Applications of Stochastic Calculus in Finance Chapter 9: Intensity-based approach

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1 A simple reduced-form model with constant intensity

Intensity-based (or reduced-form) approach is silent about why a firm defaults, and instead default is exogenously given through a default rate, i.e. the default intensity of a single jump process.

A simple example would be assuming that the default time τ as an exponential random variable independent of the Brownian filtration $\{\mathscr{F}_t\}_{t\geq 0}$, and having constant intensity λ. Then

$$\mathbf{Q}(\tau > t|\mathscr{F}_t) = \mathbf{Q}(\tau > t) = e^{-\lambda t}.$$

Remark 1. (Calibration of default intensity λ). In an ideal situation where the CDS premium is payed continuously until the default time τ, the discounted payoff of the CDS buyer at time 0 is

$$\Pi_b(0) = \mathbf{1}_{\{\tau\leq T_n\}}P(0,\tau)LGD - \int_0^{\tau\wedge T_n} P(0,s)\kappa ds$$

Given the constant default intensity λ, the CDS spread at time $t=0$ is the fixed rate κ such that $\mathbf{E}^{\mathbf{Q}}[\Pi_b(0)] = 0$, which is

$$R_{CDS}(0) = \frac{\mathbf{E}^{\mathbf{Q}}[\mathbf{1}_{\{\tau\leq T_n\}}P(0,\tau)]LGD}{\mathbf{E}^{\mathbf{Q}}[\int_0^{\tau\wedge T_n} P(0,s)ds]}.$$

Note that τ has the intensity $\lambda e^{-\lambda s}$ for $s\geq 0$. Hence,

$$\mathbf{E}^{\mathbf{Q}}[\mathbf{1}_{\{\tau\leq T_n\}}P(0,\tau)] = \int_0^{T_n} \lambda e^{-\lambda s}P(0,s)ds$$

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and

$$\mathbf{E}^{\mathbf{Q}}[\int_0^{\tau\wedge T_n} P(0,s)ds] = \int_0^{\infty} \lambda e^{-\lambda s} \int_0^{s\wedge T_n} P(0,u)duds$$
$$= -\int_0^{\infty} \left(\int_0^{s\wedge T_n} P(0,u)du \right) de^{-\lambda s}$$
$$= \int_0^{T_n} e^{-\lambda s} P(0,s)ds,$$

which yields that

$$R_{CDS}(0) = \lambda LGD.$$

In practice, this is often used to calibrate the default intensity and therefore the default probability in terms of CDS spread.

However, the constant intensity λ is insufficient for most credit risk problems. It would be desirable that the default intensity λ is stochastic. Two types of stochastic intensities will be considered in this Chapter. The first one models the *systematic factor (common noise)* while the second one models the *systemic factor (correlated noise)*.

Moreover, we also need to calculate conditional default probability and conditional price at any time t, which is a delicate issue in the reduced-form framework. Note that in the structural models, conditional values can be simply obtained from the corresponding initial values by replacing V_0 with V_t.

2 $\mathscr{F}_t$-doubly stochastic stopping times

Fix a filtered probability space $(\Omega, \mathscr{G}, \{\mathscr{G}_t\}_{t\geq 0}, \mathbf{P})$, where the filtration $\{\mathscr{G}_t\}_{t\geq 0}$ represents the flow of the complete market information. Let τ be a default time which is a $\mathscr{G}_t$-stopping time. Throughout, we will assume that there exists a sub-filtration $\{\mathscr{F}_t\}_{t\geq 0}$, usually a Brownian filtration, such that $\mathscr{F}_t \subset \mathscr{G}_t$ and $\mathscr{G}_t = \mathscr{F}_t \vee \mathscr{H}_t$, where $\mathscr{H}_t = \sigma(\{\tau \leq u\} : u \leq t)$. *When* $\mathscr{F}_t = \{\emptyset, \Omega\}$*, then* $\mathscr{G}_t = \mathscr{H}_t$.

The following lemma shows that events in $\mathscr{G}_t$ is actually $\mathscr{F}_t$-observable conditional on $\{\tau > t\}$.

Lemma 1. *For every* $A \in \mathscr{G}_t$*, there exists* $B \in \mathscr{F}_t$ *such that*

$$A \cap \{\tau > t\} = B \cap \{\tau > t\}. \tag{1}$$

Proof. Define σ-algebra:

$$\mathscr{G}_t^* = \{A \in \mathscr{G}_t : \text{ there exists } B \in \mathscr{F}_t \text{ such that (1) holds}\}$$

It is obvious that $\mathscr{G}_t^* \subset \mathscr{G}_t$. We prove the other inclusion by showing $\mathscr{F}_t, \mathscr{H}_t \subset \mathscr{G}_t^*$.

Firstly $\mathscr{F}_t \subset \mathscr{G}_t^*$ by simply taking $B = A$ for any $A \in \mathscr{F}_t$. To show that $\mathscr{H}_t \subset \mathscr{G}_t^*$, note that any $A \in \mathscr{H}_t$, $A \cap \{\tau > t\}$ is either $\emptyset$ or $\{\tau > t\}$, so we can take for B either $\emptyset$ or Ω. $\square$

Assumption 1 *There exists a nonnegative $\mathscr{F}_t$-progressively measurable process λ such that*
$$\mathbf{P}(\tau > t|\mathscr{F}_t) = e^{-\int_0^t \lambda_s ds}.$$

From the above assumption, a market participant with access to the partial market information $\mathscr{F}_t$ cannot observe whether default has occurred by time t. In other words, τ is NOT an $\mathscr{F}_t$-stopping time.

The following lemma generalizes Lemma 1 to random variables in the sense that for any r.v. $Y \in \mathscr{G}_t$, there exists $\tilde{Y}_t \in \mathscr{F}_t$ such that $\tilde{Y}_t = Y_t$ conditional on $\{\tau > t\}$. It is called filtration switching formula. *When $\mathscr{F}_t = \{\emptyset, \Omega\}$, it is formula (10) in Chapter 8.*

Lemma 2. *(Filtration switching formula) Suppose that Assumption 1 holds. For any r.v. $Y \in \mathscr{G}_\infty$,*

$$\mathbf{E}[\mathbf{1}_{\{\tau>t\}}Y|\mathscr{G}_t] = \mathbf{1}_{\{\tau>t\}}\frac{\mathbf{E}[\mathbf{1}_{\{\tau>t\}}Y|\mathscr{F}_t]}{\mathbf{P}(\tau > t|\mathscr{F}_t)} = \mathbf{1}_{\{\tau>t\}}e^{\int_0^t \lambda_s ds}\mathbf{E}[\mathbf{1}_{\{\tau>t\}}Y|\mathscr{F}_t]. \qquad (2)$$

Proof. We need to show that

$$\mathbf{E}[\mathbf{1}_{\{\tau>t\}}Y\mathbf{P}(\tau > t|\mathscr{F}_t)|\mathscr{G}_t] = \mathbf{1}_{\{\tau>t\}}\mathbf{E}[\mathbf{1}_{\{\tau>t\}}Y|\mathscr{F}_t].$$

