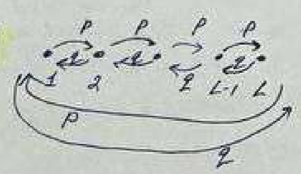
28.10.2023 Stoch Modelling HW1. Tokpeva Aleksandra

you shouldn't identify you work.

1. Simple Random Walk, $S=\{1 \dots L\}$

$p(x,y) = p\cdot\delta_{y,x+1} + q\cdot\delta_{y,x-1}$

a) $\boxed{\text{periodic boundary conditions: } p_{L,1}=p;\ p_{1,L}=q;}$



- Write down the transition matrix P.

$$P = \begin{pmatrix} 0 & p & 0 & \dots & 0 & 0 & q \\ q & 0 & p & \dots & 0 & 0 & 0 \\ & & \ddots & & \ddots & & \\ 0 & 0 & & \dots & q & 0 & p \\ p & 0 & & \dots & 0 & q & 0 \end{pmatrix}$$ ✓

- Is the corresponding Markov chain irreducible? ✓

yes, it is, because we can go from any state i to any other state j because the graph of the chain is connected (see the picture)

- Give all stationary distributions π and state whether they are reversible.

Let's find stationary distributions π: $\langle\pi| P = \langle\pi|$

so $(\pi_1 \dots \pi_L)\begin{pmatrix} 0 & p & 0 & \dots & 0 & 0 & q \\ q & 0 & p & \dots & 0 & 0 & 0 \\ 0 & 0 & 0 & \dots & q & 0 & p \\ p & 0 & 0 & \dots & 0 & q & 0 \end{pmatrix} = (\pi_1 \dots \pi_L)$

$$\Rightarrow \begin{cases} \pi_2\cdot q + \pi_L\cdot p = \pi_1 \\ \pi_{k-1}\cdot p + \pi_{k+1}\cdot q = \pi_k;\ k = 2 \dots L-1 \\ \pi_1\cdot q + \pi_{L-1}\cdot p = \pi_L \end{cases}$$

Look at $\pi_{k-1}\cdot p + \pi_{k+1}\cdot q = \pi_k$.

Let's search for the solution in the form $\pi_k = \lambda^k$

this is correct but you didn't need this much work.

$\Rightarrow \lambda^{k-1}\cdot p + \lambda^{k+1}\cdot q = \lambda^k$

$\Rightarrow \lambda^2 q - \lambda + p = 0$ it is a quadratic equation

1) First, let $q \neq 0 \Rightarrow$ then $D = 1-4pq = 1-4p(1-p) = 1-4p+4p^2 = (1-2p)^2$.

- if $p \neq q = \frac{1}{2}$, then $\lambda_{1,2} = \dfrac{1 \pm (1-2p)}{2q} \Rightarrow \left[\begin{array}{l} \lambda_1 = \frac{2-2p}{2q} = 1 \\ \lambda_2 = \frac{p}{q} \end{array}\right. \Rightarrow \pi_k = C_1 + C_2\left(\frac{p}{q}\right)^k$
- if $p = q = \frac{1}{2}$, then $\lambda_1 = \lambda_2 = \frac{1}{2q} \Rightarrow \pi_k = (C_1 + C_2 k)\left(\frac{1}{2q}\right)^k \overset{q=1/2}{=} C_1 + C_2 k$

2) Second, if $q = 0 \Rightarrow p = 1 \Rightarrow \lambda = 1 \Rightarrow \pi_2 = \dots = \pi_{L-1} = C_1$

3) If $p = 0 \Rightarrow q = 1 \Rightarrow \lambda(\lambda-1) = 0 \Rightarrow \left[\begin{array}{l} \lambda = 0 \\ \lambda = 1 \end{array}\right. \Rightarrow \pi_k = C_1 \Rightarrow \pi_2 = \dots = \pi_{L-1} = C_1$ – but this case is covered in

Now let's use two boundary conditions:

- case $p \neq q \neq 1/2$, $q \neq 0$, $\pi_k = C_1 + C_2\cdot\left(\frac{p}{q}\right)^k$

$$\begin{cases} \pi_2\cdot q + \pi_L\cdot p = \pi_1 \Rightarrow \left(C_1 + C_2\cdot\left(\frac{p}{q}\right)^2\right)q + \left(C_1 + C_2\left(\frac{p}{q}\right)^L\right)p = C_1 + C_2\cdot\left(\frac{p}{q}\right) \\ \pi_1\cdot q + \pi_{L-1}\cdot p = \pi_L \Rightarrow \left(C_1 + C_2\cdot\frac{p}{q}\right)q + \left(C_1 + C_2\left(\frac{p}{q}\right)^{L-1}\right)p = C_1 + C_2\left(\frac{p}{q}\right)^L \end{cases} \Rightarrow \begin{cases} C_2\cdot\left(\left(\frac{p}{q}\right)^2 + \left(\frac{p}{q}\right)^L\right) = C_2\cdot\frac{p}{q} \\ C_2\left(p + \frac{p^L}{q^{L-1}}\right) = C_2\cdot\left(\frac{p}{q}\right)^L \end{cases}$$

$\Rightarrow C_2 = 0 \Rightarrow \pi_k = C_1 + 0\cdot\left(\frac{p}{q}\right)^k = C_1$ ✓

$\Rightarrow \pi_1 = \dots = \pi_L = C_1$ but $\pi_1 + \dots + \pi_L = C_1 + \dots + C_1 \overset{?}{=} 1 \Rightarrow C_1 = 1/L \Rightarrow \boxed{\pi_1 = \dots = \pi_k = \frac{1}{L}}$ – stationary distribution

• case $p=1; q=0 \Rightarrow$ boundary conditions have the form: $\begin{cases} \pi_2 \cdot 0 + \pi_L = \pi_1 \\ \pi_1 \cdot 0 + \pi_{L-1} \cdot p = \pi_L \end{cases}$

$\Rightarrow$ we have $\pi_1 = \pi_2 = \dots = \pi_{L-1} = \pi_L = C_1$

but $\pi_1 + \dots + \pi_L = 1 \Rightarrow C_1 = \frac{1}{L} \Rightarrow \pi = \left(\frac{1}{L}, \dots \frac{1}{L}\right)$ is the stationary distribution

• case $p=q=1/2 \Rightarrow$ boundary conditions $\begin{cases} q(C_1+2C_2)+p(C_1+LC_2)=C_1+C_2 \\ q(C_1+C_2)+p(C_1+(L-1)C_2)=C_1+LC_2 \end{cases} \Rightarrow C_2=0 \Rightarrow \pi_k = C_1; k=1\dots L \Rightarrow C_1 = 1/L.$

So in both $p \neq q$, and $p=1; q=0$ we have only stationary distribution $\left(\frac{1}{L} \dots \frac{1}{L}\right)$

• Now let's check whether it is reversible

π is reversible if it fullfills the detailed balance condition:

$\pi(x)\, p(x,y) = \pi(y) \cdot p(y,x); \forall x,y \in S.$

in our cas $\pi(x) = \pi(y) = \frac{1}{L}$ for all $x,y \in S$.

$\Rightarrow$ we need to check: $p(x,y) \stackrel{?}{=} p(y,x), \forall x,y \in S.$

this means that transition matrix P is symmetric.

but is it so only if $p=q=1/2 \Rightarrow$ this π is reversible only for $p=q=1/2$.

$\boxed{\text{Answer: } \pi = \left(\tfrac{1}{L}; \dots \tfrac{1}{L}\right); \forall p,q; \text{chain is irreducible (periodic) and } \pi \text{ is reversible only for } p=q=1/2}$

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$\boxed{\text{Closed boundary Conditions: } p_{11} = q;\ p_{LL} = p;}$

• Write down the transition matrix P

$$P = \begin{pmatrix} q & p & 0 & \dots & 0 & 0 & 0 \\ q & 0 & p & \dots & 0 & 0 & 0 \\ & & & \ddots & & & \\ & & & & q & 0 & p \\ 0 & 0 & 0 & \dots & 0 & q & p \end{pmatrix}$$

• Is markov chain irreducible?

