Week 4(1) Chapter 2.2-2.3

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Chapter 2

The summary of short-rate models: the dynamics of the short rate r are

$$\begin{aligned} dr_t &= b_t dt + \sigma_t dW_t \quad \text{under } \mathbf{P} \\ &= (b_t - \sigma_t \Theta_t)dt + \sigma_t dW_t^{\mathbf{Q}} \quad \text{under } \mathbf{Q}. \end{aligned}$$

The dynamics of the zero-coupon bond price $P(t,T)$ are

$$\begin{aligned} \frac{dP(t,T)}{P(t,T)} &= r_t dt + h_t dW_t^{\mathbf{Q}} \quad \text{under } \mathbf{Q} \\ &= (r_t + h_t \Theta_t)dt + h_t dW_t \quad \text{under } \mathbf{P}. \end{aligned}$$

A short-rate model is not fully determined without the exogenous specification of the market price of risk Θ. Hence, it is custom to postulate the $\mathbf{Q}$-dynamics of the short rate r directly in the context of derivative pricing.

2 Affine term structure of short-rate models

In the rest of this chapter, suppose that the short rate r follows

$$dr_t = b(t,r_t)dt + \sigma(t,r_t)dW_t^{\mathbf{Q}},$$

where $b(\cdot,\cdot)$ and $\sigma(\cdot,\cdot)$ are deterministic functions, and the initial data r_0 is in an open set $\mathscr{O} \subset \mathbb{R}$. Typical choices of $\mathscr{O}$ are $\mathbb{R}$ and $(0,\infty)$.

By the Markov property:

$$P(t,T) = \mathbf{E}^{\mathbf{Q}}[e^{-\int_t^T r_s ds} \overset{\times 1}{|} \mathscr{F}_t] = \mathbf{E}^{\mathbf{Q}}[e^{-\int_t^T r_s ds}|r_t] = F(t,r_t)$$

for some function $F(\cdot,\cdot)$.

If $F(t,r) \in C^{1,2}([0,T)\times\mathscr{O})$, then by the Feynman-Kac formula, $F(t,r)$ solves the following term-structure equation on $[0,T)\times\mathscr{O}$:

$$\begin{cases} \partial_t F(t,r) + \underbrace{\tfrac{1}{2}\sigma^2(t,r)\partial_{rr}F(t,r) + b(t,t)\partial_r F(t,r)}_{\text{infinitesimal generator of } r:\ \mathcal{L} = \frac{1}{2}\sigma^2\partial_{rr} + b\partial_r} - rF(t,r) = 0, \\ F(T,r) = 1. \end{cases}$$

For the case with the state space $(0,\infty)$, a parameter condition on the coefficients need to be imposed to guarantee the above term-structure PDE is well posed even without a boundary condition on $r = 0$.

Definition 2. (Affine term structure)

A short-rate model is said to provide an affine term structure (ATS) if the corresponding zero-coupon price $P(t,T) = F(t,r)$ is of the form

$$F(t,r) = e^{-A(t)-B(t)r}$$

for some functions $A(\cdot)$ and $B(\cdot)$, where $A(T) = B(T) = 0$.

Theorem 1. *A short-rate model provides an ATS iff the volatility and drift terms are of the form:*

$$\sigma^2(t,r) = a(t) + \alpha(t)r, \quad b(t,r) = b(t) + \beta(t)r$$

for some continuous functions $a(\cdot)$, $b(\cdot)$, $\alpha(\cdot)$ and $\beta(\cdot)$, and moreover, the functions $A(\cdot)$ and $B(\cdot)$ in $F(t,r) = e^{-A(t)-B(t)r}$ solve the following ODEs:

$$\begin{cases} \dfrac{dA(t)}{dt} = \dfrac{1}{2}a(t)B^2(t) - b(t)B(t), & A(T) = 0; \\ \dfrac{dB(t)}{dt} = \dfrac{1}{2}\alpha(t)B^2(t) - \beta(t)B(t) - 1, & B(T) = 0. \end{cases}$$

Proof. Inserting $F(t,r) = e^{-A(t)-B(t)r}$ into the term-structure equation, we obtain that ATS iff

$$\frac{1}{2}\sigma^2(t,r)B^2(t) - b(t,r)B(t) = \frac{dA(t)}{dt} + (\frac{dB(t)}{dt} + 1)r \tag{1}$$

for any $t \in [0,T)$ and $r \in \mathscr{O} \subset \mathbb{R}$.

