

# Applications of Stochastic Calculus in Finance

## Chapter 0: Review of stochastic calculus

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### 1 Brownian Motion and martingales

Let  $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t \geq 0}, \mathbf{P})$  be a filtered probability space satisfying the usual conditions:

- (1) completeness,  $\mathcal{F}_0$  includes all of the  $\mathbf{P}$ -null sets;
- (2) right continuity,  $\mathcal{F}_t = \mathcal{F}_{t+}$  where  $\mathcal{F}_{t+} = \bigcap_{n \geq 1} \mathcal{F}_{t+\frac{1}{n}}$ .

**Definition 1.** A  $d$ -dimensional stochastic process  $X = (X_t)_{t \geq 0} : \Omega \times [0, \infty) \rightarrow \mathbb{R}^d$  is called (1) adapted if for any  $t \geq 0$ ,  $X_t$  is  $\mathcal{F}_t$ -measurable; (2) progressively-measurable if for any  $t \geq 0$ , the map  $(\omega, s) \mapsto X_s(\omega)$  from  $\Omega \times [0, t]$  to  $\mathbb{R}^d$  is  $\mathcal{F}_t \otimes \mathcal{B}[0, t]$ -measurable.

Progressive measurability of  $X$  is needed in order that (1) composed processes such as  $(\int_0^t X_s ds)_{t \geq 0}$ , and (2) stopped processes such as  $(X_t^\tau)_{t \geq 0}$  are both adapted. The stopped process  $X_t^\tau := X_t \mathbf{1}_{\{t \leq \tau\}} + X_\tau \mathbf{1}_{\{\tau < t\}}$  where  $\tau$  is a stopping time.

**Proposition 1.** *If  $X$  is adapted and has Càdlàg/Càglàd sample paths, then  $X$  is progressively measurable.*

Càdlàg means right continuous with left limit; Càglàd means left continuous with right limit.

**Definition 2.** Two stochastic processes  $X$  and  $Y$  are called (1) indistinguishable if  $\mathbf{P}(X_t = Y_t, t \geq 0) = 1$ ; (2) modification of each other if  $\mathbf{P}(X_t = Y_t) = 1$  for any  $t \geq 0$ .

**Proposition 2.** *If  $X$  is a modification of  $Y$ , and both  $X$  and  $Y$  have Càdlàg/Càglàd sample paths, then  $X$  and  $Y$  are indistinguishable.*

**Proposition 3.** (martingale regularity)

*Any martingale with filtration satisfying the usual conditions has a Càdlàg modification.*

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In the rest of this chapter, we always assume that

- (1) *The filtration satisfies the usual conditions;*
- (2) *Martingales (and related sub/super, local and semimartingales) are continuous.*

**Definition 3.** (Brownian motion)

An adapted process  $W = (W_t)_{t \geq 0}$  is a one-dimensional Brownian motion if it has stationary independent increments with Gaussian distribution, and continuous sample paths.

$$\begin{aligned} \mathbf{P}(W_t - W_s \in A | \mathcal{F}_s) &= \mathbf{P}(W_t - W_s \in A) \quad (\text{independent increments}) \\ &= \mathbf{P}(W_{t-s} \in A) \quad (\text{stationary}) \\ &= \int_A \frac{1}{\sqrt{2\pi(t-s)}} e^{-\frac{x^2}{2(t-s)}} dx \quad (\text{Gaussian distribution}) \end{aligned}$$

for any  $t \geq s \geq 0$ , and  $A \in \mathcal{B}(\mathbb{R})$ .

A  $d$ -dimensional Brownian motion  $W = (W^1, \dots, W^d)^T$  is such that (1) each component  $W^i$  is a one-dimensional Brownian motion, and (2) each  $W^i$  and  $W^j$  are independent.

**Definition 4.** (martingale, sub/super martingale)

An adapted process  $M$  is called a martingale if it is integrable  $\mathbf{E}[|M_t|] < \infty$  for any  $t \geq 0$ , and  $\mathbf{E}[M_t | \mathcal{F}_s] = M_s$  for any  $t \geq s \geq 0$ .

$M$  is called a supermartingale if  $\mathbf{E}[M_t | \mathcal{F}_s] \leq M_s$ . If  $M$  is a supermartingale,  $-M$  is then called a submartingale.

**Definition 5.** (local martingale)

An adapted process  $M$  is called a local martingale if there exists an increasing sequence of stopping times  $T_n \uparrow \infty$  such that  $M^{T_n}$  is a martingale.

**Definition 6.** (semimartingale)

An adapted process  $X$  is called a semimartingale if  $X_t = M_t + A_t$  where  $M$  is a local martingale, and  $A$  is a finite variation process, i.e.

$$\sup_{D_t} \sum_l |A_{t_l} - A_{t_{l-1}}| < \infty$$

where  $D_t$  is any finite partition of  $[0, t]$ .

