Week 7(1) Chapter 6.2-6.3, Chapter 7.2
15 February 2023 19:18

Structural approach: determine default by firm's structural variables, e.g. assets, liabilities.

Intensity approach / reduced form approach: Silent about reason of default
model default time exogenously.

Merton's model (1974)

Assumption 1: firm's asset value follows $\frac{dV_t}{V_t} = r\,dt + \sigma_V dW_t^Q$ under risk-neutral Q.
It can be traded in the market.

Balance sheet

Assets	Liabilities / Equities
V_t	$P_t \Leftarrow$ corporate bond (liabilities)
	$E_t \Leftarrow$ stock (equities)
V_t =	$P_t + E_t$

Corporate bond has face value K at maturity.

Assumption 2: Default allowed only at T, so default time τ is

$$\tau = \begin{cases} T & \text{if } V_T < K \quad \text{default} \\ \infty & \text{if } V_T \geq K \quad \text{no default.} \end{cases}$$

Aim to compute default prob

$$Q(\tau = T) = Q(V_T < K) = Q\left(V_0 e^{rT + \sigma_V W_T^Q - \frac{1}{2}\sigma_V^2 T} < K\right) = \Phi\left(\frac{-\ln\frac{V_0}{K} + (r - \frac{1}{2}\sigma_V^2)T}{\sigma_V\sqrt{T}}\right) \quad (*)$$

Calibration of vol σ_V by solving $(*)$:

$$\frac{1}{2}T\sigma_V^2 - \Phi^{-1}\big(\underbrace{Q(\tau=T)}_{\text{quoted in CDS market.}}\big)\sqrt{T}\,\sigma_V - rT - \ln\frac{V_0}{K} = 0$$

Aim to compute bond and stock prices

$$P_t = E^Q\left[\frac{\min(V_T, K)}{B_T}\cdot B_t \,\Big|\, \mathcal{F}_t\right]$$ $\min(V,K) = K - (K-V)^+$

$$= \underbrace{E^Q\left[\frac{K}{B_T}\cdot B_t \,\Big|\, \mathcal{F}_t\right]}_{\text{long positions of } K \text{ units of } T\text{-bond}} - \underbrace{E^Q\left[\frac{(K-V_T)^+}{B_T}\cdot B_t \,\Big|\, \mathcal{F}_t\right]}_{\text{short position of European put option.}}$$

$$E_t = E^Q\left[\frac{V_T - \min(V_T, K)}{B_T}\cdot B_t \,\Big|\, \mathcal{F}_t\right]$$

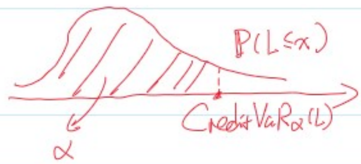
$$= \underbrace{E^Q\left[\frac{(V_T-K)^+}{B_T}\cdot B_t \,\Big|\, \mathcal{F}_t\right]}_{\text{long position of European call option.}}$$

Credit Value-at-risk (VaR)

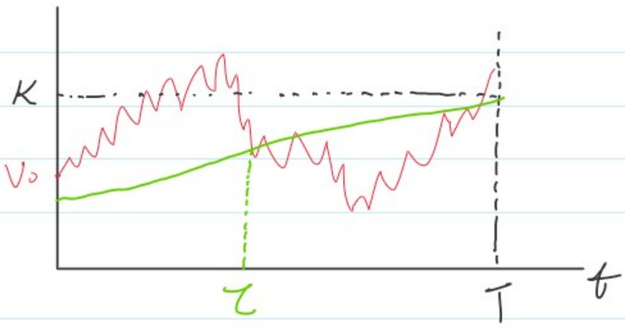
Given some confidence level α ($\alpha = 0.95$). Credit VaR of a loss L is quantile of L at level α.

quantile of L at level α:

$$\text{Credit VaR}_\alpha(L) = \inf\{x : \mathbb{P}(L \le x) \ge \alpha\} = F_L^{-1}(\alpha) \quad (\text{left inverse of } F_L)$$



First-passage-time model (Black-Cox, 1976)



Assumption 2': default barrier $D_t = K e^{-d(T-t)}$, $d \ge 0$.

