THE UNIVERSITY OF WARWICK

FOURTH YEAR EXAMINATION: Summer 2023

STOCHASTIC ANALYSIS

Time Allowed: **3 hours**

Read carefully the instructions on the answer book and make sure that the particulars required are entered on each answer book.

Calculators are not needed and are not permitted in this examination.

Candidates should answer COMPULSORY QUESTION 1 and THREE QUESTIONS out of the four optional questions 2, 3, 4 and 5.

The compulsory question is worth 40% of the available marks. Each optional question is worth 20%.

If you have answered more than the compulsory Question 1 and three optional questions, you will only be given credit for your QUESTION 1 and THREE OTHER best answers.

The numbers in the margin indicate approximately how many marks are available for each part of a question.

Unless otherwise specified, the process $(W_t : t \geq 0)$ will be a one-dimensional $(\mathcal{F}_t)$ Brownian motion, defined on probability space $(\Omega, \mathcal{F}, P)$ with a filtration $(\mathcal{F}_t : t \geq 0)$.

COMPULSORY QUESTION

1. a) (i) How is the Itô integral $\int_0^t g_s dW_s$ defined for simple adapted integrands? **[2]**

(ii) Prove the Itô isometry

$$\mathbb{E}\left[\left(\int_0^t g_s dW_s\right)^2\right] = \mathbb{E}\left[\int_0^t g_s^2 ds\right]$$

in the case that g is a simple bounded adapted integrand. **[7]**

(iii) Explain how to use approximation by simple integrands to define the integral $\int_0^t g_s dW_s$ for bounded continuous adapted integrands. **[7]**

b) State Ito's formula for $f(W^1, W^2, \ldots, W^m)$ where $W = (W^1, W^2, \ldots, W^m)$ is an m-dimensional Brownian motion and $f \in C^2(\mathbb{R}^m)$.

Show that $X = |W|^2$ solves

$$dX = m\,dt + 2\sqrt{X}d\tilde{W}$$

where $\tilde{W}$ is a 1-dimensional Brownian motion. **[8]**

c) Let X be the non-negative solution to

$$dX = a\,dt + \sqrt{X}dW, \qquad X_0 = x > 0,\ a \in \mathbb{R}.$$

Calculate $E[X_t]$ and $E[X_t^2]$ for $t \geq 0$. **[8]**

d) Use Feller's scale analysis to describe, for all values of $p > 0$, the exit of a solution to

$$dX = X dt + X^p dW$$

from the interval $(0, \infty)$. (You do not need to analyse the speed measure to decide if the exit is in finite time or not). **[8]**

OPTIONAL QUESTIONS

2. a) Define for $n = 1, 2, \ldots$

$$I_n := \sum_{k=0}^{n-1} W_{\frac{k}{n}}(W_{\frac{k+1}{n}} - W_{\frac{k}{n}}), \qquad I := \int_0^1 W_s dW_s.$$

Use the Itô isometry to show that $\mathbb{E}[|I_n - I|^2] = \frac{1}{2n}$. **[10]**

b) Suppose g are h are bounded and adapted. Define the stochastic exponential by

$$\mathcal{E}_t := \exp\left(\int_0^t g_s dW_s - \frac{1}{2}\int_0^t g_s^2 ds\right).$$

(i) Show that $(\mathcal{E}_t : t \geq 0)$ is the solution to the linear equation **[5]**

$$d\mathcal{E} = \mathcal{E} g\, dW, \quad \mathcal{E}_0 = 1.$$

(ii) By developing $d(X_t \mathcal{E}_t^{-1})$, or otherwise, find the solution to the equation **[5]**

$$dX = hdt + XgdW, \quad X_0 = x.$$

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3. a) Consider the equation $dX = \mu(X)dt + \sigma(X)dW$ with $X_0 = a$ for Lipschitz μ and σ. What does it mean that *pathwise uniqueness* holds for this equation? Suppose X and Y are two solutions. Use Ito's formula to develop $(X_t - Y_t)^2 e^{-At}$ and show for sufficiently large A that

$$\frac{d}{dt}\mathbb{E}[(X_t - Y_t)^2 e^{-At}] \leq 0$$

and deduce pathwise uniqueness. **[10]**

b) Let X be the solution to the equation

$$dX = (1 + X)dW, \quad X_0 = x \in (0, 1).$$

let $\tau = \inf\{t : X_t \notin (0, 1)\}$. Calculate $E_x[\tau]$. **[10]**

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4. a) Suppose $D \subseteq \mathbb{R}^d$ is a bounded open region. Suppose $u \in C^2(\mathbb{R}^d)$ solves

$$\begin{aligned} \frac{1}{2}\Delta u(x) + \sum_{i=1}^{d} \mu_i(x) \frac{\partial}{\partial x_i} u(x) &= g(x)u(x) \quad \text{for } x \in D, \\ u(x) &= f(x) \quad \text{for } x \in \partial D, \end{aligned}$$

where $\mu : \overline{D} \to \mathbb{R}^d$ and $g : \overline{D} \to [0, \infty)$ are continuous.
Give the proof of the probabilistic representation

$$u(x) = \mathbb{E}_x \left[f(X_\tau) \exp\left(-\int_0^\tau g(X_s) ds \right) \right]$$

for a process $X = (X_t : t \geq 0)$ and stopping time τ which you should specify. **[10]**

b) For the one dimensional process defined by $X_t = x + \mu t + W_t$ for $t \geq 0$, and the stopping time $\tau = \inf\{t : X_t \notin (0, R)\}$ where $R > 0$, calculate the expectation **[10]**

$$\mathbb{E}_x \left[\exp(-\lambda \tau) \right] \quad \text{for } x \in [0, R] \text{ and } \lambda \geq 0.$$

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5. Consider the equation

$$dX = \left(\frac{1}{X} - \frac{1}{2} \right) dt + dW.$$

(i) Find a stationary distribution on $(0, \infty)$ for this equation (you do not need to find an exact normalising constant). **[6]**

(ii) Calculate $\mathcal{L}f$ for the functions $f(x) = x^2$ and $f(x) = x^3$. **[4]**

(iii) Suppose X_0 has the stationary distribution. Find $\alpha, \beta > 0$ so that **[10]**

$$\frac{1}{\sqrt{t}} \int_0^t (X_s - \alpha) ds \xrightarrow{\mathcal{D}} N(0, \beta) \quad \text{as } t \to \infty.$$