One way to distinguish between the local martingale and the finite variation process in the semimartingale is to use quadratic variation.

**Definition 7.** (quadratic variation)

A stochastic process  $M$  has quadratic variation if for any  $t \geq 0$ , when the mesh of the partition of  $||D_t|| \rightarrow 0$ ,

$$\lim_{||D_t|| \rightarrow 0} \sum_l |M_{t_l} - M_{t_{l-1}}|^2 < \infty.$$

**Proposition 4.** *For a local martingale  $M$ , its quadratic variation  $\langle M \rangle$  can be defined equivalently as follows:*

- (1) *An increasing continuous adapted process  $\langle M \rangle$  such that  $M_t^2 - \langle M \rangle_t \geq 0$ , is a local martingale;*
- (2) *From Itô's formula,  $\langle M \rangle_t = M_t^2 - 2 \int_0^t M_s dM_s$ .*

*Note that a finite variation process always has zero quadratic variation.*

**Proposition 5.** *(Criteria for local martingales being martingales)*

- (1) *Let  $M$  be a local martingale such that  $E[\sup_{s \in [0, t]} |M_s|] < \infty$  for any  $t \geq 0$ , then  $M$  is a martingale;*
- (2) *Let  $M$  be a local martingale such that  $E\langle M \rangle_t < \infty$  for any  $t \geq 0$ , then  $M$  is a martingale;*
- (2) *A bounded local martingale  $M$  is a martingale.*

## 2 Itô's stochastic integration

Itô's stochastic integration gives a meaning for the integrals like  $\int_0^t h_s dW_s$ . First we define the spaces of integrands. Fix  $T > 0$  in the following.

Let  $\mathcal{L}^2(\mathbb{R}^d)$  be the space of  $\mathbb{R}^d$ -valued progressively measurable processes  $h = (h^1, \dots, h^d)$  with  $\mathbf{E}[\int_0^T |h_s|^2 ds] < \infty$ .

Let  $\mathcal{L}(\mathbb{R}^d)$  be the space of  $\mathbb{R}^d$ -valued progressively measurable processes  $h = (h^1, \dots, h^d)$  with  $\int_0^T |h_s|^2 ds < \infty$ .

Obviously, we have that  $\mathcal{L}^2(\mathbb{R}^d) \subset \mathcal{L}(\mathbb{R}^d)$ .

**Theorem 1.** *For any  $h \in \mathcal{L}(\mathbb{R}^d)$ , one can define the stochastic integral  $\int_0^t h_s dW_s = \int_0^t \sum_{j=1}^d h_s^j dW_s^j$ ,  $t \in [0, T]$ , with the following properties:*

- (1)  *$\int_0^t h_s dW_s$  is a local martingale;*
- (2) *(linearity) for any  $\lambda_1, \lambda_2 \in \mathbb{R}$ , and  $h(1), h(2) \in \mathcal{L}(\mathbb{R}^d)$ ,*

$$\int_0^t (\lambda_1 h_s(1) + \lambda_2 h_s(2)) dW_s = \lambda_1 \int_0^t h_s(1) dW_s + \lambda_2 \int_0^t h_s(2) dW_s;$$

- (3) *for any stopping time  $\tau$ ,*

$$\int_0^{t \wedge \tau} h_s dW_s = \int_0^t \mathbf{1}_{\{s \leq \tau\}} h_s dW_s;$$

- (4) *(dominated convergence) If  $h(n) \in \mathcal{L}(\mathbb{R}^d)$  is a sequence with  $\lim_{n \rightarrow \infty} h_t(n) = 0$  for each  $(\omega, t)$ , and such that  $|h_t(n)| \leq K$  for some  $K \geq 0$ , then*

$$\lim_{n \rightarrow \infty} \sup_{s \in [0, t]} \left| \int_0^s h_u(n) dW_u \right| = 0$$

*in probability.*

(5) (Itô's isometry) If  $h \in \mathcal{L}^2(\mathbb{R}^d)$ , then  $\int_0^\cdot h_s dW_s$  is a martingale, and moreover,

$$\mathbf{E} \left[ \left( \int_0^t h_s dW_s \right)^2 \right] = \mathbf{E} \left[ \int_0^t |h_s|^2 ds \right].$$

(6) (quadratic variation)

$$\left\langle \int_0^\cdot h_s dW_s \right\rangle_t = \int_0^t |h_s|^2 ds.$$

### 3 Itô's formula

Itô's formula says that semimartingales are invariant under  $C^2$  transformation. Since we are interested in the multi-dimensional Itô's formula, we first define the quadratic covariation  $\langle M^i, M^j \rangle_t$  as

$$\langle M^i, M^j \rangle_t = M_t^i M_t^j - \int_0^t M_s^i dM_s^j - \int_0^t M_s^j dM_s^i,$$

or, equivalently, as an increasing continuous adapted process  $\langle M^i, M^j \rangle$  such that  $M_t^i M_t^j - \langle M^i, M^j \rangle_t, t \in [0, T]$ , is a local martingale.

