Applications of Stochastic Calculus in Finance Chapter 7: Structural approach

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1 First Passage Time for Brownian Motion with Drift

Many structural models for credit risk are solvable in terms of the first passage time for Brownian motion with drift. Let

$$\overline{X}_t^\mu = \sup_{s\leq t} X_s^\mu$$

with $X_t^\mu = W_t + \mu t$ where W is a one-dimensional Brownian motion and μ is a constant drift. We are interested in the distributions of $(\overline{X}_t^\mu, X_t^\mu)$ and $\overline{X}_t^\mu$.

Let us first consider the case $\mu = 0$. Then $\overline{X}_t^0 = \overline{W}_t$ and $X_t^0 = W_t$. The key idea is the reflection principle for Brownian motion.

Theorem 1. *(Reflection principle) For $b \geq a$ and $b > 0$,*

$$\mathbf{P}\{\overline{W}_t \geq b, W_t \leq a\} = \mathbf{P}\{W_t \geq 2b - a\}$$

Proof. Define the first passage time τ^b as

$$\tau^b := \inf\{t \geq 0 : W_t \geq b\}.$$

Introduce a new process $\tilde{W}$ reflected at τ^b,

$$\tilde{W}_t := W_t^{\tau^b} - (W_t - W_t^{\tau^b}),\ t \geq 0.$$

We first show that $\tilde{W}$ is also a BM.

Indeed, the strong Markov property implies that

$$S_t := W_{\tau^b + t} - W_{\tau^b} = (W \circ \theta_{\tau^b})_t - b,\ t \geq 0,$$

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is a BM independent of $\mathscr{F}_{\tau^b}$. By symmetric property of BM, $-S$ is also a BM. Note that the first passage time τ^b and stopped process W^{τ^b} are both $\mathscr{F}_{\tau^b}$-measurable, from which we know that

$$(\tau^b, W_t^{\tau^b}, S_t) \stackrel{d}{=} (\tau^b, W_t^{\tau^b}, -S_t).$$

Moreover, note that

$$S_{(t-\tau^b)^+} = W_{(t-\tau^b)^+ + \tau^b} - W_{\tau^b} = W_t - W_t^{\tau^b}.$$

Hence, $W_t = W_t^{\tau^b} + S_{(t-\tau^b)^+}$ and $\tilde{W}_t = W_t^{\tau^b} - S_{(t-\tau^b)^+}$, $t \geq 0$, as respective functionals of $(\tau^b, W_t^{\tau^b}, S_t)$ and $(\tau^b, W_t^{\tau^b}, -S_t)$, have the same distribution, from which we conclude that $\tilde{W}$ is also a BM.

Next, we observe that

$$\begin{aligned}\{W_t \leq a\} &= \{\tilde{W}_t \geq 2b - a\}, \\ \{\overline{W}_t \geq b\} &= \{\tau^b \leq t\} = \{\tilde{\tau}^b \leq t\},\end{aligned}$$

where $\tilde{\tau}^b := \inf\{t \geq 0 : \tilde{W}_t \geq b\} = \tau^b$ as $\tilde{W}_t = W_t$ for $t \leq \tau^b$. Hence,

$$\begin{aligned}\mathbf{P}\{\overline{W}_t \geq b, W_t \leq a\} &= \mathbf{P}\{\tilde{\tau}^b \leq t, \tilde{W}_t \geq 2b - a\} \\ &= \mathbf{P}\{\tilde{W}_t \geq 2b - a\} = \mathbf{P}\{W_t \geq 2b - a\},\end{aligned}$$

where we used the assumption that $b \geq a$ so $\{\tilde{W}_t \geq 2b - a\} \subset \{\tilde{\tau}^b \leq t\}$. $\square$

