ST909

2017 Exam

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Question 1

Let $(\Omega, \mathcal{F}, \mathbb{Q})$ be a probability space supporting a 1-dimensional Brownian motion W and let $\{\mathcal{F}_t\}_{t\geq 0}$ denote the augmented natural filtration of W.

(a) Suppose that $\mathbb{F}$ and $\mathbb{Q}$ are equivalent probability measures with respect to $\mathcal{F}$. Show that M is an $(\{\mathcal{F}_t\}, \mathbb{F})$ martingale if and only if ρM is an $(\{\mathcal{F}_t\}, \mathbb{Q})$ martingale where

$$\rho_t := \left.\frac{d\mathbb{F}}{d\mathbb{Q}}\right|_{\mathcal{F}_t}.$$

[20%]

(b) State Girsanov's Theorem for a one-dimensional Brownian motion.

[20%]

(c) Consider a term structure model specified under the measure $\mathbb{Q}$ for which the value at time t of a pure discount bond paying unity at time T is of the form

$$D_{tT} = \exp\left(A_{tT} - B_{tT}y_t - C_{tT}y_t^2\right), \ t \leq T,$$

where for each T, A_{tT} and B_{tT} are deterministic functions of time $t \leq T$, and the process y satisfies the SDE

$$dy_t = -ay_t dt + \sigma dW_t, \ y_0 = 0,$$

where a and σ are strictly positive constants.
Let α_t be a differentiable deterministic function of t chosen to calibrate the model to the initial yield curve and define

$$N_t := \exp\left(\int_0^t (y_u + \alpha_u)^2 du\right).$$

In answering the following you may assume that $\mathbb{Q}$ is the equivalent martingale measure corresponding to taking N as numeraire.

(i) Use Itô's formula to show that

$$\begin{aligned} C'_{tT} - 2aC_{tT} - 2\sigma^2 C_{tT} + 1 &= 0, \\ B'_{tT} - aB_{tT} - 2\sigma^2 B_{tT}C_{tT} + 2\alpha_t &= 0, \\ A'_{tT} + \frac{1}{2}\sigma^2 B_{tT}^2 - \sigma^2 C_{tT} - \alpha_t^2 &= 0. \end{aligned}$$

with A',B' and C'representing the derivative with respect to the first parameter t.

[20%]

(ii) Define a new measure $\mathbb{F}$ on $(\Omega, \mathcal{F}_T)$ via

$$\left.\frac{d\mathbb{F}}{d\mathbb{Q}}\right|_{\mathcal{F}_t} := \exp\left(\int_0^t \nu(s,T)dW_s - \frac{1}{2}\int_0^t \nu^2(s,T)ds\right), \ t \leq T,$$

where

$$\nu(t,T) = -\sigma\left(B_{tT} + 2C_{tT}y_t\right), \ t \leq T.$$

Show that $\mathbb{F}$ is an equivalent martingale measure corresponding to taking the pure discount bond maturing at time T as numeraire.

[20%]

(iii) Show that, under the measure $\mathbb{F}$, y_t is normally distributed with mean,

$$E_{\mathbb{F}}(y_t) = -\sigma^2\phi_t^{-1}\int_0^t \phi_u B_{uT} du$$

where

$$\phi_t := \int_0^t (a + \sigma^2 C_{sT}) ds.$$

and find its variance.

Hint: You may assume that

$$\int_0^t \phi_u dW_u^T$$

is a martingale where W^T is a Brownian motion under $\mathbb{F}$.

[20%]

Question 2

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space supporting a one-dimensional Brownian motion W and let $\{\mathcal{F}_t\}_{t\geq 0}$ denote the augmented natural filtration of W.

(a) State the Martingale Representation Theorem for a one-dimensional Brownian motion.

[20%]

(b) Consider an economy defined for the finite time interval $0 \leq t \leq T < \infty$ and composed of two log-normal assets having price process denoted by $A = (S^{(1)}, S^{(2)})$ satisfying

$$\begin{aligned} dS_t^{(1)} &= \mu_t^{(1)} S_t^{(1)} dt + \sigma_t^{(1)} S_t^{(1)} dW_t. \\ dS_t^{(2)} &= \mu_t^{(2)} S_t^{(2)} dt + \sigma_t^{(2)} S_t^{(2)} dW_t, \end{aligned}$$

where $\mu^{(i)}$ and $\sigma^{(i)}$ are bounded continuous deterministic functions of time for $i = 1, 2$.

(i) Suppose $\sigma_t^{(1)} - \sigma_t^{(2)}$ is bounded away from zero for all t, and consider a deterministic function B defined via

$$B_t := \exp\left(\int_0^t \frac{(\sigma_u^{(1)}\mu_u^{(2)} - \sigma_u^{(2)}\mu_u^{(1)})}{\sigma_u^{(1)} - \sigma_u^{(2)}} du\right).$$

Show that $\phi = (\alpha^{(1)}, \alpha^{(2)})$ where $\alpha^{(i)}, i = 1, 2$ are defined by

$$\begin{aligned} \alpha_t^{(1)} &:= -\frac{B_t\sigma_t^{(2)}}{S_t^{(1)}(\sigma_t^{(1)} - \sigma_t^{(2)})}, \\ \alpha_t^{(2)} &:= \frac{B_t\sigma_t^{(1)}}{S_t^{(2)}(\sigma_t^{(1)} - \sigma_t^{(2)})}, \end{aligned}$$

is a self-financing trading strategy for this economy which replicates the deterministic process B.

[25%]

(ii) Let $\mathbb{Q}$ denote the unique equivalent martingale measure corresponding to taking B as numeraire and under which the rebased stock prices $\hat{S}^{(i)} := \frac{S^{(i)}}{B}$, $i = 1, 2$ satisfy

$$d\hat{S}_t^{(i)} = \sigma_t^{(i)} \hat{S}_t^{(i)} d\tilde{W}_t,$$

where $\tilde{W}$ is a Brownian motion under $\mathbb{Q}$. Further let X be a positive $\mathcal{F}_T$-measurable random variable satisfying $E_{\mathbb{Q}}\left[\frac{X}{D_T}\right] < \infty$.

