MA 482

THE UNIVERSITY OF WARWICK

FOURTH YEAR EXAMINATION: SUMMER 2021

STOCHASTIC ANALYSIS

Time Allowed: **3 hours**

Read all instructions carefully. Please note also the guidance you have received in advance on the departmental 'Warwick Mathematics Exams 2021' webpage.

Calculators, wikipedia and interactive internet resources are not needed and are not permitted in this examination. You are not allowed to confer with other people. You may use module materials and resources from the module webpage.

ANSWER COMPULSORY QUESTION 1 AND TWO FURTHER QUESTIONS out of the three optional questions 2, 3 and 4.

On completion of the assessment, you must upload your answer to Moodle as a single PDF document if possible, although multiple files (2 or 3) are permitted. You have an additional 45 minutes to make the upload, and instructions are available on the departmental 'Warwick Mathematics Exams 2021' webpage.
You must not upload answers to more than 3 questions, including Question 1. If you do, you will only be given credit for your Question 1 and the first two other answers.

The numbers in the margin indicate approximately how many marks are available for each part of a question. The compulsory question is worth 40 marks, while each optional question is worth 30 marks.

Throughout this exam unless stated otherwise you may assume,
$(\Omega, \mathcal{F}, \mathbb{P})$ is a probability space,
$(\mathcal{F}_t)_{t\geq 0}$ is a filtration satisfying the usual conditions,
$(B_t)_{t\geq 0}$ is a $(\mathcal{F}_t)_{t\geq 0}$ Brownian motion.

COMPULSORY QUESTION

1. This question is about the Ornstein-Uhlenbeck process. Throughout this question X is the solution to the SDE

$$\mathrm{d}X_t = -\lambda X_t \mathrm{d}t + \sigma \mathrm{d}B_t, \tag{1}$$

where $\lambda > 0, \sigma > 0$.

a) Find $\mathrm{d}(e^{\gamma t} X_t)$ for $\gamma \in \mathbb{R}$. **[3]**

b) For which values of γ is $e^{\gamma t} X_t$ a martingale? **[2]**

c) By choosing γ show that

$$X_t = e^{-\lambda t} X_0 + \int_0^t \sigma e^{-\lambda(t-s)} \mathrm{d}B_s.$$

[5]

d) When $X_0 = x$ compute $\mathbb{E}(X_t)$. **[4]**

e) Let $Y_t = e^{\lambda t} X_t$. What is the quadratic variation of Y_t? **[3]**

f) Using the Dubins-Schwarz theorem, write Y_t in the form

$$Y_0 + W_{F(t)},$$

where W is a Brownian motion and F is a function of time. **[5]**

g) Use part (f) to write X_t in the form

$$H(t, X_0) + G(t) W_{F(t)}.$$

[5]

h) Write down the Kolmogorov forward equation for the law of the solution to the SDE (1). **[5]**

i) Using the Kolmogorov forward equation find the law of a steady state solution to the SDE. How does this relate to your answer to part (g)? **[8]**

OPTIONAL QUESTIONS

2. The aim of this question is to solve the SDE

$$\mathrm{d}Y_t = \alpha(1 - Y_t)\mathrm{d}t + \beta Y_t \mathrm{d}B_t.$$

a) First, find an Ito process Z_t such that

$$\mathrm{d}(Y_t Z_t) = \alpha Z_t.$$

Find an explicit form for Z_t and write Y_t in terms of Z. **[12]**

b) Write down the equation that will be satisfied by the law of the steady state of this equation. Solve the steady state equation separately for $y > 0$ and $y < 0$. **[10]**

c) Looking at the explicit solution of the SDE, state whether Y_t is going to tend to be mostly negative, mostly positive or neither as $t \to \infty$. Use this to find a condition on α and β for the steady state to exist and write down the steady state without explicitly calculating the normalising factor. Explain how the condition relates to the behaviour of Z_t. **[8]**

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3. For a time $t > 0$ and an integer n let $t_j = (j2^{-n}) \wedge t$, $j = 0, \ldots, \lceil 2^n t \rceil$.

a) Suppose you want to approximate the Ito process $X_t = X_0 + \int_0^t U_s \mathrm{d}s + \int_0^t \sigma_s \mathrm{d}B_s$, where U, σ are continuous stochastic processes and σ is an admissible integrand and an Ito process. Which of the following approximations should you use?