If part: Substitute the ODEs for $A(\cdot)$ and $B(\cdot)$ into the RHS of the above equation:

$$RHS = \frac{1}{2}a(t)B^2(t) - b(t)B(t) + (\frac{1}{2}\alpha(t)B^2(t) - \beta(t)B(t))r.$$

Substitute $\sigma^2(t,r) = a(t) + \alpha(t)r$ and $b(t,r) = b(t) + \beta(t)r$ into its LHS:

$$\begin{aligned} LHS &= \frac{1}{2}(a(t) + \alpha(t)r)B^2(t) - (b(t) + \beta(t)r)B(t) \\ &= \frac{1}{2}a(t)B^2(t) - b(t)B(t) + (\frac{1}{2}\alpha(t)B^2(t) - \beta(t)B(t))r. \end{aligned}$$

Only if part: We only consider the case that $B_T(t)$ and $B_T^2(t)$ are linearly independent for any fixed $t \geq 0$, where we use sub T to emphasize the dependence on the maturity T. The linear dependent case is left as an exercise (see Filipovic [1] Chapter 5).

For any $T_1 > T_2 > t$,

$$\begin{pmatrix} B_{T_1}^2(t), & -B_{T_1}(t) \\ B_{T_2}^2(t), & -B_{T_2}(t) \end{pmatrix} \begin{pmatrix} \frac{1}{2}\sigma^2(t,r) \\ b(t,r) \end{pmatrix} = \begin{pmatrix} \frac{dA_{T_1}(t)}{dt} \\ \frac{dA_{T_2}(t)}{dt} \end{pmatrix} + \begin{pmatrix} \frac{dB_{T_1}(t)}{dt} + 1 \\ \frac{dB_{T_2}(t)}{dt} + 1 \end{pmatrix} r$$

Since

$$\begin{pmatrix} B_{T_1}^2(t) \\ B_{T_2}^2(t) \end{pmatrix} \text{ and } \begin{pmatrix} B_{T_1}(t) \\ B_{T_2}(t) \end{pmatrix}$$

are linearly independent by the assumption, we obtain that

$$\begin{pmatrix} \frac{1}{2}\sigma^2(t,r) \\ b(t,r) \end{pmatrix} = \begin{pmatrix} B_{T_1}^2(t), & -B_{T_1}(t) \\ B_{T_2}^2(t), & -B_{T_2}(t) \end{pmatrix}^{-1} \left(\begin{pmatrix} \frac{dA_{T_1}(t)}{dt} \\ \frac{dA_{T_2}(t)}{dt} \end{pmatrix} + \begin{pmatrix} \frac{dB_{T_1}(t)}{dt} + 1 \\ \frac{dB_{T_2}(t)}{dt} + 1 \end{pmatrix} r \right).$$

Hence, $\sigma^2(t,r)$ and $b(t,r)$ are affine functions of r. Plugging this in, LHS of (1) reads

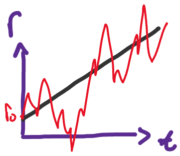
$$\frac{1}{2}a(t)B_T^2(t) - b(t)B_T(t) + (\frac{1}{2}\alpha(t)B_T^2(t) - \beta(t)B_T(t))r.$$

Terms containing t must match. This implies the two ODE. □

3 Some standard short-rate models

Vasiek is ATS $\because$ $\sigma^2(t,r) = \underset{\sigma^2}{a(t)} + \underset{0}{\alpha(t)}\, r$

$b(t,r) = \underset{a}{d(t)} + \underset{-b}{\beta(t)}\, r$



1. Vasicek Model

$$dr_t = (a - br_t)dt + \sigma dW_t^{\mathbf{Q}}$$

with $a, b, \sigma > 0$.

