Ex 1 (1) Since both $b(t,r)=a-br$ and $\sigma^2(t,r)=\sigma^2$ are affine in r, Vasicek model provides ATS, i.e. $P(t,T)=e^{-A(t)-B(t)r_t}$ #

(2) The functions $A(\cdot)$ and $B(\cdot)$ solve ODE:

$$\begin{cases} \frac{dA(t)}{dt} = \frac{1}{2}\sigma^2 B^2(t) - a B(t) \\ \frac{dB(t)}{dt} = b B(t) - 1 \end{cases} \Rightarrow \begin{cases} B(t) = -\frac{1}{b}\left(e^{-b(T-t)}-1\right) \\ A(t) = \int_t^T \left[a B(s) - \frac{1}{2}\sigma^2 B^2(s)\right] ds \end{cases}$$ #

(3) Recall $f(t,T) = -\partial_T \ln P(t,T) = \partial_T A(t) + \partial_T B(t)\cdot r_t$

From (2), we have,

$$\partial_T A(t) = a B(T) - \frac{1}{2}\sigma^2 B^2(T) + \int_t^T \left[a \partial_T B(s) - \sigma^2 B(s)\partial_T B(s)\right] ds$$

$$= \int_t^T \left[a e^{-b(T-s)} + \frac{\sigma^2}{b}\left(e^{-b(T-s)}-1\right)e^{-b(T-s)}\right] ds$$

$$\partial_T B(t) = e^{-b(T-t)}$$

In turn,

$$df(t,T) = d\left(\partial_T A(t)\right) + r_t\, d\left(\partial_T B(t)\right) + \partial_T B(t)\, dr_t$$

$$= \left[-a e^{-b(T-t)} - \frac{\sigma^2}{b}\left(e^{-2b(T-t)} - e^{-b(T-t)}\right)\right] dt$$

$$+ b e^{-b(T-t)} r_t\, dt$$

$$+ e^{-b(T-t)}\left[(a-br_t)dt + \sigma dW_t^{Q}\right]$$

$$= \frac{\sigma^2}{b}\left(e^{-b(T-t)} - e^{-2b(T-t)}\right)dt + \sigma e^{-b(T-t)} dW_t^{Q}$$ #

(4) Note that $\sigma(t,T) = \sigma e^{-b(T-t)}$

Hence $$\sigma(t,T)\int_t^T \sigma(t,s)\, ds = \sigma e^{-b(T-t)}\int_t^T \sigma e^{-b(T-s)} ds$$

$$= \underbrace{\frac{\sigma^2}{b}\left(e^{-b(T-t)} - e^{-2b(T-t)}\right)}_{\text{drift of } f(t,T) \text{ under } Q.}$$ #

Remark: Note that $f(0,T) = \left(\partial_T A(t) + \partial_T B(t)\cdot r_t\right)\big|_{t=0}$

$$= \int_0^T \left[a e^{-b(T-s)} + \frac{\sigma^2}{b}\left(e^{-2b(T-s)} - e^{-b(T-s)}\right)\right] ds + e^{-bT} r_0$$

Hence not all initial forward rate curve $f(0,T)$ can be fit by Vasicek model

$\int_0 \Big[\cdots + \frac{\cdot}{b} (\cdots - e^{\cdots}) \Big] \cdots$

Hence, not all initial forward rate curve $f(0,T)$ can be fit by Vasicek model. To match an arbitrary initial forward rate curve, the constant a need to extended to a time-dependent function.

Ex 2 (1) HJM drift condition $\alpha^*(t,T) + \frac{1}{2}|\sigma^*(t,T)|^2 = \sigma^*(t,T)\,\Theta_t$

i.e. $-\int_t^T \alpha(t,u)\,du + \frac{1}{2}\left|\int_t^T \sigma(t,u)\,du\right|^2 = -\int_t^T \sigma(t,u)\,du\,\Theta_t$

Differentiate againts T:

$$-\alpha(t,T) + \sigma(t,T)\int_t^T \sigma(t,u)\,du = -\sigma(t,T)\,\Theta_t$$

With $\sigma(t,T) = \sigma e^{-b(T-t)}$, we have

$$-\alpha(t,T) + \sigma e^{-b(T-t)}\int_t^T \sigma e^{-b(u-t)}\,du = -\sigma e^{-b(T-t)}\,\Theta_t$$

$$\Leftrightarrow -\alpha(t,T) + \frac{\sigma^2}{b}\left(e^{-b(T-t)} - e^{-2b(T-t)}\right) = -\sigma e^{-b(T-t)}\,\Theta_t$$

#

(2) $df(t,T) = \alpha(t,T)\,dt + \sigma e^{-b(T-t)}\,dW_t$

$= \alpha(t,T)\,dt + \sigma e^{-b(T-t)}(dW_t^Q - \Theta_t\,dt)$

$= \left(\alpha(t,T) - \sigma e^{-b(T-t)}\Theta_t\right)dt + \sigma e^{-b(T-t)}\,dW_t^Q$

$= \frac{\sigma^2}{b}\left(e^{-b(T-t)} - e^{-2b(T-t)}\right)dt + \sigma e^{-b(T-t)}\,dW_t^Q$

