# 2018 Exam

Joanne Kennedy

January 22, 2018

# Question 1

(a) State Girsanov's Theorem for a one-dimensional Brownian motion.

[20%]

(b) Let $(\Omega, \mathcal{F}, \mathbb{N})$ be a probability space supporting a 1-dimensional Brownian motion $W$ and let $\{\mathcal{F}_t\}_{t\geq 0}$ denote the augmented natural filtration of $W$.

Let $X$ be a process satisfying the SDE

$$dX_t = -\lambda(X_t - \theta)dt + \sigma dW_t,\; X_0 = x_0,$$

where $\lambda, \theta, \sigma$ are positive constants. Show that

$$E_{\mathbb{N}}(\exp uX_t) = \exp(\phi(t,u) + \psi(t,u)x_0)$$

where

$$\psi(t,u) = e^{-\lambda t}u,$$

and

$$\phi(t,u) = \theta u(1 - e^{-\lambda t}) + \frac{\sigma^2}{4\lambda}u^2(1 - e^{-2\lambda t}).$$

[20%]

(c) Let $0 < T < \infty$ and $u \in \mathbb{R}$. Show that the process $M^u = (M_t^u)_{0\leq t\leq T}$ defined by

$$M_t^u = \exp(\phi(T-t,u) + \psi(T-t,u)X_t)$$

is a martingale.

Hint: Consider first $M_T^u$ and you may assume without proof the identity

$$E_{\mathbb{N}}(\exp uX_{t+s}|\mathcal{F}_s) = \exp(\phi(t,u) + \psi(t,u)X_s),\; 0 \leq t+s \leq T.$$

[20%]

(d) Let $0 < T_1 < T_2 < \ldots < T_n < T_{n+1}$ be a sequence of dates and for $i = 1, \ldots, n$ let $\alpha_i = T_{i+1} - T_i$. Further let $D_{tT_i}$ denote the value at time t of a pure discount bond that pays unity at $T_i$.

Consider a term structure model in which for $i = 1, \ldots, n+1$

$$\frac{D_{tT_i}}{D_{tT_{n+1}}} := M_t^{u_i},\; t \in [0, T_i],$$

where $u_1 \geq u_2 \geq \ldots \geq u_{n+1} = 0$.

Fix $k$ such that $1 \leq k \leq n-1$. Define a new measure $\mathbb{N}^k$ on $(\Omega, \mathcal{F}_T)$ via

$$\left.\frac{d\mathbb{N}^k}{d\mathbb{N}}\right|_{\mathcal{F}_t} := \frac{M_t^{u_{k+1}}}{M_0^{u_{k+1}}},\; t \leq T.$$

(i) Show that under $\mathbb{N}^k$

$$dX_t = \left(-\lambda(X_t - \theta) + \sigma^2 e^{-\lambda(T-t)}u_{k+1}\right)dt + \sigma d\tilde{W}_t,$$

where $\tilde{W}$ is a Brownian motion under $\mathbb{N}^k$.

[20%]

(ii) Show that the $k^{\text{th}}$ forward LIBOR $L^k := L^k[T_k, T_{k+1}]$ is a martingale under the measure $\mathbb{N}^k$.

[20%]

---

## Question 2

(a) Let $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}, \mathbb{P})$ be a filtered probability space. Let $\mathbb{Q}$ be a probability measure equivalent to $\mathbb{P}$ with respect to $\mathcal{F}$. Show that $M$ is an $(\{\mathcal{F}_t\}, \mathbb{Q})$-martingale if and only if $\rho M$ is an $(\{\mathcal{F}_t\}, \mathbb{P})$-martingale where for $t \geq 0$

$$\rho_t := \frac{d\mathbb{Q}}{d\mathbb{P}}|_{\mathcal{F}_t}.$$

State any results from lectures that you use.

[20%]

(b) State the Martingale Representation Theorem for a one dimensional Brownian motion.

[20%]

(c) Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space supporting a one-dimensional Brownian motion $W$ and let $\{\mathcal{F}_t\}_{t \geq 0}$ denote the augmented natural filtration generated by $W$.

Consider an economy defined for the finite time interval $0 \leq t \leq T \leq \infty$ and composed of two assets having price process $A = (D, S)$ satisfying

$$\begin{aligned} dD_t &= rD_t dt \quad (D_T = 1, r > 0) \\ dS_t &= \mu S_t dt + \sigma S_t dW_t \quad (S_0 = 1) \end{aligned}$$

where $\mu$,$r$ and $\sigma > 0$ are constants.

(i) Show that for this economy there exists a unique equivalent martingale measure $\mathbb{Q}$ correponding to numeraire S.

[20%]

(ii) Show that there exists an $\{\mathcal{F}_t\}$-predictable self-financing strategy for replicating any given a $\mathcal{F}_T$-measurable random variable X satisfying

$$E_{\mathbb{Q}}\left[\frac{|X|}{S_T}\right] < \infty.$$

[20%]

(iii) Suppose $(N, \mathbb{N})$ is some numeraire pair for the economy. Show that for $0 \leq t \leq T$

$$\left.\frac{d\mathbb{Q}}{d\mathbb{N}}\right|_{\mathcal{F}_t} = \frac{S_t \mathbb{N}_0}{\mathbb{N}_t}.$$

20%]

---

## Question 3

Let $0 < T_1 < T_2 < ... < T_n < T_{n+1}$ be a sequence of dates and for $i = 1, \ldots, n$ write $\alpha_i = T_{i+1} - T_i$. Further let $L^{(i)}$ for $i = 1, \ldots, n$ denote a set of contiguous forward LIBORs where $L_t^{(i)} := L_t[T_i, T_{i+1}]$ and let $D_{tT}$ denote the value at time $t$ of a pure discount bond that pays unity at $T$.

