Chapter 9 Solutions

15 February 2022 10:01

Ex 1. We provide a different method which can be generalised to $n \geq 3$. **See remark.**

For $t \in [0,T]$, let $Z_t^{\tau} = \mathbb{1}_{\tau > t}\, e^{\int_0^t \mathbb{1}_{\tau > s} \lambda_s\, ds}$ and define

$$\frac{dQ^Z}{dQ} = Z_T^Z \quad \text{on } (\Omega, \mathcal{G}_T)$$

Then, $H_t^{\tau} \triangleq \mathbb{1}_{\tau \leq t}$, $t \in [0,T]$, is 0, Q^Z-a.e.

For $T > U \geq 0$, $Q(\tau > T, \bar{\tau} > U)$

$$= Q\left(Z_T^Z\, e^{-\int_0^T \mathbb{1}_{\tau > s} \lambda_s\, ds}\, \mathbb{1}_{\bar{\tau} > U}\right)$$

$$= Q\left(Z_T^Z\, e^{-\int_0^T \lambda_s ds}\, \mathbb{1}_{\bar{\tau} > U}\right)$$

$$= Q^Z\left(e^{-\int_0^T a_1 + a_2 \mathbb{1}_{\bar{\tau} \leq s}\, ds}\, \mathbb{1}_{\bar{\tau} > U}\right)$$

Since under Q^Z, $\bar{\lambda}_s = \bar{a}_1 + \bar{a}_2 \mathbb{1}_{\tau \leq s} = \bar{a}_1$ for $s \in [0,T]$, Q^Z-a.e.

it follows that $Q(\tau > T, \bar{\tau} > U)$

$$= e^{-a_1 T} \int_U^T e^{-a_2 (T-x)}\, \bar{a}_1 e^{-\bar{a}_1 x}\, dx + \int_T^{\infty} \bar{a}_1 e^{-\bar{a}_1 x}\, dx$$

In turn,

$$\frac{\partial}{\partial T}\frac{\partial}{\partial U} Q(\tau > T, \bar{\tau} > U) = \frac{\partial}{\partial T}\frac{\partial}{\partial U}\left(e^{-a_1 T}\int_U^T e^{-a_2 (T-x)}\, \bar{a}_1 e^{-\bar{a}_1 x}\, dx\right)$$

$$= \frac{\partial}{\partial T} e^{-a_1 T} \cdot \left(-e^{-a_2 (T-U)}\, \bar{a}_1 e^{-\bar{a}_1 U}\right)$$

$$= \bar{a}_1 (a_1 + a_2)\, e^{-(a_1 + a_2) T - (\bar{a}_1 - a_2) U}, \quad T > U \geq 0$$

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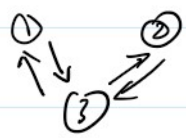
Remark: For $n = 3$, consider τ^1, τ^2, τ^3 with their corresponding intensities

$$\lambda_s^1 = a + b\, \mathbb{1}_{\tau_3 \leq t}$$

$$\lambda_s^2 = a + b\, \mathbb{1}_{\tau_3 \leq t}$$

$$\lambda_s^3 = c + d\, \mathbb{1}_{\tau_1 \leq t} + d\, \mathbb{1}_{\tau_2 \leq t}$$



Case 1. $T^3 > T^1 \vee T^2 \geq 0$, let $Z_t^3 = \mathbb{1}_{\tau^3 > t}\, e^{\int_0^t \mathbb{1}_{\tau^3 > s} \lambda_s^3\, ds}$ and define

$$\frac{dQ^3}{dQ} = Z_{T^3}^3 \quad \text{on } (\Omega, \mathcal{G}_{T^3})$$

Then, $H_t^3 \triangleq \mathbb{1}_{\ldots}$, $t \in [0, T^3]$ is 0, Q^3-a.e.

$$\frac{dQ^3}{dQ} = Z^3_{T^3} \quad \text{on } (\Omega, \mathcal{G}_{T^3})$$

Then $H^3_t \triangleq \mathbb{1}_{\tau^3 \le t}$, $t \in [0, T^3]$, is 0 Q^3-a.e.

$$Q(\tau^i > T^i, 1 \le i \le 3) = Q\left(Z^3_{T^3} e^{-\int_0^{T^3} \mathbb{1}_{\tau^3 > s} \lambda^3_s ds} \mathbb{1}_{\tau^1 > T^1} \mathbb{1}_{\tau^2 > T^2}\right)$$

$$= Q\left(Z^3_{T^3} e^{-\int_0^{T^3} \lambda^3_s ds} \mathbb{1}_{\tau^1 > T^1} \mathbb{1}_{\tau^2 > T^2}\right)$$

$$= Q^3\left(e^{-\int_0^{T^3} c + d\mathbb{1}_{\tau^1 \le s} + d\mathbb{1}_{\tau^2 \le s} ds} \mathbb{1}_{\tau^1 > T^1} \mathbb{1}_{\tau^2 > T^2}\right)$$

Since under Q^3, $\lambda^1 = \lambda^2 = a$ for $s \in [0, T^3]$, Q^3-a.e.

