Brownian Motion

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Contents

Chapter 1

Construction of Brownian motion

1.1 Stochastic processes

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space, and $(S, \mathcal{S})$ a measurable space. A random variable with values in S is a measurable map

$$X \colon \Omega \to S \ .$$

This course, instead of random variables, will focus on *stochastic processes.*

Definition 1.1.1. *Let $I \subseteq [0, \infty)$ be an index set. Then a stochastic process over I with values in S is a collection $\{X_t\}_{t \in I}$ of random variables. Namely, for every $t \in I$, X_t is a measurable map*

$$X_t \colon \Omega \to S \ .$$

If $I = \mathbb{N}$ we refer to $\{X_n\}_{n \in \mathbb{N}}$ as a discrete time *stochastic process. If $I = [0, \infty)$ we refer to $\{X_t\}_{t \in [0,\infty)}$ as a* continuous time *stochastic processes.*

For example, let $\{\eta_i\}_{i \in \mathbb{N}}$ be a collection of i.i.d. real valued random variables. Then the random walk $\{S_n\}_{n \in \mathbb{N}}$

$$S_n = \sum_{i=0}^{n} \eta_i$$

is a discrete time stochastic process. Of course, it can also be seen as a continuous time stochastic processes, by defining $S_t = S_{\lfloor t \rfloor}$ for any $t > 0$.

Kolmogorov's extension theorem

A stochastic process can be seen as a random variable taking values in a larger state space. Indeed, if $\{X_t\}_{t \in I}$ is a stochastic process, then for every $\omega \in \Omega$ the (infinite) vector $(X_t(\omega))_{t \in I}$ is an element of $S^I = \prod_{t \in I} S$. On the product space S^I one has coordinate maps $\pi_t \colon S^I \to S$ for any $t \in I$, defined by

$$\pi_t((s_r)_{r \in I}) = s_t \ .$$

Then the space S^I is a measurable space, when enriched with the product σ-field $\mathcal{S}^I$. This is the smallest sigma-field that makes finitely many coordinate maps measurable:

$$\mathcal{S}^I = \sigma(\pi_t \ : \ t \in I) \ .$$

Let us denote with $\mathcal{T}$ the set of ordered times $\mathbf{t} = (t_1, \cdots, t_n)$ with n arbitrary and $t_1 < \cdots < t_n$. Then we can equivalently characerise $\mathcal{S}^I$ as the smallest σ-field that contains all cylinder sets of the form

$$C_{\mathbf{t},A} = \{(s_t)_{t\in I} \in S^I \ : \ (s_{t_1}, \cdots, s_{t_n}) \in A\} \ , \tag{1.1}$$

for any choice of $A \in \mathcal{S}^{\mathbf{t}}$ and any $\mathbf{t} \in \mathcal{T}$.

Theorem 1.1.2 (Dynkin Systems)**.** *Let $\mathcal{D} \subseteq \Omega$ be a collection of subsets such that*

1. *$\Omega \in \mathcal{D}$.*
2. *$A, B \in \mathcal{D}$ and $B \subseteq A$ imply $A \setminus B \in \mathcal{D}$.*
3. *$\{A_n\}_{n=1}^{\infty} \subseteq \mathcal{D}$ and $A_1 \subseteq A_2 \subseteq \cdots A_n \subseteq \cdots$ imply $\bigcup_{n=1}^{\infty} A_i \in \mathcal{D}$.*

If $\mathcal{C}$ is any other collection of subsets and $\mathcal{C} \subseteq \mathcal{D}$, then also $\sigma(\mathcal{C}) \subseteq \mathcal{D}$.

Exercise 1.1.3. *Any stochastic process can be viewed as a* measurable *map $X \colon \Omega \to S^I$ (use cylinder sets and the Dynkin System theorem).*

Proof. Let $\mathcal{D}$ be the collection of subsets $A \subseteq S^I$ such that $X^{-1}(A)$ is measurable. Then $\mathcal{D}$ is a Dynkin system containing cylinder sets. □

Now consider a stochastic process X and a set of times $\mathbf{t} = (t_1, \cdots, t_n)$ such that $t_1 < \cdots < t_n$ and $t_k \in I$ for all $k \in \{1, \cdots, n\}$. To any such choice of $\mathbf{t}$ we can associate the *finite dimensional distributions* (FDDs) of X a times $\mathbf{t}$. Namely, we obtain the probability measure $\mu_{\mathbf{t}}$ on S^n characterised by

$$\mu_{\mathbf{t}}(A_1 \times \cdots \times A_n) = \mathbb{P}(X_{t_1} \in A_1, \cdots, X_{t_n} \in A_n) \ , \qquad \forall A_i \subseteq \mathcal{S} \ .$$

The family $\{\mu_{\mathbf{t}}\}_{\mathbf{t}\in\mathcal{T}}$ of FDDs of a stochastic process is *consistent*, in the following sense.

