UNIVERSITY OF WARWICK

Paper Details

Paper code: ST4030_C /MA4F70_A

Paper Title: BROWNIAN MOTION

Exam Period: April 2023

Exam Rubric

Time Allowed: **2 hours**

Exam Type: Standard Examination

Calculators may NOT be used in this examination.

Instructions

Full marks may be obtained by correctly answering THREE complete questions.

Candidates may attempt all four questions. Credit will only be given for your THREE best answers.

All questions carry an equal weight of 20 marks. There are total of **60** marks available.

A guideline to the number of marks available is shown for each question section.

Be careful in crossing out work. Crossed out work will **NOT** be marked.

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1. Path properties of Brownian motion

(a) Write the assumptions and statement of Kolmogorov's continuity criterion.

[6 marks]

(b) Use the continuity criterion and Brownian scaling to show that a Brownian motion $(B_t)_{t\geqslant 0}$ satisfies for any $\gamma \in (0, 1/2)$

$$\mathbb{P}\left(\sup_{\substack{s\neq t\\ s,t\in[0,1]}} \frac{|B_t - B_s|}{|t-s|^\gamma} < \infty\right) = 1 \ .$$

Hint: Use and prove that $\mathbb{E}|B_t|^n = t^{\frac{n}{2}} C_n$, for some constant $C_n > 0$.

[5 marks]

(c) Use Blumenthal's $0-1$ law and Brownian scaling to prove that

$$\mathbb{P}\left(\sup_{\substack{s\neq t\\ s,t\in[0,1]}} \frac{|B_t - B_s|}{\sqrt{|t-s|}} < \infty\right) = 0 \ .$$

[4 marks]

(d) Consider $t > 0$ fixed and prove that along any sequence $\{\Pi_n\}_{n\in\mathbb{N}}$ of partitions of $[0,t]$ (so $\Pi_n = \{t_i^n\}_{i=1,\dots,m_n}$ for some $m_n \in \mathbb{N}$ and some time points $0 = t_0^n < \dots < t_{m_n}^n = t$) with mesh-size $|\Pi_n| = \max_{i=0,\dots,m_n-1}\{|t_{i+1}^n - t_i^n|\}$ such that

$$\lim_{n\to\infty} |\Pi_n| = 0 \ ,$$

we have

$$\lim_{n\to\infty} \mathbb{E}\left|\left(\sum_{i=0}^{m_n-1} (B_{t_{i+1}^n} - B_{t_i^n})^2\right) - t\right|^2 = 0 \ .$$

[5 marks]

Question 2 starts on next page

2. Stopping times and the Markov property

Let $(B_t)_{t\geqslant 0}$ be a Brownian motion

(a) Consider the following times:

$$T = \inf\{t > 0 \,:\, B_t = 0\}\ , \qquad R = \inf\{t > 1 \,:\, B_t = 0\}\ ,$$
$$L = \sup\{t \in [0,1] \,:\, B_t = 0\}\ .$$

(i) Which of these are stopping times (no proof required)?

[5 marks]

(ii) Using the Markov property of B, show that for $p_t(x,y) = \sqrt{2\pi t}^{-1}\exp(-(x-y)^2/2t)$ we have

$$\mathbb{P}_0(R > 1+t) = \int_{\mathbb{R}} p_1(0,y)\mathbb{P}_y(T > t)\,\mathrm{d}y\ ,$$
$$\mathbb{P}_0(L \leqslant t) = \int_{\mathbb{R}} p_t(0,y)\mathbb{P}_y(T > 1-t)\,\mathrm{d}y\ , \qquad \forall t \in [0,1]\ .$$

[3 marks]

(b) Let $p^{\mu}_t(x,y)$ be the Markov transition pdf (of a time-homogeneous Markov process) associated to a Brownian Motion with drift $t \mapsto \mu t + B_t$, for $\mu \neq 0$.

(i) Provide an explicit expression for $p^{\mu}_t(x,y)$ (no proof required: the answer is a modification of $p_t(x,y)$ in the previous point).

