26.11.2023. Stoch. Modelling. HW 2.

1) SDEs and Gaussian processes.

B_t - standard Brownian motion

$\mu(t), \sigma(t): \mathbb{R}^+ \to \mathbb{R}$.

$$\begin{cases} dX_t = \mu(t)dt + \sigma(t)dB_t \\ X_0 = x \end{cases}$$

a) Show that X_t is a Gaussian process with increments satisfying

$$X_t - X_s \sim N\left(\int_s^t \mu(u)du; \int_s^t \Gamma(u)du\right), \text{ where } \Gamma(t) = \sigma^2(t).$$

Proof: Our SDE in the integral form is the following:

$$\begin{cases} X_t = \underbrace{X_0}_{=x} + \int_0^t \mu(u)du + \int_0^t \sigma(u)dB_u \\ X_s = x + \int_0^s \mu(u)du + \int_0^s \sigma(u)dB_u \end{cases}$$

$\Rightarrow$ for $s<t$, $X_t - X_s = \int_s^t \mu(u)du + \int_s^t \sigma(u)dB_u$; • First, let's compute its mean and variance.

$\Rightarrow E(X_t - X_s) = E\left(\int_s^t \mu(u)du + \int_s^t \sigma(u)dB_u\right) = \int_s^t \mu(u)du$, because $\int_s^t \mu(u)du$ is deterministic and hence its expectation is itself, and $E\left(\int_s^t \sigma(u)dB_u\right) = 0$ - because it is the Ito stochastic integral and according to the hint, its expectation is zero.

Then, let's calculate the variance of $X_t - X_s$:

$$Var(X_t - X_s) = E((X_t - X_s)^2) - (E(X_t - X_s))^2 = E\left(\left(\int_s^t \mu(u)du + \int_s^t \sigma(u)dB_u\right)^2\right) - \left(\int_s^t \mu(u)dB_u\right)^2 =$$

$$= E\left(\underbrace{\left(\int_s^t \mu(u)du\right)^2}_{\text{deterministic}} + 2\underbrace{\int_s^t \mu(u)du}_{\text{deterministic}} \cdot \underbrace{\int_s^t \sigma(u)dB_u}_{\text{its expectation is zero}} + \left(\int_s^t \sigma(u)dB_u\right)^2\right) - \left(\int_s^t \mu(u)dB_u\right)^2 =$$

$$= \left(\int_s^t \mu(u)du\right)^2 + 2\int_s^t \mu(u)du \cdot \underbrace{E\int_s^t \sigma(u)dB_u}_{=0} + \underbrace{E\left(\int_s^t \sigma(u)dB_u\right)^2}_{=\int_s^t \sigma^2(u)du \text{ due to Ito's isometry}} - \left(\int_s^t \mu(u)dB_u\right)^2 = \int_s^t \sigma^2(u)du$$

• Now let's understand why $X_t - X_s$ has normal distribution (with already found mean and variance).

We have: $X_t - X_s = \underbrace{\int_s^t \mu(u)du}_{\text{it is a constant}} + \underbrace{\int_s^t \sigma(u)dB_u}_{\text{it is a limit } \lim_{K\to\infty} \sum_{k=1}^K \sigma(t_{k-1})(B_{t_k} - B_{t_{k-1}})}$,

and $\sum_{k=1}^K \sigma(t_{k-1})(B_{t_k} - B_{t_{k-1}})$ has a normal distribution, because it is a sum of independent normal random variables. And the limit in L_2 of normal random variables is a normal random variable (perhaps, a constant).

To prove this, we need to look at characteristic functions.

$\xi_n \sim N(a_n; \sigma_n) \Rightarrow \varphi_{\xi_n}(t) = e^{a_n it - \frac{\sigma_n^2 t^2}{2}}$

To prove that $\xi_n \to \xi$, where ξ has normal distribution, we need to prove that $\varphi_{\xi_n}(t) \to$ to some function $\varphi_\xi(t)$ of the same form

But we know that $\xi_n \xrightarrow{L^2} \xi$.

Hence, due to continuity of scalar product in L^2,

we have
$$\begin{cases} a_n = E\xi_n = \langle \xi_n; 1\rangle \to \langle \xi; 1\rangle = E\xi =: a. \\ \sigma_n^2 = E\xi_n^2 - (E\xi_n)^2 = \langle \xi_n; \xi_n\rangle - (E\xi_n)^2 \to \langle \xi; \xi\rangle - (E\xi)^2 = E\xi^2 - (E\xi)^2 =: \sigma^2 \end{cases}$$

In other words, we know that $\forall t, \begin{cases} a_n \to a \\ \sigma_n^2 \to \sigma \end{cases}$

$$\Rightarrow \forall t: \varphi_{\xi_n}(t) = e^{a_n it - \frac{\sigma_n^2 t^2}{2}} \longrightarrow e^{ait - \frac{\sigma^2 t^2}{2}} =: \varphi_\xi(t)$$

So the limit of $\varphi_{\xi_n}(t)$ is function $\varphi_\xi(t)$ of the same form

$\Rightarrow \xi_n \to \xi$, where ξ also has normal distribution with mean a and variance σ^2 (perhaps, σ^2 may be $=0$, in that case ξ is a constant)

$\Rightarrow X_t - X_s$ has normal distribution.

And we already have computed its $\underset{=\int_s^t \mu(u)du}{\text{mean}}$ and $\underset{=\int_s^t \sigma^2(u)du}{\text{variance}}$

$$\Rightarrow \boxed{X_t - X_s \sim N\left(\int_s^t \mu(u)du; \int_s^t \sigma^2(u)du\right)}$$

- And from this we may conclude that X_t is a gaussian process.

Recall that X_t is a gaussian process if $\forall n \in \mathbb{N}_0$, $0 < t_1 < \dots < t_n$, the vector $(X_{t_1}; \dots X_{t_n})$ has a multivariate Gaussian distribution.

We have
$$\begin{pmatrix} X_{t_1} \\ \vdots \\ X_{t_n} \end{pmatrix} = \begin{pmatrix} 1 & 0 & \dots & & 0 \\ 1 & 1 & 0 & \dots & 0 \\ \vdots & & & & 0 \\ 0 & & & & 1\,1 \end{pmatrix} \begin{pmatrix} X_{t_1} \\ X_{t_2} - X_{t_1} \\ \vdots \\ X_{t_n} - X_{t_{n-1}} \end{pmatrix} = \begin{pmatrix} a_1 \\ a_1 + a_2 \\ \vdots \\ a_1 + \dots + a_n \end{pmatrix} + \begin{pmatrix} b_1 & 0 & \dots & 0 \\ b_1 & b_2 & 0 \dots & 0 \\ & & & \\ b_1 & b_2 & \dots & b_n \end{pmatrix} \begin{pmatrix} \xi_1 \\ \xi_2 \\ \vdots \\ \xi_n \end{pmatrix}$$
$\xi_i \sim N(0,1)$, $a_i = \int_{t_{i-1}}^{t_i} \mu(u)du$, $b_i = \int_{t_{i-1}}^{t_i} \sigma^2(u)du$

where $(X_{t_1}; X_{t_2} - X_{t_1}; \dots X_{t_n} - X_{t_{n-1}})$ are independent (because bounds of integration in Ito's integrals are independent) normal variables (we have already proved that $X_{t_k} - X_{t_{k-1}} \sim N\left(\int_{t_{k-1}}^{t_k} \mu(u)du; \int_{t_{k-1}}^{t_k} \sigma^2(u)du\right)$)

So $(X_{t_1} \dots X_{t_n})$ is a matrix multiplied by a vector of independent standard normal random variables $\Rightarrow$ it has multivariate normal distribution.