Definition 1.1.4. *A collection $\{\mu_{\mathbf{t}}\}_{\mathbf{t}\in\mathcal{T}}$ of probability measures such that $\mu_{\mathbf{t}}$ is a measure on S^n (where $\mathbf{t} = (t_1, \cdots, t_n)$) is said to be consistent if for any $\mathbf{t} = (t_1, \cdots, t_n) \in \mathcal{T}$ and any $k \in \{1, \cdots, n\}$ it holds that*

$$\mu_{\hat{\mathbf{t}}_k}(A_1 \times \cdots \times A_{k-1} \times A_{k+1} \times \cdots \times A_n) = \mu_{\mathbf{t}}(A_1 \times \cdots \times A_{k-1} \times S \times A_{k+1} \times \cdots \times A_n) \ ,$$

where $\hat{\mathbf{t}}_k = (t_1, \cdots, t_{k-1}, t_{k+1}, \cdots t_n)$.

The law μ of a stochastic process X is uniquely characterised by collection $\{\mu_{\mathbf{t}}\}_{\mathbf{t}\in\mathcal{T}}$ of all FDDs: this is the content of Kolmogorov's extension theorem.

Theorem 1.1.5 (Kolmogorov extension)**.** *Let $(S, \mathcal{S})$ be $(\mathbb{R}^d, \mathcal{B}(\mathbb{R}^d))$ for some $d \in \mathbb{N}$. Let $\{\mu_{\mathbf{t}}\}_{\mathbf{t}\in\mathcal{T}}$ be a consistent family of measures. Then there exists a unique probability measure on $(S^I, \mathcal{S}^I)$ such that $\{\mu_{\mathbf{t}}\}_{\mathbf{t}\in\mathcal{T}}$ is the set of its FDDs.*

Before we pass to the proof of this result, we must recall another extension theorem.

Theorem 1.1.6 (Caratheodory extension)**.** *Let $(S, \mathcal{S})$ be a measurable space and $\mathcal{C} \subseteq \mathcal{S}$ an algebra of sets (closed under complements, finite unions and finite intersections) such that $\sigma(\mathcal{C}) = \mathcal{S}$. Let μ be a map $\mu \colon \mathcal{C} \to [0, 1]$ such that for any disjoint collection of sets $\{A_n\}_{n\in\mathbb{N}} \subseteq \mathcal{C}$ satisfying $\bigcup_{n\in\mathbb{N}} A_n \in \mathcal{C}$, it holds that*

$$\mu\left(\bigcup_{n\in\mathbb{N}} A_n\right) = \sum_{n\in\mathbb{N}} \mu(A_n) \ .$$

Then there exists a unique measure $\overline{\mu} \colon \mathcal{S} \to [0, \infty]$ such that $\overline{\mu} = \mu$ when restricted to $\mathcal{C}$.

We omit the proof of Caratheodory's theorem, and pass to a proof of Kolmogorov's theorem.

Proof of Theorem 1.1.5. For any $\mathbf{t} \in \mathcal{T}$ and $A \in S^J$, let $C_{\mathbf{t},A}$ be the Cylinder set as in (1.1). Then define

$$\mu(C_{\mathbf{t},A}) = \mu_{\mathbf{t}}(A) \ .$$

Since there can exists $(\mathbf{t}, A), (\mathbf{t}', A')$ such that $C_{\mathbf{t},A} = C_{\mathbf{t}',A'}$, the definition of μ makes sense only if for any such couple we have that $\mu_{\mathbf{t}}(A) = \mu_{\mathbf{t}}(A')$. It suffices to prove the claim for product sets, namely assuming that $A = A_1 \times \cdots \times A_n$ and $A' = A'_1 \times \cdots \times A'_n$, we must show that

$$\mu_{\mathbf{t}}(A_1 \times \cdots \times A_n) = \mu_{\mathbf{t}'}(A'_1 \times \cdots A'_{n'}) \ .$$

This follows from the consistency of the measures and the observation that if $t \in \mathbf{t} \setminus \mathbf{t}'$ (when viewed as sets of times), then necessarily $A_t = \Omega$, since $C_{\mathbf{t},A} = C_{\mathbf{t}',A'}$.

