MA930 – Homework 1

Please submit answers to the questions below, which are mostly exercises taken from the lectures (with some minor amendments). Answers should be submitted online via the module webpage.

Please submit a single document. Around half of the questions require written solutions only. For questions 4ii, 6, 7 and 8i, also include in your submission any relevant computing code (where used) and a copy of the computational output (all in the same document). Marks will be assigned for clarity of the written solutions/code and presentation of the results.

1. Show that the conditionality product rule extends to three variables, so that
$$\boldsymbol{P}(\mathrm{A} \cap B \cap \mathrm{C}) = \boldsymbol{P}(\mathrm{A} \mid B \cap \mathrm{C})\boldsymbol{P}(B|C)\boldsymbol{P}(C)$$
What is the conditionality product rule for *n>3* variables? Prove it.

2. If *X* is a discrete random variable with PGF $G(z) = \frac{z}{5}(2 + 3z^2)$, then what is the distribution of *X*?

3. Use characteristic functions to derive the distributions of a sum of independent:
   i) Normally distributed random numbers?
   ii) Gamma distributed random numbers that both have the same rate parameter?

4. Neuron *A* makes *n* synaptic contacts onto neuron *B.* When neuron *A* fires, a vesicle containing neurotransmitter is released at each contact with probability *p* per contact. Each vesicle release contributes to a voltage increase in neuron *B* that is normally distributed with mean a and variance σ^2. The total increase in voltage is the sum of the effects from each contact plus some Gaussian background noise with zero mean and variance s^2. Hence, the voltage change following one such event is
$V = \phi(0, s^2) + \sum_{k=1}^{n} \delta_k\, \phi(a, \sigma^2)$, where δ_k is a Bernoulli random number with probability *p* and $\phi(\mu, \sigma^2)$ is a normal random number with mean μ and variance σ^2.

   i. Show that the distribution for *V* is given by $\boldsymbol{P}(V) = \sum_{k=0}^{n} \boldsymbol{P}_1(V|K = k)\boldsymbol{P}_2(K = k)$, where $\boldsymbol{P}_1$ and $\boldsymbol{P}_2$ are known distributions to be found and *K* is a random variable that you should define.
   ii. Using the values n = 10, p = 0.3, $a = 0.2$, $\sigma = 0.01$, s = 0.05, write code to plot an empirical distribution of 1,000,000 random simulated voltages. Compare them to the theoretical distribution. Submit code and the plotted graphs of both the empirical and theoretical distributions.

5. Find the number of non-negative integer solutions of the equation
$$x_1 + x_2 + \cdots + x_k = n$$

6. Draw $n = 30$ normally distributed random numbers with mean $\mu = 5$ and standard deviation $\sigma = 1$, and calculate the sample mean and unbiased sample variance. Repeat this 1000 times, and plot density histograms of the sample means and unbiased sample variances. Plot relevant theoretical distributions on top of these density histograms (and explain where these theoretical distributions come from).

7. In a test for intelligence, 20 MathSys students achieved the following scores:
{65, 73, 51, 67, 48, 80, 69, 83, 89, 62, 71, 67, 64, 78, 85, 49, 80, 60, 51, 70}.
Students from another course achieved the following scores:
{63, 72, 47, 63, 44, 78, 67, 52, 54, 58, 68, 65, 63, 77, 62, 46, 78, 56, 49, 65}.
Consider assessing the null hypothesis that students from both courses are equally intelligent using the two-sample (Z) test shown in lectures. For which significance levels would the null hypothesis be rejected? Is the two-sample Z test an appropriate test to use here? (If not, then which test might be more appropriate? Why?)

8.(i) For a coin with a bias of $p = 0.7$ towards heads, write code to generate 100 coin flips. Now imagine that the true value of p is unknown. For the exact sequence of coin flips observed, write down the likelihood function. Calculate posteriors corresponding to two priors: one uniform and the other proportional to $p(1 - p^4)$. Plot the priors and posteriors on the same axes.
(ii) Explain how, in general, given a unimodal posterior distribution for a model parameter α, the 90% highest density credible interval can be approximated numerically.

9. Suppose that we observe *n* independent draws from an exponential distribution with rate parameter λ (the value of which is unknown), and that we wish to estimate the value of λ. Show that, if we use a gamma distributed prior with shape parameter α and rate parameter β, then the posterior distribution for λ is also a gamma distribution. State the shape and rate parameters of the posterior distribution for λ.