frac{1}{2}\int_t^{T_{i-1}}|\lambda(s,T_{i-1})|^2 ds}.$$

Hence, we obtain that

$$C_p(t;T_{i-1},T_i) = \delta NP(t,T_i)[L(t,T_{i-1})\Phi(d_1^{T_{i-1}}) - K\Phi(d_2^{T_{i-1}})],$$

where

$$d_{1,2}^{T_{i-1}} = \frac{\ln\frac{L(t,T_{i-1})}{K} \pm \frac{1}{2}\int_t^{T_{i-1}}|\lambda(s,T_{i-1})|^2 ds}{\sqrt{\int_t^{T_{i-1}}|\lambda(s,T_{i-1})|^2 ds}}.$$

The arbitrage price of the interest rate cap at time t is

$$C_p(t) = \sum_{i=1}^M C_p(t,T_{i-1},T_i) = \delta N\sum_{i=1}^M P(t,T_i)[L(t,T_{i-1})\Phi(d_1^{T_{i-1}}) - K\Phi(d_2^{T_{i-1}})].$$

4 Exercises

Exercise 1. (Black's formula for interest rate swaption)

Recall a European payer interest rate swaption with the strike rate K is an option giving the right to enter a payer interest rate swap(IRS) with fixed rate K at a given future date, the swaption maturity. Usually, the swaption maturity is the first rest date T_0 of the underlying interest rate swap. Since the value of the payer IRS at T_0 is given by

$$\begin{aligned} & N\left[P(T_0,T_0)-P(T_0,T_n)-K\delta\sum_{i=1}^{n}P(T_0,T_i)\right] \\ =& N\delta\left[\sum_{i=1}^{n}P(T_0,T_i)(F(T_0;T_i-1,T_i)-K)\right], \end{aligned}$$

the payoff of the payer swaption is

$$N\delta\left[\sum_{i=1}^{n}P(T_0,T_i)(F(T_0;T_i-1,T_i)-K)\right]^+ = N\delta\left[\sum_{i=1}^{n}P(T_0,T_i)(L(T_0,T_i-1)-K)\right]^+$$

1. Prove the above payoff can also be written as

$$N\delta(R_{Swap}(T_0)-K)^+\sum_{i=1}^{n}P(T_0,T_i),$$

where $R_{Swap}(T_0)$ is the forward swap rate at T_0:

$$R_{Swap}(T_0)=\frac{P(T_0,T_0)-P(T_0,T_n)}{\delta\sum_{i=1}^{n}P(T_0,T_i)}.$$

2. Write down the arbitrage price of this payoff under both the spot measure $\mathbf{Q}$ and the T_0-forward measure $\tilde{\mathbf{Q}}^{T_0}$.
3. Define a numeraire:

$$D(t)=\frac{\sum_{i=1}^{n}P(t,T_i)}{P(t,T_0)}$$

for $t\in[0,T_0]$. Show that $D(t)$ is martingale under the T_0-forward measure $\tilde{\mathbf{Q}}^{T_0}$. Hence, we can use $\frac{D(t)}{D(0)}$ as the Radon-Nikodym density, and define an equivalent probability measure $\tilde{\mathbf{Q}}^{Swap}$, so called the forward swap measure,

$$\left.\frac{d\tilde{\mathbf{Q}}^{Swap}}{d\tilde{\mathbf{Q}}^{T_0}}\right|_{\mathscr{F}_{T_0}}=\frac{D(T_0)}{D(0)}.$$

Prove that the arbitrage price of this payoff under the forward swap measure $\tilde{\mathbf{Q}}^{Swap}$ is

$$N\delta\sum_{i=1}^{n}P(0,T_i)\mathbf{E}^{\tilde{\mathbf{Q}}^{Swap}}\left[(R_{Swap}(T_0)-K)^+\right]$$

4. Prove that $R_{Swap}(t)$ is a martingale under the forward swap measure $\tilde{\mathbf{Q}}^{Swap}$. Hence, similar to the LIBOR market model, suppose that there exists a bounded and deterministic function $\rho_{Swap}(t)$ such that

$$dR_{Swap}(t)=R_{Swap}(t)\rho_{Swap}(t)dW_t^{\tilde{\mathbf{Q}}^{Swap}}$$

under the forward swap measure $\tilde{\mathbf{Q}}^{Swap}$ for $t\in[0,T_0]$. Prove that the initial value of this interest rate swaption is given by the following Black's formula:

$$N\delta\sum_{i=1}^{n}P(0,T_i)\left[R_{Swap}(0)\Phi(d_1^{T_0})-K\Phi(d_2^{T_0})\right],$$

where

$$d_{1,2}^{T_0}=\frac{\ln\frac{R_{Swap}(0)}{K}\pm\frac{1}{2}\int_0^{T_0}|\rho_{Swap}(s)|^2ds}{\sqrt{\int_0^{T_0}|\rho_{Swap}(s)|^2ds}}.$$

Exercise 2. (Rebonato's analytic approximation for interest rate swaption, Filipovic [1], Chapter 11.5)

It can be shown that the volatility of forward LIBOR and the volatility of the forward swap rate can not be deterministic simultaneously. Hence, either one gets the Black's formula for interest rate cap/floor or for interest rate swaption, but not simultaneously for both. This motivates the so called Rebonato's analytic approximation for interest rate swaption given that the volatility of forward LIBOR is deterministic.

1. Prove that forward swap rate admits the following representation:

$$R_{Swap}(t)=\sum_{i=1}^{n}\omega_i(t)F(t;T_{i-1},T_i)=\sum_{i=1}^{n}\omega_i(t)L(t,T_{i-1}),$$

where the weights are given by

$$\omega_i(t)=\frac{P(t,T_i)}{\sum_{j=1}^{n}P(t,T_j)}.$$

2. From empirical evidence, the variability of those weights is small compared to that of simple forward rates (LIBOR). Therefore, the forward swap rate can be approximated as

$$R_{Swap}(t)\approx\sum_{i=1}^{n}\omega_i(0)L(t,T_{i-1})$$

By using the SDE for LIBOR $L(t,T_{i-1})$, prove that

$$\frac{dR_{Swap}(t)}{R_{Swap}(t)} \approx \left[\frac{\sum_{i=1}^{n}\omega_i(0)L(t,T_{i-1})\lambda(t,T_{i-1})\sum_{l=0}^{i-1}[\sigma^*(t,T_l)-\sigma^*(t,T_{l+1})]}{R_{Swap}(t)}\right]dt$$
$$+\left[\frac{\sum_{i=1}^{n}\omega_i(0)L(t,T_{i-1})\lambda(t,T_{i-1})}{R_{Swap}(t)}\right]dW_t^{\tilde{\mathbf{Q}}^{T_0}}$$

under the T_0 forward measure $\tilde{\mathbf{Q}}^{T_0}$.

3. In a further approximation, we replace all random variables in the above volatility by their time 0 values, so the volatility becomes

$$|\rho(t)|^2 \approx \frac{\sum_{i,j=1}^{n}\omega_i(0)\omega_j(0)L(0,T_{i-1})L(0,T_{j-1})\lambda(t,T_{i-1})\lambda(t,T_{j-1})^T}{R_{Swap}^2(0)},$$

and use $|\rho(t)|^2$ as the volatility $|\rho_{Swap}(t)|^2$ for the forward swap rate. Write down the pricing formula of the interest rate swaption.

References

1. Filipovic, Damir. *Term-Structure Models. A Graduate Course.* Springer, 2009.