UNIVERSITY OF WARWICK

April 2022

Advanced Trading Strategies

Time Allowed: 2 HOURS

Answer **Three** questions from Section 1 and **One** question from Section 2.

Full marks may be obtained by correctly answering three complete questions from Section 1 and one complete question from Section 2. Candidates may attempt all questions. Marks will be awarded for the best answer from Section 2 only.

Read carefully the instructions on the answer book and make sure that the particulars required are entered on each answer book. Please answer each section in a separate answer book.

Calculators are NOT PERMITTED.

PDAs, mobile phones and any other hand-held devices that facilitate wireless communication are NOT PERMITTED.

This is a closed book examination.
Please turn over

Question 1

In their paper, "Liquidity, Information and Infrequently Traded Stocks", Easley et al. (1996) estimate the "PIN" measure as

$$\text{PIN} = \frac{\alpha\mu}{\alpha\mu + 2\varepsilon},$$

where α is the probability of the information event, μ is the arrival rate of informed traders and ε is the arrival rate of uninformed traders.

A. Which parameter from the Glosten and Milgrom (1985) model does the PIN measure correspond to? [1]

B. Briefly comment on (at least) two assumptions that Easley et al. (1996) make to estimate the PIN measure. [2]

C. Easley et al. (1996) estimate both the arrival rate of informed traders (μ) and that of uninformed traders (ε) to be higher for more actively traded stocks. However, the PIN measure is estimated to be lower for more actively traded stocks. How can you explain this finding? [2]

D. Do you expect stocks with larger PINs to have higher or lower bid-ask spreads? Briefly explain. [2]

E. Briefly comment on (at least) three limitations of the PIN measure. [3]

Question 2 starts on next page

A one-period, discrete-time financial market consists of two assets, a riskless bond with price process $S^0 = (S_t^0)_{t\in\{0,1\}}$ and a risky asset with price process $S^1 = (S_t^1)_{t\in\{0,1\}}$. The bond value grows in a deterministic, riskless fashion: $S_1^0 = (1+r)S_0^0$. The value of the risky asset changes stochastically: $S_1^1 = ZS_0^1$. Here, S_0^0 and S_0^1 are known (and strictly positive) at time 0 and Z is a random variable, whose value is revealed at time 1, taking values in $[a, b] \subset [0, \infty)$, such that both $\mathbb{P}(Z < 1+r) > 0)$ and $\mathbb{P}(Z > 1+r) > 0$.

The decision variables for the agent are to choose c the consumption at time zero, and θ the number of shares of the risky asset held over $[0, 1)$.

A. The wealth of the agent at time 0 is x (a fixed and given quantity). If the agent follows strategy (c, θ) explain why the wealth x_1 at time 1 is given by

$$x_1 = (x - c - \theta S_0^1)(1+r) + \theta S_0^1 Z.$$

[1]

B. The objective of the agent is to maximise the expected logarithmic utility of consumption

$$\mathbb{E}\left[A_0 \log c + A_1 \log x_1\right]$$

for strictly positive constants A_0 and A_1. Find the optimal c and a representation for the optimal θ. [3]

C. Show that the optimal θ is positive if and only if $\mathbb{E}[Z] > (1+r)$. [2]

D. Give an expression for the time-0 value function and show that it is not simply given by $V(x) = A\log x$ for a constant A. [2]

E. Explain how this one period result is useful in determining the solution to the N-period problem. [2]

Question 3 starts on next page

Consider a market $(S_t^0, S_t)_{t\in[0,T]}$, where the riskless asset S^0 is constant 1 and the fundamental price of the risky asset S satisfies $S_t = S_0 + \sigma W_t$, where $S_0 \in \mathbb{R}$, $\sigma > 0$ and $W = (W_t)_{t\in[0,T]}$ is a Brownian motion. Suppose an investor holds $Q_0 > 0$ number of shares in the stock at time 0 and has to liquidate them by time $T > 0$ by choosing an trading rate $\alpha = (\alpha_t)_{t\in[0,T]} \in \mathcal{A}$, where

$$\mathcal{A} := \left\{ (\alpha_t)_{t\in[0,T]} \text{ adapted} : Q_0^\alpha = Q_0,\ \ Q_T^\alpha = 0,\ \ E\left[\int_0^T \alpha_s^2 \,\mathrm{d}s\right] < \infty,\ E\left[\int_0^t (Q_s^\alpha)^2 \,\mathrm{d}s\right] < \infty \right\},$$

