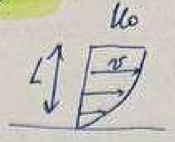
21.02.2024. MA999. HW2 (Blasius solution)



Ⓐ wind flow

① Define the Reynolds number for our problem.
Is the flow laminar or turbulent?

By definition, $Re = \frac{\rho \cdot v \cdot L}{\mu}$, where ρ is density of air, kg/m^3
v is (characteristical) speed of air, m/sec
μ is dynamic viscocity of air, $Pa \cdot sec = \frac{N \cdot sec}{m^2} = \frac{kg}{m \cdot sec}$
L is characteristical length of the profile, in our case, it is the vertical length of our wind profile.

- ρ = air density = $1.2255 \frac{kg}{m^3}$, under standard conditions and 15°C.
- $v = 5$ m/c — it is typical wind speed in West Midlands. (if we are interested in external wind speed, we can take 10 m/c)
- μ = dynamic viscosity = $18 \cdot 10^{-6} \frac{kg}{m \cdot sec}$
- $L = 40$ cm $= 0.4$ m (this quantity I took from the typical height of crops and typical height of the beam of the tractor with fertilizers, but later when I computed Blasius solution, it turned out that 30-40 cm well fits with the upper bound of obtained solution! And we can also calculate boundary layer height as $\delta \sim \frac{L}{\sqrt{Re}}$ and obtain the quantity like 30cm)

$\nu = \frac{\mu}{\rho}$ = kinematic viscocity of the air $= \frac{18 \cdot 10^{-6} \cdot \frac{kg}{m \cdot sec}}{1.2255 \frac{kg}{m^3}} = 14.68 \cdot 10^{-6} \cdot \frac{m^2}{sec} \approx 15 \cdot 10^{-6} \frac{m^2}{sec}$

$\Rightarrow Re = \frac{\rho \cdot v \cdot L}{\mu} = \frac{vL}{\nu} \approx \frac{5 \frac{m}{sec} \cdot 0.4m}{15 \cdot 10^{-6} \cdot m^2/sec} = \frac{2}{15} \cdot 10^6 = \boxed{1.3 \cdot 10^5}$ — laminar flow

if $v = 10$ m/sec, then $Re \approx \boxed{2.6 \cdot 10^5}$ — flow that is going from laminar to turbulent.

As it is known, for air if $Re < 2 \cdot 10^5$, then the flow is laminar.
if $Re \in [2 \cdot 10^5; 4 \cdot 10^5]$, then the flow is going from laminar to turbulent.
if $Re > 4 \cdot 10^5$, then the flow is turbulent.

As we will later understand from modelling and agricultural literature, winds > 5 m/c are considered as bad weather for fertilizing crops, because there are no good enough mechanizms to produce big enough droplets at good speech to prevent the drops from being blown away by the wind.
So for our industrial task speed $v = 5$ m/c is very reasonable.
Let's see at the size of boundary layer with such Reinold's number.
If the length of a field is 100m, then

$$\delta \sim \frac{L}{\sqrt{Re}} = \frac{100m}{\sqrt{1.3 \cdot 10^5}} \approx 0.27m = 27 \text{ sm.}$$

— it is well aligned with our assumption that $L = 0.4m$.

And I also want to emphasize the fact that our flow turned to be laminar is very good — because when one searches information about Blasius solution, it is written that it is in a <u>laminar</u> flow — and our flow is laminar.

(2) Introduce the stream function ψ and re-cast the system of equations and boundary conditions in view of this quantity.

Our initial problem is:

$$\begin{cases} \rho\left(u\cdot\frac{\partial u}{\partial x}+v\cdot\frac{\partial u}{\partial y}\right)=\mu\cdot\frac{\partial^2 u}{\partial y^2} & \leftarrow \text{Navier-Stocks equation in 2D with } \frac{\partial}{\partial t}=0, \text{ and } F_i=0. \\ \frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}=0 & \leftarrow \text{continuity equation} \\ u|_{y=0}=0 \\ v|_{y=0}=0 & \Big\} \leftarrow \text{slip condition: at the ground the wind's speed is zero.} \\ u(y \text{ big enough})=u_0=\text{const.} \end{cases}$$

here $u(x,y)$ is the x-component of the flow,
$v(x,y)$ is the y-component of the flow