Note that from the polarization identity

$$xy = \frac{1}{4}(|x+y|^2 - |x-y|^2)$$

and Itô's isometry in part (5) of Theorem 1, we can obtain

$$\mathbf{E} \left[ \left( \int_0^t h_s(1) dW_s \right) \left( \int_0^t h_s(2) dW_s \right) \right] = \mathbf{E} \left[ \int_0^t (h_s(1))^T h_s(2) ds \right].$$

In terms of quadratic covariation, from the polarization identity

$$\langle M^i, M^j \rangle_t = \frac{1}{4} (\langle M^i + M^j \rangle_t - \langle M^i - M^j \rangle_t)$$

and the quadratic variation result in part (6) of Theorem 1, we can obtain

$$\left\langle \int_0^\cdot h_s(1) dW_s, \int_0^\cdot h_s(2) dW_s \right\rangle_t = \int_0^t (h_s(1))^T h_s(2) ds.$$

**Theorem 2. (Itô's formula)** Let  $X = (X^1, \dots, X^n)^T$  be an  $n$ -dimensional semimartingale, and let  $f \in C^2(\mathbb{R}^n)$ . Then  $f(X_t), t \geq 0$ , is also an  $n$ -dimensional semimartingale, and moreover,

$$df(X_t) = \sum_{i=1}^n \partial_{x_i} f(X_t) dX_t^i + \frac{1}{2} \sum_{i,j=1}^n \partial_{x_i x_j} f(X_t) d\langle X^i, X^j \rangle_t.$$

As an example, suppose that  $X$  solves the following time homogenous diffusion

$$dX_t^i = \mu^i(X_t)dt + \sum_{j=1}^d \sigma^{ij}(X_t)dW_t^j, \quad 1 \leq i \leq n,$$

or in a matrix form

$$dX_t = \mu(X_t)dt + \sigma(X_t)dW_t.$$

Then from Itô's formula, we have that

$$df(X_t) = \mathcal{L}f(X_t)dt + \sum_{i=1}^n \partial_{x_i} f(X_t) \sum_{j=1}^d \sigma^{ij}(X_t)dW_t^j,$$

where  $\mathcal{L}$  is called the infinitesimal generator of  $X$ , given as

$$\mathcal{L}f(x) = \frac{1}{2} \sum_{i,j=1}^n \partial_{x_i x_j} f(x) \sum_{k=1}^d \sigma^{ik}(x) \sigma^{kj}(x) + \sum_{i=1}^n \partial_{x_i} f(x) \mu^i(x).$$

The corresponding matrix form is

$$df(X_t) = \left( \frac{1}{2} \text{Trace}\{\sigma(X_t)\sigma(X_t)^T \nabla_x^2 f(X_t)\} + \mu(X_t)^T \nabla_x f(X_t) \right) dt + (\sigma(X_t)^T \nabla_x f(X_t))^T dW_t.$$

**Proposition 6.** (*Lévy's characterization*) Let  $M = (M^1, \dots, M^d)^T$  be a  $\mathbb{R}^d$ -valued local martingale. Then it is also a  $d$ -dimensional Brownian motion if

$$\langle M^i, M^j \rangle_t = \delta_{ij}t$$

for  $1 \leq i, j \leq d$ .

## 4 Stochastic differential equation (SDE)

**Definition 8.** Let  $\mu(\cdot, x) : \Omega \times [0, \infty) \rightarrow \mathbb{R}^n$  and  $\sigma(\cdot, x) : \Omega \times [0, \infty) \rightarrow \mathbb{R}^{n \times d}$  be both progressively measurable. Given a SDE

$$dX_t = \mu(t, X_t)dt + \sigma(t, X_t)dW_t, \quad X_0 = x \in \mathbb{R}^n,$$

its solution is defined as a semimartingale  $X$  such that the corresponding integral equation is satisfied

$$X_t = x + \int_0^t \mu(s, X_s)ds + \int_0^t \sigma(s, X_s)dW_s.$$

We say  $X$  is unique if any other solution  $\tilde{X}$  is a modification of  $X$ .

Examples of SDEs include

(1) the prices of the stocks:

$$dS_t^i = S_t^i \left( \mu_t^i dt + \sum_{j=1}^d \sigma_t^{ij} dW_t^j \right), \quad 1 \leq i \leq n$$

for progressively measurable processes  $\mu^i$  and  $\sigma^{ij}$ .

(2) Time homogenous diffusion:

$$dX_t^i = \mu^i(X_t)dt + \sum_{j=1}^d \sigma^{ij}(X_t)dW_t^j, \quad 1 \leq i \leq n,$$

for deterministic functions  $\mu^i(\cdot)$  and  $\sigma^{ij}(\cdot)$ .

