CH3 Heath-Jarrow-Morton Methodology

$$df(t,T) = \alpha(t,T)dt + \sigma(t,T)dW_t \qquad \text{under } \mathbb{P}$$

where ① $\alpha(t,T)$, $\sigma(t,T)$, $t\in[0,T]$, are prog meb, smooth in T;

② $\int_0^T\int_0^T |\alpha(t,s)|dtds < \infty$, $\sup_{0\le t\le s\le T} |\sigma(t,s)| < \infty$

③ $\sigma(t,T) = (\sigma^1(t,T) \cdots \sigma^d(t,T))$, $W = (W^1 \cdots W^d)^T$

Comparison between short-rate & forward rate models

underlying assets: $B(t)$ $B(t)$, $P(t,T)$, $T\ge t$.

derivatives: Zero-coupon bond   IR cap/floor, Swaption.

Prop: $$\boxed{d\frac{P(t,T)}{B(t)} = \frac{P(t,T)}{B(t)}\left\{\left[\alpha^*(t,T) + \frac{1}{2}|\sigma^*(t,T)|^2\right]dt + \sigma^*(t,T)dW_t\right\}}$$

where $$\alpha^*(t,T) = -\int_t^T \alpha(t,s)ds \qquad \sigma^*(t,T) = -\int_t^T \sigma(t,s)ds$$

Proof: Recall $P(t,T) = e^{-\int_t^T f(t,s)ds}$

$$d\left(-\int_t^T f(t,s)ds\right) = f(t,t)dt - \int_t^T df(t,s)ds$$

$$= r(t)dt - \int_t^T(\alpha(t,s)dt + \sigma(t,s)dW_t)ds$$

Fubini   Stochastic Fubini

$$= r(t)dt \underbrace{-\int_t^T \alpha(t,s)ds}_{\alpha^*(t,T)}\, dt \underbrace{-\int_t^T \sigma(t,s)ds}_{\sigma^*(t,T)}\, dW_t$$

$$= (r(t) + \alpha^*(t,T))dt + \sigma^*(t,T)dW_t$$

Apply Itô to $e^{-\int_t^T f(t,s)ds}$: $\exp(\gamma_t)$, where $d\gamma_t$

$$dP(t,T) = P(t,T)\left(r(t) + \alpha^*(t,T) + \frac{1}{2}|\sigma^*(t,T)|^2\right)dt + P(t,T)\sigma^*(t,T)dW_t$$

$$dB(t) = B(t)r(t)dt \Rightarrow d\frac{1}{B(t)} = \frac{-r(t)}{B(t)}dt.$$

$$\Rightarrow d\frac{P(t,T)}{B(t)} = \frac{1}{B(t)}dP(t,T) + P(t,T)d\frac{1}{B(t)} \qquad \#$$

How to construct an ELMM $\mathbb{Q}$?

$$d\frac{P(t,T)}{B(t)} = \frac{P(t,T)}{B(t)}\sigma^*(t,T)\left\{\Theta(t)dt + dW_t\right\}$$

where $\Theta(t)$ is defined as solution of the system

market price of risk

$$\sigma^*(t,T)\Theta(t) = \alpha^*(t,T) + \frac{1}{2}|\sigma^*(t,T)|^2, \quad T>t$$

i.e. $$\sum_{j=1}^d \sigma^{*,j}(t,T)\Theta^j(t) = \alpha^*(t,T) + \frac{1}{2}|\sigma^*(t,T)|^2, \quad T>t. \qquad (*)$$

market price of risk system of equations

If $(*)$ admits a unique solution $\Theta(t) = (\Theta^1(t) \cdots \Theta^d(t))$, market price of risk system of equations

If $(\star)$ admits a unique solution $\Theta(t) = (\Theta^1(t) \cdots \Theta^d(t))$,
and moreover, $\Theta(t)$ satisfies Novikov

$$E^P\left[e^{\frac{1}{2}\int_0^T |\Theta(t)|^2 dt}\right] < \infty$$

(Hence, $\mathcal{E}(-\int_0^\cdot \Theta(t) dW_t)$ is a martingale)

By Girsanov. $W_t^{Q,j} \triangleq W_t^j + \int_0^t \Theta^j(t) dt$, $1 \le j \le d$, is BM under Q
where Q is defined via its RN density:

$$\left.\frac{dQ}{dP}\right|_{\mathcal{F}_t} = \mathcal{E}\left(-\int_0^\cdot \Theta(t) dW_t\right)_t.$$

$\Rightarrow$ $d\frac{P(t,T)}{B(t)} = \frac{P(t,T)}{B(t)} \sigma^*(t,T)\, dW_t^Q$ is a local martingale under Q.

