Week 8(1) Chapter 8.3

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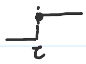
Chapter 8

Example of exponential formula. $a(s) = \mathbb{1}_{\tau \le s} - a^c(s)$ where τ is a r.v.

a^c is absolutely continuous.

$a(s)$ is BV.

$\mathbb{1}_{\tau \le s}$ 

$$Z(t) = Z(0) - \int_0^t Z(s-)\mu(s)\, d a(s)$$

$$\Leftrightarrow Z(t) = Z(0) \prod_{0<s\le t} (1-\mu(s)\Delta a(s))\, e^{-\int_0^t \mu(s) d a^c(s)} = Z(0)\left(\mathbb{1}_{\tau\le t}(1-\mu(\tau)) + \mathbb{1}_{\tau>t}\right) e^{-\int_0^t \mu(s) d a^c(s)}$$

Note that $\Delta a(s) = \begin{cases} 0 & \text{if } s<\tau \\ 1 & \text{if } s=\tau \\ 0 & \text{if } s>\tau \end{cases}$, $\prod_{0<s\le t}(1-\mu(s)\Delta a(s)) = \mathbb{1}_{\tau\le t}(1-\mu(\tau)) + \mathbb{1}_{\tau>t}$

$$\sum_{0<s\le t} g(s)\Delta f(s) = -\sum_{0<s\le t} f(s-)g(s-)\mu(s)\Delta a(s),$$

and therefore,

$$f(t)g(t) = f(0)g(0) - \int_0^t f(s-)g(s-)\mu(s)da(s).$$

That is $f(t)g(t)$, $t \in [0,T]$, is a solution to the equation (8).

The uniqueness of the solution is left as an exercise. □

3 Single jump processes and Girsanov's theorem

3.1 Stopping times

Definition 5. A filtered probability space $(\Omega, \mathscr{F}, \{\mathscr{F}_t\}_{t\ge 0}, \mathbf{P})$ is said to satisfy the usual conditions if the following conditions hold:
(1) completeness: $\mathscr{F}_0$ includes all of the **P**-null sets;
(2) right continuity: $\mathscr{F}_t = \mathscr{F}_{t+}$ where $\mathscr{F}_{t+} = \cap_{n\ge 1}\mathscr{F}_{t+\frac{1}{n}}$. □

For any "reasonable" strong Markov process X (e.g. Feller processes including Levy, Brownian and Poisson processes), its natural filtration $\mathscr{F}_t := \sigma(X_s : s \le t)$ *after augmentation* is right continuous[2].

Definition 6. A random variable $\tau : \Omega \to [0,\infty]$ is called an $\mathscr{F}_t$-stopping time if $\{\tau \le t\} \in \mathscr{F}_t$ for $t \ge 0$. The random variable τ is called an optional time if $\{\tau < t\} \in \mathscr{F}_t$ □

If τ is an $\mathscr{F}_t$-stopping time, then

$$\{\tau < t\} = \cup_{n\ge 1}\{\tau \le t - \frac{1}{n}\} \in \cup_{n\ge 1}\mathscr{F}_{t-\frac{1}{n}} \subset \mathscr{F}_t.$$

However, $\{\tau < t\} \in \mathscr{F}_t$ does not necessarily imply that $\{\tau \le t\} \in \mathscr{F}_t$ unless the filtration is right continuous. To see this,

$$\{\tau \le t\} = \cap_{n\ge 1}\{\tau < t + \frac{1}{n}\} \in \cap_{n\ge 1}\mathscr{F}_{t+\frac{1}{n}},$$

which is $\mathscr{F}_t$ only if $\{\mathscr{F}_t\}_{t\ge 0}$ is right continuous.

Example 3. Let $\{\tau_n\}_{n\ge 1}$ be a sequence of $\mathscr{F}_t$-stopping times. Then,

$$\left\{\sup_{n\ge 1}\tau_n \le t\right\} = \cap_{n\ge 1}\{\tau_n \le t\} \in \mathscr{F}_t,$$

[2] Note that the natural filtration of Poisson processes is right continuous before argumentation, and so are single jump processes.

so $\sup_{n\geq 1}\tau_n$ *is again an $\mathscr{F}_t$-stopping time*. However, since

$$\left\{\inf_{n\geq 1}\tau_n \leq t\right\} = \cap_{m\geq 1}\cup_{n\geq 1}\{\tau_n < t+\frac{1}{m}\} \in \cap_{m\geq 1}\mathscr{F}_{t+\frac{1}{m}},$$

which is $\mathscr{F}_t$ only if $\{\mathscr{F}_t\}_{t\geq 0}$ is right continuous, $\inf_{n\geq 1}\tau_n$ *is an $\mathscr{F}_t$-stopping time only if the filtration is right continuous*.

