Applications of Stochastic Calculus in Finance Chapter 4: Change of numeriare and forward measure

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1 Change of Numeraire

A numeraire $N = (N_t)_{t\geq 0}$ is the unit of account in which other assets are denominated. In principle, we can take any positively priced asset as a numeraire, and denominate other assets in terms of the chosen numeraire.

Consider a financial market with $(n+1)$ assets: $S = (B, S^1, \dots, S^n)^T$, where B is the bank account:

$$dB_t = B_t r_t dt, \quad B_0 = 1;$$

and S^i is the ith risky asset:

$$dS_t^i = S_t^i(\mu_t^i dt + \sum_{j=1}^{d} \sigma_t^{ij} dW_t^j), \quad S_0^i > 0,$$

or equivalently in a matrix form:

$$dS_t = diag[S_t](\mu dt + \sigma_t dW_t),$$

where

$$diag[S_t] = \begin{pmatrix} S_t^1 & & \\ & \ddots & \\ & & S_t^n \end{pmatrix}.$$

Assume that the short rate r, the appreciation rate μ^i and the volatility σ^{ij} are progressively-measurable processes. Moreover, both $X_t^0 = \int_0^t r_s ds$ and

$$X_t^i = \int_0^t \mu_s^i ds + \int_0^t \sum_{j=1}^{d} \sigma_s^{ij} dW_s^i$$

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are semimartingales.

Theorem 1. *(The dynamics of risky assets denominated by a numeraire)*

Suppose that there exist an ELMM $\mathbf{Q}$*, under which the discounted price of the numeraire* $\frac{N_t}{B_t}$ *follows*

$$d\frac{N_t}{B_t} = \frac{N_t}{B_t}\sum_{j=1}^{d} h_t^j dW_t^{\mathbf{Q},j} \Rightarrow \frac{N_t}{B_t} = N_0 \mathscr{E}\left(\int_0^{\cdot} \sum_{j=1}^{d} h_u^j dW_u^{\mathbf{Q},j}\right)_t$$

for some volatility process $h = (h^1, \ldots, h^d)$ *and the* d*-dimensional Brownian motion* $W^{\mathbf{Q}} = (W^{\mathbf{Q},1}, \ldots, W^{\mathbf{Q},d})^T$ *under the ELMM* $\mathbf{Q}$*.*

Suppose further that the Novikov's condition holds, i.e. $\mathbf{E}^{\mathbf{Q}}[e^{\frac{1}{2}\int_0^T |h_s|^2 ds}] < \infty$*, so the stochastic exponential* $\mathscr{E}(\int_0^{\cdot} h_s dW_s^{\mathbf{Q}})$ *is a* $\mathbf{Q}$*-martingale up to* T*. Define a new probability measure* $\mathbf{Q}^N \sim \mathbf{Q}$ *by the Radon-Nikodym density:*

$$\left.\frac{d\mathbf{Q}^N}{d\mathbf{Q}}\right|_{\mathscr{F}_t} = \mathscr{E}\left(\int_0^{\cdot} h_s dW_s^{\mathbf{Q}}\right)_t = \frac{N_t}{B_t N_0}.$$

By Girsanov's theorem, $W_t^{\mathbf{Q}^N} = W_t^{\mathbf{Q}} - \int_0^t h_s ds$*,* $t \in [0,T]$*, is a* d*-dimensional Brownian motion under* $\mathbf{Q}^N$*.*

Then the risky asset S_t^i*, in units of the numeraire* N_t*,* $S_t^{i,N} = \frac{S_t^i}{N_t}$ *is a* $\mathbf{Q}^N$*-local martingale and follows*

$$dS_t^{i,N} = S_t^{i,N}\sum_{j=1}^{d}(\sigma_t^{ij} - h_t^j)dW_t^{\mathbf{Q}^N,j} \Rightarrow S_t^{i,N} = S_0^{i,N}\mathscr{E}\left(\int_0^{\cdot}\sum_{j=1}^{d}(\sigma_u^{ij} - h_u^j)dW_u^{\mathbf{Q}^N,j}\right)_t$$