(1) The solution is Apply Ito to $e^{bt} r_t$

$$r_t = r_0 e^{-bt} + \frac{a}{b}(1 - e^{-bt}) + \sigma \int_0^t e^{b(s-t)} dW_s^{\mathbf{Q}}.$$

(2) The expectation of r_t is

$$\mathbf{E}^{\mathbf{Q}}[r_t] = r_0 e^{-bt} + \frac{a}{b}(1 - e^{-bt}) \to \frac{a}{b}, \quad \text{as } t \uparrow \infty$$

Method 1. By explicit solution,

$Var[r_t] = Var[\sigma \int_0^t e^{b(s-t)} dW_s^Q]$

$= \sigma^2 E^Q[(\int_0^t e^{b(s-t)} dW_s^Q)^2]$

$= \sigma^2 \int_0^t e^{2b(s-t)} ds$

(3) The variance of r_t is

$$Var[r_t] = \frac{\sigma^2}{2b}(1 - e^{-2bt}) \to \frac{\sigma^2}{2b}, \quad \text{as } t \uparrow \infty.$$

(4) The price of the zero-coupon bond is $P(t,T) = e^{-A(t)-B(t)r}$ where

Method 2. By def of Var.

$$\begin{cases} \frac{dA(t)}{dt} = \frac{1}{2}\sigma^2 B^2(t) - aB(t), \quad A(T) = 0; \\ \frac{dB(t)}{dt} = bB(t) - 1, \quad B(T) = 0. \end{cases}$$

$Var[r_t] = E^Q[r_t^2] - (E^Q[r_t])^2$

By Itô, $dr_t^2 = 2r_t[(a - br_t)dt + \sigma dW_t^Q] + \sigma^2 dt$

$= [\sigma^2 + 2ar_t - 2br_t^2] dt + 2\sigma r_t dW_t^Q$

The solution is

$$\begin{cases} B(t) = \frac{-1}{b}(e^{-b(T-t)} - 1) \\ A(t) = \int_t^T \left[aB(s) - \frac{1}{2}\sigma^2 B^2(s) \right] ds. \end{cases}$$

$d e^{2bt} r_t^2 = e^{2bt}[\sigma^2 + 2ar_t] dt + e^{2bt} 2\sigma r_t dW_t^Q$

(5) The drawback is that the short rate could be negative : $\mathbf{Q}(r_t < 0) > 0.$ $\Rightarrow e^{2bt} r_t^2 - r_0^2 = \int_0^t e^{2bs}[\sigma^2 + 2ar_s] ds + e^{2bs} 2\sigma r_s dW_s^Q$

2.Cox-Ingersoll-Ross (CIR) Model

$E^Q[r_t^2] = e^{-2bt}\left(r_0^2 + \int_0^t e^{2bs}(\sigma^2 + 2aE^Q[r_s]) ds\right)$

Term structure PDE for $F(t,r)$:

$\partial_t F + \frac{1}{2}\sigma^2 \partial_{rr} F + (a - br)\partial_r F - rF = 0$

Since $F(t,r) = e^{-A(t) - B(t)r}$,

$F\left(-\frac{dA(t)}{dt} - \frac{dB(t)}{dt} r\right) + \frac{1}{2}\sigma^2 F B^2 - (a - br) F B - rF = 0, \ \forall r.$

$\Rightarrow \underbrace{-\frac{dA(t)}{dt} + \frac{1}{2}\sigma^2 B^2 - aB}_{=0} + \underbrace{\left(-\frac{dB(t)}{dt} + bB - 1\right)}_{=0} r = 0, \ \forall r.$

$$dr_t = (a - br_t)dt + \sigma\sqrt{r_t} dW_t^{\mathbf{Q}}$$

CIR is ATS $\because$ $\sigma^2(t,r) = \underset{0}{a(t)} + \underset{\sigma^2}{\alpha(t)}\, r$

$b(t,r) = \underset{a}{b(t)} + \underset{-b}{\beta(t)}\, r$

with $a, b, \sigma > 0$.