Hence, Q is an ELMM #

By 1st fundamental Th of Asset Pricing, the market is arbitrage free.

How to solve $(\star)$, market price of risk system of equations.

Suppose we are given n zero-coupon bonds with maturities $T_1 \cdots T_n$:

Then, $(\star)$ reduces to

$$\underset{n\times d}{\begin{pmatrix} \sigma^{*,1}(t,T_1) & \cdots & \sigma^{*,d}(t,T_1) \\ & \ddots & \\ \sigma^{*,d}(t,T_n) & \cdots & \sigma^{*,d}(t,T_n) \end{pmatrix}} \underset{d\times 1}{\begin{pmatrix} \Theta^1(t) \\ \vdots \\ \Theta^d(t) \end{pmatrix}} = \underset{n\times 1}{\begin{pmatrix} \alpha^*(t,T_1) + \frac{1}{2}\sum_{j=1}^d |\sigma^{*,j}(t,T_1)|^2 \\ \vdots \\ \alpha^*(t,T_n) + \frac{1}{2}\sum_{j=1}^d |\sigma^{*,j}(t,T_n)|^2 \end{pmatrix}}$$

Prop: If the vol matrix $\{\sigma^{*,j}(t,T_i)\}_{\substack{1\le i\le n \\ 1\le j\le d}}$ has rank d,

then, $(\star)$ with $\{T_1 \cdots T_n\}$ admits a unique solution $\Theta(t) = (\Theta^1(t) \cdots \Theta^d(t))$.

Application of HJM:

Recall $(\star)$. $\sigma^*(t,T)\Theta(t) = \alpha^*(t,T) + \frac{1}{2}|\sigma^*(t,T)|^2$, $T > t$

i.e. $-\int_t^T \sigma(t,s) ds\, \Theta(t) = -\int_t^T \alpha(t,s) ds + \frac{1}{2}\left|\int_t^T \sigma(t,s) ds\right|^2$, $T > t$.

HJM drift condition in integral form

Differentiate against T: $\quad -\sigma(t,T)\,\Theta(t) = -\alpha(t,T) + \sigma(t,T)\left(\int_t^T \sigma(t,s)\,ds\right)^T$

$$= -\alpha(t,T) - \sigma(t,T)\left(\sigma^*(t,T)\right)^T$$

HJM drift condition in differential form

Hence.

$$df(t,T) = \alpha(t,T)\,dt + \sigma(t,T)\,dW_t \quad \text{under } \mathbb{P}$$

$$= \alpha(t,T)\,dt + \sigma(t,T)\left(dW_t^{\mathbb{Q}} - \Theta(t)\,dt\right) \quad \text{under } \mathbb{Q}.$$

$$= \left[\alpha(t,T) - \sigma(t,T)\,\Theta(t)\right]dt + \sigma(t,T)\,dW_t^{\mathbb{Q}}.$$

By HJM drift condition in differential form

$$= -\sigma(t,T)\left(\sigma^*(t,T)\right)^T dt + \sigma(t,T)\,dW_t^{\mathbb{Q}}$$

$$= \sigma(t,T)\left(\int_t^T \sigma(t,s)\,ds\right)^T dt + \sigma(t,T)\,dW_t^{\mathbb{Q}}.$$

Remark: Under $\mathbb{Q}$.

$$r(t) = f(t,t) \underset{=}{\text{HJM drift}} f(0,t) - \int_0^t \sigma(s,t)\left(\sigma^*(s,t)\right)^T ds + \int_0^t \sigma(s,t)\,dW_s^{\mathbb{Q}}.$$

Hence.

$\mathbb{E}^{\mathbb{Q}}[r(t)] \neq f(0,t)$, but $\mathbb{E}^{\tilde{\mathbb{Q}}}[r(t)] = f(0,t)$ under forward measure $\tilde{\mathbb{Q}}$.