That is, RHS of the above equality is the conditional expectation of $\mathbf{1}_{\{\tau>t\}}Y\mathbf{P}(\tau > t|\mathscr{F}_t)$ on $\mathscr{G}_t$: For any $A \in \mathscr{G}_t$,

$$\mathbf{E}[\mathbf{1}_A\mathbf{1}_{\{\tau>t\}}Y\mathbf{P}(\tau > t|\mathscr{F}_t)] = \mathbf{E}[\mathbf{1}_A\mathbf{1}_{\{\tau>t\}}\mathbf{E}[\mathbf{1}_{\{\tau>t\}}Y|\mathscr{F}_t]].$$

From Lemma 1, there exists $B \in \mathscr{F}_t$ such that $\mathbf{1}_A\mathbf{1}_{\{\tau>t\}} = \mathbf{1}_B\mathbf{1}_{\{\tau>t\}}$. Therefore, it is equivalent to show that

$$\mathbf{E}[\mathbf{1}_B\mathbf{1}_{\{\tau>t\}}Y\mathbf{P}(\tau > t|\mathscr{F}_t)] = \mathbf{E}[\mathbf{1}_B\mathbf{1}_{\{\tau>t\}}\mathbf{E}[\mathbf{1}_{\{\tau>t\}}Y|\mathscr{F}_t]].$$

But this follows by taking the conditional expectations of both sides on $\mathscr{F}_t$:

$$\begin{aligned} RHS &= \mathbf{E}[\mathbf{E}[\mathbf{1}_B\mathbf{1}_{\{\tau>t\}}\mathbf{E}[\mathbf{1}_{\{\tau>t\}}Y|\mathscr{F}_t]|\mathscr{F}_t]] \\ &= \mathbf{E}[\mathbf{1}_B\mathbf{E}[\mathbf{1}_{\{\tau>t\}}|\mathscr{F}_t]\mathbf{E}[\mathbf{1}_{\{\tau>t\}}Y|\mathscr{F}_t]], \end{aligned}$$

and

$$\begin{aligned} LHS &= \mathbf{E}[\mathbf{E}[\mathbf{1}_B\mathbf{1}_{\{\tau>t\}}Y\mathbf{E}[\mathbf{1}_{\{\tau>t\}}|\mathscr{F}_t]|\mathscr{F}_t]] \\ &= \mathbf{E}[\mathbf{1}_B\mathbf{E}[\mathbf{1}_{\{\tau>t\}}Y|\mathscr{F}_t]\mathbf{E}[\mathbf{1}_{\{\tau>t\}}|\mathscr{F}_t]]. \quad \square \end{aligned}$$

We have now the following expressions for conditional default probabilities.

Proposition 1. *Suppose that Assumption 1 holds. Then*

$$\mathbf{P}(\tau > T|\mathscr{G}_t) = \mathbf{1}_{\{\tau>t\}}\mathbf{E}[e^{-\int_t^T \lambda_s ds}|\mathscr{F}_t] \qquad (3)$$

$$\mathbf{P}(t<\tau\leq T|\mathscr{G}_t)=\mathbf{1}_{\{\tau>t\}}\mathbf{E}[1-e^{-\int_t^T\lambda_s ds}|\mathscr{F}_t] \tag{4}$$

$$\mathbf{P}(t<\tau\leq t+\Delta|\mathscr{G}_t)=\mathbf{1}_{\{\tau>t\}}\lambda_t\Delta \quad \textit{for } \Delta=o(1). \tag{5}$$

Proof. We only establish the first one. The other two then follow from (3).

Note that $\mathbf{1}_{\{\tau>T\}}=\mathbf{1}_{\{\tau>t\}}\mathbf{1}_{\{\tau>T\}}$. Then from Lemma 2,

$$\begin{aligned}\mathbf{P}(\tau>T|\mathscr{G}_t)&=\mathbf{E}[\mathbf{1}_{\{\tau>t\}}\mathbf{1}_{\{\tau>T\}}|\mathscr{G}_t]\\&=\mathbf{1}_{\{\tau>t\}}e^{\int_0^t\lambda_s ds}\mathbf{E}[\mathbf{1}_{\{\tau>T\}}|\mathscr{F}_t]\\&=\mathbf{1}_{\{\tau>t\}}e^{\int_0^t\lambda_s ds}\mathbf{E}[\mathbf{E}[\mathbf{1}_{\{\tau>T\}}|\mathscr{F}_T]|\mathscr{F}_t]\\&=\mathbf{1}_{\{\tau>t\}}e^{\int_0^t\lambda_s ds}\mathbf{E}[e^{-\int_0^T\lambda_s ds}|\mathscr{F}_t].\quad\square\end{aligned}$$

Remark 2. Note that in the simple example where τ is an exponential r.v. independent of $\mathscr{F}_t$, then (3) and (4) reduce to formulaes (11) and (12) in Chapter 8, respectively.

Based on Proposition 1, we have the following Doob-Meyer decomposition for the single jump process $\mathbf{1}_{\{\tau\leq t\}}$ for $t\geq 0$.

Proposition 2. *Suppose that Assumption 1 holds. Then the process*

$$N_t=\mathbf{1}_{\{\tau\leq t\}}-\int_0^t\lambda_s\mathbf{1}_{\{\tau>s\}}ds \ \textit{ for } t\geq 0$$

is a $\mathscr{G}_t$-martingale.

Proof. From Proposition 1 and Lemma 2, we obtain that

$$\begin{aligned}\mathbf{E}[N_T|\mathscr{G}_t]=&1-\mathbf{E}[\mathbf{1}_{\{\tau>T\}}|\mathscr{G}_t]-\int_0^t\lambda_s\mathbf{1}_{\{\tau>s\}}ds-\int_t^T\mathbf{E}[\lambda_s\mathbf{1}_{\{\tau>s\}}|\mathscr{G}_t]ds\\=&1-\mathbf{1}_{\{\tau>t\}}\mathbf{E}[e^{-\int_t^T\lambda_s ds}|\mathscr{F}_t]-\int_0^t\lambda_s\mathbf{1}_{\{\tau>s\}}ds\\&-\int_t^T\mathbf{1}_{\{\tau>t\}}e^{\int_0^t\lambda_u du}\mathbf{E}[\lambda_s\mathbf{1}_{\{\tau>s\}}|\mathscr{F}_t]ds.\end{aligned}$$

But note that the last term can be further simplified as

$$\begin{aligned}\int_t^T\mathbf{1}_{\{\tau>t\}}e^{\int_0^t\lambda_u du}\mathbf{E}[\lambda_s\mathbf{1}_{\{\tau>s\}}|\mathscr{F}_t]ds&=\int_t^T\mathbf{1}_{\{\tau>t\}}e^{\int_0^t\lambda_u du}\mathbf{E}[\lambda_s\mathbf{E}[\mathbf{1}_{\{\tau>s\}}|\mathscr{F}_s]|\mathscr{F}_t]ds\\&=\mathbf{1}_{\{\tau>t\}}\mathbf{E}[\int_t^T\lambda_s e^{-\int_t^s\lambda_u du}ds|\mathscr{F}_t]\\&=\mathbf{1}_{\{\tau>t\}}\mathbf{E}[1-e^{-\int_t^T\lambda_s ds}|\mathscr{F}_t].\end{aligned}$$

Hence, $\mathbf{E}[N_T|\mathscr{G}_t]=N_t$. $\square$

We further impose the following assumption, so called Hypothesis H in the literature. *This assumption will be used for the construction of the default time τ later on.*

Assumption 2 *(Hypothesis H) Every $\mathscr{F}_t$-martingale is also a $\mathscr{G}_t$-martingale, or equivalently*[1]*,*

$$\mathbf{P}(\tau > t|\mathscr{F}_\infty) = \mathbf{P}(\tau > t|\mathscr{F}_t) \quad \textit{for } t \geq 0.$$

The equivalent conditions of Hypothesis H can be established as follows.