Show that there exists an $\{\mathcal{F}_t\}$-predictable self-financing trading strategy

$$\varphi^* = (\phi^{(1)}, \phi^{(2)}),$$

of the form

$$\begin{aligned} \phi_t^{(1)} &= \frac{H_t}{\sigma_t^{(1)} \hat{S}_t^{(1)}} + \alpha_t^{(1)} \left(M_t - \frac{H_t}{\sigma_t^{(1)}} \right), \\ \phi_t^{(2)} &= \alpha_t^{(2)} \left(M_t - \frac{H_t}{\sigma_t^{(1)}} \right), \end{aligned}$$

for replicating the payoff X at time T where

$$M_t := E_{\mathbb{Q}}\left[\frac{X}{D_T} | \mathcal{F}_t\right],$$

and H is an $\{\mathcal{F}_t\}$-predictable process.

[45%]

(c) Consider the set

$$\Lambda = \{X : X \text{ is a positive } \mathcal{F}_T\text{-measurable random variable satisfying } E_{\mathbb{Q}}\left[\tfrac{X}{D_T}\right] < \infty\}.$$

Does the result in (b)(ii) establish that the economy is complete for the set Λ? Justify your answer.

[10%]

Question 3

Let $0 < T_1 < T_2 < ... < T_n < T_{n+1}$ be a sequence of dates and for $i = 1, \dots, n$ let $\alpha_i = T_{i+1} - T_i$. Further let D_{tT_i} denote the value at time t of a pure discount bond that pays unity at T_i.

(a) Define $L_t[T_i, T_{i+1}]$ the forward LIBOR at time t for the period $[T_i, T_{i+1}]$ and write down an expression for $L_t[T_i, T_{i+1}]$ in terms of the values of pure discount bonds.

[15%]

(b) Let $L_t^{(i)} := L_t[T_i, T_{i+1}]$. Fix k where $1 < k < n$. In a LIBOR market model working in the equivalent martingale measure corresponding to the numeraire $D_{\cdot T_{k+1}}$ suppose the forward LIBOR rate processes $L^{(i)}$ satisfy an SDE of the form

$$dL_t^{(i)} = \mu_t^{(i)} dt + \sigma_t^{(i)} L_t^{(i)} dW_t^{(i)} \qquad i = 1, \dots, n$$

where each $\sigma^{(i)}$ is a deterministic function of time t, each $\mu^{(i)}$ is some general process to be determined and $W = (W^{(1)}, \dots, W^{(n)})$ is an n dimensional Brownian motion having

$$dW_t^{(i)} dW_t^{(j)} = \rho_{ij} dt$$

where the ρ_{ij} are (appropriate) constants in $(-1, 1)$ with $\rho_{ii} = 1, \forall i$.

(i) Why is $\mu_t^{(k)} = 0$?

(ii) Show that for $1 \le i < k$

$$\mu_t^{(i)} = -\sum_{j=i+1}^{k} \frac{\alpha_j L_t^{(j)}}{1 + \alpha_j L_t^{(j)}} \sigma_t^{(j)} \sigma_t^{(i)} \rho_{ij} L_t^{(i)}.$$

(iii) Show that for $k < i \le n$

$$\mu_t^{(i)} = \sum_{j=k+1}^{i} \frac{\alpha_j L_t^{(j)}}{1 + \alpha_j L_t^{(j)}} \sigma_t^{(j)} \sigma_t^{(i)} \rho_{ij} L_t^{(i)}.$$

Hint: For $i = 1, \dots, n+1$ write

$$M_t^{(i)} := \frac{D_{tT_i}}{D_{tT_{k+1}}}$$

and show that for $i = 1, \dots, k-1$

$$M_t^{(i)} := \prod_{j=i}^{k} (1 + \alpha_j L_t^{(j)}),$$

and for $i = k+2, \dots, n+1$

$$M_t^{(i)} := \prod_{j=k+1}^{i-1} (1 + \alpha_j L_t^{(j)})^{-1}.$$

[65%]

(c) Briefly discuss the advantages and disadvantages of the model described in (b).

[20%]

Question 4

Let $0 < T_1 < T_2 < ... < T_n < T_{n+1}$ be a sequence of dates and for $i = 1, \ldots, n$ write $\alpha_i = T_{i+1} - T_i$. Further let $L^{(i)}$ for $i = 1, \ldots, n$ denote a set of contiguous forward LIBORs where $L_t^{(i)} := L_t[T_i, T_{i+1}]$ and let D_{tT} denote the value at time t of a pure discount bond that pays unity at T.

(a) Suppose that the market value at time $t = 0$ for a digital caplet with start date T_i, cashflow at time T_{i+1} and strike K is given by

$$V_0^{(i)}(K) = D_{0T_{i+1}} N(d^{(i)}(K)),$$

where

$$d^{(i)}(K) = \frac{\log\left(\frac{D_{0T_i}}{D_{0T_{i+1}}}(1+\alpha_i K)^{-1}\right)}{\Sigma^{(i)}} - \frac{1}{2}\Sigma^{(i)},$$

$N(.)$ denotes the standard cumulative normal distribution and $\Sigma^{(i)}$ is a positive constant.
Further suppose that an arbitrage free term structure model has been defined which is consistent with the above formula for all strikes for each of the digital caplets.
Show that for this model the distribution of $L_{T_i}^{(i)} + {\alpha_i}^{-1}$ is lognormal under an equivalent martingale measure corresponding to numeraire $D_{.T_{i+1}}$.

[35%]

(b) Show that for any arbitrage-free term structure model defined on the filtered space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}, \mathbb{N})$ under an equivalent martingale measure $\mathbb{N}$ corresponding to the terminal bond $D_{.T_{n+1}}$ as numeraire we have, for all $1 \le k \le n$ and $0 \le t \le s \le T_k$

$$\begin{aligned}
\prod_{j=k}^{n}(1+\alpha_j L_t^{(j)}) &= E_{\mathbb{N}}\left[\prod_{j=k}^{n}(1+\alpha_j L_s^{(j)})|\mathcal{F}_t\right] \\
&= E_{\mathbb{N}}\left[\prod_{j=k}^{n}(1+\alpha_j L_{T_j}^{(j)})\,|\mathcal{F}_t\right].
\end{aligned}$$

[25%]

(c) Consider an arbitrage-free term structure model for which for each $i = 1, \ldots, n$ the ith LIBOR at its setting date is taken to be of the form

$$(1 + \alpha_i L_{T_i}^{(i)}) = f^i(x_{T_i})$$

where

$$x_t = \sigma \int_0^t e^{au} dW_u,$$

W is a one-dimensional Brownian motion under $\mathbb{N}$, σ and a are positive constants and the functions f^i for $i = 1, \ldots, n$ are chosen so that the model calibrates to the digital caplets in part (a) with

$$\Sigma^{(i)} = \frac{e^{-aT_i} - e^{-aT_{i+1}}}{a}\sqrt{\mathrm{var}(x_{T_i})}.$$

(i) Show that for $0 < t < T_i$

$$(1 + \alpha_i L_t^{(i)}) = \frac{D_{0T_i}}{D_0 T_{i+1}} \exp\left((c_i - c_{i+1})x_t - \frac{1}{2}(c_i^2 - c_{i+1}^2)\mathrm{var}(x_t)\right)$$

where

$$c_i := \frac{e^{-aT_i} - e^{-aT_{n+1}}}{a}.$$

You may assume the model is complete.

