MA 482

THE UNIVERSITY OF WARWICK

FOURTH YEAR EXAMINATION: 2019

Stochastic Analysis

Time Allowed: **3 hours**

Read carefully the instructions on the answer book and make sure that the particulars required are entered on each answer book.

Calculators are not needed and are not permitted in this examination.

Candidates should answer COMPULSORY QUESTION 1 and THREE QUESTIONS out of the four optional questions 2, 3, 4 and 5.

The compulsory question is worth 40% of the available marks. Each optional question is worth 20%.

If you have answered more than the compulsory Question 1 and three optional questions, you will only be given credit for your QUESTION 1 and THREE OTHER best answers.

The numbers in the margin indicate approximately how many marks are available for each part of a question.

For all questions, when not stated otherwise, you may suppose:

$(\Omega, \mathcal{F}, P)$ is a probability space,
$(\mathcal{F}_t : t \geq 0)$ is a filtration,
$(B_t : t \geq 0)$ is a d-dimensional $(\mathcal{F}_t)$ Brownian motion.

COMPULSORY QUESTION

1. a) What does it mean that $(\mathcal{F}_t : t \geq 0)$ is a filtration on $(\Omega, \mathcal{F}, P)$?

What does it mean that $(h_t : t \geq 0)$ is adapted to the filtration $(\mathcal{F}_t : t \geq 0)$? [2]

Consider a simple bounded and adapted integrand defined by

$$h_t = \sum_{j=0}^{n-1} H_j I(t \in (\tfrac{j}{n}, \tfrac{j+1}{n}]) \text{ for } t \in [0,1].$$

How is the integral $\int_0^1 h_s dB_s$ defined?

Give the proof, for this integrand, of the isometry [8]

$$\mathbb{E}\left[\left(\int_0^1 h_s dB_s\right)^2\right] = \mathbb{E}\left[\int_0^1 h_s^2 ds\right].$$

b) If $(B_t : t \geq 0)$ is a d-dimensional $(\mathcal{F}_t)$ Brownian motion, explain what it means that $X = (X_t : t \geq 0)$ is an *Ito process.*

State Levy's Theorem giving a criterion for an Ito process to be a one dimensional Brownian motion. [4]

For a d-dimensional Brownian motion $B = (B_1, \ldots, B_d)$, where $d \geq 2$, let X be the square of of the radius, namely $X = |B|^2 = B_1^2 + \ldots + B_d^2$.

Develop dX by Ito's formula.

Show that X solves the SDE $dX = 2\sqrt{X}dW + d\,dt$. [6]

c) (i) Find the scale function for the SDE $dX = \left(\frac{1}{X} - 1\right)dt + dB$ on $(0, \infty)$.
Describe the behaviour of the process X on $(0, \infty)$. [5]

(ii) Using Feller's exit analysis, describe how the solution to the SDE

$$dX = \sqrt{X^2(1-X)}dB, \qquad X_0 = x \in (0,1)$$

exits the interval $(0,1)$. [5]

d) Consider the SDE $dX = \mu(X)dt + \sigma(X)dB$.

What does it mean that this SDE has *pathwise uniqueness*?

What does it mean that this SDE has *uniqueness in law*? [3]

State Girsanov's Theorem and explain how it can be used to construct a solution of the equation

$$dX_t = h(t, X_t)dt + dB_t, \qquad X_0 = 0, \quad t \in [0,1],$$

when $h : [0, \infty) \times \mathbb{R} \to \mathbb{R}$ is bounded and continuous. [4]

Explain why uniqueness in law holds for this equation. [3]

OPTIONAL QUESTIONS

2. a) Suppose $0 < s < t$ and Z is a bounded $\mathcal{F}_s$ measurable variable. Prove for a continuous adapted integrand h that

$$\int_s^t Z h_r dB_r = Z \int_s^t h_r dB_r.$$

[10]

b) Find the solution to $dX = (c - \mu X)dt + \sigma dB$ with $X_0 = x$.
Find the limit in distribution $\lim_{t\to\infty} X_t$. [10]

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3. a) Suppose h is a continuous adapted integrand, bounded by a constant C. Show that

$$\mathbb{E}\left[\left|\int_0^t h_r dB_r\right|^2\right] \leq C^2 t \quad \text{and} \quad \mathbb{E}\left[\left|\int_0^t h_r dB_r\right|^4\right] \leq 3C^4t^2.$$

[10]

b) Consider the SDE

$$dX_t = \frac{1 - X_t}{1 - t} dt + dB, \ X_0 = 0 \quad \text{for } t \in [0, 1).$$

Expand $d(X_t/(1-t))$ using Ito's formula and deduce a formula for X_t.
What is the DDS time change representation for X_t?
Show that $X_t \to 1$ almost surely as $t \uparrow 1$. [10]

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4. a) Suppose $X = x + B$ is a d-dimensional Brownian motion started at x and that $D \subseteq \mathbb{R}^d$ is a bounded open domain. Let τ be the exit time $\tau = \inf\{t : X_t \in \partial D\}$. Suppose $h \geq 0$ and that $u \in C^2(D) \cap C(\overline{D})$ solves

$$\frac{1}{2}\Delta u = hu \ \text{ on } D \text{ and } u = f \text{ on } \partial D$$

Give a proof of the probabilistic representation [10]

$$u(x) = \mathbb{E}\left[f(X_\tau)e^{-\int_0^\tau h(X_s)ds}\right].$$

b) Suppose $X = x + B$ is a one dimensional Brownian motion started at x. Let $\tau_a = \inf\{t : X_t = a\}$ and $\tau = \min\{\tau_a, \tau_b\}$. For $a < x < b$ show that

$$\mathbb{E}[e^{-\lambda\tau} I(\tau_b < \tau_a)] = \sinh(\sqrt{2\lambda}(x-a))/\sinh(\sqrt{2\lambda}(b-a)).$$

Deduce a formula for the conditional expectation $\mathbb{E}[\tau_b | \tau_b < \tau_a]$. [10]

Question 5 continued overleaf

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5. a) What is an invariant measure for the SDE $dX = \mu(X)dt + \sigma(X)dW$?

For a one dimensional equation, give the formula for the adjoint $\mathcal{L}^*$ of the infinitesimal generator. Explain how to use $\mathcal{L}^*$ to find an invariant measure. [6]

b) Consider the SDE $dX = (2 - X)dt + \sqrt{2X}dB$ on $(0, \infty)$.

(i) Find the invariant measure. [5]

(ii) Show that, when the SDE starts according to the invariant measure, [9]

$$\frac{1}{t}\int_0^t X_s ds \to 2 \quad \text{and} \quad \frac{\int_0^t X_s ds - 2t}{\sqrt{t}} \to N(0,4)$$

in distribution as $t \to \infty$.

END