Ex 1 (1) Suppose $P(t,S) > P(t,T)P(T,S)$ for $t \le T \le$. Consider the following portfolio:

At time t: Pay $P(T,S)P(t,T)$ to buy $P \quad S)$ units of T-bonds
receive $P(T,S)P(t,T)$ to sell $\frac{P \quad S)P(t,T)}{(t,S)}$ units of S-bonds } zero net investment

At time T: receive $P(T,S)$ for T-bonds
pay $P(T,S)$ to buy 1 unit S-bond } zero net investment

At time S, pay $\frac{P(T,S)P(t,T)}{P(t,S)}$ for S-bonds **sold at t**
receive 1 for S-bond purchased at T } net profit $1 - \frac{P(T,S)P(t,T)}{P(t,S)} > 0$

This is an arbitrage opportunity.

If $P(t,S) < P(t,T)P(T,S)$, the same profit can be realized by reversing the sign in the above portfolio

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(2) Take log: $\ln P(t,S) = \ln P(t,T) + \ln P(T,S)$

$$\Rightarrow -\int_t^S f(t,u)\,du = -\int_t^T f(t,u)\,du - \int_T^S f(T,u)\,du$$

$$\Rightarrow \int_T^S f(t,u)\,du = \int_T^S f(T,u)\,du$$

Differentiate against S: $f(t,S) = f(T,S)$ for $t \le T \le S$

The above holds for any $t \le T \le S$. In particular,

$f(t,S) = f(T,S) = f(S,S) = r(S)$ #

(3) Since it is in a deterministic world, the fair value of FRN can be calculated

$$P(t) = \sum_{i=1}^{n} P(t,T_i)\,C_i + P(t,T_n)\,N$$

$$= \sum_{i=1}^{n} P(t,T_i)\left(\frac{1}{P(T_{i-1},T_i)} - 1\right)N + P(t,T_n)\,N$$

$$= \sum_{i=1}^{n} \left(P(t,T_{i-1}) - P(t,T_i)\right)N + P(t,T_n)\,N = P(t,T_0)\,N \qquad \#$$

Ex 2 Since $R_{swap}(T_0) = \frac{P(T_0,T_0) - P(T_0,T_n)}{\delta \sum_j P(T_0,T_j)}$

$$= \frac{1}{\quad} \sum^{n} P(T_0,T_i)\,F(T_0,T_{i-1},T_i)$$

$$= \frac{1}{\sum_j P(T_0 T_j)} \sum_{i=1}^{n} P(T_0, T_i) F(T_0; T_{i-1}, T_i)$$

it follows that

$$N\delta \, (Rswap(T_0) - k)^+ \sum_j P(T_0 . T_j)$$

$$= N\delta \left(\frac{1}{\sum_j P(T_0 T_j)} \sum_{i=1}^{n} P(T_0 T_i) F(T_0; T_{i-1}, T_i) - k \right)^+ \sum_j P(T_0 T_j)$$

$$= N\delta \left(\sum_{i=1}^{n} P(T_0 T_i) \left(F(T_0; T_{i-1}, T_i) - k \right)^+ \right)$$

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