(a) Suppose that the market value at time $t = 0$ for a digital caplet with start date $T_i$, cashflow at time $T_{i+1}$ and strike K is given by

$$V_0^{(i)}(K) = D_{0T_{i+1}} N(d^{(i)}(K)),$$

where

$$d^{(i)}(K) = \frac{\log\left(\frac{D_{0T_i}}{D_{0T_{i+1}}}(1 + \alpha_i K)^{-1}\right)}{\Sigma^{(i)}} - \frac{1}{2}\Sigma^{(i)},$$

$N(.)$ denotes the standard cumulative normal distribution and $\Sigma^{(i)}$ is a positive constant.
Further suppose that an arbitrage-free term structure model has been defined which is consistent with the above formula for all strikes for each of the digital caplets.
Show that for this model the distribution of $L_{T_i}^{(i)} + {\alpha_i}^{-1}$ is lognormal under an equivalent martingale measure corresponding to numeraire $D_{.T_{i+1}}$.

[35%]

(b) In a LIBOR market model working in the equivalent martingale measure $\mathbb{N}$ corresponding to numeraire $D_{.T_{n+1}}$ suppose

$$L := (L^{(1)}, L^{(2)}, \ldots, L^{(n)})$$

satisfies an SDE of the form

$$dL_t^{(i)} = \mu_t^{(i)} dt + \gamma^{(i)}(L_t^{(i)}) \sigma^i(t) dW_t, \quad i = 1, \ldots, n,$$

where $W$ is a one-dimensional Brownian motion, each $\sigma^i$ is a bounded positive function of time, each $\gamma^{(i)}$ is a known function of the $i^{\text{th}}$ LIBOR and each $\mu^{(i)}$ is some general process to be determined. Assuming that the $\gamma^{(i)}$ have been suitably chosen, show how to derive appropriate forms for $\mu^{(i)}, i = 1, \ldots, n$ so that the resulting model is arbitrage-free.

[35%]

(c) For $i = 1, \ldots, n$ specify suitable choices of the functions $\gamma^{(i)}$ and $\sigma^i$ so that the LIBOR market model described in (b) calibrates to the market values for the digital caplets given in part (a). Justify your answer.

[20%]

(d) What restrictions must be placed on the functions $\sigma^i$ to enable the model in (b) to be approximated effectively by a one dimensional model.

[10%]

# Question 4

(a) State Lévy's Theorem on the characterization of Brownian motion.

[20%]

(b) Let $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}, \mathbb{Q})$ be a probability space supporting a one-dimensional Brownian motion $W$. Let $X$ denote a local martingale of the form

$$X = \int_0^t H_u dW_u,$$

where $H$ is a deterministic function of time. Suppose $[X]_t \uparrow \infty$ as $t \uparrow \infty$ and for $t \geq 0$ define

$$\tau_t = \inf\{u > 0 : [X]_u > t\}.$$

Show that

$$X_t = \tilde{W}([X]_t),$$

where $\tilde{W}$ is a Brownian motion adapted to $\{\tilde{\mathcal{F}}_t\}$ where $\tilde{\mathcal{F}}_t = \mathcal{F}_{\tau_t}$.

Hint: Take $\tilde{W}_t = X_{\tau_t}$ and apply Lévy's Theorem.

[25%]

(c) Show that

$$t \int_0^t \sigma_u dW_u = \int_0^t \int_0^u \sigma_s dW_s du + \int_0^t u \sigma_u dW_u.$$

When.

where $\sigma$ is a deterministic function of time.

[10%]

(d) Consider a short-rate model having as its short-rate the process $r$ which under the risk-neutral measure $\mathbb{Q}$ satisfies the SDE

$$dr_t = \theta_t dt + \sigma_t dW_t,$$

where $\theta$ and $\sigma$ are deterministic functions of time. Show that for this model the value at time $t$ of a pure discount bond paying unity at time $T$ is of the form

$$D_{tT} = \exp\left(\frac{1}{2}\int_t^T \sigma_u^2 (T-u)^2 du - \int_t^T \theta_u (T-u) du - (T-t) r_t\right).$$

[30%]

(e) Let $0 < T_1 < T_2 < \ldots < T_n < T_{n+1}$ be a sequence of dates and for $i = 1, \ldots, n$ write $\alpha_i = T_{i+1} - T_i$. For $i = 1, \ldots, n$ let $L_{T_i}^{(i)} := L_{T_i}[T_i, T_{i+1}]$ denote the spot LIBOR for the period $[T_i, T_{i+1}]$. Show that for the model described in (d) under $\mathbb{Q}$, for $i < j$

$$corr(\log(1 + \alpha_i L_{T_i}^i), \log(1 + \alpha_i L_{T_j}^j)) = \frac{\sqrt{(\int_0^{T_i} \sigma_u^2 du)}}{\sqrt{(\int_0^{T_j} \sigma_u^2 du)}}.$$

[15%]

Solutions

## Question 1

(a) Girsanov's Theorem for a one-dimensional B.m

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space supporting a one dimensional Brownian motion $W$ and let $\{\mathcal{F}_t\}$ denote the augmented natural filtration generated by $W$

(i) Suppose $\mathbb{Q} \sim \mathbb{P}$ w.r.t. $\mathcal{F}$. Then $\exists$ an $\{\mathcal{F}_t\}$-predictable $\mathbb{R}$-valued process $C$ such that

$$\rho_t := \frac{d\mathbb{Q}}{d\mathbb{P}}\Big|_{\mathcal{F}_t} = \exp\left(\int_0^t C_u \, dW_u - \frac{1}{2}\int_0^t C_u^2 \, du\right) \quad (+)$$

