UNIVERSITY OF WARWICK

Paper Details

Paper Code: ST9090_A

Paper Title: Applications of Stochastic Calculus in Finance

Exam Period: April 2022

Exam Rubric

Time Allowed: 2 hours

Answer THREE questions. All questions carry equal marks of 20. The numbers in the margin indicate how many marks are available for each part of a question.

Full marks will be obtained by correctly answering THREE complete questions. Candidates may attempt all questions. Marks will be awarded for the best three answers only.

Read carefully the instructions on the answer book and make sure that the particulars required are entered on each answer book.

Calculators are not permitted.

This is a closed book examination.

CONTINUED…/

1: Let $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, \mathbf{Q})$ be a filtered probability space supporting a one dimensional Brownian motion $W^{\mathbf{Q}}$ with its augmented filtration $\{\mathcal{F}_t\}_{t\geq 0}$, where $\mathbf{Q}$ is an equivalent martingale measure (EMM). Suppose that the short rate $(r_t)_{t\geq 0}$ follows the one-dimensional Vasicek model.

(a) Write down the stochastic differential equation (SDE) for $(r_t)_{t\geq 0}$, and the conditions on its coefficients. [2 marks]

(b) State the definition of an affine term structure (ATS), and explain why the Vasicek model provides an ATS. [3 marks]

(c) Write down the partial differential equation (PDE) for the price of the zero-coupon bond $P(t,T)$ given by

$$P(t,T) = \mathbf{E}^{\mathbf{Q}}[e^{-\int_t^T r_s ds} | \mathcal{F}_t],$$

and solve this PDE by making use of the ATS of $(r_t)_{t\geq 0}$. [5 marks]

(d) Derive the SDE for the corresponding forward rate $f(t,T)$.
[Hint: Use the definition of the forward rate $f(t,T)$ in terms of the zero-coupon bond price $P(t,T)$.] [5 marks]

(e) State the HJM drift condition for the forward rate $f(t,T)$, and verify it under the Vasicek model.
[5 marks]

Continued...

2: Let $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, \mathbf{P})$ be a filtered probability space supporting a two dimensional standard Brownian motion $W = (W^1, W^2)$ with its augmented filtration $\{\mathcal{F}_t\}_{t\geq 0}$, where $\mathbf{P}$ is the physical probability measure. Consider a foreign exchange model where the market consists of the British currency £, the American currency \$, and three assets: a stock S^d, a British bank account B^d and an American bank account B^f. Assume that S^d follows

$$dS_t^d = S_t^d(\mu^d dt + \sigma^d dW_t^1)$$

under £, B^d follows

$$dB^d(t) = B^d(t) r^d dt$$

under £, B^f follows

$$dB^f(t) = B^f(t) r^f dt$$

under \$, and the exchange rate $E = £/\$$ follows

$$dE_t = E_t\{\mu^E dt + \sigma^E(\rho dW_t^1 + \sqrt{1-\rho^2} dW_t^2)\}.$$

Assume that all the coefficients are constant satisfying $\mu^d, \mu^E \in \mathbb{R}$, $\sigma^d, \sigma^E, r^d, r^f > 0$, and $-1 < \rho < 1$.

(a) Let B^d in £ be the numeraire. State the definition of an equivalent local martingale measure (ELMM) $\mathbf{Q}$. [2 marks]

(b) Derive the market price of risk equations and the ELMM $\mathbf{Q}$, and under $\mathbf{Q}$ derive the stochastic differential equations (SDEs) for $\frac{S_t^d}{B_t^d}$ and $\frac{B_t^f E_t}{B_t^d}$. [10 marks]

(c) Prove that the ELMM $\mathbf{Q}$ is in fact not only an ELMM but even an equivalent martingale measure (EMM). [3 marks]

(d) Write down the pricing equation for the forward price F for one dollar, to be delivered at some future date T. Prove that the forward price at initial time 0 is given by

$$F = E_0 e^{(r^d - r^f)T}$$

with E_0 being the initial exchange rate. [5 marks]

Continued...