under the probability $\mathbf{Q}^N$ *induced by the numeraire* N*.*

Proof. Let $\tilde{N}_t = N_t/B_t$ and $\tilde{S}_t^i = S_t^i/B_t$. Then $S_t^{i,N} = \tilde{S}_t^i/\tilde{N}_t$. Under the ELMM $\mathbf{Q}$, we have

$$d\tilde{S}_t^i = \tilde{S}_t^i \sum_{j=1}^{d} \sigma_t^{ij} dW_t^{\mathbf{Q},j}$$

and

$$d\tilde{N}_t = \tilde{N}_t \sum_{j=1}^{d} h_t^j dW_t^{\mathbf{Q},j}.$$

Apply Itô's formula to $\frac{1}{\tilde{N}_t}$,

$$\begin{aligned} d\frac{1}{\tilde{N}_t} &= -\frac{1}{(\tilde{N}_t)^2}d\tilde{N}_t + \frac{1}{(\tilde{N}_t)^3}d\langle \tilde{N}\rangle_t \\ &= \frac{1}{\tilde{N}_t}(-\sum_{j=1}^{d} h_t^j dW_t^{\mathbf{Q},j}) + \frac{1}{\tilde{N}_t}(\sum_{j=1}^{d}|h_t^j|^2 dt). \end{aligned}$$

Apply Itô's formula to $\tilde{S}_t^i \cdot \frac{1}{\tilde{N}_t}$,

$$
\begin{aligned}
dS_t^{i,N} &= d(\tilde{S}_t^i \cdot \frac{1}{\tilde{N}_t}) \\
&= \tilde{S}_t^i d\frac{1}{\tilde{N}_t} + \frac{1}{\tilde{N}_t} d\tilde{S}_t^i + d\langle \tilde{S}^i, \frac{1}{\tilde{N}} \rangle_t \\
&= \frac{\tilde{S}_t^i}{\tilde{N}_t}(-\sum_{j=1}^d h_t^j dW_t^{\mathbf{Q},j} + \sum_{j=1}^d |h_t^j|^2 dt) + \frac{\tilde{S}_t^i}{\tilde{N}_t}\sum_{j=1}^d \sigma_t^{ij} dW_t^{\mathbf{Q},j} + \frac{\tilde{S}_t^i}{\tilde{N}_t}(-\sum_{j=1}^d \sigma_t^{ij} h_t^j dt) \\
&= \frac{\tilde{S}_t^i}{\tilde{N}_t}\sum_{j=1}^d (\sigma_t^{ij} - h_t^j)(dW_t^{\mathbf{Q},j} - h_t^j dt) \\
&= S_t^{i,N} \sum_{j=1}^d (\sigma_t^{ij} - h_t^j) dW_t^{\mathbf{Q}^N,j}
\end{aligned}
$$

which concludes the proof. □

Theorem 2. *(Arbitrage price under a numeraire)*
Suppose that there exists a unique ELMM $\mathbf{Q}$. *Then the arbitrage price of the contingent claim* $X \in \mathscr{F}_T$ *is given by*

$$
X_t = \mathbf{E}^{\mathbf{Q}^N}\left[\frac{X}{N_T} N_t | \mathscr{F}_t\right].
$$

Proof. By using Bayes rule, we obtain that

$$
\begin{aligned}
X_t &= \mathbf{E}^{\mathbf{Q}}\left[\frac{X}{B_T} B_t | \mathscr{F}_t\right] \\
&= \mathbf{E}^{\mathbf{Q}}\left[\frac{X}{N_T} N_t \frac{\frac{N_T}{B_T N_0}}{\frac{N_t}{B_t N_0}} \middle| \mathscr{F}_t\right] \\
&= \mathbf{E}^{\mathbf{Q}}\left[\frac{X}{N_T} N_t \frac{\left.\frac{d\mathbf{Q}^N}{d\mathbf{Q}}\right|_{\mathscr{F}_T}}{\left.\frac{d\mathbf{Q}^N}{d\mathbf{Q}}\right|_{\mathscr{F}_t}} \middle| \mathscr{F}_t\right] \\
&= \mathbf{E}^{\mathbf{Q}^N}\left[\frac{X}{N_T} N_t | \mathscr{F}_t\right].
\end{aligned}
$$