Let's introduce ψ such that $\begin{cases} u=\psi'_y \\ v=-\psi'_x. \end{cases}$ ψ is called a stream function

then:

$$\begin{cases} u\cdot\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}=\nu\frac{\partial^2 u}{\partial y^2} & \Rightarrow \psi'_y\cdot\psi''_{yx}-\psi'_x\cdot\psi''_{yy}=\nu\cdot\psi'''_{yyy} \\ \frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}=0 & \Rightarrow \psi''_{yx}-\psi'_{xy}=0. \\ u|_{y=0}=0 & \Rightarrow \psi'_y|_{y=0}=0 \\ v|_{y=0}=0 & \Rightarrow \psi'_x|_{y=0}=0. \\ u(x,+\infty)=u_0 & \Rightarrow \psi'_y(x,+\infty)=u_0\equiv\text{const}; \text{ in particular, } \psi'_y(x,y) \text{ should not depend on } x, \text{ and we will use this when} \end{cases}$$

(3) Consider $x=\beta^a\tilde{x}$; $y=\beta^b\tilde{y}$, $\psi=\beta^c\cdot\tilde{\psi}$, use similarity solutions and obtain blasius solution: $\begin{cases} f'''+\frac{1}{2}ff''=0 \\ f(0)=f'(0)=0 \\ f'(\eta\to+\infty)\to 1. \end{cases}$

Let's make change of variables:

$\begin{cases} \tilde{x}=\varepsilon^a\cdot x \\ \tilde{y}=\varepsilon^b\cdot y \\ \tilde{\psi}=\varepsilon^c\cdot\psi \end{cases}$ and plug it into our equation $\psi'_y\cdot\psi''_{yx}-\psi'_x\cdot\psi''_{yy}=\nu\cdot\psi'''_{yyy}$

$\Rightarrow$ 1) $\tilde{\psi}'_{\tilde{y}}=\tilde{\psi}'_\psi\cdot\psi'_{\tilde{y}}=\varepsilon^c\cdot\psi'_{\tilde{y}}=\varepsilon^c\cdot\psi'_y\cdot y'_{\tilde{y}}=\varepsilon^c\psi'_y\cdot\varepsilon^{-b}=\varepsilon^{c-b}\cdot\psi'_y$

2) $\tilde{\psi}''_{\tilde{y}\tilde{x}}=(\tilde{\psi}'_{\tilde{y}})'_{\tilde{x}}=(\varepsilon^{c-b}\cdot\psi'_y)'_{\tilde{x}}=\varepsilon^{c-b}\cdot\psi''_{yx}\cdot x'_{\tilde{x}}=\varepsilon^{c-b}\cdot\psi''_{yx}\cdot\varepsilon^{-a}=\varepsilon^{c-b-a}\cdot\psi''_{yx}$

3) $\tilde{\psi}'_{\tilde{x}}=\tilde{\psi}'_\psi\cdot\psi'_{\tilde{x}}=\varepsilon^c\cdot\psi'_x\cdot x'_{\tilde{x}}=\varepsilon^{c-a}\cdot\psi'_x$

4) $\tilde{\psi}''_{\tilde{y}\tilde{y}}=(\tilde{\psi}'_{\tilde{y}})'_{\tilde{y}}=(\varepsilon^{c-b}\cdot\psi'_y)'_{\tilde{y}}=\varepsilon^{c-b}\cdot\psi''_{yy}\cdot y'_{\tilde{y}}=\varepsilon^{c-2b}\cdot\psi''_{yy}$

5) $\tilde{\psi}'''_{\tilde{y}\tilde{y}\tilde{y}}=(\tilde{\psi}''_{\tilde{y}\tilde{y}})'_{\tilde{y}}=(\varepsilon^{c-2b}\cdot\psi''_{yy})'_{\tilde{y}}=\varepsilon^{c-2b}\cdot\psi'''_{yyy}\cdot y'_{\tilde{y}}=\varepsilon^{c-3b}\cdot\psi'''_{yyy}$