with $Q_t^\alpha := Q_0 + \int_0^t \alpha_u \,\mathrm{d}u$. We assume that the investor faces both temporary and permanent linear price impact, i.e., when choosing $\alpha \in \mathcal{A}$, the dynamics of the mid-price $S^\alpha = (S_t^\alpha)_{t\in[0,T]}$ are given by

$$\mathrm{d}S_t^\alpha = \mathrm{d}S_t + \beta\alpha_t \,\mathrm{d}t, \quad S_0^\alpha = S_0,$$

for some $\beta > 0$, and the corresponding execution price $\hat{S}^\alpha = (\hat{S}_t^\alpha)_{t\in[0,T]}$ is given by

$$\hat{S}_t^\alpha = S_t^\alpha + \lambda\alpha_t,$$

for some $\lambda > 0$. We assume that the investor holds initially no cash, and hence the dynamics of the cash process $X^\alpha = (X_t^\alpha)_{t\in[0,T]}$ when using the trading rate $\alpha \in \mathcal{A}$ is given by

$$\mathrm{d}X_t^\alpha = -\hat{S}_t^\alpha \,\mathrm{d}Q_t^\alpha, \quad X_0^\alpha = 0.$$

We assume that the goal functional $J : \mathcal{A} \to \mathbb{R}$ of the investor is given by

$$J(\alpha) := E\left[X_T^\alpha\right].$$

The investor seeks to maximise $J(\alpha)$ over all $\alpha \in \mathcal{A}$.

A. Define the value function $v(t, x, s, q)$ corresponding to the conditional version of the investor's control problem. **[2]**

   Hint: Do not forget to define the restarted set of admissible trading rates $\mathcal{A}_{(t,x,s,q)}$ and the restarted (controlled) processes $(S_u^{t,x,s,q,\alpha})_{u\in[t,T]}$, $(\hat{S}_u^{t,x,s,q,\alpha})_{u\in[t,T]}$, $(X_u^{t,x,s,q,\alpha})_{u\in[t,T]}$, $(Q_u^{t,x,s,q,\alpha})_{u\in[t,T]}$ for $t \in [0,T)$, $x, s, q \in \mathbb{R}$.

B. State (but do not derive) the HJB equation corresponding to the investor's control problem and give a brief motivation for the non-standard terminal condition. **[3]**

C. Show that for each $\alpha \in \mathcal{A}$,

$$E\left[X_T^\alpha\right] = Q_0 S_0 - \frac{\beta}{2} Q_0^2 - \lambda E\left[\int_0^T (\alpha_t)^2 \,\mathrm{d}t\right]. \qquad \textbf{[3]}$$

   Hint: Use an integration by parts, the fact that $\int_0^t Q_u^\alpha \,\mathrm{d}Q_u^\alpha = \frac{1}{2}((Q_t^\alpha)^2 - (Q_0^\alpha)^2)$ for $t \in [0,T]$ and the definition of $\mathcal{A}$.

D. Using C., show by a direct argument that $\alpha^* = (\alpha_t^*)_{t\in[0,T]}$ defined by $\alpha_t^* := \frac{Q_0}{T}$ is the optimal trading rate. **[2]**

Section 2 starts on next page

Question 4

Consider the static model by Kyle (1985), where (i) market makers are risk neutral and perfectly competitive, (ii) the asset value is $v \sim N(\mu, \sigma_v^2)$, (iii) the informed investor's order is $x = \beta(v - \mu)$, and the noise traders' order is $u \sim N(0, \sigma_u^2)$; (iv) market makers only observe the total net order $q = x + u$ and post the prices according to the following equation: $p = \mu + \lambda q$. Suppose the informed investor is risk averse (with constant coefficient of absolute risk aversion b) and that he liquidates any amount of the security that he buys at a liquidation value $v + \varepsilon$, where $\varepsilon \sim N(0, \sigma_\varepsilon^2)$. This noise in his signal implies that in taking long or short positions based on his privileged information, he bears some risk. At the time of trading the informed investor knows the realization of v, and he is only uncertain about ε as well as the impact λu of the noise traders on the market price.

