ST403X / MA4F70
UNIVERSITY OF WARWICK
FINAL YEAR EXAMINATIONS: SUMMER 2016

**BROWNIAN MOTION**
Time Allowed: 2 Hours

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Full marks may be gained by correctly answering 3 complete questions. Candidates may attempt all questions. Marks will be awarded for the best 3 answers only.

Read carefully the instructions on the answer book and make sure that the particulars required are entered on each answer book

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1. Let $\beta(\cdot)$ be standard, one dimensional Brownian motion starting from zero and $P_0$ denote its distribution.

   (a) State and prove Blumenthal's 0-1 law. [ 6 marks]

   (b) Define the $\sigma-$algebra of tail events for Brownian motion. Show that any event in this $\sigma-$algebra has probability zero or one. [6 marks]

   (c) Show that a.s.

$$\limsup_{t\to 0} \frac{\beta(t)}{\sqrt{t}} = +\infty \qquad \text{and} \qquad \liminf_{t\to 0} \frac{\beta(t)}{\sqrt{t}} = -\infty.$$

[8 marks]

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2. Let $P_x, x \in \mathbb{R}$, denote the Wiener measure corresponding to standard, one dimensional Brownian motion $\beta(\cdot)$ starting from $x$. For $t > 0$, consider the $\sigma$-algebra $\sigma(\beta(t))$ generated by the value of Brownian motion at time $t$ and the natural filtration $\mathcal{F}_t := \sigma\big(\beta(s) \colon s \leq t\big)$ generated by the values of Brownian motion up to time $t$.

   (a) Consider a function $f \in C_b^2(\mathbb{R})$, i.e. twice continuously differentiable function with bounded derivatives up to order two. Show that

$$M_t := f(\beta(t)) - \frac{1}{2}\int_0^t f''(\beta(s))\, ds, \qquad t \geq 0,$$

   is a martingale with respect to the filtration $\mathcal{F}_t$. [10 marks]

   (b) Compute the following conditional expectations

      (i) $E_0\big[(\beta(t))^4 \,|\, \mathcal{F}_s\big]$, with $s < t$, [5 marks]

      (ii) $E_0\big[\beta(t) \,|\, \sigma(\beta(1))\big]$. [5 marks]

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Continued ...

3. (a) State and prove the Feynman-Kac formula related to parabolic PDEs. [10 marks]
   (b) Assume that $d = 1$, i.e. that the Brownian motion $\beta(\cdot)$ is one dimensional. Let $\gamma > 0$ and $\tau := \inf\{t\colon |\beta(t)| \geq 1\}$
      (i) Compute $E_0[e^{-\gamma\tau}]$. [7 marks]
      (ii) Compute $E_0[\tau]$. [3 marks]

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4. (a) State Donsker's Theorem. [5 marks]
   (b) Let $(S_n)_{n\geq 0}$ be a simple, symmetric one dimensional random walk with $S_0 = 0$.
      (i) Show that the number of sites the walk has visited by time $n$ is given by

$$R_n = 1 + \max_{k\leq n} S_k - \min_{k\leq n} S_k.$$

[5 marks]

      (ii) Show that $R_n/\sqrt{n}$ converges in distribution, as $n$ tends to infinity. Characterise the limit. [5 marks]
      (iii) Compute explicitly (i.e. give an integral formula) the limiting distribution of $n^{-1/2}\max_{k\leq n} S_k$. [5 marks]

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**END**

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1. Let $\beta(\cdot)$ be standard, one dimensional Brownian motion starting from zero and $P_0$ denote its distribution.

   (a) State and prove Blumenthal's 0-1 law. [ 6 marks]

   **Answer.** [BOOKWORK] Let $\mathcal{F}^+(t) := \cap_{s>t}\mathcal{F}_s$ be the germ $\sigma$ algebra. Then any event $A \in \mathcal{F}^+(0)$ has probability zero or one **(2 mark)**. For the proof we show first that $\beta(\cdot + t) - \beta(t)$ is Brownian motion independent of $\mathcal{F}^+(t)$. **(2 marks)**Then letting $t \to 0$ and using continuity of Brownian motion, shows that any $A \in \mathcal{F}^+(0)$ is independent of itself **(2 marks)**.