□

Next, we present two examples of numeraire N_t:

(1) Bank account $N_t = B_t$, so $\tilde{N}_t = 1$ and $h_t^j = 0$. In this case, $\mathbf{Q}^N = \mathbf{Q}$, $W_t^{\mathbf{Q}^N} = W_t^{\mathbf{Q}}$, and $S_t^{i,N} = \frac{S_t^i}{N_t}$ follows

$$
dS_t^{i,N} = S_t^{i,N} \sum_{j=1}^d \sigma_t^{ij} dW_t^{\mathbf{Q},j}.
$$

(2) Risky asset $N_t = S_t^k$ for some $1 \leq k \leq n$. Hence, $\tilde{N}_t = \frac{S_t^k}{B_t}$ and $h_t^j = \sigma_t^{kj}$. In this case, if the Novikov's condition holds, then

$$\left.\frac{d\mathbf{Q}^N}{d\mathbf{Q}}\right|_{\mathscr{F}_t} = \mathscr{E}\left(\int_0^{\cdot} \sum_{j=1}^{d} \sigma_s^{kj} dW_s^{\mathbf{Q},j}\right)_t = \frac{S_t^k}{B_t S_0^k},$$

and

$$W_t^{\mathbf{Q}^N,j} = W_t^{\mathbf{Q},j} - \int_0^t \sigma_s^{kj} ds$$

Finally, $S_t^{i,N} = \frac{S_t^i}{N_t}$ follows

$$dS_t^{i,N} = S_t^{i,N} \sum_{j=1}^{d} (\sigma_t^{ij} - \sigma_t^{kj}) dW_t^{\mathbf{Q}^N,j}.$$

2 T-Forward Measure

If we take the price of zero-coupon bond price $P(t,T)$ as the numeraire $N_t = P(t,T)$, then the probability measure $\mathbf{Q}^N \sim \mathbf{Q}$ induced by the numeraire $P(t,T)$ is called T-forward measure, which is denoted as $\tilde{\mathbf{Q}}^T$.

We repeat the no arbitrage assumption in the interest rate modeling below (see Proposition 3 in Chapter 3 for its verification).

Assumption 1 *(No arbitrage): There exists an EMM* $\mathbf{Q}$ *such that the discounted zero-coupon bond price process* $P(t,T)/B_t$ *is a martingale under* $\mathbf{Q}$.

Under the above assumption,

$$\left.\frac{d\tilde{\mathbf{Q}}^T}{d\mathbf{Q}}\right|_{\mathscr{F}_t} = \mathscr{E}\left(\int_0^{\cdot} \sum_{j=1}^{d} \sigma^{*,j}(s,T) dW_s^{\mathbf{Q},j}\right)_t = \frac{P(t,T)}{B_t P(0,T)},$$

and

$$W_t^{\tilde{\mathbf{Q}}^T,j} = W_t^{\mathbf{Q},j} - \int_0^t \sigma^{*,j}(s,T) ds.$$

By Theorem 1, the dynamics of the stock price S^i follows $S_t^{i,N} = \frac{S_t^i}{N_t}$ follows

$$dS_t^{i,N} = S_t^{i,N} \sum_{j=1}^{d} (\sigma_t^{ij} - \sigma^{*,j}(t,T)) dW_t^{\tilde{\mathbf{Q}}^T,j}.$$

Moreover, we can also obtain the dynamics of the S-bond price, which is formulated as the following theorem.