Plug all that into equation: $\underbrace{\psi'_y}_{=\varepsilon^{b-c}\cdot\tilde{\psi}'_{\tilde{y}}}\cdot\underbrace{\psi''_{yx}}_{=\varepsilon^{a+b-c}\cdot\tilde{\psi}''_{\tilde{y}\tilde{x}}}-\underbrace{\psi'_x}_{=\varepsilon^{a-c}\cdot\tilde{\psi}'_{\tilde{x}}}\cdot\underbrace{\psi''_{yy}}_{=\varepsilon^{2b-c}\cdot\tilde{\psi}''_{\tilde{y}\tilde{y}}}=\nu\cdot\underbrace{\psi'''_{yyy}}_{=\varepsilon^{3b-c}\cdot\tilde{\psi}'''_{\tilde{y}\tilde{y}\tilde{y}}}$

$$\Rightarrow \varepsilon^{a+2b-2c} \cdot \tilde{\psi}'_{\tilde{y}} - \varepsilon^{a+2b-2c} \cdot \tilde{\psi}'_{\tilde{x}} \cdot \tilde{\psi}''_{\tilde{y}\tilde{y}} = \nu \cdot \varepsilon^{3b-c} \cdot \tilde{\psi}'''_{\tilde{y}\tilde{y}\tilde{y}}$$

And in order for the equation to be preserved in form,
we should have $a+2b-2c = 3b-c$

$$\Rightarrow a = b+c$$

And another constant on a,b,c we obtain from boundary condition, but later.

Now let's make sure that expressions $\psi \cdot x^{-\frac{c}{a}}$ and $y \cdot x^{-\frac{b}{a}}$ are both preserved in form under this change of variables:

- $\tilde{\psi} \cdot \tilde{x}^{-\frac{c}{a}} = (\varepsilon^c \cdot \psi) \cdot (\varepsilon^a \cdot x)^{-\frac{c}{a}} = \psi \cdot x^{-\frac{c}{a}}$
- $\tilde{y} \cdot \tilde{x}^{-\frac{b}{a}} = \varepsilon^b y \cdot (\varepsilon^a \cdot x)^{-\frac{b}{a}} = y \cdot x^{-\frac{b}{a}}$

$\Rightarrow$ let's look for a solution in the form: $\psi \cdot x^{-\frac{c}{a}} = f(y \cdot x^{-\frac{b}{a}})$, where $a = b+c$.

$\Rightarrow \psi = x^{\frac{c}{a}} \cdot f(y \cdot x^{-\frac{b}{a}}) = x^{\frac{c}{b+c}} \cdot f(y \cdot x^{-\frac{b}{b+c}})$; and denote $\eta := y \cdot x^{-\frac{b}{b+c}}$

Now look at three boundary conditions:

- $\psi'_y|_{y=0} = 0$: $\psi'_y = x^{\frac{c}{b+c}} \cdot f'_\eta \cdot x^{-\frac{b}{b+c}} = x^{\frac{c-b}{b+c}} \cdot f'_\eta \Big|_{\eta=0} \overset{?}{=} 0.$
- $\psi'_x|_{y=0} = 0$: $\psi'_x = \frac{c}{b+c} \cdot x^{\left(\frac{c}{b+c}-1\right)} \cdot f(\eta)\Big|_{y=0} + \underbrace{x^{\frac{c}{b+c}} \cdot f'_\eta \cdot \overset{0}{y} \cdot \left(-\frac{b}{b+c}\right) \cdot x^{\left(\frac{-b}{b+c}-1\right)}}_{=0 \text{ at } y=0}\Big|_{y=0} \overset{?}{=} 0.$
- $\psi'_y(x, +\infty) \overset{?}{=} u_0$: $\psi'_y = x^{\frac{c-b}{b+c}} \cdot f'_\eta \overset{?}{\longrightarrow} u_0$ as $y \to +\infty$, $\forall x$

This should not depend on $x \Rightarrow b = c$ $\Rightarrow$ $\psi'_y|_{y=0} \overset{?}{=} 0$ gives $f'_\eta|_{\eta=0} = 0$. $\psi'_x|_{y=0} \overset{?}{=} 0$ gives $f(\eta)|_{\eta=0} = 0$.

