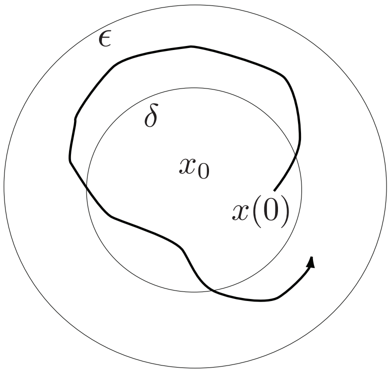
Stability

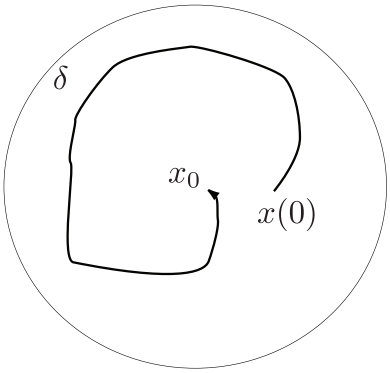
A fixed point x_0 is an attracting fixed point if all trajectories that start near x_0 approach it as $t \to \infty$. If x_0 attracts all trajectories it is called globally attracting.

A fixed point x_0 is **Lyapunov** (neutrally) stable if for all $\epsilon > 0$ there exists $\delta > 0$ such that $|x(0) - x_0| < \delta$ implies that $|x(t) - x_0| < \epsilon$ for all $t > 0$.



In other words, if a solution starts near an equilibrium x_0 then it stays near x_0 (for example harmonic oscillator).

A fixed point is **asymptotically stable** if it is Lyapunov stable and there exists $\delta > 0$ such that if $|x(0) - x_0| < \delta$ then $|x(t) - x_0| \to 0$ as $t \to \infty$.



Linearisation

Consider the system

$$\dot{x} = f(x, y),$$
$$\dot{y} = g(x, y)$$

and suppose that (x^*, y^*) is a fixed point. Considering a small disturbance from the fixed point

$$u = x - x^*, \qquad v = y - y^*$$

we have (by Taylor series expansion)

$$\dot{u} = \dot{x} = f(u + x^*, v + y^*) = f(x^*, y^*) + \left.\frac{\partial f}{\partial x}\right|_{(x^*,y^*)} \cdot u + \left.\frac{\partial f}{\partial y}\right|_{(x^*,y^*)} \cdot v + O(u^2, v^2, uv).$$

This leads to

$$\dot{u} = \left.\frac{\partial f}{\partial x}\right|_{(x^*,y^*)} \cdot u + \left.\frac{\partial f}{\partial y}\right|_{(x^*,y^*)} \cdot v + O(u^2, v^2, uv)$$

and similarly

$$\dot{v} = \left.\frac{\partial g}{\partial x}\right|_{(x^*,y^*)} \cdot u + \left.\frac{\partial g}{\partial y}\right|_{(x^*,y^*)} \cdot v + O(u^2, v^2, uv).$$

Hence

$$\begin{pmatrix} \dot{u} \\ \dot{v} \end{pmatrix} = A \begin{pmatrix} u \\ v \end{pmatrix} \quad - \text{ the linearised system}$$

with

$$A = \begin{pmatrix} \frac{\partial f}{\partial x} & \frac{\partial f}{\partial y} \\ \frac{\partial g}{\partial x} & \frac{\partial g}{\partial y} \end{pmatrix}_{(x^*,y^*)} \quad - \text{ \underline{the Jacobian matrix}}$$

Theorem (linear stability): Suppose that $\dot{x} = f(x)$ has an equilibrium at x^* and the linearisation $\dot{x} = Ax$. If A has no zero or purely imaginary eigenvalues then the local stability of the fixed point (which is called **hyperbolic** in this case) is determined by the linear system. In particular, if all eigenvalues have a negative real part Re $(\lambda_i) < 0$ for all $i = 1, \ldots, n$ then the fixed point is asymptotically stable.

Hartman-Grobman theorem: The local phase-portrait near a hyperbolic fixed point is topologically equivalent to the phase-portrait of the linearisation.

