Consider the contact process $(\eta_t : t \geq 0)$ on the complete graph $q(i,j) = \lambda$ for all $i \neq j$, with state spaces $S = \{0,1\}$ and transition rates $c(\eta, \eta^i) = \eta(i) + \lambda(1-\eta(i)) \sum_{j \neq i} \eta(j)$.

This process has generator $(\mathcal{L}f)(\eta) = \sum_{i \in \Lambda} c(\eta, \eta^i)\left(f(\eta^i) - f(\eta)\right)$ for $f : \{0,1\}^L \to \mathbb{R}$.

Let $N = \sum_{i \in \Lambda} \eta(i)$, the number of infected individuals. We aim to find the generator of N. Let $g : \{0,1,\dots,L\} \to \mathbb{R}$, so $(g \circ N)(\eta) : \{0,1\}^L \to \mathbb{R}$.

$$
\begin{aligned}
\mathcal{L}(g \circ N)(\eta) &= \sum_{i \in \Lambda} c(\eta, \eta^i)\left[g(N(\eta^i)) - g(N(\eta))\right] \\
&= \sum_{i \in \Lambda} \left[\eta(i) + \lambda(1-\eta(i)) \sum_{j \neq i} \eta(j)\right]\left[g(N(\eta^i)) - g(N(\eta))\right] && \text{using the rates given} \\
&= \sum_{\substack{i \in \Lambda \\ \eta(i)=0}} \left[\eta(i) + \lambda(1-\eta(i)) \sum_{j \neq i} \eta(j)\right]\left[g(N(\eta^i)) - g(N(\eta))\right] + \sum_{\substack{i \in \Lambda \\ \eta(i)=1}} \left[\eta(i) + \lambda(1-\eta(i)) \sum_{j \neq i} \eta(j)\right]\left[g(N(\eta^i)) - g(N(\eta))\right] && \text{splitting according to state} \\
&= \sum_{\substack{i \in \Lambda \\ \eta(i)=0}} \left[\lambda \sum_{j \neq i} \eta(j)\right]\left[g(N(\eta^i)) - g(N(\eta))\right] + \sum_{\substack{i \in \Lambda \\ \eta(i)=1}} \left[1\right]\left[g(N(\eta^i)) - g(N(\eta))\right] && \text{simplifying rates} \\
&= \sum_{\substack{i \in \Lambda \\ \eta(i)=0}} \lambda N(\eta)\underbrace{\left[g(N(\eta^i)) - g(N(\eta))\right]}_{\substack{\text{if } \eta(i)=0 \text{ then} \\ N(\eta^i) = N(\eta)+1}} + \sum_{\substack{i \in \Lambda \\ \eta(i)=1}} \underbrace{\left[g(N(\eta^i)) - g(N(\eta))\right]}_{\substack{\text{if } \eta(i)=1 \text{ then} \\ N(\eta^i) = N(\eta)-1}} && \text{simplifying rates} \\
&= \sum_{\substack{i \in \Lambda \\ \eta(i)=0}} \lambda N(\eta)\left[g(N(\eta)+1) - g(N(\eta))\right] + \sum_{\substack{i \in \Lambda \\ \eta(i)=1}} \left[g(N(\eta)-1) - g(N(\eta))\right] && \text{things being summed are independent of } i. \\
&= \lambda N(\eta)\left[g(N(\eta)+1) - g(N(\eta))\right]\Big(\sum_{\substack{i \in \Lambda \\ \eta(i)=0}} 1\Big) + \left[g(N(\eta)-1) - g(N(\eta))\right]\Big(\sum_{\substack{i \in \Lambda \\ \eta(i)=1}} 1\Big) && \text{using } N(\eta) \text{ to count the number of elements in each sum} \\
&= \lambda N(\eta)\left[g(N(\eta)+1) - g(N(\eta))\right](L - N(\eta)) + \left[g(N(\eta)-1) - g(N(\eta))\right] N(\eta) && \text{writing } N \text{ rather than } N(\eta). \\
&= \lambda N(L-N)\left[g(N+1) - g(N)\right] + N\left[g(N-1) - g(N)\right]
\end{aligned}
$$

Hence $(N_t : t \geq 0)$ is a jump process with state space $\{0,1,\dots,L\}$ and rates

$c(N, N+1) = \lambda N(L-N)$

$c(N, N-1) = N$.

N_t has a unique absorbing state at $N=0$, as this is the only state with both transition rates equal to 0.

Hence the only stationary distribution is $\pi = (1,0,0,\dots,0)$.

Now let $\rho_t = \frac{\mathbb{E}[N_t]}{L}$.

$$
\begin{aligned}
\frac{d\rho}{dt} &= \frac{1}{L}\frac{d}{dt}\mathbb{E}[N_t] \\
&= \frac{1}{L}\mathbb{E}[(\mathcal{L}g)(N_t)] && \text{with } g(N) = N \\
&= \frac{1}{L}\mathbb{E}[\lambda N(L-N) - N] \\
&= \frac{1}{L}\mathbb{E}[\lambda NL - \lambda N^2 - N] \\
&= \frac{1}{L}\left(\lambda L\,\mathbb{E}[N] - \lambda\,\mathbb{E}[N^2] - \mathbb{E}[N]\right) && \text{mean-field assumption} \\
&= \frac{1}{L}\left(\lambda L\,\mathbb{E}[N] - \lambda\,\mathbb{E}[N]^2 - \mathbb{E}[N]\right) \\
&= \lambda L\frac{\mathbb{E}[N]}{L} - \lambda L\left(\frac{\mathbb{E}[N]}{L}\right)^2 - \frac{\mathbb{E}[N]}{L} \\
&= \lambda L\rho - \lambda L\rho^2 - \rho \\
&= \lambda L\rho(1-\rho) - \rho \\
&= \bar{\lambda}\rho(1-\rho) - \rho.
\end{aligned}
$$