Exercise 1.1.7. *Prove that the collection of cylinder sets $\{C_{\mathbf{t},A}\}_{\mathbf{t},A}$ is an algebra, namely that it is closed under taking complements, finite unions and finite intersections.*

In view of this exercise, the existence of the measure μ (that is the extension of its definition to the entirety of $\mathcal{S}^I$) follows from Theorem 1.1.6, if we show that the measure is countably additive. Namely, let $C_n, n \in \mathbb{N}$ be a disjoint collection of sets in $\mathcal{C}$ such that $C = \bigcup_{n\in\mathbb{N}} C_n \in \mathcal{C}$. We would like to prove that

$$\mu(C) = \sum_{n\in\mathbb{N}} \mu(C_n) \ .$$

This is equivalent to proving that

$$\lim_{n\to\infty} \mu(Q_n) = 0 \ , \qquad \text{with } Q_n = \bigcup_{m\geqslant n} C_m \ .$$

Since $\mu(Q_n)$ is decreasing in n its limit exists. Assume therefore that

$$\lim_{n\to\infty} \mu(Q_n) = \varepsilon > 0 \ ,$$

and let us find a contradiction. Let us assume in addition that for each $n \in \mathbb{N}$ there exist times $(t_1, \cdots, t_n)$ and a set $A_n \in \mathcal{B}(\mathbb{R}^n)$ such that

$$Q_n = \{(s_{t_1}, \cdots, s_{t_n}) \in A_n\} \ .$$

Exercise 1.1.8. *How can we always reduce ourselves to this setting, up to choosing a slightly different sequence of decreasing sets Q'_n satisfying $\bigcap_{n\in\mathbb{N}} Q'_n = \bigcap_{n\in\mathbb{N}} Q_n$?*

If the sets A_n were compact we could now complete the proof. Indeed, since $\mu(Q_n) > \varepsilon$ for every $n \in \mathbb{N}$ the sets Q_n cannot be empty and we would find a point $(s^n_{t_1}, \cdots, s^n_{t_n}) \in Q_n$. Since the sets are also decreasing, the sequence $(s^n_{t_1})_{n\in\mathbb{N}} \in A_1$, which is compact, so there exists a limit point $r_{t_1} \in A_1$. Similarly for $(s^n_{t_1}, s^n_{t_2})_{n\in\mathbb{N}}$, which must admit a limit point of the form $(r_{t_1}, r_{t_2}) \in A_2$. Iterating this procedure we can construct a point in $p \in \bigcap_n Q_n$ satisfying

$$\pi_{t_k}(p) = r_k \ ,$$

which contradicts the fact that $\bigcap_n Q_n = \varnothing$.

In general the sets A_n are not compact, but this problem can be overcome as follows. For each $n \in \mathbb{N}$ one can find a compact subset $K_n \subseteq A_n$ such that $\mu_{\mathbf{t_n}}(A_n \setminus K_n) \leqslant \frac{\varepsilon}{2^n}$ (Borel sets in $\mathbb{R}^d$ are inner regular). Then define $\widetilde{Q}_n = \{(s_{t_1}, \cdots, s_{t_n}) \in K_n\}$ and $\overline{Q}_n = \bigcap_{m \leqslant n} \widetilde{Q}_m$. In this way $\overline{Q}_n$ is again a decreasing sequence of sets. In addition, we can lower bound

$$\begin{aligned}
\mu(\overline{Q}_n) &= \mu(Q_n) - \mu(Q_n \setminus \overline{Q}_n) \\
&= \mu(Q_n) - \mu\left(Q_n \setminus \bigcap_{m \leqslant n} \widetilde{Q}_n\right) \\
&\geqslant \mu(Q_n) - \sum_{m=1}^{n} \mu\left(Q_m \setminus \widetilde{Q}_m\right) \\
&\geqslant \varepsilon - \sum_{m=1}^{n} \frac{\varepsilon}{2^n} \geqslant \frac{\varepsilon}{2} \, .
\end{aligned}$$

Hence, we can apply the reasoning above to the sets $\widetilde{Q}_n$ and the proof is complete. □

1.2 Gaussian processes

Now let us focus on a particular class of stochastic processes, namely Gaussian processes. These are uniquely characterised by their covariance structure.

