Stochastic Modelling and Random Processes

Example sheet 5

1. **Contact process** [14]
Consider the CP $(\eta_t : t \geq 0)$ on the complete graph $\Lambda = \{1, \ldots, L\}$ (i.e. $q(i,j) = \lambda$ for all $i \neq j$) with state space $S = \{0,1\}^L$ and transition rates

$$c(\eta, \eta^i) = \eta(i) + \lambda\big(1 - \eta(i)\big) \sum_{j \neq i} \eta(j) \ ,$$

and generator given by $\quad (\mathcal{L}f)(\eta) = \sum_{i\in\Lambda} c(\eta, \eta^i)\big(f(\eta^i) - f(\eta)\big)$.

Recall that $\eta, \eta^i \in S$ are connected states such that the state of individual i is flipped:

$$\eta^i(k) = \begin{cases} 1 - \eta(k) & , \ k = i \\ \quad \eta(k) & , \ k \neq i \end{cases}$$

(a) Let $N(\eta) := \sum_{i\in\Lambda} \eta(i) \in \{0, \ldots, L\}$ be the number of infected individuals in configuration η. For any function $f : \{0, \ldots, L\} \to \mathbb{R}$ show that we can write for the composed function $f \circ N : S \to \mathbb{R}$

$$(\mathcal{L}f \circ N)(\eta) = \lambda(L - N)N\big[f(N+1) - f(N)\big] + N\big[f(N-1) - f(N)\big]$$

for all $\eta \in S$, where we use the simplified notation $N = N(\eta)$ on the right-hand side.
Hint: Use $N(\eta^i) = N(\eta) \pm 1$ if $\eta(i) = 0, 1$, respectively, and $\big(1 - \eta(i)\big)\eta(i) = 0$.
Convince yourself that this implies that $(N_t : t \geq 0)$ with $N_t := N(\eta_t)$ is a Markov chain on $\{0, \ldots, L\}$ and write down its generator $\mathcal{L}f(n)$.

(b) Is the process $(N_t : t \geq 0)$ irreducible, does it have absorbing states? Give all stationary distributions. Is the process ergodic?

(c) Assume that $\mathbb{E}\big(N_t^k\big) = \mathbb{E}(N_t)^k$ for all $k \geq 1$. This is called a **mean-field assumption**, meaning basically that we replace the random variable N_t by its expected value.
Use this assumption and the usual evolution equation for functions of Markov chains to derive the **mean-field rate equation** for $\rho(t) := \mathbb{E}(N_t)/L$,

$$\frac{d}{dt}\,\rho(t) = h\big(\rho(t)\big) := -\rho(t) + L\lambda\big(1 - \rho(t)\big)\rho(t) \ .$$

(d) Analyze this equation by finding the stable and unstable stationary points via $h(\rho^*) = 0$, and give the limiting behaviour of $\rho(t)$ as $t \to \infty$ depending on the parameter $\lambda > 0$. What is the prediction for the stationary density ρ^* depending on λ?

2. **Simulation of Contact Processes**

   Consider again the contact process $(\eta_t : t \geq 0)$, but now with connections only between nearest neighbours, i.e. $q(i,j) = q(j,i) = \lambda\delta_{j,i+1}$, and periodic boundary conditions.
   The critical infection rate λ_c can be defined such that the infection on the infinite lattice $\Lambda = \mathbb{Z}$ started from the fully infected lattice dies out for $\lambda < \lambda_c$, and survives for $\lambda > \lambda_c$. It is known numerically up to several digits, depends on the dimension, and is around 1.65 in our case.
   All simulations of the process should be done with initial condition $\eta_0(i) = 1$ for all $i \in \Lambda$. You should use the Gillespie algorithm or the random sequential update from handout 4.

   (a) Simulate the process for $L = 128, 256, 512, 1024$ and parameters $\lambda = 1.62, \ldots, 1.68$ with 0.01 increments (7 values) with at least 500 realizations each.

      i. For each L, plot the number of infected individuals $N_t = \sum_{i\in\Lambda_L} \eta_t(i)$ averaged over realizations as a function of time up to time $10 \times L$ for all values of λ as above in a single double-logarithmic plot. Use the curvature of the plots to estimate $\lambda_c(L)$.

      ii. Plot your estimates of $\lambda_c(L)$ with error bars ± 0.01 against $1/L$. Extrapolate to $1/L \to 0$ to get an estimate of $\lambda_c = \lambda_c(\infty)$ with a reasonable error bar.

   This approach is called **finite size scaling**, in order to correct for systematic **finite size effects** which influence the critical value.

   (b) Let T be the hitting time of the absorbing state $\eta = \mathbf{0}$, i.e. the lifetime of the infection. Measure the lifetime of the infection for $\lambda = 1$ and $\lambda = 1.8$ by running the process until extinction of the epidemic.
   For $\lambda = 1 < \lambda_c$ we expect $T \propto C \log L+$small fluctuations for some $C > 0$.

      i. Use large system sizes e.g. $L = 128, 256, 512, 1024$ (or larger), confirm that $\mathbb{E}(T)$ scales like $\log L$ and determine C by averaging at least 200 realizations of T for each L.

      ii. Then shift your data T_i for each L by $T_i - \mathbb{E}(T)$ and plot the 'empirical tail' of the distribution of the shifted data, comparing to the **Gumbel distribution** (all in one plot with log-scale on the y-axis). Look up the Gumbel distribution on Wikipedia, with mean 0 only one parameter needs fitting. Discuss why this could be a good model for the noise here (check google for **extreme value statistics**)

   (c) For $\lambda = 1.8 > \lambda_c$ we expect $T \sim Exp(1/\mu)$ to be an exponential random variable with mean $\mu = \mathbb{E}(T) \propto e^{CL}$ for some $C > 0$.

      i. Use *small* system sizes e.g. $L = 8, 10, 12, 14$ (see how far you can go), and confirm that $\mathbb{E}(T)$ scales like e^{CL}. Determine C by averaging at least 200 realizations of T.

      ii. Then rescale your data T_i for each L by $T_i/\mathbb{E}(T)$ and plot the 'empirical tail' of the distribution of the rescaled data, comparing to the theoretical tail e^{-t} (all in one plot with log-scale on the y-axis).
      *Recall: The **empirical tail** of data $T = (T_1, \ldots, T_M)$ is the statistic $\mathrm{tail}_t(T) = \frac{1}{M}\sum_{i=1}^{M} \mathbb{1}_{T_i > t}$. This decays from 1 to 0 as a (random) function of time t.*