Proposition 1. *For $b \geq a$ and $b > 0$, $(\overline{W}_t, W_t)$ has the joint distribution*

$$\mathbf{P}(\overline{W}_t \leq b, W_t \leq a) = \Phi(\frac{a}{\sqrt{t}}) - \Phi(\frac{a-2b}{\sqrt{t}})$$

with the joint density

$$f_{\overline{W}_t, W_t}(b, a) = \frac{2(2b-a)}{t\sqrt{2\pi t}} e^{-\frac{(2b-a)^2}{2t}},$$

and $\overline{W}_t$ has the distribution

$$\mathbf{P}(\overline{W}_t \leq b) = \Phi(\frac{b}{\sqrt{t}}) - \Phi(\frac{-b}{\sqrt{t}})$$

with the density

$$f_{\overline{W}_t}(b) = \frac{2}{\sqrt{2\pi t}} e^{-\frac{b^2}{2t}}.$$

Proof. Using the reflection principle for Brownian motion, we obtain that

$$\begin{aligned}\mathbf{P}(\overline{W}_t \le b, W_t \le a) &= \mathbf{P}(W_t \le a) - \mathbf{P}(\overline{W}_t \ge b, W_t \le a) \\ &= \mathbf{P}(W_t \le a) - \mathbf{P}(W_t \ge 2b - a) \\ &= \mathbf{P}\left(\frac{W_t}{\sqrt{t}} \le \frac{a}{\sqrt{t}}\right) - \mathbf{P}\left(\frac{W_t}{\sqrt{t}} \le \frac{a-2b}{\sqrt{t}}\right),\end{aligned}$$

and since $\{W_t \ge b\} \subset \{\overline{W}_t \ge b\}$,

$$\begin{aligned}\mathbf{P}(\overline{W}_t \ge b) &= \mathbf{P}(\overline{W}_t \ge b, W_t \le b) + \mathbf{P}(\overline{W}_t \ge b, W_t \ge b) \\ &= \mathbf{P}(W_t \ge 2b - b) + \mathbf{P}(W_t \ge b) \\ &= 2\mathbf{P}(W_t \ge b).\end{aligned}$$

Then

$$\begin{aligned}\mathbf{P}(\overline{W}_t \le b) &= 1 - 2\mathbf{P}(W_t \ge b) \\ &= \mathbf{P}(W_t \le b) - \mathbf{P}(W_t \ge b) \\ &= \mathbf{P}\left(\frac{W_t}{\sqrt{t}} \le \frac{b}{\sqrt{t}}\right) - \mathbf{P}\left(\frac{W_t}{\sqrt{t}} \le \frac{-b}{\sqrt{t}}\right).\end{aligned}$$

The densities follow from differentiating the respective distribution functions. □

Remark 1. The reflection principle also implies that $\overline{W}_t \overset{d}{=} |W_t|$ for any fixed $t \ge 0$. Indeed,

$$\begin{aligned}\mathbf{P}(\overline{W}_t \ge b) &= 2\mathbf{P}(W_t \ge b) \\ &= \mathbf{P}(W_t \ge b) + \mathbf{P}(W_t \le -b) = \mathbf{P}(|W_t| \ge b).\end{aligned}$$

However, as stochastic processes, $\overline{W}$ and $|W|$ behave differently as $\overline{W}$ is *non-decreasing* and $|W|$ is a *one-dimensional Bessel process*.

The case with the drift $\mu \neq 0$ can be handled by using the Girsanov's theorem.

Proposition 2. *For $b \ge a$ and $b > 0$, $(\overline{X}_t^\mu, X_t^\mu)$ has the joint distribution*

$$\mathbf{P}(\overline{X}_t^\mu \le b, X_t^\mu \le a) = \Phi(\frac{a - \mu t}{\sqrt{t}}) - e^{2\mu t}\Phi(\frac{a - 2b - \mu t}{\sqrt{t}})$$

with the joint density

$$f_{\overline{X}_t^\mu, X_t^\mu}(b,a) = \frac{2(2b-a)}{t\sqrt{2\pi t}} e^{-\frac{(2b-a)^2}{2t}} e^{\mu a - \frac{1}{2}\mu^2 t},$$