Suppose he has mean-variance preferences, such that his utility (i.e. objective function) is increasing in the expected value of his final wealth, $E[W]$, and decreasing in its riskiness, which we measure by its variance, $var[W]$:

$$E(U) = E[W] - \tfrac{b}{2} var[W].$$

A. What is the expected utility of the informed trader as a function of his order size, x? [2]

B. Derive the optimal order size, x, of the informed investor(treat the market illiquidity parameter λ as given). [5]

C. What is the equilibrium trading aggressiveness β of the risk averse informed trader? Is it higher or lower, as compared to the standard Kyle (1985) model with the risk neutral informed trader? [3]

D. In addition to λ, which other parameters impact β of the risk averse informed trader? Is β increasing/decreasing in these parameters and why? [6]

E. Does the equilibrium market depth increase or decrease in the setup with the risk averse informed investor? Show graphically how the equilibrium market depth changes in this setup, by drawing the two relationships between informed trader's aggressiveness and market depth. [4]

Question 5 starts on next page

Question 5

Fix $z > 0$ and $\lambda \in \mathbb{R} \setminus \{0\}$. Consider the controlled process X^U with dynamics

$$X_t^U = x + B_t + \lambda \int_0^t U_s ds \tag{1}$$

where $B = (B_t)_{t \geq 0}$ is Brownian motion and $x < |z|$.

Let $\tau^U = \tau^{X^U} = \inf\{s : |X_s^U| = z\}$. Consider the problem of *minimising*

$$\mathbb{E}\left[\int_0^{\tau^U} \frac{1}{2}(1 + U_s^2) ds\right]$$

over controls U such that (1) has a solution.

A. Explain why it is useful to look for $G : \mathbb{R} \mapsto \mathbb{R}_+$ such that $G(-z) = 0 = G(z)$ and

$$\inf_u \left[\frac{1}{2}G'' + \lambda u G' + \frac{1}{2}(1 + u^2)\right] = 0. \tag{2}$$

[Hint: consider the process $M^U = M = (M_t)_{t \geq 0}$ given by $M_t = \int_0^{t \wedge \tau^U} \frac{1+U_s^2}{2} ds + G(X_{t \wedge \tau^U}^U)$.] [3]

B. Solve (2) to give an expression for G. [Hint: it may help to consider the transformation $g = e^{-\lambda^2 G}$.] [4]

C. First consider the candidate optimal control U^*. Give an expression for U_t^* in terms of G and X_t.

Let $\tau^* = \tau^{X^{U^*}}$. Let $M^* = M^{U^*}$ be given by

$$M_t^* = \int_0^{t \wedge \tau^*} \frac{1 + (U_s^*)^2}{2} ds + G(X_{t \wedge \tau^*}^{U^*}).$$

Explain why M^* is a local martingale, and use this fact to deduce that $\mathbb{E}[\tau^*] < \infty$.

Find an expression for $[M^*]_t$ and deduce that M^* is an L^2 bounded martingale. Carefully explain how this can be used to show

$$G(x) = \mathbb{E}\left[\int_0^{\tau^*} \frac{1 + (U_s^*)^2}{2} ds\right].$$

[6]

D. Now consider a general control. Explain why if U is such that $\mathbb{E}\left[\tau^U\right] = \infty$ then U cannot be optimal. Deduce that we may restrict attention to controls for which $\mathbb{E}[\tau^U] < \infty$. [2]

E. By considering the process M^U given in A, or otherwise, carefully show that when $x = 0$

$$\inf_{U \in \mathcal{U}} \mathbb{E}\left[\int_0^{\tau^U} \frac{1 + U_s^2}{2} ds\right] = \frac{1}{\lambda^2} \ln \cosh \lambda z$$

[5]