3: Consider a firm issuing a corporate bond which pays K at maturity T. The firm's asset value V, under the spot measure $\mathbf{Q}$, follows

$$dV_t = V_t(rdt + \sigma dW_t),\ V_0 = v,$$

on a filtered probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, \mathbf{Q})$ which supports a one dimensional Brownian motion W. Let D be an exogenously given default barrier which follows $D_t = Ke^{-d(T-t)}$. Assume that all the coefficients are constant satisfying $r, d, v, K, \sigma > 0$.

(a) Calculate V_t/D_t, and write down the definition of the default time τ as the first passage time of D_t by V_t. [2 marks]

(b) Write down the reflection principle for one dimensional Brownian motion, and derive a closed-form expression for the joint density $f_{\bar{W}_t, W_t}(b, a)$ of $(\bar{W}_t, W_t)$ where $\bar{W}_t = \sup_{0\leq s\leq t} W_s$. [6 marks]

(c) For constant drift μ, define a stochastic process $X_t = W_t + \mu t$, and its maximum process $\bar{X}_t = \sup_{0\leq s\leq t} X_s$. Prove that

$$\mathbf{Q}(\bar{X}_t \leq b) = \int_{-\infty}^{\infty} dx \int_{\infty}^{b} dy \left(e^{\mu x - \frac{1}{2}\mu^2 t} f_{\bar{W}_t, W_t}(y, x) \right),$$

where $f_{\bar{W}_t, W_t}(\cdot, \cdot)$ is derived in (b). [6 marks]

(d) Suppose that $\mathbf{Q}(\bar{X}_t \leq b)$ in (c) has the expression

$$\mathbf{Q}(\bar{X}_t \leq b) = \Phi\left(\frac{b - \mu t}{\sqrt{t}}\right) - e^{2b\mu}\Phi\left(\frac{-b - \mu t}{\sqrt{t}}\right), \tag{1}$$

where Φ is the standard normal cumulative distribution function. Based on the expression (1), prove that the default probability $\mathbf{Q}(\tau \leq t)$ under the first-passage-time model has the expression

$$\mathbf{Q}(\tau \leq t) = 1 - \Phi\left(\frac{\ln \frac{v}{Ke^{-dT}} + (r - \frac{1}{2}\sigma^2 - d)T}{\sigma\sqrt{T}}\right) + \left(\frac{v}{Ke^{-dT}}\right)^{-\frac{2}{\sigma^2}(r - \frac{1}{2}\sigma^2 - d)} \Phi\left(\frac{-\ln \frac{v}{Ke^{-dT}} + (r - \frac{1}{2}\sigma^2 - d)T}{\sigma\sqrt{T}}\right).$$

[6 marks]

Continued...

4: Let τ be a non-negative random variable defined on a complete probability space $(\Omega, \mathcal{F}, \mathbf{P})$, and $\mathbb{F} = \{\mathcal{F}_t\}_{t\geq 0}$ be the filtration given by $\mathcal{F}_t = \sigma(\{\tau \leq u\} : u \leq t)$.

(a) For any $A \in \mathcal{F}_t$, write down two possibilities of $A \cap \{\tau > t\}$. [2 marks]

(b) Let Y be an $\mathcal{F}_\infty$-measurable and bounded random variable, where $\mathcal{F}_\infty = \sigma\left(\cup_{t\geq 0}\mathcal{F}_t\right)$. Prove that

$$\mathbf{E}[\mathbf{1}_{\{\tau>t\}}Y|\mathcal{F}_t] = \mathbf{1}_{\{\tau>t\}}\frac{\mathbf{E}[\mathbf{1}_{\{\tau>t\}}Y]}{\mathbf{P}(\tau > t)}.$$

[3 marks]

(c) Prove that τ follows exponential distribution with a constant intensity $\lambda > 0$ if and only if the process $M = (M_t)_{t\geq 0}$, where

$$M_t = 1_{\{\tau\leq t\}} - \int_0^t 1_{\{\tau>s\}}\lambda ds,$$

is an $(\mathbb{F}, \mathbf{P})$-martingale and $\mathbf{P}(\tau > 0) = 1$. [5 marks]

(d) Let $T > 0$ and $\mu \in [0, 1]$ be fixed numbers. Under the assumption in part (c), prove that the process $Z^\mu = (Z^\mu_t)_{t\in[0,T]}$, where

$$Z^\mu_t = \left(1_{\{\tau>t\}} + (1-\mu)1_{\{\tau\leq t\}}\right) e^{\int_0^t \mu 1_{\{\tau>s\}}\lambda ds},$$

is an $(\mathbb{F}, \mathbf{P})$-martingale.