Yes, it is, because the corresponding graph is connected

• Give all stationary distributions

Look for π: $\langle \pi | P = \langle \pi |$

so $(\pi_1 \dots \pi_L) \begin{pmatrix} q & p & 0 & \dots & 0 & 0 & 0 \\ q & 0 & p & 0 & \dots & 0 & 0 & 0 \\ & - & - & - & & q & 0 & p \\ 0 & 0 & 0 & \dots & 0 & q & p \end{pmatrix} = (\pi_1 \dots \pi_L)$

$$\Rightarrow \begin{cases} \pi_1 \cdot q + \pi_2 \cdot q = \pi_1 \\ p\pi_k + q \cdot \pi_{k+1} = \pi_k; \ k = 2 \dots L-1 \quad (*) \\ p\pi_{L-1} + p\pi_L = \pi_L \end{cases}$$

The equation $(*)$ we have already solved:

• $p \neq q \neq \frac{1}{2} \Rightarrow \pi_k = C_1 + C_2 \left(\frac{p}{q}\right)^k$

• $\begin{cases} q=0; p=1 \\ p=0; q=1 \end{cases} \Rightarrow \pi_2 = \dots = \pi_{L-1} = C_1.$

New let's use the boundary conditions:

1) $p \neq q \neq \frac{1}{2}; 0 \Rightarrow \begin{cases} \pi_1 \cdot q + \pi_2 \cdot q = \pi_1 \\ p \cdot \pi_{L-1} + p \cdot \pi_L = \pi_L \end{cases} \Rightarrow \begin{cases} \pi_2 \cdot q = \pi_1 \cdot p \\ \pi_{L-1} \cdot p = \pi_L \cdot q \end{cases} \Rightarrow \begin{cases} q\left(c_1 + c_2 \left(\frac{p}{q}\right)^2\right) = p \cdot \left(c_1 + c_2 \frac{p}{q}\right) \\ p \cdot \left(c_1 + c_2 \left(\frac{p}{q}\right)^{L-1}\right) = q\left(c_1 + c_2 \left(\frac{p}{q}\right)^L\right) \end{cases} \Rightarrow$

$\Rightarrow \begin{cases} c_1(q-p) + c_2 \cdot \left(\cancel{\frac{p^2}{q} - \frac{p^2}{q}}\right)^{0} = 0 \\ c_2 \cdot \cancel{\frac{p^L}{q^{L-1}}} = c_2 \cdot \cancel{\frac{p^L}{q^{L-1}}} \end{cases} \Rightarrow \begin{cases} c_1 = 0 \\ c_2 - \text{any} \end{cases} \Rightarrow \boxed{\pi_k = c_2 \cdot \left(\frac{p}{q}\right)^k ; \; k = 1 \dots L}$

but $\pi_1 + \dots + \pi_k = 1 \Rightarrow c_2 \cdot \frac{p}{q} + c_2 \cdot \left(\frac{p}{q}\right)^2 + \dots + c_2 \cdot \left(\frac{p}{q}\right)^L = 1 \Rightarrow c_2 = \frac{1}{\frac{p}{q}\left(1 + \dots + \left(\frac{p}{q}\right)^{L-1}\right)} = \frac{1 - \frac{p}{q}}{\frac{p}{q} \cdot \left(1 - \left(\frac{p}{q}\right)^L\right)}$

2) $q = 0; p = 1 \Rightarrow \begin{cases} 0 \cdot \pi_1 + 0 \cdot \pi_2 = \pi_1 \\ \pi_{L-1} + \cancel{\pi_L} = \cancel{\pi_L} \end{cases} \Rightarrow \begin{cases} \pi_1 = 0 \\ \pi_{L-1} = 0 \end{cases}$; but $\pi_2 = \dots = \pi_{L-1} = c_1 \Rightarrow \pi_1 = \dots = \pi_{L-1} = 0$.

but $\pi_1 + \dots + \pi_L = 1 \Rightarrow \pi_L = 1 \Rightarrow$ $\boxed{(0;0;\dots;1) \text{ is stationary distribution.}}$

3) $q = 1; p = 0 \Rightarrow \begin{cases} \cancel{\pi_1} + \pi_2 = \cancel{\pi_1} \\ 0 + 0 = \pi_L \end{cases} \Rightarrow \begin{cases} \pi_2 = 0 \\ \pi_L = 0 \end{cases}$ $\pi_2 = \dots = \pi_{L-1} = 0; \pi_L = 0 \Rightarrow \pi_1 = 1 \Rightarrow (1;0\dots0)$ - stationary distribution

4) $p = q = 1/2 \Rightarrow \pi_k = c_1 + c_2 k$
$\begin{cases} \pi_1 \cdot q + \pi_2 \cdot q = \pi_1 \\ \pi_{L-1} \cdot p + \pi_L \cdot p = \pi_L \end{cases} \Rightarrow \begin{cases} \pi_2 q = \pi_1 \cdot p \\ \pi_{L-1} \cdot p = \pi_L \cdot q \end{cases} \Rightarrow \begin{cases} (c_1 + 2c_2) q = (c_1 + c_2) p \\ (c_1 + (L-1)c_2) p = (c_1 + Lc_2) q \end{cases} \Rightarrow$

$\Rightarrow \begin{cases} c_1 \overbrace{(q-p)}^{0, \text{ because } p=q} = c_2 \cdot (p - 2q) \\ c_1(q-p) = c_2(p(L-1) - Lq) \end{cases} \Rightarrow \begin{cases} c_2 = 0 \\ c_1 - \text{any} \end{cases} \Rightarrow \pi_k = c_1 = \frac{1}{L}$

stationary distribution

- State whether stationary distribution is reversible

1) $p \neq q \neq \frac{1}{2}; 0$; let's check $\pi(x) p(x,y) \overset{?}{=} \pi(y) p(y,x)$, $\forall x,y \in S$.
we know that $\pi_k = c_2 \left(\frac{p}{q}\right)^k$

we know that only $p(k;k+1)$ and $p(k;k-1)$ and $p(1;1)$ and $p(L,L)$ are not zero.

for $k = 2 \dots L-1$: $\pi(k) \cdot p(k;k+1) \overset{?}{=} \pi(k+1) \cdot \pi(k+1;k)$

$c_2 \left(\frac{p}{q}\right)^k \cdot p \overset{?}{=} c_2 \cdot \left(\frac{p}{q}\right)^{k+1} \cdot q$ - yes, it is true

for $x,y \in \{1,L\}$ check separately:

$x = 1$; $\pi(1) \cdot p(1;y) \overset{?}{=} \pi(y) \cdot p(y;1)$

$y = 1$: $\pi(1) \cdot p(1,1) \overset{?}{=} \pi(1) \cdot p(1;1)$ - true

$y = 2$: $\underbrace{\pi(1)}_{= c_2 \frac{p}{q}} \cdot p \overset{?}{=} \underbrace{\pi(2)}_{= c_2 \left(\frac{p}{q}\right)^2} q$ - true

$x = L$: $y = L-1$: $\underbrace{\pi(L)}_{= c_2 \left(\frac{p}{q}\right)^L} \cdot \underbrace{p(L;L-1)}_{= q} \overset{?}{=} \underbrace{\pi(L-1)}_{= c_2 \left(\frac{p}{q}\right)^{L-1}} \cdot \underbrace{p(L-1;L)}_{= p}$ - true

$y = L$: $\pi(L) \cdot p(L;L) \overset{?}{=} \pi(L) \cdot p(L;L)$ - true $\Rightarrow$ this distribution is reversible

2) $p = q = \frac{1}{2} \Rightarrow \pi_1 = \dots = \pi_L = \frac{1}{L} \Rightarrow$ detailed balance has the form: $p(x,y) \overset{?}{=} p(y,x)$, $\forall x,y \in S$.
this means that $\underline{P}$ is symmetric matrix - and yes, for $p = q$ it is symmetric

3) $q = 1; p = 0$, $\pi = (1;0;\dots0)$

$k = 2;\dots L$: $\pi(k) = 0$; $p(k;y) = q$ only for $y = k-1$, otherwise 0; $\Rightarrow$ for $y = k-1$: $0 \cdot q \overset{?}{=} \underbrace{\pi(k-1) \cdot p(k-1;k)}_{= 0, \text{ because } \pi(k-1)=0, k \neq 2 \text{ and for } k=2: p(1;2)=0.}$

$k = 1$: only $y = k$ has $p(x,y) \neq 0 \Rightarrow 1 = 1 \Rightarrow$ true

4) $q=0; p=1; \Rightarrow \pi=(0;\dots;0;1)$

for $k=1,\dots L-1$; only $y=k+1$ has $p(x,y)\neq 0 \Rightarrow \underbrace{\pi(k)}_{=0}\cdot \underbrace{p(k;k+1)}_{=p_{=1}} \overset{?}{=} \pi(k+1)\cdot \underbrace{p(k+1;k)}_{=0} \Rightarrow$ true

for $k=L$: only $y=L$ has $p(x,y)\neq 0 \Rightarrow 1=1 \Rightarrow$ true

$\Rightarrow$ the state π is reversible

$$\boxed{\begin{array}{l} \text{Answer: } p+q\neq\frac{1}{2};0:\ \pi_k = C\left(\frac{p}{q}\right)^k, \text{ where } C=\dfrac{1-\frac{p}{q}}{\frac{p}{q}\left(1-\left(\frac{p}{q}\right)^L\right)} \text{ - reversible} \\ p=q=\frac{1}{2}:\ \pi_k=\left(\frac{1}{L};\dots\frac{1}{L}\right) \text{ - reversible} \\ p=0; q=1;\ \pi_k=(1;0\dots0) \text{ - reversible} \\ p=1; q=0,\ \pi_k=(0,0\dots0,1) \text{ - reversible} \\ \text{Chain is irreducible} \end{array}}$$

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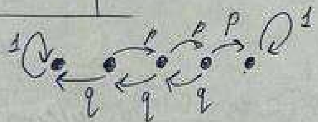
6) $\boxed{\text{absorbing boundary conditions: } p_{11}=p_{LL}=1.}$

- Write down transition matrix P



$$P=\begin{pmatrix} 1 & 0 & 0 & \dots & 0 & 0 & 0 \\ q & 0 & p & \dots & 0 & 0 & 0 \\ & & \ddots & & & & \\ & & & & q & 0 & p \\ 0 & & \dots & & & 0 & 1 \end{pmatrix}$$

- Is markov chain irreducible?