[20%]

(ii) Suppose instead the functions f^i are chosen so that the resultant model is calibrated to digital caplets satisfying Black's formula, so that $L_{T_i}^{(i)}$ is log normally distributed in the equivalent martingale measure corresponding to $D_{\cdot T_{i+1}}$ as numeraire. Is it the case that the process $L_t^{(i)}, t \leq T_i$ is a log normal martingale under this equivalent martingale measure? Justify your answer.

[20%]

Q1

(a) Recall from lectures for $X \in \mathcal{L}^1(\Omega, \mathcal{F}_t, \mathbb{F})$

$$E_{\mathbb{F}}\left[X \mid \mathcal{F}_s\right] = \rho_s^{-1} E_{\mathbb{Q}}\left[X \rho_t \mid \mathcal{F}_s\right] \qquad (*)$$

where $\rho_t := \left.\frac{d\mathbb{F}}{d\mathbb{Q}}\right|_{\mathcal{F}_t}$.

Suppose ρM is an $(\{\mathcal{F}_t\}, \mathbb{Q})$ martingale. Then
M(i) M is adapted since ρ, ρM are

$$\text{M(ii)} \quad E_{\mathbb{F}}\left[|M_t|\right] = E_{\mathbb{Q}}\left[\rho_t |M_t|\right] \overset{\text{since } \rho > 0}{=} E_{\mathbb{Q}}\left[|\rho_t M_t|\right] < \infty$$

since ρM must be in $\mathcal{L}^1(\Omega, \mathcal{F}_t, \mathbb{Q})$.

$$\text{M(iii)} \quad E_{\mathbb{F}}(M_t \mid \mathcal{F}_s) = \rho_s^{-1} E_{\mathbb{Q}}(\rho_t M_t \mid \mathcal{F}_s) \quad \text{by } (*)$$

$= \rho_s^{-1} \rho_s M_s$ using martingale property of ρM

$= M_s$

Thus properties M(i) - M(iii) hold and we have shown that ρM is an $(\{\mathcal{F}_t\}, \mathbb{Q})$ martingale $\Rightarrow$ M is an $(\{\mathcal{F}_t\}, \mathbb{F})$ martingale

Converse implication can be proved similarly, or write $\tilde{M} = \rho M$ and $M = \rho^{-1}\tilde{M}$ and not $\rho_t^{-1} = \left.\frac{d\mathbb{Q}}{d\mathbb{F}}\right|_{\mathcal{F}_t}$.

(b) Girsanov's Theorem for a one-dimensional B.M.

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space supporting a one-dimensional Brownian motion W and let $\{\mathcal{F}_t\}$ denote the augmented natural filtration generated by W.

(i) Suppose $\mathbb{Q} \sim \mathbb{P}$ w.r.t. $\mathcal{F}$. Then $\exists$ an $\{\mathcal{F}_t\}$-predictable $\mathbb{R}$-valued process C such that

$$\rho_t := \frac{d\mathbb{Q}}{d\mathbb{P}}\Big|_{\mathcal{F}_t} = \exp\left(\int_0^t C_u dW_u - \frac{1}{2}\int_0^t C_u^2 du\right) \qquad (+)$$

(ii) Conversely if ρ is a strictly positive $\left(\{\mathcal{F}_t\}_{0 \le t \le T}, \mathbb{P}\right)$ martingale some $T \in [0, \infty)$ with $E_{\mathbb{P}}[\rho_T] = 1$, then ρ has the representation in $(+)$ and define a measure $\mathbb{Q} \sim \mathbb{P}$ w.r.t $\mathcal{F}_T$.

In either of the above cases, under $\mathbb{Q}$

$$\tilde{W}_t := W_t - \int_0^t C_u du$$

is an $(\{\mathcal{F}_t\}, \mathbb{Q})$ Brownian motion (with time horizon being restricted to $[0,T]$ in the later case).

(c) (i) If $\mathbb{Q}$ is the EMM corresponding to B as numeraire then $\frac{D_{tT}}{B_t}$, $t \leq T$ is a martingale under $\mathbb{Q}$

By Itô's formula

$$dD_{tT} = \frac{\partial}{\partial t} D_{tT}\, dt + \frac{\partial}{\partial y_t} D_{tT}\, dy_t + \frac{1}{2} \frac{\partial^2}{\partial y_t^2} D_{tT}(y_t) \left(dy_t\right)^2$$

Given $D_{tT} = \exp\left(g(t, y_t)\right)$

where $g(t, y_t) := A_{tT} - B_{tT}\, y_t - C_{tT}\, y_t^2$

we have

$$g_t(t, y_t) = A'_{tT} - B'_{tT}\, y_t - C'_{tT}\, y_t^2$$

$$g_y(t, y_t) = -B_{tT} - 2\, C_{tT}\, y_t$$

$$g_{yy}(t, y_t) = -2C_{tT}$$

$$\frac{\partial D_{tT}}{\partial y_t} = g_y(t, y_t)\, D_{tT}, \quad \frac{\partial D_{tT}}{\partial t} = g_t(t, y_t)\, D_{tT}$$

$$\frac{\partial^2 D_{tT}}{\partial y_t^2} = g_{yy}(t, y_t)\, D_{tT} + \left(g_y(t, y_t)\right)^2 D_{tT}$$