[Hint: Let $H_t := 1_{\{\tau\leq t\}}$ and $V_t := 1 - H_t + (1-\mu)H_t$. You may first prove that $\Delta V_s = -\mu V_{s-}\Delta H_s$.] [5 marks]

(e) Let Z^μ_T be given as in part (d) and let $\mathbf{Q}$ be the probability measure on $(\Omega, \mathcal{F}_T)$ with the Radon-Nikodym density $\left.\frac{d\mathbf{Q}}{d\mathbf{P}}\right|_{\mathcal{F}_T} = Z^\mu_T$. Prove that the process $M^\mu = (M^\mu_t)_{t\in[0,T]}$, where $M^\mu_t = 1_{\{\tau\leq t\}} - \int_0^t (1-\mu)1_{\{\tau>s\}}\lambda ds$, is an $(\mathbb{F}, \mathbf{Q})$-martingale.

[Hint: you may use without proof the fact that M^μ is an $(\mathbb{F}, \mathbf{Q})$-martingale if and only if $M^\mu Z^\mu$ is an $(\mathbb{F}, \mathbf{P})$-martingale.] [5 marks]

End.

ST9090

UNIVERSITY OF WARWICK

April 2022

Applications of Stochastic Calculus in Finance: Sample Solutions

Question 1

The question is about Vasicek model and is based on interest rate modeling section. Students have been taught the Vasicek model and its corresponding forward rate model. Parts (a) (b) (c) are taken from the lecture notes. Parts (d) and (e) appear in the exercises.

(a) Vasicek model

$$dr_t = (a - br_t)dt + \sigma dW_t^{\mathbf{Q}}$$

for $a, b, \sigma > 0$. [2]

(b) A short rate model is said to provide an ATS if the zero-coupon bond price $P(t,T)$ is of the form $P(t,T) = F(t,r_t) = e^{-A(t,T)-B(t,T)r_t}$ for some functions A and B. The Vasicek model provides an ATS because its drift $(a - br_t)$ and volatility square σ^2 are both affine functions of r_t. [3]

(c) The pricing function $P(t,T) = F(t,r_t)$ satisfies the PDE

$$\begin{cases} \partial_t F(t,r) + \frac{1}{2}\sigma^2 \partial_{rr} F(t,r) + (a - br)\partial_r F(t,r) - rF(t,r) = 0, \\ F(T,r) = 1. \end{cases}$$

Write $A(t) = A(t,T)$ and $B(t) = B(t,T)$. Since $F(t,r) = e^{-A(t)-B(t)r}$, we have, for any $r \in \mathbb{R}$,

$$-A'(t) - aB(t) + \frac{1}{2}\sigma^2 |B(t)|^2 + (-B'(t) + bB(t) - 1)r = 0.$$

This gives us an ODE for B

$$B'(t) = bB(t) - 1.$$

Moreover, A is obtained by

$$A'(t) = -aB(t) + \frac{1}{2}\sigma^2 |B(t)|^2.$$

Together with $B(T) = A(T) = 0$, we derive

$$\begin{cases} B(t) = -\frac{1}{b}(e^{-b(T-t)-1}) \\ A(t) = \int_t^T [aB(s) - \frac{1}{2}\sigma^2 B^2(s)]ds. \end{cases}$$

[5]

