Jump process with rate $r(x,y) = q(x-y)$.

$\int_{\mathbb{R}} q(z)\, z \, dz = 0$

$\int_{\mathbb{R}} q(z)\, z^2 \, dz = \sigma^2 < \infty$.

$W(t) = \varepsilon X(t/\varepsilon^2)$. Let $(\tilde{\mathcal{L}} f)$ be the generator of $W(t)$.

$$(\tilde{\mathcal{L}} f)(w) \overset{\text{def of } \tilde{\mathcal{L}}}{=} \lim_{t \downarrow 0} \frac{1}{t}\Big(\mathbb{E}[f(W_t) \mid W_0 = w] - f(w) \Big)$$

defn. of W.

$$= \lim_{t \downarrow 0} \frac{1}{t}\Big(\mathbb{E}[f(\varepsilon X_{t/\varepsilon^2}) \mid \varepsilon X_0 = w] - f(w) \Big)$$

Letting $w = \varepsilon x$.

$$= \lim_{t \downarrow 0} \frac{1}{t}\Big(\mathbb{E}[f(\varepsilon X_{t/\varepsilon^2}) \mid \varepsilon X_0 = \varepsilon x] - f(\varepsilon x) \Big)$$

Letting $t = \varepsilon^2 s$.

$$= \lim_{s \downarrow 0} \frac{1}{\varepsilon^2 s}\Big(\mathbb{E}[f(\varepsilon X_s) \mid X_0 = x] - f(\varepsilon x) \Big)$$

$$= \frac{1}{\varepsilon^2} \lim_{s \downarrow 0} \frac{1}{s}\Big(\mathbb{E}[f(\varepsilon X_s) \mid X_0 = x] - f(\varepsilon x) \Big)$$

Letting $g(x) = f(\varepsilon x) = f(w)$.

$$= \frac{1}{\varepsilon^2} \lim_{s \downarrow 0} \frac{1}{s}\Big(\mathbb{E}[g(X_s) \mid X_0 = x] - g(x) \Big)$$

$$= \frac{1}{\varepsilon^2} (\mathcal{L} g)(x)$$

explain link

$(\mathcal{L} f)(x) = \int_{\mathbb{R}} q(y-x)\big(f(y) - f(x)\big)\, dy$ generator for $X(t)$.

$$= \frac{1}{\varepsilon^2} \int_{\mathbb{R}} q(y-x)\big(g(y) - g(x) \big)\, dy$$

using $z = y - x \Leftrightarrow z + x = y$

$$= \frac{1}{\varepsilon^2} \int_{\mathbb{R}} q(z)\big(g(x+z) - g(x) \big)\, dz$$

$$= \frac{1}{\varepsilon^2} \int_{\mathbb{R}} q(z)\Big(g(x) + z g'(x) + \frac{z^2}{2} g''(x) + \sum_{k=3}^{\infty} \frac{z^k}{k!} g^{(k)}(x) - g(x) \Big)\, dz$$

$$= \frac{1}{\varepsilon^2} \int_{\mathbb{R}} q(z)\, z\, g'(x)\, dz + \frac{1}{\varepsilon^2} \int_{\mathbb{R}} q(z) \frac{1}{2} z^2 g''(x)\, dz + \frac{1}{\varepsilon^2} \int_{\mathbb{R}} q(z) \sum_{k=3}^{\infty} \frac{z^k}{k!} g^{(k)}(x)\, dz$$

$g(x) = f(\varepsilon x)$
$g'(x) = \varepsilon f'(\varepsilon x) = \varepsilon f'(w)$

$$= \frac{1}{\varepsilon^2} \int_{\mathbb{R}} q(z)\, z\, \varepsilon f'(w)\, dz + \frac{1}{\varepsilon^2} \int_{\mathbb{R}} q(z) \frac{1}{2} z^2 \varepsilon^2 f''(w)\, dz + \frac{1}{\varepsilon^2} \int_{\mathbb{R}} q(z) \sum_{k=3}^{\infty} \frac{z^k}{k!} \varepsilon^k f^{(k)}(w)\, dz$$

$$= \frac{1}{\varepsilon} f'(w) \underbrace{\int_{\mathbb{R}} q(z)\, z\, dz}_{=0} + \frac{f''(w)}{2} \underbrace{\int_{\mathbb{R}} q(z)\, z^2\, dz}_{=\sigma^2} + \sum_{k=3}^{\infty} \varepsilon^{k-2} \frac{f^{(k)}(w)}{k!} \int_{\mathbb{R}} q(z)\, z^k\, dz$$

$$= \frac{\sigma^2}{2} f''(w) + \sum_{k=3}^{\infty} \varepsilon^{k-2} \frac{f^{(k)}(w)}{k!} \underbrace{\int_{\mathbb{R}} q(z)\, z^k\, dz}_{\text{We assume all these are finite.}}$$

$$\to \frac{\sigma^2}{2} f''(w) \quad \text{as } \varepsilon \to 0.$$