Chapter 8 Solutions

15 February 2022 10:01

Ex 1 (1) Let $Z^1(t)$ and $Z^2(t)$ be two solutions. Define $\Sigma(t) = Z^1(t) - Z^2(t)$

Then $\Sigma(t) = -\int_0^t \Sigma(s-) h(s)\, da(s)$, $t \in [0,T]$.

It follows that

$(*)$ $|\Sigma(t)| \leq \int_0^t |\Sigma(s-)|\, |h(s)|\, d|a(s)|$ where $|a| = a^+ + a^-$ is the total variation of a so $\uparrow$.

$$\leq m(T) \int_0^t |h(s)|\, d|a(s)| \quad \text{where } m(T) = \sup_{0 \leq s \leq T} |\Sigma(s)|$$

$$= m(T)\, N(t) \quad \text{where } N(t) = \int_0^t |h(s)|\, d|a(s)|$$

Plugging the above estimate into $(*)$ yields

$$|\Sigma(t)| \leq \int_0^t m(T) N(s-)\, |h(s)|\, d|a(s)|$$

$$= m(T) \int_0^t N(s-)\, dN(s)$$

Note that $N^2(t) = N^2(0) + 2\int_0^t N(s-)\, dN(s) + \underbrace{\sum_{0 \leq s \leq t} |\Delta N(s)|^2}_{\geq 0}$

$$\geq 2\int_0^t N(s-)\, dN(s)$$

Hence, $|\Sigma(t)| \leq m(T) \frac{N^2(t)}{2}$

Plugging the above estimate into $(*)$ again yields

$$|\Sigma(t)| \leq \frac{m(T)}{2} \int_0^t N^2(s-)\, dN(s)$$

Note that $N^3(t) = N^3(0) + 3\int_0^t N^2(s-)\, dN(s) + \underbrace{\sum_{0 \leq s \leq t} |\Delta N(s)|^3}_{\geq 0}$

$$\geq 3\int_0^t N^2(s-)\, dN(s)$$

Hence, $|\Sigma(t)| \leq m(T) \frac{N^3(t)}{3!}$

Hence, $|\bar{Z}(t)| \le M(T)\frac{N^3(t)}{3!}$

In general, $|\bar{Z}(t)| \le M(T)\frac{N^n(t)}{n!}$ for any $n \ge 1$.

Sending $n \to \infty$ yields that $\bar{Z}(t) = 0$. i.e. the solution is unique #

(2) $$\ln Z(t) = \ln Z(0) + \int_0^t \frac{1}{Z(s-)}\,dZ(s) + \sum_{0<s\le t} \ln Z(s) - \ln Z(s-) - \frac{1}{Z(s-)}\Delta Z(s)$$

Since $\Delta Z(s) = -Z(s-)\,\mu(s)\,\Delta a(s)$

and $$\frac{Z(s)}{Z(s-)} = \frac{Z(s-) + \Delta Z(s)}{Z(s-)} = 1 - \mu(s)\Delta a(s),$$

it follows that

$$\ln Z(t) = \ln Z(0) - \int_0^t \mu(s)\,da(s) + \sum_{0<s\le t} \ln(1 - \mu(s)\Delta a(s)) - \mu(s)\Delta a(s)$$

$$= \ln Z(0) - \int_0^t \mu(s)\,da^c(s) + \ln \prod_{0<s\le t} (1 - \mu(s)\Delta a(s))$$

So $$Z(t) = Z(0)\, e^{-\int_0^t \mu(s)\,da^c(s)} \prod_{0<s\le t} (1 - \mu(s)\Delta a(s))$$ #