[5 marks]

(ii) Show that p^{μ} solves the following partial differential equation (without need to justify the initial condition)

$$\partial_t p^{\mu} = \frac{1}{2}\partial_y^2 p^{\mu} - \mu\partial_y p^{\mu}\ , \qquad p^{\mu}_0(x,\cdot) = \delta_x(\cdot)\ .$$

[3 marks]

(c) Show that for every $A \in \mathcal{T}((B_t)_{t\geqslant 0})$ (the tail σ-algebra of a Brownian motion $(B_t)_{t\geqslant 0}$ started at $x \in \mathbb{R}$) the following hold:

(i) For fixed $x \in \mathbb{R}$ prove that $\mathbb{P}_x(A) \in \{0,1\}$. *Hint:* Use time inversion and (without proof) that Blumenthal's $0-1$ law holds for BM with drift.

[2 marks]

(ii) One of the following two must hold (note the uniformity over x!):

$$\begin{cases} \mathbb{P}_x(A) & = 1\ , \qquad \forall x \in \mathbb{R}\ , \\ \mathbb{P}_x(A) & = 0\ , \qquad \forall x \in \mathbb{R}\ . \end{cases}$$

Hint: Use that any tail event A can be seen also as a tail event for the Brownian motion started at time $t = 1$ at B_1, together with the Markov property (as in the point (a).(ii) above). Namely find a useful function φ such that

$$\mathbb{P}_x(A) = \int_{\mathbb{R}} p_1(x,y)\varphi(y)\,\mathrm{d}y\ .$$

[2 marks]

Question 3 starts on next page

3. An arcsin law.
Let $(B_t)_{t\geqslant 0}$ be a Brownian motion.

(a) Consider the hitting time

$$T_a = \inf\{t > 0 : \ B_t = a\} \ ,$$

for $a > 0$. Prove that for any $\lambda > 0$

$$\mathbb{E}[e^{-\lambda T_a}] = e^{-a\sqrt{2\lambda}} \ .$$

Hint: Use the optional stopping theorem with a suitable martingale (you do not need to prove the martingale property).

[7 marks]

(b) Define T to be the *unique* time at which the global maximum of B_t is achieved on $[0, 1]$:

$$B(T) = \sup_{t\in[0,1]} B_t \ .$$

Let $(B^1_t)_{t\geqslant 0}$ and $(B^2_t)_{t\geqslant 0}$ be two independent Brownian motions and define, for fixed $t > 0$

$$M_1 = \max_{[0,t]} B^1_s \ , \qquad M_2 = \max_{[0,1-t]} B^2_s \ .$$

Check that the following identity holds:

$$\mathbb{P}(T < t) = \mathbb{P}(M_1 > M_2) \ .$$

Hint: Use independence of Brownian increments.

[5 marks]

(c) Show, using the reflection principle, that for any $a > 0$

$$\mathbb{P}(M_1 > a) = \mathbb{P}(|B_t| > a) \ , \qquad \mathbb{P}(M_2 > a) = \mathbb{P}(|B_{1-t}| > a) \ .$$

[5 marks]

(d) Using points (b) and (c) prove that

$$\mathbb{P}(T < t) = \frac{2}{\pi} \arcsin(\sqrt{t}) \ .$$

Hint: Use Brownian scaling to rewrite $\mathbb{P}(T < t)$ as $\mathbb{P}(\varphi(N^1, N^2) \leqslant \sqrt{t})$ for some appropriate function φ and a pair (N^1, N^2) of independent standard Gaussians. Then use that $(N^1, N^2) = (R\cos\theta, R\sin\theta)$ where θ is uniformly distributed in $[0, 2\pi]$.

[3 marks]

Question 4 starts on next page

4. Hitting times and harmonic functions

(a) Let $(B_t)_{t \geqslant 0}$ be a Brownian Motion on $\mathbb{R}$. Fix $\mu \in \mathbb{R} \setminus \{0\}$ and define Brownian motion with drift by $X_t = \mu t + B_t$, for all $t \geqslant 0$.