On the other hand, if τ_n is only optional, then since $\{\inf_{n\geq 1}\tau_n \geq t\} = \cap_{n\geq 1}\{\tau^n \geq t\}$, it follows that

$$\left\{\inf_{n\geq 1}\tau_n < t\right\} = \cup_{n\geq 1}\{\tau^n < t\} \in \mathscr{F}_t,$$

$\inf_{n\geq 1}\tau_n$ is an optional time. □

Definition 7. The past at the stopping time τ is the σ-field $\mathscr{F}_\tau$ defined by

$$\mathscr{F}_\tau = \{A \in \mathscr{F}_\infty : A\cap\{\tau \leq t\} \in \mathscr{F}_t \text{ for } t \geq 0\}$$

The strict past at the stopping time τ is the σ-field $\mathscr{F}_{\tau-}$ generated by the set

$$\mathscr{F}_{\tau-} = \sigma(\{A_0 \in \mathscr{F}_0\}\cup\{A_s\cap\{\tau > s\} \text{ for } s\geq 0, A_s \in \mathscr{F}_s\})$$

Proposition 2. *Both $\mathscr{F}_\tau$ and $\mathscr{F}_{\tau-}$ are σ-fields satisfying $\mathscr{F}_{\tau-} \subset \mathscr{F}_\tau$, and τ is an $\mathscr{F}_{\tau-}$-measurable random variable (therefore also $\mathscr{F}_\tau$-measurable). When X is progressively measurable, X_τ is $\mathscr{F}_\tau$-measurable.*

Proof. The verification of $\mathscr{F}_\tau$ and $\mathscr{F}_{\tau-}$ being σ-fields is by the definition. For example, for $A \in \mathscr{F}_\tau$, $A^c\cap\{\tau\leq t\} = \{\tau \leq t\} - A\cap\{\tau\leq t\}$. Since $\{\tau\leq t\}\in\mathscr{F}_t$ and $A\cap\{\tau\leq t\}\in\mathscr{F}_t$, it follows that $A^c \in \mathscr{F}_\tau$.

To prove that $\mathscr{F}_{\tau-}\subset\mathscr{F}_\tau$, it suffices to show that the generators of $\mathscr{F}_{\tau-}$ are in $\mathscr{F}_\tau$. Indeed, $\mathscr{F}_0 \subset \mathscr{F}_\tau$. For $A_s \in \mathscr{F}_s$,

$$A_s\cap\{\tau > s\}\cap\{\tau\leq t\} = A_s\cap\{s<\tau\leq t\}\in\mathscr{F}_t.$$

The set $\{\tau = 0\}$ and $\{\tau > a\}$, $a\geq 0$, are generators of $\mathscr{F}_{\tau-}$ and therefore τ is $\mathscr{F}_{\tau-}$ measurable.

Finally, we show that X_τ is $\mathscr{F}_\tau$ measurable. For this, for fixed $t\geq 0$, we aim to show that for any Borel set V, $X_\tau^{-1}(V)\cap\{\tau\leq t\}\in\mathscr{F}_t$. Define two maps $X_\tau : \omega \mapsto X_{\tau(\omega)}(\omega)$

$$\phi_t : \{\omega : \tau(\omega)\leq t\} \to [0,t]\times\Omega, \text{ by } \phi_t(\omega) = (\tau(\omega),\omega),$$

$\omega \xmapsto{\phi_t} (\tau(\omega),\omega) \xmapsto{\phi^t} X_{\tau(\omega)}(\omega)$

and

$$\phi^t : [0,t]\times\Omega \to \mathbb{R}^d, \text{ by } \phi^t(s,\omega) = X_s(\omega).$$