and $\overline{X}_t^\mu$ has the distribution

$$\mathbf{P}(\overline{X}_t^\mu \le b) = \Phi(\frac{b - \mu t}{\sqrt{t}}) - e^{2b\mu}\Phi(\frac{-b - \mu t}{\sqrt{t}})$$

with the density

$$f_{\overline{W}_t}(b) = \frac{2}{\sqrt{2\pi t}} e^{-\frac{(b-\mu t)^2}{2t}} - 2\mu e^{2\mu t} \Phi(\frac{-b-\mu t}{\sqrt{t}}).$$

Proof. Define a new probability measure $\mathbf{Q} \sim \mathbf{P}$ by the Radon-Nikodym density

$$\left.\frac{d\mathbf{Q}}{d\mathbf{P}}\right|_{\mathscr{F}_t} = \mathscr{E}(-\mu W)_t = e^{-\mu W_t - \frac{1}{2}\mu^2 t}.$$

By Girsanov's theorem, $X_t^\mu = W_t + \mu t$ is a Brownian motion under $\mathbf{Q}$. Then

$$\begin{aligned}
\mathbf{P}(\overline{X}_t^\mu \le b, X_t^\mu \le a) &= \mathbf{E}\left[\mathbf{1}_{\{\overline{X}_t^\mu \le b, X_t^\mu \le a\}}\right] \\
&= \mathbf{E}^{\mathbf{Q}}\left[\left.\frac{d\mathbf{P}}{d\mathbf{Q}}\right|_{\mathscr{F}_t} \mathbf{1}_{\{\overline{X}_t^\mu \le b, X_t^\mu \le a\}}\right] \\
&= \mathbf{E}^{\mathbf{Q}}\left[e^{\mu W_t + \frac{1}{2}\mu^2 t} \mathbf{1}_{\{\overline{X}_t^\mu \le b, X_t^\mu \le a\}}\right] \\
&= \mathbf{E}^{\mathbf{Q}}\left[e^{\mu X_t^\mu - \frac{1}{2}\mu^2 t} \mathbf{1}_{\{\overline{X}_t^\mu \le b, X_t^\mu \le a\}}\right] \\
&= \int_{x \le a} dx \int_{y \le b} dy\, e^{\mu x - \frac{1}{2}\mu^2 t} \left(\frac{2(2y-x)}{t\sqrt{2\pi t}} e^{-\frac{(2y-x)^2}{2t}} \mathbf{1}_{\{y \ge x, y > 0\}}\right)
\end{aligned}$$

which yields the joint distribution and density of $(\overline{X}_t^\mu, X_t^\mu)$. Similarly,

$$\begin{aligned}
\mathbf{P}(\overline{X}_t^\mu \le b) &= \mathbf{E}\left[\mathbf{1}_{\{\overline{X}_t^\mu \le b\}}\right] \\
&= \mathbf{E}^{\mathbf{Q}}\left[\left.\frac{d\mathbf{P}}{d\mathbf{Q}}\right|_{\mathscr{F}_t} \mathbf{1}_{\{\overline{X}_t^\mu \le b\}}\right] \\
&= \mathbf{E}^{\mathbf{Q}}\left[e^{\mu X_t^\mu - \frac{1}{2}\mu^2 t} \mathbf{1}_{\{\overline{X}_t^\mu \le b\}}\right] \\
&= \int_{\mathbb{R}} dx \int_{y \le b} dy\, e^{\mu x - \frac{1}{2}\mu^2 t} \left(\frac{2(2y-x)}{t\sqrt{2\pi t}} e^{-\frac{(2y-x)^2}{2t}} \mathbf{1}_{\{y \ge x, y > 0\}}\right)
\end{aligned}$$

which yields the distribution and density of $\overline{X}_t^\mu$. □

2 Structural Approach

There are mainly two approaches widely used: structural (or firm value) approach and intensity-based (or reduced-form) approach. In structural approach, default is determined from the evolution of the firm's structural variables such as assets and liabilities, and occurs if the firm's assets are insufficient according to some measure.