Definition 1.2.1. *A real vector-valued random variable $(X_1, \ldots X_n)$ is a multivariate Gaussian if all linear combinations $\sum_i a_i X_i$, for any $a_i \in \mathbb{R}$ are univariate Gaussians.*

Recall that the probability density function (PDF) f_X of a (univariate) Gaussian $X \sim \mathcal{N}(\mu, \sigma^2)$ with mean $\mu \in \mathbb{R}$ and variance $\sigma^2 > 0$ is given by

$$f_X(x) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-(x-\mu)^2/(2\sigma^2)} \, .$$

Before we proceed, let us make some remarks.

- If $X_1, \ldots, X_n$ are independent Gaussians, then $(X_1, \ldots, X_n)$ is multivariate Gaussian.
- Definition 1.2.1 is stronger than the statement that each X_i is Gaussian. For example, let $X \sim \mathcal{N}(0,1)$ and

$$Y = \begin{cases} +X \, , & \text{if } |X| \leq 1 \\ -X \, , & \text{if } |X| > 1 \end{cases} \, .$$

  Then also $Y \sim \mathcal{N}(0,1)$. But since $|X + Y| \leq 2$ and $X + Y$ is not constant, $X + Y$ is not Gaussian and so (X, Y) is not bivariate Gaussian.
- Suppose $X = (X_1, \ldots, X_n)$ is Gaussian. Then for any matrix $A \in \mathbb{R}^{m \times n}$ the random vector $Y = AX \in \mathbb{R}^m$ is Gaussian.

As we mentioned, the distribution of a multivariate Gaussian is uniquely determined by its mean and its covariance structure. Let us recall this result in finite dimensions.

Proposition 1.2.2. *The distribution of a Gaussian $X = (X_1, \ldots, X_n)$ is fully determined by its mean vector $\boldsymbol{\mu} = \mathbb{E}[X] = \big(\mathbb{E}[X_1], \ldots, \mathbb{E}[X_n]\big)$ and its covariance matrix*

$$\Sigma = (\sigma_{ij})_{i,j=1,\ldots,n} \quad \textit{where} \quad \sigma_{ij} = \mathbb{E}\left((X_i - \mu_i)(X_j - \mu_j)\right) = \mathrm{Cov}[X_i, X_j] \, .$$

Σ is always positive semi-definite. If it is also positive definite, X has a PDF which is given by

$$f_X(\mathbf{x}) = \det(2\pi\Sigma)^{-1/2} \exp\left(-\frac{1}{2}(\mathbf{x}-\boldsymbol{\mu})^T \Sigma^{-1}(\mathbf{x}-\boldsymbol{\mu})\right) . \qquad (1.2)$$

Finally, for every mean vector $\boldsymbol{\mu} \in \mathbb{R}^d$ and positive semi-definite matrix $\Sigma \in \mathbb{R}^{d\times d}$ there exists a unique multivariate Gaussian distribution $\mathcal{N}(\boldsymbol{\mu}, \Sigma)$.

The previous proposition allows us to define Gaussian stochastic processes based solely on their covariance structure.

Definition 1.2.3. *A continuous-time stochastic process $(X_t)_{t\geqslant 0}$ with state space $S = \mathbb{R}$ is a Gaussian process if all FDDs are Gaussian, i.e. if for all $n \in \mathbb{N}$ and all $t_1, \ldots, t_n \geq 0$ the random vector $\big(X(t_1), \ldots, X(t_n)\big)$ has a multivariate Gaussian distribution.*

The definition extends analogously to discrete time stochastic processes. From Proposition 1.2.2 we deduce that the following holds.

Corollary 1.2.4. *Let X_t be a Gaussian process. Then its law on $(\mathbb{R}^d)^{[0,\infty)}$ is uniquely determined by its mean and covariance functions:*

$$\mu(t) := \mathbb{E}\big[X(t)\big] \quad \textit{and} \quad \sigma(s,t) := \mathrm{Cov}\big[X(s), X(t)\big] \ , \quad s,t \geq 0 \ . \qquad (1.3)$$

Moreover, for any two functions $t \mapsto \mu_t$ and $(s,t) \mapsto \sigma_{s,t}$, such that for any $\mathbf{t} = (t_1, \ldots, t_n) \in \mathcal{T}$ the matrix $(\sigma_{t_i,t_j})_{i,j}$ is positive semidefinite there exists a unique Gaussian process with mean function μ and covariance function σ.