Note that $X_\tau = \phi^t\circ\phi_t$. We verify that ϕ_t is $\mathscr{F}_t\cap\{\tau\leq t\}\to\mathscr{B}[0,t]\otimes\mathscr{F}_t$ measurable. Indeed, for $A\in\mathscr{F}_t$ and $a\in[0,t]$, since τ is a stopping time,

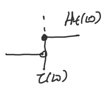
$$\phi_t^{-1}([0,a]\times A) = \{\tau\leq a\}\cap A\subset\{\tau\leq t\}\cap A\in\mathscr{F}_t\cap\{\tau\leq t\}.$$

Together with X being progressively measurable, i.e. ϕ^t is $\mathscr{B}[0,t]\otimes\mathscr{F}_t\to\mathscr{B}(\mathbb{R}^d)$ measurable, we conclude that $X_\tau = \phi^t\circ\phi_t$ is $\mathscr{F}_t\cap\{\tau\leq t\}\to\mathscr{B}(\mathbb{R}^d)$ measurable. Hence,

$$\begin{aligned} X_\tau^{-1}(V)\cap\{\tau\leq t\} &= \{\omega : \tau(\omega)\leq t, X_{\tau(\omega)}(\omega)\in V\} \\ &= \{\omega:\tau(\omega)\leq t, \phi^t\circ\phi_t(\omega)\in V\} \\ &= \{\tau\leq t\}\cap\phi^t\circ\phi_t^{-1}(V)\in\mathscr{F}_t \end{aligned}$$

□

3.2 Single jump processes



Let $\tau : \Omega\to\mathbb{R}_+$ be a non-negative random variable with property $\mathbf{P}(\tau = 0) = 0$ and $\mathbf{P}(\tau > t) > 0$ for any $t\in\mathbb{R}_+$. Introduce the corresponding single jump process $H_t = \mathbf{1}_{\{\tau\leq t\}}$, $t\geq 0$, and its natural filtration $\{\mathscr{F}_t\}_{t\geq 0}$ by

$$\mathscr{F}_t = \sigma(H_u : u\leq t)$$

with $\mathscr{F}_\infty = \sigma(H_u : u\in\mathbb{R}_+)$. It is easy to check the following properties of $\mathscr{F}_t$.

1. $\mathscr{F}_t = \sigma(\{\tau\leq u\} : u\leq t)$;
2. $\mathscr{F}_t = \sigma(\sigma(\tau)\cap\{\tau\leq t\})$;
3. $\mathscr{F}_t = \sigma(\sigma(\tau\wedge t)\cup\{\tau > t\})$;
4. $\mathscr{F}_t = \mathscr{F}_{t+}$;
5. $\mathscr{F}_\infty = \sigma(\tau)$;
6. $A\cap\{\tau\leq t\}\in\mathscr{F}_t$ for any $A\in\mathscr{F}_\infty$.

The following formulas are useful to calculate the conditional distribution of τ. The key point (which may not be obvious at the beginning) is that any $\mathscr{F}_t$ measurable r.v. X_t is of the form $X_t = x_t^0\mathbf{1}_{\{\tau>t\}} + x_t^1(\tau)\mathbf{1}_{\{\tau\leq t\}}$ (called *Jacod's decomposition for optional processes*).

if $\tau\sim\exp(\lambda)$

Lemma 1. *For any random variable $Y\in\mathscr{F}_\infty$,*

$$\mathbf{E}[Y|\mathscr{F}_t] = \frac{\mathbf{E}[\mathbf{1}_{\{\tau>t\}}Y]}{\mathbf{P}(\tau>t)}\mathbf{1}_{\{\tau>t\}} + \mathbf{E}[Y|\sigma(\tau)]\mathbf{1}_{\{\tau\leq t\}}.$$

$E[Y|\mathscr{F}_t] = \mathbb{1}_{\tau>t}E[Y\mathbb{1}_{\tau>t}]e^{\lambda t}$
$+\mathbb{1}_{\tau\leq t}E[Y|\sigma(\tau)]$

Proof. We first prove on $\{\tau\leq t\}$, $\mathbf{E}[Y|\mathscr{F}_t] = \mathbf{E}[Y|\sigma(\tau)]$, i.e.

after default

$$\mathbf{E}[\mathbf{1}_{\{\tau\leq t\}}Y|\mathscr{F}_t] = \mathbf{E}[\mathbf{1}_{\{\tau\leq t\}}Y|\sigma(\tau)]$$