2.1 Merton's model

Consider a firm issuing a corporate bond which pays K at maturity T, and the firm's asset value follows

$$dV_t = V_t(r dt + \sigma_V dW_t)$$

on a filtered probability space $(\Omega, \mathscr{F}, \{\mathscr{F}_t\}_{t\geq 0}, \mathbf{Q})$ supporting a one-dimensional Brownian motion W. We assume that $\mathbf{Q}$ is *a* risk-neutral probability measure.

Table 1: balance sheet of the firm

	Assets,	Liabilities/Equities	
at time t	V_t	P_t	Liability
		E_t	Equity
Total	V_t	V_t	

At maturity T, the bond holder will receive K if $V_T \geq K$. Otherwise, she will take over the firm and get V_T if $V_T < K$. Hence, the payoff of such a corporate bond is

$$\min(V_T, K) = K - (K - V_T)^+.$$

That is, the payoff is a long position of K units of T-bonds and a short position of a put option written on the firm assets V with strike price K.

On the other hand, the equity holder will receive

$$V_T - \min(V_T, K) = (V_T - K)^+,$$

so it is a call option with the underlying V and strike price K.

Assumption 1 *The firm assets V can be traded in the market*[1].

Under the above assumption, we can adapt the Black-Scholes model to price the corporate bond and the equity.

Proposition 3. *The value of the corporate bond at time* 0 *is*

$$P_0 = V_0 \Phi(-d_1) + Ke^{-rT}\Phi(d_2),$$

and the value of the equity is

$$E_0 = V_0 \Phi(d_1) - Ke^{-rT}\Phi(d_2),$$

where

$$d_{1,2} = \frac{\ln \frac{V_0}{K} + (r \pm \frac{1}{2}\sigma_V^2)T}{\sigma_V \sqrt{T}}.$$

[1] See Liang and Jiang [1] on how to relax such an unreasonable assumption via utility indifference valuation.

Proof. The risk-neutral pricing formula yields

$$\begin{aligned}
P_t &= \mathbf{E}^{\mathbf{Q}}\left[\frac{\min(V_T,K)}{B_T}B_t|\mathscr{F}_t\right] \\
&= \mathbf{E}^{\mathbf{Q}}[e^{-r(T-t)}(K-(K-V_T)^+)|\mathscr{F}_t] \\
&= Ke^{-r(T-t)} - \mathbf{E}^{\mathbf{Q}}[e^{-r(T-t)}(K-V_T)^+|\mathscr{F}_t],
\end{aligned}$$

and

$$\begin{aligned}
E_t &= \mathbf{E}^{\mathbf{Q}}\left[\frac{V_T-\min(V_T,K)}{B_T}B_t|\mathscr{F}_t\right] \\
&= \mathbf{E}^{\mathbf{Q}}[e^{-r(T-t)}(V_T-K)^+|\mathscr{F}_t].
\end{aligned}$$

The results then follow from the Black-Scholes formula, and the fact that $\Phi(-d_2)+\Phi(d_2)=1$. $\square$

Based on Merton's model, we can calculate the default probability of the corporate bond. The default time τ is given as

$$\tau = \begin{cases} T, & \text{if } V_T < K; \\ \infty, & \text{otherwise.} \end{cases}$$

Hence, the default probability of such a corporate bond at time 0 is

$$\mathbf{Q}(\tau=T) = \mathbf{Q}(V_T<K) = \Phi\left(-\frac{\ln\frac{V_0}{K}+(r-\frac{1}{2}\sigma_V^2)T}{\sigma_V\sqrt{T}}\right).$$