(3) Stochastic exponential: Given a local martingale  $M$ , define its stochastic exponential as  $\mathcal{E}(M)_t = e^{M_t - \frac{1}{2}\langle M \rangle_t}$ , which satisfies

$$d\mathcal{E}(M)_t = \mathcal{E}(M)_t dM_t.$$

**Theorem 3.** *If both  $\mu(\cdot, x)$  and  $\sigma(\cdot, x)$  are Lipschitz continuous in  $x$  and have at most linear growth in  $x$ , then the above SDE admits a unique solution.*

**Proposition 7.** (Novikov's condition)

Let  $M$  be a local martingale with its stochastic exponential  $\mathcal{E}(M)$ . If

$$\mathbf{E}[e^{\frac{1}{2}\langle M \rangle_T}] < \infty,$$

then  $\mathcal{E}(M)$  is a martingale up to time  $T$ .

## 5 Girsanov's theorem

**Proposition 8.** (Bayes Rule)

Let  $\mathcal{E}(M)$  be a martingale up to  $T > 0$ . Define a new probability measure  $\mathbf{Q}$  by the Radon-Nikodym density

$$\left. \frac{d\mathbf{Q}}{d\mathbf{P}} \right|_{\mathcal{F}_t} := \mathbf{E}^{\mathbf{P}}[\mathcal{E}(M)_T | \mathcal{F}_t] = \mathcal{E}(M)_t.$$

Then for any r.v.  $Y_t \in \mathcal{F}_t$ , we have that

$$\begin{aligned} \mathbf{E}^{\mathbf{Q}}[Y_t] &= \mathbf{E}^{\mathbf{P}}[Y_t \mathcal{E}(M)_t]; \\ \mathbf{E}^{\mathbf{Q}}[Y_t | \mathcal{F}_s] &= \mathbf{E}^{\mathbf{P}} \left[ \frac{Y_t \mathcal{E}(M)_t}{\mathcal{E}(M)_s} \middle| \mathcal{F}_s \right], \quad \text{for } 0 \leq s \leq t \leq T. \end{aligned}$$

*Proof.* For the first equality, it is sufficient to prove for  $Y_t = \mathbf{1}_A$  where  $A \in \mathcal{F}_t$ . Indeed,

$$\begin{aligned} \mathbf{E}^{\mathbf{Q}}[\mathbf{1}_A] &= \mathbf{Q}(A) \\ &= \int_A \mathcal{E}(M)_t d\mathbf{P} \\ &= \int_{\Omega} \mathbf{1}_A \mathcal{E}(M)_t d\mathbf{P} = \mathbf{E}^{\mathbf{P}}[\mathbf{1}_A \mathcal{E}(M)_t]. \end{aligned}$$

For the second equality, we need to show its RHS is the conditional expectation of  $Y_t$  given  $\mathcal{F}_s$  under  $\mathbf{Q}$ . Indeed, for any  $A \in \mathcal{A}_s$ ,

$$\begin{aligned} \mathbf{E}^{\mathbf{Q}} \left[ \mathbf{1}_A \mathbf{E}^{\mathbf{P}} \left[ \frac{Y_t \mathcal{E}(M)_t}{\mathcal{E}(M)_s} \middle| \mathcal{F}_s \right] \right] &= \mathbf{E}^{\mathbf{P}} \left[ \mathbf{1}_A \mathbf{E}^{\mathbf{P}} \left[ \frac{Y_t \mathcal{E}(M)_t}{\mathcal{E}(M)_s} \middle| \mathcal{F}_s \right] \mathcal{E}(M)_s \right] \\ &= \mathbf{E}^{\mathbf{P}} [\mathbf{1}_A Y_t \mathcal{E}(M)_t] \\ &= \mathbf{E}^{\mathbf{Q}} [\mathbf{1}_A Y_t] \quad \square \end{aligned}$$

**Theorem 4.** (*Girsanov's theorem*)

Let  $\mathcal{E}(M)$  be a martingale up to  $T > 0$ . Define a new probability measure  $\mathbf{Q}$  by the Radon-Nikodym density

$$\left. \frac{d\mathbf{Q}}{d\mathbf{P}} \right|_{\mathcal{F}_t} = \mathcal{E}(M)_t.$$

Let  $X$  be a semimartingale under  $\mathbf{P}$  with the decomposition:  $X_t = N_t + A_t$  where  $N$  is a  $\mathbf{P}$ -local martingale and  $A$  is a finite variation process. Then  $X$  is also a semimartingale under  $\mathbf{Q}$  with the decomposition:  $X_t = \tilde{N}_t + \tilde{A}_t$ , where  $\tilde{N}$  is a  $\mathbf{Q}$ -local martingale:

$$\tilde{N}_t = N_t - \int_0^t \frac{1}{\mathcal{E}(M)_s} d\langle \mathcal{E}(M), N \rangle_s,$$

and  $\tilde{A}$  is a finite variation process:

$$\tilde{A}_t = A_t + \int_0^t \frac{1}{\mathcal{E}(M)_s} d\langle \mathcal{E}(M), N \rangle_s.$$

*Proof.* We only need to show that  $\tilde{N}$  is a  $\mathbf{Q}$ -local martingale. The following is the key observation to prove this.

