Ex 1 (1) Using the formula in Chapter 0 Ex 1, since $a(t)=0$, $b(t) = -\frac{b}{2}$, $\sigma(t) = \frac{1}{2}\sigma$.

$\Phi(t) = e^{-\frac{1}{2}bt}$ and

$$Z_t^j = e^{-\frac{1}{2}bt}\left[x^j + \frac{\sigma}{2}\int_0^t e^{\frac{1}{2}bu}\, dW_u^j\right] \qquad \#$$

(2) $E[Z_t^j] = e^{-\frac{1}{2}bt}x^j \to 0$ as $t\to\infty$

$$\begin{aligned}
Var[Z_t^j] &= Var\left[e^{-\frac{1}{2}bt}\frac{\sigma}{2}\int_0^t e^{\frac{1}{2}bu}\, dW_u^j\right]\\
&= \frac{\sigma^2}{4}e^{-bt}\, Var\left[\int_0^t e^{\frac{1}{2}bu}\, dW_u^j\right]\\
&= \frac{\sigma^2}{4}e^{-bt}\, E\left[\left(\int_0^t e^{\frac{1}{2}bu}\, dW_u^j\right)^2\right]\\
&= \frac{\sigma^2}{4}e^{-bt}\int_0^t e^{bu}\, du = \frac{\sigma^2}{4b}(1-e^{-bt}) \to \frac{\sigma^2}{4b} \text{ as } t\to\infty
\end{aligned}$$

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(3) Apply Itô's formula to $r_t = \sum_{j=1}^d |Z_t^j|^2$.

$$\begin{aligned}
dr_t &= \sum_{j=1}^d \left(2 Z_t^j\, dZ_t^j + d\langle X^j\rangle_t\right)\\
&= \sum_{j=1}^d \left(-b|Z_t^j|^2 dt + \sigma Z_t^j\, dW_t^j + \frac{1}{4}\sigma^2 dt\right)\\
&= \left(\frac{1}{4}d\sigma^2 - b r_t\right)dt + \sigma\sqrt{r_t}\cdot \underbrace{\frac{1}{\sqrt{r_t}}\sum_{j=1}^d Z_t^j\, dW_t^j}_{dB_t}
\end{aligned}$$

Since B is a local martingale and

$$d\langle B\rangle_t = \left(\frac{1}{\sqrt{r_t}}\sum_{j=1}^d Z_t^j\, dW_t^j\right)^2 = \frac{1}{r_t}\sum_{j=1}^d |Z_t^j|^2 dt = dt$$

By Lévy characterization. B is a BM $\#$

(4) Since stochastic integral $\int_0^t e^{\frac{1}{2}bu}\, dW_u^j$ is with deterministic integrand. it is Gaussian, so is Z_t^j.

From (2). we obtain $m(t) = e^{-\frac{1}{2}bt}x^j$. $v(t) = \frac{\sigma^2}{4b}(1-e^{-bt})$.

Since $(W^1 \cdots W^d)$ are iid, **it follows that** $(Z^1 \cdots Z^d)$ are iid with $Z^j \sim N(m(t), v(t))$.

By definition, $r_t = \sum_{j=1}^d |Z_t^j|^2$ follows χ^2-distribution $\#$

(5) For any $\mu < \frac{1}{2v(t)}$,

$$E\left[e^{\mu|Z_t^j|^2}\right] = \int_R e^{\mu x^2}\frac{1}{\sqrt{2\pi v(t)}}\, e^{-\frac{(x-m(t))^2}{2v(t)}}\, dx$$

$$= \frac{1}{\sqrt{2\pi V(t)}} \int_R e^{-\frac{[1-2\mu V(t)]x^2 - 2m(t)x + |m(t)|^2}{2V(t)}} dx$$

$$= \frac{1}{\sqrt{2\pi V(t)}} \int_R \exp\left\{ - \frac{\left(x - \frac{m(t)}{1-2\mu V(t)}\right)^2 + \frac{|m(t)|^2}{1-2\mu V(t)} - \frac{|m(t)|^2}{|1-2\mu V(t)|^2}}{\frac{2V(t)}{1-2\mu V(t)}} \right\} dx$$

$$= \sqrt{\frac{1-2\mu V(t)}{2\pi V(t)}} \int_R \exp\left\{ - \frac{\left(x - \frac{m(t)}{1-2\mu V(t)}\right)^2}{\frac{2V(t)}{1-2\mu V(t)}} \right\} dx \cdot \frac{1}{\sqrt{1-2\mu V(t)}} \exp\left\{ - \frac{|m(t)|^2 (1-2\mu V(t) - 1)}{|1-2\mu V(t)|^2} \cdot \frac{1-2\mu V(t)}{2V(t)} \right\}$$

$$= \frac{1}{\sqrt{1-2\mu V(t)}} \exp\left\{ \frac{\mu |m(t)|^2}{1-2\mu V(t)} \right\} \qquad \#$$