(ii) Conversely if $\rho$ is a strictly positive $(\{\mathcal{F}_t\}_{0 \le t \le T}, \mathbb{P})$ martingale some $T \in [0,\infty)$, $E_{\mathbb{P}}[\rho_T] = 1$ then $\rho$ has the representation in $(+)$ and defines a measure $\mathbb{Q} \sim \mathbb{P}$ wrt $\mathcal{F}$.
In either of the above cases, under $\mathbb{Q}$, $\tilde{W}_t := W_t - \int_0^t C_u \, du$ is an $(\{\mathcal{F}_t\}, \mathbb{Q})$ Brownian motion (with time horizon restricted to $[0,T]$ in latter case.)

(b) Set $Y_t = e^{\lambda t} X_t$. Then by integration by parts

$$dY_t = e^{\lambda t} dX_t + X_t \lambda e^{\lambda t} dt$$

$$= -\lambda \cancel{e^{\lambda t} X_t \, dt} + \lambda \theta e^{\lambda t} dt + e^{\lambda t} \sigma \, dW_t + \cancel{\lambda X_t e^{\lambda t} dt}$$

$$= e^{\lambda t} \sigma \, dW_t + \lambda \theta e^{\lambda t} dt$$

Thus
$$Y_t = x_0 + \int_0^t \lambda \theta e^{\lambda u} du + \int_0^t e^{\lambda u} \sigma \, dW_u$$

i.e.
$$Y_t = x_0 + \theta\left(e^{\lambda t} - 1\right) + \sigma \int_0^t e^{\lambda u} dW_u$$

Finally
$$X_t = e^{-\lambda t}(x_0 - \theta) + \theta + \sigma e^{-\lambda t} \int_0^t e^{\lambda u} dW_u$$

Noting
$$\int_0^t e^{\lambda u} dW_u \sim N\left(0, \frac{(e^{2\lambda t} - 1)}{2\lambda}\right)$$

So
$$X_t \sim N\left(\theta + e^{-\lambda t}(x_0-\theta), \ \frac{\sigma^2}{2\lambda}\left(1-e^{-2\lambda t}\right)\right)$$
under $\mathbb{N}$.

Thus
$$E_{\mathbb{N}}\left(\exp(uX_t)\right) = \exp\left(u\left(\theta + e^{-\lambda t}(x_0-\theta)\right) + \frac{\sigma^2}{4\lambda}\left(1-e^{-2\lambda t}\right)u^2\right)$$

$$= \exp\left(ue^{-\lambda t}x_0 + \theta u\left(1-e^{-\lambda t}\right) + \frac{\sigma^2}{4\lambda}u^2\left(1-e^{-2\lambda t}\right)\right)$$

as required

(c) observe $m_T^u = \exp\left(\phi(0,u) + \psi(0,u)X_T\right) = \exp\left(uX_T\right)$ and from (b)

$$E_{\mathbb{N}}|m_T^u| = E_{\mathbb{N}}(m_T^u) = E_{\mathbb{N}}\left(\exp(uX_T)\right)$$

$$= \exp\left(\phi(T,u) + \psi(T,u)x_0\right)$$

$$< \infty$$

Further for $0 < t < T$
$$E_{\mathbb{N}}\left(m_T^u \mid \mathcal{F}_t\right) = E_{\mathbb{N}}\left(\exp(uX_T) \mid \mathcal{F}_t\right)$$

$$\underset{\text{hint}}{\overset{\text{using}}{=}} \exp\left(\phi(T-t,u) + \psi(T-t,u)X_t\right)$$

$$= m_t^u$$

To show $(m^u)_{0\le t\le T}$ a martingale

M(i) $X_t$ is $\mathcal{F}_t$-measurable by properties of the S.I. and for each $u$, $m_t^u$ is a smooth function of $t$ and $X_t$ so $m_t^u \in m\mathcal{F}_t \quad 0 \le t \le T$.

M(ii) For $0 \le t \le T$

$$E_{\mathbb{N}}|M_t^u| = E_{\mathbb{N}}(M_t^u) = E_{\mathbb{N}}\left(E_{\mathbb{N}}(M_T^u | \mathcal{F}_t)\right)$$

$$\overset{\text{by Tower property}}{=} E_{\mathbb{N}}(M_T^u) < \infty$$

M(iii) For $0 \le s < t \le T$

$$E_{\mathbb{N}}(M_t^u | \mathcal{F}_s) = E_{\mathbb{N}}\left(E_{\mathbb{N}}(M_T^u | \mathcal{F}_t) \middle| \mathcal{F}_s\right)$$

$$\overset{\text{Tower property}}{=} E_{\mathbb{N}}(M_T^u | \mathcal{F}_s)$$

$$= M_s^u$$

(d) Observe

$$\left.\frac{d\mathbb{N}^k}{d\mathbb{N}}\right|_{\mathcal{F}_t} = \frac{M_t^{u_{k+1}}}{M_0^{u_{k+1}}}$$

$$= \exp\Big(-\phi(T, u_{k+1}) - \psi(T, u_{k+1})x_0 + \phi(T-t, u_{k+1})$$
$$+ \psi(T-t, u_{k+1})X_t\Big)$$

Only the last term in the exponential has a local martingale part - other terms deterministic functions of $t$.