Indeed, by our definition from lecture, multivariate gaussian is a random variable with density $f(\bar{x}) = \frac{1}{\sqrt{(2\pi)^n \cdot \det(\Sigma)}} \cdot e^{-\frac{1}{2}\langle \bar{x} - \bar{\mu} | \Sigma^{-1} | \bar{x} - \bar{\mu}\rangle}$,

and if we have $\bar{X} = B\bar{\xi}$, where $\bar{\xi}$ is a vector of independent standard normal random variables,

then $p_{\bar{\xi}}(\bar{x}) = p_{\xi_1}(x_1) \dots p_{\xi_n}(x_n) = \left(\frac{1}{\sqrt{2\pi}}\right)^n \cdot e^{-\frac{1}{2}(x_1^2 + \dots + x_n^2)} = \frac{1}{(\sqrt{2\pi})^n} \cdot e^{-\frac{\|\bar{x}\|}{2}}$

ξ_i are independent

And due to change of variables formula for $\overline{X} = B\overline{\xi}$:

$$p_{\overline{X}}(\overline{x}) = \frac{1}{|\det B|} \cdot p_{\overline{\xi}}(B^{-1}\overline{x})$$

$$\Rightarrow p_{\overline{X}}(\overline{x}) = \frac{1}{|\det B|}\cdot\frac{1}{(\sqrt{2\pi})^n}\cdot e^{-\frac{1}{2}\langle B^{-1}\overline{x}; B^{-1}x\rangle} = \frac{1}{(\sqrt{2\pi})^n |\det C|}\cdot e^{-\frac{1}{2}\langle (B^{-1})^T B^{-1}\vec{x}; \vec{x}\rangle}$$

Let's denote $R = BB^T \Rightarrow R^{-1} = (B^{-1})^T B^{-1}$

$$\Rightarrow p_{\overline{X}}(\overline{x}) = \frac{1}{(\sqrt{2\pi})^n |\det R|^{1/2}} \cdot e^{-\frac{1}{2}\langle R^{-1}\vec{x};\vec{x}\rangle} \leftarrow \text{for } \overline{X} = B\overline{\xi}$$

If $\overline{X} = \overline{a} + C\overline{\xi}$, then by the same argument: $p_{\overline{X}}(\overline{x}) = \frac{1}{(\sqrt{2\pi})^n}\cdot\frac{1}{|\det R|^{1/2}}\cdot e^{-\frac{1}{2}\langle R^{-1}(\overline{x}-\overline{a});\overline{x}-\overline{a}\rangle}$.

So we see, that since $\begin{pmatrix}X_{t_1}\\ \vdots \\ X_{t_n}\end{pmatrix}$ is obtained as $\overline{X} = \overline{a} + B\overline{\xi}$ from the vector $\overline{\xi}$ of independent standard normal random variables - it has a multivariate gaussian distribution according to our definition from lecture 9.

b) Use the above result to develop an algorithm for simulating X_t for given $\mu(t), \sigma(t), x$.

Since $X_{t_n} - X_{t_{n-1}} \sim N\left(\int_{t_{n-1}}^{t_n}\mu(u)du; \int_{t_{n-1}}^{t_n}\sigma^2(u)du\right)$, we just need to compute ideally - analytically the integrals $\underbrace{\int_{t_{n-1}}^{t_n}\mu(u)du}_{a_n}$ and $\underbrace{\int_{t_{n-1}}^{t_n}\sigma^2(u)du}_{b_n}$ - and just $\overbrace{\text{sample}}^{\text{at step } n}$ $\xi_n \sim N(a_n; b_n)$, and put $X_{t_n} := x + \xi_1 + \dots + \xi_n$.

For example, for $x = 1$; $\mu(t) = 2t$; $\sigma(t) = 0.1\cos^2 t$ from question 1c,

we will have $a_n = \int_{t_{n-1}}^{t_n}\mu(u)du = \int_{t_{n-1}}^{t_n} 2u\,du = t_n^2 - t_{n-1}^2$.

$$b_n = \int_{t_{n-1}}^{t_n}\sigma^2(u)du = \int_{t_{n-1}}^{t_n} 0.01\cos^4(u)du = \frac{0.01}{4}\int_{t_{n-1}}^{t_n}(1+\cos 2u)^2 du =$$

$$= \frac{0.01}{4}\int_{t_{n-1}}^{t_n}(1+2\cos 2u + \cos^2 2u)du = \frac{0.01}{4}\left(u\Big|_{t_{n-1}}^{t_n} + \sin 2u\Big|_{t_{n-1}}^{t_n} + \frac{1}{2}\int_{t_{n-1}}^{t_n}(1+\cos 4u)du\right) =$$

$$= 0.01\left(\frac{t_n - t_{n-1}}{4} + \frac{1}{4}(\sin(2t_n) - \sin(2t_{n-1})) + \frac{1}{8}(t_n - t_{n-1}) + \frac{1}{32}(\sin(4t_n) - \sin(4t_{n-1}))\right) =$$

$$= 0.01\left(\frac{3}{8}(t_n - t_{n-1}) + \frac{1}{4}(\sin(2t_n) - \sin(2t_{n-1})) + \frac{1}{32}(\sin(4t_n) - \sin(4t_{n-1}))\right).$$

And the algorithm is the following:

1) If we want to calculate X_t, divide $[0;t]$ into $0 < t_1 < \dots < t = t_n$;

2) put $X_0 = x$

3) for k from 1 to n inclusively do: • calculate $a_k = \int_{t_{k-1}}^{t_k}\mu(u)du$; and $b_k = \int_{t_{k-1}}^{t_k}\sigma^2(u)du$.
• simulate $\xi_k \sim N(a_k; b_k)$
• put $X_{t_k} := X_{t_{k-1}} + \xi_k$; and at the end of loop we will obtain $X_{t_n} = X_t$.

Note that for some reason for needed $\mu(t)$ and $\sigma(t)$ any of the integrals

$a_k = \int_{t_{k-1}}^{t_k} \mu(u)du$; $b_k = \int_{t_{k-1}}^{t_k} \sigma^2(u)du$ can't be calculated analytically,

we will need to approximate it by some of numerical integration schemes (trapezoid, simpson, etc) - it depends on what accuracy we need.

For trapezoidal rule we will have: $a_k = \int_{t_{k-1}}^{t_k} \mu(u)du \approx \sum_{j=1}^{J} \mu\left(t_{k-1} + \frac{j(t_k - t_{k-1})}{J}\right) \cdot \frac{(t_k - t_{k-1})}{J}$

(c) Use your algorithm to simulate X_t; $t \in [0,T]$ for $\mu(t) = 2t$; $\sigma(t) = 0.1\cos^2 t$; $x = 1$. Generate $N = 1000$ samples paths and calculate the mean and variance of X_t as a function of time. Compare with theoretical results.

By theory, we have $X_T - X_0 \sim N\left(\int_0^T \mu(u)du, \int_0^T \sigma^2(u)du\right)$

$$\overset{\shortparallel}{N}\left(T^2;\ 0.01 \cdot \left(\frac{3}{8}u + \frac{1}{4}\sin(2u) + \frac{1}{32}\sin(4u)\right)\Big|_0^T\right) =$$

$$= N\left(T^2;\ 0.01 \cdot \left(\frac{3}{8}T + \frac{1}{4}\sin(2T) + \frac{1}{32}\sin(4T)\right)\right)$$

So for the sample of X_{t_k}, we expect it to have mean $t_k^2 + x$, the variance $0.01 \cdot \left(\frac{3}{8}t_k + \frac{1}{4}\sin(2t_k) + \frac{1}{32}\sin(4t_k)\right)$

On the graph we see that blue and orange lines (true and sample means and variances of X_t) are very close - that is, everything is right.

For example, for $T = 5$ (years), we expect $EX_T = 1 + T^2 = 1 + 25 = 26$, and sample_mean[-1] = 26.0551..., that is very close to 26.

(d) Solve the SDE numerically using the Euler-Maruyama scheme and compare the results with the theoretical results and (c)

Using Euler-Maruyama scheme, we have $X_k = X_{k-1} + \mu(X_{t_{k-1}})(t_k - t_{k-1}) + \sigma(X_{t_{k-1}}) \sqrt{t_k - t_{k-1}}\, \xi_k$, where $\xi_k \sim N(0,1)$

We expect this method to introduce more error, than method from 1c.

Indeed, for mean in 1c we have computed the integral $\int_{t_{k-1}}^{t_k} \mu(u)du$ theoretically, so for any amount of $(t_1 < \ldots < t_n = t)$ points on the greed, the sample mean will be almost exactly correct (and if we plug $\sigma(u) \equiv 0$ - then the sample mean will be exact as theoretical).

But in method 1d: we introduce error even when we calculate $\int_{t_{k-1}}^{t_k} \mu(u)du$, so even for $\sigma(u) \equiv 0$ but small n theoretical mean and sample mean may differ significantly.