(i) Let $M_t = \exp(\lambda X_t)$. Find λ such that $(M_t)_{t \geqslant 0}$ is a non-constant martingale.

[5 marks]

(ii) For such λ, does there exist a random variable M_∞ satisfying

$$M_\infty = \lim_{t \to \infty} M_t \quad \text{almost surely} \ , \qquad M_t = \mathbb{E}[M_\infty | \mathcal{F}_t] \ , \quad \forall t \geqslant 0?$$

Provide a proof for your answer. *Hint:* use the law of large numbers.

[2 marks]

(iii) Let $T_x = \inf\{t > 0 \ : \ X_t = x\}$ be the first hitting time of a point $x \neq 0$. For $a < 0 < b$, find $g(a, b) = \mathbb{P}(T_a < T_b)$ and compute the limits

$$\lim_{a \uparrow 0} g(a, b) \ , \quad \forall b > 0 \ , \qquad \lim_{b \downarrow 0} g(a, b) \ , \quad \forall a < 0 \ .$$

[4 marks]

(b) Let B be a Brownian motion on $\mathbb{R}^2$ (two-dimensional!). Suppose that $D \subseteq \mathbb{R}^2$ is a bounded open region and that $v \in C^2(D) \cap C(\overline{D})$ satisfies

$$\frac{1}{2} \Delta v = -1 \ , \qquad \forall x \in D \ , \qquad v(x) = 0 \ , \qquad \forall x \in \partial D \ , \tag{1}$$

(i) Show that $v(x) = \mathbb{E}_x[T]$, where $T = \inf\{t > 0 \ : \ B_t \in \partial D\}$.

[5 marks]

(ii) Using rotational symmetry (that is, assume that u is radial), find a solution

$$u \colon B_{0,r} \to \mathbb{R} \ ,$$

to (1) for $D = B_{0,r}$ (an open ball of radius r centered around 0) in the form of a second degree polynomial in $|x|$.

[3 marks]

(iii) For a general domain D such that $0 \in D$, show that $v(0) \geqslant d(0, \partial D)^2/2$. Here $d(x, \partial D)$ is the distance between the point $x \in \mathbb{R}^2$ and the set $\partial D \subseteq \mathbb{R}^2$:

$$d(x, A) = \inf\{|x - y| \ : \ y \in A\} \ .$$

[1 marks]

End

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Hint: Use and prove that $\mathbb{E}|B_t|^n = t^{\frac{n}{2}} C_n$, for some constant $C_n > 0$.

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(c) Use Blumenthal's $0 - 1$ law and Brownian scaling to prove that

$$\mathbb{P}\left(\sup_{\substack{s\neq t\\ s,t\in[0,1]}} \frac{|B_t - B_s|}{\sqrt{|t-s|}} < \infty \right) = 0 \ .$$

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(d) Consider $t > 0$ fixed and prove that along any sequence $\{\Pi_n\}_{n\in\mathbb{N}}$ of partitions of $[0, t]$ (so $\Pi_n = \{t_i^n\}_{i=1,\dots,m_n}$ for some $m_n \in \mathbb{N}$ and some time points $0 = t_0^n < \cdots < t_{m_n}^n = t$) with mesh-size $|\Pi_n| = \max_{i=0,\dots,m_n-1}\{|t_{i+1}^n - t_i^n|\}$ such that

$$\lim_{n\to\infty} |\Pi_n| = 0 \ ,$$

we have

$$\lim_{n\to\infty} \mathbb{E}\left| \left(\sum_{i=0}^{m_n-1} (B_{t_{i+1}^n} - B_{t_i^n})^2 \right) - t \right|^2 = 0 \ .$$

[5 marks]

Proof. (a) (Bookwork) For any continuous-time stochastic process $t \mapsto X_t$, suppose that there exist constants $\alpha, \beta, C, T > 0$ such that

$$\mathbb{E}|X_t - X_s|^\alpha \leqslant C|t-s|^{1+\beta} \ , \qquad \forall 0 \leqslant s \leqslant t \leqslant T \ .$$

Then there exists a continuous modification $\widetilde{X}$ of X such that

$$\mathbb{P}\left(\sup_{0\leqslant s\leqslant t\leqslant T} \frac{|\widetilde{X}_t - \widetilde{X}_s|}{|t-s|^\gamma} < \infty \right) = 1 \ ,$$

for any $\gamma \in (0, \beta/\alpha)$.