$$\psi(T-t, u_{k+1})X_t^{loc} = e^{-\lambda(T-t)} u_{k+1}\sigma e^{-\lambda t}\int_0^t e^{\lambda v}\, dW_v$$

$$= e^{-\lambda T} u_{k+1}\sigma \int_0^t e^{\lambda v}\, dW_v$$

By (b) $\left(M_t^{u_{k+1}}\right)_{0 \le t \le T}$ is a martingale so we must have (or could substitute in for $\phi, \psi$ directly)

$$\left.\frac{d\mathbb{N}^k}{d\mathbb{N}}\right|_{\mathcal{F}_t} = \exp\left(\int_0^t c_v \, dW_v - \tfrac{1}{2}\int_0^t c_v^2 \, dv\right)$$

where $c_v = e^{-\lambda T} u_{k+1} \sigma e^{\lambda v} = \sigma u_{k+1} e^{-\lambda(T-v)}$

By Girsanov's Theorem

$$\tilde{W}_t = W_t - \int_0^t c_v \, dv$$

is a Brownian motion under $\mathbb{N}^k$ and

$$dX_t = \left(-\lambda(X_t - \theta) + \sigma c_t\right)dt + \sigma d\tilde{W}_t$$

which gives the required SDE for $X$

(ii) From working in d(i) for $i = k, k+1$

$$\frac{M_t^{u_i}}{M_0^{u_i}} = \exp\left(\int_0^t c_v^i \, dW_v - \tfrac{1}{2}\int_0^t (c_v^i)^2 \, dv\right)$$

where $c_v^i = \sigma u_i e^{-\lambda(T-v)}$

$$\left(1 + \alpha_k L_t^k\right) = \frac{D_t T_k}{D_t T_{k+1}} = \frac{D_t T_k}{D_t T_{k+1}} \frac{D_t T_{k+1}}{D_t T_{k+1}} = \frac{M_t^{u_k}}{M_t^{u_{k+1}}}$$

$$= \frac{M_0^{u_k}}{M_0^{u_{k+1}}} \exp\left(\sigma(u_k - u_{k+1})\int_0^t e^{-\lambda(T-v)} dW_v - \tfrac{\sigma^2}{2}(u_k^2 - u_{k+1}^2)\int_0^t e^{-2\lambda(T-v)} dv\right)$$

$$= \frac{M_0^{u_k}}{M_0^{u_{k+1}}} \exp\left(\sigma(u_k - u_{k+1})\int_0^t e^{-\lambda(T-v)} d\tilde{W}_v + \sigma(u_k - u_{k+1})\int_0^t e^{-\lambda(T-v)} c_v^{k+1} dv - \frac{\sigma^2}{2}(u_k^2 - u_{k+1}^2)\int_0^t e^{-2\lambda(T-v)} dv\right)$$

$$= \frac{M_0^{u_k}}{M_0^{u_{k+1}}} \exp\Big( \sigma(u_k - u_{k+1}) \int_0^t e^{-\lambda(T-v)} \, d\tilde{W}_v$$

$$+ \sigma(u_k - u_{k+1}) \int_0^t e^{-\lambda(T-v)} e^{-\lambda(T-v)} u_{k+1} \sigma \, dv$$

$$- \frac{\sigma^2}{2}(u_k^2 - u_{k+1}^2) \int_0^t e^{-2\lambda(T-v)} dv \Big)$$

$$= \frac{M_0^{u_k}}{M_0^{u_{k+1}}} \exp\Big( \sigma(u_k - u_{k+1}) \int_0^t e^{-\lambda(T-v)} \, d\tilde{W}_v$$

$$- \frac{\sigma^2}{2}\left( u_k^2 + u_{k+1}^2 - 2u_k u_{k+1}\right) \int_0^t e^{-2\lambda(T-v)} dv \Big)$$

which is in the form of the Doléans exponential ~~and~~ $\mathcal{E}(X)$ with
$X_t = \sigma(u_k - u_{k+1}) \int_0^t e^{-\lambda(T-v)} d\tilde{W}_v$ and since

$$[X]_t = \sigma^2 (u_k - u_{k+1})^2 \int_0^t e^{-2\lambda(T-v)} dv$$

is a bounded deterministic function of $t$ $E_{\mathbb{P}^N}\left[\exp\frac{1}{2}[X]_t\right] < \infty$ and it follows from Novikov's condition that $1 + \delta_k L^k$ is a martingale, hence $L^k$ is a martingale under $\mathbb{P}^{N^k}$

Question 2

(a) Recall from lectures for $X \in \mathcal{L}^1(\Omega, \mathcal{F}_t, \mathbb{Q})$

$$E_{\mathbb{Q}}[X | \mathcal{F}_s] = \rho_s^{-1} E_{\mathbb{P}}[X \rho_t | \mathcal{F}_s] \qquad (+)$$

where $\rho_t := \frac{d\mathbb{Q}}{d\mathbb{P}}\Big|_{\mathcal{F}_t}$.

Suppose $\rho M$ is an $(\{\mathcal{F}_t\}, \mathbb{P})$ martingale. Then

M(i) $M$ is adapted since $\rho$, $\rho M$ are

$$M(ii) \quad E_{\mathbb{Q}}[|M_t|] = E_{\mathbb{P}}[\rho_t |M_t|] \overset{\text{since } \rho>0}{=} E_{\mathbb{P}}[|\rho_t M_t|] < \infty$$

since $\rho_t M_t$ must be in $\mathcal{L}^1(\Omega, \mathcal{F}_t, \mathbb{P})$

$$M(iii) \quad E_{\mathbb{Q}}(M_t | \mathcal{F}_s) = \rho_s^{-1} E_{\mathbb{P}}(\rho_t M_t | \mathcal{F}_s) \quad \text{by } (+)$$

$= \rho_s^{-1} \rho_s M_s$ using martingale property of $\rho M$

$= M_s$

Thus properties M(i) - M(iii) hold and we have shown that $\rho M$ is an $(\{\mathcal{F}_t\}, \mathbb{P})$ martingale $\Rightarrow$ $M$ is an $(\{\mathcal{F}_t\}, \mathbb{Q})$ martingale

Converse implication can be proved similarly or write $\tilde{M} = \rho M$ and $M = \rho^{-1}\tilde{M}$ and note $\rho_t^{-1} = \frac{d\mathbb{P}}{d\mathbb{Q}}\Big|_{\mathcal{F}_t}$.

