Stochastic Modelling and Random Processes

Handout - Itô Stochastic Integral

This is a summary of some useful properties of the Itô integral. Throughout this document, we assume that B_t is a Brownian motion.

Definition: Consider some function $f : [0,T] \times \mathbb{R} \to \mathbb{R}$ such that $\mathbb{E}\left(\int_0^T f(t,X_t)^2\, dt\right) < \infty$. The Itô integral is defined by

$$\mathcal{I}_t = \int_0^T f(t,X_t)\, dB_t = \lim_{K\to\infty} \sum_{k=0}^{K} f(t_k, X_{t_k})(B_{t_{k+1}} - B_{t_k}).$$

Note that here, we split the interval $[0,T]$ into K sub-intervals (t_k, t_{k+1}), $k = 0, \ldots, K-1$, with $t_k = k\Delta t$ (and $T = K\Delta t$). The term $(B_{t_{j+1}} - B_{t_j})$ is a Brownian increment, and therefore it is a Gaussian random variable with mean 0 and variance Δt.

The Itô stochastic integral has the following properties:

1. Additivity: $\int_0^s f(t,X_t)\, dB_t + \int_s^T f(t,X_t)\, dB_t = \int_0^T f\, dB_t$.
2. Linearity: for all $\alpha, \beta \in \mathbb{R}$ and f, g such that the integral exists, we have
$$\int_0^T (\alpha f(t,X_t) + \beta g(t,X_t))\, dB_t = \alpha \int_0^T f(t,X_t)\, dB_t + \beta \int_0^T g(t,X_t)\, dB_t.$$
3. If f is a deterministic function (i.e. it depends on time only), we have
$$\mathbb{E}\left(\int_0^T f(t)\, dB_t\right) = 0.$$
4. It verifies the Itô isometry:
$$\mathbb{E}\left(\left(\int_0^T f(s)\, dW_t\right)^2\right) = \mathbb{E}\left(\int_0^T f^2(t)\, dt\right),$$
and, more generaly
$$\mathbb{E}\left(\int_0^t f(s)\, dB_s \int_0^u g(s)\, dB_s\right) = \mathbb{E}\left(\int_0^{\min(t,u)} f(s)g(s)\, dB_s\right).$$
5. The above two properties imply that if f is deterministic, then the stochastic integral $\mathcal{I}_T$ is a Gaussian random variable with mean zero and variance $\int_0^T f^2(s)\, ds$.
6. We saw that the Itô integral **does not** verify the chain rule (meaning if we define $Y_t = f(X_t)$ for some function f, we do not have $dY_t = f'(X_t)dX_t$ which is what you would expect. However, we have the **Itô formula**: if X_t satisfies the SDE
$$dX_t = a(X_t,t)\, dt + \sigma(X_t,t)\, dB_t,$$
then $Y_t = f(X_t)$ satisfies
$$dY_t = \left(a(X_t,t)f'(X_t) + \frac{1}{2}\sigma^2(X_t,t)f''(X_t)\right)\, dt + \sigma(X_t,t)f'(t)\, dB_t.$$
(note that from the chain rule we would expect this to be true without the $\frac{1}{2}\sigma^2(X_t,t)f''(X_t)$ term. This is called the Itô correction.)

7. Some properties based on things we haven't looked at (and therefore are not likely to appear in exams or assignments) but could be useful for you in the future, depending on your research choices:
   (a) $\mathcal{I}_T$ is $\mathcal{F}_T$-measurable (where $\mathcal{F}_T$ is an appropeiate σ-algebra, meaning you can consider it as a random variable).
   (b) $\mathcal{I}_T$ admits a continuous version (remember when we discussed Brownian motion and I said the hardest part of proving it exists was to prove there was a continuous version of it?)
   (c) $\mathcal{I}_T$ is a square-integrable $\mathcal{F}_T$-martingale.

Itô's formula is useful to solve SDEs, and we saw how it can be used for the geometric Brownian motion. However, in real life a lot of these SDEs involve nonlinear functions of X_t and are not very easy to solve analytically. If we want to solve an SDE of the form

$$dX_t = a(X_t, t)\, dt + \sigma(X_t, t)\, dB_t$$

numerically, we can use the Euler-Maruyama scheme, which takes advantage of the definition of Itô integral and given an initial condition $X_0 = x$, gives

$$X_{n+1} = X_n + a(X_n, t_n)\Delta t + \sigma(X_n, t_n)\, \Delta B_n,$$

where $X_n = X_{t_n}$, $B_n = B_{t_n}$ and $\Delta B_n = B_{n+1} - B_n \sim \mathcal{N}(0, \Delta t)$ is a Brownian increment.