   (b) Define the $\sigma-$algebra of tail events for Brownian motion. Show that any event in this $\sigma-$algebra has probability zero or one. [6 marks]

   **Answer.** [BOOKWORK] The tail $\sigma$-algebra is defined as $\mathcal{T} = \cap_{t>0}\, \sigma(\beta(s) \colon s \geq t)$ **(2 marks)**. To show that any tail event is trivial consider the process

$$x(t) := \begin{cases} t\beta(1/t), & \text{if } t \neq 0, \\ 0, & \text{if } t = 0 \end{cases}$$

   By time reversal this process is also Brownian motion, so any tail event for $\beta(\cdot)$ will be a germ event for $x(\cdot)$ and by Blumenthal's law this will have probability 0 or 1 **(4 marks)**.

   (c) Show that a.s.

$$\limsup_{t\to 0} \frac{\beta(t)}{\sqrt{t}} = +\infty \qquad \text{and} \qquad \liminf_{t\to 0} \frac{\beta(t)}{\sqrt{t}} = -\infty.$$

   [8 marks]

   **Answer** [UNSEEN EXAMPLE] We have **(2 marks)**

$$P_0(\limsup_{t\to 0} \beta(t)/\sqrt{t} = +\infty) = \lim_{L\to\infty} P_0(\limsup_{t\to 0} \beta(t)/\sqrt{t} > L).$$

   Show that each probability in the RHS is equal to one. The event is in the germ $\sigma$-algebra and by Blumenthal's law its probability will be zero or one. So it suffices to show that it is positive **(2 marks)**. To do so use the monotonicity of the event therein to get **(3 marks)**

$$P_0(\limsup_{t\to 0} \beta(t)/\sqrt{t} > L) \geq \lim_{t\to 0} P_0(\beta(t)/\sqrt{t} > L) = P(\beta(1) > L) > 0.$$

   Upon changing $\beta(t)$ to $-\beta(t)$ we obtain the second statement **(1 marks)**.

Continued ...

2. Let $P_x, x \in \mathbb{R}$, denote the Wiener measure corresponding to standard, one dimensional Brownian motion starting from $x$. For $t > 0$, consider the $\sigma$-algebra $\sigma(\beta(t))$ generated by the value of Brownian motion at time $t$ and the natural filtration $\mathcal{F}_t := \sigma\big(\beta(s) \colon s \le t\big)$ generated by the values of Brownian motion up to time $t$.

   (a) Consider a function $f \in C_b^2(\mathbb{R})$, i.e. twice continuously differentiable function with bounded derivatives up to order two. Show that

$$M_t := f(\beta(t)) - \frac{1}{2}\int_0^t f''(\beta(s))\,ds, \qquad t \ge 0,$$

   is a martingale with respect to the filtration $\mathcal{F}_t$. [5 marks]
   **Answer.** [SEEN EXAMPLE] We can write $E_x[M_t \,|\, \mathcal{F}_s] = M_s + E_x[M_t - M_s \,|\, \mathcal{F}_s]$ and we need to show that the last expectation is equal to zero. First (**4 marks**)

$$\begin{aligned} E_x[f(\beta(t)) - f(\beta(s)) \,|\, \mathcal{F}_s] &= E_x[f(\beta(t)) \,|\, \mathcal{F}_s] - f(\beta(s)) \\ &= \int f(y) p_{t-s}(\beta(s), y) - f(\beta(s)), \end{aligned}$$

   where $p_t(x, y)$ is the heat kernel and where we used the Markov property. Then compute (**6 marks**)

$$\begin{aligned} \frac{1}{2}E_x\Big[\int_s^t f''(\beta(r)) \,|\, \mathcal{F}_s\Big]\,dr &= \frac{1}{2}\int_s^t E_x[f''(\beta(r)) \,|\, \mathcal{F}_s] \\ &= \frac{1}{2}\int_s^t \int f''(y) p_{r-s}(\beta(s), y)\,dy\,dr \\ &= \frac{1}{2}\int_s^t \int f(y) \frac{\partial^2}{\partial y^2} p_{r-s}(\beta(s), y)\,dy\,dr \\ &= \int_s^t \int f(y) \frac{\partial}{\partial r} p_{r-s}(\beta(s), y)\,dy\,dr \\ &= \int \int_s^t f(y) \frac{\partial}{\partial r} p_{r-s}(\beta(s), y)\,dr\,dy \\ &= \int f(y) p_{t-s}(\beta(s), y)\,dy - f(\beta(s)) \end{aligned}$$