$\Rightarrow$ we have $\begin{cases} a = b+c \\ b = c \end{cases}$

$\Rightarrow \psi = x^{\frac{1}{2}} \cdot f(y \cdot x^{-1/2}) \cdot \tilde{C}$ ← some constant

Let's denote $\delta(x) = \sqrt{\frac{\nu x}{u_0}}$; $\eta := \frac{y}{\delta(x)}$ — these coordinates will be very convenient.

$\Rightarrow \psi = u_0 \cdot \delta(x) \cdot f(\eta)$ with $f'_\eta|_{\eta=0} = 0$, $f|_{\eta=0} = 0$; and $\psi'_y = u_0 \cdot \delta(x) \cdot f'_\eta \cdot \frac{1}{\delta(x)} \overset{?}{\to} u_0$ gives $f'(+\infty) = 1$.

Now let's plug this ψ into our equation on ψ (that came from initial Navier-Stocks equation after change of variables):

$$\boxed{\psi'_y \cdot \psi''_{yx} - \psi'_x \cdot \psi''_{yy} = \nu \cdot \psi'''_{yyy}}$$

$\Rightarrow \psi'_y = u_0 \cdot \delta \cdot f'_\eta \cdot \frac{1}{\delta} = u_0 f'_\eta$

$\psi''_{yx} = (\psi'_y)'_x = (u_0 f'_\eta)'_x = u_0 \cdot f''_{\eta\eta} \cdot \eta'_x = u_0 \cdot f''_{\eta\eta} \cdot \left(-\frac{y}{\delta^2(x)} \cdot \delta'_x\right)$

$\psi'_x = u_0 \cdot \delta'_x \cdot f + u_0 \cdot \delta \cdot f'_\eta \cdot \eta'_x$

$\psi''_{yy} = (\psi'_y)'_y = (u_0 f'_\eta)'_y = u_0 \cdot f''_{\eta\eta} \cdot \frac{1}{\delta}$

$\psi'''_{yyy} = (\psi''_{yy})'_y = u_0 \cdot f'''_{\eta\eta\eta} \cdot \frac{1}{\delta^2}$

$\Rightarrow \psi'_y \cdot \psi''_{yx} - \psi'_x \cdot \psi''_{yy} = \nu \cdot \psi'''_{yyy}$ transforms into:

$$\cancel{(u_0 \cdot f'_\eta)(u_0 \cdot f''_{\eta\eta} \cdot \eta'_x)} - (u_0 \cdot \delta'_x \cdot f + \cancel{u_0 \cdot \delta \cdot f'_\eta \cdot \eta'_x})(u_0 \cdot f''_{\eta\eta} \cdot \frac{1}{\delta}) = \nu \cdot u_0 \cdot f'''_{\eta\eta\eta} \cdot \frac{1}{\delta^2}$$

$$\Rightarrow \frac{-u_0^{\cancel{2}}}{\delta} \delta'_x \cdot f f''_{\eta\eta} = \nu \cdot \cancel{u_0} \cdot f'''_{\eta\eta\eta} \cdot \frac{1}{\delta^2}$$

$$\Rightarrow \quad u_0 \delta'_x f \cdot f''_{\eta\eta} + \frac{\nu}{\delta} \cdot f'''_{\eta\eta\eta} = 0.$$

$$\left(\sqrt{\frac{\nu x}{u_0}}\right)'_x$$

$$= \sqrt{\frac{\nu}{u_0}} \frac{1}{2\sqrt{x}}$$

$$= \frac{1}{2} \cdot \frac{\nu}{u_0} \cdot \frac{1}{\delta}.$$

$$\Rightarrow \cancel{u_0}\left(\frac{1}{2} \cdot \frac{\cancel{\nu}}{\cancel{u_0}} \cdot \frac{1}{\delta}\right) \cdot f \cdot f''_{\eta\eta} + \frac{\cancel{\nu}}{\delta} \cdot f'''_{\eta\eta\eta} = 0.$$

$$\Rightarrow \boxed{\frac{1}{2} \cdot f \cdot f''_{\eta\eta} + f'''_{\eta\eta\eta} = 0.}$$

And remember the initial conditions we derived: $f|_{\eta=0}$; $f'|_{\eta=0}=0$; $f'_\eta(+\infty)=1$.

