Week 6(2) Chapter 5.2, 5.3, Chapter 6.1

14 February 2023 10:42

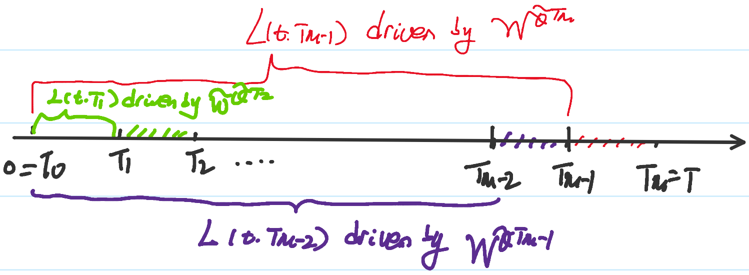
$f(t,T)$ 

$L(t,T)$ for δ period  $L(t,T) = F(t;T,T+\delta) = \frac{1}{\delta}\left(\frac{P(t,T)}{P(t,T+\delta)} - 1\right)$

$$d\,\frac{P(t,T)}{P(t,T+\delta)} = \frac{P(t,T)}{P(t,T+\delta)}\left(\sigma^*(t,T) - \sigma^*(t,T+\delta)\right) dW_t^{\mathbb{Q}^T}.$$

$$\Rightarrow dL(t,T) = \frac{1}{\delta}\, d\,\frac{P(t,T)}{P(t,T+\delta)} = \underbrace{\left(L(t,T) + \frac{1}{\delta}\right)\left(\sigma^*(t,T) - \sigma^*(t,T+\delta)\right)}_{L(t,T)\,\lambda(t,T) \text{ for vol } \lambda(t,T).} dW_t^{\mathbb{Q}^{T+\delta}}.$$

Chapter 5



Suppose that there exist an $\mathbb{R}^d$-valued deterministic and bounded function $\lambda(t,T)$ such that

$$L(t,T)\lambda(t,T) = (L(t,T) + \frac{1}{\delta})(\sigma^*(t,T) - \sigma^*(t,T+\delta)). \tag{1}$$

Then we would have a log normal model for $L(t,T)$:

$$dL(t,T) = L(t,T)\lambda(t,T)dW_t^{\hat{\mathbf{Q}}^{T+\delta}},$$

i.e.

$$L(t,T) = L(0,T)\mathscr{E}(\int_0^{\cdot} \lambda(s,T)dW_s^{\hat{\mathbf{Q}}^{T+\delta}})_t.$$

The objective is then to construct a family of log normal models for $L(t,T)$ with different settlement dates T under their respective $(T+\delta)$-forward measures.

2 LIBOR Market Model

Fix a finite time horizon $T_M = M\delta$ for $M \in \mathbb{N}$, and a filtered probability space $(\Omega, \mathscr{F}, \{\mathscr{F}_t\}_{t\geq 0}, \hat{\mathbf{Q}}^{T_M})$, which satisfies the usual conditions, and supports a d-dimensional Brownian motion $W^{\hat{\mathbf{Q}}_{T_M}}$.

Assumption 1 *We are given the following data.*
(1) For $1 \leq i \leq M$, the settlement date $T_i = i\delta$;
(2) For $1 \leq i \leq M$, the initial term structure of T_i-bond $P(0,T_i)$. Hence, we obtain the initial forward LIBOR as

$$L(0,T_i) = \frac{1}{\delta}(\frac{P(0,T_i)}{P(0,T_{i+1})} - 1)$$

for $1 \leq i \leq M-1$;
(3) For $1 \leq i \leq M-1$, the volatility $\lambda(t,T_i)$ of the forward LIBOR $L(t,T_i)$, where $\lambda(t,T_i)$ is a deterministic and bounded function for $t \in [0,T_i]$.

We construct a LIBOR market model as follows.