Thus

$$dD_{tT} = \left(A'_{tT} - B'_{tT}\, y_t - C'_{tT}\, y_t^2\right) D_{tT}\, dt$$

$$- \left(B_{tT} + 2C_{tT}\, y_t\right) D_{tT}\, dy_t$$

$$+ \frac{1}{2}\left[\left(B_{tT} + 2C_{tT}\, y_t\right)^2 - 2C_{tT}\right] \left(dy_t\right)^2$$

Noting $dy_t = -a y_t\, dt + \sigma\, dW_t$

$$\frac{dD_{tT}}{D_{tT}} = \Big[A'_{tT} - B'_{tT} y_t - C'_{tT} y_t^2 + a y_t (B_{tT} + 2C_{tT} y_t)$$

$$+ \tfrac{1}{2}\sigma^2 \left\{ (B_{tT} + 2C_{tT} y_t)^2 - 2C_{tT} \right\} \Big] dt$$

$$- (B_{tT} + 2C_{tT} y_t)\, \sigma\, dW_t^*$$

i.e.

$$\frac{dD_{tT}}{D_{tT}} = \Big[(A'_{tT} - \sigma^2 C_{tT} + \tfrac{1}{2}\sigma^2 B_{tT}^2)$$

$$+ (-B'_{tT} + aB_{tT} + 2\sigma^2 B_{tT} C_{tT})\, y_t$$

$$+ (-C'_{tT} + 2aC_{tT} + 2\sigma^2 C_{tT}^2)\, y_t^2 \Big] dt$$

$$- (B_{tT} + 2C_{tT} y_t)\, \sigma\, dW_t$$

Now

$$d\left(\frac{D_{tT}}{B_t}\right) = B_t^{-1}\, dD_{tT} + D_{tT}\, d(B_t^{-1})$$

$$= B_t^{-1}\, dD_{tT} - r_t B_t^{-1} D_{tT}\, dt$$

where $r_t := (y_t + \alpha_t)^2 = y_t^2 + 2\alpha_t y_t + \alpha_t^2$

$$d\left(\frac{D_{tT}}{B_t}\right) = \frac{D_{tT}}{B_t} \Big[(A'_{tT} - \sigma^2 C_{tT} + \tfrac{1}{2}\sigma^2 B_{tT}^2 - \alpha_t^2)$$

$$(-B'_{tT} + aB_{tT} + 2\sigma^2 B_{tT} C_{tT} - 2\alpha_t)\, y_t$$

$$(-C'_{tT} + 2aC_{tT} + 2\sigma^2 C_{tT}^2 - 1)\, y_t^2 \Big] dt$$

$$- \frac{D_{tT}}{B_t} (B_{tT} + 2C_{tT} y_t)\, \sigma\, dW_t$$

In order for $\frac{D_{tT}}{B_t}$ to be a (local) martingale the finite variation term must be zero and so the coefficients of y^i $i=0,1,2$ in the drift term must be zero. This yields the required equations.

(ii) Note from part (i)

$$d\left(\frac{D_{tT}}{B_t}\right) = -\frac{D_{tT}\,\sigma}{B_t}\left(B_{tT} + 2C_{tT}\,y_t\right)dW_t$$

and so the given Doleans exponential is

$$\left.\frac{d\mathbb{F}}{d\mathbb{Q}}\right|_{\mathcal{F}_t} = \frac{D_{tT}}{B_t D_{0T}}$$

(or could note the Radon-Nikodym derivative will be the ratio of numeraires and confirm this is haven't done (i))

Since $\mathbb{Q}$ is an EMM corresponding to B $\frac{D_{tS}}{B_t}$, $t \leq S$ is a martingale under $\mathbb{Q}$ for any S as assets/numeraire is a martingale

But by (a)

$$\frac{D_{tS}}{B_t} = \frac{D_{tS}}{D_{tT}}\frac{D_{tT}}{B_t D_{0T}}D_{0T} = \frac{D_{tS}}{D_{tT}}\left.\frac{d\mathbb{F}}{d\mathbb{Q}}\right|_{\mathcal{F}_t} D_{0T}$$

is a $\mathbb{Q}$ martingale if and only if

$\frac{D_{tS}}{D_{tT}}$ is an $\mathbb{F}$ martingale

Thus $\mathbb{F}$ is an EMM corresponding to $D_{\cdot T}$

(iii) By Girsanov's Theorem

$$dW_t^T = dW_t - \nu(t,T)\,dt = dW_t + \sigma(B_{tT} + 2C_{tT}y_t)dt$$

is a Brownian motion under $\mathbb{F}$

Thus under $\mathbb{F}$

$$dy_t = -ay_t\,dt + \sigma\left(dW_t^T - \sigma(B_{tT} + 2C_{tT}y_t)dt\right)$$

i.e.

$$dy_t = \left(-\sigma^2 B_{tT} - (a + 2\sigma^2 C_{tT})y_t\right)dt + \sigma\,dW_t^T$$

Define

$$\phi_t = \exp\left(\int_0^t (a + 2\sigma^2 C_{uT})\,du\right),$$

and set

$$z_t = \phi_t y_t$$

$$dz_t = \phi_t\,dy_t + y_t\,d\phi_t$$

$$= \left(-\sigma^2\phi_t B_{tT} - \phi_t(a + 2\sigma^2 C_{tT})y_t\right)dt + \sigma\phi_t\,dW_t^T$$

$$+ (a + 2\sigma^2 C_{tT})\phi_t y_t\,dt$$

$$= -\sigma^2\phi_t B_{tT}\,dt + \sigma\phi_t\,dW_t^T$$

So

$$z_t = z_0 - \sigma^2\int_0^t \phi_u B_{uT}\,du + \sigma\int_0^t \phi_u\,dW_u^T$$

i.e. $$y_t = \phi_t^{-1}y_0 - \sigma^2\phi_t^{-1}\int_0^t \phi_u B_{uT}\,du + \sigma\phi_t^{-1}\int_0^t \phi_u\,dW_u^T$$

Thus y_t is normally distributed as $\int_0^t \phi_u \, dW_u^T$ is normally distributed

(ϕ is deterministic and we may assume the S.I. is a martingale

$$E\, y_t = \phi_t^{-1} y_0 - \sigma^2 \phi_t^{-1} \int_0^t \phi_u B_{uT} \, du$$

$$Var(y_t) = \sigma^2 \phi_t^{-2} \int_0^t \phi_u^2 \, du$$

but, since $\phi_t = e^{\int_0^t (a + 2\sigma^2 C_{uT}) du}$, more could be said here

Why not give the value of $E(Y_t)$

Then it's clear what you're allowing as expression for $Var(Y_t)$

Question 2

(a) $(\Omega, \mathcal{F}, \mathbb{P})$ probability space supporting BM W
$\{\mathcal{F}_t\}$ augmented natural filtration generated by W

M.R.T.