(1) No explicit solution since r_t is non Gaussian. However, the short rate is always nonnegative: $r_t \geq 0$. Moreover, by using Feller's test, one can show that $r_t > 0$ if the parameters satisfy $\sigma^2 \leq 2a$ and $r_0 > 0$ (so no need to impose a boundary condition on $r = 0$ for the term-structure PDE). See Jeanblanc et al [2] Chapter 6 for the proof.

(2) The expectation of r_t. Applying Itô's formula to $e^{bt} r_t$ yields

$$d(e^{bt} r_t) = ae^{bt} dt + \sigma e^{bt} \sqrt{r_t} dW_t^{\mathbf{Q}}.$$

Hence,

$$e^{bt} r_t = r_0 + a\int_0^t e^{bs} ds + \sigma \int_0^t e^{bs} \sqrt{r_s} dW_s^{\mathbf{Q}}$$
$$= r_0 + \frac{a}{b}(e^{bt} - 1) + \sigma \int_0^t e^{bs} \sqrt{r_s} dW_s^{\mathbf{Q}}.$$

Taking expectation gives us

$$e^{bt} r_t = r_0 + a\int_0^t e^{bs} ds + \sigma \int_0^t e^{bs} \sqrt{r_s} dW_s^{\mathbf{Q}}$$

$$= r_0 + \frac{a}{b}(e^{bt} - 1) + \sigma \int_0^t e^{bs} \sqrt{r_s} dW_s^{\mathbf{Q}}.$$

Taking expectation gives us

$$\mathbf{E}^{\mathbf{Q}}[r_t] = r_0 e^{-bt} + \frac{a}{b}(1 - e^{-bt}) \to \frac{a}{b}, \quad \text{as } t \uparrow \infty.$$

(3) The variance of r_t. Introduce $X_t = e^{bt} r_t$. Then

By def. $\mathrm{Var}[r_t] = \mathbf{E}^{Q}[r_t^2] - \left(\mathbf{E}^{Q}[r_t]\right)^2$

By Itô, $dr_t^2 = \left[(2a+\sigma^2) r_t - 2b r_t^2\right] dt + 2\sigma r_t \sqrt{r_t}\, dW_t^{Q}$

$d e^{2bt} r_t^2 = e^{2bt}(2a+\sigma^2) r_t dt + e^{2bt} 2\sigma r_t \sqrt{r_t}\, dW_t^{Q}$

$\Rightarrow \mathbf{E}^{Q}[r_t^2] = e^{-2bt}\left[r_0^2 + \int_0^t e^{2bs}(2a+\sigma^2)\mathbf{E}^{Q}[r_s] ds \right]$

$$dX_t = a e^{bt} dt + \sigma e^{\frac{1}{2}bt} \sqrt{X_t} dW_t^{\mathbf{Q}}.$$

Applying Itô's formula to X_t^2 yields

$$dX_t^2 = 2a e^{bt} X_t dt + 2\sigma e^{\frac{1}{2}bt} X_t^{\frac{3}{2}} dW_t^{\mathbf{Q}} + \sigma^2 e^{bt} X_t dt.$$

Hence,

$$X_t^2 = X_0^2 + (2a + \sigma^2) \int_0^t e^{bs} X_s ds + 2\sigma \int_0^t e^{\frac{1}{2}bs} X_s^{\frac{3}{2}} dW_s^{\mathbf{Q}}.$$

Taking expectation gives us

$$\mathbf{E}^{\mathbf{Q}}[X_t^2] = X_0^2 + (2a + \sigma^2) \int_0^t e^{bs} \mathbf{E}^{\mathbf{Q}}[X_s] ds$$

$$= r_0^2 + (2a + \sigma^2) \int_0^t e^{bs} (r_0 + \frac{a}{b}(e^{bs} - 1)) ds.$$

In turns,

$$\mathbf{E}^{\mathbf{Q}}[r_t^2] = e^{-2bt} \mathbf{E}^{\mathbf{Q}}[X_t^2]$$

$$= e^{-2bt} r_0^2 + \frac{2a + \sigma^2}{b}(r_0 - \frac{a}{b})(e^{-bt} - e^{-2bt}) + \frac{a(2a + \sigma^2)}{2b^2}(1 - e^{-2bt}).$$