Step 1: For $t \in [0, T_{M-1}]$, we are given the Brownian motion $W^{\hat{\mathbf{Q}}_{T_M}}$ under the $\hat{\mathbf{Q}}^{T_M}$, the volatility $\lambda(t,T_{M-1})$, and the initial zero-coupon bond prices $P(0,T_{M-1})$, $P(0,T_M)$. We construct $L(t,T_{M-1})$ as

$$\begin{cases} dL(t,T_{M-1}) = L(t,T_{M-1})\lambda(t,T_{M-1})dW_t^{\hat{\mathbf{Q}}^{T_M}}, \\ L(0,T_{M-1}) = \frac{1}{\delta}(\frac{P(0,T_{M-1})}{P(0,T_M)} - 1) \end{cases}$$

which gives us

$$L(t,T_{M-1})=\frac{1}{\delta}(\frac{P(0,T_{M-1})}{P(0,T_M)}-1)\mathscr{E}(\int_0^\cdot \lambda(s,T_{M-1})dW_s^{\tilde{\mathbf{Q}}^{T_M}})_t.$$

Now based on (1), we define

$$\sigma^*(t,T_{M-1})-\sigma^*(t,T_M)=\frac{\delta L(t,T_{M-1})}{\delta L(t,T_{M-1})+1}\lambda(t,T_{M-1}).$$

This gives us the SDE of $\frac{P(t,T_{M-1})}{P(t,T_M)}$ under $\tilde{\mathbf{Q}}^{T_M}$:

$$d\frac{P(t,T_{M-1})}{P(t,T_M)}=\frac{P(t,T_{M-1})}{P(t,T_M)}(\sigma^*(t,T_{M-1})-\sigma^*(t,T_M))dW_t^{\tilde{\mathbf{Q}}^{T_M}}.$$

Hence, we may use $P(t,T_{M-1})$ as the numeraire, and introduce the T_{M-1}-forward measure $\tilde{\mathbf{Q}}^{T_{M-1}}$ by

$$\left.\frac{d\tilde{\mathbf{Q}}^{T_{M-1}}}{d\tilde{\mathbf{Q}}^{T_M}}\right|_{\mathscr{F}_t}=\mathscr{E}(\int_0^\cdot \sigma^*(s,T_{M-1})-\sigma^*(s,T_M))dW_s^{\tilde{\mathbf{Q}}^{T_M}})_t=\frac{P(t,T_{M-1})P(0,T_M)}{P(t,T_M)P(0,T_{M-1})}.$$

By Girsanov's theorem in BM case,

$$W_t^{\tilde{\mathbf{Q}}^{T_{M-1}}}=W_t^{\tilde{\mathbf{Q}}^{T_M}}-\int_0^t \sigma^*(s,T_{M-1})-\sigma^*(s,T_M))ds$$

is a d-dimensional Brownian motion under the T_{M-1}-forward measure $\tilde{\mathbf{Q}}^{T_{M-1}}$.

Step 2: For $t\in[0,T_{M-2}]$, we are given the Brownian motion $W^{\tilde{\mathbf{Q}}_{T_{M-1}}}$ under the $\tilde{\mathbf{Q}}^{T_{M-1}}$, the volatility $\lambda(t,T_{M-2})$, and the initial zero-coupon bond prices $P(0,T_{M-2})$, $P(0,T_{M-1})$. We construct $L(t,T_{M-2})$ as

$$\begin{cases} dL(t,T_{M-2})=L(t,T_{M-2})\lambda(t,T_{M-2})dW_t^{\tilde{\mathbf{Q}}^{T_{M-1}}}, \\ L(0,T_{M-2})=\frac{1}{\delta}(\frac{P(0,T_{M-2})}{P(0,T_{M-1})}-1) \end{cases}$$

which gives us

$$L(t,T_{M-2})=\frac{1}{\delta}(\frac{P(0,T_{M-2})}{P(0,T_{M-1})}-1)\mathscr{E}(\int_0^\cdot \lambda(s,T_{M-2})dW_s^{\tilde{\mathbf{Q}}^{T_{M-1}}})_t.$$

Now based on (1), we define

$$\sigma^*(t,T_{M-2})-\sigma^*(t,T_{M-1})=\frac{\delta L(t,T_{M-2})}{\delta L(t,T_{M-2})+1}\lambda(t,T_{M-2}).$$

