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Brownian Motion

Problem sheet 3

1. **Continuity of paths**

   (a) Let $(X_t)_{t\geqslant 0}$ be a stochastic process that satisfies the assumptions of Kolmogorov's continuity criterion:

   $$\mathbb{E}|X_t - X_s|^\alpha \leqslant C|t-s|^{1+\beta} , \quad \forall t, s \in [0,T] ,$$

   for some $\alpha, \beta, T, C > 0$. Prove that the process is continuous in probability (without using Kolmogorov's theorem), namely that

   $$\lim_{t\to t_0} \mathbb{P}(|X_t - X_{t_0}| \geqslant \varepsilon) = 0 , \qquad \forall t_0 \in (0,T) \text{ and } \varepsilon > 0 .$$

   (b) Prove (this time using Kolmogorov's continuity criterion) that sample paths of Brownian motion are almost surely γ-Hölder continuous for any $\gamma \in (0, 1/2)$, namely that:

   $$\mathbb{P}\left(\sup_{0\leqslant s\leqslant t\leqslant T} \frac{|B_t - B_s|}{|t-s|^\gamma} < \infty \right) = 1 , \qquad \forall \gamma \in (0,1/2) .$$

   (c) **(No points)** In the setting of point (a), prove that if $\beta < \alpha$, then

   $$\mathbb{P}\left(X_t = X_0 , \forall t \geqslant 0\right) = 1 .$$

2. **Geometric Brownian Motion**
   Let $\big(B(t) : t \geq 0\big)$ be a standard BM and define $\quad X(t) := e^{B(t)-at}$ for all $t \geq 0$ and $a \in \mathbb{R}$.

   (a) For which $a \in \mathbb{R}$ do we have $X(t) \to 0$ almost surely as $t \to \infty$?
   For which $a \in \mathbb{R}$ do we have $X(t) \to \infty$ almost surely? Justify your answers.

   (b) For which $a \in \mathbb{R}$ and $p > 0$ do we have $\mathbb{E}\big[X(t)^p\big] \to 0$ as $t \to \infty$?
   For which $a \in \mathbb{R}$ and $p > 0$ do we have $\mathbb{E}\big[X(t)^p\big] \to \infty$? Justify your answers.

   (c) Fix $p = 1$, then for which a do we have $\mathbb{E}[X_t] = 1$ for all $t \geqslant 0$? Let $(\mathcal{F}^0_t)_{t\geqslant 0}$ be the natural filtration generated by X_t. Can you prove that X_t is a *martingale*, namely that the following holds true:

   $$X_t \in L^1 , \quad \text{ and } \quad \mathbb{E}[X_t|\mathcal{F}^0_s] = X_s , \qquad \forall t \geqslant s \geqslant 0 , ?$$

   Hint: What can you sat about the limit $\lim_{t\to\infty} \frac{1}{t} B_t$ as $t \to \infty$?

3. **Integrated Brownian motion**
   Let $\big(B(t) : t \geq 0\big)$ be a standard BM and define $\quad X(t) := \displaystyle\int_0^t B(s)\, ds$ for all $t \geq 0$.

(a) Prove that $\big(X(t) : t \geq 0\big)$ is a Gaussian process.
Hint: For a fixed realisation $B_t(\omega)$ use the Riemann sum approximation of the time integral and a question from the previous exercise sheet.

(b) Compute the mean and covariance functions of $\big(X(t) : t \geq 0\big)$.

(c) Compute $\mathbb{E}\big[(X(t) - X(s))^2\big]$, and compare its rate of decay as $t \searrow s$ with that of $\mathbb{E}\big[(B(t) - B(s))^2\big]$.

4. **Tightness and continuity**
Suppose that a sequence of probability measures $\{\mathbb{P}_n\}_{n\in\mathbb{N}}$ on $C([0,1])$ satisfies for some $\alpha, \beta, \gamma, C > 0$

$$\sup_{n\in\mathbb{N}} \mathbb{E}_n|\omega_0|^\zeta < \infty \ ,$$
$$\sup_{n\in\mathbb{N}} \mathbb{E}_n|\omega_t - \omega_s|^\alpha \leqslant C|t-s|^{1+\beta} \ , \qquad \forall t, s \in [0,1] \ .$$

Then the sequence of probability measures $\{\mathbb{P}_n\}_{n\in\mathbb{N}}$ is tight.

Hint: Follow the steps of the proof of Komogorov's continuity criterion.

5. **Transition kernels**
Suppose B is a standard BM on $\mathbb{R}$. Suppose $x, c \in \mathbb{R}$, $c \neq 0$. Show that the following processes are time-homogeneous Markov and find their transition kernels $P(t, x, dy)$:

(a) $X(t) := x + cB(t)$; (b) $X(t) := B(t)^2$.