No, it is not, because not from any state i we can reach any other state j - for example, if $i=1$, we can't go anywhere except 1 from 1, so we can't reach $\{2;\dots L\}$

- Give all stationary distributions

We search for $\langle\pi|P=\langle\pi|$

$$\Rightarrow (\pi_1\dots\pi_L)\cdot\begin{pmatrix} 1 & 0 & 0 & \dots & 0 & 0 & 0 \\ q & 0 & p & \dots & 0 & 0 & 0 \\ 0 & q & 0 & & & & \\ & & q & \ddots & & & \\ & & & & 0 & p & 0 \\ & & & & q & 0 & p \\ & & & & 0 & 0 & 1 \end{pmatrix} = (\pi_1\dots\pi_L)$$

$$\Rightarrow \begin{cases} \cancel{\pi_1}+\pi_2\cdot q=\cancel{\pi_1} \\ \pi_3\cdot q=\pi_2 \\ p\cdot\pi_{k-1}+q\cdot\pi_{k+1}=\pi_k;\ k=3\dots L-2 \\ p\cdot\pi_{L-2}=\pi_{L-1} \\ p\cdot\pi_{L-1}+\cancel{\pi_L}=\cancel{\pi_L} \end{cases}$$

1) $p\neq q\neq\frac{1}{2};0 \Rightarrow \begin{cases} \pi_k=c_1+c_2\left(\frac{p}{q}\right)^k,\ k=3\dots L-2 \\ \pi_2=\pi_3=0 \\ \pi_{L-1}=\pi_{L-2}=0 \end{cases} \Rightarrow c_1=c_2=0 \Rightarrow \pi_2=\dots=\pi_{L-1}=0,\ \pi_1=a;\ \pi_L=1-a \Rightarrow$ $\pi=(a;0;\dots0;1-a)$ - stationary distribution

2) $p=q=\frac{1}{2}$: $\begin{cases} \pi_k=c_1+c_2\cdot k;\ k=3\dots L-2 \\ \pi_2=\pi_3=0 \\ \pi_{L-1}=\pi_{L-2}=0 \end{cases} \Rightarrow c_1=c_2=0 \Rightarrow \pi_2=\dots=\pi_{L-1}=0;\ \pi_1=a;\ \pi_L=1-a \Rightarrow$ $\pi=(a;0\dots0;1-a)$ - stationary distribution

3) $p=0; q=1$ $\begin{cases} \pi_2 = 0 \\ \pi_3 = \pi_2 = 0 \\ \pi_3 = \dots = \pi_{L-2} = c \\ \pi_{L-1} = 0 \end{cases}$ $\Rightarrow \pi = (a; 0 \dots 0; 1-a)$ - stationary distribution

$a \in [0, 1]$.

4) $p=1; q=0 \Rightarrow$ $\begin{cases} \pi_2 = 0 \\ \pi_3 = \dots = \pi_{L-2} = c \\ \pi_{L-2} = \pi_{L-1} \\ \pi_{L-1} = 0 \end{cases}$ $\Rightarrow \pi = (a; 0 \dots 0; 1-a)$ - stationary distribution.

- Are stationary distributions reversible

Let's check $\pi(x) \cdot p(x,y) \overset{?}{=} \pi(y) \cdot p(y,x), \forall x,y \in S$

if $x \in \{3 \dots L-2\}$: $\pi(x)=0$; $p(x,y) \neq 0$ only for $y=k+1$ or $k-1$, but $\pi(y)=0$ for $y=k+1$ or $k-1 \Rightarrow$ yes.

$x=2$: $\pi(x)=0$; $y=3$: $\underbrace{\pi(2)}_{0} \cdot p(2;3) \overset{?}{=} \underbrace{\pi(3)}_{0} \cdot p(3;2)$ - yes.

$y=1$: $\underbrace{\pi(2)}_{0} \cdot \underbrace{p(2;1)}_{q} \overset{?}{=} \underbrace{\pi(1)}_{a} \cdot \underbrace{p(1;2)}_{0}$ - yes.

$x=1$: $\pi(1) \cdot p(1;1) = \pi(1) \cdot p(1;1)$ - yes.

$x=L-1$: $\pi(x)=0$; $y=L$: $\underbrace{\pi(L-1)}_{0} \cdot p(L-1;L) \overset{?}{=} \underbrace{\pi(L)}_{1-a} \cdot \underbrace{p(L;L-1)}_{0}$ - yes

$y=L-2$: $\underbrace{\pi(L-1)}_{0} \cdot \underbrace{p(L-1;L-2)}_{q} \overset{?}{=} \underbrace{\pi(L-2)}_{0} \cdot p(L-2;L-1)$ - yes $\Rightarrow \pi$ is reversible

Answer: $\pi = (a; 0, \dots 0; 1-a)$ is a stationary distribution, $\forall p,q$, π is reversible
Chain is not irreducible
(Note that because it is not irreducible, it is not surprising that it has many stationary distributions - because theorem from lecture 7 guarantees the unique stationary distribution only for irreducible chains - and this chain is not irreducible

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- Give a recursion formula for h_k^L and solve it.

$h_k^L = P(X_n = L \text{ for some } n \geq 0 / X_0 = k)$

Let's condition h_k^L on the value of X_1, ~~and use that~~ and use the fact that after $X_0 = k$ only $X_1 = \left[\begin{smallmatrix} k+1 \\ k-1 \end{smallmatrix}\right.$ are possible for $k = 2 \dots L-1$, and $X_1 = k$ for $k = 1; L$

$P(A/B) = \sum_{i=1}^{n} P(A/B \cap C_i) \cdot P(C_i/B)$

markov property

for $k \neq 1, L$:

$\Rightarrow P(X_n = L \text{ for some } n \geq 0 / X_0 = k) = P(X_n = L \text{ for some } n \geq 0 / X_0 = k; X_1 = k+1) \cdot P(X_1 = k+1 / X_0 = k) + P(X_n = L \text{ for some } n \geq 0 / X_0 = k; X_1 = k-1) \cdot P(X_1 = k-1 / X_0 = k) =$

$= P(X_n = L \text{ for some } n \geq 0 / X_0 = k+1) \cdot P(X_1 = k+1 / X_0 = k)$
$+ P(X_n = L \text{ for some } n \geq 0 / X_0 = k-1) \cdot P(X_1 = k-1 / X_0 = k) = h_{k+1}^L \cdot p + h_{k-1}^L \cdot q$

$\Rightarrow \begin{cases} h_k^L = h_{k+1}^L \cdot p + h_{k-1}^L \cdot q; k = 2 \dots L-1 \\ h_1^L = 0 \\ h_L^L = 1 \end{cases}$ - boundary conditions

We seek for $h_k^L := \lambda^k \Rightarrow \lambda^2 p - \lambda + q = 0$. ; for $p=0$ this is not a quadratic equation

$\Rightarrow D = 1 - 4pq = 1 - 4p(1-p) = 1 - 4p + 4p^2 = (1-2p)^2$; $p=0; q=1 \Rightarrow h_k^L = c; k = 1 \dots L-1$

$\Rightarrow \left[\begin{array}{l} p \neq q \neq \frac{1}{2}: \lambda_1 = 1; \lambda_2 = \frac{q}{p} \Rightarrow h_k^L = c_1 + c_2 \left(\frac{q}{p}\right)^k \\ p = q = \frac{1}{2}: \lambda_1 = \lambda_2 = \frac{1}{2p} = 1 \Rightarrow h_k^L = (c_1 + c_2 k) \cdot 1 \end{array}\right.$

1) for $p \neq q \neq \frac{1}{2}, 0$: $\begin{cases} h_k^L = C_1 + C_2 \left(\frac{q}{p}\right)^k, k = 1 \ldots L-1 \\ h_1^L = 0 \\ h_L^L = 1 \end{cases} \Rightarrow \begin{cases} C_1 + C_2 \cdot \frac{q}{p} = 0 \\ C_1 + C_2 \cdot \left(\frac{q}{p}\right)^L = 1 \end{cases} \Rightarrow \begin{cases} C_2 \cdot \frac{q}{p}\left(\left(\frac{q}{p}\right)^{L-1} - 1\right) = 1 \\ C_1 = -C_2 \frac{q}{p} \end{cases} \Rightarrow \begin{cases} C_2 = \frac{p/q}{\left(\frac{q}{p}\right)^{L-1} - 1} \\ C_1 = \frac{-1}{\left(\frac{q}{p}\right)^{L-1} - 1} \end{cases}$

this needs $L \geq 3$, but if $L = 2 \Rightarrow h_1^L = 0; h_L^L = 1$ - we know the answer.

