THE UNIVERSITY OF WARWICK

FOURTH YEAR EXAMINATION: 2020

QUANTUM MECHANICS

Time Allowed: **3 hours**

Read all instructions carefully. Please note also the guidance you have received in advance on the departmental 'Warwick Mathematics Exams 2020' webpage.

Calculators, wikipedia and interactive internet resources are not needed and are not permitted in this examination. You are not allowed to confer with other people. You may use module materials and resources from the module webpage.

ANSWER COMPULSORY QUESTION 1 AND TWO FURTHER QUESTIONS out of the four optional questions 2, 3, 4 and 5.

On completion of the assessment, you must upload your answer to Moodle as a single PDF document if possible, although multiple files (2 or 3) are permitted. You have an additional 45 minutes to make the upload, and instructions are available on the departmental 'Warwick Mathematics Exams 2020' webpage.

You must not upload answers to more than 3 questions, including Question 1. If you do, you will only be given credit for your Question 1 and the first two other answers.

The numbers in the margin indicate approximately how many marks are available for each part of a question. The compulsory question is worth twice the number of marks of each optional question. Note that the marks do not sum to 100.

COMPULSORY QUESTION

1. (i) Write down the Schrödinger equation for a particle in $\mathbb{R}$ under an external potential V. **[5]**

(ii) Show that the position operator X, with domain $D(X)$ being the space of Schwartz functions, is a symmetric operator in $L^2(\mathbb{R})$. **[5]**

(iii) Show that the operator X in (ii) is unbounded. **[5]**

(iv) Give the definition of a density operator ρ on the Hilbert space $\mathbb{C}^n$. Give the definition of the state that corresponds to ρ. **[5]**

(v) State without proof the Heisenberg uncertainty principle for states in $\mathbb{C}^n$. **[5]**

 Question 1 continued overleaf

(vi) Let X be the operator $f \mapsto xf$ and P the operator $f \mapsto -\mathrm{i}f'$ on $L^2(\mathbb{R})$. Show that $[X, P] = \mathrm{i}$. **[5]**

(vii) What is the evolution operator? Give its norm, and describe its group properties. **[5]**

(viii) List all eigenvalues of the spherical Laplacian (Laplace-Beltrami operator), with their multiplicities. **[5]**

OPTIONAL QUESTIONS

2. (a) State the Heisenberg uncertainty principle for functions in $L^2(\mathbb{R})$. **[5]**

(b) Prove it. **[5]**

(c) Show that, if $\psi(x,t)$ satisfies the Schrödinger equation, the L^2 norm $\|\psi(\cdot,t)\|$ is conserved. **[5]**

Consider the functional $E(\psi) = \int(|\psi'(x)|^2 + V(x)|\psi(x)|^2)\mathrm{d}x$, where V is a function $\mathbb{R} \to [C, \infty)$, for some constant C. Assume that it has a unique minimiser among functions with norm 1, which belongs to the Sobolev space $H^2(\mathbb{R})$.

(d) Show that the minimiser is an eigenvector of the operator $H = -\Delta + V$ with the lowest eigenvalue. **[5]**

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3. (a) Give a definition of the Sobolev space $H^2(\mathbb{R})$. **[4]**

(b) Give the example of a function $f \in L^2(\mathbb{R})$ that does not belong to $H^2(\mathbb{R})$. **[2]**

(c) Prove that the Laplacian $-\Delta$ in $L^2(\mathbb{R})$, with domain $D(-\Delta) = H^2(\mathbb{R})$, is self-adjoint. **[6]**

(d) Let $H = -\Delta - \frac{1}{\|x\|}$ in $L^2(\mathbb{R}^3)$. Show that there exist $\varepsilon < 1$ and $c > 0$ such that

$$\left\| \tfrac{1}{\|x\|} f \right\| \leq \varepsilon \|\Delta f\| + c\|f\|$$

for all $f \in H^2(\mathbb{R}^3)$. (Hint: for the norm of $\frac{1}{\|x\|}f$, consider separately the domain $\|x\| \leq a$ and $\|x\| > a$. The Sobolev inequality $\|f\|_\infty \leq C\|f\|_{H^2}$, with C a constant, could be useful.) **[8]**

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4. Recall the Hamiltonian for the hydrogen atom: $H = -\Delta - \frac{1}{\|x\|}$ on $L^2(\mathbb{R}^3)$ (with domain $\mathcal{D}(H) = H^2(\mathbb{R}^3)$ so that it is self-adjoint).

(a) Prove that H is bounded below. (Hint: You can use the Sobolev inequality $\|\nabla f\|_2^2 \geq 3(\frac{\pi}{2})^{4/3}\|f\|_6^2$.) **[6]**

(b) Show that a function of the form $f(x) = \mathrm{e}^{-a\|x\|}$ with $a > 0$ (to be found) is an eigenvector, and find the corresponding eigenvalue. **[4]**

(c) Describe the full spectrum of H, including the multiplicity of all eigenvalues (no proof required). **[5]**

(d) Prove that the spectrum of H contains the half-line $[0, \infty)$. **[5]**

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5. (a) Let A be the integral operator $(Af)(x) = \int_{\mathbb{R}} a(x,y)f(y)\mathrm{d}y$. We assume that the integral kernel $a(x,y)$ is continuous. Prove that

$$\|A\|^2 \leq \left[\sup_{x\in\mathbb{R}} \int_{\mathbb{R}} |a(x,y)|\mathrm{d}y\right] \left[\sup_{y\in\mathbb{R}} \int_{\mathbb{R}} |a(x,y)|\mathrm{d}x\right].$$

[5]

(b) Sketch the construction of the operator $\mathrm{e}^{\frac{1}{2}t\Delta}$, where Δ is the Laplacian. **[5]**

(c) Show that $\mathrm{e}^{\frac{1}{2}t\Delta}$ is an integral operator. Write down its integral kernel. **[5]**

(d) Let $H = -\frac{1}{2}\Delta + V$ in $L^2(\mathbb{R}^d)$ with V a bounded continuous function $\mathbb{R}^d \to \mathbb{R}$. Write down the Feynman-Kac formula for the integral of the operator e^{-tH}. Define all the quantities. **[5]**
