Week 3 (2) Chapter 0.7, Chapter 2.1

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Chapter 0.7 Feynman-Kac formula

Prop. If $dZ_t = \mu(t, Z_t)dt + \sigma(t, Z_t)dW_t$ where μ, σ deterministic,

then, Z admits Markov property. i.e. $E[f(Z_t) | \mathcal{F}_s] = E[f(Z_t) | \underset{\uparrow \sigma(Z_s)}{Z_s}]$

Proof: Consider $\mu \equiv 0$, $\sigma \equiv 1$, verify $E[f(W_t) | \mathcal{F}_s] = E[f(W_t) | \underset{\uparrow \sigma(W_s) \subset \mathcal{F}_s}{W_s}]$, $\mathcal{F}_s = \sigma(W_r, 0 \le r \le s)$

We claim that $E[f(W_t) | \mathcal{F}_s] = g(W_s)$ $(\star)$

where $g(x) = E[f(W_t - W_s + x)]$

(Note that $g(W_s) \neq E[f(W_t - W_s + W_s)] = E[f(W_t)]$)

Then, $E[f(W_t) | W_s] = E[E[f(W_t) | \mathcal{F}_s] | W_s]$

$= E[g(W_s) | W_s] = g(W_s)$

Hence, $E[f(W_t) | \mathcal{F}_s] = E[f(W_t) | W_s]$

To prove $(\star)$, it is sufficient to consider $f(x) = e^{i\mu x}$, $\mu \in \mathbb{R}$.

Note that $e^{i\mu W_t} = e^{i\mu(W_t - W_s)} \cdot e^{i\mu W_s}$.

We aim to show $E[e^{i\mu(W_t - W_s)} \cdot e^{i\mu W_s} | \mathcal{F}_s] = g(W_s)$, i.e.

$$\underbrace{E[e^{i\mu(W_t - W_s)} \cdot e^{i\mu W_s} \mathbb{1}_A]}_{LHS} = \underbrace{E[g(W_s) \mathbb{1}_A]}_{RHS} \quad \forall A \in \mathcal{F}_s.$$

$LHS = E[E[e^{i\mu(W_t - W_s)} | \mathcal{F}_s] \cdot e^{i\mu W_s} \cdot \mathbb{1}_A]$

$= E[e^{i\mu(W_t - W_s)}] \cdot e^{i\mu W_s} \cdot \mathbb{1}_A]$

Recall $g(x) = E[e^{i\mu(W_t - W_s)} \cdot e^{i\mu x}]$

$RHS = E[g(W_s) \cdot \mathbb{1}_A] = LHS$

W_t is $\sigma(W_s, W_r - W_s, r \in [s,t])$ - m/b

For general case.

Z_t is $\sigma(Z_s, W_r - W_s, r \in [s,t])$ - m/b #

Theorem If $dZ_t = \mu(t, Z_t)dt + \sigma(t, Z_t)dW_t$,

then, $E[e^{-\int_t^T q(Z_s)ds} f(Z_T) | \mathcal{F}_t] = E[e^{-\int_t^T q(Z_s)ds} f(Z_T) | Z_t] = F(t, Z_t)$ (Markov)

If $F \in C^{1,2}([0,T] \times \mathbb{R})$, then F solves PDE:

$$
\boxed{\begin{cases} \partial_t F(t,x) + \mathcal{L} F(t,x) - q(x) F(t,x) = 0, \\ F(T,x) = f(x) \qquad (t,x) \in [0,T] \times R \end{cases}}
$$

(Feynman-Kac).

Proof: By Markov. $E\left[e^{-\int_0^T q(z_s)ds} f(z_T) \middle| \mathcal{F}_t\right] = e^{-\int_0^t q(z_s)ds} F(t,z_t) \quad \forall t \in [0,T]$.

(Note that $z_t = E[\xi \mid \mathcal{F}_t]$, $t \geq 0$, is a martingale. $E[z_t \mid \mathcal{F}_s] = z_s$, $t \geq s \geq 0$)

Apply Itô's formula to $e^{-\int_0^t q(z_s)ds} F(t,z_t)$.

$$
d e^{-\int_0^t q(z_s)ds} F(t,z_t) = -q(z_t) e^{-\int_0^t q(z_s)ds} F(t,z_t)\,dt + e^{-\int_0^t q(z_s)ds}\left[(\partial_t F + \mathcal{L} F)(t,z_t)\,dt + \sigma(z_t)\partial_x F(t,z_t)\,dW_t\right]
$$

$$
d e^{-\int_0^t q(z_s)ds} F(t,z_t) = e^{-\int_0^t q(z_s)ds} \underbrace{\left(\partial_t F(t,z_t) + \mathcal{L} F(t,z_t) - q(z_t) F(t,z_t)\right)}_{0 \text{ by mart property of } e^{-\int_0^t q(z_s)ds} F(t,z_t),\ t \in [0,T]} dt + \underbrace{e^{-\int_0^t q(z_s)ds} \sigma(z_t)\partial_x F(t,z_t)\,dW_t}_{\text{martingale}}.
$$

