15.12.2023. Stoch. modelling HW 3.

① Voter model

$c(\eta,\eta^i)=\sum_{j\neq i} q(j,i)\,(\eta(i)(1-\eta(j))+\eta(j)(1-\eta(i))),\ \forall i\in\Lambda$

a) The process will not be ergodic, because if Λ is finite and $q(i,j)$ is irreducible there is not unique stationary distribution, because there are two absorbing states: $\eta\equiv 0$ and $\eta\equiv 1$.

$\Rightarrow$ any linear combination of these states is a stationary distribution.

If $q(i,j)$ is not irreducible, then not all people are communicating in one big group, but instead there are some small groups (connected components of q), each of which should be treated as a separate subgraph Λ_j, $j=1\ldots\#$components. So for each component we have $\eta_{(j)}\equiv 0$ and $\eta_{(j)}\equiv 1$, where $\eta_{(j)}$ denotes the coordinates of points in the j-th connected component. So any combination $\eta=\eta_{(1)}\ldots\eta_{(j)}$, where $\eta(i)\equiv 0$ or $\eta(i)\equiv 1$ is a stationary distribution (this means that in each group people should move to consensus, and the consensus may be either 0 or 1).

b) $q(j,i)=1$.

$N_t:=\sum_{i=1}^{L}\eta_t(i)$

Derive the transition rates $g(n,m)$ for $n,m\in\{0,\ldots L\}$ for the process $(N_t: t\geq 0)$.

We know $c(\eta,\eta^i)=\sum_{\substack{i\in\Lambda\\ j\neq i}} \underbrace{q(j,i)}_{=1}\,(\eta(i)(1-\eta(j))+\eta(j)(1-\eta(i)))$

And we know the formula for the generator of jump process:

$$\Rightarrow (Lf)(N(\eta))=\sum_{i\in\Lambda} c(\eta,\eta^i)[f(N(\eta^i))-f(N(\eta))]=$$

$$=\sum_{i\in\Lambda}\sum_{j\neq i}(\eta(i)(1-\eta(j))+(1-\eta(i))\eta(j))\,[f(N(\eta^i))-f(N(\eta))]=$$

$$=\sum_{\substack{i\in\Lambda\\ \eta(i)=0}}\sum_{j\neq i}\eta(j)[f(N+1)-f(N)]+\sum_{\substack{i\in\Lambda\\ \eta(i)=1}}\sum_{j\neq i}(1-\eta(j))[f(N-1)-f(N)]=$$

$$=\sum_{\substack{i\in\Lambda\\ \eta(i)=0}} N\cdot[f(N+1)-f(N)]+\sum_{\substack{i\in\Lambda\\ \eta(i)=1}}\Big(\underbrace{\sum_{j\neq i}1}_{=L-1}-\underbrace{\sum_{j\neq i}\eta(j)}_{=N-1}\Big)[f(N-1)-f(N)]=$$

$$=\boxed{(L-N)N[f(N+1)-f(N)]+N\cdot(L-N)[f(N-1)-f(N)]}$$

So both transition rates are symmetric and equal to $(L-N)N$.

And from this we see that only cases when both transition rates are zero - it is when $N=0$ or $N=L$, so all people have opinion 0 or 1.

$\Rightarrow$ there are two ~~states~~ absorbing states $\eta\equiv 0$ and $\eta\equiv 1$ and any linear combination of them is a stationary distribution.

$$G = \begin{pmatrix} 0 & \cdots & 0 & & \\ (L-N)N & -2N(L-N) & N(L-N) & 0 & \\ & \ddots & \ddots & \ddots & \\ 0 & & N(L-N) & -2N(L-N) & N(L-N) \\ & & & 0 & 0 \end{pmatrix}$$ – generator.

If we solve $\langle \pi_k | G = \langle 0 |$,
we will have $\begin{cases} \pi_2 = 0 \\ -2\pi_1 + \pi_2 = 0 \\ \pi_{k-1} - 2\pi_k + \pi_{k+1} = 0, \; k = 2 \dots N-2. \\ \pi_{L-1} = 0; \; \pi_{L-2} - 2\pi_{L-1} = 0. \end{cases}$

$\Rightarrow \pi_k = C_1 + C_2 k.$

$\pi_1 = 0 \Rightarrow C_1 = 0. \Rightarrow \pi_k = C_2 k;$

$\pi_{L-1} = 0 \Rightarrow \pi_k = 0; \; k = 1 \dots L-1.$

$\Rightarrow$ stationary distributions are any $(a; 0 \dots 0; 1-a)$.

c) Give the state space S and the absorbing states of the process $(N_t; t \geq 0)$ and write master equation $p_t(i) := P(N_t = i) \; \forall i \in S$.
Give a formula for all stationary distributions.
All stationary distributions we have found looking for π_k: $\langle \pi_k | G = 0$.
master equation: $\frac{d}{dt} \langle \pi_t | = \langle \pi_t | G$

$$\Rightarrow \begin{cases} \pi_0' = N(L-N)\pi_1 \\ \pi_1' = -2N(L-N)\pi_1 + N(L-N)\pi_2 \\ \pi_k' = N(L-N)\pi_{k-1} - 2N(L-N)\pi_k + N(L-N)\pi_{k+1} \\ \pi_{L-1}' = N(N-L)\pi_{L-2} - 2N(L-N)\pi_{L-1} \\ \pi_L' = N(N-L)\pi_{L-1}. \end{cases}$$

d) Use symmetry of rates $g(n,m)$ to argue that EN_t doesn't change in time.
~~$EN_t = E(N_t | N_t)$~~ $P(N_{t+1} = N) = \underbrace{P(N_{t+1} = N | N_t = N-1)}_{=1/2} \cdot P(N_t = N-1) + \underbrace{P(N_{t+1} = N | N_t = N+1)}_{=1/2} \cdot P(N_t = N+1)$

because going from N to $N-1$ is $N+1$ has equal probabilities,
then $EN_t \rightsquigarrow \frac{1}{2}E(N_t+1) + \frac{1}{2}E(N_t-1) = EN_t \Rightarrow EN_t$ doesn't change in time.

So if $N_0 = \frac{L}{2}$, then $EN_t = EN_0 = \frac{L}{2}$

~~for example~~ $\Rightarrow$ If we look for a stationary distribution, it will be of the form $(a; 0 \dots 0; 1-a)$, but EN_t should be equal to $\frac{L}{2}$

$\Rightarrow (1-a) \cdot L = \frac{L}{2} \Rightarrow 1 - a = \frac{1}{2} \Rightarrow a = \frac{1}{2}$

$\Rightarrow$ stationary distribution is $(\frac{1}{2}; 0 \dots 0; \frac{1}{2})$.