#

(3) $r_t = f(t,t) = f(0,t) + \int_0^t \frac{\sigma^2}{b}\left(e^{-b(t-s)} - e^{-2b(t-s)}\right)ds + \underbrace{\int_0^t \sigma e^{-b(t-s)}\,dW_s^Q}_{=\sigma e^{-bt}\int_0^t e^{bs}\,dW_s^Q}$ $(*)$

Hence,

$dr_t = \Big[\partial_t f(0,t) + \frac{\sigma^2}{b}\left(e^{-b(t-t)} - e^{-2b(t-t)}\right) + \int_0^t \left(-\sigma^2 e^{-b(t-s)} + 2\sigma^2 e^{-2b(t-s)}\right)ds\Big]dt$

$- \sigma b e^{-bt}\int_0^t e^{bs}\,dW_s^Q\,dt + \sigma e^{-bt}e^{bt}\,dW_t^Q$

$= \Big[\partial_t f(0,t) + b f(0,t) + \int_0^t \sigma^2\left(2e^{-2b(t-s)} - e^{-b(t-s)}\right)ds + \int_0^t \sigma^2\left(e^{-b(t-s)} - e^{-2b(t-s)}\right)ds$

$- b\,\underbrace{\left(f(0,t) + \int_0^t \frac{\sigma^2}{b}\left(e^{-b(t-s)} - e^{-2b(t-s)}\right)ds + \int_0^t \sigma e^{-b(t-s)}\,dW_s^Q\right)}_{r_t}\Big]dt$

$+ \sigma\,dW_t^Q$

$= \underbrace{\Big[\partial_t f(0,t) + b f(0,t) + \int_0^t \sigma^2 e^{-2b(t-s)}\,ds}_{a(t) = \partial_t f(0,t) + b f(0,t) + \frac{\sigma^2}{2b} - \frac{\sigma^2}{2b}e^{-2bt}} - b r_t\Big]dt + \sigma\,dW_t^Q$

(time-dependent function) #

$$a(t) = \partial_t f(0,t) + b\, f(0,t) + \frac{\sigma^2}{2b} - \frac{\sigma^2}{2b} e^{-2bt} \quad \text{(time-dependent function)}$$ #

(4) By Proposition 4, $r_t = r_0 + \int_0^t \bar{\alpha}_u \, du + \int_0^t \sigma(u,u)\, dW_u^Q$

where $\bar{\alpha}_u = \alpha(u,u) + \partial_u f(0,u) + \int_0^u \partial_u \alpha(s,u)\, ds + \int_0^u \partial_u \sigma(s,u)\, dW_s^Q$

Since $\alpha(s,u) = \frac{\sigma^2}{b}\left(e^{-b(u-s)} - e^{-2b(u-s)}\right)$, $\sigma(s,u) = \sigma e^{-b(u-s)}$,

$$\bar{\alpha}_u = 0 + \partial_u f(0,u) + \int_0^u \partial_u \left[\frac{\sigma^2}{b}\left(e^{-b(u-s)} - e^{-2b(u-s)}\right)\right] ds + \int_0^u \partial_u \left[\sigma e^{-b(u-s)}\right] dW_s^Q$$

$$= \partial_u f(0,u) + \int_0^u \partial_u \left[\frac{\sigma^2}{b}\left(e^{-b(u-s)} - e^{-2b(u-s)}\right)\right] ds + \int_0^u -b\sigma e^{-b(u-s)}\, dW_s^Q$$

Then. $$r_t = r_0 + \int_0^t \partial_u f(0,u)\, du + \underbrace{\int_0^t \int_0^u \partial_u \left[\frac{\sigma^2}{b}\left(e^{-b(u-s)} - e^{-2b(u-s)}\right)\right] ds\, du}_{①}$$

$$+ \underbrace{\int_0^t \int_0^u -b\sigma e^{-b(u-s)}\, dW_s^Q\, du + \int_0^t \sigma\, dW_u^Q}_{②}.$$

By (Stochastic) Fubini:

For $① = \int_0^t \int_s^t \partial_u \left[\frac{\sigma^2}{b}\left(e^{-b(u-s)} - e^{-2b(u-s)}\right)\right] du\, ds$

$$= \int_0^t \frac{\sigma^2}{b}\left(e^{-b(t-s)} - e^{-2b(t-s)}\right) ds$$

For $② = \int_0^t \int_s^t -b\sigma e^{-b(u-s)}\, du\, dW_s^Q + \int_0^t \sigma\, dW_u^Q$

$$= \int_0^t \sigma\left(e^{-b(t-s)} - 1\right) dW_s^Q + \int_0^t \sigma\, dW_u^Q$$

$$= \int_0^t \sigma e^{-b(t-s)}\, dW_s^Q$$

Hence. $$r_t = \underbrace{f(0,0) + \int_0^t \partial_u f(0,u)\, du}_{f(0,t)} + \int_0^t \frac{\sigma^2}{b}\left(e^{-b(t-s)} - e^{-2b(t-s)}\right) ds + \int_0^t \sigma e^{-b(t-s)}\, dW_s^Q$$

Which is consistent with (*)

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