Summary / Cheat-sheet for discrete state space Markov Chains

Property	Discrete time Markov chain	Continuous time Markov chain	Property
Stochastic Process	Sequence $(X_n \ : n \in \mathbb{N}_0)$	Family $(X_t : t \geq 0)$	**Stochastic Process**
Markov Process	For all $A \subset S$ and all $s_0, \dots, s_n \in S$, we have $P(X_{n+1} \in A \,\vert X_n = s_n, \dots, X_0 = s_0) = \; = \; P(X_{n+1} \in A \mid X_n = s_n)$	For all $A \subset S$, $t_0 < t_1 < \cdots < t_n \in [0, \infty)$, and all $s_0, \dots, s_n \in S$, we have $P\left(X_{t_{n+1}} \in A \,\middle\vert X_{t_n} = s_n, \dots, X_{t_0} = s_0\right) = \; = \; P(X_{t_{n+1}} \in A \mid X_{t_n} = s_n)$	**Markov Process**
Homogeneous	$P(X_{n+k} \in A \,\vert X_n = s) = \quad P(X_k \in A \mid X_0 = s)$	$P(X_{t+u} \in A \,\vert X_u = s) = \quad P(X_t \in A \mid X_0 = s)$	**Homogeneous**
Path Space	$S^{\mathbb{N}}$ (Think of a sequence of integers)	Space of right-continuous paths (constant until a jump happens, continuous after the jump)	**Path Space**
Transition function	$p_n(x,y) = P(X_n = y \,\vert X_0 = x) = P(X_{n+k} = y \vert X_k = x)$	$p_t(x,y) = P(X_t = y \,\vert X_0 = x) = P(X_{t+u} = y \vert X_u = x)$	**Transition function**
Chapman-Kolmogorov equations	$p_{k+n}(x,y) = \sum_{z \in S} p_k(x,z) p_n(z,y), \quad \forall n,k \in \mathbb{N}_0, \quad x,y \in S$	$p_{t+u}(x,y) = \sum_{z \in S} p_t(x,z) p_u(z,y), \quad \forall t,u \geq 0, \quad x,y \in S$	**Chapman-Kolmogorov equations**
Transition Matrix	Matrix P with $P(x,y) = p_1(x,y) = p(x,y)$ Size of P is the same of state space	Matrix G with $G = \frac{d}{dt} P_t(x,y)$ evaluated at $t = 0$ Size of G is the same of state space	**Generator**
Eigenvalues and eigenvectors	P is stochastic and we have $P\vert\mathbf{1} >= \vert\mathbf{1} >$ so 1 is an eigenvalue with right-eigenvector $\vert\mathbf{1} >$	G verifies $G\vert\mathbf{1} >= \vert\mathbf{0} >$ so 0 is an eigenvalue with right-eigenvector $\vert\mathbf{1} >$	**Eigenvalues and eigenvectors**
Distribution at time n	$< \pi_n \vert = < \pi_0 \vert P^n$	$P_t = \exp(tG) = \sum_{k=0}^{\infty} \frac{t^k}{k!} G^k$ and $< \pi_t \vert = < \pi_0 \vert \exp(tG)$	**Distribution at time t**
Stationary distribution	Left-eigenvector, solves $< \pi \vert \, P = < \pi \vert$ Can show uniqueness if irreducible and S finite using PF Thm	Left-eigenvector, solves $< \pi \vert \, G = < \mathbf{0} \vert$ Can show uniqueness if irreducible and S finite using PF	**Stationary distribution**
Reversible distribution	$\pi(x) p(x,y) = \pi(y) p(y,x)$ for all $x, y \in S$	$\pi(x) g(x,y) = \pi(y) g(y,x)$ for all $x, y \in S$	**Reversible distribution**
Irreducible MC	For all $x, y \in S$, $p_n(x,y) > 0$ for **some** n	For all $x, y \in S$, $p_t(x,y) > 0$ for **some** t	**Irreducible MC**
Eigenvector representation	$< \pi_n \vert = < \pi_0 \vert v_1 > \lambda_1^n < u_1 \vert + \cdots + < \pi_0 \vert v_L > \lambda_L^n < u_L \vert$ And $\vert \lambda_i \vert \leq 1$ so as $n \to \infty$ $< \pi_n \vert \to < \pi \vert = < u_1 \vert$	$< \pi_t \vert = < \pi_0 \vert v_1 > e^{\lambda_1 t} < u_1 \vert + \cdots + < \pi_0 \vert v_L > e^{\lambda_L t} < u_L \vert$ And $\lambda\_i \leq 0$ so as $t \to \infty$ $< \pi_n \vert \to < \pi \vert = < u_1 \vert$	**Eigenvector representation**
Transition rates	Same as transition function	$p_{\Delta t}(x,y) = \delta_{x,y} + g(x,y) \Delta t + o(\Delta t)$	**Transition rates**
Ergodicity	$\frac{1}{N} \sum_{n=1}^{N} f(X_t) \to E_\pi(f)$ as $N \to \infty$ where expectation is taken with respect to stat. dist. π	$\int_0^T f(X_t) \, dt \to E_\pi(f)$ as $T \to \infty$ where expectation is taken with respect to stat. dist. π	**Ergodicity**
Time-reversal	If stationary, we can write a time-reversed DTMC $Y_n = X_{-n}$ $p^Y(x,y) = \frac{\pi(y)}{\pi(x)} p^X(y,x)$	If stationary, we can write a time-reversed DTMC $Y_t = X_{-t}$ $g^Y(x,y) = \frac{\pi(y)}{\pi(x)} g^X(y,x)$	**Time-reversal**