Moreover, $F(T,z_T) = E\left[e^{-\int_t^T q(z_s)ds} f(z_T) \middle| \mathcal{F}_T\right] = f(z_T)$ #

Chapter 2

Applications of Stochastic Calculus in Finance
Chapter 2: Short-rate models

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1 Arbitrage-free family of zero-coupon bond prices

Fix a filtered probability space $(\Omega, \mathscr{F}, \{\mathscr{F}_t\}_{t\geq 0}, \mathbf{P})$, which satisfies the usual conditions, and supports a one-dimensional Brownian motion W.

In short-rate models, we mainly model the dynamics of short rates. However, as we shall see later, the market under such models would never be complete.

Assumption 1 *(1) The short rate follows SDE*

$$dr_t = b_t dt + \sigma_t dW_t$$

which determines the bank account $B_t = e^{\int_0^t r_s ds}$. Moreover, the drift b and the volatility σ are both progressively measurable processes such that $\int_0^{\cdot}(b_s ds + \sigma_s dW_s)$ is a semimartingale.

(2) (No arbitrage): There exists an EMM $\mathbf{Q}$ whose Radon-Nikodym density of the form

$$\frac{d\mathbf{Q}}{d\mathbf{P}} = \mathscr{E}(-\int_0^{\cdot} \theta_s dW_s)$$

model.

such that the discounted zero-coupon bond price process $P(t,T)/B_t$ for $t \in [0,T]$ is a martingale under $\mathbf{Q}$, and $P(T,T) = 1$.

We make some comments on the above *no arbitrage* assumption. In short-rate models, the only tradeable asset is the bank account. Zero-coupon bonds or more general contingent claims are treated as derivatives as in the Black-Scholes theory, and the short rate (or the corresponding bank account) plays the role of underlying asset. Hence it is not possible to form portfolios which can replicate interesting contingent claims, not even zero-coupon bonds. Such a market is not complete, i.e. ELMM $\mathbf{Q}$ is not unique.

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$\frac{P(t,T)}{B_t}$ is mart

$\Rightarrow E^{Q}\left[\frac{P(T,T)}{B_T} \middle| \mathcal{F}_t\right] = \frac{P(t,T)}{B_t}$

$\Rightarrow P(t,T) = E^{Q}\left[\frac{B_t}{B_T} \cdot 1 \middle| \mathcal{F}_t\right] = E^{Q}\left[e^{-\int_t^T r_s ds} \cdot 1 \middle| \mathcal{F}_t\right]$

BS markets:
underlying: B_t. S^i, $1 \leq i \leq n$.
options written on S^i

Short-rate model
underlying: B_t.
zero-coupon bond written on B

HJM / LIBOR model
underlying: B_t. $P(t,T)$, $T > 0$.
Interest rate cap/floor,
Swaptions written on B, $P(\cdot,T)$

Notwithstanding the non-uniqueness of ELMM, if the *no arbitrage* assumption in Assumption 1 holds, $\mathbf{Q}$ is not only an ELMM, but also an EMM. Hence, we have

$$P(t,T) = \mathbf{E}^{\mathbf{Q}}[e^{-\int_t^T r_s ds} \times 1 | \mathscr{F}_t]$$

which coincides the risk-neutral pricing formula.

Definition 1. A family $P(t,T)$ for $0 \le t \le T < \infty$ of adapted processes is called an arbitrage-free family of zero-coupon bonds if the *no arbitrage* assumption in Assumption 1 holds.

Proposition 1. *Under Assumption 1, the short rate $r = (r_t)_{t \ge 0}$ follows SDE*

$$dr_t = (b_t - \sigma_t \Theta_t)dt + \sigma_t dW_t^{\mathbf{Q}}$$

$dr_t = b_t dt + \sigma_t dW_t$ - under $\mathbb{P}$.

$\left.\frac{dQ}{dP}\right|_{\mathscr{F}_t} = \mathcal{E}(-\int_0^{\cdot} \Theta_s dW_s)_t$

under the EMM $\mathbf{Q}$. Moreover, if the filtration $\{\mathscr{F}_t\}_{t \ge 0}$ is the Brownian filtration, then there exists a process $h \in \mathscr{L}^2(\mathbb{R})$ such that

$$\frac{dP(t,T)}{P(t,T)} = r_t dt + h_t dW_t^{\mathbf{Q}}$$
$$= (r_t + h_t \Theta_t)dt + h_t dW_t.$$

by Girsanov

$dW^Q = dW + \Theta dt$

$\Rightarrow W_t^Q = W_t + \int_0^t \Theta_s ds$ is BM under Q.