Example 1. To illustrate some of the principles covered let us do a phase-plane analysis of the Lotka-Volterra model of population dynamics of two competing species. Assume i) each species grows in the absence of the other with logistic growth ($\dot{x} = x(1-x)$) and ii) when both species are present they compete for food such that one may go hungry. A particular model of rabbits (r) and sheep (s):

$$\dot{r} = r(3 - r - 2s) \equiv f(r, s)$$
$$\dot{s} = s(2 - r - s) \equiv g(r, s)$$

Fixed points defined by $\dot{r} = \dot{s} = 0$. One finds $(\bar{r}, \bar{s}) = (0,0), (0,2), (3,0), (1,1)$. To classify them we compute

$$A = \begin{bmatrix} \frac{\partial f}{\partial r} & \frac{\partial f}{\partial s} \\ \frac{\partial g}{\partial r} & \frac{\partial g}{\partial s} \end{bmatrix} = \begin{bmatrix} 3 - 2r - 2s & -2r \\ -s & 2 - r - 2s \end{bmatrix}$$

1. $(\bar{r}, \bar{s}) = (0, 0)$

   $$A = \begin{bmatrix} 3 & 0 \\ 0 & 2 \end{bmatrix}$$

   The eigenvalues are both positive so $(0,0)$ is an unstable node. $\lambda = 2$, eigenvector $(0,1)$ - slow eigendirection; $\lambda = 3$, eigenvector $(1,0)$ - fast eigendirection. Trajectories are tangential to the slow eigendirection (i.e. smallest $|\lambda|$) at a node, so they are tangential to $(0,1)$ here.

2. $(\bar{r}, \bar{s}) = (0, 2)$

   $$A = \begin{bmatrix} -1 & 0 \\ -2 & -2 \end{bmatrix}, \qquad \Lambda = \begin{bmatrix} -1 & 0 \\ 0 & -2 \end{bmatrix}$$

   Hence $(0,2)$ is a stable node. Slow eigendirection is $(1,-2)$.

3. $(\overline{r}, \overline{s}) = (3, 0)$

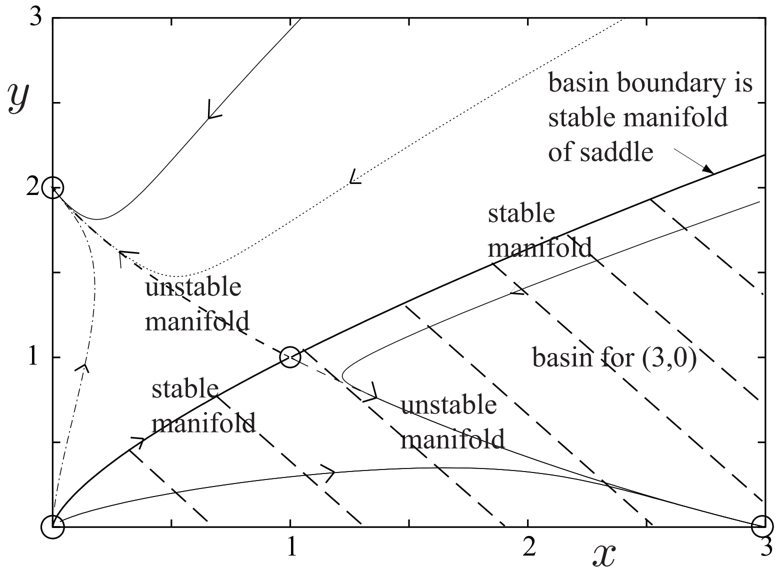
$$A = \begin{bmatrix} -3 & -6 \\ 0 & -1 \end{bmatrix}, \qquad \Lambda = \begin{bmatrix} -3 & 0 \\ 0 & -1 \end{bmatrix}$$

   Hence $(3, 0)$ is a stable node. Slow eigendirection is $(3, -1)$.

4. $(\overline{r}, \overline{s}) = (1, 1)$

$$A = \begin{bmatrix} -1 & -2 \\ 1 & -1 \end{bmatrix}, \qquad \Lambda = \begin{bmatrix} -1 + \sqrt{2} & 0 \\ 0 & -1 - \sqrt{2} \end{bmatrix}$$

   Hence, $(1, 1)$ is a saddle



The above example nicely illustrates the notion of a **basin of attraction**. Given an attracting fixed point $\overline{x}$ we define its basin of attraction to be the set of initial conditions x_0 such that $x(t) \to \overline{x}$ as $t \to \infty$. For instance the basin of attraction for the node at $(3, 0)$ consists of all points lying below the stable manifold of the saddle. Because the stable manifold separates the basins of two nodes, it is called the **basin boundary**.