2) for $p = 0, q = 1$: $\begin{cases} h_k^L = h_{k-1}^L \; k = 1 \ldots L-1 \\ h_1^L = 0 \\ h_L^L = 1 \end{cases} \Rightarrow h_k^L = 1$ for $k = L$ and $h_k^L = 0$ for $k \neq L$

3) for $p = q = \frac{1}{2}$: $\begin{cases} h_k^L = C_1 + C_2 k; k = 1 \ldots L-1 \\ h_1^L = 0 \\ h_L^L = 1 \end{cases} \Rightarrow \begin{cases} C_1 + C_2 = 0 \\ C_1 + C_2 L = 1 \end{cases} \Rightarrow \begin{cases} C_2 (L-1) = 1 \\ C_1 = -C_2 \end{cases} \Rightarrow \begin{cases} C_2 = \frac{1}{L-1} \\ C_1 = -\frac{1}{L-1} \end{cases} \Rightarrow \boxed{h_k^L = \frac{1}{L-1}(k-1); k = 1 \ldots L}$

4) for $p = 1; q = 0$: $\begin{cases} h_k^L = h_{k+1}^L \quad ; k = 2 \ldots L-1 \\ h_1^L = 0 \\ h_L^L = 1 \end{cases} \Rightarrow h_k^L = \begin{cases} 1, & k = 2 \ldots L \\ 0; & k = 1 \end{cases}$

total Q1
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Answer: for $p \neq q \neq \frac{1}{2}; 0$: $h_k^L = \dfrac{-1 + \left(\frac{q}{p}\right)^{k-1}}{-1 + \left(\frac{q}{p}\right)^{L-1}}$; $k = 1 \ldots L$

for $p = q = \frac{1}{2}$: $h_k^L = \frac{k-1}{L-1}$; $k = 1 \ldots L$

for $p = 0; q = 1$: $h_k^L = \begin{cases} 0, & k = 1 \ldots L-1 \\ 1; & k = L \end{cases}$

for $p = 1; q = 0$: $h_k^L = \begin{cases} 0, & k = 1 \\ 1, & k = 2 \ldots L \end{cases}$

5/5

c) Simulate $N = 500$ realizations of a chain with closed boundary conditions of length $T = 100$ and plot X_{10} and X_{100} as a histogram.

We see that the histogram of X_{10} is not close to the theoretical stationary distribution at all, but X_{100} is very close to it. It is ok, because the chain needs time to converge to the limit distribution (and in our case the limit distribution is the found stationary distribution)

d) Simulate one chain of length 500 (I take the first row of $N \times T$ simulations from c) and plot the histogram fraction of time spent in each state as a histogram. 15/15
after $t = 50$ and 500 steps. We see that $t = 50$ gives the distribution not close to the stationary distribution, but $t = 500$ gives a close variant. It is that we expect because due to the ergodic theorem, for ~~a chain with~~ an ergodic chain with unique stationary distribution, we have $\frac{1}{N}\sum_{n=1}^{N} f(X_n) \xrightarrow[N\to\infty]{} E_\pi(f)$, and take $f(X_n) = \mathbb{1}\{X_n = k\}$, $k = 1 \ldots L$

(2.) $G = \begin{pmatrix} -2 & 1 & 1 \\ 1 & -4 & 3 \\ 0 & 1 & -1 \end{pmatrix}$

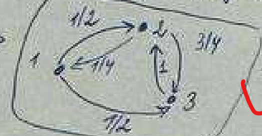
a) Draw a graph representation for the chain (connect the three states by their jump rates), and give the transition matrix P^Y of the corresponding jump chain $(Y_n : n \in \mathbb{N}_0)$

The jump chain: $Y_n = X_{J_n}$, where $J_0 = 0$; $J_{n+1} = \inf\{t > J_n : X_t \neq X_{J_n}\}$

And from lecture 6, $p^Y(x;y) = \begin{cases} \begin{cases} 0, & x = y \\ \frac{g(x,y)}{|g(x,x)|}, & x \neq y \end{cases}, & \text{if } g(x,x) \neq 0. \\ \delta_{x,y}, & \text{if } g(x,x) = 0. \end{cases}$

– and our G has $g(x,x) \neq 0$

$\Rightarrow P^Y = \begin{pmatrix} 0 & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{4} & 0 & \frac{3}{4} \\ 0 & 1 & 0 \end{pmatrix} \Rightarrow$



← the graph for the chain

5/5

b) • Consider the Taylor series of the matrix P_t and confirm that $\frac{d}{dt}P_t\big|_{t=0} = G$, $\frac{d^2}{dt^2}P_t\big|_{t=0} = G^2$.

We know that $P_t = \exp(tG) = \sum_{k=0}^{\infty} \frac{t^k}{k!} G^k$.

$\Rightarrow \frac{d}{dt}(P_t)\big|_{t=0} = \frac{d}{dt}\left(\sum_{k=0}^{\infty} \frac{t^k}{k!} G^k\right)\Big|_{t=0} = \sum_{k=0}^{\infty} k \cdot \frac{t^{k-1}}{k!} \cdot G^k \Big|_{t=0} = G$, because all other components, except $k=1$, go to zero with $t=0$.

$\frac{d^n}{dt^n}(P_t)\big|_{t=0} = \frac{d^n}{dt^n}\left(\sum_{k=0}^{\infty} \frac{t^k}{k!} G^k\right)\Big|_{t=0} = \sum_{k=0}^{\infty} \frac{n! \, t^{k-n}}{k!} G^k \Big|_{t=0} = G^n$, because all powers with $k<n$ will be eaten by differentiation, and all powers with $k>n$ will turn to zero with $t=0$.

• Assume that $G = Q^{-1}\Lambda Q$, show that $P(t) = \exp(tG) = Q^{-1} \cdot \begin{pmatrix} 1 & 0 & 0 \\ 0 & e^{\lambda_2 t} & 0 \\ 0 & 0 & e^{\lambda_3 t} \end{pmatrix} Q$, where $\Lambda = \begin{pmatrix} 0 & & \\ & \lambda_2 & \\ & & \lambda_3 \end{pmatrix}$

6/6

This is because

If $G = Q^{-1}\Lambda Q$, then $P_t = \exp(tG) = \sum_{k=0}^{\infty} \frac{t^k}{k!} G^k = \sum_{k=0}^{\infty} \frac{t^k}{k!} \cdot (Q^{-1}\Lambda Q)^k =$

$= \sum_{k=0}^{\infty} \frac{t^k}{k!} \cdot \underbrace{(Q^{-1}\Lambda \cancel{Q} \, \cancel{Q^{-1}} \Lambda Q \ldots Q^{-1}\Lambda Q)}_{k \text{ times}} = \sum_{k=0}^{\infty} \frac{t^k}{k!} \cdot Q^{-1} \cdot \Lambda^k \cdot Q = Q^{-1}\left(\sum_{k=0}^{\infty} \frac{t^k}{k!} \Lambda^k\right) Q =$

diagonal matrix commutes with Q and Q^{-1}

$= Q^{-1} \cdot \sum_{k=0}^{\infty} \frac{t^k}{k!} \cdot \begin{pmatrix} \lambda_1^k & 0 & 0 \\ 0 & \lambda_2^k & 0 \\ 0 & 0 & \lambda_3^k \end{pmatrix} Q = Q^{-1} \cdot \begin{pmatrix} \sum_{k=0}^{\infty} \frac{t^k \lambda_1^k}{k!} & 0 & 0 \\ 0 & \sum_{k=0}^{\infty} \frac{t^k \lambda_2^k}{k!} & 0 \\ 0 & 0 & \sum_{k=0}^{\infty} \frac{t^k \lambda_3^k}{k!} \end{pmatrix} Q = Q^{-1} \cdot \begin{pmatrix} e^{t\lambda_1} & 0 & 0 \\ 0 & e^{t\lambda_2} & 0 \\ 0 & 0 & e^{t\lambda_3} \end{pmatrix} Q$

And because the sum of each row of G is zero, that is, $\langle 0 | G = \langle 0 |$, zero is the eigenvalue $\Rightarrow \lambda_1 = 0 \Rightarrow e^{t\lambda_1} = 1 \Rightarrow$ $\boxed{P_t = e^{tG} = Q^{-1} \begin{pmatrix} 1 & 0 & 0 \\ 0 & e^{t\lambda_2} & 0 \\ 0 & 0 & e^{t\lambda_3} \end{pmatrix} Q}$

7/7

c) Compute λ_2 and λ_3. Use this to compute $p_{11}(t)$.

