Stochastic Modelling and Random Processes - Class test

The class test counts $80/100$ module marks, [x] indicates weight of each question.
At the end of this paper, you can find some supporting material.
Attempt all 4 questions and justify all your answers

1. **General concepts** **[30]**

   (a) Consider a discrete time stochastic process $(X_n : n \in \mathbb{N}_0)$ with state space S. When is this process a Markov chain? And when is it time-homogeneous?

   (b) For a **discrete-time, time-homogeneous Markov chain** $(X_n : n \in \mathbb{N}_0)$ with state space S, define the transition function, and then derive the Chapman-Kolmogorov equations using the law of total probability, the Markov property and time homogeneity.

   (c) Consider a **continuous-time Markov chain** (CTMC) $(X_t : t \geq 0)$ with state space S. State (without proof) the Chapman-Kolmogorov equations and explain how you can use them to define the generator of the CTMC and thus the transition matrix $P_t = (p_t(x,y) : x, y \in S)$, where $p_t(x,y)$ is the probability of transitioning from x to y after time t.

   (d) Give the definition of a diffusion process $(X_t : t \geq 0)$ on $\mathbb{R}$ and write down its generator and the corresponding stochastic differential equation (SDE).

   (e) Consider the linear voter model $(\eta_t : t \geq 0)$ where individual i influences the opinion of individual j with rate $q(i,j) \geq 0$. Give the state space of the model and the transition rates $c(\eta, \eta^i)$ using the standard notation: if the opinion of individual i is changed, then

   $$\eta^i(k) = \begin{cases} \eta(k) & , \ k \neq i \\ 1 - \eta(k) \, , & k = i \end{cases}$$

   Is this process ergodic (justify your answer)? Give a formula for all stationary distributions of the process, assuming that $q(i,j)$ is irreducible and the state space is finite.

   (f) Choose two of the models of random graphs that we discussed in lectures. Define these graphs and discuss their main properties, including properties which make them appropriate (or not) to model real-life networks.

2. **Markov chains** **[20]**

   Let $(X_t : t \geq 0)$ be a continuous-time random walk on $S = \{1, 2, 3, 4, 5, 6\}$ with generator

   $$\mathcal{L}f(n) = \big(f(n+2) - f(n)\big) + 2\big(f(n-1) - f(n)\big) \, , \quad f : S \to \mathbb{R},$$

   i.e., jumping two steps to the right with rate 1, and one step to the left with rate 2. Consider periodic boundary conditions (i.e., if $n + 2 > 6$ or $n - 1 < 1$ take these modulo 6).

   (a) Write the generator in matrix form and give the transition matrix P^Y of the corresponding jump chain $(Y_n : n \in \mathbb{N}_0)$. Draw a graph representation for this random walk.
   Hint: I will accept whether you decide to write the discrete chain with the transition rates for Y_n or the continuous one with rates from the generator, as long as you state which case you are sketching and the sketch is correct.

   (b) Is this chain irreducible? Is it ergodic? Find its stationary distribution(s) and state whether it is (they are) reversible. Justify your answers.

(c) Describe how you would simulate sample paths of this random walk numerically, including defining all the auxilliary variables you would need, starting from $X_0 = 1$.

(d) Consider a DTMC $(Z_n : n \in \mathbb{N}_0)$, where now the chain jumps two steps to the right with probability p and one step to the left with probability $q = 1 - p$, with periodic boundary conditions. Is this chain irreducible? Is it ergodic? Find its stationary distribution(s) and state whether it is (they are) reversible. Justify your answers.
Hint: if necessary consider different cases for p and q.

3. **Diffusion processes** **[20]**

(a) Define the standard Brownian motion $(B_t : t \geq 0)$ and write down its generator $\mathcal{L}_B f(x)$.

(b) State Itô's formula for $(X_t : t \geq 0)$ and a smooth function $f : \mathbb{R} \to \mathbb{R}$.

(c) Let $(B_t : t \geq 0)$ be a standard Brownian motion and consider the SDE

$$dX_t = \theta X_t \, dt + \sigma X_t dB_t, \quad X_0 = x_0.$$

Solve this equation. What is the resulting stochastic process?