default time $\tau = \inf\{t \ge 0 : V_t \le D_t\}$

$$= \inf\{t \ge 0 : \frac{V_t}{D_t} \le 1\}$$

$$\left.\begin{aligned} dV_t &= V_t(r\,dt + \sigma_V\,dW_t^{\mathbb{Q}}) \\ dD_t &= D_t\,d\,dt \end{aligned}\right\} \Rightarrow$$

$$d\frac{V_t}{D_t} = \frac{V_t}{D_t}\left[(r-d)\,dt + \sigma_V\,dW_t^{\mathbb{Q}}\right] \Rightarrow$$

$$\frac{V_t}{D_t} = \frac{V_0}{D_0} e^{\sigma_V W_t^{\mathbb{Q}} + (r-d)t - \frac{1}{2}\sigma_V^2 t}$$

$$\{\tau \le t\} = \{\inf_{0 \le s \le t} \frac{V_s}{D_s} \le 1\}$$

for $V_0 > D_0$

$$\{\tau \le t\} = \{\max_{0 \le s \le t} \frac{V_s}{D_s} \ge 1\}$$

for $V_0 < D_0$

Q: Compute distribution of maximum/minimum of Bm with drift!

Chapter 6

The CDS spread/CDS premium at initial time 0, denoted as $R_{CDS}(0)$, is the fixed rate κ such that $\mathbf{E}^{\mathbf{Q}}[\Pi_b(0)] = 0$ under the risk neutral probability measure $\mathbf{Q}$, which is

$$R_{CDS}(0) = \frac{\mathbf{E}^{\mathbf{Q}}[\mathbf{1}_{\{\tau \le T\}} e^{-r\tau}] LGD}{\sum_{i=1}^{n} \left(\mathbf{Q}\{\tau > T_i\} e^{-rT_i} \delta + \mathbf{E}^{\mathbf{Q}}[\mathbf{1}_{\{T_{i-1} < \tau \le T_i\}} e^{-r\tau}(\tau - T_{i-1})] \right)}$$

2.1 CDS with continuum payments and counterparty default

In an ideal situation where the CDS premium is payed continuously until the default time τ, the discounted payoff of the CDS buyer at time 0 is

$$\Pi_b(0) = \mathbf{1}_{\{\tau \le T\}} e^{-r\tau} LGD - \int_0^{\tau \wedge T} e^{-rs} \kappa ds.$$

Thus, the CDS spread at initial time 0 is

$$R_{CDS}(0) = \frac{\mathbf{E}^{\mathbf{Q}}[\mathbf{1}_{\{\tau \le T\}} e^{-r\tau}] LGD}{\mathbf{E}^{\mathbf{Q}}[\int_0^{\tau \wedge T} e^{-rs} ds]} = r \frac{\mathbf{E}^{\mathbf{Q}}[\mathbf{1}_{\{\tau \le T\}} e^{-r\tau}] LGD}{1 - \mathbf{E}^{\mathbf{Q}}[e^{-r(\tau \wedge T)}]}.$$

In practice, the CDS seller may also default, so we also need to take account of the default risk of the CDS seller as the counterparty (so called *counterparty risk*). Let $\bar{\tau}$ be the default time of the CDS seller, then the discounted payoff of the CDS buyer at time 0 is modified as

$$\Pi_b(0) = \mathbf{1}_{\{\tau \le T\}} \mathbf{1}_{\{\bar{\tau} > \tau\}} e^{-r\tau} LGD - \int_0^{\tau \wedge \bar{\tau} \wedge T} e^{-rs} \kappa ds.$$

Thus, the CDS spread at initial time 0 is

$$R_{CDS}(0) = \frac{\mathbf{E}^{\mathbf{Q}}[\mathbf{1}_{\{\tau \le T\}} \mathbf{1}_{\{\bar{\tau} > \tau\}} e^{-r\tau}] LGD}{\mathbf{E}^{\mathbf{Q}}[\int_0^{\tau \wedge \bar{\tau} \wedge T} e^{-rs} ds]} = r \frac{\mathbf{E}^{\mathbf{Q}}[\mathbf{1}_{\{\tau \le T\}} \mathbf{1}_{\{\bar{\tau} > \tau\}} e^{-r\tau}] LGD}{1 - \mathbf{E}^{\mathbf{Q}}[e^{-r(\tau \wedge \bar{\tau} \wedge T)}]}.$$

3 Basket CDS

Basket CDS: A CDS that refers to multiple reference entities, which may correlate with each other. The contract specifies the number of defaults after which the payoff is generated, based on which basket CDS is classified as first-to-default CDS (F2D), second-to-default CDS (S2D) or more generally mth-to-default CDS (M2D).