This gives us the SDE of $\frac{P(t,T_{M-2})}{P(t,T_{M-1})}$ under $\tilde{\mathbf{Q}}^{T_{M-1}}$:

$$d\frac{P(t,T_{M-2})}{P(t,T_{M-1})}=\frac{P(t,T_{M-2})}{P(t,T_{M-1})}(\sigma^*(t,T_{M-2})-\sigma^*(t,T_{M-1}))dW_t^{\tilde{\mathbf{Q}}^{T_{M-1}}}.$$

Hence, we may use $P(t,T_{M-2})$ as the numeraire, and introduce the T_{M-2}-forward measure $\tilde{\mathbf{Q}}^{T_{M-2}}$ by

$$\frac{d\tilde{\mathbf{Q}}^{T_{M-2}}}{d\tilde{\mathbf{Q}}^{T_{M-1}}}\bigg|_{\mathscr{F}_t} = \mathscr{E}(\int_0^\cdot \sigma^*(s,T_{M-2})-\sigma^*(s,T_{M-1}))dW_s^{\tilde{\mathbf{Q}}^{T_{M-1}}})_t = \frac{P(t,T_{M-2})P(0,T_{M-1})}{P(t,T_{M-1})P(0,T_{M-2})}.$$

By Girsanov's theorem in BM case,

$$W_t^{\tilde{\mathbf{Q}}^{T_{M-2}}} = W_t^{\tilde{\mathbf{Q}}^{T_{M-1}}} - \int_0^t \sigma^*(s,T_{M-2})-\sigma^*(s,T_{M-1}))ds$$

is a d-dimensional Brownian motion under the T_{M-2}-forward measure $\tilde{\mathbf{Q}}^{T_{M-2}}$.

...

Step $M-1$: For $t\in[0,T_1]$, we are given the Brownian motion $W^{\tilde{\mathbf{Q}}_{T_2}}$ under the $\tilde{\mathbf{Q}}^{T_2}$, the volatility $\lambda(t,T_1)$, and the initial zero-coupon bond prices $P(0,T_1)$, $P(0,T_2)$. We construct $L(t,T_1)$ as

$$\begin{cases} dL(t,T_1) = L(t,T_1)\lambda(t,T_1)dW_t^{\tilde{\mathbf{Q}}^{T_2}}, \\ L(0,T_1) = \frac{1}{\delta}(\frac{P(0,T_1)}{P(0,T_2)}-1) \end{cases}$$

which gives us

$$L(t,T_1) = \frac{1}{\delta}(\frac{P(0,T_1)}{P(0,T_2)}-1)\mathscr{E}(\int_0^\cdot \lambda(s,T_1)dW_s^{\tilde{\mathbf{Q}}^{T_2}})_t.$$

Based on (1), we may define $\sigma^*(t,T_1)-\sigma^*(t,T_2)$, and therefore the T_1-forward measure $\tilde{\mathbf{Q}}^{T_1}$ and the corresponding Brownian motion $W^{\tilde{\mathbf{Q}}^{T_1}}$. However, they will not be used.

Step M: Obviously, we have that

$$L(0,T_0) = \delta(\frac{1}{P(0,T_1)}-1).$$

In summary, we have constructed a family of log normal models for $L(t,T_i)$ under their respective T_{i+1}-forward measure $\tilde{\mathbf{Q}}^{T_{i+1}}$, for $1\le i\le M-1$, and have defined an arbitrage free market for the zero coupon bonds with maturities $T_1,T_2,\ldots,T_M$.