If  $\tilde{N}_t \mathcal{E}(M)_t$ ,  $t \in [0, T]$ , is a  $\mathbf{P}$ -local martingale, then  $\tilde{N}_t$ ,  $t \in [0, T]$ , is a  $\mathbf{Q}$ -local martingale.

Hence, we first prove that  $\tilde{N}_t \mathcal{E}(M)_t$  is a  $\mathbf{P}$ -local martingale. Note that

$$\begin{aligned} d\tilde{N}_t &= dN_t - \frac{1}{\mathcal{E}(M)_t} d\langle \mathcal{E}(M), N \rangle_t, \\ d\mathcal{E}(M)_t &= \mathcal{E}(M)_t dM_t, \end{aligned}$$

Applying Itô's formula to  $\tilde{N}_t \mathcal{E}(M)_t$  yields that

$$\begin{aligned}
d\tilde{N}_t \mathcal{E}(M)_t &= \mathcal{E}(M)_t d\tilde{N}_t + \tilde{N}_t d\mathcal{E}(M)_t + d\langle \tilde{N}, \mathcal{E}(M) \rangle_t \\
&= \mathcal{E}(M)_t dN_t - d\langle \mathcal{E}(M), N \rangle_t + \tilde{N}_t d\mathcal{E}(M)_t + d\langle N, \mathcal{E}(M) \rangle_t \\
&= \mathcal{E}(M)_t dN_t + \tilde{N}_t d\mathcal{E}(M)_t
\end{aligned}$$

which is obviously a  $\mathbf{P}$ -local martingale. That is, there exists an increasing sequence of stopping times  $T_n \uparrow \infty$  such that  $(\tilde{N}^{\mathcal{E}(M)})^{T_n}$  is a martingale.

Next we prove that  $\tilde{N}$  is a  $\mathbf{Q}$ -local martingale. Consider the above stopping time sequence  $T_n$  such that  $\{s \leq T_n\}$ , by Bayes rule, since  $\tilde{N}_t^{T_n} \in \mathcal{F}_t$ ,

$$\mathbf{E}^{\mathbf{Q}} [\tilde{N}_t^{T_n} | \mathcal{F}_s] = \mathbf{E}^{\mathbf{P}} \left[ \frac{\tilde{N}_t^{T_n} \mathcal{E}(M)_t}{\mathcal{E}(M)_s} | \mathcal{F}_s \right].$$

On the event  $\{t > T_n\} \in \mathcal{F}_{T_n}$  (since  $T_n \in \mathcal{F}_{T_n}$ ), the optional stopping theorem implies that

$$\mathbf{E}^{\mathbf{P}} [\mathcal{E}(M)_t | \mathcal{F}_{T_n}] = \mathcal{E}(M)_{T_n}.$$

On the other hand,  $\tilde{N}_t^{T_n} = \tilde{N}_{T_n} \in \mathcal{F}_{T_n}$  and the tower property yields

$$\mathbf{E}^{\mathbf{P}} [\tilde{N}_t^{T_n} \mathcal{E}(M)_t | \mathcal{F}_s] = \mathbf{E}^{\mathbf{P}} [\tilde{N}_t^{T_n} \mathbf{E}^{\mathbf{P}} [\mathcal{E}(M)_t | \mathcal{F}_{T_n}] | \mathcal{F}_s] = \mathbf{E}^{\mathbf{P}} [\tilde{N}_{T_n} \mathcal{E}(M)_{T_n} | \mathcal{F}_s].$$

Hence,

$$\begin{aligned}
\mathbf{E}^{\mathbf{Q}} [\tilde{N}_t^{T_n} | \mathcal{F}_s] &= \frac{1}{\mathcal{E}(M)_s} \mathbf{E}^{\mathbf{P}} [\tilde{N}_t^{T_n} \mathcal{E}(M)_t \mathbf{1}_{\{t \leq T_n\}} + \tilde{N}_t^{T_n} \mathcal{E}(M)_t \mathbf{1}_{\{t > T_n\}} | \mathcal{F}_s] \\
&= \frac{1}{\mathcal{E}(M)_s} \mathbf{E}^{\mathbf{P}} [\tilde{N}_t \mathcal{E}(M)_t \mathbf{1}_{\{t \leq T_n\}} + \tilde{N}_{T_n} \mathcal{E}(M)_{T_n} \mathbf{1}_{\{t > T_n\}} | \mathcal{F}_s] \\
&= \frac{1}{\mathcal{E}(M)_s} \mathbf{E}^{\mathbf{P}} [(\tilde{N}^{\mathcal{E}(M)})_t^{T_n} | \mathcal{F}_s] = \frac{1}{\mathcal{E}(M)_s} (\tilde{N}^{\mathcal{E}(M)})_s^{T_n} = \tilde{N}_s
\end{aligned}$$

on the event  $\{s \leq T_n\}$ .