For **Continuous time Markov chains only**, we have a few more definitions:

- Holding time $W_x = \inf\{t > 0 : X_t \neq x\}$ is the time the chain spends in x
  - We proved that W_x is exponentially distributed with mean $\frac{1}{|g(x,x)|}$.
  - We also showed that if $g(x,x) \neq 0$ the chain jumps from x to y after time W_x with probability $\frac{g(x,y)}{|g(x,x)|}$.
- Jump times $J_0, J_1, \ldots$ are the times at which the chain jumps.
  - We saw that $J_n = \sum_{k=0}^{n-1} W_k$.
- Jump chain $(Y_n : n \in \mathbb{N}_0)$ with $Y_n = X_{J_n}$ is a discrete time Markov chain that tells us where the chain is before each jump.
  - If $g(x,x) < 0$ it has transition function $p(x,y) = \frac{g(x,y)}{|g(x,x)|}$ if $y \neq x$ and 0 otherwise (Y_{n+1} is **always** different that Y_n because the jump chain only changes when the original chain jumps).
  - If $g(x,x) < 0$ then $p(x,y) = \delta_{x,y}$.
- We saw how to use this to simulate CTMCs

For countably infinite state spaces, we can also define the following:

- Return time $T_x = \inf\{t > J_1 : X_t = x\}$ or for DTMCs $T_x = \inf\{n > 1 : X_n = x\}$
  - A state is transient if $P(T_x = \infty | X_0 = x) > 0$ (once the chain leaves x, it never returns)
  - A state is null recurrent if $P(T_x < \infty | X_0 = x) = 1$ and $E(T_x | X_0 = x) = \infty$ (once the chain leaves x it can return but very likely after an infinite time)
  - A state is positive recurrent if $P(T_x < \infty | X_0 = x) = 1$ and $E(T_x | X_0 = x) < \infty$ (once the chain leaves x it can return and will after a finite time
- We saw that in this case we need a chain to be positive recurrent (in addition to irreducible) to guarantee existence and uniqueness of a stationary distribution.
- Explosion time $J_\infty = \lim_{n\to\infty} J_n$ is the time after giving an infinite amount of jumps
  - We said that the chain is non-explosive if $P(J_\infty = \infty) = 1$ and explosive otherwise
  - Being explosive means the chain can jump an infinite amount of times in finite time (which is hard to simulate) and I showed you an example.