$G = \begin{pmatrix} -2 & 1 & 1 \\ 1 & -4 & 3 \\ 0 & 1 & -1 \end{pmatrix}$ ← let's find the eigenvalues

$G - \lambda E = \begin{pmatrix} -2-\lambda & 1 & 1 \\ 1 & -4-\lambda & 3 \\ 0 & 1 & -1-\lambda \end{pmatrix}$

$\Rightarrow \det(G - \lambda E) = -(2+\lambda)\left((4+\lambda)(1+\lambda) - 3\right) - 1 \cdot (-1-\lambda) + 1 \cdot (1-0) = -(2+\lambda)(4 + 5\lambda + \lambda^2 - 3) + 1 + \lambda + 1 =$

$= -(2+\lambda)(\lambda^2 + 5\lambda + 1) + \lambda + 2 = (\lambda+2)(1 - \lambda^2 - 5\lambda - 1) = -(\lambda+2)(\lambda^2 + 5\lambda) = -\lambda(\lambda+2)(\lambda+5) \Rightarrow$ $\lambda_1 = 0$, $\lambda_2 = -2$, $\lambda_3 = -5$

then, because $P_t = Q^{-1}\begin{pmatrix} 1 & 0 & 0 \\ 0 & e^{\lambda_2 t} & 0 \\ 0 & 0 & e^{\lambda_3 t}\end{pmatrix} Q$, we see that $p_{11}(t) = a + be^{-2t} + ce^{-5t}$, for some a, b, c.

~~Bcc~~ And we need 3 equations, to find a, b, c.

These equations are
$$\begin{cases} P(0) = E \Rightarrow p_{11}(0) = 1 \\ \frac{d}{dt}P_t\Big|_{t=0} = G \Rightarrow \frac{d}{dt}p_{11}(t)\Big|_{t=0} = -2 \\ \frac{d^2}{dt^2}P_t\Big|_{t=0} = G^2 \Rightarrow \frac{d^2}{dt^2}p_{11}(t)\Big|_{t=0} = (-2; 1; 1)\begin{pmatrix}-2\\1\\0\end{pmatrix} = 4+1 = 5 \end{cases}$$

$$\Rightarrow \begin{cases} a+b+c = 1 \\ -2b-5c = -2 \\ 4b+25c = 5 \end{cases} \Rightarrow \left(\begin{array}{ccc|c} 1 & 1 & 1 & 1 \\ 0 & -2 & -5 & -2 \\ 0 & 4 & 25 & 5 \end{array}\right) \sim \left(\begin{array}{ccc|c} 1 & 1 & 1 & 1 \\ 0 & -2 & -5 & -2 \\ 0 & 0 & 15 & 1 \end{array}\right) \Rightarrow \begin{cases} c = \frac{1}{15} \\ b = \frac{2-5c}{2} = \frac{2-\frac{1}{3}}{2} = \frac{5}{6} \\ a = 1-b-c = 1-\frac{5}{6}-\frac{1}{15} = \frac{1}{10} \end{cases}$$

$$\Rightarrow \boxed{p_{11}(t) = \frac{1}{10} + \frac{5}{6}\cdot e^{-2t} + \frac{1}{15}e^{-5t}}$$

7/7

d) What is the stationary distribution π of X?

The stationary distribution: $\langle\pi| P = \langle\pi|$, or, equivalently, $\langle\pi| G = \langle 0|$

$$\Rightarrow (\pi_1\ \pi_2\ \pi_3)\begin{pmatrix} -2 & 1 & 1 \\ 1 & -4 & 3 \\ 0 & 1 & -1 \end{pmatrix} = (0\ 0\ 0)$$

$$\Rightarrow \begin{cases} -2\pi_1 + \pi_2 = 0 \\ \pi_1 - 4\pi_2 + \pi_3 = 0 \\ \pi_1 + 3\pi_2 - \pi_3 = 0 \end{cases} \Rightarrow \left(\begin{array}{ccc|c} -2 & 1 & 0 & 0 \\ 1 & -4 & 1 & 0 \\ 1 & 3 & -1 & 0 \end{array}\right) \sim \left(\begin{array}{ccc|c} 1 & 3 & -1 & 0 \\ 1 & -4 & 1 & 0 \\ -2 & 1 & 0 & 0 \end{array}\right) \sim \left(\begin{array}{ccc|c} 1 & 3 & -1 & 0 \\ 0 & -7 & 2 & 0 \\ 0 & 7 & -2 & 0 \end{array}\right) \Rightarrow \begin{cases} \pi_3 - \text{arbitrary} = a \\ \pi_2 = \frac{2}{7}\pi_3 \\ \pi_1 = \pi_3 - 3\pi_2 = \pi_3 - \frac{6}{7}\pi_3 = \frac{\pi_3}{7} \end{cases}$$

$$\Rightarrow (\pi_1, \pi_2, \pi_3) = \left(\frac{\pi_3}{7}; \frac{2\pi_3}{7}; \pi_3\right) = \left(\frac{a}{7}; \frac{2}{7}a; a\right)$$

but $\pi_1 + \pi_2 + \pi_3 = 1 \Rightarrow \frac{10}{7}a = 1 \Rightarrow a = \frac{7}{10} \Rightarrow \boxed{\pi = \left(\frac{1}{10}; \frac{2}{10}; \frac{7}{10}\right)}$ - stationary distribution

5/5

Q2 - 30/30

__Note:__ We can find P_t explicitly and look at it.

$$G = \begin{pmatrix} -2 & 1 & 1 \\ 1 & -4 & 3 \\ 0 & 1 & -1 \end{pmatrix} \qquad \begin{matrix} \lambda_1 = 0 \\ \lambda_2 = -2 \\ \lambda_3 = -5. \end{matrix}$$

$$\lambda = 0 \Rightarrow \begin{pmatrix} -2 & 1 & 1 \\ 1 & -4 & 3 \\ 0 & 1 & -1 \end{pmatrix} \sim \begin{pmatrix} 1 & -4 & 3 \\ 0 & -7 & 7 \\ 0 & 1 & -1 \end{pmatrix} \Rightarrow \begin{matrix} b = c \\ a = 4b - 3c = c \end{matrix} \Rightarrow e_1 = (c; c; c) \sim (1; 1; 1)$$

$$\lambda = -2 \Rightarrow \begin{pmatrix} 0 & 1 & 1 \\ 1 & -2 & 3 \\ 0 & 1 & 1 \end{pmatrix} \sim \begin{pmatrix} 1 & -2 & 3 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{pmatrix} \Rightarrow \begin{matrix} b = -c \\ a = 2b - 3c = -5c \end{matrix} \Rightarrow e_2 = (-5c; -c; c) \sim (-5; -1; 1)$$

$$\lambda = -5 \Rightarrow \begin{pmatrix} 3 & 1 & 1 \\ 1 & 1 & 3 \\ 0 & 1 & 4 \end{pmatrix} \sim \begin{pmatrix} 1 & 1 & 3 \\ 0 & -2 & -8 \\ 0 & 1 & 4 \end{pmatrix} \Rightarrow \begin{matrix} b = -4c \\ a = -b - 3c = c \end{matrix} \Rightarrow e_3 = (c; -4c; c) \sim (1; -4; 1)$$

no need

$$\Rightarrow P(t) = e^{tG} = c_1 e^{0t}\begin{pmatrix}1\\1\\1\end{pmatrix} + c_2 \cdot e^{-2t}\begin{pmatrix}-5\\-1\\1\end{pmatrix} + c_3 \cdot e^{-5t}\begin{pmatrix}1\\-4\\1\end{pmatrix}$$

And now we need to find three triples of $(c_1; c_2; c_3)$, that are needed to obtain $P_0 = \begin{pmatrix}1\\0\\0\end{pmatrix}$; $P_0 = \begin{pmatrix}0\\1\\0\end{pmatrix}$; $P_0 = \begin{pmatrix}0\\0\\1\end{pmatrix}$ – the columns of $P(0) = E$.

Seen.