On the other hand, on the event  $\{s > T_n\}$ , since  $\tilde{N}_t^{T_n} = \tilde{N}_{T_n}$ , it follows that

$$\mathbf{E}^{\mathbf{Q}} [\tilde{N}_t^{T_n} | \mathcal{F}_s] = \tilde{N}_{T_n},$$

which shows that  $\tilde{N}$  is indeed a  $\mathbf{Q}$ -local martingale.  $\square$

**Theorem 5.** (*Girsanov's theorem in Brownian motion case*)

Let  $M_t = \int_0^t h_s dW_s$  with

$$\mathbf{E}^{\mathbf{P}} [\exp(\frac{1}{2} \int_0^T |h_s|^2 ds)] < \infty,$$

so that  $\mathcal{E}(M)$  is a martingale. Define a new probability measure  $\mathbf{Q}$  by the Radon-Nikodym density

$$\left. \frac{d\mathbf{Q}}{d\mathbf{P}} \right|_{\mathcal{F}_t} = \mathcal{E}(M)_t.$$



Then  $W_t^{\mathbf{Q}} := W_t - \int_0^t h_s ds$  is a  $d$ -dimensional Brownian motion under  $\mathbf{Q}$ .

*Proof.* From Theorem 4, the Brownian motion  $W$ , as a  $\mathbf{P}$ -local martingale, is modified as

$$W_t - \int_0^t \frac{1}{\mathcal{E}(M)_s} d\langle \mathcal{E}(M), W \rangle_s = W_t - \int_0^t h_s ds,$$

which is a  $\mathbf{Q}$ -local martingale. Since

$$\langle W^{\mathbf{Q},i}, W^{\mathbf{Q},j} \rangle_t = \langle W^i, W^j \rangle_t = \delta_{ij}t,$$

we conclude that  $W^{\mathbf{Q}}$  is a Brownian motion under  $\mathbf{Q}$  by using Lévy's characterization.  $\square$

## 6 Martingale representation

### Theorem 6. (Martingale representation)

Assume that the filtration  $\{\mathcal{F}_t\}$  is generated by the  $d$ -dimensional Brownian motion  $W = (W^1, \dots, W^d)^T$ . For any local martingale  $M$ , there exists a density process  $h = (h^1, \dots, h^d)$  in  $\mathcal{L}(\mathbb{R}^d)$  such that

$$M_t = M_0 + \int_0^t h_s dW_s.$$

Moreover, if  $M$  is a martingale, then  $h \in \mathcal{L}^2(\mathbb{R}^d)$ .

## 7 Feynman-Kac formula

### Proposition 9. (Markov property)

Suppose that  $X$  solves the following SDE:

$$dX_t = \mu(t, X_t)dt + \sigma(t, X_t)dW_t,$$

where  $\mu(\cdot, \cdot) : [0, T] \times \mathbb{R}^n \rightarrow \mathbb{R}^n$  and  $\sigma(\cdot, \cdot) : [0, T] \times \mathbb{R}^n \rightarrow \mathbb{R}^{n \times d}$  are both deterministic functions. Then  $X$  admits the Markov property: For any measurable function  $f(\cdot)$  defined on  $\mathbb{R}^n$ ,

$$\mathbf{E}[f(X_t) | \mathcal{F}_s] = \mathbf{E}[f(X_t) | X_s] = F(s, X_s)$$

for some measurable function  $F(\cdot, \cdot)$ .

*Proof.* We first show the case  $X_t = W_t$ , i.e. a BM admits the Markov property. We claim that

$$\mathbf{E}[f(W_t) | \mathcal{F}_s] = g(W_s) \tag{1}$$

with  $g(x) = \mathbf{E}[f(W_t - W_s + x)]$ .