Suppose there are M reference entities in a basket CDS, and we consider the discounted payoff of the mth-to-default CDS. Let $\tau(m)$ represent the mth default time and $N(t)$ be the number of defaults by time t. Then

$$\{N(t) = m\} = \{\tau(m) \le t < \tau(m+1)\}, \quad \{\tau(m) > t\} = \cup_{j=0}^{m-1} \{N(t) = j\},$$

and conditional on $\tau(m)$ being known, the discounted payoff of the mth-to-default CDS at time 0 is

$$\begin{aligned} &\mathbf{1}_{\{\tau(m) \le T_n\}} P(0, \tau(m)) LGD(m) \\ &- \sum_{i=1}^{n} \left(\mathbf{1}_{\{\tau(m) > T_i\}} P(0, T_i) \delta \kappa + \mathbf{1}_{\{T_{i-1} < \tau(m) \le T_i\}} P(0, \tau(m)) (\tau(m) - T_{i-1}) \kappa \right), \end{aligned}$$

where $LGD(m)$ is the loss given default for the mth default. Note that if the spot rate is a constant, then $P(0,T) = e^{-rT}$.

Similar to the determination of the spread/premium of CDS, we define the basket CDS spread at time $t = 0$ as the fixed rate κ such that

$$R_{BCDS}(0) = \frac{\mathbf{E}^{\mathbf{Q}}[\mathbf{1}_{\{\tau(m) \le T_n\}} P(0, \tau(m))] LGD(m)}{\sum_{i=1}^{n} \left(\mathbf{Q}\{\tau(m) > T_i\} P(0, T_i) \delta + \mathbf{E}^{\mathbf{Q}}[\mathbf{1}_{\{T_{i-1} < \tau(m) \le T_i\}} P(0, \tau(m)) (\tau(m) - T_{i-1})] \right)}$$

The rest of this credit risk section is to develop stochastic models for the default times τ, $\bar{\tau}$ and $\tau(m)$, and calculate their distributions. This will in turn provide us with CDS pricing formulas.

Exercise 1. (Collateralized Debt Obligation (CDO))

A CDO is a structured finance product that is backed by a pool of loans and other assets and sold to institutional investors. A CDO is a particular type of derivative because, as its name implies, its value is derived from another underlying asset. If the underlying is bond/loan, it is called *cash CDO*. If the underlying is CDS, it is called *synthetic CDO*.

The CDO is sliced into sections known as *tranches*. If some underlying defaults and the cash collected by the CDO is insufficient to pay all of its investors, those in the lowest, most junior tranches suffer losses first. The last to lose payment from default are the safest, most senior tranches. Consequently, coupon payments vary by tranche with the safest/most senior tranches receiving the lowest rates and the lowest tranches receiving the highest rates to compensate for higher default risk.

Consider a synthetic CDO written on M CDSs with maturity T and with 3 tranches (*junior, mezzanine and senior*). Let τ_m be the default time of the mth CDS (note that it is NOT $\tau(m)$). The total loss is

$$L(t) = \sum_{m=1}^{M} L_m(t) = \sum_{m=1}^{M} \mathbf{1}_{\{\tau_m \le t\}} LGD_m$$

where LGD_m is the loss given default for the mth CDS. Then, the loss of the jth tranche (for $j = 1, 2, 3$) at time t is

$$Z_j(t) = \begin{cases} 0, & L(t) \le Z_{Lj}; \\ L(t) - Z_{Lj}, & Z_{Lj} < L(t) \le Z_{Uj}; \\ Z_{Uj} - Z_{Lj}, & L(t) > Z_{Uj}, \end{cases}$$

where Z_{Lj} and Z_{Uj} are the *attachment* and *detachment* points of the jth tranche. It is clear that we must have $Z_{Lj} = Z_{U(j-1)}$.

The discounted payoff of the premium leg that the jth tranche CDO holder will receive is

$$\sum_{i=1}^{n} \delta \kappa_j [Z_{Uj} - Z_{Lj} - Z_j(t_i)] e^{-rt_i}.$$

The corresponding discounted payoff of the default leg is

$$\sum_{i=1}^{n} [Z_j(t_i) - Z_j(t_{i-1})] e^{-rt_i}.$$

Compute the CDO premium at initial time 0 for the jth tranche.

References

1. Bielecki, Tomasz, and Marek Rutkowski. *Credit risk: modeling, valuation and hedging*. Springer, 2013.