Theorem 3. *(The dynamic of the T-bond discounted S-bond price process)*

Take T-bond price $P(t,T)$ as the numeraire. Then $\frac{P(t,S)}{P(t,T)}$, $t \in [0, S \wedge T]$, is martingale under the T-forward measure $\tilde{Q}^T$. Moreover,

$$d\frac{P(t,S)}{P(t,T)} = \frac{P(t,S)}{P(t,T)}(\sigma^*(t,S) - \sigma^*(t,T))dW_t^{\tilde{\mathbf{Q}}^T},$$

i.e.

$$\frac{P(t,S)}{P(t,T)} = \frac{P(0,S)}{P(0,T)}\mathscr{E}(\int_0^{\cdot}(\sigma^*(u,S) - \sigma^*(u,T))dW_u^{\tilde{\mathbf{Q}}^T})_t$$

From the above Theorem 3, the T-froward measure $\tilde{\mathbf{Q}}^T$ and the S-forward measure $\tilde{\mathbf{Q}}^S$ are related by

$$\begin{aligned}
\left.\frac{d\tilde{\mathbf{Q}}^S}{d\tilde{\mathbf{Q}}^T}\right|_{\mathscr{F}_t} &= \left.\frac{d\tilde{\mathbf{Q}}^S}{d\mathbf{Q}}\right|_{\mathscr{F}_t} \Big/ \left.\frac{d\tilde{\mathbf{Q}}^T}{d\mathbf{Q}}\right|_{\mathscr{F}_t} \\
&= \frac{P(t,S)}{B_t P(0,S)} \Big/ \frac{P(t,T)}{B_t P(0,T)} \\
&= \frac{P(t,S)P(0,T)}{P(t,T)P(0,S)} \\
&= \mathscr{E}\left(\int_0^{\cdot}(\sigma^*(u,S) - \sigma^*(u,T))dW_u^{\tilde{\mathbf{Q}}^T}\right)_t
\end{aligned}$$

We thus receive an entire collection of EMMs. Each $\tilde{\mathbf{Q}}^T$ corresponds to a different numeraire, namely T-bond. Since the EMM $\mathbf{Q}$ is related to the bank account B_t, it is also called the spot measure/risk-neutral measure.

Under the spot measure $\mathbf{Q}$, the arbitrage price of $X \in \mathscr{F}_T$ is

$$X_t = B_t \mathbf{E}^{\mathbf{Q}}[\frac{X}{B_T}|\mathscr{F}_t],$$

so we have to know the joint distribution of X and $1/B_T$, and integrate with respect to that joint distribution. However, if we choose the T-bond as a numeraire, then by Theorem 2, we only need to know the distribution of X under the T-forward measure $\tilde{\mathbf{Q}}^T$, and the price $P(t,T)$ can be observed at time t. This result is formulated as the following theorem.

Theorem 4. *(Arbitrage price under T-forward measure)*

Suppose that there exists a unique EMM $\mathbf{Q}$. Then the arbitrage price of the contingent claim $X \in \mathscr{F}_T$ is given by

$$X_t = P(t,T)\mathbf{E}^{\tilde{\mathbf{Q}}^T}[X|\mathscr{F}_t].$$

Next, we discuss the connection between the future spot rate at time t and the current forward rate at time t.

Theorem 5. *(Expectation hypothesis)*

The expectation of the future short rate equals to the current value of the forward rate Under the T-forward measure, i.e.

$$f(t,T) = \mathbf{E}^{\tilde{\mathbf{Q}}^T}[r_T|\mathscr{F}_t]$$

Proof. Under the spot measure, we have $\mathbf{Q}$:

$$\begin{aligned} df(t,T) &= -\sigma(t,T)(\sigma^*(t,T))^T dt + \sigma(t,T) dW_t^{\mathbf{Q}} \\ &= \sigma(t,T)(-\sigma^*(t,T)^T dt + dW_t^{\mathbf{Q}}) \\ &= \sigma(t,T) dW_t^{\tilde{\mathbf{Q}}^T}, \end{aligned}$$

so the conclusion follows. □

We conclude this section by an interesting result about the yield curve for the continuously compounded short rate $R(t,T)$.

Theorem 6. *(Dybvig-Ingersoll-Ross Theorem)*
Long rates never fall. That is, if $s<t$, then $R_\infty(s) \leq R_\infty(t)$, where $R_\infty(t) := \lim_{T\uparrow\infty} R(t,T)$.

Proof. Recall that $R(t,T) = -\frac{1}{T-t}\ln P(t,T) = \frac{1}{T-t}\int_t^T f(t,u)du$, and that $T \to R(t,T)$ is called yield curve.