Theorem 1. *Given the Brownian motion $W^{\tilde{\mathbf{Q}}^{T_M}}$ under $\tilde{\mathbf{Q}}^{T_M}$, the dynamics of the forward LIBOR $L(t,T_i)$ for $i=1,\cdots,M-1$ is given by*

$$dL(t,T_i) = L(t,T_i)\lambda(t,T_i)\left(dW_t^{\tilde{\mathbf{Q}}^{T_M}} - \sum_{j=i+1}^{M-1}\frac{\delta L(t,T_j)}{\delta L(t,T_j)+1}\lambda(t,T_j)dt\right).$$

Proof. We proceed backwards. It is obvious for $i=M-1$, since $\sum_M^{M-1}=0$ by convention. Now for $i=M-2$, by our construction, we have that

$$dL(t,T_{M-2}) = L(t,T_{M-2})\lambda(t,T_{M-2})dW_t^{\hat{\mathbf{Q}}^{T_{M-1}}},$$

where

$$W_t^{\hat{\mathbf{Q}}^{T_{M-1}}} = W_t^{\hat{\mathbf{Q}}^{T_M}} - \int_0^t \sigma^*(s,T_{M-1}) - \sigma^*(s,T_M))ds$$

The conclusion then follows from (1). □

3 Black's Formula for Interest Rate Caps

Recall that, for $1 \le i \le M$, the holder of the interest rate cap at the settlement date T_i (with the reset date T_{i-1}) receives

$$\delta N(F(T_{i-1},T_i) - K)^+ = \delta N(L(T_{i-1},T_{i-1}) - K)^+$$

$t=0,$

$$C_p(0,T_{i-1},T_i) = E^{Q}\left[\frac{\delta N(L(T_{i-1},T_{i-1})-K)^+}{B_{T_i}}\right]$$

The time t value of the above payoff is

$$C_p(t;T_{i-1},T_i) = \mathbf{E}^{\mathbf{Q}}[\frac{B_t}{B_{T_i}}\delta N(L(T_{i-1},T_{i-1}) - K)^+|\mathscr{F}_t].$$

$$= E^{P}\left[\frac{dQ}{dP}\Big|_{\mathcal{F}_{T_i}} \frac{\delta N(L(T_{i-1},T_{i-1})-K)^+}{B_{T_i}}\right]$$

From Bayes formula, the above conditional expectation can also be calculated under the T_i-forward measure $\hat{\mathbf{Q}}^{T_i}$:

$$C_p(t;T_{i-1},T_i) = \delta NP(t,T_i)\mathbf{E}^{\hat{\mathbf{Q}}^{T_i}}[(L(T_{i-1},T_{i-1}) - K)^+|\mathscr{F}_t].$$

Stochastic discount factor / deflator at T_i:

$$\frac{dQ}{dP}\Big|_{\mathcal{F}_{T_i}} \frac{1}{B_{T_i}}$$

But under the LIBOR market model,

$$L(T_{i-1},T_{i-1}) = L(t,T_{i-1})e^{\int_t^{T_{i-1}}\lambda(s,T_{i-1})dW^{\hat{\mathbf{Q}}^{T_i}} - \frac{1}{2}\int_t^{T_{i-1}}|\lambda(s,T_{i-1})|^2 ds}.$$

Hence, we obtain that

$$C_p(t;T_{i-1},T_i) = \delta NP(t,T_i)[L(t,T_{i-1})\Phi(d_1^{T_{i-1}}) - K\Phi(d_2^{T_{i-1}})],$$

In particular, in B-S markets with constant interest rate r

where

$$d_{1,2}^{T_{i-1}} = \frac{\ln\frac{L(t,T_{i-1})}{K} \pm \frac{1}{2}\int_t^{T_{i-1}}|\lambda(s,T_{i-1})|^2 ds}{\sqrt{\int_t^{T_{i-1}}|\lambda(s,T_{i-1})|^2 ds}}.$$

$$\mathcal{E}\left(\int_0^{\cdot}\theta_t dW_t\right)_{T_i} \cdot \frac{1}{B_{T_i}}$$

The arbitrage price of the interest rate cap at time t is

$$C_p(t) = \sum_{i=1}^{M} C_p(t,T_{i-1},T_i) = \delta N\sum_{i=1}^{M} P(t,T_i)[L(t,T_{i-1})\Phi(d_1^{T_{i-1}}) - K\Phi(d_2^{T_{i-1}})].$$

$$= \exp\left(\int_0^{T_i}\frac{\mu-r}{\sigma}dW_t - \frac{1}{2}\int_0^{T_i}\left(\frac{\mu-r}{\sigma}\right)^2 dt\right)\cdot e^{-rT_i}$$