Question 2 starts on next page

(b) (In an exercise) We use that from the scaling of Brownian motion

$$\mathbb{E}|B_t - B_s|^n = C_n|t-s|^{\frac{n}{2}}$$

and apply the theorem with $\alpha = n$ and $\beta = \frac{n}{2} - 1$, for $n > 2$. Then we get to any γ such that

$$\gamma < \frac{n/2 - 1}{n} = \frac{1}{2} - \frac{1}{n} \ .$$

Since n can be chosen arbitrarily large the result follows.

(c) (Bookwork and similar to examples seen in class, plus the TA addressed this question) It suffices to prove that for every $n \in \mathbb{N}$

$$\mathbb{P}(\limsup_{t\to 0} \frac{|B_t|}{\sqrt{t}} < n) = 0 \ .$$

In particular

$$\begin{aligned}\mathbb{P}(\limsup_{t\to 0} \frac{|B_t|}{\sqrt{t}} < n) &= \lim_{t\to 0} \mathbb{P}(\sup_{0<s\leqslant t} \frac{|B_s|}{\sqrt{s}} < n) \leqslant \lim_{t\to 0} \mathbb{P}(|B_s|/\sqrt{s} < n) \\ &= \mathbb{P}(|\mathcal{N}(0,1)| < n) < 1 \ .\end{aligned}$$

By Blumenthal's $0-1$ law, the event on the lhs has probability either zero or one, hence it must be zero.

(d) (Bookwork) We have that

$$\begin{aligned}\mathbb{E}\left|\left(\sum_{i=0}^{m_n-1}(B_{t^n_{i+1}} - B_{t^n_i})^2\right) - t\right|^2 &= \mathbb{E}\left|\sum_{i=0}^{m_n-1}\left\{(B_{t^n_{i+1}} - B_{t^n_i})^2 - (t^n_{i+1} - t^n_i)\right\}\right|^2 \\ &= \sum_{i=0}^{m_n-1}\mathbb{E}\left|(B_{t^n_{i+1}} - B_{t^n_i})^2 - (t^n_{i+1} - t^n_i)\right|^2 \ ,\end{aligned}$$

because the sum is over independent, centered random variables. Then since

$$\mathbb{E}\left|(B_{t^n_{i+1}} - B_{t^n_i})^2 - (t^n_{i+1} - t^n_i)\right|^2 = (t^n_{i+1} - t^n_i)^2(\mathbb{E}B_1^4 - 1) \ ,$$

we obtain

$$\mathbb{E}\left|\left(\sum_{i=0}^{m_n-1}(B_{t^n_{i+1}} - B_{t^n_i})^2\right) - t\right|^2 \leqslant |\Pi_n|(\mathbb{E}B_1^4 - 1)\sum_{i=1}^{m_n-1} t^n_{i+1} - t^n_i = |\Pi_n|(\mathbb{E}B_1^4 - 1)t \ ,$$

from which the result follows.

$\square$

Question 2 continued overleaf

2. Stopping times and the Markov property
Let $(B_t)_{t\geqslant 0}$ be a Brownian motion

(a) Consider the following times:

$$T = \inf\{t > 0 \,:\, B_t = 0\} \,, \qquad R = \inf\{t > 1 \,:\, B_t = 0\} \,,$$
$$L = \sup\{t \in [0,1] \,:\, B_t = 0\} \,.$$

(i) Which of these are stopping times (no proof required)?

