Introduction to Dynamical Systems

Differential equs

ODEs

$$m\frac{d^2x}{dt^2} + b\frac{dx}{dt} + kx = 0$$

$$m\ddot{x} + kx = 0$$

PDEs

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}$$

$u(x,t)$

$\frac{dx}{dt}$ $\dot{x}$ $\ddot{x}$

Difference equs (iterated maps)

$$x_{n+1} = f(x_n)$$

$$\boxed{\begin{aligned} \dot{x}_1 &= f_1(x_1, .., x_n) \\ &\vdots \\ \dot{x}_n &= f_n(x_1, .., x_n) \end{aligned}}$$

n-dimensional system

$$\dot{x} = f(x)$$

$f_i(x_1, x_2, .., x_n)$

autonomous systems

$f_i(x_1, x_2, .., x_n, t)$

nonautonomous systems

$x_{n+1} = t$

$\dot{x}_{n+1} = 1$

n-order ODE $\Rightarrow$

$\Rightarrow$ can write as n-dim system

$$m\ddot{x} + b\dot{x} + kx = 0$$

$x_1 = x$ - position

$x_2 = \dot{x}$ - velocity

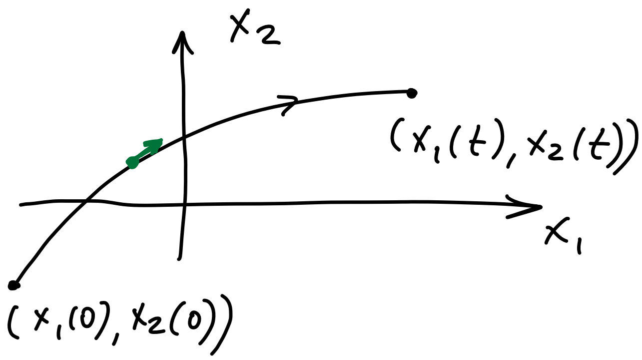
$$\Downarrow$$

$$\dot{x}_1 = x_2$$

$$\dot{x}_2 = \frac{1}{m}\left[-b x_2 - k x_1\right]$$

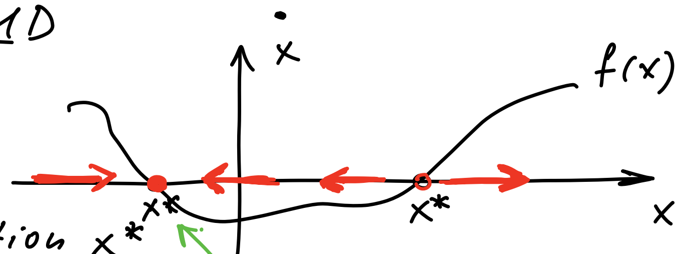
$$\dot{x}_1 = f_1(x_1, x_2)$$

$$\dot{x}_2 = f_2(x_1, x_2)$$



Systems in 1D

$$\dot{x} = f(x)$$



$f(x) = 0 \Rightarrow$ solution x^*

x^* - fixed points (equilibriums, steady-states)

locally stable (not globally)

$$\dot{x} = \frac{dx}{dt} \approx \frac{x_i - x_{i-1}}{\Delta t}$$

$\dot{x} > 0 \qquad x_i > x_{i-1} \quad \rightarrow$

$\dot{x} < 0 \qquad x_i < x_{i-1} \quad \leftarrow$

Linear stability analysis $\quad \dot{x} = f(x)$

x^* - f.p. $\quad \eta(t) = x(t) - x^*$ $\quad \eta(t)$ - grow or decay?

small perturbation

$$\dot{\eta} = \frac{d}{dt}(x(t) - x^*) = \dot{x}$$

$$\dot{\eta} = f(\eta(t) + x^*) =$$

Taylor's series expansion

$$= \underbrace{f(x^*)}_{=0} + f'(x^*)\eta + O(\eta^2) \qquad \eta << 1$$

$$\boxed{\dot{\eta} = f'(x^*)\eta}$$

linear diff. equ.