$\Rightarrow$ we obtain:

$$\boxed{\begin{cases} f'''_{\eta\eta\eta} + \frac{1}{2} f f''_{\eta\eta} = 0 \\ f|_{\eta=0} = 0 \\ f'_\eta|_{\eta=0} = 0 \\ f'_\eta(+\infty) = 1. \end{cases}}$$

4. Solve the Blasius solution

We first need to do smth with initial condition $f'_\eta(+\infty)=1$.

Standard initial conditions for 3rd order ODE will be:

$f(0)=0$; $f'(0)=0$; $f''(0)=a$, where a is some number, that will give such a solution that has $f'(+\infty)=1$.

To find this a we can use a shooting method:

find two solutions, one with initial conditions $(0)=0$; $'(0)=0$; $''(0)=a$,

and the other with init. conditions $(0)=0$; $'(0)=0$ $''(0)=b$

And then say that since if we take a linear combination of them:

$\Rightarrow f = w \cdot g + (1-w) \cdot h$

$\Rightarrow f'(+\infty) = w \cdot g'(+\infty) + (1-w) \cdot h'(+\infty) \stackrel{?}{=} 1.$

$$\Rightarrow w(g'(+\infty) - h'(+\infty)) \stackrel{?}{=} 1 - h'(+\infty)$$

$$\Rightarrow w^* = \frac{1 - h'(+\infty)}{g'(+\infty) - h'(+\infty)} \Rightarrow \text{take } f = w^* g + (1-w^*) \cdot h.$$

But since we are using python, we can numerically integrate our ~~equati~~ ODE, by reducing it to a system in 3D:

$y_1 := y \quad \Rightarrow y_1' = y_2$

$y_2 := y' \quad \Rightarrow y_2' = y_3$

$y_3 := y'' \quad \Rightarrow y_3' = -\frac{1}{2} \cdot y \cdot y'' = -\frac{1}{2} \cdot y_1 \cdot y_3$

$$\Rightarrow \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix}' = \begin{pmatrix} y_2 \\ y_3 \\ -\frac{1}{2} y_1 y_3 \end{pmatrix} \rightsquigarrow \text{plug into odeint and get } y, y', y''(a), \text{ where } a \text{ is a parameter.}$$

And now we can use scipy.optimize.root$(y'(+\infty)-1;\ [1.0])$ to find such a, (3)
that will give $y'(a)\big|_{\eta=+\infty} - 1 \stackrel{?}{=} 0$, that is, $y'(+\infty)=1$.

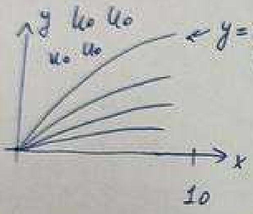
And we will get $a = 0.33205736$.

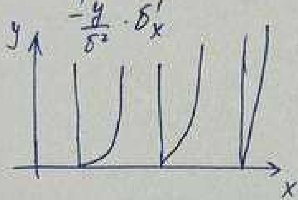
Then, when we get f, f', f'' for all etas (in truth, we got f, f', f'' only on a discretized grid of etas, but since in physics all functions are good and continuous, we set f, f', f'' at other etas to be interpolated f, f', f'' from the grid), we should obtain back $u(x,y)$ and $v(x,y)$:

$$\psi = u_0 \cdot \delta(x) \cdot f(\eta)$$

$$\Rightarrow u = \psi'_y = u_0 \cdot \delta \cdot f'_\eta \cdot \frac{1}{\delta} = u_0 \cdot f'_\eta$$

$$v = -\psi'_x = -u_0 \cdot (\delta'_x \cdot f(\eta) + \delta \cdot f'_\eta \cdot \underbrace{\eta'_x}_{= -\frac{y}{\delta^2}\cdot\delta'_x}) = -u_0 \cdot \delta' (f - f'_\eta \cdot \underbrace{\left(\frac{y}{\delta}\right)}_{=\eta}) = -u_0 \cdot \delta' \cdot (f - f'_\eta \cdot \eta)$$





$\leftarrow$ developing Blasius boundary layer (see in jupyter notebook) My picture looks like the picture from wikipedia.

