Stochastic Modelling and Random Processes

Example sheet 4

1. **Sampling from Gaussian processes**
One of the main things we need to do for continuous time, continuous state space Markov processes (especially for diffusions) is to be able to simulate Gaussian processes - we will see that if we can do that, we can simulate these very efficiently. We will mostly work with 1D problems in this module, but we will see that it is worth being able to do multivariate random variables too.

   (a) Sampling a 2D multivariate Gaussian process (So that we can visualise things. Higher dimensional problems are similar). Suppose we have a Gaussian random vector $\mathbf{X}_t = (X_1(t), X_2(t)) \sim \mathcal{N}(\mu, \Sigma)$ where

   $$\mu = (\mu_1, \mu_2), \qquad \Sigma = \begin{bmatrix} \sigma_{11} & \sigma_{12} \\ \sigma_{21} & \sigma_{22} \end{bmatrix}.$$

   Draw $N = 1000$ samples from this distribution.

   - Plot their marginal histograms to see that $X_1 \sim \mathcal{N}(\mu_1, \sigma_{11})$ and $X_2 \sim \mathcal{N}(\mu_2, \sigma_{22})$ for a couple of chosen values for μ and Σ.
   - Plot the 2-dimensional histogram to see the behaviour of the vector and compare with the 2-dimensional Gaussian pdf. Try a few values of σ_{12}, σ_{21} to see the effect of the correlation on the behaviour of the distribution.

   (b) A useful property of multivariate Gaussian r.v.s is that if we condition on part of the random vector, the resulting distribution remains Gaussian. To see this, suppose that

   $$\mathbf{X} = (X_1, X_2)^\top \sim \mathcal{N}(\mu, \Sigma), \qquad \Sigma = \begin{bmatrix} \Sigma_{11} & \Sigma_{12} \\ \Sigma_{21} & \Sigma_{22} \end{bmatrix}.$$

   Then we know (see previous question) that

   $$X_1 \sim \mathcal{N}(\mu_1, \Sigma_{11}), \quad \text{and} \quad X_2 \sim \mathcal{N}(\mu_2, \Sigma_{22}).$$

   Furthermore, it can be proved that the conditional distribution of X_2 conditional on X_1 is a multivariate normal with

   $$\mathbb{E}[X_2 \,|\, X_1] = \mu_2 + \Sigma_{21}\Sigma_{11}^{-1}(X_1 - \mu_1), \quad \text{and} \quad \text{Var}(X_2 \,|\, X_1) = \Sigma_{22} - \Sigma_{21}\Sigma_{11}^{-1}\Sigma_{12}.$$

   Using these properties we can develop an efficient scheme to simulate Gaussian processes:

      i. Suppose we wish to generate X_{n+1} at time t_{n+1} given that we have already generated $X_0, \ldots, X_n$.
         A. Specify the conditional distribution of X_{n+1} given $X_0, \ldots, X_n$.
         B. Use this to construct a numerical scheme to simulate a Gaussian stochastic process.

      ii. Suppose additionally that the Gaussian process $X(t)$ is Markovian, so that in particular, you only need to know the value of $X(t_n)$ to generate $X(t_{n+1})$. Construct a scheme to iteratively sample $X(t_i)$ over a sequence of points $t_0 < t_1 < t_2 < \ldots$.

A. In the case of Brownian motion, show that the update formula can be written as:

$$X(t_{i+1}) = X(t_i) + \left(\sqrt{t_{i+1} - t_i}\right) Z, \text{ where } Z \sim \mathcal{N}(0,1)$$

(think about what this means in terms of the Euler-Maruyama scheme for solving SDEs numerically)

B. Derive a similar update formula for the stationary Ornstein-Uhlenbeck process with mean 0 and covariance $C(s,t) = \exp(\alpha|t-s|/2)$.

Use the above to simulate sample paths of a Brownian motion and of the OU process.