$$\eta(t) = e^{f'(x^*)t}$$

$f'(x^*) > 0 \Rightarrow \eta(t)$ grows exponentially

$\Rightarrow x^*$ - unstable

$f'(x^*) < 0 \Rightarrow \eta(t)$ decays $\Rightarrow$

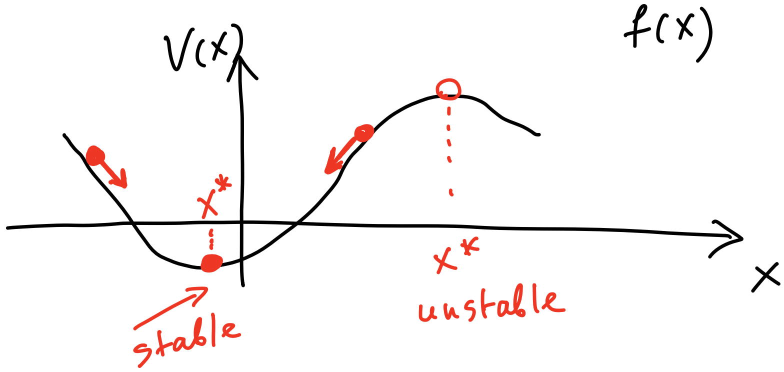
x^* - stable

characteristic time $\dfrac{1}{|f'(x^*)|}$

$\dot{x} = f(x)$ $\quad$ Potential

$$V(x): \quad \boxed{f(x) = -\frac{dV}{dx}}$$

$$\frac{dV(x(t))}{dt} = \frac{dV}{dx}\cdot\underbrace{\frac{dx}{dt}}_{f(x)} = -\left(\frac{dV}{dx}\right)^2 \leq 0$$



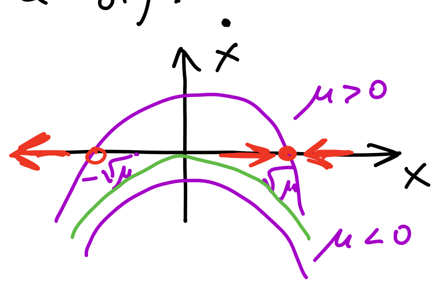
$$\frac{dV}{dx} = 0 \iff f(x) = 0$$

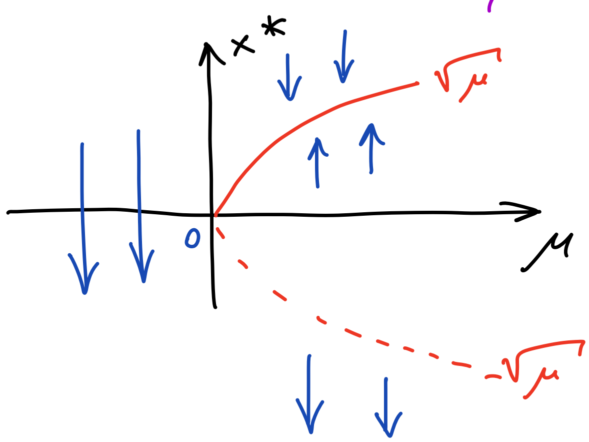
local min/max $\iff$ x^* - f.p.

Bifurcations

1D ① Saddle-node bif.

$$\boxed{\dot{x} = \mu - x^2}$$

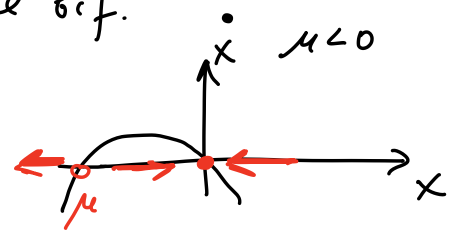


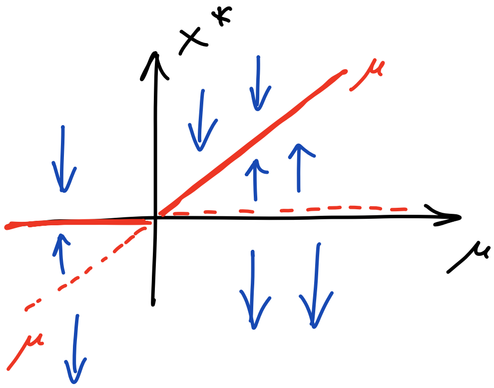


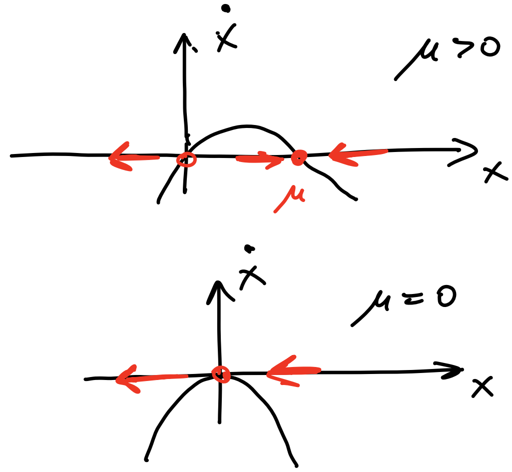
$\mu = 0$ - bif. point

② Transcritical bif.