it follows that $Q(\tau^i > T^i, 1 \le i \le 3)$

$$= e^{-cT^3}\Bigg(\int_{T^1}^{T_3} dx_1 \int_{T^2}^{T_3} dx_2 \, e^{-d(T_3 - x_1)} e^{-d(T_3 - x_2)} \cdot f(x_1) f(x_2)$$

$$+ \int_{T^1}^{T_3} dx_1 \int_{T^3}^{\infty} dx_2 \, e^{-d(T_3 - x_1)} f(x_1) f(x_2)$$

$$+ \int_{T_3}^{\infty} dx_1 \int_{T_2}^{T_3} dx_2 \, e^{-d(T_3 - x_2)} f(x_1) f(x_2)$$

$$+ \int_{T_3}^{\infty} dx_1 \int_{T^3}^{\infty} dx_2 \, f(x_1) f(x_2)\Bigg) \quad \text{with } f(x) = a e^{-ax}$$

In turn, $\frac{\partial}{\partial T_1} \frac{\partial}{\partial T_2} \frac{\partial}{\partial T_3} Q(\tau^i > T^i, 1 \le i \le 3)$

$$= \frac{\partial}{\partial T_1} \frac{\partial}{\partial T_2} \frac{\partial}{\partial T_3} \left(e^{-cT^3} \int_{T^1}^{T^3} dx_1 \int_{T^2}^{T^3} dx^2 \, e^{-d(T_3 - x_1)} e^{-d(T_3 - x_2)} a e^{-ax_1} a e^{-ax_2}\right)$$

$$= \frac{\partial}{\partial T^3} e^{-cT^3} \cdot \left(-e^{-d(T_3 - T_1)} a e^{-aT_1}\right) \cdot \left(-e^{-d(T_3 - T_2)} a e^{-aT_2}\right)$$

$$= \frac{\partial}{\partial T^3} (-1)^2 a^2 e^{-(c+2d)T_3} e^{-(a-d)(T_1 + T_2)}$$

$$= (-1)^3 a^2 (c+2d) e^{-(c+2d)T_3 - (a-d)(T_1 + T_2)} \quad \text{for } T^3 > T^1 \vee T^2 \ge 0$$

Case 2 $T^1 \vee T^2 > T^3 \ge 0$. Without loss of generality, consider $T^1 > T^2$.

Let $Z^1_t = \mathbb{1}_{\tau^1 > t} e^{-\int_0^t \mathbb{1}_{\tau^1 > s} \lambda^1_s ds}$ and define

$$\frac{dQ^1}{dQ} = Z^1_{T_1} \quad \text{on } (\Omega, \mathcal{G}_{T_1})$$

Then $H^1_t \triangleq \mathbb{1}_{\ldots}$, $t \in [0, T^1]$, is 0 Q^1-a.e.

$$\frac{dQ'}{dQ} = Z'_{T_1} \text{ on } (\Omega, \mathcal{F}_{T_1})$$

Then $H'_t \hat{=} \mathbb{1}_{\tau^1 \le t}$, $t \in [0, T^1]$, is 0 Q'-a.e.

Note that under Q': $\lambda^2_s = a + b\,\mathbb{1}_{\tau^3 \le s}$

$\lambda^3_s = c + d\,\mathbb{1}_{\tau^2 \le s}$

Hence, $Q(\tau^i > T^i,\ 1 \le i \le 3) = Q'\left(e^{-\int_0^{T_1} a + b\mathbb{1}_{\tau^3 \le s} ds}\ \mathbb{1}_{\tau^2 > T^2}\mathbb{1}_{\tau^3 > T^3} \right)$

$$= e^{-aT^1}\left(\int_{T^2}^{\infty} dx_2 \int_{T^3}^{T_1} dx_3\ e^{-b(T_1 - x_3)} f_{23}(x_2, x_3) + \int_{T^2}^{\infty} dx_2 \int_{T^1}^{\infty} dx_3\ f_{23}(x_2, x_3) \right).$$

with $f_{23}(x_2, x_3)$ joint density of (τ^2, τ^3) derived in Ex 1.

In turn. $\frac{\partial}{\partial T_1}\frac{\partial}{\partial T_2}\frac{\partial}{\partial T_3} Q(\tau^i > T^i,\ 1 \le i \le 3)$

$$= \frac{\partial}{\partial T_1}\frac{\partial}{\partial T_2}\frac{\partial}{\partial T_3}\left(e^{-aT^1}\int_{T^2}^{\infty} dx_2 \int_{T^3}^{T_1} dx_3\ e^{-b(T_1 - x_3)} f_{23}(x_2, x_3) \right)$$

$$= \frac{\partial}{\partial T_1}\frac{\partial}{\partial T_3}\ e^{-aT^1}\cdot\left(- \int_{T^3}^{T_1} dx_3\ e^{-b(T_1 - x_3)} f_{23}(T^2, x^3) \right)$$

$$= \frac{\partial}{\partial T_1}\ e^{-aT^1}\cdot\ e^{-b(T_1 - T_3)} f_{23}(T^2, T^3)$$

$$= (-1)\ (a+b)\ e^{-(a+b)T_1 + bT_3}\ f_{23}(T^2, T^3) \qquad T^1 > T^2 \vee T^3 \ge 0$$

Similarly, for $T^2 \ge T^1$,

$$\frac{\partial}{\partial T_1}\frac{\partial}{\partial T_2}\frac{\partial}{\partial T_3} Q(\tau^i > T^i,\ 1 \le i \le 3) = (-1)\ (a+b)\ e^{-(a+b)T_2 + bT_3} f_{13}(T^1, T^3)$$

$$T^2 > T^1 \vee T^3 \ge 0.$$

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