Example 1.2.5. *Let $\big(X(t) : t \in \mathbb{N}\big)$ be a family of i.i.d. (independent, identically distributed) standard Gaussians $X(t) \sim \mathcal{N}(0,1)$. Then $\big(X(t) : t \in \mathbb{N}\big)$ is a Gaussian process with mean $\mu(t) \equiv 0$ and covariance*

$$\sigma(s,t) = \begin{cases} 1 \ , & \text{if } s = t \\ 0 \ , & \text{if } s \neq t \end{cases} \ .$$

1.2.1 Brownian motion

We are now ready to introduce (one-dimensional) Brownian motion as a continuous time stochastic process on $\mathbb{R}^{[0,\infty)}$.

Definition 1.2.6. *A Brownian motion $(B_t)_{t\geqslant 0}$ started in $B_0 = 0$ is a stochastic process satisfying*

1. (Stationary increments) *For any $t \geqslant s \geqslant 0$ the increment $B_t - B_s$ has the same distribution as B_{t-s}.*

2. (Independent increments) *For any $t \geqslant s \geqslant r \geqslant u$ the increments $B_t - B_s$ and $B_r - B_u$ are independent.*

3. (Variance and mean) *For any $t \geqslant 0$ we have* $\mathrm{Var}(B_t) = t$, *and* $\mathbb{E}[B_t] = 0$. *Or better $B_t \sim \mathcal{N}(0,t)$.*

4. (Continuous sample paths) *Almost surely, sample paths are continuous:*

$$\mathbb{P}\left(\{\omega : \ t \mapsto B_t(\omega) \ \textit{is continuous on} \ [0,\infty)\}\right) = 1 \ .$$

Observe that it is yet unclear that Brownian motion exists, as we have no tools to prove that the fourth property holds. But the tools we have introduced so far allow us to construct a unique (in law, on $\mathbb{R}^{[0,\infty)}$) stochastic process satisfying properties (1) – (3).

For the previous definition to make sense (in view of Proposition 1.2.2 and Kolmogorov's consistency criterion), we observe that for ordered times $0 \leqslant t_1 < \ldots < t_n$ the FDDs of a stochastic process $(X_t)_{t \geqslant 0}$ at times $\mathbf{t} = (t_1, \ldots, t_n)$ are fully characterized by the initial distribution of X_0 and distribution of the increments $\Delta(t_{i-1}, t_i) := X(t_i) - X(t_{i-1})$ (with the convention that $t_0 = 0$), since we can write:

$$\big(X(t_1), \ldots, X(t_n)\big) = X(0) + \Big(\Delta(t_0, t_1), \Delta(t_0, t_1) + \Delta(t_1, t_2), \ldots, \sum_{i=1}^{n} \Delta(t_{i-1}, t_i)\Big) \ . \quad (1.4)$$

Proposition 1.2.7. *The following are equivalent for a stochastic process $\big(X(t) : t \geq 0\big)$ on $\mathbb{R}$ with a fixed initial condition $X(0) = x$.*

(a) $\big(X(t) : t \geq 0\big)$ has stationary independent increments with

$$X(t) - X(s) \sim \mathcal{N}(0, t-s) \quad \textit{for all } t > s \geq 0 \ .$$

(b) $\big(X(t) : t \geq 0\big)$ is a Gaussian process with constant mean $\mathbb{E}[X(t)] = x$ and covariance

$$\mathrm{Cov}[X_s, X_t] = s \wedge t \ (:= \min\{s, t\}) \ . \quad (1.5)$$

Proof. Suppose (a) holds. Pick $n \geq 1$, $a_1, \ldots, a_n \in \mathbb{R}$ and $t_0 = 0 < t_1 < \cdots < t_n$ which are (wlog) ordered. Then, using (1.4), we find b_k such that

$$\sum_{k=1}^{n} a_k X(t_k) = b_1 x + \sum_{k=1}^{n} b_k \big(X(t_k) - X(t_{k-1})\big) \ . \quad (1.6)$$

