Under $\mathbb{Q}$. $df(t,T) = \sigma(t,T)\left(\int_t^T \sigma(t,s)ds\right)dt + \sigma(t,T)dW_t^{\mathbb{Q}}$.

Let $x := T-t$, i.e. x is time to maturity.

$r(t,x) := f(t,t+x)$

Note that $dr(t,x) = df(t,t+x) + \partial_T f(t,t+x)dt$

$$= \sigma(t,t+x)\left(\int_t^{t+x}\sigma(t,s)ds\right)dt + \sigma(t,t+x)dW_t^{\mathbb{Q}}$$

$$+ \partial_x r(t,x)dt$$

$$= \left[\sigma_0(t,x)\left(\int_0^x \sigma_0(t,s)ds\right) + \partial_x r(t,x)\right]dt + \sigma_0(t,x)dW_t^{\mathbb{Q}}$$

where $\sigma_0(t,x) := \sigma(t,t+x)$ Musiela SPDE.

Chapter 4 Change of numeraire. N_t

Consider a B-S market, $dB_t = B_t r_t dt$, $B_0 = 1$

$$dS_t^i = S_t^i\left(\mu_t^i dt + \sum_{j=1}^d \sigma_t^{ij} dW_t^j\right), \ S_0^i = s^i, \quad 1 \le i \le n$$

Assumption:

Suppose $\exists$ an ELMM $\mathbb{Q}$, under which the discounted price of the numeraire N follows:
$$d\frac{N_t}{B_t} = \frac{N_t}{B_t}\sum_{j=1}^d h_t^j dW_t^{\mathbb{Q}j} \quad \text{for some r.d process } h = (h^1 \dots h^d).$$

$$\Leftrightarrow \frac{N_t}{B_t} = N_0 \ \mathcal{E}\left(\int_0^\cdot \sum_{j=1}^d h_s^j dW_s^{\mathbb{Q}j}\right)_t$$

Suppose $\frac{N}{B}$ is a martingale under $\mathbb{Q}$. (e.g. if Novikov condition holds $E^{\mathbb{Q}}\left[e^{\frac{1}{2}\int_0^T |h_s|^2 ds}\right] < \infty$)

Define $\mathbb{Q}^N \sim \mathbb{Q}$ by R-N density $\frac{d\mathbb{Q}^N}{d\mathbb{Q}}\Big|_{\mathcal{F}_t} = \mathcal{E}\left(\int_0^\cdot \sum_{j=1}^d h_s^j dW_s^{\mathbb{Q}j}\right)_t = \frac{N_t}{B_t N_0}$

By Girsanov, $W_t^{\mathbb{Q}^N} = W_t^{\mathbb{Q}} - \int_0^t h_s ds$, $t \in [0,T]$, is BM under $\mathbb{Q}^N$.

Prop: the risky asset S_t^i in units of the numeraire N_t,
$$S_t^{i,N} = S_t^i / N_t$$
follows
$$dS_t^{i,N} = S_t^{i,N}\sum_{j=1}^d \left(\sigma_t^{ij} - h_t^j\right) dW_t^{\mathbb{Q}^N}$$

$$\Leftrightarrow S_t^{i,N} = S_0^{i,N}\,\mathcal{E}\left(\int_0^\cdot \sum_{j=1}^d \left(\sigma_s^{ij} - h_s^j\right) dW_s^{\mathbb{Q}^N}\right)_t$$

$$\mathbb{P} \xleftrightarrow{\text{no arbitrage}} \mathbb{Q} \xleftrightarrow{\text{change of numeraire}} \mathbb{Q}^N$$

real-world prob measure | risk-neutral prob measure | prob measure induced by N.

Proof: $S_t^{i,N} = \frac{S_t^i}{N_t} = \frac{S_t^i / B_t}{N_t / B_t}$

Proof: $S_t^{i,N} = \frac{S_t^i}{N_t} = \frac{S_t^i/B_t}{N_t/B_t}$

By no arbitrage condition, $d\frac{S_t^i}{B_t} = \frac{S_t^i}{B_t}\sum_{j=1}^d \sigma_t^{ij}\,dW_t^{Q,j}$

By numeraire assumption, $d\frac{N_t}{B_t} = \frac{N_t}{B_t}\sum_{j=1}^d h_t^j\,dW_t^{Q,j}$

By Itô, $dS_t^{i,N} = S_t^{i,N}\left(\sum_{j=1}^d (\sigma_t^{ij} - h_t^j)\underbrace{(dW_t^{Q,j} - h_t^j dt)}_{dW_t^{Q^N,j}}\right)$ #

Application to no-arbitrage price for a payoff $X \in \mathcal{F}_T$.

$$X_t = E^Q\left[\frac{X}{B_T}\cdot B_t \,\middle|\, \mathcal{F}_t\right]$$

$$= E^Q\left[\frac{X}{N_T}\cdot N_t \cdot \frac{\frac{N_T}{B_T N_0}}{\frac{N_t}{B_t N_0}} \,\middle|\, \mathcal{F}_t\right]$$

$$= E^Q\left[\frac{X}{N_T}\cdot N_t \cdot \frac{\frac{dQ^N}{dQ}\big|_{\mathcal{F}_T}}{\frac{dQ^N}{dQ}\big|_{\mathcal{F}_t}} \,\middle|\, \mathcal{F}_t\right]$$