(b) Since $r_t = \sum_{j=1}^{d} |\delta_t^j|^2$, it follows that

$$E\left[e^{\mu r_t} \right] = \left(E\left[e^{\mu |\delta_t^j|^2} \right] \right)^d = \frac{1}{\left(\sqrt{1-2\mu V(t)}\right)^d} \exp\left\{ \frac{d\mu |m(t)|^2}{1-2\mu V(t)} \right\}$$

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Ex 2. (1) $\begin{cases} \partial_t F(t,r) + \frac{1}{2}\sigma^2 \partial_{rr} F(t,r) + b(t)\partial_r F(t,r) - r F(t,r) = 0 \\ F(T,r) = 1 \end{cases}$ $\#$

(2) Both drift and vol are independent of r, so Ho-Lee model provides ATS, i.e. $F(t,r) = e^{-A(t) - B(t) r}$, where

$$\begin{cases} \frac{dA(t)}{dt} = \frac{\sigma^2}{2} |B(t)|^2 - b(t) B(t), & A(T) = 0 \\ \frac{dB(t)}{dt} = -1, & B(T) = 0 \end{cases}$$

$$\Rightarrow \begin{cases} B(t) = T - t \\ A(t) = -\frac{\sigma^2}{6}(T-t)^3 + \int_t^T b(s)(T-s) ds \end{cases} \qquad \#$$

(3) For $b(t) = \partial_t f(0,t) + \sigma^2 t$

$$r_t = r_0 + \int_0^t b(s) ds + \sigma W_t^Q$$
$$= \underline{r_0} + f(0,t) - \underline{f(0,0)} + \frac{\sigma^2}{2} t + \sigma W_t^Q$$
$$= f(0,t) + \frac{\sigma^2}{2} t + \sigma W_t^Q$$

$$f(t,T) = -\frac{\partial}{\partial T} \ln P(t,T) = \frac{\partial}{\partial T} A(t) + \frac{\partial}{\partial T} B(t) r_t$$
$$= -\frac{\sigma^2}{2}(T-t)^2 + \int_t^T b(s) ds + r_t$$

However, $\int_t^T b(s) ds = f(0,T) - f(0,t) + \frac{1}{2}\sigma^2 (T^2 - t^2)$

However, $\displaystyle\int_t^T b(s)ds = f(0,T) - f(0,t) + \frac{1}{2}\sigma^2(T^2-t^2)$

Hence, $f(t,T) = -\frac{\sigma^2}{2}(T-t)^2 + \underbrace{f(0,T) - f(0,t) + \frac{\sigma^2}{2}(T^2-t^2)}_{\int_t^T b(s)ds} + \underbrace{f(0,t) + \frac{\sigma^2}{2}t + \sigma W_t^Q}_{r_t}$

$$= f(0,T) + \sigma^2 t\left(T - \frac{t}{2}\right) + \sigma W_t^Q$$

$\Rightarrow df(t,T) = \left(\sigma^2\left(T-\frac{t}{2}\right) - \frac{1}{2}\sigma^2 t\right)dt + \sigma dW_t^Q = \underbrace{\sigma^2(T-t)}_{\alpha(t,T)}dt + \underbrace{\sigma}_{\sigma(t,T)} dW_t^Q$ #

(4) Since $\sigma(t,T) = \sigma$, it is clear that

$$\sigma(t,T)\int_t^T \sigma(t,s)ds = \sigma\int_t^T \sigma ds = \sigma^2(T-t) = \alpha(t,T)$$ #

Remark: Note that only the above choice of $b(t)$ will match any initial forward rate curve $f(0,T)$. Indeed,

$$f(0,T) = \left(\frac{\partial}{\partial T}A(t) + \frac{\partial}{\partial T}B(t)\, r_t\right)\Big|_{t=0}$$

$$= -\frac{1}{2}\sigma^2 T^2 + \int_0^T b(s)ds + r_0$$

Differentiate against T on both sides yields

$\partial_T f(0,T) = -\sigma^2 T + b(T)$ which explains the choice of $b(\cdot)$.