But for big n (amount of steps on time grid) both 1c and 1d are close to theoretical results.

② **Ornstein-Uhlenbeck process**

$$\begin{cases} (\mathcal{L}f)(x) = -\alpha x f'(x) + \frac{1}{2}\sigma^2 f''(x); \ \alpha, \sigma^2 > 0. \\ X_0 = x_0. \end{cases}$$

(a) Use the evolution equation of expectation values of test functions $f: \mathbb{R} \to \mathbb{R}$
$\frac{d}{dt} E[f(X_t)] = E[\mathcal{L}f(X_t)]$ to derive ODEs for the mean $m(t) = EX_t$
and the variance $v(t) = E[X_t^2] - m(t)^2$

Solution: • For $m(t) = EX_t$ let's take $f(x) = x$

$\Rightarrow f'(x) = 1; \ f''(x) = 0.$

$\Rightarrow \mathcal{L}f(x) = -\alpha x \cdot 1 + 0$

$\Rightarrow \frac{d}{dt} EX_t = E[-\alpha X_t] = -\alpha EX_t$

$\Rightarrow \frac{d}{dt} m_t = -\alpha m_t$

$\Rightarrow m_t = m_0 \cdot e^{-\alpha t}$, but $m_0 = EX_0 = x_0$

$\Rightarrow \boxed{m_t = x_0 \cdot e^{-\alpha t}}$ ← In the lecture there was the exact solution
$X_t = X_0 e^{-\alpha t} + \sigma \int_0^t e^{-\alpha(t-s)} dB_s$ – and $EX_t = x_0 \cdot e^{-\alpha t}$ ⇒ good.

• For $v(t) = E[X_t^2] - m(t)^2 = E[X_t^2] - (EX_t)^2 = E[(X_t - m(t))^2]$
let's take $f(x) = (x - m_t)^2$

$\Rightarrow f'_x = 2(X_t - m_t); \ f''_{xx} = 2.$

$$\Rightarrow \frac{d}{dt} v_t = E[\mathcal{L}f(x)] = E[-\alpha X_t \cdot 2 \cdot (X_t - m_t) + \sigma^2] = -2\alpha \cdot E[(X_t \pm m_t)(X_t - m_t)] + \sigma^2 =$$

$$= -2\alpha \cdot E[(X_t - m_t)^2] - 2\alpha E[m_t(X_t - m_t)] + \sigma^2 =$$

$$= -2\alpha \cdot v_t - 2\alpha m_t \cdot \underbrace{(EX_t - m_t)}_{0} + \sigma^2 = -2\alpha v_t + \sigma^2.$$

$\Rightarrow \frac{d}{dt} v_t = -2\alpha v_t + \sigma^2.$

First solve $\frac{d}{dt} v_t = -2\alpha v_t$ (homogeneous equation)

$\Rightarrow v_t = c \cdot e^{-2\alpha t}$

Then use method of varying of arbitrary c to solve inhomogeneous equation:

$v_t = c(t) \cdot e^{-2\alpha t}$

$\Rightarrow \frac{d}{dt} v_t = c' e^{-2\alpha t} \cancel{- 2\alpha c \cdot e^{-2\alpha t}} \overset{?}{=} \cancel{-2\alpha \overset{c \cdot e^{-2\alpha t}}{v_t}} + \sigma^2$

$\Rightarrow c' = \sigma^2 e^{2\alpha t}$

$\Rightarrow c(t) = \sigma^2 \int_0^t e^{2\alpha s} ds + C = \frac{\sigma^2}{2\alpha}(e^{2\alpha t} - 1) + C$

$\Rightarrow v_t = c(t) \cdot e^{-2\alpha t} = \frac{\sigma^2}{2\alpha}(1 - e^{-2\alpha t}) + C e^{-2\alpha t}$

Now use the initial conditions: $v_0 = E[X_0^2] - (m(0))^2 = x_0^2 - x_0^2 = 0.$

$\Rightarrow v_0 = \frac{\sigma^2}{2\alpha}(1 - 1) + C \overset{?}{=} 0 \Rightarrow C = 0 \Rightarrow \boxed{v_t = \frac{\sigma^2}{2\alpha}(1 - e^{-2\alpha t})}$ – and it is the same answer that we had in lectures.

б) Give the distribution of X_t for all $t \geq 0$ (using the fact that X_t is a Gaussian process). What is the stationary distribution of the process?

Solution: We know that $\begin{cases} EX_t = x_0 \cdot e^{-\alpha t} \\ Var X_t = \frac{\sigma^2}{2\alpha}(1-e^{-2\alpha t}) \\ X_t \text{ is gaussian} \end{cases} \Rightarrow \boxed{X_t \sim N\left(x_0 e^{-\alpha t}, \frac{\sigma^2}{2\alpha}(1-e^{-2\alpha t})\right)}$

Now to find the stationary distribution we use Fokker-Planck equation:

$$\underbrace{\frac{\partial}{\partial t} p_t(x,y)}_{=0,\ \text{because it is a stationary distribution}} = -\frac{\partial}{\partial y}(\underbrace{a(y,t)}_{=-\alpha y} p_t) + \frac{1}{2}\frac{\partial^2}{\partial y^2}(\underbrace{\sigma^2(y,t)}_{=\sigma^2} p_t)$$

$$\Rightarrow 0 = -\frac{\partial}{\partial y}(-\alpha y p) + \frac{\sigma^2}{2} \cdot \frac{\partial^2}{\partial y^2}(p)$$

$\Rightarrow$ integrating both parts from 0 to y, we have:

$$-\alpha y p = \frac{1}{2}\frac{\partial}{\partial y}(\sigma^2 p) + C$$

$$\Rightarrow p' = -\frac{2\alpha}{\sigma^2} y \cdot p + \tilde{C} \quad (*)$$

First solve the homogeneous equation $p' = -\frac{2\alpha}{\sigma^2} \cdot y \cdot p$.

$$\Rightarrow \frac{dp}{p} = -\frac{2\alpha}{\sigma^2} y\,dy$$

$$\Rightarrow \ln p = -\frac{\alpha}{\sigma^2} y^2 + \tilde{\tilde{C}}$$

$$\Rightarrow p = \tilde{\tilde{\tilde{C}}} \cdot e^{-\frac{\alpha}{\sigma^2} y^2}$$

Now use the method of varying arbitrary constant to obtain solution of inhomogeneous equation (*):

$$p(y) = \tilde{\tilde{\tilde{C}}}(y) \cdot e^{-\frac{\alpha}{\sigma^2} \cdot y^2}$$

$$\Rightarrow p'_y = \tilde{\tilde{\tilde{C}}}' \cdot e^{-\frac{\alpha}{\sigma^2} y^2} - \cancel{\frac{2\alpha}{\sigma^2} y \tilde{\tilde{\tilde{C}}} \cdot e^{-\frac{\alpha}{\sigma^2} y^2}} \overset{?}{=} -\cancel{\frac{2\alpha}{\sigma^2} \cdot y \cdot \tilde{\tilde{\tilde{C}}} \cdot e^{-\frac{\alpha}{\sigma^2} \cdot y^2}} + \tilde{C}$$

$$\Rightarrow \tilde{\tilde{\tilde{C}}}' = \tilde{C} \cdot e^{\frac{\alpha}{\sigma^2} y^2}$$

$$\Rightarrow \tilde{\tilde{\tilde{C}}} = \tilde{C} \cdot \int_0^y e^{\frac{\alpha z^2}{\sigma^2}} dz + \tilde{\tilde{\tilde{\tilde{C}}}}$$

$$\Rightarrow p(y) = \tilde{\tilde{\tilde{\tilde{C}}}} \cdot e^{-\frac{\alpha}{\sigma^2} y^2} + \tilde{C} \cdot \int_0^y e^{\frac{\alpha}{\sigma^2}(z^2 - y^2)} dz$$

And we now must choose such constants $\tilde{\tilde{\tilde{\tilde{C}}}}$ and $\tilde{C}$, that $p(y)$ will be a density, that is, it will integrate into one.

