Applications of Stochastic Calculus in Finance Chapter 8: Stochastic calculus for single jump processes

Gechun Liang

1 Functions with one-sided limits

Definition 1. (One-sided limits) For $T > 0$, a function $f : [0,T] \to \mathbb{R}$ is said to have right limits at $t \in [0,T)$ if

$$f(t+) := \lim_{s \downarrow t} f(s)$$

exists, and is said to have left limits at $t \in (0,T]$ if

$$f(t-) := \lim_{s \uparrow t} f(s)$$

exists. f is said to have one-sided limits at $t \in (0,T)$ if both $f(t+)$ and $f(t-)$ exist. Furthermore, if $f(t) = f(t+)$ holds, then f is called Cadlag (right continuous with left limits); if $f(t) = f(t-)$ holds, then f is called Caglad (left continuous with right limits). □

For convenience, *we set $f(T+) = f(T)$ and $f(0-) = f(0)$ in the rest of this chapter*. Functions with one-sided limits are more well behaved than one might initially expect.

Theorem 1. *If $f : [0,T] \to \mathbb{R}$ has one-sided limits, then f is bounded on $[0,T]$.*

Proof. Fix $t \in [0,T]$. Since $f(t+)$ exists, there exists $\delta_{t+} > 0$ such that $|f(s) - f(t+)| < 1$ for all $s \in (t, t+\delta_{t+})$. Thus,

$$|f(s)| \leq |f(s) - f(t+)| + |f(t+)| < 1 + |f(t+)|, \quad \text{for } s \in (t, t+\delta_{t+}).$$

Similarly, there exists $\delta_{t-} > 0$, such that

$$|f(s)| \leq |f(s) - f(t-)| + |f(t-)| < 1 + |f(t-)|, \quad \text{for } s \in (t-\delta_{t-}, t).$$

Gechun Liang
Department of Statistics, University of Warwick, U.K. e-mail: g.liang@warwick.ac.uk

Hence, we have, for $s \in O_t := (t - \delta_{t-}, t + \delta_{t+})$,

$$|f(s)| \leq \max\{1 + |f(t+)|, 1 + |f(t-)|, |f(t)|\} =: r_t.$$

Since $\{O_t : t \in [0,T]\}$ is an open cover of $[0,T]$, there exists a finite set $\{t_1, \ldots, t_n\} \subset [0,T]$ such that $[0,T] \subset \cup_{i=1}^n O_{t_i}$. It follows that

$$\sup_{s \in [0,T]} |f(s)| \leq \max\{r_{t_1}, \ldots, r_{t_n}\}.$$

The proof is complete. □

Functions with one-sided limits cannot have large jumps which accumulate, i.e. for any point, there exists a neighborhood of that point such that the function jumps with infinitesimal size in that neighborhood except at that point.

Theorem 2. *If $f : [0,T] \to \mathbb{R}$ has one-sided limits, then for any $t \in [0,T]$ and $\varepsilon > 0$, there exits $\delta_t > 0$ such that*

$$|f(s+) - f(s)| + |f(s-) - f(s)| < \varepsilon$$

for $s \in (t - \delta_t, t) \cup (t, t + \delta_t)$.

Proof. Suppose not, there exists $t \in [0,T]$, $\varepsilon > 0$, and a sequence $\{s_n\}_{n \geq 1}$ such that

$$|f(s_n+) - f(s_n)| + |f(s_n-) - f(s_n)| \geq \varepsilon$$

for $s_n \in (t - \frac{1}{n}, t) \cup (t, t + \frac{1}{n})$.

Introduce the following four sets

$$\begin{aligned}
S_1 &:= \left\{ n : s_n > t, |f(s_n+) - f(s_n)| \geq \frac{\varepsilon}{2} \right\} \\
S_2 &:= \left\{ n : s_n > t, |f(s_n) - f(s_n-)| \geq \frac{\varepsilon}{2} \right\} \\
S_3 &:= \left\{ n : s_n < t, |f(s_n+) - f(s_n)| \geq \frac{\varepsilon}{2} \right\} \\
S_4 &:= \left\{ n : s_n < t, |f(s_n) - f(s_n-)| \geq \frac{\varepsilon}{2} \right\}
\end{aligned}$$

Then, it is clear that $\cup_{i=1}^4 S_i$ covers the set of natural numbers $\mathbb{N}$.

Firstly, for $n \in S_1$, since $f(s_n+)$ exists, we may choose $u_n \in (s_n, s_n + \frac{1}{n})$ such that $|f(u_n) - f(s_n+)| < \frac{\varepsilon}{4}$. In turn,

$$\begin{aligned}
\frac{\varepsilon}{2} \leq |f(s_n+) - f(s_n)| &\leq |f(s_n+) - f(u_n)| + |f(u_n) - f(s_n)| \\
&\leq \frac{\varepsilon}{4} + |f(u_n) - f(s_n)|.
\end{aligned}$$

However, since $f(t+)$ exists, and $u_n, s_n \downarrow t$, we have

$$\lim_{n\to\infty} |f(u_n) - f(s_n)| \le \lim_{n\to\infty} |f(u_n) - f(t+)| + \lim_{n\to\infty} |f(s_n) - f(t+)| = 0,$$

which is a contradiction.

Similarly, we also get contradiction for the other three sets S_2, S_3 and S_4. $\square$

One of the most fundamental properties of functions with one-sided limits is the following:

Theorem 3. *If $f : [0,T] \to \mathbb{R}$ has one-sided limits, then it has at most countably many jumps, and finite many jumps with size larger than* 1.

Proof. Note that f is continuous at $t \in [0,T]$ iff $f(t+) = f(t-) = f(t)$. Let

$$A_n = \{t \in [0,T] : |f(t+) - f(t)| + |f(t-) - f(t)| \ge \frac{1}{n}\}.$$

Then, $A = \cup_{n\ge1} A_n$ is the set of jumps of the function f, and A_1 is the set of jumps with size larger than 1.