(i)

$$A_t^{(1)} = X_0 + \sum_{j=1}^{\lceil 2^n t \rceil} U_{t_j}(t_j - t_{j-1}) + \sum_{j=1}^{\lceil 2^n t \rceil} \sigma_{t_{j-1}}(B_{t_j} - B_{t_{j-1}})$$

(ii)

$$A_t^{(2)} = X_0 + \sum_{j=1}^{\lceil 2^n t \rceil} U_{t_{j-1}}(t_j - t_{j-1}) + \sum_{j=1}^{\lceil 2^n t \rceil} \sigma_{t_j}(B_{t_j} - B_{t_{j-1}})$$

Justify your answer. **[5]**

b) For *both* approximations, identify the limit (without proving the convergence) as $n \to \infty$ of the approximation in $L^2(\Omega)$ in the following cases:

(i) When $U_s = 0$ and $\sigma_s = B_s, \forall s \leq t$. **[10]**

(ii) When $U_s = -1$ and $\sigma_s = B_s \forall s \leq t$. **[3]**

(iii) For general U, σ leaving your answer in terms of time integrals, Ito integrals and quadratic covariations. **[5]**

(iv) When $U_s = 0$ and $\sigma_s = B_s^2 - s, \forall s \leq t$. Find any quadratic covariation terms explicitly. **[7]**

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4. a) Explain why $\int_0^t B_s \mathrm{d}s$ is a Gaussian random variable. **[2]**

b) Show that $cov\left(B_t, \int_0^t B_s \mathrm{d}s\right) = t^2/2$ and that $\mathbb{E}\left(\left(\int_0^t B_s \mathrm{d}s\right)^2\right) = t^3/3$. Hence find the joint density $f_{B_t, \int_0^t B_s \mathrm{d}s}(y,z)$ of $\left(B_t, \int_0^t B_s \mathrm{d}s\right)$. Leave your answer in terms of matrix determinants and inverses without explicitly computing them. **[5]**

c) For $\gamma > 0$, and $\phi \in C_0^2(\mathbb{R})$ (twice continuously differentiable and vanishing at infinity), consider the PDE

$$\partial_t h(t,x,v) = \frac{1}{2}\partial^2_{v^2} h(t,x,v) + \gamma v \partial_x h(t,x,v), \quad h(0,x,v) = \phi(v)e^{-x}.$$

Where $x \in \mathbb{R}, v \in \mathbb{R}$, and $t > 0$. Write the solution $h(t,x,v)$ in the form $\mathbb{E}_{x,v}(Z_t)$ where Z_t is a stochastic process you should specify. Give the name of the theorem you are using. **[10]**

d) Using your answer to parts (b) and (c) write $h(t,x,v)$ in the form

$$h(t,x,v) = \int_{\mathbb{R}^2} \psi(t,x,v,y,z) f_{B_t, \int_0^t B_s \mathrm{d}s}(y,z)\mathrm{d}y\mathrm{d}z,$$

where you should find ψ explicitly. **[8]**

e) Using the fact that the conditional density of $\int_0^t B_s \mathrm{d}s$ given B_t is

$$g(z|B_t) = \sqrt{\frac{6}{\pi t^3}} \exp\left(-\frac{6}{t^3}\left(z - \frac{tB_t}{2}\right)^2\right),$$

show that

$$h(t,0,v) = \mathbb{E}\left(\phi(v+B_t)\exp\left(-\gamma t(v + B_t/2) + \frac{1}{24}\gamma^2 t^3\right)\right).$$

. **[5]**