Use the Euler-Poisson integral: $\int_{-\infty}^{+\infty} e^{-x^2} dx = \sqrt{\pi}$

$$\Rightarrow \int_{-\infty}^{+\infty} e^{-\frac{\alpha}{\sigma^2} y^2} dy = \int_{-\infty}^{+\infty} e^{-\left(\frac{\sqrt{\alpha}}{\sigma} y\right)^2} dy \underset{\frac{\sqrt{\alpha}}{\sigma} y = z}{=} \frac{\sigma}{\sqrt{\alpha}} \cdot \int_{-\infty}^{+\infty} e^{-z^2} dz = \frac{\sigma}{\sqrt{\alpha}} \cdot \sqrt{\pi}$$

$$\Rightarrow p(y) = \frac{\sqrt{\alpha}}{\sigma\sqrt{\pi}} e^{-\frac{\alpha}{\sigma^2} y^2}$$ – it is the stationary density.

And note that it is density of normal random variable $N(0, \frac{\sigma^2}{2\alpha})$

And it corresponds to what we found in 2a,

that $X_t - X_0 \sim N\left(x_0 e^{-\alpha t}; \frac{\sigma^2}{2\alpha}(1-e^{-2\alpha t})\right)$,

because as $t \to \infty$, then $e^{-\alpha t}$ and $e^{-2\alpha t} \to 0$, hence $X_\infty - x \sim N(0, \frac{\sigma^2}{2\alpha}) \Rightarrow X_\infty \sim N(x; \frac{\sigma^2}{2\alpha})$

and it is exactly what we found in 2b.

c) $\alpha = 1; \sigma^2 = 1; X_0 = 5$.

Integrate SDE numerically with $\Delta t = 0.1$ and $\Delta t = 0.01$.

We have the generator $\mathcal{L}f(x) = -\alpha x \cdot f'(x) + \frac{\sigma^2}{2} f''(x)$

$$\Rightarrow \mu(x) = -\alpha x; \; \sigma(x) = \sigma$$

$$\Rightarrow \text{SDE looks like } \begin{cases} dX_t = \underbrace{-\alpha X_t}_{=a(x_t)} dt + \underbrace{\sigma}_{=\sigma(x_t)} dB_t \\ X_0 = x \end{cases}$$

And we use Euler-Maruyama scheme:

$$X_{t_k} = X_{t_{k-1}} + \underbrace{a(X_{t_{k-1}})}_{= -\alpha X_{t_{k-1}}} \Delta t + \sigma(X_{t_{k-1}}) \cdot \sqrt{\Delta t} \cdot \underbrace{\xi_k}_{\sim N(0,1)}$$

And theoretical mean and variance are $m(t) = x_0 e^{-\alpha t}$; $v(t) = \frac{\sigma^2}{2\alpha}(1-e^{-2\alpha t})$

We see that sample mean is close to $m(t)$ for both $\Delta t = 0.1$ and $\Delta t = 0.01$, while for sample variance $\Delta t = 0.01$ gives better result (namely, the problem that for $\Delta t = 0.1$ sample variance is greater than theoretical variance, disappears). It ~~is~~ happens because the less Δt, the less error we introduce into the integral while we integrate it numerically.

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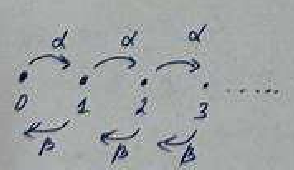
3) Birth-death processes

$S = \mathbb{N}_0 = \{0, 1, 2 \dots\}$

$$\begin{cases} x \xrightarrow{\alpha_x} x+1; \text{ for all } x \in S \\ x \xrightarrow{\beta_x} x-1; \text{ for } x \geq 1. \end{cases}$$

a) $\begin{cases} \alpha_x = \alpha > 0 \\ \beta_x = \beta > 0 \end{cases}$



$$G = \begin{pmatrix} -\alpha & \alpha & 0 & 0 \\ \beta & -(\alpha+\beta) & \alpha & 0 \\ 0 & \beta & -(\alpha+\beta) & \alpha \\ & & \dots & & \ddots \end{pmatrix}$$

i) Is X irreducible? Give all communicating classes in $\mathbb{N}_0$ and state whether they are transient or null/positive recurrent.

Answer: 1) $\alpha = 0; \beta = 0 \Rightarrow$ not irreducible, all transient, all separate classes
2) $\alpha = 0; \beta \neq 0 \Rightarrow$ not irreducible, all transient, all separate classes
3) $\alpha \neq 0; \beta = 0 \Rightarrow$ not irreducible, all transient, all separate classes
4) $\alpha \neq 0; \beta \neq 0$: is irreducible, one class; $\alpha < \beta$ positive recurrent; $\alpha = \beta$ null-recurrent; $\alpha > \beta$ transient

1) $\alpha = 0; \beta = 0 \Rightarrow$

$\Rightarrow$ chain is not irreducible, because we can't get from any x to any y, for example, from $x = 0$ we can't go anyway, except 0.

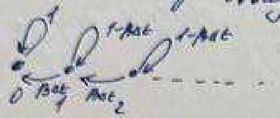
So each $x = 0, 1, 2 \dots$ is a separate communicating class.

Recall that $T_x = \inf\{t > J_1 : X_t = x\}$, and a state $x \in S$ is called:

- transient, if $P(T_x = \infty \mid X_0 = x) > 0$
- null recurrent, if $P(T_x < \infty \mid X_0 = x) = 1$ and $E(T_x \mid X_0 = x) = \infty$.
- positive recurrent, if $P(T_x < \infty \mid X_0 = x) = 1$ and $E(T_x \mid X_0 = x) < \infty$.

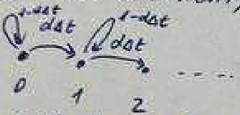
We see that all states x are transient, because $J_1 = \infty$, since we will never go away from $x_0 \Rightarrow \inf\{t > J_1 : X_t = x\} = \infty \Rightarrow$ transient

2) $\alpha = 0;\ \beta \neq 0$



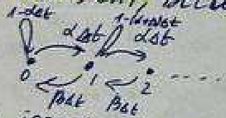
The chain is not irreducible, because from $x = k$ we can't go to $x = k+1$, since we are allowed to either stay at k, or go left.
So again each $x = k$ is a separate communicating class (because for i and j to be in one communicating state, i must be reachable from j and vice versa).
All states, except 0, are transient, because once we leave them - we will never return.
And $x = 0$ is also transient, because $J_1 = \infty \Rightarrow \inf\{t > J_1 : X_t = x\} = T_0 = \infty \Rightarrow$ transient

3) $\alpha \neq 0;\ \beta = 0$



The chain is not irreducible, because from $x = k$ we can't go into $x = k-1$, since we are allowed to either stay at k, or go right.
So again each $x = k$ is a separate communicating class.
All states are transient, because once we leave them, we will never return.

4) $\alpha \neq 0,\ \beta \neq 0$



The chain is irreducible, because we can go from any x to any y.
All $\{0, 1, 2, \dots\}$ are in the same communicating class. And they all are recurrent / transient at the same time.
According to wikipedia page, birth-death process is recurrent, if and only if $\sum_{i=1}^{\infty} \prod_{n=1}^{i} \frac{\mu_n}{\lambda_n} = \infty$.

In our case $\sum_{i=1}^{\infty} \prod_{n=1}^{i} \frac{\mu_n}{\lambda_n} = \sum_{i=1}^{\infty} \left(\frac{\beta}{\alpha}\right)^i$ $\to < \infty$, if $\beta < \alpha$ $\Rightarrow$ not recurrent, that is transient; $\to = \infty$, if $\beta \geq \alpha$.

And birth-death process is null-recurrent, if $\begin{cases} \sum_{i=1}^{\infty} \prod_{n=1}^{i} \frac{\mu_n}{\lambda_n} = \infty \\ \sum_{i=1}^{\infty} \prod_{n=1}^{i} \frac{\lambda_{n-1}}{\mu_n} = \infty. \end{cases}$

In our case $\sum_{i=1}^{\infty} \prod_{n=1}^{i} \frac{\lambda_{n-1}}{\mu_n} = \sum_{i=1}^{\infty} \left(\frac{\alpha}{\beta}\right)^i$ $\to < \infty$, if $\alpha < \beta$ $\Rightarrow$ not null-recurrent; $\to = \infty$, if $\alpha \geq \beta$.