Suppose that every $\mathscr{F}_t$-martingale is a $\mathscr{G}_t$-martingale. Then, for $X \in \mathscr{F}_\infty$, $M_t := \mathbf{E}[X|\mathscr{F}_t]$ is an $\mathscr{F}_t$-martingale. It follows that

$$\mathbf{E}[X|\mathscr{G}_t] = \mathbf{E}[M_\infty|\mathscr{G}_t] = M_t = \mathbf{E}[X|\mathscr{F}_t]. \tag{6}$$

Since $\mathbf{1}_{\tau>t} \in \mathscr{G}_t$, by the definition of conditional expectation,

$$\mathbf{E}[\mathbf{1}_{\tau>t}X] = \mathbf{E}[\mathbf{1}_{\tau>t}\mathbf{E}[X|\mathscr{F}_t]].$$

However, conditioning on the RHS w.r.t $\mathscr{F}_t$ yields

$$\begin{aligned}\mathbf{E}[\mathbf{1}_{\tau>t}\mathbf{E}[X|\mathscr{F}_t]] &= \mathbf{E}[\mathbf{E}[\mathbf{1}_{\tau>t}|\mathscr{F}_t]\mathbf{E}[X|\mathscr{F}_t]] \\ &= \mathbf{E}[\mathbf{E}[\mathbf{1}_{\tau>t}|\mathscr{F}_t]X].\end{aligned}$$

By taking $X = \mathbf{1}_C$ for $C \in \mathscr{F}_\infty$, we have

$$\mathbf{E}[\mathbf{1}_{\tau>t}\mathbf{1}_C] = \mathbf{E}[\mathbf{E}[\mathbf{1}_{\tau>t}|\mathscr{F}_t]\mathbf{1}_C],$$

that is

$$\mathbf{P}(\tau > t|\mathscr{F}_\infty) = \mathbf{P}(\tau > t|\mathscr{F}_t).$$

On the other hand, suppose the above equality holds. We first prove (6). Note that for $A \in \mathscr{G}_t$, there exists $B \in \mathscr{F}_t$ such that $\mathbf{1}_A\mathbf{1}_{\tau>t} = \mathbf{1}_B\mathbf{1}_{\tau>t}$. Hence, it is sufficient to show that

$$\mathbf{E}[X\mathbf{1}_B\mathbf{1}_{\tau>t}] = \mathbf{E}[\mathbf{E}[X|\mathscr{F}_t]\mathbf{1}_B\mathbf{1}_{\tau>t}].$$

However, by conditioning w.r.t. $\mathscr{F}_\infty$ and $\mathscr{F}_t$ respectively, the LHS is equal to

$$\mathbf{E}[X\mathbf{1}_B\mathbf{1}_{\tau>t}] = \mathbf{E}[X\mathbf{1}_B\mathbf{E}[\mathbf{1}_{\tau>t}|\mathscr{F}_\infty]],$$

and the RHS is equal to

$$\mathbf{E}[\mathbf{E}[X|\mathscr{F}_t]\mathbf{1}_B\mathbf{1}_{\tau>t}] = \mathbf{E}[X\mathbf{1}_B\mathbf{E}[\mathbf{1}_{\tau>t}|\mathscr{F}_t]],$$

which are equal by the assumption. Next, we prove that every $\mathscr{F}_t$-martingale is a $\mathscr{G}_t$-martingale from (6). Let M be an $\mathscr{F}_t$-martingale. Then, for any $T > t$, since $M_T \in \mathscr{F}_\infty$, $\mathbf{E}[M_T|\mathscr{G}_t] = \mathbf{E}[M_T|\mathscr{F}_t] = M_t$, which means M is also a $\mathscr{G}_t$-martingale.

Definition 1. A $\mathscr{G}_t$-stopping time τ that satisfies Assumptions 1 and 2 is called an $\mathscr{F}_t$-doubly stochastic stopping time.

[1] One can make a comparison with Markov property with $\mathscr{F}_\infty$ represents 'future', $\mathscr{F}_t$ represents 'present', $\mathscr{H}_t$ represents 'past' and $\mathscr{G}_t = \mathscr{F}_t \vee \mathscr{H}_t$ represents 'present and past'. Then, the Hypothesis H also implies the following (1) $\mathbf{E}[X|\mathscr{G}_t] = \mathbf{E}[X|\mathscr{F}_t]$; (2) $\mathbf{E}[XY|\mathscr{F}_t] = \mathbf{E}[X|\mathscr{F}_t]\mathbf{E}[Y|\mathscr{F}_t]$ for $X \in \mathscr{F}_\infty$ and $Y \in \mathscr{H}_t$.

Proposition 3. *Suppose that Assumptions 1 and 2 hold. Then the process*

$$N_t = \mathbf{1}_{\{\tau\leq t\}} - \int_0^t \lambda_s \mathbf{1}_{\{\tau>s\}} ds \ \ for\ t\geq 0$$

is an $(\mathscr{F}_\infty \vee \mathscr{H}_t)$-martingale.

Proof. First, analogous to the proof of Lemma 2, it can be proved that the following generalized filtration switching formula holds

$$\mathbf{E}[\mathbf{1}_{\{\tau>t\}}Y|\mathscr{F}_\infty \vee \mathscr{H}_t] = \mathbf{1}_{\{\tau>t\}} \frac{\mathbf{E}[\mathbf{1}_{\{\tau>t\}}Y|\mathscr{F}_\infty]}{\mathbf{P}(\tau>t|\mathscr{F}_\infty)} = \mathbf{1}_{\{\tau>t\}} e^{\int_0^t \lambda_s ds} \mathbf{E}[\mathbf{1}_{\{\tau>t\}}Y|\mathscr{F}_\infty].$$