Any local martingale N wrt $\{\mathcal{F}_t\}$ can be written in the form

$$N_t = N_0 + \int_0^t H_u dW_u$$

for some $\{\mathcal{F}_t\}$-predictable H s.t. $\int_0^t H_u^2 du < \infty$ a.s. all t

(b)
(i) For $\phi = (\alpha^{(1)}, \alpha^{(2)})$ to be a self-financing strategy replicating B we require

$$(\dagger) \quad B_t = \alpha_t^{(1)} S_t^{(1)} + \alpha_t^{(2)} S_t^{(2)} = 1 + \int_0^t \alpha_u^{(1)} dS_u^{(1)} + \int_0^t \alpha_u^{(2)} dS_u^{(2)}$$

Noting B is of finite variation we must have

$$0 = \alpha_t^{(1)} \sigma_t^{(1)} S_t^{(1)} dW_t + \alpha_t^{(2)} \sigma_t^{(2)} S_t^{(2)} dW_t$$

$$\alpha_t^{(1)} \sigma_t^{(1)} S_t^{(1)} = -\alpha_t^{(2)} \sigma_t^{(2)} S_t^{(2)}$$

i.e. $\alpha_t^{(1)} = -\alpha_t^{(2)} \dfrac{\sigma_t^{(2)}}{\sigma_t^{(1)}} \dfrac{S_t^{(2)}}{S_t^{(1)}}$

Substituting in $(\dagger)$ and solving for $\alpha^{(2)}$ yields

$$\alpha_t^{(2)} = \frac{B_t \quad \sigma_t^1}{S_t^{(2)} (\sigma_t^1 - \sigma_t^{(2)})}$$

and thus

$$\alpha_t^{(1)} = \frac{-B_t \quad \sigma_t^{(2)}}{S_t^{(1)} (\sigma_t^{(1)} - \sigma_t^{(2)})}$$

as required

(ii) Self-financing strategy ϕ^* must satisfy

$$\phi_t^{(1)} S_t^{(1)} + \phi_t^{(2)} S_t^{(2)} = \phi_0^{(1)} S_0^{(1)} + \phi_0^{(2)} S_0^{(2)} + \int_0^t \phi_u^{(1)} dS_u^{(1)} + \int_0^t \phi_u^{(2)} dS_u^{(2)}$$

and

$$X = \phi_0^{(1)} S_0^{(1)} + \phi_0^{(2)} S_0^{(2)} + \int_0^T \phi_u^{(1)} dS_u^{(1)} + \int_0^T \phi_u^{(2)} dS_u^{(2)}$$

By numeraire invariance this is equivalent to

$$\frac{X}{B_T} = \phi_0^{(1)} \hat{S}_0^{(1)} + \phi_0^{(2)} \hat{S}_0^{(2)} + \int_0^T \phi_u^{(1)} d\hat{S}_u^{(1)} + \int_0^T \phi_u^{(2)} d\hat{S}_u^{(2)}$$

Define $(\{\mathcal{F}_t\}, \mathbb{Q})$ martingale

$$M_t := E_{\mathbb{Q}}\left(\frac{X}{B_T} \,\middle|\, \mathcal{F}_t\right), \quad t \leq T$$

By MRT

$$M_t = E_{\mathbb{Q}}\left(\frac{X}{B_T}\right) + \int_0^t H_u \, d\tilde{W}_u$$

Observe $\quad d\tilde{W}_t = \dfrac{d\hat{S}_t^{(i)}}{\sigma^{(i)} \hat{S}_t^{(i)}} \quad i=1,2$

and so

$$(4) \quad M_t = E_{\mathbb{Q}}\left(\frac{X}{B_T}\right) + \int_0^t \frac{H_u \, d\hat{S}_t^{(1)}}{\sigma^{(1)} \hat{S}_t^{(1)}}$$

Taking $t=T$

$$\frac{X}{B_T} = E_{\mathbb{Q}}\left(\frac{X}{B_T}\right) + \int_0^t \frac{H_u \, d\hat{S}_t^{(1)}}{\sigma^{(1)} \hat{S}_t^{(1)}}$$

Need ϕ satisfying

$$M_t = E_{\mathbb{Q}}\left(\frac{X}{B_T}\right) + \int_0^t \phi_u^{(1)} d\hat{S}_u^{(1)} + \int_0^t \phi_u^{(2)} d\hat{S}_u^{(2)}$$

$$= \phi_t^{(1)} \hat{S}_t^{(1)} + \phi_t^{(2)} \hat{S}_t^{(2)}$$

self-financing condition

Use (+) to first find self-financing strategy $\bar{\phi} = (\bar{\phi}_1, \bar{\phi}_2)$ specifying holdings in $S^{(1)}$ and B to replicate M and then convert to a self-financing strategy in $S^{(1)}$ and $S^{(2)}$ using part (a)

Look for $\bar{\phi}$ satisfying

$$M_t = E_{\mathbb{Q}}\left(\frac{X}{B_T}\right) + \int_0^t \bar{\phi}^{(1)}_u d\hat{S}^{(1)}_u + \int_0^t \bar{\phi}^{(2)}_u \underbrace{d\hat{B}_u}_{=0}$$

$$= \bar{\phi}^{(1)}_t \hat{S}^{(1)}_t + \bar{\phi}^{(2)}_t \hat{B}_t$$

$$= \bar{\phi}^{(1)}_t \hat{S}^{(1)}_t + \bar{\phi}^{(2)}_t$$

Take $\bar{\phi}^{(1)}_t = \dfrac{H_t}{\sigma^{(1)} \hat{S}^{(1)}_t}$, $\bar{\phi}^{(2)}_t = M_t - \bar{\phi}^{(1)}_t \hat{S}^{(1)}_t = M_t - \dfrac{H_t}{\sigma^{(1)}_t}$

