Second and higher order systems

$\dot{x} = f(x) \qquad x \in R^n \qquad (x \in R^2)$

Linear system $\dot{x} = Ax \qquad A_{2\times 2}$

$x = (x_1, x_2)^T$

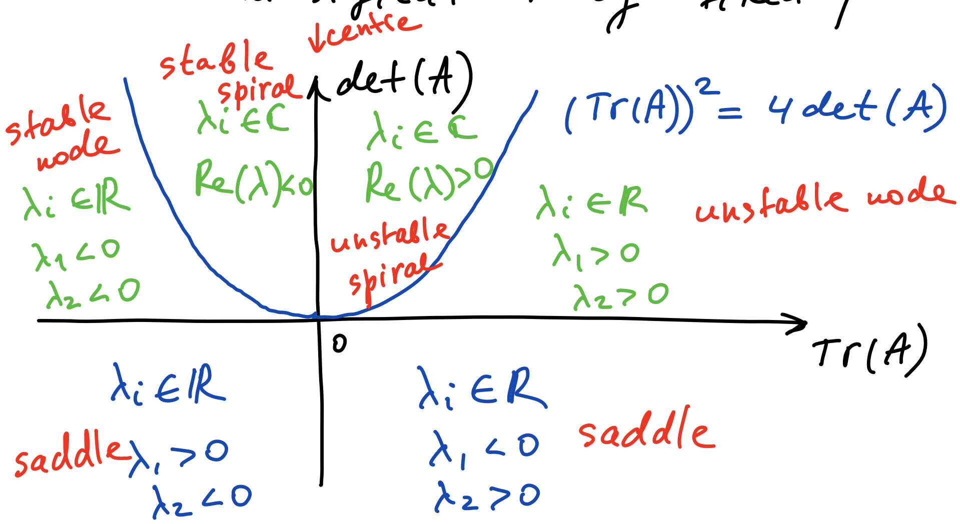
Eigenvalues of A: λ_1, λ_2

$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$

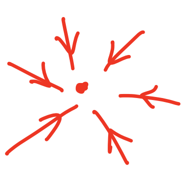
$$\lambda_{1,2} = \frac{1}{2}\left[Tr(A) \pm \sqrt{(Tr(A))^2 - 4\det(A)}\right]$$

$$x(t) = c_1 e^{\lambda_1 t} v_1 + c_2 e^{\lambda_2 t} \mathbf{v}_2$$

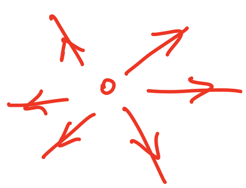
Classification of fixed points in R^2



stable node

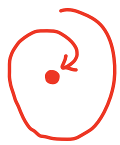


unstable node



$$\lambda_{1,2} = \alpha \pm i\omega \qquad e^{(\alpha + i\omega)t} = e^{\alpha t} \cdot \underbrace{e^{i\omega t}}_{\cos\omega t + i\sin\omega t}$$

stable spiral



unstable spiral

centre $\quad T = \frac{2\pi}{\omega}$

neutrally stable

Lyapunov stable

$m\ddot{x} + kx = 0$

harmonic osc.

Saddle

Nonlinear systems in $\mathbb{R}^2$ (in $\mathbb{R}^4$)

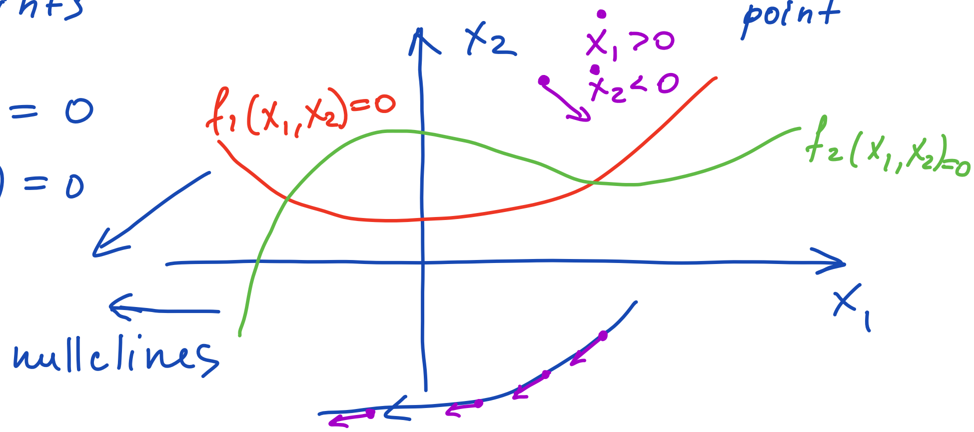
$$\dot{x}_1 = f_1(x_1, x_2)$$
$$\dot{x}_2 = f_2(x_1, x_2)$$

$\mathbf{x} = (x_1, x_2)$ – point in the phase-plane

$\dot{\mathbf{x}} = (\dot{x}_1, \dot{x}_2)$ – velocity vector at that point

Fixed points

$$\begin{cases} f_1(x_1, x_2) = 0 \\ f_2(x_1, x_2) = 0 \end{cases}$$



Existence & uniqueness theorem in $\mathbb{R}^n$

$\dot{x} = f(x) \quad f: \mathbb{R}^n \to \mathbb{R}^n$

$\left.\dfrac{\partial f_i}{\partial x_j} \text{ exist \& continuous}\right| \Rightarrow$ solution exists and unique