Then similar to the proof of Proposition 2, we obtain that

$$\begin{aligned}
&\mathbf{E}[N_T|\mathscr{F}_\infty \vee \mathscr{H}_t]\\
=&1-\mathbf{E}[\mathbf{1}_{\{\tau>T\}}|\mathscr{F}_\infty \vee \mathscr{H}_t] - \int_0^t \lambda_s \mathbf{1}_{\{\tau>s\}} ds - \int_t^T \mathbf{E}[\lambda_s \mathbf{1}_{\{\tau>s\}}|\mathscr{F}_\infty \vee \mathscr{H}_t] ds\\
=&1-\mathbf{1}_{\{\tau>t\}} e^{-\int_t^T \lambda_s ds} - \int_0^t \lambda_s \mathbf{1}_{\{\tau>s\}} ds - \int_t^T \mathbf{1}_{\{\tau>t\}} e^{\int_0^t \lambda_u du} \mathbf{E}[\lambda_s \mathbf{1}_{\{\tau>s\}}|\mathscr{F}_\infty] ds\\
=&1-\mathbf{1}_{\{\tau>t\}} e^{-\int_t^T \lambda_s ds} - \int_0^t \lambda_s \mathbf{1}_{\{\tau>s\}} ds - \mathbf{1}_{\{\tau>t\}} \int_t^T e^{\int_0^t \lambda_u du} \lambda_s e^{-\int_0^s \lambda_u du} ds\\
=&1-\mathbf{1}_{\{\tau>t\}} e^{-\int_t^T \lambda_s ds} - \int_0^t \lambda_s \mathbf{1}_{\{\tau>s\}} ds - \mathbf{1}_{\{\tau>t\}} (1-e^{-\int_t^T \lambda_s ds})\\
=&\mathbf{1}_{\{\tau\leq t\}} - \int_0^t \lambda_s \mathbf{1}_{\{\tau>s\}} ds. \quad \square
\end{aligned}$$

3 Intensity-based Approach

Construction of intensity-based models: We want to construct a default time τ which satisfies Assumptions 1 and 2, so it is a $\mathscr{F}_t$-doubly stochastic stopping time.

(1) Given a probability space $(\Omega,\mathscr{G},\mathbf{Q})$, start with a filtration $\{\mathscr{F}_t\}_{t\geq 0}$ (usually Brownian filtration) satisfying the usual conditions, and $\mathscr{F}_\infty \subset \mathscr{G}$.

(2) Let λ_t for $t\geq 0$ be a nonnegative $\mathscr{F}_t$-progressively measurable process such that $\int_0^t \lambda_s ds < \infty$, *a.s.* for $t\geq 0$.

(3) Fix an exponential r.v. E with intensity 1 and independent of $\mathscr{F}_\infty$, and define

$$\tau = \inf\{t\geq 0 : \int_0^t \lambda_s ds \geq E\}.$$

(4) Finally define $\mathscr{G}_t = \mathscr{F}_t \vee \mathscr{H}_t$ where $\mathscr{H}_t = \sigma(\{\tau\leq u\} : u\leq t)$.

Proposition 4. *Under the above canonical construction, the random time τ is a $\mathscr{F}_t$-doubly stochastic stopping time.*

Proof.

$$\mathbf{Q}(\tau > t|\mathscr{F}_\infty) = \mathbf{Q}(E > \int_0^t \lambda_s ds|\mathscr{F}_\infty) = \mathbf{Q}(E > x)|_{x=\int_0^t \lambda_s ds} = e^{-\int_0^t \lambda_s ds}.$$

Conditioning both sides on $\mathscr{F}_t$ then yields

$$\mathbf{Q}(\tau > t|\mathscr{F}_t) = e^{-\int_0^t \lambda_s ds}.$$

Hence, both Assumptions 1 and 2 hold. $\square$

Assumption 3 *Suppose that under the spot measure* $\mathbf{Q}$*, the short rate* r *is* $\mathscr{F}_t$*-progressively measurable, and there exists a nonnegative* $\mathscr{F}_t$*-progressively measurable intensity* λ *such that*

$$\int_0^t (|r_s| + \lambda_s)ds, \ a.s. \, for \, t \geq 0.$$

and Assumptions 1 and 2 hold.

Conditional default probability can be easily computed under intensity-based models.

$$\begin{aligned}\mathbf{Q}(\tau \leq T|\mathscr{G}_t) &= 1 - \mathbf{Q}(\tau > T|\mathscr{G}_t) \\ &= 1 - \mathbf{1}_{\{\tau > t\}}\mathbf{E}[e^{-\int_t^T \lambda_s ds}|\mathscr{F}_t].\end{aligned}$$

Consider a corporate bond which is supposed to pay 1 at maturity T. When defaults occurs for such a corporate bond, the recovery could be delivered either at maturity T (correspond to Merton's model), or at the default time τ (correspond to first-passage-time model).

If the recovery is delivered at maturity T, the discounted payoff at time t is then $\frac{B_t}{B_T}(\mathbf{1}_{\{\tau > T\}} + \delta \mathbf{1}_{\{t < \tau \leq T\}})$ for some recovery rate $\delta \in [0,1]$

Proposition 5. *Under the intensity-based model, if the recovery is delivered at maturity* T*, the conditional value of the corporate bond at time* t *is*

$$P_t = \delta \mathbf{1}_{\{\tau > t\}}\mathbf{E}^{\mathbf{Q}}[e^{-\int_t^T r_s ds}|\mathscr{F}_t] + (1-\delta)\mathbf{1}_{\{\tau > t\}}\mathbf{E}^{\mathbf{Q}}[e^{-\int_t^T (r_s + \lambda_s)ds}|\mathscr{F}_t].$$

where $P(t,T)$ *is the time* t *value of* T*-bond.*

Proof. The arbitrage price the corporate bond at any time $t \leq T$ is

$$\begin{aligned}P_t &= \mathbf{E}^{\mathbf{Q}}[\frac{B_t}{B_T}(\mathbf{1}_{\{\tau > T\}} + \delta \mathbf{1}_{\{t < \tau \leq T\}})|\mathscr{G}_t] \\ &= \mathbf{E}^{\mathbf{Q}}[\frac{B_t}{B_T}(\delta \mathbf{1}_{\{\tau > t\}} + (1-\delta)\mathbf{1}_{\{\tau > T\}})|\mathscr{G}_t]\end{aligned}$$

For the first term, by the Hypothesis H, we have

$$
\begin{aligned}
\mathbf{E}^{\mathbf{Q}}[\frac{B_t}{B_T}\delta\mathbf{1}_{\{\tau>t\}}|\mathscr{G}_t] &= \delta\mathbf{1}_{\{\tau>t\}}\mathbf{E}^{\mathbf{Q}}[\frac{B_t}{B_T}|\mathscr{G}_t] \\
&= \delta\mathbf{1}_{\{\tau>t\}}\mathbf{E}^{\mathbf{Q}}[e^{-\int_t^T r_s ds}|\mathscr{F}_t].
\end{aligned}
$$

Hence, we only need to compute

$$
\begin{aligned}
\mathbf{E}^{\mathbf{Q}}[e^{-\int_t^T r_s ds}\mathbf{1}_{\{\tau>T\}}|\mathscr{G}_t] &= \mathbf{1}_{\{\tau>t\}}e^{\int_0^t \lambda_s ds}\mathbf{E}^{\mathbf{Q}}[e^{-\int_t^T r_s ds}\mathbf{1}_{\{\tau>T\}}|\mathscr{F}_t] \\
&= \mathbf{1}_{\{\tau>t\}}e^{\int_0^t \lambda_s ds}\mathbf{E}^{\mathbf{Q}}[e^{-\int_t^T r_s ds}\mathbf{E}^{\mathbf{Q}}[\mathbf{1}_{\{\tau>T\}}|\mathscr{F}_T]|\mathscr{F}_t] \\
&= \mathbf{1}_{\{\tau>t\}}\mathbf{E}^{\mathbf{Q}}[e^{-\int_t^T (r_s+\lambda_s) ds}|\mathscr{F}_t],
\end{aligned}
$$

where the first equality follows from the filtration switching formula in Lemma 2. □

If the recovery is delivered at default time τ, the discounted payoff at time t is then $\frac{B_t}{B_T}\mathbf{1}_{\{\tau>T\}} + \frac{B_t}{B_\tau}\delta\mathbf{1}_{\{t<\tau\leq T\}}$.