If (1) holds, then the tower property yields

$$\begin{aligned}\mathbf{E}[f(W_t)|W_s] &= \mathbf{E}[\mathbf{E}[f(W_t)|\mathcal{F}_s]|W_s] \\ &= \mathbf{E}[g(W_s)|W_s] = g(W_s).\end{aligned}$$

To show (1), we aim to show that for any  $A \in \mathcal{F}_s$ ,

$$\mathbf{E}[f(W_t)\mathbf{1}_A] = \mathbf{E}[g(W_s)\mathbf{1}_A].$$

We first approximate  $f(W_t)$  pointwisely by the functions of the form

$$f(W_t - W_s + W_s) = \lim_{m \rightarrow \infty} \sum_{n=1}^m \phi_n^1(W_s) \phi_n^2(W_t - W_s).$$

Then, the DCT/MCT yields

$$\begin{aligned}g(x) &= \mathbf{E}\left[\sum_n \phi_n^1(x) \phi_n^2(W_t - W_s)\right] \\ &= \sum_n \phi_n^1(x) \mathbf{E}[\phi_n^2(W_t - W_s)],\end{aligned}$$

and using the fact that  $W_t - W_s$  is independent of  $\mathcal{F}_s$ ,

$$\begin{aligned}\mathbf{E}[f(W_t)\mathbf{1}_A] &= \mathbf{E}\left[\sum_n \phi_n^1(W_s) \phi_n^2(W_t - W_s) \mathbf{1}_A\right] \\ &= \sum_n \mathbf{E}[\phi_n^1(W_s) \mathbf{1}_A \mathbf{E}[\phi_n^2(W_t - W_s)|\mathcal{F}_s]] \\ &= \sum_n \mathbf{E}[\phi_n^1(W_s) \mathbf{1}_A \mathbf{E}[\phi_n^2(W_t - W_s)]] \\ &= \mathbf{E}\left[\left(\sum_n \phi_n^1(x) \mathbf{E}[\phi_n^2(W_t - W_s)]\right) \Big|_{x=W_s} \mathbf{1}_A\right] = \mathbf{E}[g(W_s)\mathbf{1}_A].\end{aligned}$$

The general case uses the fact that  $X_t$  is  $\sigma(X_s, W_r - W_s, r \in [s, t])$ -measurable, so there exists a measurable function  $F$  such that

$$X_t = F(X_s, W_r - W_s, r \in [s, t]).$$

See Oksendal [2] Chapter 7.1 for a detailed proof.  $\square$

**Theorem 7. (Feynman-Kac Formula)**

Suppose that  $X$  solves the SDE as in Proposition 9, so that it admits the Markov property: For any measurable functions  $f(\cdot)$  and  $q(\cdot)$  on  $\mathbb{R}^n$ ,

$$\mathbf{E}[e^{-\int_t^T q(X_s)ds} f(X_T)|\mathcal{F}_t] = \mathbf{E}[e^{-\int_t^T q(X_s)ds} f(X_T)|X_t] = F(t, X_t)$$

for some measurable function  $F(\cdot, \cdot)$ .

If  $F \in C^{1,2}([0, T] \times \mathbb{R}^n)$  then  $F(t, x)$  solves the following PDE on  $[0, T] \times \mathbb{R}^n$ :

$$\begin{cases} \partial_t F(t, x) + \mathcal{L}F(t, x) - q(x)F(t, x) = 0, \\ F(T, x) = f(x) \end{cases} \quad (2)$$

where  $\mathcal{L}$  is the infinitesimal generator of  $X$ .

*Proof.* We first show the Markov property. The general idea is to replace the underlying Markov process  $X$  with a new Markov process which corresponds to the “killing” of the sample paths of  $X$  at the rate  $q(\cdot)$ . However, we give a more direct proof by introducing an auxiliary process

$$dY_t = -q(X_t)Y_t dt$$

Then, it is clear that  $Y_T = Y_t e^{-\int_t^T q(X_s) ds}$ . We may view  $(X, Y)$  as a pair of Markov processes, so  $e^{-\int_t^T q(X_s) ds} f(X_T) = Y_T f(X_T) / Y_t$ . By Proposition 9,

$$\mathbf{E}\left[\frac{Y_T f(X_T)}{Y_t} \middle| \mathcal{F}_t\right] = \frac{\mathbf{E}[Y_T f(X_T) | (X_t, Y_t)]}{Y_t} = \mathbf{E}[e^{-\int_t^T q(X_s) ds} f(X_T) | (X_t, Y_t)].$$

However, the term  $e^{-\int_t^T q(X_s) ds} f(X_T)$  does not depend on  $Y_t$ , from which we conclude.

To derive the PDE, the key observation is that

$$e^{-\int_0^t q(X_s) ds} F(t, X_t) = \mathbf{E}[e^{-\int_0^T q(X_s) ds} F(T, X_T) | \mathcal{F}_t]$$

so  $e^{-\int_0^t q(X_s) ds} F(t, X_t)$ ,  $t \in [0, T]$ , is a martingale. Applying Itô’s formula to  $e^{-\int_0^t q(X_s) ds} F(t, X_t)$  gives us

$$\begin{aligned} de^{-\int_0^t q(X_s) ds} F(t, X_t) &= e^{-\int_0^t q(X_s) ds} dF(t, X_t) - e^{-\int_0^t q(X_s) ds} q(X_t) F(t, X_t) dt \\ &= e^{-\int_0^t q(X_s) ds} (\partial_t F(t, X_t) + \mathcal{L}F(t, X_t) - q(X_t) F(t, X_t)) dt \\ &\quad + e^{-\int_0^t q(X_s) ds} (\sigma(X_t)^T \nabla_x F(t, X_t))^T dW_t \end{aligned}$$