Case 1. $Y = \mathbb{1}_{\tau>T}$

In other words, $\mathbf{E}[\mathbf{1}_{\{\tau\leq t\}}Y|\mathscr{F}_t]$ *is the conditional expectation of $\mathbf{1}_{\{\tau\leq t\}}Y$ on $\sigma(\tau)$*. Indeed, for any $A\in\sigma(\tau)$, $A\cap\{\tau\leq t\}\in\mathscr{F}_t$, it follows that

$E[\mathbb{1}_{\tau>T}|\mathscr{F}_t] = \mathbb{1}_{\tau>t}E[\mathbb{1}_{\tau>T}]e^{\lambda t}$

$$\mathbf{E}[\mathbf{1}_A\mathbf{E}[\mathbf{1}_{\{\tau\leq t\}}Y|\mathscr{F}_t]] = \mathbf{E}[\mathbf{1}_{A\cap\{\tau\leq t\}}\mathbf{E}[Y|\mathscr{F}_t]] = \mathbf{E}[\mathbf{1}_A\mathbf{1}_{\{\tau\leq t\}}Y].$$

$= \mathbb{1}_{\tau>t}e^{-\lambda(T-t)}$.

Case 2 $Y = \mathbb{1}_{\tau\leq T}$

$E[\mathbb{1}_{\tau\leq T}|\mathscr{F}_t] = \mathbb{1}_{\tau>t}E[\mathbb{1}_{t<\tau\leq T}]e^{\lambda t}$
$+\mathbb{1}_{\tau\leq t}E[\mathbb{1}_{\tau\leq T}|\sigma(\tau)]$
$= \mathbb{1}_{\tau>t}(1-e^{-\lambda(T-t)})$

$+\mathbb{1}_{\tau\le t}\,\mathbb{E}[\mathbb{1}_{\tau\le T} \mid \sigma(\tau)]$

before default

Next, we show on $\{\tau > t\}$, $\mathbf{E}[Y|\mathscr{F}_t] = \frac{\mathbf{E}[\mathbf{1}_{\{\tau>t\}}Y]}{\mathbf{P}(\tau>t)}$, i.e.

$$\mathbf{E}[\mathbf{1}_{\{\tau>t\}}Y|\mathscr{F}_t] = \mathbf{1}_{\{\tau>t\}}\frac{\mathbf{E}[\mathbf{1}_{\{\tau>t\}}Y]}{\mathbf{P}(\tau>t)}. \tag{10}$$

$\Leftrightarrow \mathbb{E}[\mathbb{1}_{\tau>t} Y P(\tau>t) \mid \mathcal{F}_t] = \mathbb{1}_{\tau>t}\,\mathbb{E}[\mathbb{1}_{\tau>t} Y]$

Hence, we need to verify

$\int_A \mathbb{1}_{\tau>t} Y P(\tau>t)\,d\mathbb{P} = \int_A \mathbb{1}_{\tau>t}\,\mathbb{E}[\mathbb{1}_{\tau>t} Y]\,d\mathbb{P}$ for $\forall A \in \mathcal{F}_t$.

In other words, $\mathbf{1}_{\{\tau>t\}}\mathbf{E}[\mathbf{1}_{\{\tau>t\}}Y]$ *is the conditional expectation of* $\mathbf{1}_{\{\tau>t\}}Y\mathbf{P}(\tau>t)$ *on* $\mathscr{F}_t$. For this, for any $A \in \mathscr{F}_t$, it is sufficient to consider $A = \{\tau \le s\}$ for $s \le t$ which yields $A \cap \{\tau > t\} = \emptyset$, and $A = \{\tau > t\}$ which yields $A \cap \{\tau > t\} = \{\tau > t\}$.

For the case $A \cap \{\tau > t\} = \emptyset$,

$$\mathbf{E}\left[\mathbf{1}_A\mathbf{1}_{\{\tau>t\}}\mathbf{E}[\mathbf{1}_{\{\tau>t\}}Y]\right] = \mathbf{E}\left[\mathbf{1}_A\mathbf{1}_{\{\tau>t\}}Y\mathbf{P}(\tau>t)\right] = 0,$$

so that (10) holds.