Proposition 6. *Under the intensity-based model, if the recovery is delivered at the default time τ, the conditional value of the corporate bond at time t is*

$$
\begin{aligned}
P_t = &\mathbf{1}_{\{\tau>t\}}\mathbf{E}^{\mathbf{Q}}[e^{-\int_t^T (r_s+\lambda_s) ds}|\mathscr{F}_t] \\
&+ \delta\mathbf{1}_{\{\tau>t\}}\int_t^T \mathbf{E}^{\mathbf{Q}}[\lambda_u e^{-\int_t^u (r_s+\lambda_s) ds}|\mathscr{F}_t]du.
\end{aligned}
$$

Proof. The arbitrage price the corporate bond at any time $t \leq T$ is

$$
\begin{aligned}
P_t &= \mathbf{E}^{\mathbf{Q}}[\frac{B_t}{B_T}\mathbf{1}_{\{\tau>T\}} + \frac{B_t}{B_\tau}\delta\mathbf{1}_{\{t<\tau\leq T\}}|\mathscr{G}_t] \\
&= \mathbf{E}^{\mathbf{Q}}[e^{-\int_t^T r_s ds}\mathbf{1}_{\{\tau>T\}}|\mathscr{G}_t] + \delta\mathbf{E}^{\mathbf{Q}}[e^{-\int_t^\tau r_s ds}\mathbf{1}_{\{t<\tau\leq T\}}|\mathscr{G}_t]
\end{aligned}
$$

The first conditional expectation has been computed in the last proposition. Hence, we only need to compute the second one:

$$
\mathbf{E}^{\mathbf{Q}}[e^{-\int_t^\tau r_s ds}\mathbf{1}_{\{t<\tau\leq T\}}|\mathscr{G}_t] = \mathbf{E}^{\mathbf{Q}}[\mathbf{E}^{\mathbf{Q}}[e^{-\int_t^\tau r_s ds}\mathbf{1}_{\{t<\tau\leq T\}}|\mathscr{F}_\infty \vee \mathscr{H}_t]|\mathscr{G}_t]
$$

Similar to (4) in Proposition 1, it can be shown that for $t \leq u$,

$$
\mathbf{Q}(t < \tau \leq u|\mathscr{F}_\infty \vee \mathscr{H}_t) = \mathbf{1}_{\{\tau>t\}}(1 - e^{-\int_t^u \lambda_s ds})
$$

so the conditional density of τ on $\mathscr{F}_\infty \vee \mathscr{H}_t$ is

$$
1_{\{\tau>t\}}\lambda_u e^{-\int_t^u \lambda_s ds}
$$

for $u \geq t$. Therefore,

$$\begin{aligned}\mathbf{E}^{\mathbf{Q}}[e^{-\int_t^\tau r_s ds}\mathbf{1}_{\{t<\tau\le T\}}|\mathscr{G}_t] &= \mathbf{E}^{\mathbf{Q}}[\int_t^T 1_{\{\tau>t\}}\lambda_u e^{-\int_t^u \lambda_s ds}e^{-\int_t^u r_s ds}du|\mathscr{G}_t] \\ &= 1_{\{\tau>t\}}\int_t^T \mathbf{E}^{\mathbf{Q}}[\lambda_u e^{-\int_t^u (r_s+\lambda_s)ds}|\mathscr{G}_t]du\end{aligned}$$

We are left to show that

$$1_{\{\tau>t\}}\mathbf{E}^{\mathbf{Q}}[\lambda_u e^{-\int_t^u (r_s+\lambda_s)ds}|\mathscr{G}_t] = 1_{\{\tau>t\}}\mathbf{E}^{\mathbf{Q}}[\lambda_u e^{-\int_t^u (r_s+\lambda_s)ds}|\mathscr{F}_t]$$

which follows from Assumption 2. Indeed, it is sufficient to show that for any $A\in\mathscr{G}_t$,

$$\mathbf{E}^{\mathbf{Q}}[\mathbf{1}_A 1_{\{\tau>t\}}\mathbf{E}^{\mathbf{Q}}[\lambda_u e^{-\int_t^u (r_s+\lambda_s)ds}|\mathscr{F}_t]] = \mathbf{E}^{\mathbf{Q}}[\mathbf{1}_A 1_{\{\tau>t\}}\lambda_u e^{-\int_t^u (r_s+\lambda_s)ds}]$$

From Lemma 1, there exits $B\in\mathscr{F}_t$ such that $\mathbf{1}_A 1_{\{\tau>t\}} = \mathbf{1}_B 1_{\{\tau>t\}}$. Therefore, it is equivalent to show that

$$\mathbf{E}^{\mathbf{Q}}[\mathbf{1}_B 1_{\{\tau>t\}}\mathbf{E}^{\mathbf{Q}}[\lambda_u e^{-\int_t^u (r_s+\lambda_s)ds}|\mathscr{F}_t]] = \mathbf{E}^{\mathbf{Q}}[\mathbf{1}_B 1_{\{\tau>t\}}\lambda_u e^{-\int_t^u (r_s+\lambda_s)ds}]$$

Take conditional expectations of RHS first on $\mathscr{F}_\infty$ and then on $\mathscr{F}_t$, and of LHS on $\mathscr{F}_t$:

$$\begin{aligned}RHS =&\mathbf{E}^{\mathbf{Q}}[\mathbf{E}^{\mathbf{Q}}[\mathbf{1}_B 1_{\{\tau>t\}}\lambda_u e^{-\int_t^u (r_s+\lambda_s)ds}|\mathscr{F}_\infty]] \\ =&\mathbf{E}^{\mathbf{Q}}[\mathbf{1}_B\mathbf{Q}(\tau>t|\mathscr{F}_\infty)\lambda_u e^{-\int_t^u (r_s+\lambda_s)ds}] \\ =&\mathbf{E}^{\mathbf{Q}}[\mathbf{1}_B\mathbf{Q}(\tau>t|\mathscr{F}_t)\lambda_u e^{-\int_t^u (r_s+\lambda_s)ds}] \\ =&\mathbf{E}^{\mathbf{Q}}[\mathbf{E}^{\mathbf{Q}}[\mathbf{1}_B\mathbf{Q}(\tau>t|\mathscr{F}_t)\lambda_u e^{-\int_t^u (r_s+\lambda_s)ds}|\mathscr{F}_t]] \\ =&\mathbf{E}^{\mathbf{Q}}[\mathbf{1}_B\mathbf{Q}(\tau>t|\mathscr{F}_t)\mathbf{E}^{\mathbf{Q}}[\lambda_u e^{-\int_t^u (r_s+\lambda_s)ds}|\mathscr{F}_t]],\end{aligned}$$

and

$$\begin{aligned}LHS =&\mathbf{E}^{\mathbf{Q}}[\mathbf{E}^{\mathbf{Q}}[\mathbf{1}_B 1_{\{\tau>t\}}\mathbf{E}^{\mathbf{Q}}[\lambda_u e^{-\int_t^u (r_s+\lambda_s)ds}|\mathscr{F}_t]|\mathscr{F}_t]] \\ =&\mathbf{E}^{\mathbf{Q}}[\mathbf{1}_B\mathbf{Q}(\tau>t|\mathscr{F}_t)\mathbf{E}^{\mathbf{Q}}[\lambda_u e^{-\int_t^u (r_s+\lambda_s)ds}|\mathscr{F}_t]]. \quad \square\end{aligned}$$

