Proof from today (20/10/2022 → holding times).

<u>Part 1:</u> $W_x \sim Exp(|g(x,x)|)$

→ we want to prove that $\mathbb{P}(W_x \leq t) = 1 - e^{\gamma t}$,
or equivalently $P(W_x > t) = e^{\gamma t}$, for
$\gamma = 1/g(x,x))$.

i) Prove memory-less property:

$$P(W_x > t+u \mid W_x > t) = P(W_x > t+u \mid X_t = x)$$
$$= P(W_x > u)$$

if $W_x > t$
then $X_t = x$

if $X_t = x$, we can use homogeneity.

ii) Use this to ~~too~~ obtain something like an ODE for $\mathbb{P}(W_x > t)$:

$$\mathbb{P}(W_x > t+u) = \mathbb{P}(W_x > t+u \mid W_x > t)\,\mathbb{P}(W_x > t) + \underbrace{\mathbb{P}(W_x > t+u \mid W_x \leq t)}_{=0 \text{ because } > t+u \text{ and } \leq t \text{ is not possible.}}\,\mathbb{P}(W_x \leq t)$$

LTP

$$= \mathbb{P}(W_x > u)\,\mathbb{P}(W_x > t)$$

memory-less.

iii) use exponential argument (make $u = \Delta t$)

$$\mathbb{P}(W_x > t+\Delta t) = \mathbb{P}(W_x > t)\,\mathbb{P}(W_x > \Delta t)$$

$$\Rightarrow \mathbb{P}(W_x > t+\Delta t) - \mathbb{P}(W_x > t) = \mathbb{P}(W_x > t)\,(\mathbb{P}(W_x > \Delta t) - 1)$$

$$\Rightarrow \boxed{\frac{\mathbb{P}(W_x > t+\Delta t) - \mathbb{P}(W_x > t)}{\Delta t}} = \mathbb{P}(W_x > t)\,\boxed{\frac{\mathbb{P}(W_x > \Delta t) - 1}{\Delta t}}$$

(as $\Delta t \to 0$) $\frac{d}{dt}\mathbb{P}(W_x > t)$ call this γ

$$\Rightarrow \frac{d}{dt}\mathbb{P}(W_x > t) = \gamma\,\mathbb{P}(W_x > t) \Rightarrow \mathbb{P}(W_x > t) = e^{\gamma t}$$

constant here.

Since $\mathbb{P}(W_x > 0) = 1$, $\gamma = \lim_{\Delta t \to 0} \frac{\mathbb{P}(W_x > \Delta t) - 1}{\Delta t} = \frac{d}{dt}\mathbb{P}(W_x > t)\Big|_{t=0}$

$W_x > \Delta t$ if $X_{\Delta t} = x \mid X_0 = x$, so we can have

$$\lim_{\Delta t \to 0} \frac{p(x,x) - 1}{\Delta t} = \lim_{\Delta t \to 0} \frac{1 + g(x,x)\Delta t + o(\Delta t) - 1}{\Delta t} = g(x,x).$$

Part 2 → We want to prove that the probability of jumping from x to y after the holding time W_x is $\frac{g(x,y)}{|g(x,x)|}$

We know: $\boxed{\mathbb{P}(X_{t+\Delta t} = y \mid X_t = x) = p_{\Delta t}(x,y) = \Delta t\, g(x,y)}$

$\boxed{\text{We also know that } \mathbb{P}(W_x < \Delta t \mid X_t = x) = 1 - p_{\Delta t}(x,x)}$ ↳ because if $W_x < \Delta t$) and $X_t = x$, then $X_{t+\Delta t} \neq x$, so the probability of this happening is ~~$1 - \mathbb{P}(X_{t+\Delta t}$~~ $1 - \mathbb{P}(X_{t+\Delta t} = x \mid X_t = x) = 1 - p_{\Delta t}(x,x)$.

last friday ← Also, $1 - p_{\Delta t}(x,x) = 1 - (1 + \Delta t\, g(x,x) + o(\Delta t))$
$= -\Delta t\, g(x,x) + o(\Delta t)$.

Now we use

$$\mathbb{P}(X_{t+\Delta t} = y \mid X_t = x) \overset{LTP}{=} \mathbb{P}(X_{t+\Delta t} = y \mid X_t = x, W_x < \Delta t)\,\mathbb{P}(W_x < \Delta t) + \underbrace{\mathbb{P}(X_{t+\Delta t} = y \mid X_t = x, W_x \geq \Delta t)}_{=0 \text{ because if } W_x \geq \Delta t,\ X_{t+\Delta t} = x\,!}\mathbb{P}(W_x \geq \Delta t)$$

and so

$$\mathbb{P}(X_{t+\Delta t} = y \mid X_t = x, W_x < \Delta t) = \frac{\mathbb{P}(X_{t+\Delta t} = y \mid X_t = x)}{\mathbb{P}(W_x < \Delta t)}$$

and you can put it together

$$\mathbb{P}(X_{t+\Delta t} = y \mid X_t = x, W_x < \Delta t) = \frac{\cancel{\Delta t}\, g(x,y)}{-\cancel{\Delta t}\, g(x,x)} = -\frac{g(x,y)}{g(x,x)}$$

as we wanted