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Brownian Motion

Problem sheet 7 (valid as assignment)

1. **Martingales** [24]
   Let $\big(B(t) : t \geq 0\big)$ be a Brownian Motion. Show that the following are martingales:
   $$\big(B(t)^2 - t : t \geq 0\big) \;, \quad \big(B(t)^3 - 3tB(t) : t \geq 0\big) \quad \text{and} \quad \big(B(t)^4 - 6tB(t)^2 + 3t^2 : t \geq 0\big) \;.$$

2. **Laplacian in generalised polar coordinates** [14]
   Let $u\colon \mathbb{R}^d \to \mathbb{R}$ be a smooth *radial* function of the form $u(x) = \widetilde{u}(r)$ with $r = |x|$.

   (a) Show that for $|x| > 0$
   $$\Delta u(x) = \frac{1}{r^{d-1}} \partial_r \left(r^{d-1} \partial_r \widetilde{u} \right) (|x|) \;.$$
   Hint: use the identity $\partial_{x_i} |x| = \frac{x_i}{|x|}$ and similar formulas.

   (b) Use the formula above to prove Stokes' theorem on the disc for radial functions:
   $$\int_{B_r(0)} \Delta u(x) \,\mathrm{d}x = \int_{\partial B_r(0)} \partial_{\mathbf{n}} u(z) \,\mathrm{d}\sigma(z) \;.$$
   Hint: Use that $\partial_{\mathbf{n}} u(x) = \partial_r \widetilde{u}(|x|)$.

3. **Conformal invariance** [26]
   Let $\mathbf{C}_t = B^1_t + \iota B^2_t$ be a Brownian motion on the complex plane (here B^1 and B^2 are independent 1D BMs: note that this is equivalent to a 2D Brownian motion).

   (a) If $D \subseteq \mathbf{C}$ is a domain and $f\colon D \to \mathbf{C}$ is analytic, prove that $f(\mathbf{C}_t) = \mathbf{D}_t$ is a martingale.
   Hint: Use the Cauchy–Riemann equations
   $$\partial_x u = \partial_y v \;, \qquad \partial_y u = -\partial_x v \;,$$
   for $u = \mathfrak{Re}(f)$ and $v = \mathfrak{Im}(f)$ the real and imaginary parts of f (together with the parametrisation $z = x + \iota y$ of points in $\mathbf{C}$).

   (b) If $f\colon \mathbf{C} \to \mathbb{R}$ is harmonic (meaning $(\partial_x^2 + \partial_y^2) f = 0$), then
   $$f(z_0) = \int_0^{2\pi} f(z_0 + \varrho e^{\iota \vartheta}) \,\mathrm{d}\vartheta \;, \qquad \forall z_0 \in \mathbf{C};, \varrho > 0 \;.$$
   Hint: use that the law of Brownian motion started at z_0 and stopped at $\partial B_\varrho(z_0)$ is the uniform distribution on the boundary $\partial B_\varrho(z_0)$ (this follows from the rotational symmetry of Brownian motion).

4. **Ornstein–Uhlenbeck** [26]

Recall the transition density of the Ornstein–Uhlenbeck process $\big(X(t) : t \geq 0\big)$:

$$q(t,x,y) = \frac{1}{\sqrt{2\pi(1-e^{-2ct})}} \exp\left(\frac{-(y-xe^{-ct})^2}{2(1-e^{-2ct})}\right) \quad \text{with } c > 0 \ .$$

(a) Consider the limit $\overline{q}(y) = \lim_{t\to\infty} q(t,x,y)$ and show that it does not depend on x. Identify $y \mapsto a(y)$ (*Hint*: it's a polynomial) and $D > 0$ such that

$$0 = -\frac{\partial}{\partial y}\Big(a(y)\overline{q}(y)\Big) + \frac{\partial^2}{\partial y^2}\Big(D \cdot \overline{q}(y)\Big) \ .$$

(b) Derive the generator $\mathcal{L}$ of the Ornstein–Uhlenbeck process, using Dynkin's formula

$$\frac{d}{dt}\mathbb{E}_x\big[f(X(t)\big] = \mathbb{E}_x\big[\mathcal{L}f(X(t))\big] \quad \text{for all } f \in C_0^2(\mathbb{R},\mathbb{R}) \text{ with compact support} \ .$$

5. **Poisson problem** [10]

Suppose $D \subseteq \mathbb{R}^d$ is a bounded open region and $\psi : \overline{D} \to \mathbb{R}$ is continuous. Suppose that $u \in \mathcal{C}^2(D) \cap C(\overline{D})$ solves the Poisson problem

$$\Delta u = \psi \text{ on } D \quad \text{and} \quad u = 0 \text{ on } \partial D \ .$$

Show that, for a Brownian motion $\big(B(t) : t \geq 0\big)$ on $\mathbb{R}^d$ started at x,

$$u(x) = -\frac{1}{2}\mathbb{E}_x\left[\int_0^T \psi\big(B(s)\big)\, ds\right] \ , \quad \text{where } T = \inf\{t : B_t \in \partial D\} \ .$$