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Brownian Motion

Problem sheet 6

1. **Conditional expectations**
   Suppose $\big(B(t) : t \geq 0\big)$ is a Brownian motion.

   (a) Find, for $s \leq t$,
   $$\text{(a)} \quad \mathbb{E}\big[\exp(-\textstyle\int_0^t B(r)\,dr)\big|\mathcal{F}^0(s)\big] \ , \qquad \text{(b)} \quad \mathbb{E}\big[B(t)\big|\sigma(B(1))\big] \ .$$

   (b) Show that $t \mapsto B(t)^3 - 3\int_0^t B(s)\,ds$ and $t \mapsto \int_0^t \big(B(t) - B(s)\big)\,ds$ are martingales. Use this to conclude that also $t \mapsto B(t)^3 - 3tB(t)$ is a martingale.

   (c) Recall fractional Brownian motion $\big(B_H(t) : t \geq 0\big)$ from Q4.2 with $H \in (0, 1)$. Is B_H a martingale? Justify your answer.

   (d) Recall integrated Brownian motion from Q3.2 given by $X(t) = \int_0^t B(s)\,ds$ where $\big(B(t) : t \geq 0\big)$ is standard BM. Is X a martingale? Justify your answer.

2. **Positive martingales**

   (a) Recall the exponential martingales
   $$M_t^\theta = \exp\left\{\theta B_t - \frac{\theta^2 t}{2}\right\} \ ,$$
   for $\theta \in \mathbb{R}$. Show that for all $\theta \neq 0$, $M^\theta(t) \to 0$ almost surely as $t \to \infty$.
   Show that for all $\theta \neq 0$, $\mathrm{Var}\big[M^\theta(t)\big] \to \infty$ as $t \to \infty$.

   (b) Let $\big(M(t) : t \geq 0\big)$ be a non-negative continuous martingale with $M(0) = x \geq 0$ and $M(t) \to 0$ almost surely as $t \to \infty$. Show by optional stopping that
   $$\mathbb{P}_x\left[\sup_{t\geq 0} M(t) \geq a\right] = \min\left\{\frac{x}{a}, 1\right\} .$$
   Use this to find $E\big[\sup_{t\geq 0} M(t)^p\big]$ for all $p > 0$. This result applies also to absorbed Brownian motion $M(t) = x + B(t \wedge T)$ where $T = \inf\{t : x + B(t) = 0\}$.

3. **Law of exit time from an interval**
   Suppose $\big(B(t) : t \geq 0\big)$ is a standard BM. Fix $a < 0 < b$ and consider $T = T_a \wedge T_b$ the first exit time from the interval (a, b).

   (a) Use the two martingales $\exp\big(\pm\theta B(t) - \theta^2 t/2\big)$ from the previous question to find
   $$\mathbb{E}\left[e^{-\theta^2 T/2}\mathbb{1}_{T_a < T_b}\right] \quad \text{and} \quad \mathbb{E}\left[e^{-\theta^2 T/2}\mathbb{1}_{T_b < T_a}\right] .$$
   Combine these to find the moment generating function
   $$m(\lambda) := \mathbb{E}\big[\exp(\lambda T)\big] \quad \text{for } \lambda \leq 0 \ .$$

(b) Check your answer by deriving the value for $E[T]$, using left derivatives w.r.t. λ of the moment generating function.

4. **Markov property**
   Prove that reflected Brownian Motion $X_t = |B_t|$ is a time-homogeneous Markov process and compute its Markov transition density.

5. **Fractional moments of hitting times**
   Let $\big(B(t) : t \geq 0\big)$ be a Brownian Motion and define $T_1 = \inf\{t \geq 0 : B(t) = 1\}$, so that $B(T_1) = 1$ almost surely. Show that $E[T_1^\alpha] < \infty$ if and only if $\alpha < 1/2$.

   Hint: Perhaps use that $E[T_1^\alpha] = \int_0^\infty P[T_1^\alpha > t]\,\mathrm{d}t$ and write $\{T_1^\alpha > t\}$ in terms of the maximum of Brownian Motion.