Second (and higher) order systems

We shall consider equations of the form

$$\dot{x} = f(x), \quad x \in \mathbb{R}^2, \qquad (x \in \mathbb{R}^n)$$

Linear systems in $\mathbb{R}^2$

$$\begin{aligned}\dot{x}_1 &= ax_1 + bx_2 \\ \dot{x}_2 &= cx_1 + dx_2\end{aligned}$$

Introducing the vector $x = (x_1, x_2)^T$ we have

$$\dot{x} = Ax, \qquad A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

Try a solution of the form

$$x = \mathrm{e}^{\lambda t} v$$

This leads to the linear homogeneous equation

$$Av = \lambda v.$$

v is an *eigenvector* of A with corresponding *eigenvalue* λ. For the system above to have a non-trivial solution we require that

$$\boxed{\det(A - \lambda I) = 0}$$

which is called the *characteristic equation*. Here I is the 2×2 identity matrix. Substituting the components of A into the characteristic equation gives

$$\lambda^2 - (a + d)\lambda + (ad - bc) = 0$$

or

$$\lambda^2 - \mathrm{Tr}\, A\ \lambda + \det A = 0$$

so that

$$\boxed{\lambda_\pm = \frac{1}{2}\left[\mathrm{Tr}\, A \pm \sqrt{(\mathrm{Tr}\, A)^2 - 4 \det A}\right]}$$

The general solution for $x(t)$:

$$x(t) = c_1 \mathrm{e}^{\lambda_1 t} v_1 + c_2 \mathrm{e}^{\lambda_2 t} v_2.$$

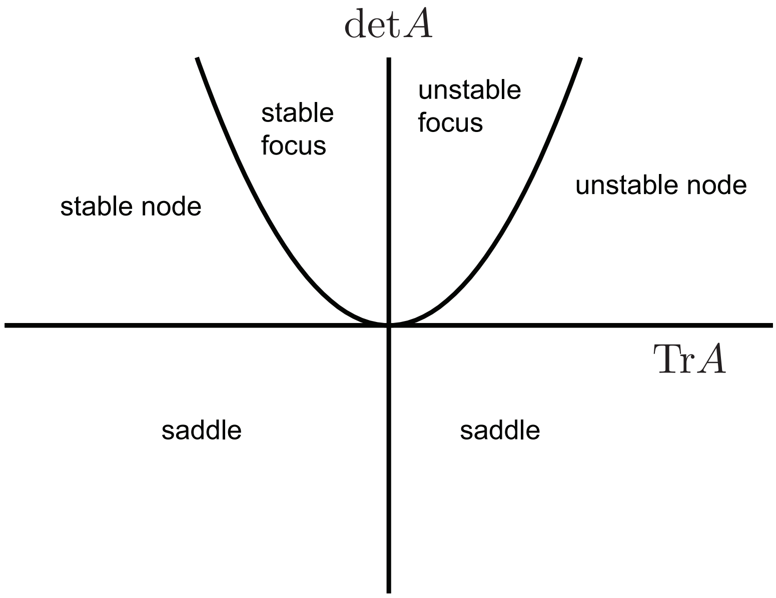
If $\lambda_{1,2}$ are complex ($\lambda_{1,2} = \alpha \pm i\omega$), the fixed point is either a *centre* or a *spiral*. Since $x(t)$ involves linear combinations of $\mathrm{e}^{\alpha \pm i\omega}$, $x(t)$ is a combination of terms involving $\mathrm{e}^{\alpha t} \cos(\omega t)$ and $\mathrm{e}^{\alpha t} \sin(\omega t)$ (by Euler's formula $\mathrm{e}^{i\omega t} = \cos(\omega t) + i \sin(\omega t)$).

- If $\alpha < 0 \Rightarrow$ stable focus (or stable spiral)
- If $\alpha > 0 \Rightarrow$ unstable focus (or unstable spiral)
- If $\alpha = 0 \Rightarrow$ a centre (periodic solution with period $T = 2\pi/\omega$), marginally stable.

Classification of fixed points

We classify the different types of behaviour according to the values of Tr A and det A.

- $\lambda_\pm$ are real if $(\text{Tr } A)^2 > 4 \det A$.
- Real eigenvalues have the same sign if $\det A > 0$ and are positive if $\text{Tr } A > 0$ (negative if $\text{Tr } A < 0$) — **stable and unstable nodes**.
- Real eigenvalues have opposite signs if $\det A < 0$ — **saddle node**.
- Eigenvalues are complex if $(\text{Tr } A)^2 < 4 \det A$ — **focus**.