[5 marks]

(ii) Using the Markov property of B, show that for $p_t(x,y) = \sqrt{2\pi t}^{-1} \exp(-(x-y)^2/2t)$ we have

$$\mathbb{P}_0(R > 1+t) = \int_{\mathbb{R}} p_1(0,y)\mathbb{P}_y(T > t)\,\mathrm{d}y \,,$$
$$\mathbb{P}_0(L \leqslant t) = \int_{\mathbb{R}} p_t(0,y)\mathbb{P}_y(T > 1-t)\,\mathrm{d}y \,, \qquad \forall t \in [0,1] \,.$$

[3 marks]

(b) Let $p^\mu_t(x,y)$ be the Markov transition pdf (of a time-homogeneous Markov process) associated to a Brownian Motion with drift $t \mapsto \mu t + B_t$, for $\mu \neq 0$.

(i) Provide an explicit expression for $p^\mu_t(x,y)$ (no proof required: the answer is a modification of $p_t(x,y)$ in the previous point).

[5 marks]

(ii) Show that p^μ solves the following partial differential equation (without need to justify the initial condition)

$$\partial_t p^\mu = \frac{1}{2}\partial_y^2 p^\mu - \mu \partial_y p^\mu \,, \qquad p^\mu_0(x,\cdot) = \delta_x(\cdot) \,.$$

[3 marks]

(c) Show that for every $A \in \mathcal{T}((B_t)_{t\geqslant 0})$ (the tail σ-algebra of a Brownian motion $(B_t)_{t\geqslant 0}$ started at $x \in \mathbb{R}$) the following hold:

(i) For fixed $x \in \mathbb{R}$ prove that $\mathbb{P}_x(A) \in \{0,1\}$. *Hint:* Use time inversion and (without proof) that Blumenthal's $0-1$ law holds for BM with drift.

[2 marks]

(ii) One of the following two must hold (note the uniformity over x!):

$$\begin{cases} \mathbb{P}_x(A) & = 1 \,, \qquad \forall x \in \mathbb{R} \,, \\ \mathbb{P}_x(A) & = 0 \,, \qquad \forall x \in \mathbb{R} \,. \end{cases}$$

Hint: Use that any tail event A can be seen also as a tail event for the Brownian motion started at time $t = 1$ at B_1, together with the Markov property (as in the point (a).(ii) above). Namely find a useful function φ such that

$$\mathbb{P}_x(A) = \int_{\mathbb{R}} p_1(x,y)\varphi(y)\,\mathrm{d}y \,.$$

[2 marks]

Question 3 starts on next page

Proof. (a) (New, but similar to seen)

(i) (Yes, yes, no)

(ii) We have that (first one is similar to second one, so we simply consider the latter)

$$\begin{aligned}\mathbb{P}(L \leqslant t) = \mathbb{P}(B_s \neq 0 \forall s \in (t,1]) &= \int_{\mathbb{R}} p_t(0,y)\mathbb{P}_y(B_s \neq 0 \forall s \in [0, 1-t])\,\mathrm{d}y \\ &= \int_{\mathbb{R}} p_t(0,y)\mathbb{P}_y(T > 1-t)\,\mathrm{d}y\ ,\end{aligned}$$

by using the Markov property of Brownian motion.

(b) (New, but similar to seen) The pdf is given by

$$p_t(x,y) = \frac{1}{\sqrt{2\pi t}} \exp\left\{ -\frac{(y-x-\mu t)^2}{2t} \right\}\ .$$

Then we have that

$$\begin{aligned}\partial_t p &= \frac{1}{\sqrt{2\pi}} \left\{ -\frac{1}{2} t^{-\frac{3}{2}} e^{-\frac{(y-x-\mu t)^2}{2t}} + t^{-\frac{1}{2}} e^{-\frac{(y-x-\mu t)^2}{2t}} \left(\frac{(y-x-\mu t)^2}{2t^2} + \frac{(y-x-\mu t)\mu}{t} \right) \right\} \\ \partial_y p &= \frac{1}{\sqrt{2\pi t}} \left\{ e^{-\frac{(y-x-\mu t)^2}{2t}} \frac{-(y-x-\mu t)}{t} \right\} \\ \partial_y^2 p &= \frac{1}{\sqrt{2\pi t}} \left\{ e^{-\frac{(y-x-\mu t)^2}{2t}} \frac{-1}{t} + e^{-\frac{(y-x-\mu t)^2}{2t}} \frac{(y-x-\mu t)^2}{t^2} \right\}\ ,\end{aligned}$$

from which the claim follows.

