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Brownian Motion

Problem sheet 5

1. **Stopping time exercises**
   (a) If $S \leq T$ are stopping times, check that $\mathcal{F}_S \subseteq \mathcal{F}_T$.
   (b) If S, T are stopping times, check that the event $\{S < T\}$ is both in $\mathcal{F}(S)$ and $\mathcal{F}(T)$.
   (c) If S, T are stopping times, check that $S \wedge T$ is a stopping time and that $\mathcal{F}(S \wedge T) = \mathcal{F}_S \cap \mathcal{F}_T$.
   (d) Show that there exists a stopping time T for Brownian Motion with $\mathbb{E}[T] = \infty$ but $\mathbb{E}\big[B(T)^2\big] < \infty$.

2. **Reflection principle**
   Suppose $\big(B(t) : t \geq 0\big)$ is a Brownian Motion on $\mathbb{R}$. For $M(t) := \sup_{s \leq t} B(s)$, use the reflection principle to see for $a, b > 0$ that

$$\mathbb{P}\big[M(t) \geq a,\, B(t) \leq a - b\big] = \mathbb{P}\big[M(t) \geq a,\, B(t) \geq a + b\big] = \mathbb{P}\big[B(t) \geq a + b\big] \ .$$

   Use this to find the joint density $\mathbb{P}\big[M(t) \in dx,\, B(t) \in dy\big]$.

3. **Hitting times**
   For $X(t) = x + B(t)$ let $T = \inf\{t : X(t) = 0\}$, let Brownian motion absorbed at 0 be defined as the process $t \mapsto X(t \wedge T)$.
   Show that this is a time-homogeneous Markov process and find the transition kernel $P(t, x, dy)$ for $(X(t \wedge T) : t \geq 0)$.
   (Hint: use the reflection principle; the answer will have an atom at the origin.)

4. **Modulus of continuity at a point**
   Prove that a Brownian motion satisfies that for all $t \geq 0$, almost surely,

$$\limsup_{h \searrow 0} \frac{|B(t+h) - B(t)|}{\sqrt{2h \log\log(1/h)}} = 1 \ .$$

   Does it also hold that, almost surely, for all $t \geqslant 0$

$$\limsup_{h \searrow 0} \frac{|B(t+h) - B(t)|}{\sqrt{2h \log\log(1/h)}} = 1 \ ?$$

   Motivate your answer.

5. **Hitting time for the Ornstein Uhlenbeck process**
   Recall the Ornstein-Uhlenbeck process on $\mathbb{R}$ given by $t \mapsto X(t) = e^{-ct} B(e^{2ct})$ for all $t \in \mathbb{R}$ and some $c > 0$, where $\big(B(t) : t \geq 0\big)$ is a standard BM.

(a) What is the distribution of $X(0)$?

(b) By conditioning on $\mathcal{F}^+(1)$, find $\quad \mathbb{P}\big[B(s) \neq 0 \text{ for all } s \in [1,t]\big]$.

(c) Let $T_0 := \inf\big\{t \geq 0 : X(t) = 0\big\}$. Find $\mathbb{P}[T_0 > t]$.

(d) Show $\mathbb{E}\big[T_0^p\big] < \infty$ for all $p > 0$.