Then we have $(B_0 = 1)$

$$\bar{\phi}^{(1)}_t S^{(1)}_t + \bar{\phi}^{(2)}_t B_t = \tilde{\phi}^{(1)}_0 S^{(1)}_0 + \tilde{\phi}^{(2)}_0 + \int_0^t \bar{\phi}^{(1)}_u dS^{(1)}_u + \int_0^t \tilde{\phi}^{(2)}_u dB_u$$

and

$$X = \bar{\phi}^{(1)}_0 S^{(1)}_0 + \phi^{(2)}_0 + \int_0^T \bar{\phi}^{(1)}_u dS^{(1)}_u + \int_0^T \bar{\phi}^{(2)}_u dB_u$$

Substituting for $B_t = \alpha_t S^{(1)}_t + \alpha^{(2)}_t S^{(2)}_t$ and $dB_t = \alpha^{(1)}_t dS^{(1)}_t + \alpha^{(2)}_t dS^{(2)}_t$

$$\bar{\phi}^{(1)}_t S^{(1)}_t + \bar{\phi}^{(2)}_t \left(\alpha^{(1)}_t S^{(1)}_t + \alpha^{(2)}_t S^{(2)}_t\right)$$

$$= \bar{\phi}^{(1)}_0 S^{(1)}_0 + \bar{\phi}^{(2)}_0 B_0 + \int_0^t \bar{\phi}^{(1)}_u dS^{(1)}_u + \int_0^t \bar{\phi}^{(2)}_u \alpha^{(1)}_u dS^{(1)}_u + \int_0^t \bar{\phi}^{(2)}_u \alpha^{(2)}_u dS^{(2)}_u$$

i.e.

$$\left(\bar{\phi}^{(1)}_t + \alpha^{(1)}_t \bar{\phi}^{(2)}_t\right) S^{(1)}_t + \alpha^{(2)}_t \bar{\phi}^{(2)}_t S^{(2)}_t$$

$$= \bar{\phi}^{(1)}_0 S^{(1)}_0 + \bar{\phi}^{(2)}_0 B_0 + \int_0^t \left(\bar{\phi}^{(1)}_u + \alpha^{(1)}_u \bar{\phi}^{(2)}_u\right) dS^{(1)}_u + \int_0^t \bar{\phi}^{(2)}_u \alpha^{(2)}_u dS^{(2)}_u$$

So take

$$\phi_t^{(1)} = \tilde{\phi}_t^{(1)} + \alpha_t^{(1)} \tilde{\phi}_t^{(2)}$$

$$\phi_t^{(2)} = \alpha_t^{(2)} \tilde{\phi}_t^{(2)}$$

i.e. $$\phi_t^{(1)} = H_t \left(\sigma^{(1)} \hat{S}_t^{(1)}\right)^{-1} \cdot \alpha_t^{(1)} \left(M_t - H_t / \sigma_t^{(1)}\right)$$

$$\phi_t^{(2)} = \alpha_t^{(2)} \left(M_t - H_t / \sigma_t^{(1)}\right)$$

(c) The result in (b)(ii) provides a replicating strategy for any $X \in \Lambda$. To show the economy is complete we need to check that this strategy is admissible. For the definition of admissible given in lectures this means we need to check the rebased gain process is a martingale w.r.t. any numeraire pair (N, M)

Solutions

Q3

(a) $L_{T_i}[T_i, T_{i+1}]$ = rate of interest payable at T_{i+1} for a unit deposit at T_i for period $[T_i, T_{i+1}]$

payment made is $\alpha_i L_{T_i}[T_i, T_{i+1}]$, $\alpha_i = T_{i+1} - T_i$

$L_t[T_i, T_{i+1}]$ = rate of interest that will be paid as above if contract agreed at t

$$L_t[T_i, T_{i+1}] = \frac{D_{tT_i} - D_{tT_{i+1}}}{\alpha_i D_{tT_{i+1}}} \qquad (*)$$

(b) Under $\mathbb{N}$, each $\dfrac{D_{tT_i}}{D_{tT_{k+1}}}$ is a martingale

(i) It follows from (*) that $L_t^{(k)}$ is a martingale and so the drift in the SDE for $L_t^{(k)}$ must be zero i.e. $\mu_t^{(k)} \equiv 0$.

(ii) From (*)

$$D_{tT_j} = \left(1 + \alpha_j L_t^{(j)}\right) D_{tT_{j+1}}$$

For $1 \le i < k$

$$M_t^{(i)} := \frac{D_{tT_i}}{D_{tT_{k+1}}} = \prod_{j=i}^{k} \frac{D_{tT_j}}{D_{tT_{j+1}}} = \prod_{j=i}^{k}\left(1 + \alpha_j L_t^{(j)}\right) \qquad (\dagger)$$

and so

$$M_t^{(i)} = M_t^{(i+1)} + \alpha_i L_t^{(i)} M_t^{(i+1)}$$

$M_t^{(i)}$ a martingale $\Rightarrow$ $\alpha_i L_t^{(i)} M_t^{(i+1)}$ a martingale
for all i

Now

$$d\left(L_t^{(i)} M_t^{(i+1)}\right) = L_t^{(i)} dM_t^{(i+1)} + M_t^{(i+1)} dL_t^{(i)} + dM_t^{(i+1)} dL_t^{(i)}$$

and equating finite variation terms to zero $\Rightarrow$

$$M_t^{(i+1)} \mu_t^{(i)} dt + dM_t^{(i+1)} \sigma^{(i)} L_t^{(i)} dW_t^{(i)} = 0 \qquad (\dagger)$$

From $(\dagger)$

$$dM_t^{(i+1)} = \sum_{j=i+1}^{k} \frac{\partial M_t^{(i+1)}}{\partial L^{(j)}} \sigma^{(j)} L_t^{(j)} dW_t^{(j)} + F.V.$$

$$= \sum_{j=i+1}^{k} \frac{\alpha_j M_t^{(i+1)}}{1 + \alpha_j L_t^{(j)}} \sigma^{(j)} L_t^{(j)} dW_t^{(j)} + F.V.$$