(c) (New and difficult) If $B_t = x + W_t$, then by time-inversion we have

$$X_t = tB_{1/t} = xt + \widetilde{W}_t\ ,$$

for $\widetilde{W}$ a new Brownian motion and $A \in \mathcal{F}_0^+$, the germ-sigma filed of W. Then by Blumenthal's $0-1$ law the result follows. By time inversion we have $\mathbb{P}_x(A) \in \{0,1\}$ for all $x \in \mathbb{R}$ (but not necessarily the same value for every x), this was seen in class. Next, we can write

$$\mathbb{P}_x(A) = \frac{1}{\sqrt{2\pi}} \int e^{-(x-y)^2/2} \mathbb{P}_y^1(A)\,\mathrm{d}y$$

Here $\mathbb{P}_y^1(\cdot)$ is the law of Brownian motion started at time 1 at point y. By the $0-1$ law for tail events we have $\mathbb{P}_y^1(A) \in \{0,1\}$ for all $y \in \mathbb{R}$. If $\mathbb{P}_x(A) = 0$ for one $x \in \mathbb{R}$, then it must be that $\mathbb{P}_y^1(A) = 0$ for almost all $x \in \mathbb{R}$. Hence $\mathbb{P}_x(A)$ must be zero for all $x \in \mathbb{R}$.

□

Question 3 starts on next page

3. An arcsin law.
Let $(B_t)_{t\geqslant 0}$ be a Brownian motion.

(a) Consider the hitting time

$$T_a = \inf\{t > 0 : \ B_t = a\} \ ,$$

for $a > 0$. Prove that for any $\lambda > 0$

$$\mathbb{E}[e^{-\lambda T_a}] = e^{-a\sqrt{2\lambda}} \ .$$

Hint: Use the optional stopping theorem with a suitable martingale (you do not need to prove the martingale property).

[7 marks]

(b) Define T to be the *unique* time at which the global maximum of B_t is achieved on $[0,1]$:

$$B(T) = \sup_{t\in[0,1]} B_t \ .$$

Let $(B_t^1)_{t\geqslant 0}$ and $(B_t^2)_{t\geqslant 0}$ be two independent Brownian motions and define, for fixed $t > 0$

$$M_1 = \max_{[0,t]} B_s^1 \ , \qquad M_2 = \max_{[0,1-t]} B_s^2 \ .$$

Check that the following identity holds:

$$\mathbb{P}(T < t) = \mathbb{P}(M_1 > M_2) \ .$$

Hint: Use independence of Brownian increments.

[5 marks]

(c) Show, using the reflection principle, that for any $a > 0$

$$\mathbb{P}(M_1 > a) = \mathbb{P}(|B_t| > a) \ , \qquad \mathbb{P}(M_2 > a) = \mathbb{P}(|B_{1-t}| > a) \ .$$

[5 marks]

(d) Using points (b) and (c) prove that

$$\mathbb{P}(T < t) = \frac{2}{\pi}\arcsin(\sqrt{t}) \ .$$

Hint: Use Brownian scaling to rewrite $\mathbb{P}(T < t)$ as $\mathbb{P}(\varphi(N^1, N^2) \leqslant \sqrt{t})$ for some appropriate function φ and a pair (N^1, N^2) of independent standard Gaussians. Then use that $(N^1, N^2) = (R\cos\theta, R\sin\theta)$ where θ is uniformly distributed in $[0, 2\pi]$.