   (b) Compute the following conditional expectations

      (i) $E_0\big[(\beta(t))^4 \,|\, \mathcal{F}_s\big]$, with $s < t$, [5 marks]
      **Answer.** [UNSEEN EXAMPLE] If $s > t$ then the conditional expectation is equal to $(\beta(t))^4$ (**1 mark**). If $s < t$, consider the function $f(x) = x^4$ and apply the Markov property"

$$\begin{aligned} E_0\big[(\beta(t))^4 \,|\, \mathcal{F}_s\big] &= E_{\beta(s)}[(\beta(t-s))^4] = \int (y + \beta(s))^4 p_{t-s}(y) \\ &\int y^4 p_{t-s}(y) + 6(\beta(s))^2 \int y^2 p_{t-s}(y) + (\beta(s))^4 \\ &= 3(t-s)^2 + 6(\beta(s))^2(t-s) + (\beta(s))^4, \end{aligned}$$

      where $p_t(y)$ is the heat kernel (**4 marks**).

(ii) $E_0[\beta(t) \,|\, \sigma(\beta(1))]$. [5 marks]
**Answer.** [UNSEEN EXAMPLE] If $t \geq 1$, then the conditional expectation equals $\beta(1)$ (**1 mark**). If $t < 1$,

$$E_0[\beta(t) \,|\, \sigma(\beta(1))] = E_0[\beta(t) \,|\, \beta(1)] = \int y \, \frac{p_{1-t}(y, \beta(1))\, p_t(y)}{p_1(\beta(1))} \, dy$$

(**2 marks**) After a couple of easy computations this equals (**2 marks**)

$$\left(\frac{1}{2\pi t(1-t)}\right)^{1/2} \int y e^{-\frac{(y-t\beta(1))}{2(t(1-t))}} \, dy = t\beta(1)$$

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3. (a) State the Feynman-Kac formula related to parabolic PDEs. [5 marks]

**Answer.** [BOOKWORK] Let $V(t,x) \in C_b^{1,2}$, a bounded function with bounded derivatives and $f \in C_b(\mathbb{R})$. The solution to the equation

$$\begin{aligned} u_t &= \frac{1}{2}\Delta u + Vu, \qquad && t > 0, x \in \mathbb{R}^d, \\ u(0,x) &= f(x), && x \in \mathbb{R}. \end{aligned}$$

Then the solution of this problem is given by

$$u(t,x) = E_x\left[f(\beta(t)) e^{\int_0^t V(t-s,\beta(s))ds}\right].$$

(b) State the Feynman-Kac formula related to the Dirichlet problem of a bounded domain in $\mathbb{R}^d$. State any further assumptions needed. [5 marks]

**Answer** [BOOKWORK] Assume that the boundary $\partial D$ is smooth. Let $V(x) \in C_b^2 D$, a bounded function with bounded derivatives and $f \in C_b(\partial D)$. The solution to the equation

$$\begin{aligned} \frac{1}{2}\Delta u + Vu &= 0, \qquad && x \in D, \\ u(x) &= f(x), && x \in \partial D. \end{aligned}$$

Then the solution of this problem is given by

$$u(x) = E_x\left[f(\beta(\tau)) e^{\int_0^\tau V(\beta(s))ds}\right], \qquad x \in D$$

and $\tau$ is the hitting time of the boundary.