(b) Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space supporting a one dimensional Brownian motion $W$, $\{\mathcal{F}_t\}$ the augmented natural filtration generated by $W$.

M.R.T. :

Any local martingale $N$ w.r.t $\{\mathcal{F}_t\}$ can be written in the form

$$N_t = N_0 + \int_0^t H_u \, dW_u$$

for some $\{\mathcal{F}_t\}$-predictable $H$ s.t. $\int_0^t H_u^2 \, du < \infty$ a.s. all $t$.

(c)(i) Set $D^S := \frac{D}{S}$

$$dD_t^S = D_t \, dS_t^{-1} + S_t^{-1} dD_t \qquad \text{since } D \text{ has finite variation}$$

$$dS_t^{-1} = -S_t^{-2} dS_t + \tfrac{1}{2} 2 S_t^{-3} (dS_t)^2 = -S_t^{-1} \mu dt - \sigma S_t^{-1} dW_t + \sigma^2 S_t^{-1} dt$$
$$= (\sigma^2 - \mu) S_t^{-1} dt - \sigma S_t^{-1} dW_t$$

$$dD_t^S = r S_t^{-1} D_t dt + (\sigma^2 - \mu) S_t^{-1} D_t dt - \sigma S_t^{-1} D_t dW_t$$

So under $\mathbb{P}$

$$dD_t^S = (\sigma^2 + (r - \mu)) D_t^S dt - \sigma D_t^S dW_t$$

Let $\mathbb{Q}$ denote the EMM corresponding to numeraire $S$. We need to prove $\mathbb{Q}$ exists and is unique. Girsanov's Theorem suggests we take

$$\frac{d\mathbb{Q}}{d\mathbb{P}}\Big|_{\mathcal{F}_t} = \exp\left(\phi W_t - \tfrac{1}{2}\phi^2 t\right) := \rho_t \qquad (*)$$

where $\phi = \dfrac{\sigma^2 + r - \mu}{\sigma}$.

This was shown to be a martingale in lectures or follows from an easy application of Novikov's condition as $\mathbb{E}_{\mathbb{P}}\left(\exp(\tfrac{1}{2}\phi^2 t)\right) = \exp(\tfrac{1}{2}\phi^2 t) < \infty$.

So $(*)$ with $t = T$ defines a measure $\mathbb{Q} \sim \mathbb{P}$ on $\mathcal{F}_T$ by Girsanov's Theorem (Part (ii)). Under $\mathbb{Q}$ Girsanov's Theorem tell us that

$$\tilde{W}_t = W_t - \int_0^t \phi \, du = W_t - \phi t$$

is an $(\{\mathcal{F}_t\}, \mathbb{Q})$ Brownian motion on $[0, T]$ and so under $\mathbb{Q}$

$$dD_t^S = -\sigma D_t^S d\tilde{W}_t.$$

This SDE has unique solution

$$D_t^S = \exp\left(-\sigma \tilde{W}_t - \tfrac{1}{2}\sigma^2 t\right)$$

which again (Novikov) is a martingale.

Under $\mathbb{Q}$ $S^S := \frac{S}{S} \equiv 1$ is trivially a martingale

We have shown an EMM corresponding to the numeraire $S$ exists.

To show $\mathbb{Q}$ is unique let $\mathbb{Q}^*$ be some other EMM for the numeraire $S$. Then by part (i) of Girsanov's Theorem there exists an $\{\mathcal{F}_t\}$-predictable process $\gamma$ such that under $\mathbb{Q}^*$

$$W^*_t = \tilde{W}_t - \int_0^t \gamma_u du$$

will be a Brownian motion. But then

$$dD^S_t = -\sigma D_t^S (dW^*_t + \gamma_t dt)$$

For $D^S$ to be a martingale under $\mathbb{Q}^*$ we must have $\int_0^t D^S_u \gamma_u du \equiv 0$ all $t \leq T$, so $\gamma \equiv 0$, $\mathbb{Q}^* \equiv \mathbb{Q}$ on $\mathcal{F}_T$

(ii) Under $\mathbb{Q}$

$$d\tilde{W}_t = -\sigma^{-1}(D^S_t)^{-1} dD^S_t \qquad (+)$$

Define $\quad M_t = E_{\mathbb{Q}}\left[\frac{X}{S_T} \middle| \mathcal{F}_t\right] \qquad 0 \leq t \leq T$

Then $M$ is an $(\{\mathcal{F}_t\}, \mathbb{Q})$ martingale.
By the MRT we can find some $\{\mathcal{F}_t\}$-predictable $H$ such that

$$M_t = M_0 + \int_0^t H_u d\tilde{W}_u$$

In particular taking $t = T$

$$\frac{X}{S_T} = E_{\mathbb{Q}}\left[\frac{X}{S_T}\right] + \int_0^T H_u d\tilde{W}_u$$

Using (+) we have

$$M_t = E_{\mathbb{Q}}\left[\frac{X}{S_T}\right] - \int_0^T H_u \sigma^{-1}(D^S_u)^{-1} dD^S_u \quad (*)$$