[3 marks]

Question 4 starts on next page

Proof. (a) (Seen in an exercise sheet) We use that the process

$$t \mapsto e^{\sqrt{2\lambda}B_t - \lambda t}$$

is a martingale. We want to apply the optional stopping theorem with T_a. Note that $M_{t\wedge T_a} \leqslant e^{\sqrt{2\lambda}a}$, so we have a uniformly dominating random variable and we deduce that

$$e^{\sqrt{2\lambda}a}\mathbb{E}[e^{-\lambda T_a}] = \mathbb{E}[M_{T_a}] = \mathbb{E}[M_0] = 1$$

(b) (Similar to seen examples) We have that

$$\mathbb{P}\left[T < t\right] = \mathbb{P}\left[\max_{s\in[0,t]} B(s) > \max_{s\in[t,1]} B(s)\right] = \mathbb{P}\left[\max_{s\in[0,t]} B(s) - B(t) > \max_{s\in[t,1]} B(s) - B(t)\right] ,$$

from which the result follows by independence of Brownian increments.

(c) (Bookwork) We have that for any $a > 0$

$$\mathbb{P}(M_1 > a) = \mathbb{P}(M_1 > a, B_t > a) + \mathbb{P}(M_1 > a, B_t \leqslant a) .$$

Then consider $\widetilde{B}_t$ the Brownian motion reflected at time

$$\tau_a = \inf\{t > 0 : \ B_t \geqslant a\} ,$$

so that we can rewrite the second statement as

$$\mathbb{P}(M_1 > a) = \mathbb{P}(M_1 > a, B_t > a) + \mathbb{P}(M_1 > a, \widetilde{B}_t \geqslant a) = 2\mathbb{P}(B_t > a) ,$$

because we have

$$M_1 > a \Leftarrow B_t > a , \qquad M_1 > a \Leftarrow \widetilde{B}_t > a .$$

(d) (New) We have

$$\mathbb{P}(T < t) = \mathbb{P}(\sqrt{t}|N^1| > \sqrt{1-t}N^2) = \mathbb{P}(\frac{|N^2|}{\sqrt{(N^1)^2 + (N^2)^2}} < \sqrt{t}) = \mathbb{P}(|\sin\theta| < \sqrt{t})$$
$$= 4\mathbb{P}(\theta < \arcsin(\sqrt{t})) = \frac{2}{\pi}\arcsin(\sqrt{t}) .$$

□

Question 4 starts on next page

4. Hitting times and harmonic functions

(a) Let $(B_t)_{t\geqslant 0}$ be a Brownian Motion on $\mathbb{R}$. Fix $\mu \in \mathbb{R} \setminus \{0\}$ and define Brownian motion with drift by $X_t = \mu t + B_t$, for all $t \geqslant 0$.

(i) Let $M_t = \exp(\lambda X_t)$. Find λ such that $(M_t)_{t\geqslant 0}$ is a non-constant martingale.

[5 marks]

(ii) For such λ, does there exist a random variable M_∞ satisfying

$$M_\infty = \lim_{t\to\infty} M_t \quad \text{almost surely} \ , \qquad M_t = \mathbb{E}[M_\infty|\mathcal{F}_t] \ , \quad \forall t \geqslant 0?$$

Provide a proof for your answer. *Hint:* use the law of large numbers.

[2 marks]

(iii) Let $T_x = \inf\{t > 0 \ : \ X_t = x\}$ be the first hitting time of a point $x \neq 0$. For $a < 0 < b$, find $g(a, b) = \mathbb{P}(T_a < T_b)$ and compute the limits

$$\lim_{a\uparrow 0} g(a, b) \ , \quad \forall b > 0 \ , \qquad \lim_{b\downarrow 0} g(a, b) \ , \quad \forall a < 0 \ .$$

[4 marks]

(b) Let B be a Brownian motion on $\mathbb{R}^2$ (two-dimensional!). Suppose that $D \subseteq \mathbb{R}^2$ is a bounded open region and that $v \in C^2(D) \cap C(\overline{D})$ satisfies

$$\frac{1}{2}\Delta v = -1 \ , \qquad \forall x \in D \ , \qquad v(x) = 0 \ , \qquad \forall x \in \partial D \ , \tag{1}$$

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(ii) Using rotational symmetry (that is, assume that u is radial), find a solution

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to (1) for $D = B_{0,r}$ (an open ball of radius r centered around 0) in the form of a second degree polynomial in $|x|$.