So if $\alpha > \beta$, then all states are transient,
If $\alpha = \beta$, then all states are null-recurrent
If $\alpha < \beta$, then all states are positive-recurrent (because they satisfy the condition for recurrence, but don't satisfy condition for null-recurrence)

ii) We look for stationary distributions

Recall that stationary distribution satisfy the condition $\langle \pi_t | G = 0$.

$$\Rightarrow (\pi_0\ \pi_1\ \pi_2 \dots) \begin{pmatrix} -\alpha & \alpha & 0 & 0 & 0 \\ \beta & -(\alpha+\beta) & \alpha & 0 & 0 \\ 0 & \beta & -(\alpha+\beta) & \alpha & 0 \\ 0 & 0 & \beta & -(\alpha+\beta) & \alpha \end{pmatrix} = (-\alpha\pi_0 + \beta\pi_1;\ \underbrace{\alpha\pi_0 - (\alpha+\beta)\pi_1 + \beta\pi_2}_{\text{that is } \alpha\pi_{k-1} - (\alpha+\beta)\pi_k + \beta\pi_{k+1};\ k \geq 1};\ \dots) \stackrel{?}{=} (0, 0 \dots 0 \dots)$$

1) if $\alpha = \beta = 0$, then $\pi = (\pi_0; \pi_1 \dots)$ is stationary - because matrix G is zero,
and it is reversible, because $P_t = E$, so $p(x,y) = 0\ \forall y \neq x$, and $\pi(x) \cdot p(x,y) = \pi(y) \cdot p(y,x)\ \forall y = x$.

2) if $\alpha = 0; \beta \neq 0 \Rightarrow \begin{cases} \beta\pi_1 = 0. \\ -\beta\pi_k + \beta\pi_{k+1} = 0 \end{cases} \Rightarrow \pi_1 = \pi_2 = \dots = 0; \Rightarrow \pi_0 = 1 \Rightarrow (1, 0; \dots 0 \dots)$ is only stationary distribution
and it is reversible, because only for $x=0$, we have $\pi(x) \neq 0$, and $\forall y > 0$: and $\pi(x) \cdot p(x,y) = 0$, because $\alpha = 0$, so $\underbrace{\pi(x)}_{\neq 0} \cdot p(x,y) = 0 = \underbrace{\pi(y)}_{=0} \cdot p(y,x)$

3) if $\alpha \neq 0, \beta = 0 \Rightarrow \begin{cases} -\alpha\pi_0 = 0. \\ \alpha\pi_{k-1} - \alpha\pi_k = 0 \end{cases} \Rightarrow \begin{cases} \pi_0 = 0 \\ \pi_1 = 0 \\ \dots \end{cases} \Rightarrow \sum_{k=0}^{\infty} \pi_k = 0 \neq 1 \Rightarrow$ there is no stationary distribution (that is, we got lost in infinity)

4) $\alpha \neq 0; \beta \neq 0 \Rightarrow \begin{cases} -\alpha\pi_0 = -\beta\pi_1 \\ \alpha\pi_{k-1} - (\alpha+\beta)\pi_k + \beta\pi_{k+1} = 0. \end{cases}$

$\Rightarrow$ seek for solution in the form $\pi_k = \lambda^k$

$\Rightarrow \beta\lambda^2 - (\alpha+\beta)\lambda + \alpha = 0.$

we have $\alpha \neq 0; \beta \neq 0 \Rightarrow$ it is a quadratic (nondegenerate) equation

$\Rightarrow D = (\alpha+\beta)^2 - 4\alpha\beta = (\alpha-\beta)^2.$

4.1) if $\alpha \neq \beta$ then $\lambda_{1;2} = \dfrac{(\alpha+\beta) \pm (\alpha-\beta)}{2\beta}$

$\Rightarrow \lambda_1 = 1; \lambda_2 = \dfrac{\alpha}{\beta}.$

$\Rightarrow \pi_k = C_1 + C_2 \cdot \left(\dfrac{\alpha}{\beta}\right)^k$

But $\alpha\pi_0 = \beta\pi_1 \Rightarrow \alpha(C_1 + C_2) = \beta(C_1 + C_2 \dfrac{\alpha}{\beta})$

$\Rightarrow \alpha C_1 = \beta C_1$

But $\alpha \neq \beta$ in 4.1 $\Rightarrow C_1 = 0. \Rightarrow \pi_k = C_2 \cdot \left(\dfrac{\alpha}{\beta}\right)^k$ — if $\alpha < \beta$

Now we need $\sum_{k=0}^{\infty} \pi_k \stackrel{?}{=} 1 \Rightarrow C_2 = \dfrac{1}{1 - \frac{\alpha}{\beta}}$, if $\alpha < \beta$,

and there is no stationary distribution if $\alpha > \beta$.

4.2) if $\alpha = \beta$ then $\pi_k = C_1 + C_2 k$

but $\alpha\pi_0 = \beta\pi_1 \underset{\alpha=\beta}{\Rightarrow} \pi_0 = \pi_1 \Rightarrow \cancel{C_1} = \cancel{C_1} + C_2 \Rightarrow C_2 = 0 \Rightarrow \pi_k = C_1.$

but we need $\sum_{k=0}^{\infty} \pi_k \stackrel{?}{=} 1$ - but $\sum_{k=0}^{\infty} \pi_k = C_1 \cdot \infty \neq 1 \Rightarrow$ no stationary distribution.

So we see that for $\alpha \geq \beta$ there is no stationary distribution,
and for $\alpha < \beta$ we have only stationary distribution $\pi_k = \dfrac{\beta}{\beta - \alpha} \cdot \left(\dfrac{\alpha}{\beta}\right)^k;\ k = 0, 1, \dots$

Let's check whether it is reversible.

Let's $x = s;\ y = s + m \Rightarrow \pi_y = C_2 \cdot \left(\dfrac{\alpha}{\beta}\right)^{s+m};\ \pi_x = C_2 \left(\dfrac{\alpha}{\beta}\right)^s$

$\Rightarrow \underbrace{\pi(x)}_{= C_2 \left(\frac{\alpha}{\beta}\right)^s} \cdot p(x,y) \stackrel{?}{=} \underbrace{\pi(y)}_{= C_2 \left(\frac{\alpha}{\beta}\right)^m} \cdot p(y,x) \Rightarrow p(x,y) \stackrel{?}{=} p(y,x) \left(\dfrac{\alpha}{\beta}\right)^m$, where $x = s;\ y = s + m.$

So we need to check: $p(s;s+m) \overset{?}{=} \left(\frac{\alpha}{\beta}\right)^m \cdot p(s+m;s)$

Yes, that is true, because $\begin{cases} P(X+1;X) \sim \alpha \\ P(X-1;X) \sim \beta \end{cases}$

And $\begin{cases} p(s;s+m) = p^m, \text{ where } (p,q,r) \text{ are probabilities of success in 3-binomial distribution, and } p \text{ is the probability of going right} \\ p(s+m;s) = r^m, \text{ where } r \text{ is the propability of going left, and } q = 1-p-r \text{ is the probability of staying at the same state where are you} \end{cases}$

And we know that $\frac{p}{r} = \frac{\alpha}{\beta}$.

$\Rightarrow \frac{p(s,s+m)}{p(s+m,s)} = \left(\frac{\alpha}{\beta}\right)^m$ – is true $\Rightarrow$ this stationary distribution is reversible.

Answer:
1) $\alpha=0; \beta=0$ $\Rightarrow$ any π is stationary and reversible
2) $\alpha=0; \beta\neq0$ $\Rightarrow$ $\pi = (1;0\dots)$ is stationary and reversible
3) $\alpha\neq0; \beta=0$ $\Rightarrow$ no stationary distribution
4) $\alpha\neq0; \beta\neq0$ $\longrightarrow$ $\alpha\geq\beta$ – no stationary distribution
$\searrow$ $\alpha<\beta$ $\Rightarrow$ $\pi_k = \frac{1}{1-\frac{\alpha}{\beta}}\left(\frac{\alpha}{\beta}\right)^k$ – stationary and reversible

iii) Is the process ergodic?

According to the wikipedia page, birth-death process is ergodic, if and only if $\sum_{i=1}^{\infty}\prod_{n=1}^{i}\frac{\mu_n}{\lambda_n} = \infty$ and $\sum_{i=1}^{\infty}\prod_{n=1}^{i}\frac{\lambda_{n-1}}{\mu_n} < \infty$.