Remark 3. Note that in the above calculation, using filtration switching formula directly on $\mathbf{E}^{\mathbf{Q}}[e^{-\int_t^\tau r_s ds}\mathbf{1}_{\{t<\tau\le T\}}|\mathscr{G}_t]$ does not help:

$$\mathbf{E}^{\mathbf{Q}}[e^{-\int_t^\tau r_s ds}\mathbf{1}_{\{t<\tau\le T\}}|\mathscr{G}_t] = \mathbf{1}_{\{\tau>t\}}e^{\int_0^t \lambda_s ds}\mathbf{E}^{\mathbf{Q}}[e^{-\int_t^\tau r_s ds}\mathbf{1}_{\{t<\tau\le T\}}|\mathscr{F}_t],$$

as this needs the conditional density of τ on $\mathscr{F}_t$, which is difficult to obtain.

4 Reduced-form models with correlated intensities

We model a pair of interactive default times τ and $\bar{\tau}$, where τ represents the default time of the reference entity, and $\bar{\tau}$ represents the default time of the CDS seller. Another interpretation is $(\tau,\bar{\tau})$ are a pair of default times for a basket CDS with two reference names.

Assumption 4 *Let $(\tau,\bar{\tau})$ be a pair of non-negative random variables defined on a complete probability space $(\Omega,\mathscr{G},\mathbf{Q})$, and $\{\mathscr{G}_t\}_{t\geq 0}$ be the natural filtration of $(H_t,\bar{H}_t)=\left(\mathbf{1}_{\{\tau\leq t\}},\mathbf{1}_{\{\bar{\tau}\leq t\}}\right)$, $t\geq 0$. i.e. $\mathscr{G}_t=\sigma(H_s,\bar{H}_s,s\leq t)$, such that*

$$M_t:=H_t-\int_0^t \mathbf{1}_{\{\tau>s\}}\lambda_s ds,\ t\geq 0,$$

and

$$\bar{M}_t:=\bar{H}_t-\int_0^t \mathbf{1}_{\{\bar{\tau}>s\}}\bar{\lambda}_s ds,\ t\geq 0,$$

are both $(\mathscr{G}_t,\mathbf{Q})$-martingales. Moreover, λ and $\bar{\lambda}$ are given by

$$\lambda_s=a_1+a_2\mathbf{1}_{\{\bar{\tau}\leq s\}}, \tag{7}$$

$$\bar{\lambda}_s=\bar{a}_1+\bar{a}_2\mathbf{1}_{\{\tau\leq s\}}, \tag{8}$$

for constants $a_1,\bar{a}_1>0$ and $a_2,\bar{a}_2\geq 0$. Finally, we assume that $\mathbf{Q}(\tau>0)=\mathbf{Q}(\bar{\tau}>0)=1$.

We aim to calculate the joint distribution of $(\tau,\bar{\tau})$ using the Girsanov's theorem introduced in Chapter 8. The difficulty herein is the looping feature of the two intensity processes. If τ happens (the reference entity defaults), the intensity of $\bar{\tau}$ will increase from $\bar{a}_1$ to $\bar{a}_1+\bar{a}_2$. Similarly, if $\bar{\tau}$ happens (the CDS seller defaults), the intensity of τ will increase from a_1 to a_1+a_2. By changing the probability measure $\mathbf{Q}$ to another probability measure, we open this loop, which will in turn facilitate the calculation of the joint distribution of $(\tau,\bar{\tau})$.

Theorem 1. *Suppose that Assumption 4 holds. Then, for $T,U\geq 0$,*

$$\mathbf{Q}(\tau>T)=\frac{a_2e^{-(a_1+\bar{a}_1)T}-\bar{a}_1e^{-(a_1+a_2)T}}{a_2-\bar{a}_1}; \tag{9}$$

$$\mathbf{Q}(\bar{\tau}>U)=\frac{\bar{a}_2e^{-(\bar{a}_1+a_1)U}-a_1e^{-(\bar{a}_1+\bar{a}_2)U}}{\bar{a}_2-a_1}. \tag{10}$$

Proof. We first prove (9). For $t\in[0,T]$, let

$$Z_t=\mathbf{1}_{\{\tau>t\}}e^{\int_0^t \mathbf{1}_{\{\tau>s\}}\lambda_s ds}$$

Then, Theorem 8 in Chapter 8 (with $\mu=1$) implies that Z is an $(\mathscr{G}_t,\mathbf{Q})$-martingale. Define a new probability measure $\mathbf{Q}^1$ on $\mathscr{G}_T$ by

$$\frac{d\mathbf{Q}^1}{d\mathbf{Q}} = Z_T.$$

Then, Theorem 9 in Chapter 8 (with $\mu = 1$) implies that $H_t = \mathbf{1}_{\{\tau \le t\}}$, $t \in [0,T]$, is an $(\mathscr{G}_t, \mathbf{Q}^1)$-positive martingale, which is 0, $\mathbf{Q}^1$-a.e.

By Bayes' formula, we have

$$\begin{aligned}
\mathbf{Q}(\tau > T) &= \mathbf{E}^{\mathbf{Q}}[Z_T e^{-\int_0^T \mathbf{1}_{\{\tau > s\}}\lambda_s ds}] \\
&= \mathbf{E}^{\mathbf{Q}}[Z_T e^{-\int_0^T \lambda_s ds}] \\
&= \mathbf{E}^{\mathbf{Q}^1}[e^{-\int_0^T \lambda_s ds}] \\
&= \mathbf{E}^{\mathbf{Q}^1}[e^{-\int_0^T a_1 + a_2 \mathbf{1}_{\{\bar{\tau} \le s\}} ds}]
\end{aligned}$$

Note that under $\mathbf{Q}^1$, for $s \in [0,T]$, $\bar{\lambda}_s = \bar{a}_1 + \bar{a}_2 \mathbf{1}_{\{\tau \le s\}} = \bar{a}_1$, $\mathbf{Q}^1$-a.e., so we further have

$$\begin{aligned}
\mathbf{Q}(\tau > T) &= e^{-a_1 T}\left(\mathbf{E}^{\mathbf{Q}^1}[\mathbf{1}_{\{\bar{\tau} > T\}}] + \mathbf{E}^{\mathbf{Q}^1}[\mathbf{1}_{\{\bar{\tau} \le T\}} e^{-a_2(T - \bar{\tau})}]\right) \\
&= e^{-a_1 T}\left(e^{-\bar{a}_1 T} + \int_0^T e^{-a_2(T-s)} \bar{a}_1 e^{-\bar{a}_1 s} ds\right) \\
&= e^{-a_1 T} \frac{(a_2 - \bar{a}_1)e^{-\bar{a}_1 T} + \bar{a}_1 e^{-\bar{a}_1 T} - \bar{a}_1 e^{-a_2 T}}{a_2 - \bar{a}_1},
\end{aligned}$$

which proves (9).