(5.) Extend the model to include z-component.

look at 3D Navier-Stokes equations:

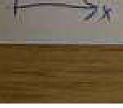
$$\rho \cdot \frac{d\bar{v}}{dt} = -\bar{\nabla} p + \mu \nabla^2 \bar{u} + \bar{F_e} \cdot \rho$$

We assume that
- $\frac{\partial p}{\partial y} = 0$, that means $p = p(x, z)$ - pressure;
- region is sufficiently large and wind is sufficiently strong to neglect gravity $\Rightarrow (F_{ex}, F_{ey}, F_{ez}) = 0$.
- The wind front is stable and has reached steady state $\Rightarrow \frac{\partial}{\partial t} = 0$.
- Region is spatially infinite in x and z-directions $\Rightarrow \frac{\partial}{\partial x}, \frac{\partial}{\partial z} \ll \frac{\partial}{\partial y}$

$$\Rightarrow \begin{cases} \rho \cdot \left(u \cdot \frac{\partial u}{\partial x} + v \cdot \frac{\partial u}{\partial y} + w \cdot \frac{\partial u}{\partial z}\right) = -\frac{\partial p}{\partial x} + \mu \left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}\right) \\ \rho \cdot \left(u \cdot \frac{\partial v}{\partial x} + v \cdot \frac{\partial v}{\partial y} + w \cdot \frac{\partial v}{\partial z}\right) = -\frac{\partial p}{\partial y} + \mu \cdot \left(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} + \frac{\partial^2 v}{\partial z^2}\right) \\ \rho \left(u \cdot \frac{\partial w}{\partial x} + v \cdot \frac{\partial w}{\partial y} + w \cdot \frac{\partial w}{\partial z}\right) = -\frac{\partial p}{\partial z} + \mu \left(\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} + \frac{\partial^2 w}{\partial z^2}\right) \\ \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0. \\ u|_{y=0} = 0 \\ v|_{y=0} = 0 \\ w|_{y=0} = 0 \\ u(y=+\infty) = u_0. \end{cases}$$

And we can also assume that pressure is constant $\Rightarrow \frac{\partial p}{\partial x} = \frac{\partial p}{\partial y} = \frac{\partial p}{\partial z} = 0.$

But to make this 3D system solvable, let's introduce the model that is a multiplication of two independent boundary layers, one in xOy, and the other in yOz



$\Rightarrow$ we have 6 equations solutions and can solve the system

⑥ Make three suggestions to make the flow more realistic.

1) Clearly make it 3D, for example, think of it as a multiplication of two independent Blasius solutions in XOY and YOZ planes. Because if we review the winds in West Midlands, we will see that the wind is never totally north, or totally east/west/south, and it can change its direction, and for us not to recalculate the whole model (we definitely can put the x-axis in the direction of the wind), we should take into account that the wind is in 3d.

2) Also we should take into consideration the fact that winds depend on time, because winds in the morning and winds in the midday are not the same, and also they have seasonality: winds in summer and in winter differ.

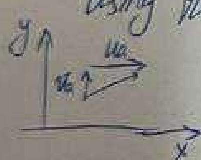
$\Rightarrow$ better model will be $u(x,y,z,t) = u(x,y) \cdot F_2(z) \cdot F_3(t)$, where $F_3(t)$ may be taken sinusoidal, for example.

3) Gravity should also taken into account, because it is negligible only for very-very strong winds, but what if we want to calculate and model our fertilization campaign in a day with almost no wind $\Rightarrow F_{gy} = -m_d g \neq 0$.

4) We must take into account the fact that close to earth there is grass $\Rightarrow$ temperature is 1-2°C higher than in the air $\Rightarrow$ there will be an upward force, which should be taken into account.

Ⓑ Particle modelling

① Develop a simple model of movement for spherical particles using the Blasius solution flow as hosting flow.