And by definition, process is ergodic, if it has a stationary distribution, it is unique, and $\pi_t \to \pi^*$ ← stationary distribution.

1) $\alpha=0;\beta=0$: stationary distribution is not unique $\Rightarrow$ not ergodic

2) $\alpha=0;\beta\neq0$ – has unique stationary distribution,
$\begin{cases} \sum_{i=1}^{\infty}\prod_{n=1}^{i}\frac{\mu_n}{\lambda_n} = \sum_{i=1}^{\infty}\left(\frac{\beta}{0}\right) \overset{?}{=} \infty \text{ – yes,} \\ \sum_{i=1}^{\infty}\prod_{n=1}^{i}\frac{\lambda_{n-1}}{\mu_n} = \sum_{i=1}^{\infty}\frac{0}{\beta} = 0 < \infty \text{ yes} \end{cases}$ $\Rightarrow$ ergodic

3) $\alpha\neq0;\beta=0$ – no stationary distribution $\Rightarrow$ not ergodic

4) $\alpha\neq0;\beta\neq0$ $\longrightarrow$ $\alpha\geq\beta$ – no stationary distribution $\Rightarrow$ not ergodic
$\searrow$ $\alpha<\beta$ $\Rightarrow$ unique stationary distribution,
$\begin{cases} \sum_{i=1}^{\infty}\prod_{n=1}^{i}\frac{\mu_n}{\lambda_n} = \sum_{i=1}^{\infty}\left(\frac{\beta}{\alpha}\right)^i = \infty, \text{ because } \beta>\alpha. \\ \sum_{i=1}^{\infty}\prod_{n=1}^{i}\frac{\lambda_{n-1}}{\mu_n} = \sum_{i=1}^{\infty}\left(\frac{\alpha}{\beta}\right)^i = \frac{1}{1-\frac{\alpha}{\beta}} < \infty, \text{ because } \beta>\alpha \end{cases}$ $\Rightarrow$ is ergodic

Answer:
1) $\alpha=0;\beta=0$: not ergodic
2) $\alpha=0;\beta\neq0$: ergodic
3) $\alpha\neq0;\beta=0$: not ergodic
4) $\alpha\neq0;\beta\neq0$ $\longrightarrow$ $\alpha\geq\beta$ – not ergodic
$\longrightarrow$ $\alpha<\beta$ – ergodic

iv) Write down the Generator G of this process and the master equation. Using this, write a differential equation for $\mu_t = EX_t$ and solve it for $X_0 = 1$.

$$G = \begin{pmatrix} -\alpha & \alpha & 0 & 0 & 0 & 0 & \dots \\ \beta & -(\alpha+\beta) & \alpha & 0 & 0 & 0 & \dots \\ 0 & \beta & -(\alpha+\beta) & \alpha & 0 & \dots \\ \dots \end{pmatrix}$$

Master equation: $\frac{d}{dt}\langle \pi_t | = \langle \pi_t | G$, that is $\begin{cases} \pi_0' = -\alpha \pi_0 + \beta \pi_1 \\ \pi_k' = \alpha \pi_{k-1} - (\alpha+\beta)\pi_k + \beta \pi_{k+1};\ k \geq 1 \end{cases}$

$$\Rightarrow \frac{d}{dt}\mu_t = \frac{d}{dt}(EX_t) = \frac{d}{dt}\left(\sum_{k=1}^{\infty} k \cdot \pi_k^{(t)}\right) = \sum_{k=1}^{\infty} k \cdot \pi_k' = \sum_{k=1}^{\infty} k \cdot \left(\alpha \pi_{k-1} - (\alpha+\beta)\pi_k + \beta \pi_{k+1}\right) =$$

$$= \alpha \cdot \sum_{k=1}^{\infty} k \cdot \pi_{k-1} - (\alpha+\beta) \cdot \sum_{k=1}^{\infty} k \cdot \pi_k + \beta \sum_{k=1}^{\infty} k \cdot \pi_{k+1} =$$

$= \sum_{k=1}^{\infty} (k-1)\pi_k$

$$= \alpha \cdot \sum_{k=0}^{\infty} (k+1)\pi_k - (\alpha+\beta)\sum_{k=1}^{\infty} k \cdot \pi_k + \beta \sum_{k=2}^{\infty} (k-1)\pi_k =$$

$= -\beta \cdot \sum_{k=1}^{\infty} \pi_k = -\beta(1-\pi_0)$

$$= \alpha \underbrace{\sum_{k=0}^{\infty} k \cdot \pi_k}_{= \sum_{k=1}^{\infty} k\pi_k} + \alpha \underbrace{\sum_{k=0}^{\infty} \pi_k}_{= 1} - (\alpha+\beta)\sum_{k=1}^{\infty} k\pi_k + \beta\Big(\underbrace{\sum_{k=2}^{\infty} k\pi_k + \pi_1}_{= \sum_{k=1}^{\infty} k\pi_k} - \pi_1\Big) - \beta \sum_{k=2}^{\infty} \pi_k =$$

$$= \cancel{\alpha \cdot \sum_{k=1}^{\infty} k\pi_k} + \alpha - \cancel{(\alpha+\beta)\sum_{k=1}^{\infty} k\pi_k} + \cancel{\beta \sum_{k=1}^{\infty} k\pi_k} - \beta(1-\pi_0) = (\alpha-\beta) + \beta\pi_0(t)$$

$$\Rightarrow \boxed{\frac{d}{dt}\mu_t = (\alpha-\beta) + \beta\pi_0(t)}$$

And we want to solve it for $X_0 = 1$, that is $\pi(0) = (0, 1, 0 \dots 0) \Rightarrow \pi_0(0) = 0$.

$\Rightarrow \frac{d}{dt}\mu_t = (\alpha-\beta) + \beta\pi_0(t)$ – but we don't know $\pi_0(t)$.

So we will treat $\pi_0(t)$ as an unknown function $f(t)$.

So: $\alpha = 0; \beta = 0 \Rightarrow \frac{d\mu_t}{dt} = 0 \Rightarrow \mu_t = \mu_0 = const = 1$

$\alpha = 0; \beta \neq 0 \Rightarrow \frac{d}{dt}\mu_t = \beta(\pi_0(t) - 1) \Rightarrow \mu_t = \mu_0 + \beta\int_0^t (\pi_0(s) - 1)\,ds = 1 + \beta\int_0^t (\pi_0(s) - 1)\,ds$

$\alpha \neq 0; \beta = 0 \Rightarrow \frac{d}{dt}\mu_t = \alpha \Rightarrow \mu_t = \mu_0 + \alpha t = 1 + \alpha t$.

$\alpha \neq 0; \beta \neq 0 \Rightarrow \frac{d}{dt}\mu_t = (\alpha-\beta) + \beta\pi_0(t) \Rightarrow \mu_t = \mu_0 + (\alpha-\beta)t + \beta\int_0^t \pi_0(s)\,ds = 1 + (\alpha-\beta)t + \beta\int_0^t \pi_0(s)\,ds$.

Answer: $\alpha = 0; \beta = 0 \Rightarrow \mu_t \equiv 1$

$\alpha = 0; \beta \neq 0 \Rightarrow \mu_t = 1 + \beta\int_0^t (\pi_0(s) - 1)\,ds$

$\alpha \neq 0; \beta = 0 \Rightarrow \mu_t = 1 + \alpha t$

$\alpha \neq 0; \beta \neq 0 \Rightarrow \mu_t = 1 + (\alpha-\beta)t + \beta\int_0^t \pi_0(s)\,ds$

b) $\alpha_x = x\alpha$; $\beta_x = x\beta$; $x \geq 0$; $\alpha, \beta \geq 0$, $X_0 = 1$.

$$\begin{tikzcd} 0 \arrow[loop above, "1"] & 1 \arrow[l, "\beta"] \arrow[r, bend left, "\alpha"] & 2 \arrow[l, "2\beta"] \arrow[r, bend left, "2\alpha"] & 3 \arrow[l, "3\beta"] \end{tikzcd}$$

i) Is X irreducible? Give all communicating classes in $\mathbb{N}_0$ and state whether they are transient or null/positive recurrent.