① $A = \{\tau \le s\}$ for $s \le t$

$\{\tau>t\} \cap A = \phi \Rightarrow \text{LHS} = \text{RHS} = 0$

For the case $A \cap \{\tau > t\} = \{\tau > t\}$,

$$\mathbf{E}\left[\mathbf{1}_A\mathbf{1}_{\{\tau>t\}}\mathbf{E}[\mathbf{1}_{\{\tau>t\}}Y]\right] = \mathbf{P}(\tau>t)\mathbf{E}[\mathbf{1}_{\{\tau>t\}}Y],$$

and

$$\mathbf{E}\left[\mathbf{1}_A\mathbf{1}_{\{\tau>t\}}Y\mathbf{P}(\tau>t)\right] = \mathbf{E}[\mathbf{1}_{\{\tau>t\}}Y]\mathbf{P}(\tau>t),$$

from which we conclude. □

② $A = \{\tau > t\}$.

$\text{LHS} = \mathbb{E}[\mathbb{1}_{\tau>t} Y]\,P(\tau>t)$

$\text{RHS} = \mathbb{E}[\mathbb{1}_{\tau>t} Y]\,\mathbb{P}(\tau>t)$. #

One of the most typical examples of the stopping time τ used to model default time is generated by an exponential random variable with constant intensity $\lambda > 0$, as shown in the following example.

Example 4. If τ follows exponential distribution with constant intensity $\lambda > 0$, then formula (10) implies that

$$\mathbf{E}[\mathbf{1}_{\{\tau>t\}}Y|\mathscr{F}_t] = \mathbf{1}_{\{\tau>t\}}e^{\lambda t}\mathbf{E}[\mathbf{1}_{\{\tau>t\}}Y].$$

In particular, taking $Y = \mathbf{1}_{\{\tau>T\}}$ yields

$$\mathbf{P}(\tau > T|\mathscr{F}_t) = \mathbf{1}_{\{\tau>t\}}e^{-\lambda(T-t)}. \tag{11}$$

Taking $Y = \mathbf{1}_{\{t<\tau\le T\}}$ yields

$$\mathbf{P}(t < \tau \le T|\mathscr{F}_t) = \mathbf{1}_{\{\tau>t\}}(1-e^{-\lambda(T-t)}). \tag{12}$$

We also have the martingale characterisation of the single jump process $H_t := \mathbf{1}_{\{\tau\le t\}}$, $t \ge 0$, when τ follows exponential distribution.

Lemma 2. *The $\mathscr{F}_t$-stopping time τ follows exponential distribution with constant intensity $\lambda > 0$ iff*

$$M_t := H_t - \int_0^t \mathbf{1}_{\{\tau>s\}}\lambda ds,\ t \ge 0,$$

is an $(\mathscr{F}_t, \mathbf{P})$-martingale and $\mathbf{P}(\tau > 0) = 1$.

Proof. Only if part: For any $T \ge t \ge 0$, by the formula (11),

$$\begin{aligned}\mathbf{E}[M_T|\mathscr{F}_t] =& 1 - \mathbf{E}[1_{\{\tau>T\}}|\mathscr{F}_t] - \int_0^t \mathbf{1}_{\{\tau>s\}}\lambda ds - \int_t^T \mathbf{E}[\mathbf{1}_{\{\tau>s\}}\lambda|\mathscr{F}_t]ds\\ =& 1 - \mathbf{1}_{\{\tau>t\}}e^{-\lambda(T-t)} - \int_0^t \mathbf{1}_{\{\tau>s\}}\lambda ds - \mathbf{1}_{\{\tau>t\}}\int_t^T \lambda e^{-\lambda(s-t)}ds\\ =& \mathbf{1}_{\{\tau\le t\}} - \int_0^t \mathbf{1}_{\{\tau>s\}}\lambda ds = M_t.\end{aligned}$$

Since τ follows exponential distribution, it follows that $\mathbf{P}(\tau > 0) = e^{-\lambda 0} = 1$.