Fix $t \in [0,T]$. It follows from Theorem 2 that there exists $\delta_t > 0$ such that, for $s \in (t-\delta_t, t) \cup (t, t+\delta_t)$,

$$|f(s+) - f(s)| + |f(s-) - f(s)| < \frac{1}{n}.$$

Thus, $(t-\delta_t, t) \cup (t, t+\delta_t) \cap A_n = \emptyset$. In turn, for $O_t := (t-\delta_t, t+\delta_t)$, we have $O_t \cap A_n \subset \{t\}$.

Since $\{O_t : t \in [0,T]\}$ is an open cover of $[0,T]$, there exists a finite set $\{t_1, \dots, t_m\} \subset [0,T]$ such that $[0,T] \subset \cup_{i=1}^m O_{t_i}$. In turn,

$$A_n = [0,T] \cap A_n \subset \cup_{i=1}^m O_{t_i} \cap A_n \subset \{t_1, \dots, t_m\},$$

which means A_n is finite. In particular, A_1 is finite. Finally, since $A = \cup_{n\ge1} A_n$, it follows that A is countable. $\square$

Proposition 1. *If $f : [0,T] \to \mathbb{R}$ is increasing, then f has one-sided limits. Therefore, an increasing function has at most countably many jumps, and finite many jumps with size larger than* 1.

Proof. Fix $t \in [0,T]$. We show that $f(t-)$ exists. Let $t_n \uparrow t$. By the increasing property of f, $f(t_n) \le f(t)$. Thus, there exists $L \in \mathbb{R}$, such that $f(t_n) \uparrow L$.

Consider another sequence $s_m \uparrow t$. Similarly, there exists $L' \in \mathbb{R}$, such that $f(s_m) \uparrow L'$. For $m \in \mathbb{N}$, since $s_m < t$ and $t_n \uparrow t$, there exists N such that $t_n > s_m$ for $n \ge N$. Hence, $f(s_m) \le f(t_n)$. Sending $n \to \infty$ yields $f(s_m) \le L$. Sending $m \to \infty$ further yields $L' \le L$. By interchanging the rule of the sequences $\{s_m\}_{m\ge1}$ and $\{t_n\}_{n\ge1}$, we also have $L \le L'$, so $L = L'$. The existence of $f(t+)$ is left as an exercise. $\square$

2 Cadlag Functions with bounded variation

Definition 2. (Bounded variation) Let $f : [0,T] \to \mathbb{R}$ be a function such that

$$v_f(T) := \sup_D \sum_{i=1}^{N} |f(t_i) - f(t_{i-1})| < \infty,$$

where D ranges over all the partitions of $[0,T]$: $0 = t_0 < t_1 < \cdots < t_N = T$. Then f is said to have bounded variation over $[0,T]$, and $v_f(T)$ is called the variation of the function f over $(0,T]$.[1] □

Example 1. • Lebesgue integral is of bounded variation.

$$f(t) = \int_0^t g(s)ds, \quad \text{for } t \in [0,T],$$

where $\int_0^T |g(s)|ds < \infty$. This is because

$$v_f(T) = \sup_D \sum_{i=1}^{N} |\int_{t_{i-1}}^{t_i} g(s)ds| \le \sup_D \sum_{i=1}^{N} \int_{t_{i-1}}^{t_i} |g(s)|ds = \int_0^T |g(s)|ds < \infty.$$

• Increasing function is of bounded variation.

$$f(t) = a(t), \quad \text{for } t \in [0,T],$$

where a is increasing. This is because

$$v_f(T) = \sup_D \sum_{i=1}^{N} |a(t_i) - a(t_{i-1})| = a(T) - a(0) < \infty. \ \square$$

Definition 3. (Canonical decomposition) Let $f : [0,T] \to \mathbb{R}$ be of bounded variation. Its canonical decomposition is defined as

$$f(t) = f(0) + a(t) - b(t),$$

where a and b are two increasing functions satisfying $a(0) = b(0) = 0$, and are given by

$$\begin{aligned} a(t) &= v_f(t), \\ b(t) &= a(t) - f(t) + f(0). \ \square \end{aligned}$$

It is clear that in the canonical decomposition, a is increasing, and $a(0) = b(0) = 0$. To prove that b is increasing, note that for $t' \ge t$, we have

---

[1] Recall a process A is said to have *finite variation* if it has *bounded variation* over every finite time interval, say $[0,t]$.

$$a(t') - a(t) = v_f(t') - v_f(t) \geq |f(t') - f(t)| \geq f(t') - f(t).$$

It follows that

$$b(t') - b(t) = a(t') - a(t) - (f(t') - f(t)) \geq 0.$$

If a function f is of bounded variation, by the canonical decomposition, it can be written as the difference of two increasing functions. It thus follows from Proposition 1 that f has one-sided limits and, therefore, has at most countably many jumps.

If, furthermore, f is Cadlag, we call such a function a BV function. Then, a and b in the canonical decomposition of f are also Cadlag.

Definition 4. (Lebesgue-Stieltjes measure and integral) Let a and b be the canonical decomposition of a BV function f. Define a finite measure on $((0,T], \mathscr{B}(0,T])$ induced by a by

$$\mu_a(t_1,t_2] = a(t_2) - a(t_1),$$

for $(t_1,t_2] \subset (0,T]$, and by convention $\mu_a\{0\} = 0$.