(d) Use the evolution equation of expectation values of test functions $f : \mathbb{R} \to \mathbb{R}$ to derive ODEs for the mean $m(t) := \mathbb{E}(X_t)$ and the second moment $\mathbb{E}(X_t^2)$. Solve these ODEs and obtain expressions for the mean $m(t)$ and variance $v(t)$ of the solution of this SDE.

(e) Write down the Fokker-Planck equation for the density of this diffusion process.

4. **Graphs and Networks** **[30]**

(a) Define the Erdős-Rényi random graph model $\mathcal{G}_{N,p}$, including the set of all possible graphs and the corresponding probability distribution.

(b) At the end of this paper there is a sequence of iid samples of a uniform distribution in the interval $(0, 1)$. Use it to sample an (undirected) ER random graph with $N = 8$ and a value of $p \in [0.3, 0.8]$ of your choice ($G = \mathcal{G}_{8,p}$). Draw this graph, write down its adjacency matrix and explain how you could use it to find useful quantities such as the number of triangles of the graph.
Hint: State the value of p you chose and make sure you say how you ordered your edges for sampling. In order to make the following questions easier, you might not want to choose a very large value.

(c) Identify all cliques of vertices in the graph and draw a spanning tree of this graph.
Note: if there is no spanning tree, explain why.

(d) Write down a matrix whose entries are the vertex distances $\big(d_{ij} : i, j = 1, \ldots, 8\big)$ (you can use that $d_{ii} = 0 \ \forall i$). Compute the characteristic path length $L(G)$, the diameter $\mathrm{diam}(G)$ of G and the closeness centrality for each node.

(e) Compute the degree distribution $p(k)$ and the average degree $\langle k \rangle$ of G.

(f) Compute the local clustering coefficients C_i and their average $\langle C_i \rangle$. Use these to compute the global clustering coefficient C.

(g) Give all non-zero entries of the joint degree distribution $q(k, k')$ and compute the marginal distribution $q(k) = \sum_{k'} q(k, k')$. For all k' with $q(k') > 0$ compute the conditional distribution $q(k|k')$ and the corresponding expectation $k_{nn}(k')$.

Supporting material

Here is a list of iid samples of $U([0,1])$. You might not need to use them all. When writing down your ER random graph, make sure you say how you ordered your edges for sampling.

01 to **10**	0.4561	0.0620	0.4648	0.3596	0.2089	0.1040	0.2972	0.0715	0.8972	0.1017
11 to **20**	0.3308	0.7069	0.7640	0.0567	0.9052	0.7455	0.2999	0.2425	0.1967	0.9954
21 to **30**	0.8985	0.9995	0.8182	0.5219	0.6754	0.7363	0.1341	0.0538	0.0934	0.3321
31 to **40**	0.1182	0.2878	0.1002	0.3358	0.4685	0.5619	0.2126	0.4417	0.3074	0.2973
41 to **50**	0.9884	0.4145	0.1781	0.1757	0.9121	0.1842	0.8949	0.0133	0.6628	0.5400

If you need to, (and you may not need to), at any point, you can use the folowing properties of the Itô integral:

1. Additivity: $\int_0^s f(t,X_t)\,dB_t + \int_s^T f(t,X_t)\,dB_t = \int_0^T f\,dB_t$.
2. Linearity: for all $\alpha, \beta \in \mathbb{R}$ and f, g such that the integral exists, we have
$$\int_0^T (\alpha f(t,X_t) + \beta g(t,X_t))\,dB_t = \alpha \int_0^T f(t,X_t)\,dB_t + \beta \int_0^T g(t,X_t)\,dB_t.$$
3. If f is a deterministic function (i.e. it depends on time only), we have
$$\mathbb{E}\left(\int_0^T f(t)\,dB_t\right) = 0.$$
4. It verifies the Itô isometry:
$$\mathbb{E}\left(\left(\int_0^T f(s)\,dW_t\right)^2\right) = \mathbb{E}\left(\int_0^T f^2(t)\,dt\right),$$
and, more generaly
$$\mathbb{E}\left(\int_0^t f(s)\,dB_s \int_0^u g(s)\,dB_s\right) = \mathbb{E}\left(\int_0^{\min(t,u)} f(s)g(s)\,dB_s\right).$$
5. The above two properties imply that if f is deterministic, then the stochastic integral $\mathcal{I}_T$ is a Gaussian random variable with mean zero and variance $\int_0^T f^2(s)\,ds$.