1) $\alpha = 0; \beta = 0$ – the chain is not irreducible, because we can't go anywhere from $x=0$, except $x=0$.
all x are separate communication classes
all x are transient, because we can't go away from them $\Rightarrow T_x = \infty$.

2) $\alpha = 0; \beta \neq 0$ – the chain is not irreducible, because we can't go anywhere from $x=0$
3) $\alpha \neq 0; \beta = 0$ all x are separate communication classes except $x=0$.
all x are transient, because when we go away from them, we can't get back

4) $\alpha \neq 0; \beta \neq 0$ – the chain is not irreducible, because we can't go anywhere from $x=0$ except $x=0$.
$\{0\}$ and $\{1,2,3\dots\}$ are two communication classes.
$\{0\}$ is transient, because we can't go away from it $\Rightarrow T_x = \infty$.
$\{1,2,3\dots\}$ are: if $\alpha > \beta \Rightarrow$ transient, because $\sum_{i=1}^{\infty} \left(\frac{\beta}{\alpha}\right)^i < \infty$
if $\alpha = \beta \Rightarrow$ null-recurrent, because $\sum_{i=1}^{\infty} \left(\frac{\beta}{\alpha}\right)^i = \infty$ and $\sum_{i=1}^{\infty} \left(\frac{\alpha}{\beta}\right)^i = \infty$
if $\alpha < \beta \Rightarrow$ positive-recurrent, because $\sum_{i=1}^{\infty} \left(\frac{\beta}{\alpha}\right)^i = \infty$ and $\sum_{i=1}^{\infty} \left(\frac{\alpha}{\beta}\right)^i < \infty$.

Answer:
1) $\alpha = 0; \beta = 0$
2) $\alpha = 0; \beta \neq 0$
3) $\alpha \neq 0; \beta = 0$ $\Rightarrow$ the chain is not irreducible
all states are separate classes
all states are transient

4) $\alpha \neq 0; \beta \neq 0 \Rightarrow$ chain is not irreducible
$\{0\}$ and $\{1,2,3\dots\}$ are two communicating classes; $\{0\}$ is transient
$\alpha > \beta \Rightarrow$ transient $\{1,2,3\dots\}$
$\alpha = \beta \Rightarrow \{1,2,3\dots\}$ null-recurrent
$\alpha < \beta \Rightarrow \{1,2,3\dots\}$ positive-recurrent

ii) give all stationary distributions and state whether they are reversible
We search for $\langle \pi_t |$: $\langle \pi_t | G = \langle 0 |$

$$G = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ \beta & -(\alpha+\beta) & \alpha & 0 & 0 \\ 0 & 2\beta & -2(\alpha+\beta) & 2\alpha & \dots \end{pmatrix} \Rightarrow \begin{cases} \pi_1 \beta = 0 \\ \alpha(k-1)\pi_{k-1} - (\alpha+\beta)k\pi_k + \beta(k+1)\pi_{k+1} = 0, \quad k \geq 1 \end{cases}$$

1) $\alpha = 0; \beta = 0 \Rightarrow$ any π is stationary, because $G \equiv 0$.
and any π is reversible, because $P_t = E$, so $p(x,y) = 0, \forall y \neq x$,
and for $y = x$: $\pi(x)p(x,y) = \pi(y)p(y,x)$ is true

2) $\alpha = 0; \beta \neq 0 \Rightarrow \begin{cases} \pi_1 \beta = 0. \\ -\beta k \pi_k + \beta(k+1)\pi_{k+1} = 0, \ k \geq 1 \end{cases} \Rightarrow \pi_1 = \pi_2 = \dots = 0 \Rightarrow \pi_0 = 1 \Rightarrow (1, 0 \dots)$ is only stationary distribution
and it is reversible, because only for $x=0$ we have $\pi(x) \neq 0$, hence $\underbrace{\pi(0)}_{1} \cdot p(0,y) \stackrel{?}{=} \underbrace{\pi(y)}_{0} \cdot p(y,0)$ is true, because $p(0,y) = 0 \ \forall y \neq 0$.

3) $\alpha \neq 0; \beta = 0 \Rightarrow \begin{cases} \pi_1 \cdot 0 = 0. \\ 0 - \alpha\pi_1 = 0. \Rightarrow \pi_1 = 0 \end{cases}$
$\alpha(k-1) \cdot \pi_{k-1} - \alpha k \pi_k = 0 \Rightarrow \pi_k = 0; k \geq 2$ $\Rightarrow$ two cases can happen:
if $\pi_0 = 1$, then $(1, 0 \dots)$ is stationary $\Rightarrow$ extinction
if $\pi_0 < 1 \Rightarrow P(X_\infty = \infty) = 1 - p_0 \Rightarrow$ unbounded growth – not stationary

4) $\alpha \neq 0; \beta \neq 0 \Rightarrow \begin{cases} \pi_0' = \beta\pi_1 = 0 \\ \pi_n' = \alpha(n-1)\pi_{n-1} - (\alpha+\beta)n\pi_n + \beta(n+1)\pi_{n+1} = 0. \ ; n \geq 1 \end{cases}$

$\Rightarrow \begin{cases} \pi_1 = 0. \\ -(\alpha+\beta)\pi_1 + \beta\cdot 2\pi_2 = 0. \Rightarrow \pi_2 = 0. \\ \alpha(n-1)\pi_{n-1} - (\alpha+\beta)n\pi_n + \beta(n+1)\pi_{n+1} = 0 \Rightarrow \pi_3 = \pi_4 = \ldots = 0. \end{cases}$

$\Rightarrow$ two cases can be $\begin{cases} \text{if } \pi_0(\infty) = 1 \Rightarrow \text{other } \pi_n = 0, \forall n > 1 \Rightarrow \text{extinction} \\ \text{if } \pi_0(\infty) = \pi_0 < 1 \Rightarrow P(X_t = \infty) = 1 - \pi_0 > 0 \Rightarrow \text{population increases without a bound.} \end{cases}$

State $(1,0,0.)$ is reversible,
because $\underbrace{\pi(x)}_{1}\underbrace{p(x,y)}_{0} = \underbrace{\pi(y)}_{0} \Rightarrow 0 = 0.$

Answer: 1) $\alpha = 0; \beta = 0 \Rightarrow$ any π is stationary and reversible
2) $\alpha = 0; \beta \neq 0 \Rightarrow \pi = (1,0,0.)$ is the only stationary, it is reversible
3) $\alpha \neq 0; \beta = 0 \Rightarrow \pi = (1,0\ldots)$ is the only stationary, it is reversible
4) $\alpha \neq 0; \beta \neq 0 \Rightarrow \pi = (1,0\ldots)$ is the only stationary, it is reversible

but this stationary distribution is not attainable with probability 1.

iii) Write down the generator G of this process and the master equation. Using this, write a differential equation for $\mu_t = EX_t$ and solve it for $X_0 = 1$.

$$G = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ \beta & -(\alpha+\beta) & \alpha & 0 & 0 \\ 0 & 2\beta & -2(\alpha+\beta) & 2\alpha & \ldots \end{pmatrix}$$

Master equation is $\langle \pi_k |' = \langle \pi_k | G$, that is $\begin{cases} \pi_0' = \beta\pi_1 \\ \pi_k' = \alpha(k-1)\pi_{k-1} - (\alpha+\beta)k\pi_k + \beta(k+1)\pi_{k+1}; k \geq 1. \end{cases}$

$$\Rightarrow \frac{d}{dt}\mu_t = \sum_{k=1}^{\infty} k\cdot\pi_k' = \sum_{k=1}^{\infty} k\left(\alpha(k-1)\pi_{k-1} - (\alpha+\beta)k\pi_k + \beta(k+1)\pi_{k+1}\right) =$$