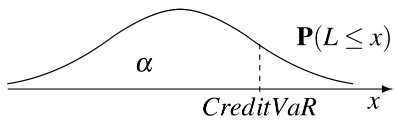
Remark 2. (Calibration of the volatility σ_V). From the above calculation of the default probability, we can also recover the volatility σ_V of the firm assets by solving the following quadratic equation:

$$\frac{1}{2}T\sigma_V^2 - \Phi^{-1}(\mathbf{Q}(\tau\le T))\sqrt{T}\sigma_V - rT - \ln\frac{V_0}{K} = 0. \tag{1}$$

Remark 3. (Credit VaR) A widely used risk measure for credit risk is Credit VaR, which is defined analogously to the way one defines VaR for market risk. Given some confidence level $\alpha\in(0,1)$, *Credit VaR of a loss L at confidence level α* is the smallest number x such that the probability that the loss L controlled by x is larger than or equal to α, i.e. *the left-continuous inverse of CDF $F_L(x)=\mathbf{P}(L\le x)$*,

$$CreditVaR_{\alpha}(L) = \inf\{x : \mathbf{P}(L\le x)\ge\alpha\},$$

where $\mathbf{P}$ is the physical probability measure (NOT the risk-neutral probability measure $\mathbf{Q}$ as it is about risk management).



2.2 First-passage-time model

One drawback of the above Merton's model is that the firm's asset value V can dwindle to zero without triggering the default before maturity T. This is unfavorable to the bondholder. Bond indenture provisions often include safety covenants that give the bondholder the right to reorganize the firm if its asset value falls below some default barrier. This is the so called *first-passage-time model*.

Let $D_t = Ke^{-d(T-t)}$ be an exogenous default barrier, then the default time is given by the first passage time of D_t by V_t:

$$\tau = \inf\{t \geq 0 : V_t \leq D_t\} = \inf\{t \geq 0 : \frac{V_t}{D_t} \leq 1\}.$$

Hence, the payoff is $\mathbf{1}_{\{\tau>T\}}K + \mathbf{1}_{\{\tau\leq T\}}V_\tau$. This can be compared with *down-and-out barrier options*.

Proposition 4. *Under the first-passage-time model, the default probability at time* 0 *is*

$$\mathbf{Q}(\tau \leq t) = \Phi\left(-\frac{\ln\frac{V_0}{Ke^{-dT}} + (r - \frac{1}{2}\sigma_V^2 - d)t}{\sigma_V\sqrt{t}}\right) + \left(\frac{V_0}{Ke^{-dT}}\right)^{-\frac{2}{\sigma_V^2}(r-\frac{1}{2}\sigma_V^2-d)} \Phi\left(-\frac{\ln\frac{V_0}{Ke^{-dT}} - (r - \frac{1}{2}\sigma_V^2 - d)t}{\sigma_V\sqrt{t}}\right)$$

for $t \leq T$.

Proof. First note that $\{\tau \leq t\} = \{\inf_{s\leq t}(\frac{V_s}{D_s}) \leq 1\}$ and

$$\frac{V_t}{D_t} = \frac{V_0}{Ke^{-dT}} e^{\sigma_V W_t^{\mathbf{Q}} + (r-\frac{1}{2}\sigma_V^2-d)t}.$$

for the Brownian motion $W^{\mathbf{Q}}$ under the spot measure $\mathbf{Q}$. Hence,