Bayes rule

$$= E^{Q^N}\left[\frac{X}{N_T}\cdot N_t \,\middle|\, \mathcal{F}_t\right]$$

Example of numeraire N

① $N_t = B_t$, so $\tilde{N}_t = \frac{N_t}{B_t} = 1$ and $h_t^j = 0$

In this case, $Q^N = Q$, $W^{Q^N} = W^Q$

② $N_t = S_t^1$, so $\tilde{N}_t = \frac{S_t^1}{B_t}$ and $h_t^j = \sigma_t^{1,j}$

In this case, RN density is $\frac{dQ^N}{dQ}\Big|_{\mathcal{F}_t} = \frac{S_t^1/B_t}{S_0^1} = \mathcal{E}\left(\int_0^{\cdot} \sum_{j=1}^d \sigma_s^{1,j}\,dW_s^{Q,j}\right)_t$

$W_t^{Q^N,j} = W_t^{Q,j} - \int_0^t \sigma_s^{1,j}ds$, $1\le j\le d$, is BM under Q^N

$\Rightarrow S_t^{i,N} = \frac{S_t^i}{S_t^1}$ follows $dS_t^{i,N} = S_t^{i,N}\sum_{j=1}^d (\sigma_t^{ij} - \sigma_t^{1,j})\,dW_t^{Q^N,j}$

③ $N_t = P(t,T)$, so $\tilde{N}_t = \frac{P(t,T)}{B_t} = P(0,T)\,\mathcal{E}\left(\int_0^{\cdot}\sum_{j=1}^d \underbrace{\sigma^{*j}(s,T)}_{h_s^j}\,dW_s^{Q,j}\right)_t$

In this case, R-N density is $\frac{dQ^N}{dQ}\Big|_{\mathcal{F}_t} = \frac{P(t,T)/B_t}{P(0,T)}$

Def: Q^N is called T-forward measure, denoted as $\tilde{Q}^T$.

$$W_t^{\tilde{Q}^T, j} = W_t^{Q,j} - \int_0^t \sigma^{*,j}(s,T)\,ds, \quad 1 \le j \le d, \text{ is BM under } \tilde{Q}^T.$$

$$\Rightarrow S_t^{iN} = \frac{S_t^i}{P(t,T)} \text{ follows } dS_t^{iN} = S_t^{iN} \sum_{j=1}^{d} \left(\sigma_t^{ij} - \sigma^{*,j}(t,T)\right) dW_t^{\tilde{Q}^T, j}$$

Application 1.
$$d\,\frac{P(t,S)}{P(t,T)} = \frac{P(t,S)}{P(t,T)} \left[\sigma^*(t,S) - \sigma^*(t,T)\right] dW_t^{\tilde{Q}^T}, \quad t \in [0, T \wedge S]$$

Application 2.
$$\left.\frac{d\tilde{Q}^S}{d\tilde{Q}^T}\right|_{\mathcal{F}_t} = \left.\frac{d\tilde{Q}^S}{dQ}\right|_{\mathcal{F}_t} \Big/ \left.\frac{d\tilde{Q}^T}{dQ}\right|_{\mathcal{F}_t}$$

$$= \frac{P(t,S)/B_t}{P(0,S)} \Big/ \frac{P(t,T)/B_t}{P(0,T)}$$

$$= \frac{P(t,S)\,P(0,T)}{P(t,T)\,P(0,S)}$$

Remark: We receive a family of EMMs; Each $\tilde{Q}^T$ corresponds to a different numeraire T-bond. Since EMM Q corresponds to bank account B_t as numeraire, it is also called spot measure / risk-neutral measure.

Application 3. Expectation hypothesis

By HJM Condition,
$$df(t,T) = \sigma(t,T) \int_t^T \sigma(t,s)\,ds\,dt + \sigma(t,T)\,dW_t^Q$$

$$= -\sigma(t,T)\,\sigma^*(t,T)\,dt + \sigma(t,T)\,dW_t^Q \quad \text{under spot measure } Q.$$

$$= \sigma(t,T)\left(-\sigma^*(t,T)\,dt + dW_t^Q\right)$$

$$= \sigma(t,T)\,dW_t^{\tilde{Q}^T} \quad \text{under } T\text{-forward measure } \tilde{Q}^T$$

Hence, $f(t,T)$, $0 \le t \le T$, is a martingale under $\tilde{Q}^T$,

i.e. $f(t,T) = E^{\tilde{Q}^T}[f(T,T) \mid \mathcal{F}_t] = E^{\tilde{Q}^T}[r_T \mid \mathcal{F}_t]$

Application 4 Dybvig–Ingersoll–Ross Theorem.

Define $R_\infty(t) := \lim_{T \uparrow \infty} R(t,T)$. Then $R_\infty(s) \le R_\infty(t)$ for $s \le t$.