Proof. We only show that

$$\frac{dP(t,T)}{P(t,T)} = r_t dt + h_t dW_t^{\mathbf{Q}}.$$

The other two equations follow from the Girsanov's theorem.

Indeed, since $P(t,T)e^{-\int_0^t r_s ds}$ is a $\mathbf{Q}$-martingale, by the martingale representation, there exists a process $\tilde{h} \in \mathscr{L}^2(\mathbb{R})$ such that

$$P(t,T)e^{-\int_0^t r_s ds} = P(0,T) + \int_0^t \tilde{h}_s dW_s^{\mathbf{Q}}.$$

Since $\underbrace{P(t,T)e^{-\int_0^t r_s ds}}_{\text{martingale.}} = E^Q[P(T,T)e^{-\int_0^T r_s ds} | \mathscr{F}_t]$

Itô's formula yields

$$\frac{dP(t,T)}{P(t,T)} = r_t dt + \frac{\tilde{h}_t B_t}{P(t,T)} dW_t^{\mathbf{Q}}.$$

We conclude by letting $h_t = \tilde{h}_t B_t / P(t,T)$. $\square$

Apply Itô to $P(t,T)e^{-\int_0^t r_s ds}$:

$d\,P(t,T)e^{-\int_0^t r_s ds} = e^{-\int_0^t r_s ds} dP(t,T)$

$- r_t e^{-\int_0^t r_s ds} P(t,T)dt$

Apply mart. representation to $P(t,T)e^{-\int_0^t r_s ds}$:

$d\,P(t,T)e^{-\int_0^t r_s ds} = \tilde{h}_t dW_t^Q$ for some $\tilde{h}_t \in L^2$

$e^{-\int_0^t r_s ds} dP(t,T) = r_t e^{-\int_0^t r_s ds} P(t,T)dt + \tilde{h}_t dW_t^Q.$

$\Rightarrow \frac{dP(t,T)}{P(t,T)} = r_t dt + \boxed{\frac{\tilde{h}_t}{e^{-\int_0^t r_s ds} P(t,T)}} dW_t^Q.$

$\| $

h_t

The summary of short-rate models: the dynamics of the short rate r are

$$dr_t = b_t dt + \sigma_t dW_t \quad \text{under } \mathbf{P}$$
$$= (b_t - \sigma_t \Theta_t)dt + \sigma_t dW_t^{\mathbf{Q}} \quad \text{under } \mathbf{Q}.$$

The dynamics of the zero-coupon bond price $P(t,T)$ are

$$\frac{dP(t,T)}{P(t,T)} = r_t dt + h_t dW_t^{\mathbf{Q}} \quad \text{under } \mathbf{Q}$$
$$= (r_t + h_t \Theta_t)dt + h_t dW_t \quad \text{under } \mathbf{P}.$$

A short-rate model is not fully determined without the exogenous specification of the market price of risk Θ. Hence, it is custom to postulate the $\mathbf{Q}$-dynamics of the short rate r directly in the context of derivative pricing.

2 Affine term structure of short-rate models

In the rest of this chapter, suppose that the short rate r follows

$$dr_t = b(t, r_t)dt + \sigma(t, r_t)dW_t^{\mathbf{Q}},$$

where $b(\cdot,\cdot)$ and $\sigma(\cdot,\cdot)$ are deterministic functions, and the initial data r_0 is in an open set $\mathscr{O} \subset \mathbb{R}$. Typical choices of $\mathscr{O}$ are $\mathbb{R}$ and $(0,\infty)$.

By the Markov property:

$$P(t,T) = \mathbf{E}^{\mathbf{Q}}[e^{-\int_t^T r_s ds} | \mathscr{F}_t] = \mathbf{E}^{\mathbf{Q}}[e^{-\int_t^T r_s ds} | r_t] = F(t, r_t)$$

for some function $F(\cdot,\cdot)$.

If $F(t,r) \in C^{1,2}([0,T) \times \mathscr{O})$, then by the Feynman-Kac formula, $F(t,r)$ solves the following term-structure equation on $[0,T) \times \mathscr{O}$:

$$\begin{cases} \partial_t F(t,r) + \frac{1}{2}\sigma^2(t,r)\partial_{rr}F(t,r) + b(t,t)\partial_r F(t,r) - rF(t,r) = 0, \\ F(T,r) = 1. \end{cases}$$

For the case with the state space $(0,\infty)$, a parameter condition on the coefficients need to be imposed to guarantee the above term-structure PDE is well posed even without a boundary condition on $r = 0$.

Definition 2. (Affine term structure)

A short-rate model is said to provide an affine term structure (ATS) if the corresponding zero-coupon price $P(t,T) = F(t,r)$ is of the form

$$F(t,r) = e^{-A(t) - B(t)r}$$