Finally, we obtain

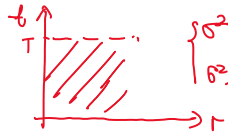
$$
\begin{aligned}
Var[r_t] &= \mathbf{E}^{\mathbf{Q}}[r_t^2] - (\mathbf{E}^{\mathbf{Q}}[r_t])^2 \\
&= \frac{\sigma^2}{b} r_0 (e^{-bt} - e^{-2bt}) + \frac{a\sigma^2}{2b^2}(1 - 2e^{-bt} + e^{-2bt}) \to \frac{a\sigma^2}{2b^2}, \quad \text{as } t \uparrow \infty.
\end{aligned}
$$

(4) The price of the zero-coupon bond is $P(t,T) = e^{-A(t)-B(t)r}$ where

$$
\begin{cases}
\dfrac{dA(t)}{dt} = -aB(t), \quad A(T) = 0; \\
\dfrac{dB(t)}{dt} = \dfrac{1}{2}\sigma^2 B^2(t) + bB(t) - 1, \quad B(T) = 0.
\end{cases}
$$

Term structure PDE for $F(t,r)$

$$
\begin{cases}
\partial_t F + \frac{1}{2}\sigma^2 r \partial_{rr} F + (a - br)\partial_r F - rF = 0 \\
F(T,r) = 1
\end{cases}
\qquad (t,r) \in [0,T) \times (0,\infty)
$$



$\begin{cases} \sigma^2 \le 2a & \text{no boundary on } r=0 \\ \sigma^2 > 2a, & \text{need to impose a boundary at } r=0. \end{cases}$

The equation for $B(\cdot)$ is a Riccati ODE. The solution is

$$
\begin{cases}
B(t) = \dfrac{\sinh(\gamma(T-t))}{\gamma\cosh(\gamma(T-t)) + \frac{b}{2}\sinh(\gamma(T-t))} \\
A(t) = -\dfrac{2a}{\sigma^2} \ln\left[\dfrac{\gamma e^{\frac{1}{2}b(T-t)}}{\gamma\cosh(\gamma(T-t)) + \frac{b}{2}\sinh(\gamma(T-t))}\right]
\end{cases}
$$

where $\gamma = \frac{1}{2}\sqrt{b^2 + 2\sigma^2}$ and

$$
\sinh(\gamma(T-t)) = \frac{1}{2}(e^{\gamma(T-t)} - e^{-\gamma(T-t)}); \quad \cosh(\gamma(T-t)) = \frac{1}{2}(e^{\gamma(T-t)} + e^{-\gamma(T-t)}).
$$

See Shreve [3] Chapter 6 for the proof.
3. Extended Vasicek Model

$$
dr_t = (a(t) - b(t)r_t)dt + \sigma(t)dW_t^{\mathbf{Q}}
$$

with deterministic functions $a(t), b(t), \sigma(t) > 0$.

4. Extended CIR Model

$$
dr_t = (a(t) - b(t)r_t)dt + \sigma(t)\sqrt{r_t}dW_t^{\mathbf{Q}}
$$

with deterministic functions $a(t), b(t), \sigma(t) > 0$.

4 Exercises

Exercise 1. (CIR model)
Let $W = (W^1, \cdots, W^d)$ be a d-dimensional Brownian motion. For $1 \le j \le d$, let X^j be the solution of the following Ornstein-Uhlenbeck SDE:

$$dX_t^j = -\frac{b}{2}X_t^j dt + \frac{\sigma}{2}dW_t^j, \quad X_0^j = x^j$$

where $b > 0$, $\sigma > 0$.