Define

$$\begin{aligned} p(t) &= \lim_{T\uparrow\infty} (P(t,T))^{\frac{1}{T}} \\ &= \lim_{T\uparrow\infty} (P(t,T))^{\frac{1}{T-t}} \\ &= \lim_{T\uparrow\infty} (e^{-(T-t)R(t,T)})^{\frac{1}{T-t}} = e^{-R_\infty(t)}. \end{aligned}$$

Hence, we only need to show that $p(s) \geq p(t)$.

Note that under the t-forward measure $\tilde{\mathbf{Q}}^t$, $\frac{P(s,T)}{P(s,t)}$ is a martingale, so

$$\frac{P(s,T)}{P(s,t)} = \mathbf{E}^{\tilde{\mathbf{Q}}^t}[\frac{P(t,T)}{P(t,t)}|\mathscr{F}_s] = \mathbf{E}^{\tilde{\mathbf{Q}}^t}[P(t,T)|\mathscr{F}_s]$$

which gives

$$\frac{P(s,T)^{\frac{1}{T}}}{P(s,t)^{\frac{1}{T}}} = (\mathbf{E}^{\tilde{\mathbf{Q}}^t}[P(t,T)|\mathscr{F}_s])^{\frac{1}{T}}.$$

Let $T \uparrow \infty$, we obtain that

$$p(s) = \lim_{T\uparrow\infty} (\mathbf{E}^{\tilde{\mathbf{Q}}^t}[P(t,T)|\mathscr{F}_s])^{\frac{1}{T}}.$$

Now let $X \geq 0$ be any bounded r.v. with $\mathbf{E}^{\tilde{\mathbf{Q}}^t}[X] = 1$, so X can be used as a Radon-Nikodym density.

$$
\begin{aligned}
\mathbf{E}^{\tilde{\mathbf{Q}}^t}[Xp(t)] &= \mathbf{E}^{\tilde{\mathbf{Q}}^t}[X\lim_{T\uparrow\infty}(P(t,T))^{\frac{1}{T}}] \\
&= \mathbf{E}^{\tilde{\mathbf{Q}}^t}[\liminf_{T\uparrow\infty} X(P(t,T))^{\frac{1}{T}}] \\
&\leq \mathbf{E}^{\tilde{\mathbf{Q}}^t}[\liminf_{T\uparrow\infty} \mathbf{E}^{\tilde{\mathbf{Q}}^t}[X(P(t,T))^{\frac{1}{T}}|\mathscr{F}_s]] \quad \text{(Fatou)} \\
&\leq \mathbf{E}^{\tilde{\mathbf{Q}}^t}[\liminf_{T\uparrow\infty}(\mathbf{E}^{\tilde{\mathbf{Q}}^t}[X^{\frac{T}{T-1}}|\mathscr{F}_s])^{\frac{T-1}{T}}(\mathbf{E}^{\tilde{\mathbf{Q}}^t}[P(t,T)|\mathscr{F}_s])^{\frac{1}{T}}] \quad \text{(Holder)} \\
&\leq \mathbf{E}^{\tilde{\mathbf{Q}}^t}[Xp(s)] \quad \text{(Dominated convergence)}
\end{aligned}
$$

Since X is arbitrary, we can conclude that $p(t) \leq p(s)$. $\square$

3 Black-Scholes Model with Random Interest Rates

Assumption 2 *(1) The volatility of the forward rate $\sigma(t,T) = (\sigma^1(t,T),\ldots,\sigma^d(t,T))$ is deterministic. Hence the forward rate $f(t,T)$ is Gaussian distributed.*

(2) There is one risky asset S following

$$dS_t = S_t(r_t dt + \sigma dW_t^{\mathbf{Q}})$$

under the spot measure $\mathbf{Q}$, where σ is a constant volatility vector.