1) $P(0) = \begin{pmatrix}1\\0\\0\end{pmatrix}$: $\begin{cases} c_1 - 5c_2 + c_3 = 1 \\ c_1 - c_2 - 4c_3 = 0 \\ c_1 + c_2 + c_3 = 0 \end{cases} \Rightarrow \left(\begin{array}{ccc|c} 1 & -5 & 1 & 1 \\ 1 & -1 & -4 & 0 \\ 1 & 1 & 1 & 0 \end{array}\right) \sim \left(\begin{array}{ccc|c} 1 & -5 & 1 & 1 \\ 0 & 4 & -5 & -1 \\ 0 & 6 & 0 & -1 \end{array}\right) \sim \left(\begin{array}{ccc|c} 1 & -5 & 1 & 1 \\ 0 & 1 & 0 & -1/6 \\ 0 & 0 & -5 & -\frac{1}{3} \end{array}\right)$

$$\Rightarrow \begin{cases} c_3 = \frac{1}{15} \\ c_2 = -\frac{1}{6} \\ c_1 = 1 + 5c_2 - c_3 = 1 - \frac{5}{6} - \frac{1}{15} = \frac{3}{30} = \frac{1}{10} \end{cases} \Rightarrow \underbrace{\begin{pmatrix} p_{11}(t) \\ p_{21}(t) \\ p_{31}(t) \end{pmatrix}}_{\text{first column of } P_t} = \frac{1}{10}\begin{pmatrix}1\\1\\1\end{pmatrix} - \frac{1}{6}e^{-2t}\begin{pmatrix}-5\\-1\\1\end{pmatrix} + \frac{1}{15}e^{-5t}\begin{pmatrix}1\\-4\\1\end{pmatrix}$$

2) $P(0) = \begin{pmatrix}0\\1\\0\end{pmatrix}$: $\left(\begin{array}{ccc|c} 1 & -5 & 1 & 0 \\ 1 & -1 & -4 & 1 \\ 1 & 1 & 1 & 0 \end{array}\right) \sim \left(\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 0 & -6 & 0 & 0 \\ 0 & -2 & -5 & 1 \end{array}\right) \sim \left(\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -5 & 1 \end{array}\right) \Rightarrow \begin{cases} c_3 = -1/5 \\ c_2 = 0 \\ c_1 = -c_2 - c_3 = \frac{1}{5} \end{cases} \Rightarrow \begin{pmatrix} p_{12}(t) \\ p_{22}(t) \\ p_{32}(t) \end{pmatrix} = \frac{1}{5}\begin{pmatrix}1\\1\\1\end{pmatrix} - \frac{1}{5}e^{-5t}\begin{pmatrix}1\\-4\\1\end{pmatrix}$

3) $P(0) = \begin{pmatrix}0\\0\\1\end{pmatrix}$ $\left(\begin{array}{ccc|c} 1 & -5 & 1 & 0 \\ 1 & -1 & -4 & 0 \\ 1 & 1 & 1 & 1 \end{array}\right) \sim \left(\begin{array}{ccc|c} 1 & -5 & 1 & 0 \\ 0 & 4 & -5 & 0 \\ 0 & 6 & 0 & 1 \end{array}\right) \Rightarrow \begin{cases} c_2 = 1/6 \\ c_3 = \frac{4}{5}c_2 = \frac{4}{5}\cdot\frac{1}{6} = \frac{2}{15} \\ c_1 = 5c_2 - c_3 = \frac{5}{6} - \frac{2}{15} = \frac{21}{30} = \frac{7}{10} \end{cases} \Rightarrow \begin{pmatrix} p_{13}(t) \\ p_{23}(t) \\ p_{33}(t) \end{pmatrix} = \frac{7}{10}\begin{pmatrix}1\\1\\1\end{pmatrix} + \frac{1}{6}e^{-2t}\begin{pmatrix}-5\\-1\\1\end{pmatrix} + \frac{2}{15}e^{-5t}\begin{pmatrix}1\\-4\\1\end{pmatrix}$

$\Rightarrow$ with $t \to \infty$: $P_t \to \begin{pmatrix} \frac{1}{10} & \frac{1}{5} & \frac{7}{10} \\ \frac{1}{10} & \frac{1}{5} & \frac{7}{10} \\ \frac{1}{10} & \frac{1}{5} & \frac{7}{10} \end{pmatrix}$ – and this is our stationary distribution from d) so we checked that it is correct.

---

3 a) $\underline{X}(n)$ – is the state of the contents as a function of discrete time n.

Give the state space S, the initial condition $\underline{X}(0)$ and the transition probabilities.

- State space $S = \{(x_1; \dots x_k)\}$, where x_i is an integer $1, 2, 3 \dots$, showing the amount of balls of color k that are in the urn after n operations of replacement

✓ $\mathbb{N}^k$ ← but okay

- $\underline{X}(0) = (1; 1; \dots 1)$ – we have one ball of each color at $t=0$.
- From state $(x_1; x_2 \dots x_k)$ the chain can move only to
  - $(x_1+1; x_2; \dots x_k)$ with prob. $\frac{x_1}{x_1 + \dots + x_k}$
  - $(x_1; x_2+1 \dots x_k)$ with prob. $\frac{x_2}{x_1 + \dots + x_k}$
  - $(x_1; x_2; \dots x_k+1)$ with prob. $\frac{x_k}{x_1 + \dots + x_k}$

$$\Rightarrow \boxed{p(\underline{x}; \underline{y}) = \frac{x_1}{x_1 + \dots + x_k} \cdot \delta_{x_1+1, x_2 \dots x_k}(\underline{y}) + \dots + \frac{x_k}{x_1 + \dots + x_k} \cdot \delta_{x_1; x_2; \dots x_k+1}(\underline{y})}$$

✓

4/4

b) For $k=2$ sketch the state space and the transition probabilities between states and show that $\forall (x_1; x_2) \in S$: $P[\underline{X}(n) = (x_1; x_2)] = \frac{1}{n+1} \cdot \delta_{n+2}(x_1 + x_2)$

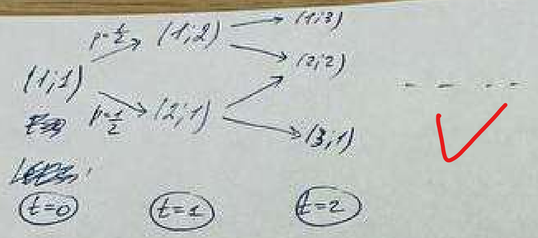
We see that after $t=0$ we have $n=2$ balls, and with each tick of time one new ~~one~~ ball is added $\Rightarrow$ after $t=n$ in the urn there will be $n+2$ balls.

Let's prove the needed equation with induction

~~[illegible]~~

The state space for $k=2$ is $\{(x_1; x_2)\}$, and the transition probabilities

$$p((x_1; x_2); (y_1; y_2)) = \frac{x_1}{x_1 + x_2} \delta_{x_1+1; x_2}((y_1; y_2)) + \frac{x_2}{x_1 + x_2} \cdot \delta_{x_1; x_2+1}((y_1; y_2))$$



2/2

Let's prove the induction base

$P(\underline{X}(0) = (x_1;x_2)) \overset{?}{=} \frac{1}{0+1} \cdot \delta_2(x_1+x_2)$ — yes, it is true, because $\underline{X}(0)=(1;1)$

Then let's assume that we ~~[illegible]~~ already know that

$P(\underline{X}(n) = (x_1;x_2)) = \frac{1}{n+1} \cdot \delta_{n+1;x_1+x_2}$, 2 not 1 and let's prove that $P(\underline{X}(n+1) = (y_1;y_2)) = \frac{1}{n+2} \cdot \delta_{n+2;y_1+y_2}$ 3, not 2

We know from understanding the process, that $(y_1;y_2)$ after $t=n+1$ could have emerged ~~[illegible]~~ only from $(y_1-1;y_2)$ or $(y_1;y_2-1)$. So let's condition $P(\underline{X} = (y_1;y_2))$ on the value of $\underline{X}(n)$ (use the law of total probability)

$\Rightarrow P(\underline{X}(n+1) = (y_1;y_2)) = \sum_{s \in S} P(\underline{X}(n+1) = (y_1;y_2) \mid \underline{X}(n) = s) \cdot P(\underline{X}(n) = s) =$

$= P(\underline{X}(n+1) = (y_1;y_2) \mid \underline{X}(n) = (y_1-1;y_2)) \cdot \underbrace{P(\underline{X}(n) = (y_1-1;y_2))}_{\text{we know from presumption of induction}} + P(\underline{X}(n+1) = (y_1;y_2) \mid \underline{X}(n) = (y_1;y_2-1)) \, P(\underline{X}(n) = (y_1;y_2-1))$

$= \frac{y_1-1}{y_1+y_2-1} \cdot \frac{1}{n+1} + \frac{y_2-1}{y_1+y_2-1} \cdot \frac{1}{n+1} = \frac{1}{n+1} \cdot \frac{y_1+y_2-2}{y_1+y_2-1} \underset{y_1+y_2=n+3}{\uparrow}{=} \frac{n+1}{n+1 \; n+2} \cdot \delta \quad = \frac{1}{n+2} \delta_{n+3;y_1+y_2}$

correct here

$\Rightarrow$ the step of induction is proved.