- $(u_a; v_a)$ is the air x- and y-velocities. In our case $u_a(x,y)$ and $v_a(x,y)$ are the flow field, found in Ⓐ as a Blasius boundary layer solution.
- (u_d, v_d) is the x- and y-velocity of a droplet (of water)
- the droplet is spherical
- the model is 2D for simplicity
- $\begin{cases} u_{rel} = u_d - u_a \\ v_{rel} = v_d - v_a \end{cases}$ is the relative speed of the droplet with respect to air.
- $|u_{rel}| = \sqrt{(u_{rel})^2 + (v_{rel})^2}$ — the absolute value of u_{rel}.

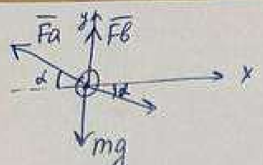
- On a drop acts 3 forces: gravitation force (downwards), buoyancy force (upwards), and aerodynamical force, that is proportional to the square of speed and acts in the direction that is opposite to speed.

- We can write second Newton's law and project it onto 2 axes: x and y.

$$m_d \bar{a} = \bar{F} = F_a + F_g + F_B$$

$$\Rightarrow m_d \ddot{x} = \bar{F} \; ; \; \dot{x} = (u_d; v_d)$$

$$\Rightarrow \begin{cases} \dfrac{du_d}{dt} = (F_{ax} + \overset{=0}{F_{gx}} + \overset{=0}{F_{Bx}}) / m_d \\ \dfrac{dv_d}{dt} = (F_{ay} + F_{gy} + F_{By}) / m_d. \end{cases}$$



$m\vec{g} + \vec{F_b}$ projected on y axis give:

$$-m_d g + \rho_a \cdot g \cdot V_d = -m_d \cdot g + \rho_a \cdot g \cdot \frac{V_d \cdot \rho_d}{\rho_d} =$$

$$= -m_d \cdot g + \frac{\rho_a}{\rho_d} \cdot g \cdot m_d = -m_d g \left(1 - \frac{\rho_a}{\rho_d}\right).$$

- $|F_a| = \frac{1}{2} \cdot C_d \cdot \rho_a \cdot A_d \cdot |U_{res}|^2$

where
- C_d is the drag coefficient, it depends only on the form of the droplet, and for a sphere the drag coefficient is 0.47 (Wikipedia)
- ρ_a is the density of air, $1.2255 \frac{kg}{m^3}$
- $A_d = \pi R^2 = \frac{\pi D^2}{4}$ is the frontal area of the drop, where D is the diameter of the drop
- $U^2_{res} = (u_d - u_a)^2 + (v_d - v_a)^2$.
- $m_d = \rho_d \cdot V_d = \rho_d \cdot \frac{4}{3}\pi R^3 = \rho_d \frac{4}{3}\pi \cdot \frac{D^3}{8}$ is the mass of the droplet; $\rho_d = 1000 \frac{kg}{m^3}$ is the density of water.

See that $F_{ax} = -|F_a| \cdot \cos\alpha \cdot \underbrace{\frac{\vec{U}_{res}}{|U_{res}|}}_{\text{unit vector pointing in the direction of } U_{res}} = \frac{1}{2} \cdot C_d \cdot \rho_a \frac{\pi D^2}{4} \underbrace{\cos\alpha \frac{|U_{res}|^2 \, U_{res}}{|U_{res}|}}_{=(U_{res})_{x\ component}}$

$$\Rightarrow F_{ax} = -\frac{1}{8} \cdot C_d \cdot \rho_a \cdot \pi D^2 \cdot (u_d - u_a).$$

Analogously, $F_{ay} = -|F_a| \sin\alpha \frac{\vec{U}_{res}}{|U_{res}|} = -\frac{1}{8} C_d \cdot \rho_a \cdot \pi D^2 \cdot (v_d - v_a)$