Consider a European call option written on the stock S, with maturity T and strike price K. Its arbitrage price at time 0 is

$$
\begin{aligned}
X_0 &= \mathbf{E}^{\mathbf{Q}}[\frac{1}{B_T}(S_T - K)^+] \\
&= \mathbf{E}^{\mathbf{Q}}[\frac{S_T}{B_T}\mathbf{1}_{\{S_T \geq K\}}] - K\mathbf{E}^{\mathbf{Q}}[\frac{1}{B_T}\mathbf{1}_{\{S_T \geq K\}}]
\end{aligned}
$$

For the first integral, if we choose S as the numeraire, then the induced probability measure $\mathbf{Q}^S$ is defined as

$$\left.\frac{d\mathbf{Q}^S}{d\mathbf{Q}}\right|_{\mathscr{F}_t} = \frac{S_t}{B_t S_0}.$$

Bayes rule then implies that

$$
\begin{aligned}
\mathbf{E}^{\mathbf{Q}}[\frac{S_T}{B_T}\mathbf{1}_{\{S_T \geq K\}}] &= S_0\mathbf{E}^{\mathbf{Q}}[\frac{S_T}{B_T S_0}\mathbf{1}_{\{S_T \geq K\}}] \\
&= S_0\mathbf{E}^{\mathbf{Q}^S}[\mathbf{1}_{\{S_T \geq K\}}] \\
&= S_0\mathbf{Q}^S(\frac{P(T,T)}{S_T} \leq \frac{1}{K})
\end{aligned}
$$

Note that under the spot measure $\mathbf{Q}$,

$$d\frac{P(t,T)}{B_t} = \frac{P(t,T)}{B_t}\sigma^*(t,T)dW_t^{\mathbf{Q}};$$
$$d\frac{S_t}{B_t} = \frac{S_t}{B_t}\sigma dW_t^{\mathbf{Q}}.$$

Hence, under the induced probability measure $\mathbf{Q}^S$,

$$d\frac{P(t,T)}{S_t} = \frac{P(t,T)}{S_t}(\sigma^*(t,T)-\sigma)dW_t^{\mathbf{Q}^S},$$

where $W_t^{\mathbf{Q}^S} = W_t^{\mathbf{Q}} - \sigma t$ is the Brownian motion under $\mathbf{Q}^S$. In turn,

$$\begin{aligned}
&\mathbf{E}^{\mathbf{Q}}[\frac{S_T}{B_T}\mathbf{1}_{\{S_T\geq K\}}] \\
&= S_0\mathbf{Q}^S\left(\frac{P(0,T)}{S_0}e^{\int_0^T(\sigma^*(t,T)-\sigma)dW_t^{\mathbf{Q}^S}-\frac{1}{2}\int_0^T|\sigma^*(t,T)-\sigma|^2dt}\leq\frac{1}{K}\right) \\
&= S_0\mathbf{Q}^S\left(\frac{\int_0^T(\sigma^*(t,T)-\sigma)dW_t^{\mathbf{Q}^S}}{\sqrt{\int_0^T|\sigma^*(t,T)-\sigma|^2dt}}\leq\frac{\ln\frac{S_0}{P(0,T)K}+\frac{1}{2}\int_0^T|\sigma^*(t,T)-\sigma|^2dt}{\sqrt{\int_0^T|\sigma^*(t,T)-\sigma|^2dt}}\right).
\end{aligned}$$

Since

$$\frac{\int_0^T(\sigma^*(t,T)-\sigma)dW_t^{\mathbf{Q}^S}}{\sqrt{\int_0^T|\sigma^*(t,T)-\sigma|^2dt}}\sim N(0,1),$$

we obtain that

$$\mathbf{E}^{\mathbf{Q}}[\frac{S_T}{B_T}\mathbf{1}_{\{S_T\geq K\}}] = S_0\Phi(d_1)$$

where $\Phi(\cdot)$ is the CDF of standard normal distribution, and

$$d_1 = \frac{\ln\frac{S_0}{P(0,T)K}+\frac{1}{2}\int_0^T|\sigma^*(t,T)-\sigma|^2dt}{\sqrt{\int_0^T|\sigma^*(t,T)-\sigma|^2dt}}.$$

