ST403X / MA4F70
UNIVERSITY OF WARWICK
FINAL YEAR EXAMINATIONS: SUMMER 2018

BROWNIAN MOTION
Time Allowed: 2 Hours

Full marks may be gained by correctly answering 3 complete questions. Candidates may attempt all questions. Marks will be awarded for the best 3 answers only.

Read carefully the instructions on the answer book and make sure that the particulars required are entered on each answer book

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1. (a) Give a definition of one-dimensional, standard Brownian motion.

[4 marks]

(b) Construct a process which has the same finite dimensional distributions as one-dimensional Brownian motion but is *almost surely* not continuous. You need to include full justifications of the properties of the process that you construct.

[8 marks]

(c) For a one-dimensional, standard Brownian motion $\beta(\cdot)$, starting from zero, denote by P_0 its distribution. Consider the process $x(t) := \int_0^t \beta(s)\, ds$ and calculate $E_0\big[x(t)\big]$, $E_0\Big[\big(x(t)\big)^2\Big]$ and $E_0\Big[\big(x(t)\big)^3\Big]$.

[8 marks]

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2. Let $\beta(\cdot)$ be one-dimensional Brownian motion starting at 0 and denote by P_0 its distribution. For $n, k_n \geq 1$ let

$$0 = t_0^n \leq t_1^n \leq \cdots \leq t_{k_n}^n = t,$$

define partitions of the interval $(0,t)$, such that $\max_{i=1,\ldots,k_n} |t_j^n - t_{j-1}^n| \leq e^{-n}$. Define

$$D_n := \sum_{j=1}^{k_n} \big(\beta(t_j^n) - \beta(t_{j-1}^n)\big)^2.$$

(a) Compute the mean of D_n. [3 marks]

(b) Compute the variance of D_n. [6 marks]

(c) Show that D_n converges *almost surely* as $n \to \infty$ and compute its limit. [6 marks]

(d) Show that

$$\sup_{n\geq 1} \sum_{i=1}^{k_n} \big|\, \beta(t_i^n) - \beta(t_{i-1}^n) \,\big| = +\infty, \qquad P_0 - a.s.$$

[5 marks]

Continued ...

3. (a) Consider a cylinder $\mathcal{C} := B(0,1) \times \mathbb{R}_+$, where $B(0,1)$ is the disk on $\mathbb{R}^2$, with unit radius and centred at zero. Assume that there is a "smooth" solution to the boundary value problem

$$\begin{aligned} u_t &= \frac{1}{2}\Delta u, && \text{in } \ \mathcal{C}, \\ u(0,x) &= f(x), && \text{for } \ x \in B(0,1), \\ u(t,x) &= g(t,x), && \text{for } \ t>0, \quad x \in \partial B(0,1). \end{aligned}$$

Write a representation of the solution as a functional of Brownian motion. Justify all your steps and state the "smoothness" assumptions that you need.

[6 marks]

(b) Let $P_x, x \in \mathbb{R}$, denote the Wiener measure corresponding to standard, one-dimensional Brownian motion starting from x. Denote $u(t,x) := P_x\big(\max_{0\le s\le t} \beta(s) < b\big)$. Set up the boundary value problem that $u(t,x)$ satisfies. Justify your derivation. [6 marks]

(c) Let $\beta_1(\cdot)$ and $\beta_2(\cdot)$ be two independent, standard, one-dimensional Brownian motions starting from x_1, x_2, respectively. Compute the probability that they do not meet in the time interval $[0,t]$, $t>0$. [8 marks]

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4. (a) Consider the parabolic problem

$$\begin{aligned} \frac{\partial u}{\partial t} &= \frac{1}{2}\Delta u + V(x)u, && t>0, x\in\mathbb{R}^d, \\ u(0,x) &= f(x), && x \in \mathbb{R}^d. \end{aligned}$$

Show that

$$u(t,x) := E_x\left[f(\beta(t)) \exp\Big(\int_0^t V(\beta(s))\, ds \Big) \right]$$

solves the above boundary value problem. State any assumptions you might need on V, f. [10 marks]

(b) Let $\beta(\cdot)$ be a one-dimensional Brownian motion. Let $\gamma, \alpha > 0$ and $\tau := \inf\{t \colon |\beta(t)| \ge \alpha\}$. Compute $E_0[e^{\gamma\tau}]$. Does this exist for all $\gamma > 0$? Justify your answer. [10 marks]

THE END

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1. (a) Give the definition of one dimensional, standard Brownian motion.

