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Stochastic Modelling and Random Processes

Example sheet 3

1. **Symmetric random walk and absorbing times** Let $(X_n : n \in \mathbb{N}_0)$ be a simple symmetric random walk (i.e. a simple random walk with $p = q = 1/2$) in discrete time, on the state space $S = \{-N, \ldots, N\}$ with absorbing boundary conditions.

   (a) Sketch the one-step transition matrix P.

   (b) Give a formula for all stationary distributions π. Are they reversible? Is this process ergodic? Justify your answers.

   (c) For $A = \{-N, N\}$ we know (from lectures) that the process gets absorbed with probability 1 in a point of A for all initial conditions k, i.e.

   $$h_k^A := \mathbb{P}[X_n \in A \text{ for some } n \geq 0 | X_0 = k] = 1.$$

   Let $T^A = \min\{n \geq 0 : X_n \in A\}$ be the corresponding absorption time, and $\tau_k^A = \mathbb{E}[T^A | X_0 = k]$ its expected value starting in k. Show that

   $$\tau_k^A = \tfrac{1}{2}\,\tau_{k-1}^A + \tfrac{1}{2}\,\tau_{k+1}^A + 1\,, \quad k = -N+1, \ldots, N-1\,.$$

   What are the boundary conditions of this recursion?

   (d) The solution of the above recursion is of the form $\tau_k^A = ak^2 + bk + c$. Use the symmetry of the problem to determine $a, b, c \in \mathbb{R}$ and compute τ_0^A.

   (e) To confirm your findings from the previous parts, simulate 500 realisations of this random walk starting from $X_0 = 0$ for a few values of N (say $N = 5$, $N = 7$ and $N = 10$), and for an appropriately long period of time (you should justify your choice for the final time, and this need not be the same for all values of N).
   For each realisation, keep track of the absorption time T^A and use this to compute τ_0^A. Plot the empirical distribution in the form of a histogram after 10 time steps, and at your chosen final time.

   (f) Repeat part (e) for $X_0 = 2$ and $X_0 = -2$. What do you observe?

2. **Simulating a continuous time Markov Chain**
   Consider the CTMC from Problem 2 in assignment 1, $(X_t : t \geq 0)$ with generator

$$G = \begin{pmatrix} -2 & 1 & 1 \\ 1 & -4 & 3 \\ 0 & 1 & -1 \end{pmatrix}.$$

   Suppose the state space of this chain is $S = \{1, 2, 3\}$. We will use the "algorithm" from lectures to generate paths of this CTMC.

Step 1. Start from $X_0 = 2$ and define $Y_0 = X_0$.

Step 2. Compute the current holding time by sampling $W_2 \sim \mathrm{Exp}(|g(2,2)|)$ to define $J_0 = W_{Y_0} = W_2$. We have $X_t = X_0$ for $0 \leq t < J_0$

Step 3. Compute Y_1 by sampling from the DTMC Y_n with transition matrix (cf slide 7, lecture 5)

$$p^Y(x, y) = (1 - \delta_{x,y}) \frac{g(x, y)}{|g(x, x)|}$$

Step 4. Compute the current holding time by sampling $W_{Y_1} \sim \mathrm{Exp}(|g(Y_1, Y_1)|)$ to define $J_1 = W_{Y_1}$. We have $X_t = Y_1$ for $J_0 \leq t < J_1$

Step 5 Repeat for the number of steps required.

Use this algorithm to sample and visualise a few paths of this CTMC (e.g. up to $T = 10, 100, ...$ so you can see a few jumps).

3. **Time reversed Markov Chains**

We discussed earlier in the module some of the implications of reversibility (including that reversible distributions are stationary). Another of its implications is that we can actually look at what happens when we reverse time.

Let $(X_t : t \geq 0)$ be a homogeneous CTMC with finite state space S and let G^X be its generator. Assume that X_t has a unique stationary distribution π. Let us also consider that we are working on a compact interval $t \in [0, T]$ in which X_t is stationary, i.e. $X_t \sim \pi$ (we can do this by assuming we have started counting time once the chain has reached its stationary state – this is okay because the chain is homogeneous).

We define the *time-reversed chain* $(Y_t : t \in [0, T])$ to be a CTMC with state space S and where $Y_t := X_{T-t}$.

(a) Prove that the generator of the CTMC for Y_t is given by

$$g^Y(x, y) = \frac{\pi(y)}{\pi(x)} g^X(y, x).$$

Hint: Use the transition probability function and the detinition of conditional distribution.

(b) Show that Y_t is stationary and its stationary distribution is π.

Note that: This means that stationary chains can be extended to negative times by taking $Y_t = X_{-t}$.

(c) Use item (a) to show that stationary chains where the stationary distributon π are time-reversible (i.e. $g^Y(x, y) = g^X(x, y)$ or $p^Y(x, y) = p^X(x, y)$).

(d) **[Harder, non-examinable question]** You can also show that the time-reversal of a *non-stationary* Marov chain is in general **not** a homogeneous Markov chain. This means that its transition function depends on time. Using Bayes' theorem, you can actually show that, in this case (and for DTMCs), we have

$$p^Y(x, y; n) = \frac{\pi_{N-n-1}(y)}{\pi_{N-n-1N-n}(x)} p^X(y, x).$$