For the second integral, if we choose $P(t,T)$ as the numeraire, then the corresponding induced measure is the T-forward measure defined by

$$\left.\frac{d\tilde{\mathbf{Q}}^T}{d\mathbf{Q}}\right|_{\mathscr{F}_t} = \frac{P(t,T)}{B_tP(0,T)}.$$

Bayes rule then implies that

$$\begin{aligned}\mathbf{E}^{\mathbf{Q}}[\frac{1}{B_T}\mathbf{1}_{\{S_T\geq K\}}] &= P(0,T)\mathbf{E}^{\mathbf{Q}}[\frac{P(T,T)}{B_T P(0,T)}\mathbf{1}_{\{S_T\geq K\}}]\\ &= P(0,T)\mathbf{E}^{\tilde{\mathbf{Q}}^T}[\mathbf{1}_{\{S_T\geq K\}}]\\ &= P(0,T)\tilde{\mathbf{Q}}^T(\frac{S_T}{P(T,T)}\geq K)\end{aligned}$$

Note that under the spot measure $\mathbf{Q}$,

$$\begin{aligned} d\frac{P(t,T)}{B_t} &= \frac{P(t,T)}{B_t}\sigma^*(t,T)dW_t^{\mathbf{Q}};\\ d\frac{S_t}{B_t} &= \frac{S_t}{B_t}\sigma dW_t^{\mathbf{Q}}.\end{aligned}$$

Hence, under the T-froward measure $\tilde{\mathbf{Q}}^T$,

$$d\frac{S_t}{P(t,T)} = \frac{S_t}{P(t,T)}(\sigma-\sigma^*(t,T))dW_t^{\tilde{\mathbf{Q}}^T},$$

where $W_t^{\tilde{\mathbf{Q}}^T} = W_t^{\mathbf{Q}} - \int_0^t \sigma^*(s,T)ds$ is the Brownian motion under $\tilde{\mathbf{Q}}^T$. In turn,

$$\begin{aligned}&\mathbf{E}^{\mathbf{Q}}[\frac{1}{B_T}\mathbf{1}_{\{S_T\geq K\}}]\\ &= P(0,T)\tilde{\mathbf{Q}}^T\left(\frac{S_0}{P(0,T)}e^{\int_0^T(\sigma-\sigma^*(t,T))dW_t^{\tilde{\mathbf{Q}}^T}-\frac{1}{2}\int_0^T|\sigma^*(t,T)-\sigma|^2dt}\geq K\right)\\ &= P(0,T)\tilde{\mathbf{Q}}^T\left(\frac{\int_0^T(\sigma-\sigma^*(t,T))dW_t^{\tilde{\mathbf{Q}}^T}}{\sqrt{\int_0^T|\sigma^*(t,T)-\sigma|^2dt}}\geq\frac{-\ln\frac{S_0}{P(0,T)K}+\frac{1}{2}\int_0^T|\sigma^*(t,T)-\sigma|^2dt}{\sqrt{\int_0^T|\sigma^*(t,T)-\sigma|^2dt}}\right)\\ &= P(0,T)\tilde{\mathbf{Q}}^T\left(\frac{\int_0^T(\sigma^*(t,T)-\sigma)dW_t^{\tilde{\mathbf{Q}}^T}}{\sqrt{\int_0^T|\sigma^*(t,T)-\sigma|^2dt}}\leq\frac{\ln\frac{S_0}{P(0,T)K}-\frac{1}{2}\int_0^T|\sigma^*(t,T)-\sigma|^2dt}{\sqrt{\int_0^T|\sigma^*(t,T)-\sigma|^2dt}}\right)\end{aligned}$$

Since

$$\frac{\int_0^T(\sigma^*(t,T)-\sigma)dW_t^{\mathbf{Q}^S}}{\sqrt{\int_0^T|\sigma^*(t,T)-\sigma|^2dt}}\sim N(0,1),$$

we obtain that

$$\mathbf{E}^{\mathbf{Q}}[\frac{1}{B_T}\mathbf{1}_{\{S_T\geq K\}}] = P(0,T)\Phi(d_2)$$

where

$$d_2 = \frac{\ln\frac{S_0}{P(0,T)K}-\frac{1}{2}\int_0^T|\sigma^*(t,T)-\sigma|^2dt}{\sqrt{\int_0^T|\sigma^*(t,T)-\sigma|^2dt}}.$$