The r.h.s. is a sum of independent Gaussians, and hence Gaussian (see the remarks after Definition 1.2.1), so $\big(X(t) : t \geq 0\big)$ is a Gaussian process. With $X(t) - X(s) \sim \mathcal{N}(0, t-s)$ we obviously have

$$\mathbb{E}\big[X(t)\big] = x + \mathbb{E}\big[X(t) - X(0)\big] = x \ ,$$

and to compute covariances pick $s < t$ and write

$$\begin{aligned} \mathrm{Cov}\big[X(s), X(t)\big] &\overset{(I)}{=} \mathbb{E}\big[\big(X(s) - x\big)\big(X(t) - x\big)\big] \\ &= \mathbb{E}\big[\big(X(s) - x\big)\big(X(t) - X(s)\big)\big] + \mathbb{E}\big[\big(X(s) - x\big)^2\big] \overset{(II)}{=} 0 + s = s \wedge t \ . \end{aligned}$$

Equality (I) is due to the invariance of Cov w. r. t. a constant shift, equality (II) is due to the independence of increments. The final formula holds for any $s, t \geq 0$ due to the symmetry of covariance, $\mathrm{Cov}[X_t, X_s] = \mathrm{Cov}[X_s, X_t]$. Now assume (b) holds. Then for all $s < t$, $X(t) - X(s)$ is Gaussian with mean 0 and

$$\mathrm{Var}\big[X(t) - X(s)\big] = \mathrm{Var}\big[X(t)\big] + \mathrm{Var}\big[X(s)\big] - 2\mathrm{Cov}\big[X(t), X(s)\big] = t + s - 2(s \wedge t) = t - s \ ,$$

so the process has stationary increments. To check independence of increments, take $0 \leq t_1 < \cdots < t_n$ and note that (1.6) implies that increments

$$\big(X(t_2) - X(t_1), \ldots, X(t_n) - X(t_{n-1})\big)$$

are Gaussian, so by Proposition 1.2.2 it is enough to show that increments are uncorrelated. To do so, take $u < v \leq s < t$ and compute

$$\mathrm{Cov}\big[X(v) - X(u), X(t) - X(s)\big] = v \wedge t - v \wedge s - u \wedge t + u \wedge s = v - v - u + u = 0 \ .$$

□

1.3 Construction of Brownian motion

While the FDDs fix the distributional properties of a process via its law on the product space $(S^I, \mathcal{S}^I)$, we might be interested in path properties of Brownian motion. For example we might wonder whether the typical path $t \mapsto B_t(\omega)$ is continuous (or Hölder continuous, or even differentiable). For all these purposes, the infinite produce space $\mathbb{R}^{[0,\infty)}$ is insufficient and we have to consider the law of B on the space of continuous functions.

Exercise 1.3.1. *The set $C = \{t \mapsto \omega(t) : \omega \text{ is continuous}\}$ is not an element of $\mathcal{B}(\mathbb{R}^{[0,\infty)})$.*

1.3.1 Brownian motion

The problem in constructing Brownian motion is not just the technicality observed in Exercise 1.3.1. Indeed, even if we were to assign a reasonable "pretend-probability" to the event C, it would have to be 0 rather than 1. Assume that $\mathbb{P}(C) \in [0,1]$ exists and define for each $k \in \mathbb{N}$

$$B_k(t) := \begin{cases} B(t) & , \text{ if } t \neq \tau_k \\ B(t)+1 & , \text{ if } t = \tau_k \end{cases}$$

with independent random variables $\tau_k \sim U\big([k-1,k)\big)$. Then the processes $B_k = \big(B_k(t) : t \geq 0\big)$ are defined on $(\mathbb{R}^{[0,\infty)}, \mathcal{B}(\mathbb{R}^{[0,\infty)}))$ and have the same FDDs as B since

$$\mathbb{P}\big[B_k(t) = B(t)\big] = \mathbb{P}[\tau_k \neq t] = 1 \quad \text{for all } t \geq 0 \ , \tag{1.7}$$

due to the continuous distribution of the τ_k. Since via Theorem 1.1.5 the FDDs uniquely determine the law of a process on $(\mathbb{R}^{[0,\infty)}, \mathcal{B}(\mathbb{R}^{[0,\infty)}))$, we must have $\mathbb{P}[C_k] = \mathbb{P}[C]$ for all $k \geq 1$ where $C_k = \{X_k \text{ is continuous}\}$. Since the $C_k \subseteq \Omega_0$ are mutually disjoint this implies $\mathbb{P}[C] = 0$.