2. **Idea of the proof, Brownian motion as scaling limit of a jump process**
Recently we stated that a jump process with translation invariant rates $r(x,y) = q(y-x)$ with the properties

$$\int_{\mathbb{R}} q(z)z\, dz = 0, \qquad \int_{\mathbb{R}} q(z)z^2\, dz = \sigma^2 < \infty$$

converges in distribution to a Brownian motion when rescaled appropriately. Part of the proof of this proposition involved Taylor expanding the generators. Recall that we have

Generator of the jump process: $(\mathcal{L}f)(x) = \int_{\mathbb{R}} q(y-x)(f(y) - f(x))\, dy,$

Generator of the Brownian motion: $(\mathcal{L}f)(x) = \frac{\sigma^2}{2} f''(x)$.

Do the change of variables $t \to t/\epsilon^2$, $x \to \epsilon x$ and Taylor expand $f(y)$ around x to convince yourself that the (rescaled) generators should be the same.

3. **Geometric Brownian motion**
Let $(X_t : t \geq 0)$ be a Brownian motion with constant drift on $\mathbb{R}$ with generator

$$(\mathcal{L}f)(x) = \mu\, f'(x) + \frac{1}{2}\sigma^2\, f''(x)\,, \quad \mu \in \mathbb{R}, \sigma^2 > 0\,,$$

with initial condition $X_0 = 0$. As we saw in the lectures (week 5), **Geometric Brownian motion** is defined as

$$(Y_t : t \geq 0) \quad \text{with} \quad Y_t = e^{X_t}\,.$$

(a) Show that $(Y_t : t \geq 0)$ is a diffusion process on $[0, \infty)$ and compute its generator. Write down the SDE that Y_t solves. (you can repeat what we did in our lectures).

(b) Use the evolution equation of expectation values of test functions $f : \mathbb{R} \to \mathbb{R}$

$$\frac{d}{dt}\mathbb{E}\big[f(Y_t)\big] = \mathbb{E}\big[\mathcal{L}f(Y_t)\big]\,,$$

to derive ODEs for the mean $m(t) := \mathbb{E}[Y_t]$ and the variance $v(t) := \mathbb{E}[Y_t^2] - m(t)^2$. Solve these ODEs and derive expressions for the mean $m(t)$ and variance $v(t)$ of the gBM as a function of time (cf the values from lectures).
Is $(Y_t : t \geq 0)$ a Gaussian process?

(c) Under which conditions on μ and σ^2 does the process have a stationary distribution, and what is it? Does it converge to the stationary distribution with $Y_0 = 1$?

(d) For $\sigma^2 = 1$ choose $\mu = -1/2$ and two other values $\mu < -1/2$ and $\mu > -1/2$. Simulate and plot a few sample paths of the process with $Y_0 = 1$ up to time $t = 10$, by numerically integrating the corresponding SDE with time steps $\Delta t = 0.1$ and 0.01.

4. **Extra exercises if there is time**

(a) Use what you learned in 3. to simulate sample paths of the following processes:

  i. Fractional Brownian Motion (see lectures) for a few values of H to see the different behaviours. Try the limits we looked at ($H \to 0$, $H \to 1$).

  ii. Brownian bridge **(see link)**, which is a process that is "pinned" at the same value for the start and end times: e.g. if $T = 1$ you have $X_0 = X_T = 0$. It is defined as $X_t = (B_t | B_T = 0)$. For the purposes of this exercise, all you need to know is that it is a Gaussian process with mean $\mu(t) = 0$ and covariance $C(t, s) = \min(t, s) - ts$. Simulate it for $t \in [0, 1]$.

(b) Simulate a Gaussian process with mean $\mu(t)$ of your choice.

(c) Consider the Metropolis-Hastings MCMC algorithm (Week 4) but where the acceptance probability is

$$a(x, y) = \left(1 + \frac{\pi(x)q(x, y)}{\pi(y)q(y, x)}\right)^{-1}$$

Show that the scheme you obtain is reversible with respect to π.