(d) Using $f(t,T) = -\partial_T \ln P(t,T)$ and $P(t,T) = e^{-A(t,T)-B(t,T)r_t}$, we derive

$$f(t,T) = \partial_T A(t,T) + \partial_T B(t,T) r_t.$$

From the expressions of A and B, we have

$$\begin{aligned}
\partial_T A(t,T) &= aB(T,T) - \frac{1}{2}\sigma^2 B^2(T,T) + \int_t^T [a\partial_T B(s,T) - \sigma^2 B(s,T)\partial_T B(s,T)]ds \\
&= \int_t^T [ae^{-b(T-s)} + \frac{\sigma^2}{b}(e^{-b(T-s)} - 1)e^{-b(T-s)}]ds \\
\partial_T B(t,T) &= e^{-b(T-t)}.
\end{aligned}$$

In turn,

$$\begin{aligned}
df(t,T) &= d(\partial_T A(t,T)) + r_t d(\partial_T B(t,T)) + \partial_T B(t,T)dr_t \\
&= [-ae^{-b(T-t)} - \frac{\sigma^2}{b}(e^{-2b(T-t)} - e^{-b(T-t)})]dt \\
&\quad + be^{-b(T-t)} r_t dt + e^{-b(T-t)}[(a - br_t)dt + \sigma dW_t^{\mathbf{Q}}] \\
&= \frac{\sigma^2}{b}(e^{-b(T-t)} - e^{-2b(T-t)})dt + \sigma e^{-b(T-t)} dW_t^{\mathbf{Q}}.
\end{aligned}$$

[5]

(e) Given the volatility $\sigma(t,T)$ of the forward rate $f(t,T)$, the HJB drift condition indicates that the drift of the forward rate must be of the form $\sigma(t,T)\int_t^T \sigma(t,s)ds$ under the EMM $\mathbf{Q}$. In the Vasicek model, we have

$$\begin{aligned}
\sigma(t,T) &= \sigma e^{-b(T-t)} \\
\sigma(t,T)\int_t^T \sigma(t,s)ds &= \frac{\sigma^2}{b}(e^{-b(T-t)} - e^{-2b(T-t)}).
\end{aligned}$$

They indeed satisfy the HJM drift condition. [5]

Question 2 The question is about change of numeraire and is based on interest rate modeling section. It is taken from the exercises.

(a) An ELMM $\mathbf{Q} \sim \mathbf{P}$ has the property that the discounted price process (denominated by the British bank account B^d): $\frac{S_t^d}{B^d(t)}$, $\frac{B^d(t)}{B^d(t)}$ and $\frac{B^f(t)E_t}{B^d(t)}$ are local martingales under $\mathbf{Q}$. [2]

(b) The stock price S^d, in units of the British bank account B^d, follows

$$
\begin{aligned}
d\frac{S_t^d}{B^d(t)} &= \frac{S_t^d}{B^d(t)}[(\mu^d - r^d)dt + \sigma^d dW_t^1] \\
&= \frac{S_t^d}{B^d(t)}\sigma^d(\Theta^1 dt + dW_t^1),
\end{aligned}
$$

with Θ^1 given by the first market price of risk equation

$$\sigma^d\Theta^1 = \mu^d - r^d.$$

On the other hand, the American bank account B^f, in units of the British bank account B^d has the price $\frac{B^f(t)E_t}{B^d(t)}$. Since

$$d\frac{B^f(t)}{B^d(t)} = \frac{B^f(t)}{B^d(t)}(r^f - r^d)dt,$$

it follows that

$$
\begin{aligned}
d(\frac{B^f(t)E_t}{B^d(t)}) &= E_t d(\frac{B^f(t)}{B^d(t)}) + (\frac{B^f(t)}{B^d(t)})dE_t \\
&= (\frac{B^f(t)E_t}{B^d(t)})[(r^f - r^d)dt + \mu^E dt + \sigma^E(\rho dW_t^1 + \sqrt{1-\rho^2}dW_t^2)] \\
&= (\frac{B^f(t)E_t}{B^d(t)})[\sigma^E\rho(\Theta^1 dt + dW_t^1) + \sigma^E\sqrt{1-\rho^2}(\Theta^2 dt + dW_t^2)],
\end{aligned}
$$

with Θ^2 given by the second market price of risk equation

$$\sigma^E\rho\Theta^1 + \sigma^E\sqrt{1-\rho^2}\Theta^2 = r^f - r^d + \mu^E.$$

Define the Radon-Nikodym density by

$$\left.\frac{d\mathbf{Q}}{d\mathbf{P}}\right|_{\mathcal{F}_t} := \mathcal{E}(-\Theta^1 W^1 - \Theta^2 W^2)_t.$$