$$
\begin{aligned}
\mathbf{Q}(\tau \le t) =&\mathbf{Q}(\inf_{s\le t}(\frac{V_s}{D_s}) \le 1)\\
=&\mathbf{Q}\left(\inf_{s\le t}(\frac{V_0}{Ke^{-dT}}e^{\sigma_V W_s^{\mathbf{Q}}+(r-\frac{1}{2}\sigma_V^2-d)s}) \le 1\right)\\
=&\mathbf{Q}\left(\sup_{s\le t}(-W_s^{\mathbf{Q}}-\frac{1}{\sigma_V}(r-\frac{1}{2}\sigma_V^2-d)s) \ge \frac{1}{\sigma_V}\ln\frac{V_0}{Ke^{-dT}}\right)\\
=&1-\mathbf{Q}\left(\sup_{s\le t}(-W_s^{\mathbf{Q}}\underbrace{-\frac{1}{\sigma_V}(r-\frac{1}{2}\sigma_V^2-d)}_{\mu}s) \le \underbrace{\frac{1}{\sigma_V}\ln\frac{V_0}{Ke^{-dT}}}_{b}\right)\\
=&1-\Phi\left(\frac{\ln\frac{V_0}{Ke^{-dT}}+(r-\frac{1}{2}\sigma_V^2-d)t}{\sigma_V\sqrt{t}}\right)\\
&+\left(\frac{V_0}{Ke^{-dT}}\right)^{-\frac{2}{\sigma_V^2}(r-\frac{1}{2}\sigma_V^2-d)}\Phi\left(\frac{-\ln\frac{V_0}{Ke^{-dT}}+(r-\frac{1}{2}\sigma_V^2-d)t}{\sigma_V\sqrt{t}}\right),
\end{aligned}
$$

where the last equality follows from the first passage time of Brownian motion with drift in Proposition 2. □

Next, we derive the value of the corporate bond under first-passage-time model.

Proposition 5. *Under the first-passage-time model, the value of the corporate bond at time* 0 *is*

$$
\begin{aligned}
P_0 = V_0\Phi&\left(-\frac{\ln\frac{V_0}{Ke^{-dT}}+(r+\frac{1}{2}\sigma_V^2-d)T}{\sigma_V\sqrt{T}}\right)\\
&+Ke^{-rT}\Phi\left(\frac{\ln\frac{V_0}{Ke^{-dT}}+(r-\frac{1}{2}\sigma_V^2-d)T}{\sigma_V\sqrt{T}}\right)\\
&+V_0\left(\frac{V_0}{Ke^{-dT}}\right)^{-\frac{2}{\sigma_V^2}(r+\frac{1}{2}\sigma_V^2-d)}\Phi\left(-\frac{\ln\frac{V_0}{Ke^{-dT}}-(r+\frac{1}{2}\sigma_V^2-d)T}{\sigma_V\sqrt{T}}\right)\\
&-Ke^{-rT}\left(\frac{V_0}{Ke^{-dT}}\right)^{-\frac{2}{\sigma_V^2}(r-\frac{1}{2}\sigma_V^2-d)}\Phi\left(-\frac{\ln\frac{V_0}{Ke^{-dT}}-(r-\frac{1}{2}\sigma_V^2-d)T}{\sigma_V\sqrt{T}}\right)
\end{aligned}
$$

for $t \le T$.

Proof. First note that the price of the corporate bond at any time $t \le T$ is

$$P_t = \mathbf{E}^{\mathbf{Q}}\left[\mathbf{1}_{\{\tau>T\}}\frac{KB_t}{B_T}+\mathbf{1}_{\{\tau\le T\}}\frac{V_\tau B_t}{B_\tau}|\mathscr{F}_t\right].$$

We only need to evaluate the following two expectations:

$$\begin{aligned}
\mathbf{E}^{\mathbf{Q}}[\mathbf{1}_{\{\tau>T\}}\frac{K}{B_T}] &= Ke^{-rT}\mathbf{Q}(\tau>T)\\
&= Ke^{-rT}\Phi\left(\frac{\ln\frac{V_0}{Ke^{-dT}}+(r-\frac{1}{2}\sigma_V^2-d)T}{\sigma_V\sqrt{T}}\right)\\
&\quad -Ke^{-rT}\left(\frac{V_0}{Ke^{-dT}}\right)^{-\frac{2}{\sigma_V^2}(r-\frac{1}{2}\sigma_V^2-d)}\Phi\left(-\frac{\ln\frac{V_0}{Ke^{-dT}}-(r-\frac{1}{2}\sigma_V^2-d)T}{\sigma_V\sqrt{T}}\right),
\end{aligned}$$