Change Q to $\hat{Q}^{T_i}$.

$$C_p(0,T_{i-1},T_i) = P(0,T_i)\,E^{Q}\left[\frac{P(T_i,T_i)}{B_{T_i}}\Big/P(0,T_i)\cdot \delta N(L(T_{i-1},T_{i-1})-K)^+\right]$$

$$= P(0,T_i)\,E^{\hat{Q}^{T_i}}\left[\delta N(L(T_{i-1},T_{i-1})-K)^+\right]$$

With $dL(t,T_{i-1}) = L(t,T_{i-1})\lambda(t,T_{i-1})\,d\widetilde{W}_t^{Q^{T_i}}$

4 Exercises

Exercise 1. (Black's formula for interest rate swaption)

Recall a European payer interest rate swaption with the strike rate K is an option giving the right to enter a payer interest rate swap(IRS) with fixed rate K at a given future date, the swaption maturity. Usually, the swaption maturity is the first rest date T_0 of the underlying interest rate swap. Since the value of the payer IRS at T_0 is given by

$$N\left[P(T_0,T_0) - P(T_0,T_n) - K\delta\sum_{i=1}^{n}P(T_0,T_i)\right]$$
$$= N\delta\left[\sum_{i=1}^{n}P(T_0,T_i)(F(T_0;T_i-1,T_i) - K)\right],$$

the payoff of the payer swaption is

$$N\delta\left[\sum_{i=1}^{n}P(T_0,T_i)(F(T_0;T_i-1,T_i) - K)\right]^+ = N\delta\left[\sum_{i=1}^{n}P(T_0,T_i)(L(T_0,T_i-1) - K)\right]^+$$

1. Prove the above payoff can also be written as

$$N\delta(R_{Swap}(T_0) - K)^+\sum_{i=1}^{n}P(T_0,T_i),$$

where $R_{Swap}(T_0)$ is the forward swap rate at T_0:

$$R_{Swap}(T_0) = \frac{P(T_0,T_0) - P(T_0,T_n)}{\delta\sum_{i=1}^{n}P(T_0,T_i)}.$$

2. Write down the arbitrage price of this payoff under both the spot measure $\mathbf{Q}$ and the T_0-forward measure $\hat{\mathbf{Q}}^{T_0}$.
3. Define a numeraire:

$$D(t) = \frac{\sum_{i=1}^{n}P(t,T_i)}{\ldots}$$

2. Write down the arbitrage price of this payoff under both the spot measure $\mathbf{Q}$ and the T_0-forward measure $\tilde{\mathbf{Q}}^{T_0}$.
3. Define a numeraire:

$$D(t) = \frac{\sum_{i=1}^{n} P(t,T_i)}{P(t,T_0)}$$

for $t \in [0,T_0]$. Show that $D(t)$ is martingale under the T_0-forward measure $\tilde{\mathbf{Q}}^{T_0}$. Hence, we can use $\frac{D(t)}{D(0)}$ as the Radon-Nikodym density, and define an equivalent probability measure $\tilde{\mathbf{Q}}^{Swap}$, so called the forward swap measure,

$$\left.\frac{d\tilde{\mathbf{Q}}^{Swap}}{d\tilde{\mathbf{Q}}^{T_0}}\right|_{\mathscr{F}_{T_0}} = \frac{D(T_0)}{D(0)}.$$

Prove that the arbitrage price of this payoff under the forward swap measure $\tilde{\mathbf{Q}}^{Swap}$ is

$$N\delta \sum_{i=1}^{n} P(0,T_i)\mathbf{E}^{\tilde{\mathbf{Q}}^{Swap}}\left[(R_{Swap}(T_0)-K)^+\right]$$