To prove (10), with

$$\bar{Z}_t = \mathbf{1}_{\{\bar{\tau} > t\}} e^{\int_0^t \mathbf{1}_{\{\bar{\tau} > s\}} \bar{\lambda}_s ds}$$

for $t \in [0,U]$, we define a new probability measure $\bar{\mathbf{Q}}^1$ on $\mathscr{G}_T$ by

$$\frac{d\bar{\mathbf{Q}}^1}{d\mathbf{Q}} = \bar{Z}_U.$$

Then, $\bar{H}_t = \mathbf{1}_{\{\bar{\tau} \le t\}}$, $t \in [0,U]$, is an $(\mathscr{G}_t, \bar{\mathbf{Q}}^1)$-positive martingale, which is 0, $\bar{\mathbf{Q}}^1$-a.e. The rest of the proof follows along the same argument as the first case. □

An interesting observation is that the reference entity's survival probability $\mathbf{Q}\{\tau > T\}$ is independent of $\bar{a}_2$. This is because for $s \in [0,T]$, $\{\tau \le s\}$ is an zero event on the set $\{\tau > T\}$, so $\bar{\lambda}_s$ will keep as the constant $\bar{a}_1$ and is independent of $\bar{a}_2$. Similarly, the CDS seller's survival probability $\mathbf{Q}\{\bar{\tau} > U\}$ is independent of a_2.

When $a_2 = 0$, i.e. the CDS seller's default will not affect the reference entity, then (9) reduces to $\mathbf{Q}(\tau > T) = e^{-aT}$. When $a_2 = \bar{a}_1$, then the L'Hopital's rule implies that

$$\mathbf{Q}(\tau > T) = (1 + \bar{a}_1 T)e^{-(a_1 + \bar{a}_1)T}.$$

Theorem 2. *Suppose that Assumption 4 holds. Then, for $U > T \ge 0$,*

$$\mathbf{Q}(\tau > T, \bar{\tau} > U) = e^{-(a_1+\bar{a}_1)T}\,\frac{\bar{a}_2 e^{-(a_1+\bar{a}_1)(U-T)} - a_1 e^{-(\bar{a}_1+\bar{a}_2)(U-T)}}{\bar{a}_2 - a_1}. \qquad (11)$$

Proof. We take conditional expectation on $\mathscr{G}_T$ and get

$$\mathbf{Q}(\tau > T, \bar{\tau} > U) = \mathbf{E}^{\mathbf{Q}}\left[\mathbf{1}_{\{\tau > T\}}\mathbf{Q}(\bar{\tau} > U|\mathscr{G}_T)\right].$$

Note that

$$\mathbf{Q}(\bar{\tau} > U|\mathscr{G}_T) = \mathbf{1}_{\{\bar{\tau} > T\}}\left(\mathbf{1}_{\{\tau > T\}}\mathbf{Q}(\bar{\tau} > U - T) + \mathbf{1}_{\{\tau \le T\}}e^{-(\bar{a}_1+\bar{a}_2)(U-T)}\right).$$

Hence,

$$\begin{aligned}
\mathbf{Q}(\tau > T, \bar{\tau} > U) &= \mathbf{E}^{\mathbf{Q}}[\mathbf{1}_{\{\tau > T\}}\mathbf{1}_{\{\bar{\tau} > T\}}] \times \mathbf{Q}(\bar{\tau} > U - T)\\
&= \mathbf{E}^{\mathbf{Q}}[\mathbf{1}_{\{\tau > T\}}\mathbf{1}_{\{\bar{\tau} > T\}}e^{\int_0^T \mathbf{1}_{\{\bar{\tau}>s\}}\bar{\lambda}_s ds}e^{-\int_0^T \mathbf{1}_{\{\bar{\tau}>s\}}\bar{\lambda}_s ds}] \times \mathbf{Q}(\bar{\tau} > U - T)\\
&= \mathbf{E}^{\mathbf{Q}}[\mathbf{1}_{\{\tau > T\}}\bar{Z}_T e^{-\int_0^T \bar{\lambda}_s ds}] \times \mathbf{Q}(\bar{\tau} > U - T)\\
&= \mathbf{E}^{\bar{\mathbf{Q}}^1}[\mathbf{1}_{\{\tau > T\}}e^{-\int_0^T \bar{a}_1 + \bar{a}_2\mathbf{1}_{\{\tau\le s\}}ds}] \times \mathbf{Q}(\bar{\tau} > U - T).
\end{aligned}$$

Note that under $\bar{\mathbf{Q}}^1$, for $s \in [0,T]$, $\lambda_s = a_1 + a_2\mathbf{1}_{\{\bar{\tau}\le s\}} = a_1$, $\bar{\mathbf{Q}}^1$-a.e., so we further have

$$\mathbf{Q}(\tau > T, \bar{\tau} > U) = \bar{\mathbf{Q}}^1(\tau > T)e^{-\bar{a}_1 T} \times \mathbf{Q}(\bar{\tau} > U - T) = e^{-(a_1+\bar{a}_1)T} \times \mathbf{Q}(\bar{\tau} > U - T),$$

which completes the proof. □

Note that the above joint probability is independent of a_2. This is because on the set $\{\bar{\tau} > U\}$, we also have $\bar{\tau} > T$, so $\{\bar{\tau} \le s\}$ for $s \in [0,U]$ is a zero event on $\{\bar{\tau} > U\}$. Thus, λ will keep as the constant a_1, and is independent of a_2.

