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Brownian Motion

Problem sheet 8

1. **Stopping times**
   Let $(B_t)_{t\geqslant 0}$ be a Brownian motion and consider the hitting time
   $$T_a = \inf\{t > 0 \,:\; B_t = a\}\ ,$$
   for $a > 0$. Prove that for any $\lambda > 0$
   $$\mathbb{E}[e^{-\lambda T_a}] = e^{-a\sqrt{2\lambda}}\ .$$
   Hint: Use the optional stopping theorem with a suitable (exponential) martingale.

2. **Expected occupation measures**
   Consider the expected occupation measure associated to d-dimensional Brownian motion
   $$\nu_{x,t}(A) = \mathbb{E}_x \int_0^t 1_A(B_s)\,\mathrm{d}s\ .$$
   (a) Show that $\nu_{x,y}$ has a density with respect to the Lebesgue measure given by
   $$g_t(x,y) = \int_0^t p_s(x,y)\,\mathrm{d}s\ ,$$
   where $p_t(x,y)$ is the transition probability density of Brownian motion.
   (b) Prove that in $d \geqslant 3$
   $$\lim_{t\to\infty} g_t(x,y) = g(x,y) = \frac{\Gamma(d/2-1)}{2\pi^{\frac{d}{2}}}|x-y|^{2-d}$$
   Hint: Use the change of variables $r = |x-y|^2/2s$.
   (c) Deduce that
   $$\lim_{t\to\infty} \nu_{x,t}(A) = \begin{cases} \int_A g(x,y)\,\mathrm{d}y \text{ if } d\geqslant 3\ ,\\ +\infty \text{ if } d \leqslant 2\ .\end{cases}$$

3. **Bessel process** Let $\big(\mathbf{B}(t) : t \geq 0\big)$ be a d-dimensional BM with generator $\frac{1}{2}\Delta$ defined for all $f \in C_0^2(\mathbb{R}^d, \mathbb{R})$, $d \geq 2$.
   Define the **d-dimensional Bessel process** as $t \mapsto X(t) := \big\|\mathbf{B}(t)\big\|_2 \in \mathbb{R}$. As it turns out, X is a Feller Markov process.
   (a) For a radially-symmetric $f \in C_0^2(\mathbb{R}^d, \mathbb{R})$, compute $\frac{1}{2}\Delta f\big(X(t)\big)$ to derive the generator of the Bessel process
   $$\mathcal{L}f(x) = \frac{d-1}{2x}f'(x) + \frac{1}{2}f''(x)\ .$$
   (Radial symmetry means that f depends on the Euclidean norm $x := ||\mathbf{x}||_2$ only). :c

(b) Show that $X(t)^2 - d\,t$ is a martingale.

(c) For $d = 3$ show that $t \mapsto M(t) := X(t)^4 - 10tX(t)^2 + 15t^2$ is a martingale.

4. **Harmonic functions**
Let B be a Brownian motion on $\mathbb{R}^2$. Suppose that $D \subseteq \mathbb{R}^2$ is a bounded open region and that $v \in C^2(D) \cap C(\overline{D})$ satisfies

$$\frac{1}{2}\Delta v = -1 \,, \qquad \forall x \in D \,, \qquad v(x) = 0 \,, \qquad \forall x \in \partial D \,,$$

(a) Show that $v(x) = \mathbb{E}_x[T]$, where $T = \inf\{t > 0 : \ B_t \in \partial D\}$.

(b) Using rotational symmetry (that is, assume that u is radial), find a solution

$$u \colon B_{0,r} \to \mathbb{R} \,,$$

to the above equation for $D = B_{0,r}$ (an open ball of radius r centered around 0) in the form of a second degree polynomial in $|x|$.

(c) For a general domain D such that $0 \in D$, show that $v(0) \geqslant d(0, \partial D)^2/2$. Here $d(x, \partial D)$ is the distance between the point $x \in \mathbb{R}^2$ and the set $\partial D \subseteq \mathbb{R}^2$:

$$d(x, A) = \inf\{|x - y| \ : \ y \in A\} \ .$$