4. Prove that $R_{Swap}(t)$ is a martingale under the forward swap measure $\tilde{\mathbf{Q}}^{Swap}$. Hence, similar to the LIBOR market model, suppose that there exists a bounded and deterministic function $\rho_{Swap}(t)$ such that

$$dR_{Swap}(t) = R_{Swap}(t)\rho_{Swap}(t)dW_t^{\tilde{\mathbf{Q}}^{Swap}}$$

under the forward swap measure $\tilde{\mathbf{Q}}^{Swap}$ for $t \in [0,T_0]$. Prove that the initial value of this interest rate swaption is given by the following Black's formula:

$$N\delta \sum_{i=1}^{n} P(0,T_i)\left[R_{Swap}(0)\Phi(d_1^{T_0}) - K\Phi(d_2^{T_0})\right],$$

where

$$d_{1,2}^{T_0} = \frac{\ln\frac{R_{Swap}(0)}{K} \pm \frac{1}{2}\int_0^{T_0}|\rho_{Swap}(s)|^2 ds}{\sqrt{\int_0^{T_0}|\rho_{Swap}(s)|^2 ds}}.$$

Exercise 2. (Rebonato's analytic approximation for interest rate swaption, Filipovic [1], Chapter 11.5)

It can be shown that the volatility of forward LIBOR and the volatility of the forward swap rate can not be deterministic simultaneously. Hence, either one gets the Black's formula for interest rate cap/floor or for interest rate swaption, but not simultaneously for both. This motivates the so called Rebonato's analytic approximation for interest rate swaption given that the volatility of forward LIBOR is deterministic.

1. Prove that forward swap rate admits the following representation:

$$R_{Swap}(t) = \sum_{i=1}^{n} \omega_i(t)F(t;T_{i-1},T_i) = \sum_{i=1}^{n} \omega_i(t)L(t,T_{i-1}),$$

where the weights are given by

$$\omega_i(t) = \frac{P(t,T_i)}{\sum_{j=1}^{n} P(t,T_j)}.$$

2. From empirical evidence, the variability of those weights is small compared to that of simple forward rates (LIBOR). Therefore, the forward swap rate can be approximated as

$$R_{Swap}(t) \approx \sum_{i=1}^{n} \omega_i(0)L(t,T_{i-1})$$

By using the SDE for LIBOR $L(t,T_{i-1})$, prove that

$$\frac{dR_{Swap}(t)}{R_{Swap}(t)} \approx \left[\frac{\sum_{i=1}^{n}\omega_i(0)L(t,T_{i-1})\lambda(t,T_{i-1})\sum_{l=0}^{i-1}[\sigma^*(t,T_l)-\sigma^*(t,T_{l+1})]}{R_{Swap}(t)}\right]dt + \left[\frac{\sum_{i=1}^{n}\omega_i(0)L(t,T_{i-1})\lambda(t,T_{i-1})}{R_{Swap}(t)}\right]dW_t^{\tilde{\mathbf{Q}}^{T_0}}$$

under the T_0 forward measure $\tilde{\mathbf{Q}}^{T_0}$.

3. In a further approximation, we replace all random variables in the above volatility by their time 0 values, so the volatility becomes

$$|\rho(t)|^2 \approx \frac{\sum_{i,j=1}^{n}\omega_i(0)\omega_j(0)L(0,T_{i-1})L(0,T_{j-1})\lambda(t,T_{i-1})\lambda(t,T_{j-1})^T}{R_{Swap}^2(0)},$$

and use $|\rho(t)|^2$ as the volatility $|\rho_{Swap}(t)|^2$ for the forward swap rate. Write down the pricing formula of the interest rate swaption.

References

1. Filipovic, Damir. *Term-Structure Models. A Graduate Course.* Springer, 2009.

Chapter 6

Applications of Stochastic Calculus in Finance
Chapter 6: Credit risk and credit derivatives

Gechun Liang

1 Credit Risk

So far, the zero coupon bond price $P(t,T)$ is assumed to have the property

$$P(T,T) = 1.$$

That is, the payoff at maturity T is certain, and there is no risk of default by the issuer. This is the case for *treasury bonds*. On the other hand, *corporate bonds*, may be subject to a substantial risk of default.