If part: For $t \ge 0$, define $\Phi(t) = \mathbf{P}(\tau > t)$. Then, following the martingale property of M,

$$\begin{aligned}\Phi(t) =& \mathbf{E}\left[1 - M_t - \int_0^t \mathbf{1}_{\{\tau>s\}}\lambda ds\right]\\ =& 1 - M_0 - \lambda\int_0^t \Phi(s)ds.\end{aligned}$$

It follows from the condition $\tau > 0$ a.s. that $M_0 = H_0 = 0$, a.s., so

$$\Phi(t) = 1 - \lambda\int_0^t \Phi(s)ds$$

which implies that $\Phi(t) = e^{-\lambda t}$, i.e. τ follows exponential distribution with intensity λ. □

In practice, we often need to model λ as an $\mathscr{F}_t$-prog measurable stochastic process. Based on the above martingale characterisation, we impose the following assumption on the $\mathscr{F}_t$-stopping time τ through its corresponding single jump process $H_t = \mathbf{1}_{\{\tau\le t\}}$, $t \ge 0$. It is clear that for each ω, $H_\cdot(\omega)$ is a BV function (recall BV means Cadlag with bounded variation).

Assumption 1 *Let τ be a non-negative random variable defined on $(\Omega, \mathscr{F}, \mathbf{P})$, and $\{\mathscr{F}_t\}_{t\ge0}$ be the natural filtration of $H_t = \mathbf{1}_{\{\tau\le t\}}$, $t \ge 0$. i.e. $\mathscr{F}_t = \sigma(H_s : s \le t)$, such that*

$$M_t := H_t - \int_0^t \mathbf{1}_{\{\tau>s\}}\lambda_s ds,\ t \ge 0,$$

is an $(\mathscr{F}_t, \mathbf{P})$-martingale, for λ being an $\mathscr{F}_t$-prog measurable, strictly positive and bounded process. Moreover, we assume that $\mathbf{P}(\tau > 0) = 1$.

Since $H_\cdot(\omega)$ is BV, it is obvious that $M_\cdot(\omega)$ is also BV, that is, *M is a Cadlag martingale with bounded variation, and moreover, $\Delta M_t = \Delta H_t$.*

3.3 Girsanov's theorem

We next discuss the Girsanov's theorem for the single jump process H under Assumption 1.

Theorem 7. *Let $\mu \in [0,1]$ be a constant, and suppose that Assumption 1 is satisfied. For $T>0$, define $Z_t^\mu = C_t^\mu V_t^\mu$ for $t \in [0,T]$, where*

$$C_t^\mu = e^{\int_0^t \mu \mathbf{1}_{\{\tau>s\}}\lambda_s ds},$$

and

$$V_t^\mu = \mathbf{1}_{\{\tau>t\}} + (1-\mu)\mathbf{1}_{\{\tau\le t\}}.$$

Then, Z^μ is an $(\mathscr{F}_t, \mathbf{P})$-martingale, and satisfies,

$$Z_t^\mu = 1 - \int_0^t Z_{s-}^\mu \mu dM_s, \quad \text{for } t \in [0,T].$$

Proof. Note that for $T>0$, $\int_0^T |\mu| dM_s = \mu M_T < \infty$. We decompose the martingale M into its continuous part and pure jump part as

$$\begin{aligned} M_t &= M_t^c + \sum_{0<s\le t} \Delta M_s \\ &= -\int_0^t \mathbf{1}_{\{\tau>s\}}\lambda_s ds + H_t \end{aligned}$$

In turn, we have

$$e^{-\int_0^t \mu dM_s^c} = e^{\int_0^t \mu \mathbf{1}_{\{\tau>s\}}\lambda_s ds} = C_t^\mu,$$

and since $\Delta M_s = \Delta H_s$,

$$\prod_{0<s\le t}(1-\mu\Delta M_s) = \prod_{0<s\le t}(1-\mu\Delta H_s) = \mathbf{1}_{\{\tau>t\}} + (1-\mu)\mathbf{1}_{\{\tau\le t\}} = V_t^\mu.$$

Theorem 6 then implies that $C_t^\mu V_t^\mu$ satisfies, for $t \in [0,T]$,

$$C_t^\mu V_t^\mu = 1 - \int_0^t C_{s-}^\mu V_{s-}^\mu \mu dM_s,$$

so $Z_t^\mu = C_t^\mu V_t^\mu$, $t \in [0,T]$, is an $(\mathscr{F}_t, \mathbf{P})$-local martingale. Since both C_t^μ and V_t^μ are bounded for $t \in [0,T]$, we conclude that Z^μ is also an $(\mathscr{F}_t, \mathbf{P})$-martingale. □