Substituting in $(\dagger)$

$$M_t^{(i+1)} \mu_t^{(i)} dt + \sum_{j=i+1}^{k} \frac{\alpha_j M_t^{(i+1)}}{1 + \alpha_j L_t^{(j)}} \sigma^{(j)} L_t^{(j)} \sigma^{(i)} L_t^{(i)} \rho_{ij} dt = 0$$

and thus for $1 \le i < k$

$$\mu_t^{(i)} = -\sigma^{(i)} L_t^{(i)} \sum_{j=i+1}^{k} \frac{\alpha_j \sigma^{(j)} L_t^{(j)} \rho_{ij} \sigma^{(i)} L_t^{(i)}}{1 + \alpha_j L_t^{(j)}}$$

as required

(iii) For $i = k+2, \dots, n+1$

$$M_t^{(i)} := \frac{D_t T_i}{D_t T_{k+1}} = \prod_{j=k+1}^{i-1} \frac{D_t T_{j+1}}{D_t T_j} = \prod_{j=k+1}^{i-1} \left(1 + \alpha_j L_t^{(j)}\right)^{-1} \quad (\#\#\#)$$

and so

$$M_t^{(i)} = \prod_{j=k+1}^{i} \left(1 + \alpha_j L_t^{(j)}\right)^{-1} \left(1 + \alpha_i L_t^{(i)}\right)$$

i.e. $M_t^{(i)} = M_t^{(i+1)} + \alpha_i L_t^{(i)} M_t^{(i+1)}$

as before.

Equation (##) follows as before for $i = k+1, \dots, n$ but now from (###) we have

$$dM_t^{(i+1)} = \sum_{j=k+1}^{i-1} \frac{\partial M_t^{(i+1)}}{\partial L^{(j)}} \sigma^{(j)} L_t^{(j)} dW_t^{(j)} + \text{F.V.}$$

$$= \sum_{j=k+1}^{i} \frac{-\alpha_j M_t^{(i+1)}}{1 + \alpha_j L_t^{(j)}} \sigma^{(j)} L_t^{(j)} dW_t^{(j)} + \text{F.V.}$$

Substituting in (##)

$$M_t^{(i+1)} \mu_t^{(i)} dt - \sum_{j=k+1}^{i} \frac{\alpha_j M_t^{(i+1)}}{1 + \alpha_j L_t^{(j)}} \sigma^{(j)} L_t^{(j)} \sigma^{(i)} L_t^{(i)} \rho_{ij} dt = 0$$

and thus for $i = k+1, \dots, n$

$$\mu_t^{(i)} = \sum_{j=k+1}^{i} \frac{\alpha_j L_t^{(j)}}{1 + \alpha_j L_t^{(j)}} \sigma^{(j)} \sigma^{(i)} \rho_{ij} L_t^{(i)}$$

as required.

(c)

Allows for a very flexible fit to market correlations between LIBOR rates

Calibrates to caplet prices as given by Black's formula

High dimensional so difficult to implement in practice especially for path dependent products,

A separability assumption would reduce effective dimension (so close approximation could be implemented via a Markov functional approach) but this would reduce flexibility.

Q4

(a) Taking $D_{\cdot T_{i+1}}$ as numeraire let $\mathbb{N}^i$ denote the corresponding EMM. The value of the i^{th} digital caplet is given by

$$V_0^{(i)}(K) = D_{0T_{i+1}} E_{\mathbb{N}^i}\left[\frac{V_{T_i}^{(i)}}{D_{T_iT_{i+1}}}\right]$$

$$= D_{0T_{i+1}} \mathbb{N}^i\left(L_{T_i}^{(i)} > K\right).$$

Thus $\mathbb{N}^i(L_{T_i}^i > K) = N(d^{(i)})$ for $K \geq 0$ (+)

where $N(\ )$ is the cumulative normal distribution. This specifies the distribution of $L_{T_i}^{(i)}$ under $\mathbb{N}^i$

To identify the distribution note

$$\mathbb{N}^i(L_{T_i}^{(i)} > K) = \mathbb{N}^i\left(\log(L_{T_i}^i + \alpha_i^{-1}) > \log(K + \alpha_i^{-1})\right)$$

$$= \mathbb{N}^i\left(\frac{\log(L_{T_i}^i+\alpha_i^{-1}) - \log(L_0^{(i)}+\alpha_i^{-1}) + \frac{1}{2}(\Sigma^i)^2}{\Sigma^i}\right.$$

$$\left. > \frac{\log(K+\alpha_i^{-1}) - \log(L_0^i + \alpha_i^{-1}) + \frac{1}{2}(\Sigma^i)^2}{\Sigma^i}\right)$$

Note
$\alpha_i L_0^{(i)} + 1 = \frac{D_{0T_i}}{D_{0T_{i+1}}}$

$$= \mathbb{N}^i\left(\frac{\log(L_{T_i}^i+\alpha_i^{-1}) - \left[\log(L_0^{(i)}+\alpha_i^{-1}) - \frac{1}{2}(\Sigma^i)^2\right]}{\Sigma^i} > -d^{(i)}\right)$$

$$= \mathbb{N}^i\left(Z > -d^{(i)}\right) \quad \text{using (+)} \quad Z \sim N(0,1)$$

Hence

$$\log(L_{T_i}^{(i)} + \alpha_i^{-1}) \sim N\left(\log(L_0^{(i)}+\alpha_i^{-1}) - \tfrac{1}{2}(\Sigma^i)^2, (\Sigma^i)^2\right)$$

(b) Observe

$$M_t^k := \prod_{j=k}^{n}\left(1+\alpha_j L_t^{(j)}\right) = \prod_{j=k}^{n} \frac{D_{tT_j}}{D_{tT_{j+1}}} = \frac{D_{tT_k}}{D_{tT_{n+1}}}$$

Since of the form $\frac{\text{asset}}{\text{numeraire}}$ this must be a

martingale under the EMM $\tilde{N}$. This gives the first equality which is just the martingale property

$$(+) \qquad E^{\tilde{N}}\left(M_s^k \middle| \mathcal{F}_t\right) = M_t^k, \quad k=1,\dots,n, \quad 0\le t\le s\le T_k$$

Next observe

$$M_t^k = E^{\tilde{N}}\left(M_s^k \middle| \mathcal{F}_t\right) = E^{\tilde{N}}\left[E^{\tilde{N}}\left(M_{T_k}^k \middle| \mathcal{F}_s\right) \middle| \mathcal{F}_t\right]$$ using mg property (+) at $s=T_k$

$$= E^{\tilde{N}}\left(M_{T_k}^k \middle| \mathcal{F}_t\right)$$ by Tower property

$$= E^{\tilde{N}}\left(M_{T_k}^{k+1}\left(1+\alpha_k L_{T_k}^{(k)}\right) \middle| \mathcal{F}_t\right)$$ by definition M^k, M^{k+1}

$$= E^{\tilde{N}}\left[E\left(M_{T_{k+1}}^{k+1} \middle| \mathcal{F}_{T_k}\right)\left(1+\alpha_k L_{T_k}^{(k)}\right) \middle| \mathcal{F}_t\right]$$ by mg property for M^{k+1}

$$= E^{\tilde{N}}\left[M_{T_{k+1}}^{k+1}\left(1+\alpha_k L_{T_k}^{k}\right) \middle| \mathcal{F}_t\right]$$ using "taking out what is known" and tower property