1. Show that
$$X_t^j = e^{-\frac{b}{2}t}[x^j + \frac{\sigma}{2}\int_0^t e^{\frac{b}{2}u}dW_u^j]$$
2. Calculate $m(t) = \mathbf{E}[X_t^j]$ and $v(t) = Var(X_t)$ and their limites when $t \to \infty$
3. Define $r_t = \sum_{j=1}^d |X_t^j|^2$. Show that
$$dr_t = (\frac{d\sigma^2}{4} - br_t)dt + \sigma\sqrt{r_t}dB_t$$
where
$$B_t = \int_0^t \frac{1}{\sqrt{r_s}}\sum_{j=1}^d X_s^j dW_s^j$$
is a Brownian motion.
4. Prove that $X_t^1 \cdots, X_t^d$ are iid normal random variables $N(m(t), v(t))$. Therefore, $r_t = \sum_{j=1}^d |X_t^j|^2$ is the sum of square of iid normal random variables, and hence r_t has χ^2-distribution.
5. Prove that the moment generating function of $|X_t^j|^2$ is given by
$$\mathbf{E}[\exp\{\mu|X_t^j|^2\}] = \frac{1}{\sqrt{1-2v(t)\mu}}\exp\left\{\frac{\mu|m(t)|^2}{1-2v(t)\mu}\right\}$$
for any $\mu < \frac{1}{2v(t)}$.
6. Based on (5), prove that the moment generating function of r_t is given by
$$\mathbf{E}[\exp\{\mu r_t\}] = \frac{1}{(\sqrt{1-2v(t)\mu})^d}\exp\left\{\frac{d\mu|m(t)|^2}{1-2v(t)\mu}\right\}$$
for any $\mu < \frac{1}{2v(t)}$.

Exercise 2. (Ho-Lee model and corresponding forward rate)

The one dimensional Ho-Lee model is given by

$$dr_t = b(t)dt + \sigma dW_t^{\mathbf{Q}}$$

under the EMM $\mathbf{Q}$, where $b(\cdot)$ is some deterministic function, and $\sigma > 0$. The corresponding zero-coupon bond price $P(t,T)$ is calculated as

$$P(t,T) = \mathbf{E}^{\mathbf{Q}}[e^{-\int_t^T r_s ds}|\mathscr{F}_t].$$

1. Since $(r_t)_{t\geq 0}$ admits Markov property, there exists some measurable function $F(t,r)$ on $[0,T]\times\mathbb{R}$ such that $P(t,T) = F(t,r_t)$. Suppose that $F(t,r)$ is in

$C^{1,2}([0,T) \times \mathbb{R})$. Write down the PDE for $F(t,r)$ by using the Feynman-Kac formula.

2. Explain why $F(t,r)$ has the affine form:

$$F(t,r) = e^{-A(t)-B(t)r}.$$

Prove that $A(t)$ and $B(t)$ satisfy the following ODE system:

$$\frac{dA(t)}{dt} = -b(t)B(t) + \frac{1}{2}\sigma^2|B(t)|^2;$$
$$\frac{dB(t)}{dt} = -1,$$

and solve the above ODE system to get the expressions for $A(t)$ and $B(t)$.

3. Recall that the forward rate $f(t,T)$ is defined as

$$f(t,T) = -\frac{\partial \ln P(t,T)}{\partial T}.$$

Now suppose $b(t)$ has the form $b(t) = \partial_t f(0,t) + \sigma^2 t$. Prove that the short rate r_t has the dynamic

$$r_t = f(0,t) + \frac{\sigma^2 t^2}{2} + \sigma W_t^{\mathbf{Q}},$$

and the forward rate has $f(t,T)$ has the dynamic

$$f(t,T) = f(0,T) + \sigma^2 t(T - \frac{t}{2}) + \sigma W_t^{\mathbf{Q}}.$$

Therefore the volatility $\sigma(t,T)$ of the forward rate is a constant: $\sigma(t,T) = \sigma$.

4. Show that the drift of the forward rate is nothing but $\sigma(t,T)\int_t^T \sigma(t,s)ds$.

References

1. Filipovic, Damir. *Term-Structure Models. A Graduate Course.* Springer, 2009.
2. Jeanblanc, Monique, Marc Yor, and Marc Chesney. *Mathematical methods for financial markets.* Springer, 2009.
3. Shreve, Steven E. *Stochastic calculus for finance II: Continuous-time models.* Springer, 2004.