Theorem 8. *Let $T>0$ be fixed. Given the $(\mathscr{F}_t, \mathbf{P})$-martingale Z^μ as in Theorem 7, define a new probability measure $\mathbf{Q}^\mu$ by the Radon-Nikodym density*

$$\left.\frac{d\mathbf{Q}^\mu}{d\mathbf{P}}\right|_{\mathscr{F}_t} = Z_t^\mu.$$

Then,

$$M_t^\mu = H_t - \int_0^t (1-\mu)\mathbf{1}_{\{\tau>s\}}\lambda_s ds,\ t \in [0,T],$$

is an $(\mathscr{F}_t, \mathbf{Q}^\mu)$-martingale.

Proof. Note that by the Bayes' formula, M_t^μ, $t \in [0,T]$, *is an $(\mathscr{F}_t, \mathbf{Q}^\mu)$-martingale iff $M_t^\mu Z_t^\mu$, $t \in [0,T]$ is an $(\mathscr{F}_t, \mathbf{P})$-martingale.*

Hence, it is sufficient to show that $M_t^\mu Z_t^\mu$, $t \in [0,T]$, is an $(\mathscr{F}_t, \mathbf{P})$-martingale. Using the integration by parts formula (3), we obtain

$$M_t^\mu Z_t^\mu = \int_0^t M_{s-}^\mu dZ_s^\mu + \int_0^t Z_{s-}^\mu dM_s^\mu + \sum_{0<s\le t} \Delta M_s^\mu \Delta Z_s^\mu. \tag{13}$$

Note that M^μ can be rewritten as

$$M_s^\mu = M_s + \int_0^t \mu \mathbf{1}_{\{\tau>s\}}\lambda_s ds,$$

so

$$\int_0^t Z_{s-}^\mu dM_s^\mu = \int_0^t Z_{s-}^\mu dM_s + \int_0^t Z_{s-}^\mu \mu \mathbf{1}_{\{\tau>s\}}\lambda_s ds. \tag{14}$$

On the other hand, since $\Delta Z_s^\mu = -Z_{s-}^\mu \mu \Delta M_s$, we have

$$\sum_{0<s\le t} \Delta M_s^\mu \Delta Z_s^\mu = -\sum_{0<s\le t} Z_{s-}^\mu \mu |\Delta M_s|^2.$$

But $\Delta M_s = \Delta H_s$ and $|\Delta H_s|^2 = \Delta H_s$, it follows that

$$\sum_{0<s\le t} \Delta M_s^\mu \Delta Z_s^\mu = -\sum_{0<s\le t} Z_{s-}^\mu \mu \Delta H_s = -\int_0^t Z_{s-}^\mu \mu dH_s. \tag{15}$$

Plugging (14) and (15) into (13), we get

$$M_t^\mu Z_t^\mu = \int_0^t M_{s-}^\mu dZ_s^\mu + \int_0^t Z_{s-}^\mu dM_s - \int_0^t Z_{s-}^\mu \mu dM_s,$$

which implies that $M_t^\mu Z_t^\mu$, $t \in [0,T]$, is an $(\mathscr{F}_t, \mathbf{P})$-local martingale. Finally, since $M^\mu Z^\mu$ is bounded, it is also an $(\mathscr{F}_t, \mathbf{P})$-martingale. □

Note that when $\mu = 0$, then $Z^0 = 1$. Therefore, $\mathbf{Q}^0 = \mathbf{P}$, and $M^0 = M$ is an $(\mathscr{F}_t, \mathbf{P})$-martingale following from Assumption 1.

On the other hand, when $\mu = 1$, $\mathbf{Q}^1$ is only absolutely continuous w.r.t. $\mathbf{P}$. Therefore, for $A \subset \Omega$, $\mathbf{P}(A) = 0 \Rightarrow \mathbf{Q}^1(A) = 0$. However, for the sets $B_t = \{\tau \le t\}$, $t \in [0,T]$, we have $\mathbf{Q}^1(B_t) = 0$ but $\mathbf{P}(B_t) \neq 0$, so $\mathbf{Q}^1$ and $\mathbf{P}$ are not equivalent.