(c) Assume that $d = 1$, i.e. that the Brownian motion $\beta(\cdot)$ is one dimensional. Let $\gamma, \alpha > 0$ and $\tau := \inf\{t \colon |\beta(t)| \geq a\}$ Compute $E_0[e^{\gamma\tau}]$. [7 marks]

**Answer.** [UNSEEN EXAMPLE] Apply Feynman-Kac formula to see that $u(x) := E_x[e^{-\gamma\tau}], x \in (-1,1)$, solves the problem

$$\begin{aligned} \frac{1}{2}u_{xx} - \gamma u &= 0, \qquad && x \in (-1,1), \\ u(1) = u(-1) &= 1, && \end{aligned}$$

(**3 marks**)

The general solution to this equation is $u(x) = A\cosh(\sqrt{2\gamma}x) + B\sinh(\sqrt{2\gamma}x)$. Satisfying the boundary conditions gives

$$1 = A\cosh(\sqrt{2\gamma}) + B\sinh(\sqrt{2\gamma})$$
$$1 = A\cosh(\sqrt{2\gamma}) - B\sinh(\sqrt{2\gamma})$$

**(3 marks)**
Adding and subtracting the equations gives $A\cos(\sqrt{2\gamma}a) = 1$ and $B\sin(\sqrt{2\gamma}a) = 0$. From this we can choose $B = 0$ and we have that

$$u(x) = \frac{\cos(\sqrt{2\gamma}x)}{\cos(\sqrt{2\gamma}a)} \quad \text{or} \quad E_0[e^{-\gamma\tau}] = \frac{1}{\cosh(\sqrt{2\gamma})}.$$

**(1 marks)**

(d) Compute $E_0[\tau]$. [3 marks]

**Answer.** Differentiate $E_0[e^{-\gamma\tau}]$ with respect to $\gamma$ and set $\gamma = 0$ to get

$$E_0[\tau] = \frac{1}{(2\gamma)^{1/2}} \frac{\sinh(\sqrt{2\gamma})}{(\cosh(\sqrt{2\gamma}))^2}\Big|_{\gamma=0} = 1.$$

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Continued ...

4. (a) State Donsker's Theorem. [5 marks]

**Answer.** [BOOKWORK] Let $(S_n)_{n\geq 1}$ be random walk with jump increments having mean zero and variance one. Set $S_N^*(t)$ be the continuous function, which equals $S_{tN}$, for $t \in N^{-1}\mathbb{N}$ and linearly interpolated between these values, otherwise. Then $S_N^*(\cdot)$ converges in distribution, on the space of continuous function equipped with the *sup* topology to Brownian motion.

(b) Let $(S_n)_{n\geq 0}$ be a simple, symmetric one dimensional random walk with $S_0 = 0$.

(i) Show that the number of sites the walk has visited by time $n$ is given by

$$R_n = 1 + \max_{k\leq n} S_k - \min_{k\leq n} S_k.$$

[5 marks]

**Answer.** [UNSEEN EXAMPLE] Since the walk is one dimensional all the sites visited will be exactly those in the interval $[\min_{k\leq n} S_k, \max_{k\leq n} S_k]$. The number of sites in this interval is the requested one.

(ii) Show that $R_n/\sqrt{n}$ converges in distribution, as $n$ tends to infinity. [5 marks]

**Answer.** [UNSEEN EXAMPLE] The limit of $R_n/\sqrt{n}$ is equal to the limit of

$$\frac{1}{\sqrt{n}}\max_{k\leq n} S_k - \frac{1}{\sqrt{n}}\min_{k\leq n} S_k$$

This is a continuous function of the (interpolated path) of $(S_n)_{n\geq 0}$ and therefore, by Donsker's theorem, it will converge in distribution to the distribution of

$$\max_{t\leq 1} \beta(t) - \min_{t\leq 1} \beta(t)$$

(iii) Compute explicitly the limiting distribution of $n^{-1/2}\max_{k\leq n} S_k$. [5 marks]

**Answer.** [UNSEEN EXAMPLE] By Donsker's thm. this distribution converges to the distribution of $\max_{t\leq 1}\beta(t)$ with B.M. starting at zero. The law is given by the reflection principle and equals $2P_0(\beta(t) > x)$.

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