Exercise 1. For $T > U \ge 0$, show the joint probability

$$\mathbf{Q}(\tau > T, \bar{\tau} > U) = e^{-(a_1+\bar{a}_1)U}\mathbf{Q}(\tau > T - U).$$

Thus, $(\tau, \bar{\tau})$ admit the joint density

$$f_{(\tau,\bar{\tau})}(T,U) = \begin{cases} (\bar{a}_1 + \bar{a}_2)e^{-(\bar{a}_1+\bar{a}_2)U}a_1 e^{(\bar{a}_2 - a_1)T}, & \text{if } U > T;\\ (a_1 + a_2)e^{-(a_1+a_2)T}\bar{a}_1 e^{(a_2 - \bar{a}_1)U}, & \text{if } U < T. \end{cases} \qquad (12)$$

Exercise 2. (Total hazard construction of correlated default times [1])

In general, when there are M default times $(\tau^i)_{1\le i\le M}$ as in the basket CDS, we may simulate those default times via the so called *total hazard construction*. It is based on the following simple observation: For a single jump process $H_t = \mathbf{1}_{\{\tau\le t\}}$, $t \ge 0$, with a constant intensity $a > 0$, we can simulate its associated jump time τ as

$$\hat{\tau} = \inf\{t \ge 0 : \Lambda_t \ge E\}$$

where E is a standard exponential r.v. and $\Lambda_t := \int_0^t ads$, the so called hazard of τ. Then, $\hat{\tau} = \tau$ in distribution. Indeed, for any $T > 0$,

$$\mathbf{Q}(\hat{\tau} > T) = \mathbf{Q}(\Lambda_T < E) = \mathbf{Q}(aT < E) = e^{-aT}.$$

Assumption 5 *Let $(\tau^i)_{1\le i\le M}$ be a sequence of non-negative random variables defined on a complete probability space $(\Omega, \mathscr{G}, \mathbf{Q})$, and $\{\mathscr{G}_t\}_{t\ge 0}$ be the natural filtration of $H_t^i = \mathbf{1}_{\{\tau^i \le t\}}$, $t \ge 0$, $1 \le i \le M$, i.e. $\mathscr{G}_t = \sigma(H_s^i, s \le t, 1 \le i \le M)$, such that*

$$M_t^i := H_t^i - \int_0^t \mathbf{1}_{\{\tau^i > s\}} \lambda_s^i ds,\ t \ge 0,\ 1 \le i \le M,$$

are $(\mathscr{G}_t, \mathbf{Q})$-martingales. Moreover, λ^i, $1 \le i \le M$, are given by

$$\lambda_s^i = a_{ii} + \sum_{j \ne i} a_{ij} \mathbf{1}_{\{\tau^j \le s\}}, \tag{13}$$

with a constant matrix $A = (a_{ij})_{1\le i,j\le M}$ satisfying $a_{ij} > 0$ and $a_{ij} \ge 0$ for $i \ne j$. Finally, we assume that $\mathbf{Q}(\tau^i > 0) = 1$ for $1 \le i \le M$.

Note that there are $M!$ orders of $(\tau^i)_{1\le i\le M}$. Let us consider one of the orders: $\{\tau^1 < \tau^2 < \cdots < \tau^M\}$. In such a case, we construct the M default times as follows.

Step 0: Let $(E^i)_{1\le i\le M}$ be a sequence of mutually independent standard exponential random variables.

Step 1: Since there is no default prior to τ^1, its intensity is a_{11} and, therefore,

$$\hat{\tau}^1 = \inf\left\{ t \ge 0 : \int_0^t a_{11} ds \ge E^1 \right\} = \frac{E^1}{a_{11}}.$$

Step 2: Prior to τ^2, τ^1 has already occurred, so the intensity of τ^2 is $a_{22} + a_{21}\mathbf{1}_{\{\tau^1 \le s\}}$ and it can be constructed by

$$\begin{aligned}\hat{\tau}^2 =& \inf\left\{ t \ge 0 : \int_0^{\hat{\tau}^1} a_{22} ds + \int_{\hat{\tau}^1}^t (a_{22} + a_{21}) ds \ge E^2 \right\} \\ =& \frac{E^1}{a_{11}} + \frac{a_{22}}{a_{22} + a_{21}} \left(\frac{E^2}{a_{22}} - \frac{E^1}{a_{11}} \right).\end{aligned}$$

Step k: In general, we construct τ^k for $1 \le k \le M$ recursively by

$$\hat{\tau}^k = \inf\left\{ t \ge 0 : \sum_{i=1}^{k-1} \int_{\hat{\tau}^{i-1}}^{\hat{\tau}^i} (a_{kk} + \sum_{j=1}^{i-1} a_{kj}) ds + \int_{\hat{\tau}^{k-1}}^t (a_{kk} + \sum_{j=1}^{k-1} a_{kj}) ds \ge E^k \right\},$$

where $\hat{\tau}^0 := 0$ and $\sum_{i=1}^0 := 1$.

Then, on the set $\{\tau^1 < \tau^2 < \cdots < \tau^M\}$, we have $(\tau_i)_{1\le i\le M} = (\hat{\tau}_i)_{1\le i\le M}$ in distribution. The same result holds for the other $(M!-1)$ situations via permutation.

Consider the intensity model in section 4. Thus, $M = 2$, $a_{11} = a_1$, $a_{12} = a_2$, $a_{22} = \bar{a}_1$ and $a_{21} = \bar{a}_2$. Show that $(\tau, \bar{\tau})$ have the expressions (equal in distribution)

$$\tau = \begin{cases} \dfrac{E^1}{a_{11}}, & \text{if } \frac{E^1}{a_{11}} < \frac{E^2}{a_{22}}; \\ \dfrac{E^2}{a_{22}} + \dfrac{a_{11}}{a_{11}+a_{12}}(\dfrac{E^1}{a_{11}} - \dfrac{E^2}{a_{22}}), & \text{if } \frac{E^1}{a_{11}} > \frac{E^2}{a_{22}}, \end{cases}$$

and

$$\bar{\tau} = \begin{cases} \dfrac{E^1}{a_{11}} + \dfrac{a_{22}}{a_{22}+a_{21}}(\dfrac{E^2}{a_{22}} - \dfrac{E^1}{a_{11}}), & \text{if } \frac{E^1}{a_{11}} < \frac{E^2}{a_{22}}; \\ \dfrac{E^2}{a_{22}}, & \text{if } \frac{E^1}{a_{11}} > \frac{E^2}{a_{22}}. \end{cases}$$

Equivalently, on $\{\tau < \bar{\tau}\}$,

$$\begin{cases} E^1 = a_{11}\tau; \\ E^2 = a_{22}\tau + (\bar{\tau} - \tau)(a_{22} + a_{21}), \end{cases}$$

and on the set $\{\tau > \bar{\tau}\}$,

$$\begin{cases} E^2 = a_{22}\bar{\tau}; \\ E^1 = a_{11}\bar{\tau} + (\tau - \bar{\tau})(a_{11} + a_{12}). \end{cases}$$

The above total hazard construction procedure also yields the joint density of default times. To see this, since (E^1, E^2) have the joint density $f_{(E^1,E^2)}(t,u) = e^{-(t+u)}$, and the Jacobi determinant of (E^1, E^2) with respect to $(\tau, \bar{\tau})$ is

$$\left| \frac{\partial(E^1, E^2)}{\partial(\tau, \bar{\tau})} \right| = \begin{cases} a_{11}(a_{22} + a_{21}), \text{ if } \tau < \bar{\tau}; \\ a_{22}(a_{11} + a_{12}), \text{ if } \tau > \bar{\tau}. \end{cases}$$

Hence, we can obtain the joint density of $(\tau, \bar{\tau})$ by

$$f_{(\tau,\bar{\tau})}(T,U) = f_{(E^1,E^2)}(t,u) \left| \frac{\partial(E^1, E^2)}{\partial(\tau, \bar{\tau})} \right|,$$

which yields exactly the same joint density formula (12).

References

1. Yu, Fan. Correlated defaults in intensity-based models. *Mathematical Finance* 17(2) (2007): 155-173.