Credit risk is the potential default risk that an obligor fails to honor its obligation, resulting in the losses of its counterparty. For the question of assessment of credit risk, we have not only to consider default probability, but also the probability of transitions between credit ratings. There are mainly three rating agencies: *Moody's Investors Service (Moody's)*, *Standard & Poor's (S&P)* and *Fitch Ratings*. They assign a credit rating that reflect the creditworthiness of an obligor, and adjust its ratings regularly.

In this course, we mainly assess credit risk in terms of default probability. Transitions of credit ratings can be studied by using Markov chain theory (see Bielecki and Rutkowski [1] Chapter 12). We mainly consider two types of credit derivatives in this course: *CDS* and *basket CDS*. Notorious *Collateralized Debt Obligation (CDO)*, which played a major role in the recent financial crisis, will be discussed in exercise.

2 Credit default swap (CDS)

Credit default swap (CDS) is a swap contract that the seller will compensate the buyer in a credit event, such as bankruptcy or failure to pay a debt. The buyer makes

Gechun Liang
Department of Statistics, University of Warwick, U.K. e-mail: g.liang@warwick.ac.uk

a series of fixed payments (CDS spread, CDS premium) to the seller, and in exchange, receives a payoff if the credit event occurs.

A CDS may refer to a specified loan or bond obligation of a reference entity, which may have nothing to do with the buyer. So the buyer does not need to own the underlying debt, and does not even have to suffer a loss from the default. In such a case, the CDS is called the naked CDS. *It is like buying fire insurance on your neighbor's house*.

The CDS buyer pays its seller a fixed rate $\delta\kappa$ with $\delta = T_i - T_{i-1}$ at a sequence of settlements dates $T_1 < T_2 < \cdots < T_n = T$ until the default time τ of the reference entity. Usually, the premium is paid every three months, so $\delta = 0.25$. In return, if the default time $\tau \in (0, T]$, then the seller pays the buyer a deterministic cash amount LGD (*loss given default*).

Conditional on $\{T_{i-1} < \tau \leq T_i\}$, the discounted payoff of the buyer's cash flow at initial time 0 is

$$\underbrace{e^{-r\tau}LGD}_{default\ leg} - \underbrace{\left(\sum_{j=1}^{i-1} e^{-rT_j}\delta\kappa + e^{-r\tau}(\tau - T_{i-1})\kappa\right)}_{premium\ leg}$$

with the convention $\sum_{j=1}^{0} = 0$, and $r \geq 0$ being the constant interest rate.

See Fig 1 for the cash flow of a CDS buyer conditional on $\{T_{i-1} < \tau \leq T_i\}$.

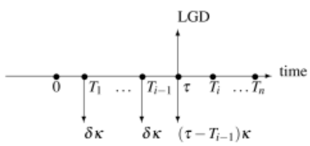


Fig 1: cash flow of a CDS buyer conditional on $\{T_{i-1} < \tau \leq T_i\}$.

Summing up from $i = 1$ to n, we obtain the discounted payoff (conditional on we knowing the default time τ):

$$\begin{aligned}\Pi_b(0) = &\sum_{i=1}^{n} \mathbf{1}_{\{T_{i-1}<\tau\leq T_i\}}\left(e^{-r\tau}LGD - \sum_{j=1}^{i-1} e^{-rT_j}\delta\kappa - e^{-r\tau}(\tau - T_{i-1})\kappa\right)\\ &- \mathbf{1}_{\{\tau>T_n\}}\sum_{j=1}^{n} e^{-rT_j}\delta\kappa\\ = &\mathbf{1}_{\{\tau\leq T\}}e^{-r\tau}LGD\\ &- \sum_{i=1}^{n}\left(\mathbf{1}_{\{\tau>T_i\}}e^{-rT_i}\delta\kappa + \mathbf{1}_{\{T_{i-1}<\tau\leq T_i\}}e^{-r\tau}(\tau - T_{i-1})\kappa\right)\end{aligned}$$