In general, for any $E \subset \cup_{i\geq 1} A_i$ with A_i being the interval of the form $(t_1,t_2] \subset (0,T]$, we define the outer measure of E by

$$\mu_a^*(E) = \inf_{(A_i)_{i\geq 1}} \left\{ \sum_{i\geq 1} \mu_a(A_i) \right\}.$$

The Caratheodory theorem then implies that the collection $\mathscr{A}$ of μ_a^*-measurable sets forms a σ-algebra and, moreover, μ_a^* is a measure on $([0,T], \mathscr{A})$ containing all the null sets. Every Borel set in $\mathscr{B}(0,T]$ is μ_a^*-measurable. We now drop asterisk from μ_a^* and call μ_a Lebesgue-Stieltjes measure induced by a. Note that since $\{t\} = \cap_{n\geq 1}(t - \frac{1}{n}, t]$, we have

$$\begin{aligned}
\mu_a\{t\} &= \lim_{\varepsilon\to 0} \mu_a(t-\varepsilon, t] \\
&= \lim_{\varepsilon\to 0} (a(t) - a(t-\varepsilon)) = a(t) - a(t-),
\end{aligned}$$

for $t \in (0,T]$. We will be using the notations $d\mu_a(t) = \mu_a(dt)$ interchangeably.

Similarly, we also define a finite measure $d\mu_b(t)$ on $((0,T], \mathscr{B}(0,T])$ induced by b. Then, $d\mu_b(t) \leq 2(d\mu_a(t))$, and the Lebesgue-Stieltjes measure induced by the BV function f is defined as

$$df(t) = d\mu_a(t) - d\mu_b(t).$$

The corresponding Lebesgue-Stieltjes integral on any interval $(t_1,t_2] \subset (0,T]$ is defined as

$$\int_{(t_1,t_2]} df(t) = \int_{t_1}^{t_2} df(t) = f(t_2) - f(t_1). \ \square$$

In general, for any function g such that $\int_0^T |g(s)| d\mu_a(s) < \infty$ and $A \in \mathscr{B}(0,T]$,

$$\int_A g(t)df(t) = \int_0^T \mathbf{1}_A(t)g(t)df(t)$$
$$= \int_0^T \mathbf{1}_A(t)g(t)d\mu_a(t) - \int_0^T \mathbf{1}_A(t)g(t)d\mu_b(t).$$

Note that if g is a pure jump Cadlag function (so g is of bounded variation and a BV function), i.e. $g(t) = g(0) + \sum_{0<s\leq t} \Delta g(s)$, with $\Delta g(s) = g(s) - g(s-)$, then

$$\int_0^t \Delta g(s)df(s) = \sum_{0<s\leq t} \Delta g(s)\Delta f(s),$$

and

$$\int_0^t f(s)dg(s) = \sum_{0<s\leq t} f(s)\Delta g(s).$$

Theorem 4. *For any two BV functions f and $g : [0,T] \to \mathbb{R}$, the following integration by parts formulas hold:*

$$f(t)g(t) = f(0)g(0) + \int_0^t f(s)dg(s) + \int_0^t g(s-)df(s) \tag{1}$$
$$= f(0)g(0) + \int_0^t f(s-)dg(s) + \int_0^t g(s)df(s) \tag{2}$$
$$= f(0)g(0) + \int_0^t f(s-)dg(s) + \int_0^t g(s-)df(s) + \sum_{0<s\leq t} \Delta f(s)\Delta g(s). \tag{3}$$

Proof. We first prove (1). Note that

$$(f(t) - f(0))(g(t) - g(0)) = \int_0^t df(x)\int_0^t dg(y)$$
$$= \int_D df(x)dg(y) \tag{4}$$

with $D = (0,t] \times (0,t]$. Introduce

$$D_1 := \{(x,y) \in D : x \leq y\},$$
$$D_2 := \{(x,y) \in D : x > y\}.$$

Then, using Fubini's theorem, we obtain

$$\int_{D_1} df(x)dg(y) = \int_{(0,t]} \left(\int_{(0,y]} df(x)\right) dg(y)$$
$$= \int_{(0,t]} (f(y) - f(0))\, dg(y)$$
$$= \int_0^t f(y)dg(y) - f(0)(g(t) - g(0)). \tag{5}$$

Likewise,

$$\begin{aligned}\int_{D_2} df(x)dg(y) &= \int_{(0,t]} \left(\int_{(0,x)} dg(y) \right) df(x) \\ &= \int_{(0,t]} (g(x-) - g(0))\, df(x) \\ &= \int_0^t g(x-)df(x) - g(0)(f(t) - f(0)). \end{aligned} \tag{6}$$

The integration by parts formula (1) then follows by combining (4), (5) and (6). □

The proof of the formula (2) is similar to (1). Finally, to prove (3), we note that

$$\begin{aligned}\int_0^t g(s)df(s) &= \int_0^t g(s-)df(s) + \int_0^t \Delta g(s)df(s) \\ &= \int_0^t g(s-)df(s) + \sum_{0<s\le t} \Delta g(s)\Delta f(s),\end{aligned}$$

and (3) follows from (2). □

In general, we also have the following change of variables formula, whose proof is omitted.

Theorem 5. *Let $F \in C^1(\mathbb{R})$. For any BV function $f : [0,T] \to \mathbb{R}$, the following* <u>*change of variables formula*</u> *holds:*

$$\begin{aligned}F(f(t)) - F(f(0)) = &\int_0^t F'(f(s-))df(s) \\ &+ \sum_{0<s\le t} \left[F(f(s)) - F(f(s-)) - F'(f(s-))\Delta f(s) \right]. \end{aligned} \tag{7}$$

Example 2. For BV function f, we calculate $(f(t))^2$. Applying the integration by parts formula (3) with $g = f$ yields

$$(f(t))^2 = (f(0))^2 + 2\int_0^t f(s-)df(s) + \sum_{0<s\le t} |\Delta f(s)|^2.$$

Note that this implies that the infinite sum $\sum_{0<s\le t} |\Delta f(s)|^2 < \infty$.