By Girsanov's theorem, $W^{\mathbf{Q}} = (W^{\mathbf{Q},1}, W^{\mathbf{Q},2})$, where $W_t^{\mathbf{Q},i} := W_t^i + \Theta^i t$, $i = 1, 2$, is a two-dimensional BM under the ELMM $\mathbf{Q}$. Moreover,

$$
\begin{aligned}
d\frac{S_t^d}{B^d(t)} &= \frac{S_t^d}{B^d(t)}\sigma^d dW_t^{\mathbf{Q},1} \\
d(\frac{B^f(t)E_t}{B^d(t)}) &= (\frac{B^f(t)E_t}{B^d(t)})\sigma^E(\rho dW_t^{\mathbf{Q},1} + \sqrt{1-\rho^2}dW_t^{\mathbf{Q},2}).
\end{aligned}
$$

[10]

(c) Note that

$$\frac{S^d_t}{B^d(t)} = S^d_0 \mathcal{E}\left(\sigma^d W^{\mathbf{Q},1}\right)_t$$
$$\frac{B^f(t)E_t}{B^d(t)} = E_0 \mathcal{E}\left(\sigma^E(\rho W^{\mathbf{Q},1} + \sqrt{1-\rho^2} W^{\mathbf{Q},2})\right)_t.$$

Since

$$\mathbf{E}^{\mathbf{Q}}[e^{\frac{1}{2}\langle \sigma^d W^{\mathbf{Q},1}\rangle_t}] = e^{\frac{1}{2}|\sigma^d|^2 t} < \infty$$
$$\mathbf{E}^{\mathbf{Q}}[e^{\frac{1}{2}\langle \sigma^E(\rho dW^{\mathbf{Q},1}_t + \sqrt{1-\rho^2} dW^{\mathbf{Q},2}_t\rangle_t}] = e^{\frac{1}{2}|\sigma^E|^2 t} < \infty,$$

by the Novikov's condition, both $\frac{S^d_t}{B^d(t)}$ and $\frac{B^f(t)E_t}{B^d(t)}$ are martingales under $\mathbf{Q}$, so $\mathbf{Q}$ is an EMM. [3]

(d) The pricing equation is $\mathbf{E}^{\mathbf{Q}}[\frac{E_T - F}{B^d(T)}] = 0$. Since

$$dE_t = E_t[(r^d - r^f)dt + \sigma^E(\rho d_t W^{\mathbf{Q},1} + \sqrt{1-\rho^2} dW^{\mathbf{Q},2}_t)],$$

it follows that $E_t e^{(r^f - r^d)t}$ is a martingale under $\mathbf{Q}$. Hence,

$$F = \mathbf{E}^{\mathbf{Q}}[E_T] = e^{-(r^f-r^d)T}\mathbf{E}^{\mathbf{Q}}[E_T e^{(r^f-r^d)T}] = e^{(r^f-r^d)T}E_0.$$

[5]

Question 3 This is about the first-passage-time model and is based on the credit risk modeling section. It is based on the lecture notes. Students have seen a more general case. Parts (a) (b) (c) are lecture materials. Part (d) is new.