Use (*) to form a self-financing strategy s.t.

$$X = \phi_0 \cdot A_0 + \int_0^T \phi_u \cdot dA_u \qquad A = (D\ S)$$

$$\phi = (\phi^{(1)}, \phi^{(2)})$$

By numeraire invariance this is equivalent to

$$X/S_T = \phi_0^{(1)} D_0^S + \phi_0^{(2)} + \int_0^T \underset{\overset{\shortparallel}{0}}{\phi_u^{(2)} dS_u^S} + \int_0^T \phi_u^{(1)} dD_u^S$$

Find $\phi$ satisfying

self-financing property

$$M_t = E_{\mathbb{Q}}\left[\frac{X}{S_T}\right] + \int_0^t \phi_u^{(2)} dS_u^S + \int_0^t \phi_u^{(1)} dD_u^S = \phi_t^{(1)} D_t^S + \phi_t^{(2)}$$

Take $\quad \phi_t^{(1)} = -\sigma^{-1} H_t (D_t^S)^{-1}$

$$\phi_t^{(2)} = M_t - \phi_t^{(1)} D_t^S$$

For replication doesn't matter what $\phi^{(2)}$ is taken to be but we want self-financing property to hold

(iii) Since $(N, \mathbb{N})$ is a numeraire pair under $\mathbb{N}$

$\frac{S}{N}$ and $\frac{D}{N}$ are (strictly positive) martingales

Define a measure $\hat{\mathbb{Q}} \sim \mathbb{N}$ on $\mathcal{F}_T$ via

$$\left.\frac{d\hat{\mathbb{Q}}}{d\mathbb{N}}\right|_{\mathcal{F}_t} = \frac{S_t N_0}{N_t} \qquad t \leq T.$$

Then by part (a)

$\frac{D}{S}$ is a $\hat{\mathbb{Q}}$ martingale iff $\left.\frac{d\hat{\mathbb{Q}}}{d\mathbb{N}}\right| \cdot \frac{D}{S} = \frac{D}{N}$ is an $\mathbb{N}$ martingale

But $\mathbb{Q}$ is the unique EMM on $\mathcal{F}_T$ corresponding to numeraire $S$ so $\hat{\mathbb{Q}} \equiv \mathbb{Q}$ on $\mathcal{F}_T$

Question 3

(a) Let $\mathbb{N}^i$ denote the EMM corresponding to numeraire $D_{\cdot T_{i+1}}$. The value of the $i^{th}$ digital caplet is given by

$$V_0^{(i)}(K) = D_{0T_{i+1}} E_{\mathbb{N}^i}\left[\frac{V_{T_i}^{(i)}}{D_{T_iT_{i+1}}}\right]$$

$$= D_{0T_{i+1}} \mathbb{N}^i\left(L_{T_i}^{(i)} > K\right)$$

If the model is consistent with the specified market values we have

$$(+) \qquad \mathbb{N}^i\left(L_{T_i}^{(i)} > K\right) = N\left(d^{(i)}(K)\right) \qquad \text{for} \quad K \geq -\alpha_i^{-1}$$

where $N(\ )$ is the cumulative normal distribution
(Note : model allows negative rates)
This specifies the distribution of $L^{(i)}_{T_i}$ under $\mathbb{N}^i$

Observe

$$\mathbb{N}\left(L_{T_i}^{i} > K\right) = \mathbb{N}^i\left(\log\left(L_{T_i}^{(i)} + \alpha_i^{-1}\right) > \log\left(K + \alpha_i^{-1}\right)\right)$$

$$= \mathbb{N}^i\left(\frac{\log\left(L_{T_i}^{(i)} + \alpha_i^{-1}\right) - \log\left(L_0^{(i)} + \alpha_i^{-1}\right) + \frac{1}{2}\left(\Sigma^{(i)}\right)^2}{\Sigma^{(i)}}\right.$$

Note $\quad 1 + \alpha_i L_0^{(i)} = \dfrac{D_{0T_i}}{D_{0T_{i+1}}}$

$$\left. > \frac{\log\left(K + \alpha_i^{-1}\right) - \log\left(L_0^{(i)} + \alpha_i^{-1}\right) + \frac{1}{2}\left(\Sigma^{(i)}\right)^2}{\Sigma^{(i)}}\right)$$

$$= \mathbb{N}^i\left(Z > -d^{(i)}(K)\right)$$

Using (+) $\quad Z = \dfrac{\log\left(L_{T_i}^{(i)} + \alpha_i^{-1}\right) - \left(\log\left(L_0^{(i)} + \alpha_i^{-1}\right) - \frac{1}{2}\left(\Sigma^{(i)}\right)^2\right)}{\Sigma^{(i)}} \sim N(0,1)$

Hence $\quad \log\left(L_{T_i}^{(i)} + \alpha_i^{-1}\right) \sim N\left(\log\left(L_0^{(i)} + \alpha_i^{-1}\right) - \frac{1}{2}\left(\Sigma^{(i)}\right)^2, \left(\Sigma^{(i)}\right)^2\right)$

(b) For the model to be arbitrage free we require for $i=1,\dots,n$

$$M_t^{(i)} := \frac{D_{tT_i}}{D_{tT_{n+1}}}$$

to be martingales under $N$, the EMM corresponding to $D_{\cdot T_{n+1}}$ as numeraire.