On the other hand, we may also apply the change of variables formula (7) with $F(x) = x^2$, and have

$$\begin{aligned}(f(t))^2 - (f(0))^2 = &2\int_0^t f(s-)df(s) \\ &+ \sum_{0<s\le t} \left[(f(s))^2 - (f(s-))^2 - 2f(s-)\Delta f(s) \right].\end{aligned}$$

Note that the last term is

$$\begin{aligned}&(f(s))^2-(f(s-))^2-2f(s-)\Delta f(s)\\=&(f(s)-f(s-))^2+2f(s)f(s-)-2(f(s-))^2-2f(s-)\Delta f(s)\\=&|\Delta f(s)|^2+2f(s-)(f(s)-f(s-))-2f(s-)\Delta f(s)\end{aligned}$$

which is nothing but $|\Delta f(s)|^2$. □

Theorem 6. *Let $a:[0,T]\to\mathbb{R}$ be a BV function, with $\int_0^T|\mu(s)|da(s)<\infty$ for some function μ. Then the equation*

$$Z(t)=Z(0)-\int_0^t Z(s-)\mu(s)da(s) \tag{8}$$

admits a (unique) solution given by

$$Z(t)=Z(0)\prod_{0<s\leq t}(1-\mu(s)\Delta a(s))e^{-\int_0^t\mu(s)da^c(s)},$$

where $\Delta a(s)=a(s)-a(s-)$ and $a^c(s)=a(s)-\sum_{0<s\leq t}\Delta a(s)$.

Proof. Let $f(t)=Z(0)\prod_{0<s\leq t}(1-\mu(s)\Delta a(s))$ and $g(t)=e^{-\int_0^t\mu(s)da^c(s)}$. It is obvious that g is a BV function. We next verify that f is also a BV function. We first verify that the infinite product in f is finite. Indeed, using the elementary inequality $(1-x)\leq e^{-x}$, we get

$$\prod_{0<s\leq t}(1-\mu(s)\Delta a(s))\leq e^{\sum_{0<s\leq t}-\mu(s)\Delta a(s)}.$$

By the assumption on the function μ,

$$\int_0^T|\mu(s)|da(s)=\int_0^T|\mu(s)|da^c(s)+\sum_{0<s\leq t}|\mu(s)|\Delta a(s)<\infty.$$

This implies that $\sum_{0<s\leq t}-\mu(s)\Delta a(s)<\infty$, so f is well defined. It is obvious that f, as a pure jump function, is of bounded variation and Cadlag, so it is BV function. Furthermore, since $f(t)=f(t-)(1-\mu(t)\Delta a(t))$, we have

$$\begin{aligned}\Delta f(t)&=f(t)-f(t-)\\&=-f(t-)\mu(t)\Delta a(t).\end{aligned} \tag{9}$$

Using the integration by parts formula (2) in Theorem 4, we obtain

$$\begin{aligned}f(t)g(t)&=f(0)g(0)+\int_0^t f(s-)dg(s)+\int_0^t g(s)df(s)\\&=f(0)g(0)-\int_0^t f(s-)g(s-)\mu(s)da^c(s)+\sum_{0<s\leq t}g(s)\Delta f(s).\end{aligned}$$

Note that the last term, by (9), is equal to

$$\sum_{0<s\le t} g(s)\Delta f(s) = -\sum_{0<s\le t} f(s-)g(s-)\mu(s)\Delta a(s),$$

and therefore,

$$f(t)g(t) = f(0)g(0) - \int_0^t f(s-)g(s-)\mu(s)da(s).$$

That is $f(t)g(t)$, $t\in[0,T]$, is a solution to the equation (8).

The uniqueness of the solution is left as an exercise. $\square$

3 Single jump processes and Girsanov's theorem

3.1 Stopping times

Definition 5. A filtered probability space $(\Omega, \mathscr{F}, \{\mathscr{F}_t\}_{t\ge 0}, \mathbf{P})$ is said to satisfy the usual conditions if the following conditions hold:
(1) completeness: $\mathscr{F}_0$ includes all of the $\mathbf{P}$-null sets;
(2) right continuity: $\mathscr{F}_t = \mathscr{F}_{t+}$ where $\mathscr{F}_{t+} = \cap_{n\ge 1}\mathscr{F}_{t+\frac{1}{n}}$. $\square$

For any "reasonable" strong Markov process X (e.g. Feller processes including Levy, Brownian and Poisson processes), its natural filtration $\mathscr{F}_t := \sigma(X_s : s\le t)$ *after augmentation* is right continuous[2].

Definition 6. A random variable $\tau : \Omega \to [0,\infty]$ is called an $\mathscr{F}_t$-stopping time if $\{\tau \le t\} \in \mathscr{F}_t$ for $t\ge 0$. The random variable τ is called an optional time if $\{\tau < t\} \in \mathscr{F}_t$ $\square$

If τ is an $\mathscr{F}_t$-stopping time, then

$$\{\tau < t\} = \cup_{n\ge 1}\{\tau \le t - \frac{1}{n}\} \in \cup_{n\ge 1}\mathscr{F}_{t-\frac{1}{n}} \subset \mathscr{F}_t.$$

However, $\{\tau < t\}\in\mathscr{F}_t$ does not necessarily imply that $\{\tau\le t\}\in\mathscr{F}_t$ unless the filtration is right continuous. To see this,

$$\{\tau \le t\} = \cap_{n\ge 1}\{\tau < t + \frac{1}{n}\} \in \cap_{n\ge 1}\mathscr{F}_{t+\frac{1}{n}},$$

which is $\mathscr{F}_t$ only if $\{\mathscr{F}_t\}_{t\ge 0}$ is right continuous.

Example 3. Let $\{\tau_n\}_{n\ge 1}$ be a sequence of $\mathscr{F}_t$-stopping times. Then,

$$\left\{\sup_{n\ge 1}\tau_n \le t\right\} = \cap_{n\ge 1}\{\tau_n \le t\} \in \mathscr{F}_t,$$

[2] Note that the natural filtration of Poisson processes is right continuous before argumentation, and so are single jump processes.

so $\sup_{n\geq 1}\tau_n$ *is again an* $\mathscr{F}_t$*-stopping time*. However, since

$$\left\{\inf_{n\geq 1}\tau_n \leq t\right\} = \cap_{m\geq 1}\cup_{n\geq 1}\{\tau_n < t+\frac{1}{m}\} \in \cap_{m\geq 1}\mathscr{F}_{t+\frac{1}{m}},$$

which is $\mathscr{F}_t$ only if $\{\mathscr{F}_t\}_{t\geq 0}$ is right continuous, $\inf_{n\geq 1}\tau_n$ *is an* $\mathscr{F}_t$*-stopping time only if the filtration is right continuous*.

On the other hand, if τ_n is only optional, then since $\{\inf_{n\geq 1}\tau_n \geq t\} = \cap_{n\geq 1}\{\tau^n \geq t\}$, it follows that

$$\left\{\inf_{n\geq 1}\tau_n < t\right\} = \cup_{n\geq 1}\{\tau^n < t\} \in \mathscr{F}_t,$$

$\inf_{n\geq 1}\tau_n$ is an optional time. □

Definition 7. The past at the stopping time τ is the σ-field $\mathscr{F}_\tau$ defined by

$$\mathscr{F}_\tau = \{A \in \mathscr{F}_\infty : A\cap\{\tau \leq t\} \in \mathscr{F}_t \text{ for } t \geq 0\}$$

The strict past at the stopping time τ is the σ-field $\mathscr{F}_{\tau-}$ generated by the set

$$\mathscr{F}_{\tau-} = \sigma\left(\{A_0 \in \mathscr{F}_0\}\cup\{A_s\cap\{\tau > s\} \text{ for } s\geq 0, A_s \in \mathscr{F}_s\}\right)$$

Proposition 2. *Both* $\mathscr{F}_\tau$ *and* $\mathscr{F}_{\tau-}$ *are* σ*-fields satisfying* $\mathscr{F}_{\tau-}\subset\mathscr{F}_\tau$*, and* τ *is an* $\mathscr{F}_{\tau-}$*-measurable random variable (therefore also* $\mathscr{F}_\tau$*-measurable). When* X *is progressively measurable,* X_τ *is* $\mathscr{F}_\tau$*-measurable.*

Proof. The verification of $\mathscr{F}_\tau$ and $\mathscr{F}_{\tau-}$ being σ-fields is by the definition. For example, for $A \in \mathscr{F}_\tau$, $A^c\cap\{\tau\leq t\} = \{\tau \leq t\} - A\cap\{\tau\leq t\}$. Since $\{\tau \leq t\}\in\mathscr{F}_t$ and $A\cap\{\tau\leq t\}\in\mathscr{F}_t$, it follows that $A^c \in \mathscr{F}_\tau$.

To prove that $\mathscr{F}_{\tau-}\subset\mathscr{F}_\tau$, it suffices to show that the generators of $\mathscr{F}_{\tau-}$ are in $\mathscr{F}_\tau$. Indeed, $\mathscr{F}_0\subset\mathscr{F}_\tau$. For $A_s\in\mathscr{F}_s$,

$$A_s\cap\{\tau > s\}\cap\{\tau\leq t\} = A_s\cap\{s<\tau\leq t\}\in\mathscr{F}_t.$$

The set $\{\tau = 0\}$ and $\{\tau > a\}$, $a\geq 0$, are generators of $\mathscr{F}_{\tau-}$ and therefore τ is $\mathscr{F}_{\tau-}$ measurable.

Finally, we show that X_τ is $\mathscr{F}_\tau$ measurable. For this, for fixed $t\geq 0$, we aim to show that for any Borel set V, $X_\tau^{-1}(V)\cap\{\tau\leq t\}\in\mathscr{F}_t$. Define two maps

$$\phi_t : \{\omega : \tau(\omega)\leq t\}\to[0,t]\times\Omega, \text{ by } \phi_t(\omega) = (\tau(\omega),\omega),$$

and

$$\phi^t : [0,t]\times\Omega\to\mathbb{R}^d, \text{ by } \phi^t(s,\omega) = X_s(\omega).$$

Note that $X_\tau = \phi^t\circ\phi_t$. We verify that ϕ_t is $\mathscr{F}_t\cap\{\tau\leq t\}\to\mathscr{B}[0,t]\otimes\mathscr{F}_t$ measurable. Indeed, for $A\in\mathscr{F}_t$ and $a\in[0,t]$, since τ is a stopping time,

$$\phi_t^{-1}([0,a]\times A) = \{\tau\leq a\}\cap A\subset\{\tau\leq t\}\cap A\in\mathscr{F}_t\cap\{\tau\leq t\}.$$

Together with X being progressively measurable, i.e. ϕ^t is $\mathscr{B}[0,t]\otimes\mathscr{F}_t \to \mathscr{B}(\mathbb{R}^d)$ measurable, we conclude that $X_\tau = \phi^t \circ \phi_t$ is $\mathscr{F}_t \cap \{\tau \le t\} \to \mathscr{B}(\mathbb{R}^d)$ measurable. Hence,

$$\begin{aligned} X_\tau^{-1}(V)\cap\{\tau\le t\} &= \{\omega : \tau(\omega)\le t, X_{\tau(\omega)}(\omega)\in V\} \\ &= \{\omega : \tau(\omega)\le t, \phi^t\circ\phi_t(\omega)\in V\} \\ &= \{\tau\le t\}\cap\phi^t\circ\phi_t^{-1}(V)\in\mathscr{F}_t \end{aligned}$$

□

3.2 Single jump processes

Let $\tau : \Omega \to \mathbb{R}_+$ be a non-negative random variable with property $\mathbf{P}(\tau = 0) = 0$ and $\mathbf{P}(\tau > t) > 0$ for any $t \in \mathbb{R}_+$. Introduce the corresponding single jump process $H_t = \mathbf{1}_{\{\tau\le t\}}$, $t \ge 0$, and its natural filtration $\{\mathscr{F}_t\}_{t\ge 0}$ by

$$\mathscr{F}_t = \sigma(H_u : u \le t)$$

with $\mathscr{F}_\infty = \sigma(H_u : u \in \mathbb{R}_+)$. It is easy to check the following properties of $\mathscr{F}_t$.

1. $\mathscr{F}_t = \sigma(\{\tau \le u\} : u \le t)$;
2. $\mathscr{F}_t = \sigma(\sigma(\tau)\cap\{\tau \le t\})$;
3. $\mathscr{F}_t = \sigma(\sigma(\tau\wedge t)\cup\{\tau > t\})$;
4. $\mathscr{F}_t = \mathscr{F}_{t+}$;
5. $\mathscr{F}_\infty = \sigma(\tau)$;
6. $A\cap\{\tau \le t\}\in\mathscr{F}_t$ for any $A\in\mathscr{F}_\infty$.

The following formulas are useful to calculate the conditional distribution of τ. The key point (which may not be obvious at the beginning) is that any $\mathscr{F}_t$ measurable r.v. X_t is of the form $X_t = x_t^0\mathbf{1}_{\{\tau>t\}} + x_t^1(\tau)\mathbf{1}_{\{\tau\le t\}}$ (called *Jacod's decomposition for optional processes*).

Lemma 1. *For any random variable $Y \in \mathscr{F}_\infty$,*

$$\mathbf{E}[Y|\mathscr{F}_t] = \frac{\mathbf{E}[\mathbf{1}_{\{\tau>t\}}Y]}{\mathbf{P}(\tau>t)}\mathbf{1}_{\{\tau>t\}} + \mathbf{E}[Y|\sigma(\tau)]\mathbf{1}_{\{\tau\le t\}}.$$

Proof. We first prove on $\{\tau\le t\}$, $\mathbf{E}[Y|\mathscr{F}_t] = \mathbf{E}[Y|\sigma(\tau)]$, i.e.

$$\mathbf{E}[\mathbf{1}_{\{\tau\le t\}}Y|\mathscr{F}_t] = \mathbf{E}[\mathbf{1}_{\{\tau\le t\}}Y|\sigma(\tau)]$$

In other words, $\mathbf{E}[\mathbf{1}_{\{\tau\le t\}}Y|\mathscr{F}_t]$ *is the conditional expectation of* $\mathbf{1}_{\{\tau\le t\}}Y$ *on* $\sigma(\tau)$. Indeed, for any $A\in\sigma(\tau)$, $A\cap\{\tau\le t\}\in\mathscr{F}_t$, it follows that

$$\mathbf{E}[\mathbf{1}_A\mathbf{E}[\mathbf{1}_{\{\tau\le t\}}Y|\mathscr{F}_t]] = \mathbf{E}[\mathbf{1}_{A\cap\{\tau\le t\}}\mathbf{E}[Y|\mathscr{F}_t]] = \mathbf{E}[\mathbf{1}_A\mathbf{1}_{\{\tau\le t\}}Y].$$

Next, we show on $\{\tau > t\}$, $\mathbf{E}[Y|\mathscr{F}_t] = \frac{\mathbf{E}[\mathbf{1}_{\{\tau>t\}}Y]}{\mathbf{P}(\tau>t)}$, i.e.

$$\mathbf{E}[\mathbf{1}_{\{\tau>t\}}Y|\mathscr{F}_t] = \mathbf{1}_{\{\tau>t\}}\frac{\mathbf{E}[\mathbf{1}_{\{\tau>t\}}Y]}{\mathbf{P}(\tau>t)}. \tag{10}$$

In other words, $\mathbf{1}_{\{\tau>t\}}\mathbf{E}[\mathbf{1}_{\{\tau>t\}}Y]$ *is the conditional expectation of* $\mathbf{1}_{\{\tau>t\}}Y\mathbf{P}(\tau>t)$ *on* $\mathscr{F}_t$. For this, for any $A \in \mathscr{F}_t$, it is sufficient to consider $A = \{\tau \le s\}$ for $s \le t$ which yields $A \cap \{\tau > t\} = \emptyset$, and $A = \{\tau > t\}$ which yields $A \cap \{\tau > t\} = \{\tau > t\}$.

For the case $A \cap \{\tau > t\} = \emptyset$,

$$\mathbf{E}\left[\mathbf{1}_A\mathbf{1}_{\{\tau>t\}}\mathbf{E}[\mathbf{1}_{\{\tau>t\}}Y]\right] = \mathbf{E}\left[\mathbf{1}_A\mathbf{1}_{\{\tau>t\}}Y\mathbf{P}(\tau>t)\right] = 0,$$

so that (10) holds.

For the case $A \cap \{\tau > t\} = \{\tau > t\}$,

$$\mathbf{E}\left[\mathbf{1}_A\mathbf{1}_{\{\tau>t\}}\mathbf{E}[\mathbf{1}_{\{\tau>t\}}Y]\right] = \mathbf{P}(\tau>t)\mathbf{E}[\mathbf{1}_{\{\tau>t\}}Y],$$

and

$$\mathbf{E}\left[\mathbf{1}_A\mathbf{1}_{\{\tau>t\}}Y\mathbf{P}(\tau>t)\right] = \mathbf{E}[\mathbf{1}_{\{\tau>t\}}Y]\mathbf{P}(\tau>t),$$

from which we conclude. □

One of the most typical examples of the stopping time τ used to model default time is generated by an exponential random variable with constant intensity $\lambda > 0$, as shown in the following example.

Example 4. If τ follows exponential distribution with constant intensity $\lambda > 0$, then formula (10) implies that

$$\mathbf{E}[\mathbf{1}_{\{\tau>t\}}Y|\mathscr{F}_t] = \mathbf{1}_{\{\tau>t\}}e^{\lambda t}\mathbf{E}[\mathbf{1}_{\{\tau>t\}}Y].$$

In particular, taking $Y = \mathbf{1}_{\{\tau>T\}}$ yields

$$\mathbf{P}(\tau > T|\mathscr{F}_t) = \mathbf{1}_{\{\tau>t\}}e^{-\lambda(T-t)}. \tag{11}$$

Taking $Y = \mathbf{1}_{\{t<\tau\le T\}}$ yields

$$\mathbf{P}(t < \tau \le T|\mathscr{F}_t) = \mathbf{1}_{\{\tau>t\}}(1-e^{-\lambda(T-t)}). \tag{12}$$

We also have the martingale characterisation of the single jump process $H_t := 1_{\{\tau\le t\}}$, $t \ge 0$, when τ follows exponential distribution.

Lemma 2. *The* $\mathscr{F}_t$*-stopping time* τ *follows exponential distribution with constant intensity* $\lambda > 0$ *iff*

$$M_t := H_t - \int_0^t \mathbf{1}_{\{\tau>s\}}\lambda ds,\ t \ge 0,$$

is an $(\mathscr{F}_t, \mathbf{P})$*-martingale and* $\mathbf{P}(\tau > 0) = 1$.

Proof. Only if part: For any $T \geq t \geq 0$, by the formula (11),

$$\begin{aligned}\mathbf{E}[M_T|\mathscr{F}_t] =& 1-\mathbf{E}[1_{\{\tau>T\}}|\mathscr{F}_t] - \int_0^t \mathbf{1}_{\{\tau>s\}}\lambda ds - \int_t^T \mathbf{E}[\mathbf{1}_{\{\tau>s\}}\lambda|\mathscr{F}_t]ds \\ =& 1-\mathbf{1}_{\{\tau>t\}}e^{-\lambda(T-t)} - \int_0^t \mathbf{1}_{\{\tau>s\}}\lambda ds - \mathbf{1}_{\{\tau>t\}}\int_t^T \lambda e^{-\lambda(s-t)}ds \\ =& 1_{\{\tau\leq t\}} - \int_0^t \mathbf{1}_{\{\tau>s\}}\lambda ds = M_t.\end{aligned}$$

Since τ follows exponential distribution, it follows that $\mathbf{P}(\tau > 0) = e^{-\lambda 0} = 1$.

If part: For $t \geq 0$, define $\Phi(t) = \mathbf{P}(\tau > t)$. Then, following the martingale property of M,

$$\begin{aligned}\Phi(t) =& \mathbf{E}\left[1 - M_t - \int_0^t \mathbf{1}_{\{\tau>s\}}\lambda ds\right] \\ =& 1 - M_0 - \lambda\int_0^t \Phi(s)ds.\end{aligned}$$

It follows from the condition $\tau > 0$ a.s. that $M_0 = H_0 = 0$, a.s., so

$$\Phi(t) = 1 - \lambda\int_0^t \Phi(s)ds$$

which implies that $\Phi(t) = e^{-\lambda t}$, i.e. τ follows exponential distribution with intensity λ. □

In practice, we often need to model λ as an $\mathscr{F}_t$-prog measurable stochastic process. Based on the above martingale characterisation, we impose the following assumption on the $\mathscr{F}_t$-stopping time τ through its corresponding single jump process $H_t = \mathbf{1}_{\{\tau\leq t\}}$, $t \geq 0$. It is clear that for each ω, $H_\cdot(\omega)$ is a BV function (recall BV means Cadlag with bounded variation).

Assumption 1 *Let τ be a non-negative random variable defined on $(\Omega, \mathscr{F}, \mathbf{P})$, and $\{\mathscr{F}_t\}_{t\geq 0}$ be the natural filtration of $H_t = \mathbf{1}_{\{\tau\leq t\}}$, $t \geq 0$. i.e. $\mathscr{F}_t = \sigma(H_s : s \leq t)$, such that*

$$M_t := H_t - \int_0^t \mathbf{1}_{\{\tau>s\}}\lambda_s ds,\ t \geq 0,$$

is an $(\mathscr{F}_t, \mathbf{P})$-martingale, for λ being an $\mathscr{F}_t$-prog measurable, strictly positive and bounded process. Moreover, we assume that $\mathbf{P}(\tau > 0) = 1$.

Since $H_\cdot(\omega)$ is BV, it is obvious that $M_\cdot(\omega)$ is also BV, that is, *M is a Cadlag martingale with bounded variation, and moreover, $\Delta M_t = \Delta H_t$.*

3.3 Girsanov's theorem

We next discuss the Girsanov's theorem for the single jump process H under Assumption 1.

Theorem 7. *Let $\mu \in [0,1]$ be a constant, and suppose that Assumption 1 is satisfied. For $T>0$, define $Z_t^\mu = C_t^\mu V_t^\mu$ for $t \in [0,T]$, where*

$$C_t^\mu = e^{\int_0^t \mu \mathbf{1}_{\{\tau>s\}}\lambda_s ds},$$

and

$$V_t^\mu = \mathbf{1}_{\{\tau>t\}} + (1-\mu)\mathbf{1}_{\{\tau\le t\}}.$$

Then, Z^μ is an $(\mathscr{F}_t, \mathbf{P})$-martingale, and satisfies,

$$Z_t^\mu = 1 - \int_0^t Z_{s-}^\mu \mu dM_s, \quad \text{for } t \in [0,T].$$

Proof. Note that for $T>0$, $\int_0^T |\mu| dM_s = \mu M_T < \infty$. We decompose the martingale M into its continuous part and pure jump part as

$$\begin{aligned} M_t =& M_t^c + \sum_{0<s\le t} \Delta M_s \\ =& -\int_0^t \mathbf{1}_{\{\tau>s\}}\lambda_s ds + H_t \end{aligned}$$

In turn, we have

$$e^{-\int_0^t \mu dM_s^c} = e^{\int_0^t \mu \mathbf{1}_{\{\tau>s\}}\lambda_s ds} = C_t^\mu,$$

and since $\Delta M_s = \Delta H_s$,

$$\prod_{0<s\le t}(1-\mu\Delta M_s) = \prod_{0<s\le t}(1-\mu\Delta H_s) = \mathbf{1}_{\{\tau>t\}} + (1-\mu)\mathbf{1}_{\{\tau\le t\}} = V_t^\mu.$$

Theorem 6 then implies that $C_t^\mu V_t^\mu$ satisfies, for $t \in [0,T]$,

$$C_t^\mu V_t^\mu = 1 - \int_0^t C_{s-}^\mu V_{s-}^\mu \mu dM_s,$$

so $Z_t^\mu = C_t^\mu V_t^\mu$, $t \in [0,T]$, is an $(\mathscr{F}_t, \mathbf{P})$-local martingale. Since both C_t^μ and V_t^μ are bounded for $t \in [0,T]$, we conclude that Z^μ is also an $(\mathscr{F}_t, \mathbf{P})$-martingale. $\square$

Theorem 8. *Let $T>0$ be fixed. Given the $(\mathscr{F}_t, \mathbf{P})$-martingale Z^μ as in Theorem 7, define a new probability measure $\mathbf{Q}^\mu$ by the Radon-Nikodym density*

$$\left.\frac{d\mathbf{Q}^\mu}{d\mathbf{P}}\right|_{\mathscr{F}_t} = Z_t^\mu.$$

Then,

$$M_t^\mu = H_t - \int_0^t (1-\mu)\mathbf{1}_{\{\tau>s\}}\lambda_s ds,\ t\in[0,T],$$

is an $(\mathscr{F}_t, \mathbf{Q}^\mu)$*-martingale.*

Proof. Note that by the Bayes' formula, M_t^μ, $t\in[0,T]$, *is an* $(\mathscr{F}_t, \mathbf{Q}^\mu)$*-martingale iff* $M_t^\mu Z_t^\mu$, $t\in[0,T]$ *is an* $(\mathscr{F}_t, \mathbf{P})$*-martingale.*

Hence, it is sufficient to show that $M_t^\mu Z_t^\mu$, $t\in[0,T]$, is an $(\mathscr{F}_t, \mathbf{P})$-martingale. Using the integration by parts formula (3), we obtain

$$M_t^\mu Z_t^\mu = \int_0^t M_{s-}^\mu dZ_s^\mu + \int_0^t Z_{s-}^\mu dM_s^\mu + \sum_{0<s\le t}\Delta M_s^\mu \Delta Z_s^\mu. \tag{13}$$

Note that M^μ can be rewritten as

$$M_s^\mu = M_s + \int_0^t \mu\mathbf{1}_{\{\tau>s\}}\lambda_s ds,$$

so

$$\int_0^t Z_{s-}^\mu dM_s^\mu = \int_0^t Z_{s-}^\mu dM_s + \int_0^t Z_{s-}^\mu \mu\mathbf{1}_{\{\tau>s\}}\lambda_s ds. \tag{14}$$

On the other hand, since $\Delta Z_s^\mu = -Z_{s-}^\mu \mu\Delta M_s$, we have

$$\sum_{0<s\le t}\Delta M_s^\mu \Delta Z_s^\mu = -\sum_{0<s\le t} Z_{s-}^\mu \mu|\Delta M_s|^2.$$

But $\Delta M_s = \Delta H_s$ and $|\Delta H_s|^2 = \Delta H_s$, it follows that

$$\sum_{0<s\le t}\Delta M_s^\mu \Delta Z_s^\mu = -\sum_{0<s\le t} Z_{s-}^\mu \mu\Delta H_s = -\int_0^t Z_{s-}^\mu \mu dH_s. \tag{15}$$

Plugging (14) and (15) into (13), we get

$$M_t^\mu Z_t^\mu = \int_0^t M_{s-}^\mu dZ_s^\mu + \int_0^t Z_{s-}^\mu dM_s - \int_0^t Z_{s-}^\mu \mu dM_s,$$

which implies that $M_t^\mu Z_t^\mu$, $t\in[0,T]$, is an $(\mathscr{F}_t, \mathbf{P})$-local martingale. Finally, since $M^\mu Z^\mu$ is bounded, it is also an $(\mathscr{F}_t, \mathbf{P})$-martingale. □

Note that when $\mu = 0$, then $Z^0 = 1$. Therefore, $\mathbf{Q}^0 = \mathbf{P}$, and $M^0 = M$ is an $(\mathscr{F}_t, \mathbf{P})$-martingale following from Assumption 1.

On the other hand, when $\mu = 1$, $\mathbf{Q}^1$ is only absolutely continuous w.r.t. $\mathbf{P}$. Therefore, for $A\subset\Omega$, $\mathbf{P}(A) = 0 \Rightarrow \mathbf{Q}^1(A) = 0$. However, for the sets $B_t = \{\tau\le t\}$, $t\in[0,T]$, we have $\mathbf{Q}^1(B_t) = 0$ but $\mathbf{P}(B_t)\neq 0$, so $\mathbf{Q}^1$ and $\mathbf{P}$ are not equivalent.

Exercise 1. (Exponential formula)

1. Prove the solution to the equation (8) is unique.
2. Apply the change of variables formula in Theorem 5 to $\ln Z(t)$ to derive the solution of the equation (8) directly.

References

1. Bremaud, Pierre. *Point processes and queues: martingale dynamics*. Springer-Verlag, 1981.