In order to guarantee that  $e^{-\int_0^t q(X_s) ds} F(t, X_t)$  is a martingale, we must have

$$\partial_t F(t, X_t) + \mathcal{L}F(t, X_t) - q(X_t) F(t, X_t) = 0,$$

which gives us the PDE for  $F(\cdot, \cdot)$ .  $\square$

*Remark 1.* Note that from the martingale property of  $e^{-\int_0^t q(X_s) ds} F(t, X_t)$ , we also have

$$\int_0^\cdot e^{-\int_0^s q(X_s) ds} (\sigma(X_t)^T \nabla_x F(t, X_t))^T dW_t \quad (3)$$

is a martingale. Hence, we can obtain a reverse statement, which also refers to the Feynman-Kac formula in the literature as it provides a formula to compute conditional expectation. Suppose  $F \in C^{1,2}([0, T] \times \mathbb{R}^n)$  solves (1) and the martingale

condition (3) holds, then

$$F(t, X_t) = \mathbf{E}[e^{-\int_t^T q(X_s)ds} f(X_T) | \mathcal{F}_t].$$

## 8 Exercises

### Exercise 1. (Ornstein-Uhlenbeck Process and Vasicek Model)

1. Consider the following linear SDE system:

$$dX_t = (a(t) + b(t)X_t)dt + \sigma(t)dW_t, \quad X_0 = x,$$

where  $W = (W^1, \dots, W^d)^T$  is a  $d$ -dimensional Brownian motion. Assume that  $a(t), b(t) : [0, \infty) \rightarrow \mathbb{R}^d$  and  $\sigma(t) : [0, \infty) \rightarrow \mathbb{R}^{d \times d}$  are deterministic and bounded functions. By using integration by parts, prove that

$$X_t = \Phi(t)[x + \int_0^t \Phi^{-1}(s)a(s)ds + \int_0^t \Phi^{-1}(s)\sigma(s)dW_s],$$

where  $\Phi(\cdot)$  solves the following ODE system:

$$d\Phi(t) = b(t)\Phi(t)dt, \quad \Phi(0) = 1.$$

2. Now consider the one-dimensional case, and all the coefficients are constants  $a(t) = a > 0$ ,  $b(t) = -b < 0$ , and  $\sigma(t) = \sigma > 0$ . Write down the explicit formula for  $X_t$ .
3. Under the conditions of (2), compute  $E(X_t)$  and  $Var(X_t)$  and their limits when  $t \rightarrow \infty$ .
4. Under the conditions of (2), define  $Y_t = \int_0^t e^{bs}dW_s$ , and  $B_t = \sqrt{2b}Y_{\frac{\ln(t+1)}{2b}}$ . Prove that  $B = (B_t)_{t \geq 0}$  is a Brownian motion, and  $X_t$  has the representation:

$$X_t = xe^{-bt} + \frac{a}{b}(1 - e^{-bt}) + \frac{\sigma e^{-bt}}{\sqrt{2b}}B_{e^{2bt}-1},$$

by using the fact that any zero mean continuous Gaussian process  $B = (B_t)_{t \geq 0}$  with covariance  $E[B_t B_s] = s \wedge t$  is a Brownian motion. (A stochastic process is Gaussian if its any finite dimensional distribution is Gaussian distributed.)

### Exercise 2. (Stochastic Exponential)

Let  $M = (M_t)_{t \geq 0}$  be a continuous local martingale with  $M_0 = 0$ . Its stochastic exponential is defined as  $\mathcal{E}(M)_t = e^{M_t - \frac{1}{2}\langle M \rangle_t}$ .

1. By using the Itô's formula, prove that  $\mathcal{E}(M)$  satisfies the following SDE:

$$d\mathcal{E}(M)_t = \mathcal{E}(M)_t dM_t, \quad \mathcal{E}(M)_0 = 1.$$

Therefore  $\mathcal{E}(M)$  is a nonnegative continuous local martingale.

2. Prove that  $\mathcal{E}(M)$  is a supermartingale.
3. Prove that  $\mathcal{E}(M)_t^{-1} = \mathcal{E}(-M)_t e^{\langle M \rangle_t}$ .
4. Let  $N$  be another a continuous local martingale with  $N_0 = 0$ . Prove that

$$\mathcal{E}(M)_t \mathcal{E}(N)_t = \mathcal{E}(M + N)_t e^{\langle M, N \rangle_t}.$$

## References

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2. Oksendal, Bernt. *Stochastic differential equations: an introduction with applications*. Springer, 2013.
3. Revuz, Daniel, and Marc Yor. *Continuous martingales and Brownian motion*. Springer, 2013.