- Show that $\frac{1}{n+2} \underline{X}(n) \xrightarrow[n\to\infty]{} (U;1-U)$ where $U \sim U[0;1]$

7/7

Let's denote $\xi := \frac{\underline{X}(n)}{n+2}$ and prove that $P(\xi \le x) \to x$ as $n \to \infty$

From 3b we know that $\underline{X}(n) = \begin{cases} (1;n+1) \\ (2;n) \\ (3;n-1) \\ \\ (n+1;1) \end{cases}$, each variant with probability $\frac{1}{n+1}$

$\Rightarrow$ for each $x \in [0;1]$ (let's take $x := \frac{k}{n+2}$; $k = 0 \dots n+2$)

$P(\xi^{(1)} \le x) = P\left(\frac{\underline{X}(n)^{(1)}}{n+2} \le x\right) = P\left(\frac{\underline{X}(n)^{(1)}}{n+2} \le \frac{k}{n+2}\right) = P(\underline{X}(n)^{(1)} \le k) = \sum_{i=1}^{k} P(\underline{X}(n) = i) = \frac{k}{n+1} = \frac{n+2}{n+1} \cdot x \xrightarrow[n\to\infty]{} x$

$\Rightarrow P(\xi \le x) \to x$ as $n \to \infty$. (due to the fact that x are dense in $[0;1]$)

Analoguously, for the second component of ξ, $\xi^{(2)}$:

$P(\xi^{(2)} \le x) = P\left(\frac{\underline{X}(n)^{(2)}}{n+2} \le x\right) = P(\underline{X}(n)^{(2)} \le k) = P(\underline{X}(n)^{(1)} \ge n+2-k) = 1 - P(\underline{X}(n)^{(1)} \le n+2-k) =$

$= 1 - P(\xi^{(1)} \le 1-x) = P(\xi^{(1)} \ge 1-x)$

3/3

$\Rightarrow \xi^{(1)} \sim U[0;1]$ and $\xi^{(2)} = 1 - \xi^{(1)}$

6

c) We see that graphs in column 1 have x-range ending at $n+k$, while the graphs in the second column have x-range $\in [0,1]$, because $\frac{X(n)}{n+k}$ has the fractions of balls of colour i on the i-th position, and these fractions sum up to 1.

okay

We also see that when $\gamma \geq 1$ (e.g. $\gamma = 1.5$), the balls quickly become of one color, that is the system expresses monopoly. It is why we see the graph of almost a delta-function in the right bottom corner. ✓ What about small γ? 7/8

d) The interesting thing is how the shape of graphs change when we change $f(i)$. We can change the behaviour of graph in the right upper corner by putting $f(i) = i^3$ (it means that the bigger the number, the more fit is the ball). But ~~[illegible]~~ if we put $f(i) = e^i$ – then even the right upper graph will be almost the delta-function – it means that for such a strong fit-function $f(i) = e^i$, even for gamma $= 0$ we have monopoly quickly. But also if we put $f(i) = \log i$, then we also will have the upper right graph almost as a delta-function.

nice choice and comments 6/6

Q3 – 29/30

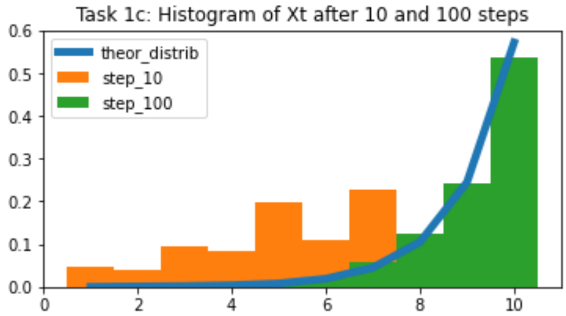
total – 100/100

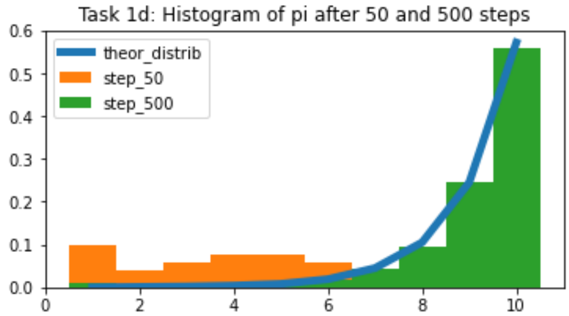
In [2]:
```
import numpy as np
import matplotlib.pyplot as plt
#np.set_printoptions(precision=3,suppress=True)
np.set_printoptions(formatter={'float': lambda x: "{0:0.3f}".format
```

In [3]:
```
# task1: c and d
#closed
L=10
p=0.7
q=1-p
T=500
N=500
otv=np.ones(N*(T+1)).reshape(N,T+1)
for n in range(N):
    Xt=1
    for t in range(1,T+1):
        r=np.random.rand()
        if r<=p: #go right
            Xt=min(Xt+1,10)
        else: #go left
            Xt=max(Xt-1,1)
        otv[n][t]=Xt



print("After step 10 :", np.histogram(otv[:,10],density=True,bins=[
print("After step 100:", np.histogram(otv[:,100],density=True,bins=
lambd=p/q
c=(1-lambd)/(lambd*(1-lambd**10))
theor_distrib=list(map(lambda x: float("{0:0.3f}".format(x)), [c*la
print("Theoretical stationary distribution is:", theor_distrib)

fig = plt.figure(figsize=(13,3))
ax = fig.subplots(nrows=1, ncols=2)

ax[0].plot(range(1,L+1), theor_distrib,label='theor_distrib', linew
ax[0].hist(otv[:,10],density=True,label='step_10',bins=[0.5+i for i
ax[0].hist(otv[:,100],density=True,label='step_100',bins=[0.5+i for
ax[0].set_title('Task 1c: Histogram of Xt after 10 and 100 steps')
ax[0].legend()

ax[1].plot(range(1,L+1), theor_distrib,label='theor_distrib', linew
ax[1].hist(otv[0,0:51],density=True,label='step_50',bins=[0.5+i for
ax[1].hist(otv[0,0:501],density=True,label='step_500',bins=[0.5+i f
ax[1].set_title('Task 1d: Histogram of pi after 50 and 500 steps')
ax[1].legend()

plt.show()
```

```
After step 10 : [0.048 0.038 0.094 0.084 0.198 0.108 0.228 0.060
0.118 0.024]
```

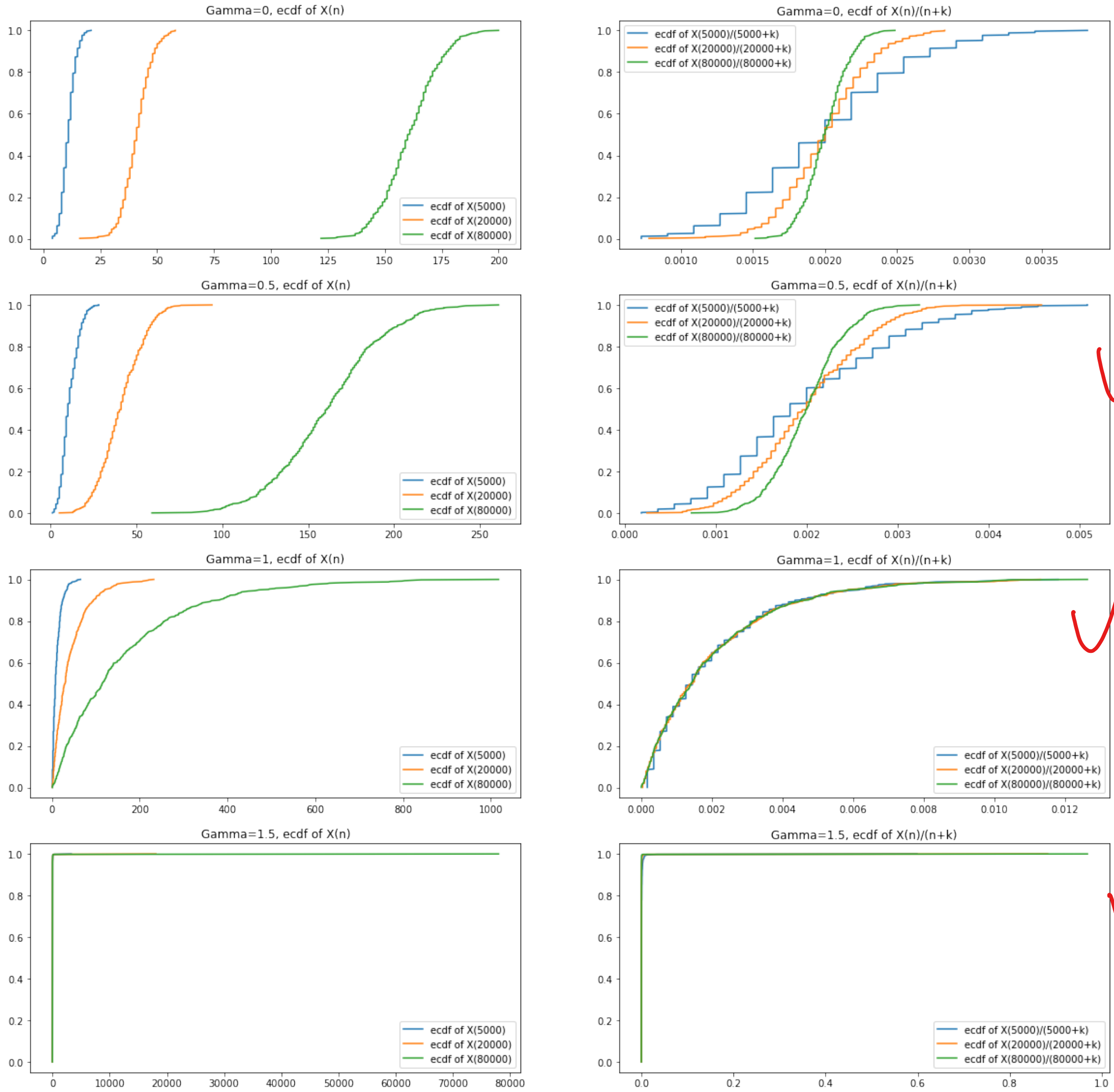
```
After step 100: [0.000 0.002 0.002 0.002 0.010 0.022 0.058 0.124
0.242 0.538]
Theoretical stationary distribution is: [0.0, 0.001, 0.002, 0.004,
0.008, 0.019, 0.045, 0.105, 0.245, 0.572]
```