Question 6 starts on next page

Question 6

Consider a market $(S_t^0, S_t)_{t\in[0,T]}$, where the riskless asset S^0 is constant 1 and the fundamental price of the risky asset S satisfies $S_t = S_0 + \mu t + \sigma W_t$, where $S_0, \mu \in \mathbb{R}$, $\sigma > 0$ and $W = (W_t)_{t\in[0,T]}$ is a Brownian motion. (Note the drift term!) Suppose an investor holds $Q_0 > 0$ number of shares in the stock at time 0 and aims to liquidate them if possible by time $T > 0$ by choosing a trading rate $\alpha = (\alpha_t)_{t\in[0,T]} \in \mathcal{A}$, where

$$\mathcal{A} := \left\{(\alpha_t)_{t\in[0,T]} \text{ adapted} : Q_0^\alpha = Q_0,\ \ E\left[\int_0^T \alpha_s^2 \,\mathrm{d}s\right] < \infty,\ E\left[\int_0^t (Q_s^\alpha)^2 \,\mathrm{d}s\right] < \infty\right\},$$

with $Q_t^\alpha := Q_0 + \int_0^t \alpha_u \,\mathrm{d}u$. We assume that the investor faces temporary linear price impact, i.e., when choosing $\alpha \in \mathcal{A}$, the corresponding execution price $\hat{S}^\alpha = (\hat{S}_t^\alpha)_{t\in[0,T]}$ is given by

$$\hat{S}_t^\alpha = S_t + \lambda\alpha_t,$$

for some $\lambda > 0$. We assume that the investor holds initially no cash, and hence the dynamics of the cash process $X^\alpha = (X_t^\alpha)_{t\in[0,T]}$ when using the trading rate $\alpha \in \mathcal{A}$ is given by

$$\mathrm{d}X_t^\alpha = -\hat{S}_t^\alpha \,\mathrm{d}Q_t^\alpha, \quad X_0^\alpha = 0.$$

We assume that the goal functional $J : \mathcal{A} \to \mathbb{R}$ of the investor is given by

$$J(\alpha) := E\left[X_T^\alpha + Q_T^\alpha S_T - \eta(Q_T^\alpha)^2 - \gamma\int_0^T (Q_t^\alpha)^2 \,\mathrm{d}t\right],$$

where $\eta, \gamma > 0$. The investor seeks to maximise $J(\alpha)$ over all $\alpha \in \mathcal{A}$.

A. Explain the meaning of the third and fourth term in the investor's goal functional. **[2]**

B. Define the value function $v(t,x,s,q)$ corresponding to the conditional version of the investor's control problem. **[2]**

Hint: Do not forget to define the restarted set of admissible trading rates $\mathcal{A}_{(t,x,s,q)}$ and the restarted (controlled) processes $(S_u^{t,x,s,q,\alpha})_{u\in[t,T]}$, $(\hat{S}_u^{t,x,s,q,\alpha})_{u\in[t,T]}$, $(X_u^{t,x,s,q,\alpha})_{u\in[t,T]}$, $(Q_u^{t,x,s,q,\alpha})_{u\in[t,T]}$ for $t \in [0,T)$, $x, s, q \in \mathbb{R}$.

C. State the dynamic programming principle (DPP) for the investor's control problem. **[2]**

D. Starting from the DPP and assuming that the value function is sufficiently regular, carefully show that the corresponding HJB equation is given by

$$v_t + \mu v_s + \frac{1}{2}\sigma^2 v_{ss} + \sup_{a\in\mathbb{R}}\left(-a(s+\lambda a)v_x + a v_q\right) - \gamma q^2 = 0, \quad t \in [0,T),$$
$$v(T,x,s,q) = x + qs - \eta q^2.$$

[6]

E. Making the ansatz $v(t,x,s,q) := x + sq + h(t,q)$, show that h satisfies the PDE

$$h_t(t,q) + \frac{1}{4\lambda}h_q(t,q)^2 + \mu q - \gamma q^2 = 0, \quad t \in [0,T),$$
$$h(T,q) = -\eta q^2.$$

[3]

F. Making the ansatz $h(t,q) := j_0(t) + j_1(t)q + j_2(t)q^2$, show that j_0, j_1, j_2 satisfy the ODEs

$$j_0'(t) = -\frac{1}{4\lambda}j_1(t)^2, \qquad j_0(T) = 0$$
$$j_1'(t) = -\frac{1}{\lambda}j_2(t)j_1(t) - \mu, \qquad j_1(T) = 0$$
$$j_2'(t) = -\frac{1}{\lambda}j_2(t)^2 + \gamma, \qquad j_2(T) = -\eta$$

[3]

G. Explain in which order you would solve the ODE system in F. **[2]**

End of Exam