THE UNIVERSITY OF WARWICK

FOURTH YEAR EXAMINATION: 2021

QUANTUM MECHANICS

Time Allowed: **3 hours**

Read all instructions carefully. Please note also the guidance you have received in advance on the departmental 'Warwick Mathematics Exams 2021' webpage.

Calculators, wikipedia and interactive internet resources are not needed and are not permitted in this examination. You are not allowed to confer with other people. You may use module materials and resources from the module webpage.

ANSWER ALL THREE QUESTIONS.

On completion of the assessment, you must upload your answer to Moodle as a single PDF document if possible, although multiple files (2 or 3) are permitted. You have an additional 45 minutes to make the upload, and instructions are available on the departmental 'Warwick Mathematics Exams 2021' webpage.

The numbers in the margin indicate approximately how many marks are available for each part of a question.

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1. (i) If A is an operator in $\mathbb{C}^2$, consider the map $w: \ A \mapsto \omega(A) = \mathrm{Tr} A\rho$, where

$$\rho = \tfrac{1}{2}\begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix}.$$

Is ω a state? If yes, is it a pure or mixed state? **[5]**

(ii) Same question as (i), but with matrix

$$\rho = \tfrac{1}{2}\begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}.$$

Is ω a state? If yes, is it a pure or mixed state? **[5]**

(iii) Show that the Laplacian $\Delta = \frac{\mathrm{d}^2}{\mathrm{d}x^2}$ is a symmetric operator in $L^2(\mathbb{R})$ (for the domain, take the space of Schwartz functions). **[5]**

(iv) Let X be the position operator in $L^2(\mathbb{R})$, i.e. $(Xf)(x) = xf(x)$. Show that X is unbounded. **[5]**

 Question 1 continued overleaf

(v) Show that the Schrödinger operator $-\Delta - |x|^{-1/2}$, with domain $H^2(\mathbb{R})$, is bounded below. **[5]**

(vi) On the interval $[0,1]$, consider the Laplacian $\Delta = \frac{\mathrm{d}^2}{\mathrm{d}x^2}$ with domain $\mathcal{D}(\Delta) = C^2([0,1])$. Is Δ symmetric? Self-adjoint? **[5]**

(vii) Let X be the operator $f \mapsto xf$ and P the operator $f \mapsto -\mathrm{i}f'$ on $L^2(\mathbb{R})$. Show that $[X,P] = \mathrm{i}$. **[5]**

(viii) Find $[X^2, P]$. **[5]**

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2. Part I. Let ρ be a density operator on $\mathbb{C}^n$.

(a) Describe the possible eigenvalues of ρ. **[3]**

(b) Assuming that ρ is the density matrix of a pure state, describe its possible eigenvalues. **[3]**

(c) Assuming that ρ is the density matrix of a mixed state, describe its possible eigenvalues. **[3]**

On $\mathbb{C}^2$, let $H = H^*$ be the Hamiltonian and let $U_t = \mathrm{e}^{-\mathrm{i}tH}$ be the evolution operator. The evolved density operator is $\rho_t = U_t \rho_0 U_t^*$.

(d) Can you find an example where ρ_0 is a pure state, but ρ_t is a mixed state for some $t > 0$? **[3]**

Part II. Recall the Hamiltonian for the hydrogen atom: $H = -\Delta - \frac{1}{\|x\|}$ on $L^2(\mathbb{R}^3)$ (with domain $\mathcal{D}(H) = H^2(\mathbb{R}^3)$ so that it is self-adjoint).

(e) Prove that $H \geq -1/3$. **[6]**

(f) Prove that the spectrum of H contains the half-line $[0,\infty)$. **[6]**

(g) Prove that $-1/4$ is an eigenvalue of H. **[6]**

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3. Let A be a bounded operator in the Hilbert space $\mathcal{H}$.

(a) Give the definition of the operator e^{sA} where $s \in \mathbb{C}$ (prove the existence of this operator). **[4]**

(b) Assume that $A = A^*$ and that $A \geq a$ for some $a \in \mathbb{R}$. Prove that $\|e^{-A}\| \leq e^{-a}$. **[4]**

(c) Let H be a (possibly unbounded, but bounded below) self-adjoint operator in $\mathcal{H}$. Give the full construction of the operator e^{-tH} for $t \geq 0$. **[8]**

(d) Prove that $e^{-sH}e^{-tH} = e^{-(s+t)H}$ for all $s, t \geq 0$. **[4]**

Now we consider the operator $e^{\frac{1}{2}t\Delta}$ in $L^2(\mathbb{R})$ where $t \geq 0$.

(e) What is the norm of $e^{\frac{1}{2}t\Delta}$ for $t \geq 0$? **[4]**

(f) Find the integral kernel of $e^{\frac{1}{2}t\Delta}$, that is, the function $a_t(x, y)$ such that

$$e^{\frac{1}{2}t\Delta} f(x) = \int_{-\infty}^{\infty} a_t(x, y) f(y) \mathrm{d}y$$

for all f in a dense subspace of $L^2(\mathbb{R})$. Give a full proof. **[6]**