(a)
$$\frac{V_t}{D_t} = \frac{v}{Ke^{-dT}} e^{\sigma W_t + (r - \frac{1}{2}\sigma^2 - d)t}.$$
The default time $\tau = \inf\{t \geq 0 : V_t \leq D_t\} = \inf\{t \geq 0 : \frac{V_t}{D_t} \leq 1\}$. [2]

(b) The reflection principle

$$\mathbf{Q}(\bar{W}_t \geq b, W_t \leq a) = \mathbf{Q}(\bar{W}_t \geq b, W_t \geq 2b - a) = \mathbf{Q}(W_t \geq 2b - a)$$

for $b \geq \max(a, 0)$. Hence,

$$\begin{aligned}
\mathbf{Q}(\bar{W}_t \leq b, W_t \leq a) &= \mathbf{Q}(W_t \leq a) - \mathbf{Q}(\bar{W}_t \geq b, W_t \leq a) \\
&= \mathbf{Q}(W_t \leq a) - \mathbf{Q}(W_t \geq 2b - a) \\
&= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{a/\sqrt{t}} e^{-\frac{x^2}{2}} dx - \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{(a-2b)/\sqrt{t}} e^{-\frac{x^2}{2}} dx.
\end{aligned}$$

In turn,

$$\begin{aligned}
f_{\bar{W}_t, W_t}(b, a) &= \frac{\partial^2}{\partial b \partial a} \mathbf{Q}(\bar{W}_t \leq b, W_t \leq a) \\
&= \frac{2(2b - a)}{t\sqrt{2\pi t}} e^{-\frac{(2b-a)^2}{2t}}.
\end{aligned}$$

[6]

(c) Define a new probability measure $\mathbf{P}$ by the Radon-Nikodym density

$$\left.\frac{d\mathbf{P}}{d\mathbf{Q}}\right|_{\mathcal{F}_t} := \mathcal{E}(-\mu W)_t,$$

and
$$\mathbf{P}(A) = \mathbf{E}[\mathbf{1}_A e^{-\mu W_T - \frac{1}{2}\mu^2 t}], \quad A \in \mathcal{F}_T.$$
By Girsanov's theorem, $X_t = W_t + \mu t$, $t \in [0, T]$, is a BM under $\mathbf{P}$. Hence,

$$\begin{aligned}
\mathbf{Q}(\bar{X}_t \leq b) &= \mathbf{E}[\mathbf{1}_{\bar{X}_t \leq b}] \\
&= \mathbf{E}^{\mathbf{P}}[e^{\mu W_t + \frac{1}{2}\mu^2 t} \mathbf{1}_{\bar{X}_t \leq b}] \\
&= \mathbf{E}^{\mathbf{P}}[e^{\mu X_t - \frac{1}{2}\mu^2 t} \mathbf{1}_{\bar{X}_t \leq b}] \\
&= \int_{-\infty}^{\infty} dx \int_{\infty}^{b} dy \left(e^{\mu x - \frac{1}{2}\mu^2 t} f_{\bar{W}_t, W_t}(y, x) \right).
\end{aligned}$$

[6]

(d) We first calculate

$$
\begin{aligned}
\mathbf{Q}(\inf_{t\in[0,T]}\frac{V_t}{D_t}\le 1) &= \mathbf{Q}\left(\inf_{t\in[0,T]}\left(\frac{v}{Ke^{-dT}}e^{\sigma W_t+(r-\frac{1}{2}\sigma^2-d)t}\right)\le 1\right)\\
&= \mathbf{Q}\left(\sup_{t\in[0,T]}\left(-W_t-\frac{1}{\sigma}(r-\frac{1}{2}\sigma^2-d)t\right)\ge \frac{1}{\sigma}\log\frac{v}{Ke^{-dT}}\right)\\
&= 1-\mathbf{Q}\left(\sup_{t\in[0,T]}\left(-W_t\underbrace{-\frac{1}{\sigma}(r-\frac{1}{2}\sigma^2-d)}_{\mu}t\right)\le \underbrace{\frac{1}{\sigma}\ln\frac{v}{Ke^{-dT}}}_{b}\right).
\end{aligned}
$$

