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Brownian Motion

Problem sheet 4 (valid as an assignment)

1. **Transition kernels** [12]
Recall the **Ornstein-Uhlenbeck process**:

$$\big(X(t) : t \in \mathbb{R}\big)$$

as defined in Q2.3 with $X(t) = e^{-ct}B(e^{2ct})$ for all $t \in \mathbb{R}$ and some $c > 0$. Show that $\big(X(t) : t \geqslant 0\big)$ and $\big(X(-t) : t \geqslant 0\big)$ are Markov processes (with respect to their natural filtration) and find their Markov transition kernels.

2. **Markov processes** [17]
   (a) Let $X = \big(X(t) : t \geq 0\big)$ be a process on $\mathbb{R}$ with stationary, independent increments. Show that this implies that X is time-homogeneous Markov.
   (b) Give an example of a time-homogeneous Markov process that does not have independent increments.
   (c) Give an example of a process with stationary increments that is not Markov.
   (d) Recall the Brownian bridge $\big(X(t) : t \in [0, 1]\big)$ defined in Q1.6 with $X(t) = B(t) - tB(1)$ for all $t \in [0, 1]$. Is this a Markov process? Is it a time-homogeneous Markov process? Justify your answers.

3. **Fractional Brownian motion** [35]
Consider a Gaussian processes with continuous paths $\big(B_H(t) : t \geq 0\big)$, for a parameter $H \in (0, 1)$, with $B_H(0) = 0$, mean zero and covariance

$$E\left[B_H(s)B_H(t)\right] = \frac{1}{2}\left(|s|^{2H} + |t|^{2H} - |t-s|^{2H}\right) .$$

   (a) Check that for $H = 1/2$ the process is a BM.
   (b) For which $H \in (0, 1)$ does the process have stationary increments?
   For which $H \in (0, 1)$ does the process have independent increments?
   Justify your answers and give the distribution of increments.
   (c) Show that the process satisfies a scaling relation such that $t \mapsto \lambda^{-\alpha}B_H(\lambda t)$ is again a fractional BM for some $\alpha > 0$ and identify α.
   (d) What degree of Hölder continuity can you guarantee for the paths of B_H by using Kolmogorov's criterion?
   (e) Compute $\displaystyle\sum_{n=1}^{\infty} \mathbb{E}\big[B_H(1)(B_H(n+1) - B_H(n))\big]$.
   The result can be interpreted that B_H has positive 'long-range' correlations for $H > 1/2$ and negative correlations for $H < 1/2$.

(f) Show that for fractional BM $\big(B_H(t+s) - B_H(s) : t \geq 0\big)$ is independent of $\mathcal{F}^0(s)$ if and only if $H = 1/2$.

4. **'Derivative' of Brownian motion/white noise** [36]
Let $B = \big(B(t) : t \geq 0\big)$ be a Brownian motion on $\mathbb{R}$.
For fixed $h > 0$ consider the process $t \mapsto X_h(t) := \frac{1}{h}\big(B(t+h) - B(t)\big)$.

(a) What is the distribution of the time marginal $X_h(t)$ for $t \geq 0$?
For fixed $t \geq 0$, does the limit $\lim_{h \to 0} X_h(t)$ exist almost surely?
Does the limit exist in distribution? Justify your answers.

(b) Show that $\big(X_h(t) : t \geq 0\big)$ is a Gaussian process and compute its mean and covariances.

(c) What is the distribution of the increments $X_h(t) - X_h(s)$ for $0 < s < t$?
Does $\big(X_h(t) : t \geq 0\big)$ have stationary increments?
Does $\big(X_h(t) : t \geq 0\big)$ have independent increments?
Are the increments stationary? Are the increments independent?
(Hint: Where appropriate, consider $|t-s| < h$ and $|t-s| \geq h$ separately.) Is $\big(X_h(t) : t \geq 0\big)$ a martingale w.r.t. the filtration $\big(\mathcal{F}^0(t+h) : t \geq 0\big)$?
Is $\big(X_h(t) : t \geq 0\big)$ a Markov process?

Justify your answers.