[3 marks]

(iii) For a general domain D such that $0 \in D$, show that $v(0) \geqslant d(0, \partial D)^2/2$. Here $d(x, \partial D)$ is the distance between the point $x \in \mathbb{R}^2$ and the set $\partial D \subseteq \mathbb{R}^2$:

$$d(x, A) = \inf\{|x - y| \ : \ y \in A\} \ .$$

[1 marks]

Proof. (a) (i) (Seen exercise) We can compute

$$\mathbb{E}e^{\lambda B_1} = \int e^{\lambda x} \frac{1}{\sqrt{2\pi}} e^{-x^2/2}\, \mathrm{d}x = e^{\lambda^2/2} \int \frac{1}{\sqrt{2\pi}} e^{-(x-\lambda)^2/2}\, \mathrm{d}x = e^{\lambda^2/2} \ .$$

Therefore we obtain a martingale (check all the definitions) for

$$\lambda^2/2 + \lambda\mu = 0 \iff \lambda = -2\mu \ .$$

Question 5 starts on next page

(ii) (New in this context) No, because we have

$$\lim_{t\to\infty} M_t = 0 ,$$

by the LLN.

(iii) (Similar exercises already seen) If we denote with $p = \mathbb{P}(\tau_a < \tau_b)$, then we find

$$1 = \mathbb{E}[M_{\tau_a \wedge \tau_b}] = e^{\lambda a} p + e^{\lambda b}(1-p) ,$$

from which

$$p = g(a,b) = \frac{1 - e^{\lambda b}}{e^{\lambda a} - e^{\lambda b}} = \frac{e^{\lambda b} - 1}{e^{\lambda b} - e^{\lambda a}} .$$

We find

$$g(0,b) = 1 , \qquad g(a,0) = 0 .$$

(b) (i) (Seen) We have that

$$t \mapsto v(B_t) + t$$

is a martingale on $[0,T]$. Hence

$$\mathbb{E}_x[T] = v(x) .$$

(ii) (Similar to previously seen example) We can write $v(x) = \varphi(r)$. Then

$$\begin{aligned} \partial_{x_1} v &= \varphi'(r)\partial_{x_1} r = \varphi'(r)\frac{x_1}{r} \\ \partial^2_{x_1} v &= \varphi''(r)\frac{x_1^2}{r^2} + \varphi'(r)\left\{\frac{1}{r} - \frac{x_1}{r^2}\partial_{x_1} r\right\} \\ &= \varphi''(r)\frac{x_1^2}{r^2} + \varphi'(r)\left\{\frac{1}{r} - \frac{x_1^2}{r^3}\right\} , \end{aligned}$$

so that

$$\Delta u = \varphi'' + \frac{1}{r}\varphi'(r)$$

Then we must solve

$$\frac{1}{2}\left\{\varphi'' + \frac{1}{r}\varphi'(r)\right\} = -1 , \qquad \varphi(r_0) = 0 .$$

Then if $ar^2 + b = \varphi(r)$ we obtain

$$-2 = 4a \iff a = -\frac{1}{2} ,$$

so that

$$\varphi(r) = -\frac{1}{2}r^2 + \frac{1}{2}r_0^2 .$$

(iii) (New) To conclude, let $S = \inf\{t > 0 : \ B_t \notin B(x, d(x, \partial D))\}$. Then $\mathbb{E}_x[T] \geqslant \mathbb{E}_x[S] = \frac{1}{2} d(x,\partial D)^2$ from the previous point.

$\square$

End