Solving linear systems

- Real eigenvalue $\lambda \quad \Rightarrow \quad C\mathrm{e}^{\lambda t}$
- Real eigenvalue λ of multiplicity $r \quad \Rightarrow \quad C_1\mathrm{e}^{\lambda t} + C_2 t\mathrm{e}^{\lambda t} + \cdots + C_r t^{r-1}\mathrm{e}^{\lambda t}$
- Pair of complex eigenvalues $\lambda = \rho \pm i\omega \quad \Rightarrow \quad \mathrm{e}^{\rho t}(B \cos \omega t + C \sin \omega t)$
- Pair of complex eigenvalues $\lambda = \rho \pm i\omega$, each with multiplicity $r \quad \Rightarrow$
  $\mathrm{e}^{\rho t}(B_1 \cos \omega t + C_1 \sin \omega t + B_2 t \cos \omega t + C_2 t \sin \omega t + \cdots + B_r t^{r-1} \cos \omega t + C_r t^{r-1} \sin \omega t)$

Nonlinear systems in $\mathbb{R}^2$ (in $\mathbb{R}^n$)

We shall consider equations of the form

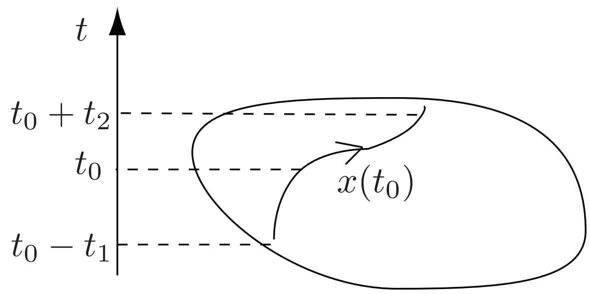
$$\dot{x}_1 = f_1(x_1, x_2),$$
$$\dot{x}_2 = f_2(x_1, x_2).$$

This system can be wirtten in vector notation

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}),$$

where $\mathbf{x}(x_1, x_2)$, $\mathbf{f}(\mathbf{x}) = (f_1(\mathbf{x}), f_2(\mathbf{x}))$. $\mathbf{x}$ represents a point in the phase plane, and $\dot{\mathbf{x}}$ is the velocity vector at that point.

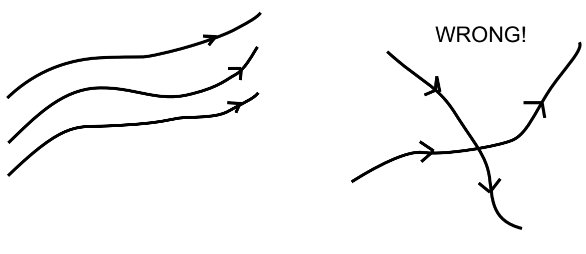
Existence and uniqueness theorem (in $\mathbb{R}^n$): Suppose $\dot{x} = f(x)$ and $f : \mathbb{R}^n \to \mathbb{R}^n$ is continuously differentiable (i.e. $\partial f_i / \partial x_j$, $i,\ j = 1, ..., n$ exist and are continuous for all x). Then there exits $t_1 > 0$ and $t_2 > 0$ such that the solution with $x(t_0) = x_0$ exists and is unique for all $t \in (t_0 - t_1, t_0 + t_2)$.



Phase-space and flows. Refer to local solution through x_0 as a *solution curve* or *trajectory*. Suppose that $\dot{x} = f(x)$, $x \in \mathbb{R}^n$, $f : \mathbb{R}^n \to \mathbb{R}^n$. We define a flow $\phi(x, t)$ such that $\phi(x, t)$ is the solution of the ODE at time t with initial value x_0 at $t = 0$. The solution $x(t)$ with $x(0) = x_0$ is now written as $\phi(x_0, t)$

$$\frac{\mathrm{d}\phi(x, t)}{\mathrm{d}t} = f(\phi(x, t)), \qquad \phi(x, 0) = x_0$$

By varying initial condition x_0 we generate a family of trajectories called the *flow* generated by Φ.



Note that uniqueness imples that trajectories cannot cross.

An **equilibrium** or fixed point satisfies $\Phi(x, t) = x$ for all t. Thus $f(x) = 0$. An important feature of nonlinearities is that there can exist more than one (isolated) fixed point.