Note all processes positive so "taking out what is known" applies assuming CE exist.

$$= E^{\tilde{N}}\left(M_{T_{k+2}}^{k+1} \prod_{j=k}^{k+1}\left(1+\alpha_j L_{T_j}^{(j)}\right) \middle| \mathcal{F}_t\right)$$ repeating the above steps

$$= E^{\tilde{N}}\left(M_{T_n}^{n} \prod_{j=k}^{n-1}\left(1+\alpha_j L_{T_j}^{(j)}\right) \middle| \mathcal{F}_t\right)$$ by iteration

$$= E^{\tilde{N}}\left(\prod_{j=k}^{n}\left(1+\alpha_j L_{T_j}^{j}\right) \middle| \mathcal{F}_t\right)$$

(c)

(i) For $L^{(i)}$ as given we have

$$\frac{D_{tT_i}}{D_{tT_{n+1}}} = \prod_{j=i}^{n}\left(1 + \alpha_j L_t^{(j)}\right)$$

$$= \left[\prod_{j=i}^{n}\left(\frac{D_{0T_j}}{D_{0T_{j+1}}}\right)\right] \exp\left(\sum_{j=i}^{n}(c_j - c_{j+1})\, x_t - \frac{1}{2}\sum_{j=i}^{n}(c_j^2 - c_{j+1}^2)\, \mathrm{var}\, x_t\right)$$

$$= \frac{D_{0T_i}}{D_{0T_{n+1}}} \exp\left((c_i - c_{n+1})\, x_t - \frac{1}{2}(c_i^2 - c_{n+1}^2)\, \mathrm{var}(x_t)\right)$$

$$= \frac{D_{0T_i}}{D_{0T_{n+1}}} \exp\left(c_i x_t - \frac{1}{2} c_i^2\, \mathrm{var}(x_t)\right)$$

as $c_{n+1} = 0$. Novikov's condition holds

and so $\dfrac{D_{tT_i}}{D_{tT_{n+1}}}$ $t \le T_i$ is a martingale under N

It remains to check that $L^i_{T_i}$ has the appropriate distribution under N^i, the EMM corresponding to $D_{\cdot T_{i+1}}$ as numeraire.

Assuming the model is complete

$$\left.\frac{dN^i}{dN}\right|_{\mathcal{F}_t} = \frac{D_{tT_{i+1}}}{D_{tT_{n+1}}}\,\frac{D_{0T_{n+1}}}{D_{0T_{i+1}}}$$

$$= \exp\left(c_{i+1}\,\sigma \int_0^t e^{au}\, dW_u - \frac{1}{2} c_i^2 \sigma^2 \int_0^t e^{2au}\, du\right)$$

By Girsanov's Theorem $dW^i_t = dW_t - c_{i+1}\,\sigma e^{at}\, dt$ is a Brownian motion under N^i

Thus under $\mathbb{N}^i$

$$\left(1+\alpha_i L_t^{(i)}\right) = \frac{D_{0T_i}}{D_{0T_{i+1}}} \exp\left((c_i - c_{i+1})\,\sigma \int_0^t e^{au}\left(dW_u^i + c_{i+1}\sigma e^{au}\,du\right) - \tfrac{1}{2}\left(c_i^2 - c_{i+1}^2\right)\sigma^2 \int_0^t e^{2au}\,du\right)$$

$$= \frac{D_{0T_i}}{D_{0T_{i+1}}} \exp\left(\sigma(c_i - c_{i+1}) \int_0^t e^{au}\,dW_u^i + \sigma^2 (c_i - c_{i+1})\, c_{i+1} \int_0^t e^{2au}\,du - \tfrac{1}{2}\left(c_i^2 - c_{i+1}^2\right)\sigma^2 \int_0^t e^{2au}\,du\right)$$

$$= \frac{D_{0T_i}}{D_{0T_{i+1}}} \exp\left(\sigma(c_i - c_{i+1}) \int_0^t e^{au}\,dW_u^i - \tfrac{1}{2}\left(c_i - c_{i+1}\right)^2 \sigma^2 \int_0^t e^{2au}\,du\right)$$

and so

$$\log\left(1+\alpha_i L_{T_i}^{(i)}\right) = \log\left(\frac{D_{0T_i}}{D_{0T_{i+1}}}\right) - \tfrac{1}{2}\left(c_i - c_{i+1}\right)^2 \mathrm{Var}\left(x_{T_i}\right) + \left(c_i - c_{i+1}\right) x_{T_i}$$

i.e. $\log\left(\alpha_i^{-1} + L_{T_i}^{(i)}\right)$

$$\sim N\left(\log\left(\frac{D_{0T_i}}{D_{0T_{i+1}}}\right) - \log \alpha_i - \tfrac{1}{2}\left(\Sigma^{(i)}\right)^2, \left(\Sigma^{(i)}\right)^2\right)$$

Note $\log \frac{D_{0T_i}}{D_{0T_{i+1}}} - \log \alpha_i = \log\left(L_0^{(i)} + \alpha_i^{-1}\right)$

(ii) From part (b) we can see that $L_t^{(i)}$ will be a function of the one-dimensional process x_t for all t.

If $L^{(i)}$ is a log-normal martingale under the measure $\mathbb{N}^i$ corresponding to $D_{T_{i+1}}$ as numeraire we would have

$$dL_t^i = \sigma^i L_t^{(i)} \sigma e^{at} d\tilde{W}_t$$

$\tilde{W}$ a BM under $D_{T_{i+1}}$.

But we know from lectures that the only way the model could then be arbitrage free is if the drift of L^i under the terminal measure is as in the LIBOR-Market model and so a function of L^j $j=i+1,\ldots,n$. But this means the model would be high-dimensional - a contradiction.

Note The model in (ii) is the LIBOR Markov-functional model calibrated to Black's formula