In summary, we obtain the initial value of the European option as

$$X_0 = S_0\Phi(d_1) - KP(0,T)\Phi(d_2).$$

Exercise 1. (Forward exchange rate, Shreve [1] Chapter 9)

A forward contract is an agreement to pay a specified delivery price at a future delivery date T for a risky asset. The T-forward price of this asset is the value that makes the forward contract have arbitrage price 0.

Consider a forward contract for foreign exchange rates E_t, which gives units of domestic currency per unit of foreign currency (domestic/foreign):

$$dE_t = E_t(\mu_t^E dt + \sigma_t^E(\rho_t dW_t^1 + \sqrt{1-\rho_t^2}dW_t^2)).$$

Suppose there are three assets in the market. The first one is a stock S^d, priced in domestic currency, following

$$dS_t^d = S_t^d(\mu_t^d dt + \sigma_t^d dW_t^1).$$

The second one is a domestic short rate r^d, which leads to a domestic bank account B^d following

$$dB_t^d = B_t^d r_t^d dt.$$

The third one is a foreign short rate r^f, which leads to a foreign bank account B^f following

$$dB_t^f = B_t^f r_t^f dt.$$

Assume that r_t^d, r_t^f, μ_t^d, σ_t^d, μ_t^f, σ_t^f, μ_t^E, σ_t^E and ρ_t are all *bounded and deterministic* functions. Moreover, $\sigma_t^d > 0$, $\sigma_t^f > 0$ and $-1 < \rho_t < 1$.

1. The (domestic currency) forward price F^d for a unit of foreign currency, to be delivered at time T, is determined by the equation

$$\mathbf{E}^{\mathbf{Q}^d}\left[\frac{E_T - F^d}{B^d(T)}\right] = 0.$$

Prove that

$$F^d = E_0 e^{\int_0^T (r_t^d - r_t^f)dt}.$$

(Hint: First derive the dynamic of E_t under the domestic ELMM $\mathbf{Q}^d$.)

2. The (foreign currency) forward price F^f for a unit of domestic currency, to be delivered at time T, is determined by the equation

$$\mathbf{E}^{\mathbf{Q}^f}\left[\frac{\frac{1}{E_T} - F^f}{B^f(T)}\right] = 0.$$

Prove that

$$F^f = \frac{1}{E_0}e^{\int_0^T (r_t^f - r_t^d)dt} = \frac{1}{F^d}.$$

(Hint: First derive the dynamic of $1/E_t$ under the foreign ELMM $\mathbf{Q}^f$.)

Exercise 2. (Bond option)

Assume that the volatility of the forward rate $\sigma(t,T)$ is some *bounded and deterministic* function. Recall in the HJM framework, the discounted price of the zero-coupon bond $P(t,T)$ follows:

$$d\frac{P(t,T)}{B(t)} = \frac{P(t,T)}{B(t)}\sigma^*(t,T)dW_t^{\mathbf{Q}}.$$

under the EMM $\mathbf{Q}$. Consider a call option with maturity T and strike price K, written on a zero-coupon bond with maturity $S > T$. Prove that the arbitrage price of this bond option

$$\mathbf{E}^{\mathbf{Q}}\left[\frac{(P(T,S)-K)^+}{B(T)}\right]$$

is given by the formula:

$$P(0,S)\Phi(d_1) - KP(0,T)\Phi(d_2),$$

where Φ is the standard Gaussian CDF, and

$$d_{1,2} = \frac{\ln\frac{P(0,S)}{KP(0,T)} \pm \frac{1}{2}\int_0^T |\sigma^*(u,S)-\sigma^*(u,T)|^2du}{\sqrt{\int_0^T |\sigma^*(u,S)-\sigma^*(u,T)|^2du}}.$$

References

1. Shreve, Steven E. *Stochastic calculus for finance II: Continuous-time models*. Springer, 2004.