which follows from Proposition 4, and

$$\begin{aligned}
\mathbf{E}^{\mathbf{Q}}[\mathbf{1}_{\{\tau\le T\}}\frac{V_\tau}{B_\tau}] =& V_0\mathbf{E}^{\mathbf{Q}}[\mathbf{1}_{\{\tau\le T\}}\frac{V_\tau}{B_\tau V_0}]\\
=& V_0\mathbf{E}^{\mathbf{Q}^V}[\mathbf{1}_{\{\tau\le T\}}],
\end{aligned}$$

where $\mathbf{Q}^V\sim\mathbf{Q}$ is defined via the Radon-Nikodym density

$$\left.\frac{d\mathbf{Q}^V}{d\mathbf{Q}}\right|_{\mathscr{F}_t} = \mathscr{E}(\sigma_V W^{\mathbf{Q}})_t = \frac{V_t}{B_t V_0}.$$

By Girsanov's theorem, $W_t^{\mathbf{Q}^V} = W_t^{\mathbf{Q}} - \sigma_V t$ is a Brownian motion under $\mathbf{Q}^V$. Hence, using the same argument as in Proposition 4, we obtain that

$$\begin{aligned}
\mathbf{Q}^V(\tau\le T) =& \mathbf{Q}^V(\inf_{s\le T}(\frac{V_s}{D_s})\le 1)\\
=& \mathbf{Q}^V\left(\inf_{s\le T}(\frac{V_0}{Ke^{-dT}}e^{\sigma_V W_s^{\mathbf{Q}^V}+(r+\frac{1}{2}\sigma_V^2-d)s})\le 1\right)\\
=& \Phi\left(-\frac{\ln\frac{V_0}{Ke^{-dT}}+(r+\frac{1}{2}\sigma_V^2-d)T}{\sigma_V\sqrt{T}}\right)\\
&+\left(\frac{V_0}{Ke^{-dT}}\right)^{-\frac{2}{\sigma_V^2}(r+\frac{1}{2}\sigma_V^2-d)}\Phi\left(-\frac{\ln\frac{V_0}{Ke^{-dT}}-(r+\frac{1}{2}\sigma_V^2-d)T}{\sigma_V\sqrt{T}}\right). \quad \square
\end{aligned}$$

3 Exercises

Exercise 1. (Factor structural model for portfolio credit risk)
Suppose there are M firms and the assets value of the ith firm follows

$$dV_t^i = V_t^i(rdt+\sigma_i dW_t^i)$$

for $i=1,2,\ldots,M$, with

$$\mathbf{E}^{\mathbf{Q}}[dW_t^i dW_t^j] = \rho_i\rho_j dt.$$

where $\rho_i\in[-1,1]$ and $\rho_i^2=1$.

Since $X^i := W_T^i/\sqrt{T}$ is standard normal with correlation $\mathbf{E}^{\mathbf{Q}}[X^i X^i] = \rho_i \rho_j$, we can introduce the following factor decomposition

$$X^i = \rho_i \xi + \sqrt{1-\rho_i^2}\,\xi^i.$$

where $(\xi^1, \ldots, \xi^M, \xi)$ are iid standard normal.

Based on the above factor decomposition, show the joint default probability in Merton's model has the form

$$\mathbf{Q}\left(V_T^1 < K_1, \cdots, V_T^M < K_M\right) = \int \prod_{i=1}^{M} \Phi\left(\frac{-d_i - \rho_i x}{\sqrt{1-\rho_i^2}}\right) \phi(x)dx$$

where $\Phi(\cdot)$ and $\phi(\cdot)$ are the CDF and density of standard normal, respectively, and

$$d_i = \frac{\ln \frac{V_0^i}{K_i} + \left(r - \sigma_i^2/2\right)T}{\sigma_i \sqrt{T}}.$$

References

1. Liang, Gechun, and Lishang Jiang. A modified structural model for credit risk. *IMA Journal of Management Mathematics*. 23(2) (2012): 147-170.