From expression (1), we further have

$$
\begin{aligned}
\mathbf{Q}(\tau\le t) &= \mathbf{Q}(\inf_{t\in[0,T]}\frac{V_t}{D_t}\le 1)\\
&= 1-\Phi\left(\frac{b-\mu T}{\sqrt{T}}\right)+e^{2b\mu}\Phi\left(\frac{-b-\mu T}{\sqrt{T}}\right)\\
&= 1-\Phi\left(\frac{\ln\frac{v}{Ke^{-dT}}+(r-\frac{1}{2}\sigma^2-d)T}{\sigma\sqrt{T}}\right)\\
&\quad +(\frac{v}{Ke^{-dT}})^{-\frac{2}{\sigma^2}(r-\frac{1}{2}\sigma^2-d)}\Phi\left(\frac{-\ln\frac{v}{Ke^{-dT}}+(r-\frac{1}{2}\sigma^2-d)T}{\sigma\sqrt{T}}\right).
\end{aligned}
$$

[6]

Question 4 This question is about reduced form model and is based on the credit risk modeling section. Parts (a), (b), (c) are the lecture materials. Parts (d) (e) are new.

(a) For any $A \in \mathcal{F}_t$, we have $A \cap \{\tau > t\} = \{\tau > t\}$ or $\emptyset$. [2]

(b) We show that $\mathbf{1}_{\{\tau>t\}}\mathbf{E}[\mathbf{1}_{\{\tau>t\}}Y]$ is the conditional expectation of $\mathbf{1}_{\{\tau>t\}}Y\mathbb{P}(\tau > t)$ w.r.t $\mathcal{F}_t$. Let $A \in \mathcal{F}_t$.

If $A \cap \{\tau > t\} = \emptyset$, obviously,

$$\mathbf{E}\left[\mathbf{1}_A\mathbf{1}_{\{\tau>t\}}\mathbf{E}[\mathbf{1}_{\{\tau>t\}}Y]\right] = \mathbf{E}\left[\mathbf{1}_A\mathbf{1}_{\{\tau>t\}}Y\mathbf{P}(\tau > t)\right] = 0.$$

If $A \cap \{\tau > t\} = \{\tau > t\}$, then,

$$\mathbf{E}\left[\mathbf{1}_A\mathbf{1}_{\{\tau>t\}}\mathbf{E}[\mathbf{1}_{\{\tau>t\}}Y]\right] = \mathbf{P}(\tau > t)\mathbf{E}[\mathbf{1}_{\{\tau>t\}}Y],$$

and

$$\mathbf{E}\left[\mathbf{1}_A\mathbf{1}_{\{\tau>t\}}Y\mathbf{P}(\tau > t)\right] = \mathbf{E}[\mathbf{1}_{\{\tau>t\}}Y]\mathbf{P}(\tau > t),$$

from which we conclude. [3]

(c) Only if part: For any $T \geq t \geq 0$, we have

$$\begin{aligned}\mathbf{E}[M_T|\mathcal{F}_t] =& 1 - \mathbf{E}[1_{\{\tau>T\}}|\mathcal{F}_t] - \int_0^t \mathbf{1}_{\{\tau>s\}}\lambda ds - \int_t^T \mathbf{E}[\mathbf{1}_{\{\tau>s\}}\lambda|\mathcal{F}_t]ds \\ =& 1 - \mathbf{1}_{\{\tau>t\}}e^{-\lambda(T-t)} - \int_0^t \mathbf{1}_{\{\tau>s\}}\lambda ds - \mathbf{1}_{\{\tau>t\}}\int_t^T \lambda e^{-\lambda(s-t)}ds \\ =& 1_{\{\tau\leq t\}} - \int_0^t \mathbf{1}_{\{\tau>s\}}\lambda ds = M_t.\end{aligned}$$

Since τ follows exponential distribution, it follows that $\mathbf{P}(\tau > 0) = e^{-\lambda 0} = 1$.