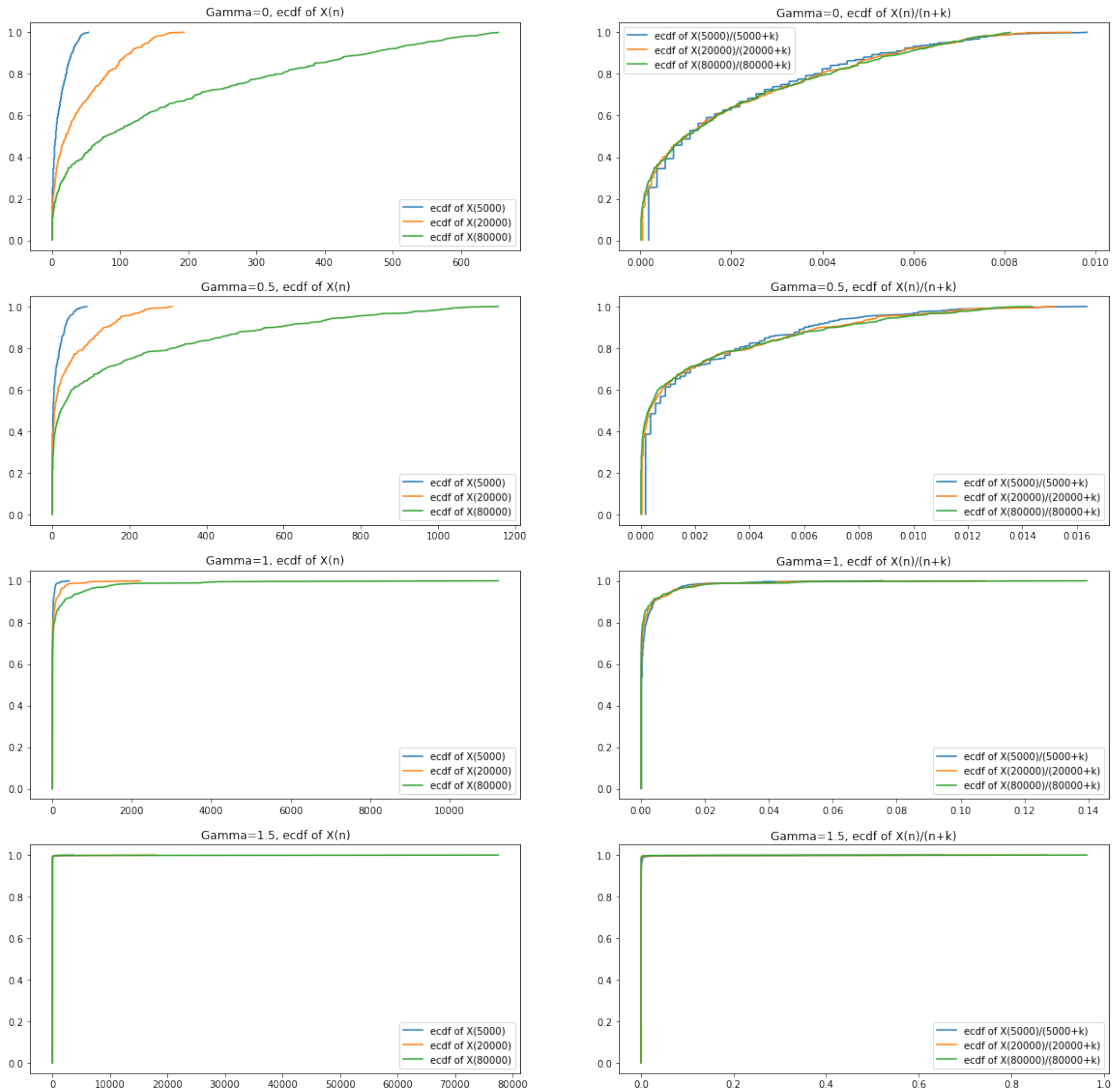
In [91]:
```
#3
from statsmodels.distributions.empirical_distribution import ECDF

k=500
def f(i):
    return 1

def get_p(mas_x,gamma):
    otv=np.array([f(i+1)*xi**gamma for i,xi in enumerate(mas_x)])
    return otv/sum(otv)

gammas=[0,0.5,1,1.5]
T=80000

fig,ax=plt.subplots(4,2, figsize=(20,20))

for i,gamma in enumerate(gammas):
    otv=np.ones(k*(T+1)).reshape(k,T+1)
    for t in range(1,T+1):
        tek_mas_x=otv[:,t-1]
        mas_p=get_p(tek_mas_x,gamma)
        tek_ball_number=np.random.choice(a=np.arange(1,k+1),size=1,
        #print("gamma=",gamma, tek_ball_number,tek_mas_x,mas_p)
        otv[:,t]=otv[:,t-1]
        otv[tek_ball_number-1,t]+=1
    #print("last state=",otv[:,T])

    ax[i,0].plot(ECDF(otv[:,5000]).x, ECDF(otv[:,5000]).y,label='ec
    ax[i,0].plot(ECDF(otv[:,20000]).x, ECDF(otv[:,20000]).y,label='
    ax[i,0].plot(ECDF(otv[:,80000]).x, ECDF(otv[:,80000]).y,label='
    ax[i,0].legend()
    ax[i,0].set_title("Gamma={}, ecdf of X(n)".format(gamma))

    ax[i,1].plot(ECDF(otv[:,5000]/(5000+k)).x, ECDF(otv[:,5000]/(50
    ax[i,1].plot(ECDF(otv[:,20000]/(20000+k)).x, ECDF(otv[:,20000]/
    ax[i,1].plot(ECDF(otv[:,80000]/(80000+k)).x, ECDF(otv[:,80000]/
    ax[i,1].legend()
    ax[i,1].set_title("Gamma={}, ecdf of X(n)/(n+k)".format(gamma))
```

```
plt.show()
```



In [97]:
```
#3
from statsmodels.distributions.empirical_distribution import ECDF

k=500
def f(i):
    return i**3

def get_p(mas_x,gamma):
    otv=np.array([f(i+1)*xi**gamma for i,xi in enumerate(mas_x)])
    return otv/sum(otv)

gammas=[0,0.5,1,1.5]
T=80000

fig,ax=plt.subplots(4,2, figsize=(20,20))

for i,gamma in enumerate(gammas):
    otv=np.ones(k*(T+1)).reshape(k,T+1)
```

```
    for t in range(1,T+1):
        tek_mas_x=otv[:,t-1]
        mas_p=get_p(tek_mas_x,gamma)
        tek_ball_number=np.random.choice(a=np.arange(1,k+1),size=1,
        #print("gamma=",gamma, tek_ball_number,tek_mas_x,mas_p)
        otv[:,t]=otv[:,t-1]
        otv[tek_ball_number-1,t]+=1
    #print("last state=",otv[:,T])

    ax[i,0].plot(ECDF(otv[:,5000]).x, ECDF(otv[:,5000]).y,label='ec
    ax[i,0].plot(ECDF(otv[:,20000]).x, ECDF(otv[:,20000]).y,label='
    ax[i,0].plot(ECDF(otv[:,80000]).x, ECDF(otv[:,80000]).y,label='
    ax[i,0].legend()
    ax[i,0].set_title("Gamma={}, ecdf of X(n)".format(gamma))

    ax[i,1].plot(ECDF(otv[:,5000]/(5000+k)).x, ECDF(otv[:,5000]/(50
    ax[i,1].plot(ECDF(otv[:,20000]/(20000+k)).x, ECDF(otv[:,20000]/
    ax[i,1].plot(ECDF(otv[:,80000]/(80000+k)).x, ECDF(otv[:,80000]/
    ax[i,1].legend()
    ax[i,1].set_title("Gamma={}, ecdf of X(n)/(n+k)".format(gamma))
plt.show()
```



In [ ]: