MA999: Fundamentals of Mathematical Modelling

Part 1 Introduction to Dynamical Systems

1. Consider an nonautonomous system $m\ddot{x} + b\dot{x} + kx = F\cos t$. Re-write it as a first-order system of higher dimensionality.

2. Solve the logistic equation $\dot{N} = rN(1 - N/K), \quad N(0) = N_0$.

3. Find fixed points and classify their stability (by different methods):

   (a) $\dot{x} = 4x^2 - 16$, by graphical analysis or linear stability analysis;

   (b) $\dot{x} = 1 - x^{14}$, by graphical analysis or linear stability analysis;

   (c) $\dot{x} = 1 + \frac{1}{2}\cos x$, by graphical analysis or linear stability analysis;

   (d) $\dot{x} = 1 - 2\cos x$, by linear stability analysis;

   (e) $\dot{x} = e^x - \cos x$, by graphical analysis;

   (f) $\dot{N} = rN(1 - N/K)$, by linear stability analysis;

4.

> The Law of Mass Action: the rate of the chemical reaction is proportional to the product of the concentrations of the molecular species involved in the reaction
>
> $$A + B \xrightarrow{k} C \qquad \Rightarrow \qquad \frac{d[C]}{dt} = k[A][B]$$
>
> k is called the *rate constant*.
>
> The main principle for constructing dynamic equations:
> rate of change = inflow rate - outflow rate
>
> $$A + B \underset{k_-}{\overset{k_+}{\rightleftharpoons}} C \qquad \Rightarrow \qquad \frac{d[A]}{dt} = k_-[C] - k_+[A][B]$$
>
> The Law of Mass Action for the more general reaction:
>
> $$A + mB \xrightarrow{k} nB + pC \qquad \Rightarrow$$
>
> $$\frac{d[A]}{dt} = -kAB^m, \ \frac{d[B]}{dt} = (n - m)kAB^m, \ \frac{d[C]}{dt} = pkAB^m$$

Consider the following reaction of autocatalysis

$$A + X \underset{k_{-1}}{\overset{k_1}{\rightleftharpoons}} 2X.$$

Assuming the concentration $[A] = a$ is constant, write down an ODE for this reaction and explain the system's behaviour.

5. Consider the following reactions

$$A + X \underset{k_{-1}}{\overset{k_1}{\rightleftharpoons}} 2X, \qquad\qquad B + X \overset{k_2}{\rightarrow} C.$$

Assuming the concentrations $[A] = a$, $[B] = b$ and $[C] = c$ are constant, write down an ODE for this model and classify the stability of fixed points as a parameter $p = k_1 a - k_2 b$ varies.

6. Construct the bifurcation diagrams (a diagram of values of fixed points as a function of a parameter) for the following systems:

   (a) $\dot{x} = \mu - |x|$
   (b) $\dot{x} = \mu - x^2 + 4x^4$
   (c) $\dot{x} = r + x - x^3$

7. Sketch the phase portrait of each of the following linear systems:

   (a) $\dot{x} = -3x, \qquad \dot{y} = -2y$
   (b) $\dot{x} = -x + y, \qquad \dot{y} = -x - y$
   (c) $\dot{x} = 3y, \qquad \dot{y} = -3x$