If part: For $t \geq 0$, define $\Phi(t) = \mathbf{P}(\tau > t)$. Then, the martingale property of M yields that

$$\begin{aligned}\Phi(t) =& \mathbf{E}\left[1 - M_t - \int_0^t \mathbf{1}_{\{\tau>s\}}\lambda ds\right] \\ =& 1 - M_0 - \lambda\int_0^t \Phi(s)ds.\end{aligned}$$

The condition $\tau > 0$ a.s. further yields that $M_0 = 0$, a.s. Thus,

$$\Phi(t) = 1 - \lambda\int_0^t \Phi(s)ds,$$

which implies that $\Phi(t) = e^{-\lambda t}$, i.e. τ follows exponential distribution with intensity λ. [5]

(d) Let $H_t := 1_{\{\tau\le t\}}$, $V_t := 1 - H_t + (1-\mu)H_t$ and $C_t := e^{\int_0^t \mu 1_{\{\tau>s\}}\lambda ds}$. Note that for each ω, both $V_t(\omega)$ and $C_t(\omega)$ are BV functions. Using integration by parts formula, we obtain

$$\begin{aligned} V_tC_t &= V_0C_0 + \int_0^t V_s dC_s + \int_0^t C_{s-}dV_s \\ &= 1 + \int_0^t V_sC_s\mu 1_{\{\tau>s\}}\lambda ds + \sum_{0<s\le t} C_{s-}\Delta V_s. \end{aligned}$$

Note that

$$\Delta V_s = V_s - V_{s-} = -\Delta H_s + (1-\mu)\Delta H_s = -\mu\Delta H_s.$$

Moreover, since $H_{s-}\Delta H_s = 0$, we have

$$V_{s-}\Delta H_s = (1 - H_{s-} + (1-\mu)H_{s-})\Delta H_s = \Delta H_s.$$

In turn,

$$\Delta V_s = -\mu V_{s-}\Delta H_s.$$

Therefore,

$$\begin{aligned} V_tC_t &= 1 + \int_0^t \mu V_sC_s 1_{\{\tau>s\}}\lambda ds - \sum_{0<s\le t}\mu V_{s-}C_{s-}\Delta H_s \\ &= 1 + \int_0^t \mu V_{s-}C_{s-}1_{\{\tau>s\}}\lambda ds - \int_0^t \mu V_{s-}C_{s-}dH_s \\ &= 1 - \int_0^t \mu V_{s-}C_{s-}(dH_s - 1_{\{\tau>s\}}\lambda ds), \end{aligned}$$

from which we deduce that $Z = VC$ is a local martingale. Since Z is bounded, it is a martingale. [5]

(e) It suffices to show that $M^\mu Z^\mu$ is a martingale under $\mathbf{P}$. Using the integration by parts formula, we obtain

$$M_t^\mu Z_t^\mu = \int_0^t M_{s-}^\mu dZ_s^\mu + \int_0^t Z_{s-}^\mu dM_s^\mu + \sum_{0<s\le t}\Delta M_s^\mu \Delta Z_s^\mu. \tag{1}$$

Note that M^μ has the decomposition $M_s^\mu = M_s + \int_0^t \mu\mathbf{1}_{\{\tau>s\}}\lambda ds$, so

$$\int_0^t Z_{s-}^\mu dM_s^\mu = \int_0^t Z_{s-}^\mu dM_s + \int_0^t Z_{s-}^\mu \mu\mathbf{1}_{\{\tau>s\}}\lambda ds. \tag{2}$$

On the other hand, from part (d), we have $\Delta Z_s^\mu = -\mu Z_{s-}^\mu \Delta M_s$, and therefore,

$$\sum_{0<s\le t}\Delta M_s^\mu\Delta Z_s^\mu = -\sum_{0<s\le t} Z_{s-}^\mu \mu|\Delta M_s|^2.$$

But $\Delta M_s = \Delta H_s$ and $|\Delta H_s|^2 = \Delta H_s$, it follows that

$$\sum_{0<s\le t}\Delta M_s^\mu \Delta Z_s^\mu = -\sum_{0<s\le t} Z_{s-}^\mu \mu\Delta H_s = -\int_0^t Z_{s-}^\mu \mu dH_s. \tag{3}$$

Plugging (2) and (3) into (1), we get

$$M_t^\mu Z_t^\mu = \int_0^t M_{s-}^\mu dZ_s^\mu + \int_0^t Z_{s-}^\mu dM_s - \int_0^t Z_{s-}^\mu \mu dM_s,$$

from which we deduce that $M^\mu Z^\mu$ is a local martingale. Since $M^\mu Z^\mu$ is bounded, it is also a martingale. [5]