[4 marks]

Solution [Bookwork]. Brownian motion is a one dimensional, continuous process with independent increments, which are distributed as normals with mean zero and variance equal to the time increment.

(b) Construct a process which has the same finite dimensional distributions like one dimensional Brownian motion but it is *almost surely* not continuous. You need to include full justifications of the properties of the process that you will construct.

Solution [Similar example seen]: Let U be a uniform random variable on $[0,1]$ and $\beta(\cdot)$ a standard one dimensional Brownian motion. Define the process $\big(x(t) : t \in [0,1]\big)$ which equals $\beta(t)$ for any $t \in [0,1]$ different than U and it is equal to zero for $t = U$. For any collection of times $0 \le t_1 < t_2 < \cdots < t_k \le 1$, we have that *a.s.* $U \notin \{t_1, ..., t_k\}$ and thus $(x(t_1), ..., x(t_k)) = (\beta(t_1), ..., \beta(t_k))$, which implies that the finite dimensional distributions of $x(\cdot)$ agree with those of $\beta(\cdot)$. However, given U the probability that $\beta(U) = 0$ is zero and the continuity of Brownian motion implies that $x(\cdot)$ is *a.s.* not continuous at U **[4 marks for the process + 4 for the properties]**. [8 marks]

(c) Given a one dimensional, standard Brownian motion $\beta(\cdot)$, starting from zero, consider the process $x(t) := \int_0^t \beta(s)\, ds$. Calculate $E_0[x(t)]$, $E_0\Big[\big(x(t)\big)^2\Big]$ and $E_0\Big[\big(x(t)\big)^3\Big]$.

[8 marks]

Solution [Unseen]:

(i)

$$E_0 x(t) = E_0 \int_0^t \beta(s)\, ds = \int_0^t E_0[\beta(s)]\, ds = 0.$$

(ii)

$$\begin{aligned} E_0\big(x(t)\big)^2 &= E_0 \int_0^t \int_0^t \beta(r)\, \beta(s)\, dr\, ds = \int_0^t \int_0^t E_0[\beta(r)\beta(s)]\, dr\, ds \\ &= \int_0^t \int_0^t \min\{r, s\}\, dr\, ds = \int_{0<r<s<t} r\, dr\, ds + \int_{0<s<r<t} r\, dr\, ds \\ &= 2 \int_{0<r<s<t} r\, dr\, ds = \frac{1}{3} t^3. \end{aligned}$$

(d) Show that

$$V_n := \sup_{n\geq 1} \sum_{i=1}^{k_n} \left| \beta(t_i^n) - \beta(t_{i-1}^n) \right| = +\infty, \qquad P_0 - a.s.$$

[5 marks]

Solution [Unseen example]: Assume that $\sup_n V_n < \infty$. Then

$$D_n \leq \max_{i=1,\dots,k_n} \left|\beta(t_i^n) - \beta(t_{i-1}^n)\right| V_n \leq C \max_{i=1,\dots,k_n} \left|\beta(t_i^n) - \beta(t_{i-1}^n)\right|.$$

Continuity of the brownian motion and the fact that the mesh of the partition converges to zero implies that the right hand side would converge to zero if $< \infty$, which is a contradiction to the fact that D_n converges to t.

Continued ...