Recall for $i=1,\dots,n$ $\quad L_t^{(i)} = \frac{D_{tT_i} - D_{tT_{i+1}}}{\alpha_i D_{tT_{i+1}}}$, $\alpha_i = T_{i+1} - T_i$

Under $N$, $L^{(n)}$ is a martingale $\left( L_t^n = \alpha_n^{-1}\left(\frac{D_{tT_n}}{D_{tT_{n+1}}} - 1\right)\right)$ and so $\mu_t^{(n)} \equiv 0$, $t \le T_n$.

Next observe

$$M_t^{(i)} := \frac{D_{tT_i}}{D_{tT_{n+1}}} = \prod_{j=i}^{n}\left(1+\alpha_j L_t^{(j)}\right) \qquad (+)$$

$$= M_t^{(i+1)} + \alpha_i L_t^{(i)} M_t^{(i+1)}$$

$M^{(i)}, M^{(i+1)}$ martingales $\Rightarrow \alpha_i L^{(i)} M^{(i+1)}$ a martingale

Now

$$d\left(L_t^{(i)} M_t^{(i+1)}\right) = L_t^{(i)} dM_t^{(i+1)} + M_t^{(i+1)} dL_t^{(i)} + dM_t^{(i+1)} dL_t^{(i)}$$

and equating finite variation terms to zero $\Rightarrow$

$$M_t^{(i+1)} \mu_t^{(i)} dt + dM_t^{(i+1)} \sigma_t^{(i)} \gamma^{(i)}\left(L_t^{(i)}\right) dW_t = 0 \quad (\#)$$

From (+) noting $M^{(i+1)}$ a martingale

$$dM_t^{(i+1)} = \sum_{j=i+1}^{n} \frac{\partial M_t^{(i+1)}}{\partial L_t^{(j)}} \sigma_t^{(j)} \gamma^{(j)}\left(L_t^{(j)}\right) dW_t$$

$$= \sum_{j=i+1}^{n} \frac{\alpha_j M_t^{(i+1)}}{\left(1+\alpha_j L_t^{(j)}\right)} \sigma_t^{(j)} \gamma^{(j)}\left(L_t^{(j)}\right) dW_t$$

Substituting in (#)

$$M_t^{(i+1)} \mu_t^{(i)} dt + \sum_{j=i+1}^{n} \frac{\alpha_j M^{(i+1)}}{\left(1+\alpha_j L_t^{(j)}\right)} \sigma_t^{(j)} \gamma^{(j)}\left(L_t^{(j)}\right) \sigma_t^{(i)} \gamma^{(i)}\left(L_t^{(i)}\right) dt$$

and thus for $i=1,\dots,n-1$

$$\mu_t^{(i)} = -\sigma_t^{(i)} \gamma^{(i)}\left(L_t^{(i)}\right) \sum_{j=i+1}^{n} \frac{\alpha_j \gamma^{(j)}\left(L_t^{(j)}\right)}{\left(1+\alpha_j L_t^{(j)}\right)} \sigma_t^{(j)}$$

(c) As the model in (b) is arbitrage free and complete we have $L^{(i)}$ a martingale under $\mathbb{N}^i$ (as $L^{(i)}$ of form asset / numeraire) where $\mathbb{N}^i$ is the EMM corresponding to $D_{\cdot T_{i+1}}$ as numeraire. The equation for $L^{(i)}$ under $\mathbb{N}^i$ is

$$dL^{(i)}_t = \gamma^{(i)}(L^{(i)}_t)\,\sigma^{(i)}(t)\,d\tilde{W}_t$$

where $\tilde{W}$ is a B.M under $\mathbb{N}^i$.

Consider $\gamma^{(i)}(L^{(i)}_t) = L^{(i)}_t + \alpha_i^{-1}$ and set $Y^i_t = L^{(i)}_t + \alpha_i^{-1}$

Then $dL^{(i)}_t = dY^{(i)}_t = Y^{(i)}_t \sigma^{(i)}(t)\,d\tilde{W}_t$

which has unique solution at $T_i$

$$Y^{(i)}_{T_i} = Y^{(i)}_0 \exp\left(\int_0^{T_i} \sigma^{(i)}(t)\,d\tilde{W}_u - \tfrac{1}{2}\int_0^{T_i} (\sigma^{(i)}(u))^2 du\right)$$

So for this choice of $\gamma^i$ under $\mathbb{N}^{(i)}$

$$\log(L^{(i)}_{T_i} + \alpha_i^{-1}) \sim N\left(\log(L^{(i)}_0 + \alpha_i^{-1}) - \tfrac{1}{2}\int_0^{T_i} (\sigma^{(i)}(u))^2 du,\ \int_0^{T_i} (\sigma^{(i)}(u))^2 du\right)$$

Thus taking

$$\gamma^{(i)}(L^{(i)}_t) = L^{(i)}_t + \alpha_i^{-1}$$

and

$$(\Sigma^{(i)})^2 = \int_0^{T_i} (\sigma^{(i)}(u))^2 du$$

$i = 1, \ldots, n$

(d) Separable assumption $\sigma^{(i)}(t) := \sigma^i \sigma(t)$
$\sigma^i > 0$, $\sigma(t)$ positive bdd function of time enables drift approximation (eg based on Brownian bridge as in lectures) to give an approximation in which $L^{(i)}_t$ is a function of $x_t = \int_0^t \sigma_u\, dW_u$

## Question 4

(a) Let $X$ be a continuous $d$-dimensional local martingale adapted to the filtration $\{\mathcal{F}_t\}$. Then $X$ is an $\{\mathcal{F}_t\}$ Brownian motion if and only if

$$[X^{(i)}, X^{(j)}]_t = \delta_{ij}\, t \quad \text{a.s.}$$

for all $i, j$ and $t$.