All Together:

$$\begin{cases} \frac{du_d}{dt} = \frac{F_{ax} + F_{gx}^{\nearrow 0} + F_{bx}^{\nearrow 0}}{m_d} \\ \frac{dv_d}{dt} = \frac{F_{ay} + F_{gy} + F_{by}}{m_d} \end{cases}$$

$$\Rightarrow \begin{cases} \frac{du_d}{dt} = \frac{-\frac{1}{8} C_d \rho_a \cdot \pi D^2 \cdot (u_d - u_a)}{\rho_d \cdot \frac{1}{2} \cdot \frac{1}{3} \pi D^3} = -\frac{3}{4} \frac{C_d \cdot \rho_a \cdot D}{\rho_d} \cdot (u_d - u_a) \\ \frac{dv_d}{dt} = -\frac{3}{4} \frac{C_d \rho_a \cdot D}{\rho_d} \cdot (v_d - v_a) + \frac{\left(-m_d \cdot g \cdot \left(1 - \frac{\rho_a}{\rho_d}\right)\right)}{m_d} = -\frac{3}{4} \frac{C_d \cdot \rho_a \cdot D}{\rho_d} \cdot (v_d - v_a) - g\left(1 - \frac{\rho_a}{\rho_d}\right). \end{cases}$$

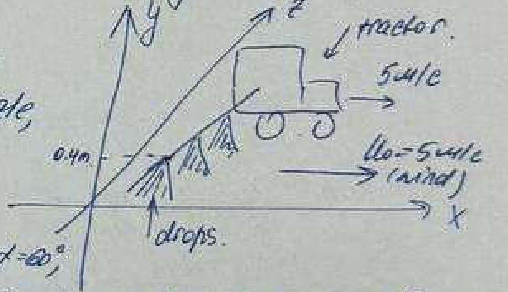
And transform this into 4 1d-ODE:

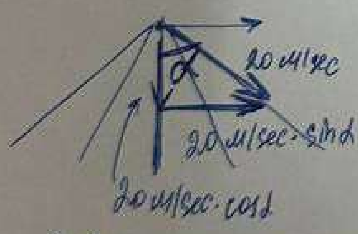
$$\begin{cases} \dot{x} = u_d \\ \dot{u}_d = -\frac{3}{4} \frac{C_d \rho_a D}{\rho_d} (u_d - u_a) \\ \dot{y} = v_d \\ \dot{v}_d = -\frac{3}{4} \frac{C_d \cdot \rho_a \cdot D}{\rho_d} \cdot (v_d - v_a) - g\left(1 - \frac{\rho_a}{\rho_d}\right) \\ + \text{initial conditions } x(0) = x_0;\ y(0) = y_0;\ \dot{x}(0) = u_0;\ \dot{y}(0) = v_0 \end{cases}$$

(2.) Justify suitable choices in initial conditions and ejected droplet characteristics

drop size we can find in the brochure about fertilization by Lechler machines.
they present different nozzles with different colors, and color corresponds to the size of the droplet that will leave the nozzle (if the pressure in the nozzle will be kept as is written and recommended)
recommended pressure is 3 bar.
Insecticides and fungicides usually are used with drop sizes 100-200 microns, gerbisides - with 100-300 microns. When there is fast wind and/or the drops are evaporating hugely, than greater drop sizes are used, up to 400 microns. 500 microns - are huge drops. And less 100 microns in diameter drops are not used, because they are hugely shifted by the wind and they also evaporate totally untill they reach the plant that we are fertilizing.

About initial conditions: x_0, y_0 is the initial position of the drop - it is clearly the position of the nozzle. Nozzles are located on a beam attached to a tractor, and the heigh of the beam is usually 0.4m-0.5m. We will use a grid of y_0 from 0 to 0.5m.
As for x, we will use $x=1$, because we assume that the field is spatially uniform, so x_0 are all having the same properties.
Now let's consider u_0, v_0.
We read from Lechler brochure that depending on pressure inside the nozzle, the drop will leave the nozzle with speed at 20-60 m/sec
and the angles also differs from $\alpha = 40°$ to $\alpha = 60°$, where α is the half of the angle of the cone of drops.





And projecting $(u_0; v_0)$ with $\sqrt{|u_0|^2+|v_0|^2} = 20$ m/sec on x and y axes, we see that $u_0 = 20\sin\alpha$, and $v_0 = 20\cos\alpha$.
We will now model trajectories of the drops with different angles, and at different $u_0 = [0.1, 5, 10, 15]$ m/sec to understand, how the wind affects the trajectories inside the cone.
And we will also realize, that the higher the beam is located, the more the wind does affect!