THE UNIVERSITY OF WARWICK

FOURTH YEAR EXAMINATION: Summer 2020

Stochastic Analysis

Time Allowed: **3 hours**

Read all instructions carefully. Please note also the guidance you have received in advance on the departmental 'Warwick Mathematics Exams 2020' webpage.

Calculators, wikipedia and interactive internet resources are not needed and are not permitted in this examination. You are not allowed to confer with other people. You may use module materials and resources from the module webpage.

ANSWER COMPULSORY QUESTION 1 AND TWO FURTHER QUESTIONS out of the four optional questions 2, 3, 4 and 5.

On completion of the assessment, you must upload your answer to Moodle as a single PDF document if possible, although multiple files (2 or 3) are permitted. You have an additional 45 minutes to make the upload, and instructions are available on the departmental 'Warwick Mathematics Exams 2020' webpage.

You must not upload answers to more than 3 questions, including Question 1. If you do, you will only be given credit for your Question 1 and the first two other answers.

The numbers in the margin indicate approximately how many marks are available for each part of a question. The compulsory question is worth twice the number of marks of each optional question. Note that the marks do not sum to 100.

For all questions, when not stated otherwise, you may suppose:

$(\Omega, \mathcal{F}, P)$ is a probability space,
$(\mathcal{F}_t : t \geq 0)$ is a filtration,
$(B_t : t \geq 0)$ is an $(\mathcal{F}_t)$ Brownian motion.

COMPULSORY QUESTION

1. a) Let $(\mathcal{F}_t : t \geq 0)$ be a filtration on a probability space $(\Omega, \mathcal{F}, P)$. What does it mean that $(B_t : t \geq 0)$ is a d-dimensional $(\mathcal{F}_t)$ Brownian motion?

Explain what it means that $X = (X_t : t \geq 0)$ is an *Ito process* with respect to this Brownian motion. **[5]**

b) Find the scale function and hence describe the behaviour of the solution to

$$dX = c\,dt + \sqrt{X}dB,$$

where $c > 0$, on the interval $(0, \infty)$ **[9]**

c) Suppose $\sigma(x) > 0$ on the interval $x \in (0, 1)$. State Feller's test to decide if the solution to

$$dX = \sigma(X)dB, \quad X_0 = x_0 \in (0, 1),$$

exits the interval $(0, 1)$ in finite time or not.

Give, without proof, three examples of σ, one where the exit time is finite almost surely, one where the exit time is infinite almost surely, and one where it has non-zero probability of being either finite or infinite. **[9]**

d) What does it mean that τ is an $(\mathcal{F}_t)$ stopping time?

What is the stopping rule for stochastic integrals?

Explain how the stopping rule is used to define the integral $\int_0^t h_s dB_s$ for continuous adapted integrands. (You may assume the integral has been defined already when h is bounded.) **[7]**

e) Suppose B is a one dimensional Brownian motion. Let $\sigma : \mathbb{R} \to \mathbb{R}$ be continuous and bounded. Prove that

$$\sum_{k=0}^{N-1} \sigma(B_{\frac{k}{N}}) \left(B_{\frac{k+1}{N}} - B_{\frac{k}{N}} \right)^2 \quad \to \quad \int_0^1 \sigma(B_s)ds \quad \text{in probability}$$

as $N \to \infty$. **[10]**

OPTIONAL QUESTIONS

2. a) Suppose $\alpha = (\alpha_t : t \geq 0)$ is deterministic and continuous. Let $X_t = \int_0^t \alpha_s dB_s$.
Explain why the process $X = (X_t : t \geq 0)$ is Gaussian.
Find its mean $E[X_t]$ and covariance $E[X_s X_t]$ in terms of α. **[8]**

b) Let $(X, Y, Z) = (x, y, z) + (B^{(1)}, B^{(2)}, B^{(3)})$ be a three-dimensional Brownian motion started at $(x, y, z) \neq 0$.
Show that $R = \sqrt{X^2 + Y^2 + Z^2}$ and $\hat{Z} = \frac{Z}{R}$ solve the equations

$$\begin{aligned} dR &= \frac{1}{R} dt + dW^{(1)}, \\ d\hat{Z} &= \frac{\hat{Z}}{R} dt + \frac{\sqrt{1 - \hat{Z}^2}}{R} dW^{(2)}, \end{aligned}$$

for a two dimensional Brownian motion $W = (W^{(1)}, W^{(2)})$. **[12]**

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3. a) Suppose X takes values in $[0, 1]$ and solves the equation

$$dX = -X dt + \sqrt{X(1 - X)} dB, \qquad X_0 = x.$$

Calculate the moments $m_1(t) = E[X_t]$ and $m_2(t) = E[X_t^2]$. **[8]**

b) Suppose $(B_t : t \geq 0)$ is a one dimensional Brownian motion. Give the proof of pathwise uniqueness for the one-dimensional equation $dX = \mu(X)dt + \sigma(X)dB_t$, with $X_0 = x_0$, where $\mu, \sigma : \mathbb{R} \to \mathbb{R}$ are Lipshitz functions. (You may quote Gronwall's Lemma without proof.) **[8]**

Explain briefly how to show that the equation

$$dX = (X - X^3)\, dt + dB, \quad X_0 = 0,$$

has a pathwise unique solution $(X_t : t \geq 0)$. **[4]**

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4. a) Suppose X solves the equation

$$dX = \mu(X)dt + dB, \qquad X_0 = x$$

where $\mu : \mathbb{R}^d \to \mathbb{R}$ is bounded and continuous and B is a d-dimensional Brownian motion. Suppose $D \subseteq \mathbb{R}^d$ is a bounded open domain and $x \in D$. Let τ be the exit time $\tau = \inf\{t : X_t \in \partial D\}$. Your may assume that $P[\tau < \infty] = 1$.

Suppose $h \geq 0$ and that $u \in C^2(D) \cap C(\overline{D})$ solves

$$\frac{1}{2}\Delta u + \sum_j \mu_j \frac{\partial u}{\partial x_j} = -h \text{ on } D, \qquad u = 0 \text{ for } x \in \partial D.$$

Give a proof of the probabilistic representation **[12]**

$$u(x) = \mathbb{E}\left[\int_0^\tau h(X_s)ds\right].$$

b) In the one dimensional case where $D = (0,1)$ and where $X_t = x + \mu t + B_t$, for some constant $\mu > 0$, find $E[\tau]$ explicitly. **[8]**

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5. Consider the SDE $dX = (1 - 2X)dt + XdB$ on $(0, \infty)$.

a) Find the invariant measure $\nu(dx)$ (you may leave an unknown normalisation constant in your answer).

Which moments $\int_0^\infty x^p \nu(dx)$ (for $p > 0$) are finite? **[6]**

b) Show that, when the SDE starts according to the invariant measure, **[8]**

$$\frac{1}{t}\int_0^t X_s^2 ds \to \frac{1}{3} \quad \text{in } L^1$$

as $t \to \infty$.

c) State the DDS time change rule for stochastic integrals.

Explain how to prove the central limit theorem:

$$\frac{\int_0^t X_s^2 ds - \frac{t}{3}}{\sqrt{t}} \to N(0, \sigma^2) \quad \text{in distribution}$$

(you do not need to calculate the variance σ^2 exactly). **[6]**