Exercise 1. (Exponential formula)

1. Prove the solution to the equation (8) is unique.
2. Apply the change of variables formula in Theorem 5 to $\ln Z(t)$ to derive the solution of the equation (8) directly.

References

1. Bremaud, Pierre. *Point processes and queues: martingale dynamics*. Springer-Verlag, 1981.

Girsanov's theorem. Given a Cadlag BV martingale $M_t = H_t - \int_0^t 1_{\tau > s} \lambda_s ds$, $\langle M \rangle_t = \int_0^t 1_{\tau > s} \lambda_s ds$. $([M]_t = H_t)$

Consider SDE $\begin{cases} dZ_t = -Z_t \mu \, dM_t, \\ Z_0 = 1 \end{cases}$

i.e. $Z_t = \mathcal{E}(-\int \mu \, dM)_t$ stochastic exponential, local martingale.

By exponential formula,

$$Z_t = \underbrace{(1_{\tau \le t}(1-\mu) + 1_{\tau > t})}_{bdd} \underbrace{e^{-\int_0^t \mu 1_{\tau > s} \lambda_s ds}}_{bdd} \quad \text{is bdd}$$

Hence, Z_t is a martingale.

Define an equivalent prob measure $Q^\mu \sim P$ by R-N density

$$\left.\frac{dQ^\mu}{dP}\right|_{\mathcal{F}_t} = Z_t = \mathcal{E}(-\int \mu \, dM)_t$$

Then, under Q^μ: by Girsanov's theorem

$$M_t^\mu \triangleq M_t - \int_0^t \frac{1}{Z_s} d\langle Z, M \rangle_s \quad \text{is a martingale}$$

$$d\langle Z, M \rangle_t = -Z_t \mu 1_{\tau > t} \lambda_t dt$$

$$= M_t + \int_0^t \mu 1_{\tau > t} \lambda_t dt \quad \text{is a martingale.} \quad (\star)$$

$$= H_t - \int_0^t 1_{\tau > s} \lambda_s ds + \int_0^t \mu 1_{\tau > s} \lambda_s ds$$

$$= H_t - \int_0^t (1-\mu) 1_{\tau > s} \lambda_s ds$$

In particular, for $\mu \equiv 1$, $H_t = 1_{\tau \le t}$ is a Q^μ-martingale.

To prove $(\star)$ is a martingale under Q^μ, it is sufficient to show $M_t^\mu Z_t$ is a martingale under $\mathbb{P}$.

Since $dM_t^\mu = dM_t + \mu 1_{\tau > t} \lambda_t dt$ is BV

$dZ_t = -Z_t \mu \, dM_t$ is BV.

By integration by parts formula,

$$M_t^\mu Z_t = \underbrace{\underset{0}{M_0^\mu} \underset{1}{Z_0}}_{0} + \underbrace{\int_0^t M_{s-}^\mu dZ_s}_{\text{local mart}} + \int_0^t Z_{s-} dM_s^\mu + \sum_{0 \le s \le t} \Delta M_s^\mu \Delta Z_s$$

$$\int_0^t Z_{s-} dM_s^\mu = \underbrace{\int_0^t Z_{s-} dM_s}_{\text{local martingale}} + \int_0^t Z_{s-} \mu 1_{\tau > s} \lambda_s ds$$

local martingale

Note that $\Delta Z_s = -\mu Z_{s-} \Delta M_s$
$\Delta M^{\mu}_s = \Delta M_s$

$\Rightarrow \sum_{0<s\le t} \Delta M^{\mu}_s \Delta Z_s = -\sum_{0<s\le t} \mu Z_{s-} (\Delta M_s)^2.$

$\Delta M_s = \Delta H_s \Rightarrow (\Delta M_s)^2 = \Delta H_s$

$= -\sum_{0<s\le t} \mu Z_{s-} \Delta H_s$

$= -\int_0^t \mu Z_{s-} \, dH_s.$

$$\Rightarrow M^{\mu}_t Z_t = \int_0^t M^{\mu}_{s-} \, dZ_s + \int_0^t Z_{s-} \, dM_s - \int_0^t \mu Z_{s-} \underbrace{(dH_s - 1_{\tau > s} \lambda_s ds)}_{dM_s}$$

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