$\begin{cases} (k-1)k = (k-1)^2 + (k-1) \\ (k+1)k = (k+1)^2 - (k+1) \end{cases}$

$$= \sum_{k=1}^{\infty} \alpha\cdot(k-1)^2\cdot\pi_{k-1} + \alpha\cdot\sum_{k=1}^{\infty}(k-1)\pi_{k-1} - (\alpha+\beta)\sum_{k=1}^{\infty}k^2\pi_k + \beta\sum_{k=1}^{\infty}(k+1)^2\pi_{k+1} - \beta\sum_{k=1}^{\infty}(k+1)\pi_{k+1} =$$

$$= \underbrace{\alpha\cdot\sum_{k=0}^{\infty}k^2\cdot\pi_k + \alpha\cdot\sum_{k=0}^{\infty}k\cdot\pi_k - (\alpha+\beta)\sum_{k=1}^{\infty}k^2\pi_k + \beta\sum_{k=2}^{\infty}k^2\cdot\pi_k \pm\beta\cdot 1\cdot\pi_1}_{0} - \beta\cdot\sum_{k=2}^{\infty}k\pi_k =$$

$$= (\alpha-\beta)\sum_{k=1}^{\infty}k\cdot\pi_k = (\alpha-\beta)\mu_t$$

$$\Rightarrow \frac{d}{dt}\mu_t = (\alpha-\beta)\mu_t$$

$\Rightarrow \mu_t = \mu_0\cdot e^{(\alpha-\beta)t}$; But $\mu_0 = EX_0 = 1 \Rightarrow \mu_t = e^{(\alpha-\beta)t}$

So if $\alpha > \beta$, then population grows unboundedly
if $\alpha = \beta$, then $\mu_t \equiv 1$.
if $\alpha < \beta$, then $\mu_t \to 0$, that mean extinction.

iv) Set up a recursion for the extinction probability $h_x = P[X_t \to 0 \mid X_0 = x]$ and give the smallest solution with boundary condition $h_0 = 1$.

Let's use law of total probability when we condition on X_1, and also use markov property.

We know that $P(A|B) = \sum_{i=1}^{n} P(A|B \cap C_i) \cdot P(C_i|B)$

And we know that from $x = k$ process can go to $k+1$, $k-1$ or stay at k.

$$\Rightarrow \underbrace{P(X_t \to 0 \mid X_0 = k)}_{= h_k} = P(X_t \to 0 \mid X_0 = k; X_1 = k-1) \cdot P(X_1 = k-1 \mid X_0 = k) +$$
$$+ P(X_t \to 0 \mid X_0 = k; X_1 = k) \cdot P(X_1 = k \mid X_0 = k) +$$
$$+ P(X_t \to 0 \mid X_0 = k; X_1 = k+1) \cdot P(X_1 = k+1 \mid X_0 = k) =$$
$$= h_{k-1} \cdot P(X_1 = k-1 \mid X_0 = k) + h_k \cdot P(X_1 = k \mid X_0 = k) + h_{k+1} \cdot P(X_1 = k+1 \mid X_0 = k)$$

$$\Rightarrow h_k \underbrace{(1 - P(X_1 = k \mid X_0 = k))}_{= P(X_1 = k-1 \mid X_0 = k) + P(X_1 = k+1 \mid X_0 = k)} = h_{k-1} \cdot P(X_1 = k-1 \mid X_0 = k) + h_{k+1} \cdot P(X_1 = k+1 \mid X_0 = k)$$

$$\Rightarrow h_k = h_{k-1} \cdot \underbrace{\frac{P(X_1 = k-1 \mid X_0 = k)}{P(X_1 = k-1 \mid X_0 = k) + P(X_1 = k+1 \mid X_0 = k)}}_{= \beta/(\alpha+\beta)} + h_{k+1} \cdot \underbrace{\frac{P(X_1 = k+1 \mid X_0 = k)}{P(X_1 = k+1 \mid X_0 = k) + P(X_1 = k+1 \mid X_0 = k)}}_{= \frac{\alpha}{(\alpha+\beta)}}$$

because going from x to $x+1$ has probability $\alpha x \cdot \Delta t + \bar{o}(\Delta t)$ and going from x to $x-1$ has probability $\beta x \cdot \Delta t + \bar{o}(\Delta t)$

$\Rightarrow h_k = h_{k-1} \cdot \frac{\beta}{\alpha+\beta} + h_{k+1} \cdot \frac{\alpha}{\alpha+\beta}$

$\Rightarrow$ look for solution in the form $h_k = \lambda^k$.

$\Rightarrow \lambda^2 \alpha - (\alpha+\beta)\lambda + \beta = 0$

$\Rightarrow D = (\alpha+\beta)^2 - 4\alpha\beta = (\alpha-\beta)^2$

If $\alpha \neq \beta \Rightarrow \lambda_{1;2} = \frac{(\alpha+\beta) \pm (\alpha-\beta)}{2\alpha} \Rightarrow \lambda_1 = 1;\ \lambda_2 = \frac{\beta}{\alpha} \Rightarrow h_k = C_1 + C_2 \left(\frac{\beta}{\alpha}\right)^k$

If $\alpha = \beta$ then $h_k = C_1 + C_2 k$.

But $h_0 = 1 \Rightarrow \begin{cases} \alpha \neq \beta: C_1 + C_2 = 1 \Rightarrow C_2 = 1 - C_1 \Rightarrow h_k = C_1 + (1 - C_1)\left(\frac{\beta}{\alpha}\right)^k \\ \alpha = \beta: C_1 = 1 \Rightarrow h_k = 1 + C_2 k. \end{cases}$

And now we choose the smallest solution:

$\begin{cases} \text{if } \alpha < \beta \Rightarrow \frac{\beta}{\alpha} > 1 \Rightarrow \text{the smallest } h_k \text{ will be with } C_1 = 1; \Rightarrow h_k = 1 \\ \text{if } \alpha > \beta \Rightarrow \left(\frac{\beta}{\alpha}\right) < 1 \Rightarrow \left(\frac{\beta}{\alpha}\right)^k \text{ decrease} \Rightarrow h_k = \left(\frac{\beta}{\alpha}\right)^k \text{ is the smallest} \\ \text{if } \alpha = \beta \Rightarrow h_k = 1 \end{cases}$

, because if $C_2 > 0$, then for some k h_k will be > 1, and if $C_2 < 0$, then for some k h_k will be < 0.

Answer:
$\alpha \leq \beta \Rightarrow h_k = 1 \ \forall k$
$\alpha > \beta \Rightarrow h_k = \left(\frac{\beta}{\alpha}\right)^k$

v) Is the process ergodic?

1) $\alpha = 0; \beta = 0 \Rightarrow$ stationary distribution is not unique $\Rightarrow$ not ergodic

2) $\alpha = 0; \beta \neq 0$ – has unique stationary distribution $(1; 0 \dots)$

$$\begin{cases} \sum_{i=1}^{\infty} \prod_{n=1}^{i} \frac{\mu_n}{\lambda_n} = \sum_{i=1}^{\infty} \left(\frac{\beta}{0}\right)^i \stackrel{?}{=} \infty \text{ – yes} \\ \sum_{i=1}^{\infty} \prod_{n=1}^{i} \frac{\lambda_{n-1}}{\mu_n} = 0 < \infty \Rightarrow \text{ergodic} \end{cases}$$

3) $\alpha \neq 0; \beta = 0$ – has unique stationary distribution $(1; 0 \dots)$

$$\left\{ \sum_{i=1}^{\infty} \prod_{n=1}^{i} \frac{\mu_n}{\lambda_n} = 0 \stackrel{?}{=} \infty \text{ – no} \Rightarrow \text{not ergodic} \right.$$

4) $\alpha \neq 0; \beta \neq 0$ – has unique stationary distribution $(1; 0 \dots)$

$$\begin{cases} \sum_{i=1}^{\infty} \prod_{n=1}^{i} \frac{\mu_n}{\lambda_n} = \sum_{i=1}^{\infty} \left(\frac{\beta}{\alpha}\right)^i = \infty \text{ iff } \beta \geq \alpha \\ \sum_{i=1}^{\infty} \prod_{n=1}^{i} \frac{\lambda_{n-1}}{\mu_n} = 0 \stackrel{?}{<} \infty \text{ – true} \end{cases} \Rightarrow \text{it is ergodic iff } \beta \geq \alpha.$$

Answer:
1) $\alpha = 0; \beta = 0$ – not ergodic
2) $\alpha = 0; \beta \neq 0$ – ergodic
3) $\alpha \neq 0; \beta = 0$ – not ergodic
4) $\alpha \neq 0; \beta \neq 0$: $\alpha \leq \beta$ – ergodic
$\alpha > \beta$ – not ergodic