(b) Observe that

$$[X]_t = \int_0^t H_u^2\, du \uparrow \infty .$$

Since $H$ is deterministic, $\tau$ is an increasing deterministic function of time. Setting

$$\tilde{W}_t := X_{\tau_t} = \int_0^{\tau_t} H_u\, dW_u$$

we have

$$\tilde{W}([X]_t) = X_{\tau_{[X]_t}} = X_t$$

It remains to show that $\tilde{W}$ is a Brownian motion adapted to $\{\mathcal{F}_{\tau_t}\}$ by applying Lévy's Theorem.

Adaptedness and continuity of $\tilde{W}$ follow from that of the stochastic integral. Define

$$S_n := \inf\{t > 0 : |X_t| > n\}$$

and

$$T_n := \tau^{-1}_{S_n} = [X]_{S_n}$$

Since $\tau$ is increasing it follows for each fixed $t$, that

$$\tau_{t \wedge T_n} = S_n \wedge \tau_t$$

and

$$\{T_n \le t\} = \{S_n \le \tau_t\} \in \mathcal{F}_{\tau_t}$$

Thus

$$\tilde{W}^{T_n}_t := \tilde{W}_{t \wedge T_n} = X^{S_n}_{\tau_t}$$

and since $\{S_n\}$ is a reducing sequence for $X$, it follows that $\tilde{W}^{T_n}$ is a martingale with respect to $\{\mathcal{F}_{\tau_t}\}$. Thus $\{T_n\}$ is a reducing sequence for the continuous local martingale $\tilde{W}$. Finally note $[\tilde{W}]_t = \int_0^{\tau_t} H_u^2\, du = [X]_{\tau_t} = t$. Thus by Lévy's

(c) For continuous semimartingales $X, Y$ by integration by parts

$$(+) \quad X_tY_t = X_0Y_0 + \int_0^t X_u\,dY_u + \int_0^t Y_u\,dX_u + [X,Y]_t$$

Set $X_t = t$, $Y_t = \int_0^t \sigma_u\,dW_u$ and note $[X,Y]_t = 0$ as $X$ finite variation. Then by (+) ($Y$ a cts local mg)

$$t\int_0^t \sigma_u\,dW_u = \int_0^t u\sigma_u\,dW_u + \int_0^t \int_0^u \sigma_s\,dW_s\,du$$

as required.

(d)

$$\int_0^t r_u\,du = \int_0^t r_0\,du + \int_0^t \int_0^u \theta_s\,ds\,du + \int_0^t \int_0^u \sigma_s\,dW_s\,du$$

$$= r_0 t + \int_0^t \int_s^t du\,\theta_s\,ds + t\int_0^t \sigma_u\,dW_u - \int_0^t u\sigma_u\,dW_u$$

$$= r_0 t + \int_0^t (t-s)\,\theta_s\,ds + \int_0^t \sigma_u(t-u)\,dW_u$$

$$\int_t^T r_u\,du = r_0(T-t) + \int_t^T (T-s)\,\theta_s\,ds + (T-t)\int_0^t \theta_s\,ds$$
$$+ \int_t^T \sigma_u(T-u)\,dW_u + (T-t)\int_0^t \sigma_u\,dW_u$$

For a short-rate model

$$D_{tT} = E_{\mathbb{Q}}\left(\exp\left(-\int_t^T r_u\,du\right) \middle| \mathcal{F}_t\right)$$

using taking out what is known and independent increment property of B.M.

$$= \exp\left(-r_0(T-t) - \int_t^T (T-s)\theta_s\,ds - (T-t)\int_0^t \theta_s\,ds - (T-t)\int_0^t \sigma_u\,dW_u\right)$$
$$E_{\mathbb{Q}}\left(\exp\left(-\int_t^T \sigma_u(T-u)\,dW_u\right)\right)$$

$$= \exp\left( -(T-t) r_t - \int_t^T (T-s)\theta_s ds + \frac{1}{2}\int_t^T \sigma_u^2 (T-u)^2 du \right)$$

since $-\int_t^T \sigma_u (T-u) dW_u \sim N\left(0, \int_t^T \sigma_u^2 (T-u)^2 du\right)$

(e) Observe

$$1 + \alpha_i L^i_{T_i} = \frac{1}{D_{T_i T_{i+1}}}$$

so using (d)

$$\operatorname{cov}\left( \log(\alpha_i L^i_{T_i} + 1) \log(\alpha_j L^j_{T_j} + 1) \right)$$

$$= \operatorname{cov}\left( (T_{i+1} - T_i) r_{T_i}, (T_{j+1} - T_j) r_{T_j} \right)$$

as other terms deterministic

$$= (T_{i+1} - T_i)(T_{j+1} - T_j) \operatorname{cov}(r_{T_i}, r_{T_j})$$

$$= (T_{i+1} - T_i)(T_{j+1} - T_j)\left( \int_0^{T_i} \sigma_u^2 du \right) \text{ as } T_i < T_j$$

$$\operatorname{var}\left( \log(1 + \alpha_i L^i_{T_i}) \right) = (T_{i+1} - T_i)^2 \int_0^{T_i} \sigma_u^2 du$$

$$\operatorname{corr}\left( \log(1 + \alpha_i L^i_{T_i}), \log(1 + \alpha_j L^j_{T_j}) \right)$$

$$= \frac{(T_{i+1} - T_i)\, T_{j+1} - T_j)\left( \int_0^{T_i} \sigma_u^2 du \right)}{(T_{i+1} - T_i)\sqrt{\int_0^{T_i} \sigma_u^2 du}\, (T_{j+1} - T_j)\sqrt{\int_0^{T_j} \sigma_u^2 du}}$$

= required result