(c) Let $\beta_1(\cdot)$ and $\beta_2(\cdot)$ be two independent, standard, one dimensional Brownian motions starting from x_1, x_2, respectively. Compute the probability that they do not meet in the time interval $[0, t]$, $t > 0$. [8 marks]

Solution [Unseen example]: The difference $\beta(s) := (\beta_1(s) - \beta_2(s))/\sqrt{2}$ is a Brownian motion **[2 marks]**. Therefore, they will not meet by time t, if $\beta(s)$ does not hit zero by time $2t$. This now reduces to the reflection principle: Start from $P(\beta(t) \geq b) = P(\beta(t) \geq b,\ \tau_b \leq t)$ where τ_b is the hitting time of b **[2 marks]**. Use the strong Markov property at $\mathcal{F}^+(\tau_b)$. Finally, use the fact that for any t, $P_0(\beta(t) \geq 0) = 1/2$.

$$\begin{aligned} P(\beta(t) \geq b) &= P(\beta(t) \geq b,\ \tau_b \leq t) = E\big[P(\beta(t) \geq b,\ \tau_b \leq t) \,|\, \mathcal{F}_{\tau_b}\big] \\ &= E[P_b(\beta(t - \tau_b) \geq b)\, 1_{\tau_b \leq t}] = \frac{1}{2} P(\tau_b \leq t) \end{aligned}$$

Apply this formula for $b = x_1 - x_2$. **[4 marks]**

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4. (a) Consider the parabolic problem

$$\begin{aligned} \frac{\partial u}{\partial t} &= \frac{1}{2}\Delta u + V(x) u, & t > 0, x \in \mathbb{R}^d, \\ u(0, x) &= f(x) & x \in \mathbb{R}^d. \end{aligned}$$

Show that

$$u(t, x) := E_x\left[f(\beta(t)) \exp\Big(\int_0^t V(\beta(s))\, ds \Big) \right]$$

solves the above boundary value problem. State any assumptions you might need on u, V, f. [10 marks]

Solution [Seen example]: Expand the exponential:

$$\begin{aligned} &1 + \sum_{n \geq 1} \frac{1}{n!} E_x\left[f(\beta(t)) \Big(\int_0^t V(\beta(s))\, ds \Big)^n \right] \\ &= 1 + \sum_{n \geq 1} \frac{1}{n!} E_x\left[f(\beta(t)) \int_0^t \cdots \int_0^t V(\beta(s_1)) \cdots V(\beta(s_n))\, ds \right], \end{aligned}$$

which by symmetry writes as

$$1 + \sum_{n \geq 1} E_x\left[f(\beta(t)) \int_{0 < s_1 < \cdots < s_n < t} V(\beta(s_1)) \cdots V(\beta(s_n))\, ds_1 \cdots ds_n \right].$$

[2 marks]

Denote the n^{th} term by I_n and change variables $s_i =: t - \eta_i$ to write the above as

$$I_n(t, x) := \int_{0 < \eta_n < \cdots < \eta_1 < t} E_x\left[f(\beta(t)) \int_0^t \cdots \int_0^t V(\beta(t - \eta_1)) \cdots V(\beta(t - \eta_n))\, d\eta_1 \cdots d\eta_n \right].$$

The general solution to this equation is $u(x) = A\cos(\sqrt{2\gamma}x) + B\sin(\sqrt{2\gamma}x)$. Satisfying the boundary conditions gives

$$\begin{aligned} 1 &= A\cos(\sqrt{2\gamma}a) + B\sin(\sqrt{2\gamma}a) \\ 1 &= A\cos(\sqrt{2\gamma}a) - B\sin(\sqrt{2\gamma}a) \end{aligned}$$

(3 marks)

Adding and subtracting the equations gives $A\cos(\sqrt{2\gamma}a) = 1$ and $B\sin(\sqrt{2\gamma}a) = 0$. From this we can choose $B = 0$. However, if $\cos(\sqrt{2\gamma}a) = 0$ then there is no choice of A that can satisfy the boundary condition. So, if $\sqrt{2\gamma}a < \pi/2$ then we can choose

$$A = \frac{\cos(\sqrt{2\gamma}x)}{\cos(\sqrt{2\gamma}a)}.$$

(2 marks)

If γ is such that $\sqrt{2\gamma}a \geq \pi/2$ then there is no solution, either because the boundary condition will force $A = B = 0$ or the above solution takes negative values. So for $\sqrt{2\gamma}a \geq \pi/2$ the exponential moment of τ is infinite **(2 marks)**.

THE END