Instead, we shall prove that it is always possible to modify Brownian motion in such a way that the FDDs are not affected, but also so that its sample paths are continuous in time.

Definition 1.3.2. *Y is a* modification *of X, if $\mathbb{P}\big[X(t) = Y(t)\big] = 1$ for all $t \geq 0$.*

Lemma 1.3.3. *If Y is a modification of X, then the laws of Y and X on $\mathbb{R}^{[0,\infty)}$ coincide.*

Proof. Since a finite union of null-sets is still a null-set, we have that for any $t_1 < t_2 < \cdots < t_n$

$$\mathbb{P}(Y_{t_1} = X_{t_1}, Y_{t_2} = X_{t_2}, \ldots, Y_{t_n} = X_{t_n}) = 1 \ .$$

The result follows. □

In the above construction the processes B and B_k, $k \geq 1$ are all modifications of Brownian motion, but at most one of them can have continuous paths. The construction in Theorem 1.1.5 does not deliver a continuous modification of BM: this is a different result.

Theorem 1.3.4. *Brownian motion exists.*

Before we prove this result, let us consider a few simple properties of Brownian motion (assuming that the process exists).

1.3.2 Simple properties of standard Brownian motion

Proposition 1.3.5. *Suppose $\big(B(t) : t \geq 0\big)$ is a standard BM started in $x = 0$, then so is the process $\big(X(t) : t \geq 0\big)$ provided that*

(a) $X(t) := B(t+s) - B(s)$ *for any fixed* $s \geq 0$, **(time translation)**

(b) $X(t) := -B(t)$, **(reflection)**

(c) $X(t) := \frac{1}{\sqrt{c}} B(ct)$ *for any fixed* $c > 0$, **(scaling)**

(d) $X(t) := \begin{cases} 0 & , \text{ for } t = 0 \\ tB(1/t) & , \text{ for } t > 0 \end{cases}$, **(time inversion)**

(e) $X(t) := B(1) - B(1-t)$ *for* $t \in [0, 1]$. **(time reversal)**

The proof is left as an exercise.

Corollary 1.3.6. *For a standard BM $\big(B(t) : t \geq 0\big)$ we have $B(t)/t \to 0$ almost surely as $t \to \infty$.*

Proof. Using $\big(X(t) : t \geq 0\big)$ from Point (d) of Proposition 1.3.5, which has continuous paths, we see that

$$B(t)/t = X(1/t) \to X(0) = 0 \;\; a.s. \quad \text{as } t \to \infty \; .$$

□

Note that this implies the following asymptotic behaviour: For $a \in \mathbb{R}$ we have

$$B(t) + at = t\Big(\frac{B(t)}{t} + a\Big) \to \begin{cases} +\infty & , \; a > 0 \\ -\infty & , \; a < 0 \end{cases} \quad \text{almost surely as } t \to \infty \; .$$

1.3.3 Kolmogorov's continuity criterion

We will now prove Theorem 1.3.4, which will follow from the following result.

Theorem 1.3.7 (Kolmogorov's continuity criterion)**.** *For any continuous-time stochastic process $t \mapsto X_t$, suppose that there exist constants $\alpha, \beta, C, T > 0$ such that*

$$\mathbb{E}|X_t - X_s|^\alpha \leqslant C|t-s|^{1+\beta} \; , \qquad \forall 0 \leqslant s \leqslant t \leqslant T \; .$$

Then there exists a modification $\widetilde{X}$ of X such that

$$\mathbb{P}\left(\sup_{0 \leqslant s \leqslant t \leqslant T} \frac{|\widetilde{X}_t - \widetilde{X}_s|}{|t-s|^\gamma} < \infty \right) = 1 \; ,$$

for any $\gamma \in (0, \beta/\alpha)$.

In particular, the conclusion of Theorem 1.3.7 provides more than mere continuity: it even guarantees $\gamma-$Hölder continuity for the modification $\widetilde{X}$, for any $\gamma \in (0, 1/2)$.

Exercise 1.3.8. *Prove Theorem 1.3.4. Hint: first show that for every $n \in \mathbb{N}$ there exists a C_n such that*

$$\mathbb{E}|B_t - B_s|^{2n} = C_n|t-s|^n \; .$$