$$\boxed{\dot{x} = \mu x - x^2}$$

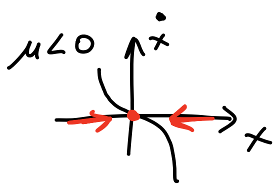


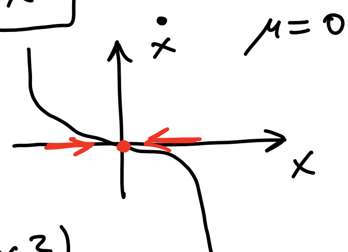


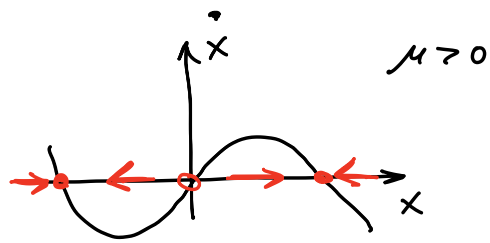


③ ⓐ Pitchfork bif. (supercritical bif.)

$$\boxed{\dot{x} = \mu x - x^3}$$







$f(x) = x(\mu - x^2)$

$f(x) = 0 \iff \quad x^* = 0 \qquad x^* = \pm\sqrt{\mu} \quad \mu > 0$

$f'(x^*) \qquad f'(x) = \mu - 3x^2$

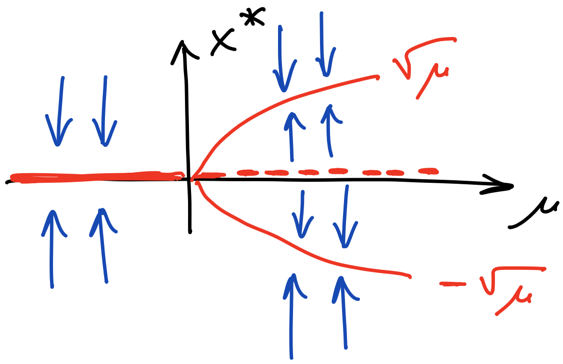
$f'(0) = \mu \quad \Rightarrow \quad \mu > 0 - \quad x^* = 0$ unstable

$\qquad\qquad\qquad\quad \mu < 0 - \quad x^* = 0$ stable

$$f'(\pm\sqrt{\mu}) = \mu - 3\mu = -2\mu < 0$$

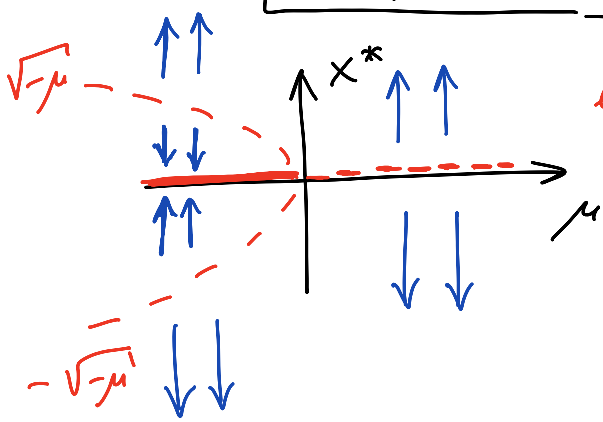
$x^* = \pm\sqrt{\mu}$ stable

$(\mu > 0)$



3 b Pitchfork bif. (subcritical)

$$\boxed{\dot{x} = \mu x + x^3}$$



blow up

Consider $\boxed{\dot{x} = \mu x + x^3 - x^5}$ HW

Consider $\boxed{\dot{x} = h + \mu x - x^3}$ HW

$h = 0$ – pitchfork bif.

$h \neq 0$ – breaks the symmetry

hint: split $f(x)$ into 2 functions

$$y = -h \qquad y = \mu x - x^3$$

$f(x) = 0$