MA999 Fundamentals of Mathematical Modelling

Continuous dynamical systems

Differential equations describe the evolution of systems in continuous time (whereas iterated maps - in discrete time). A system of ordinary differential equations (ODEs):

$$
\begin{aligned}
\dot{x}_1 &= f_1(x_1, ..., x_n), \\
&\vdots \\
\dot{x}_n &= f_n(x_1, ..., x_n).
\end{aligned}
$$

Here $\dot{x}_i \equiv \mathrm{d}x_i/\mathrm{d}t$.

A single ODE is a mathematical equation for an unknown function of one independent variable that relates the values of the function itself and of its derivative. In general, ODEs define the rates of change of the variables in terms of the current state. To solve a system of ODEs means to find continuous functions $x_1(t)$,..., $x_n(t)$ of the independent variable t that, along with their derivatives, satisfies the system of equations.

Autonomous systems: $f_i(x_1, ..., x_n)$; **nonautonomous systems**: $f_i(x_1, ..., x_n, t)$.

First-order systems

$$\dot{x} = f(x)$$

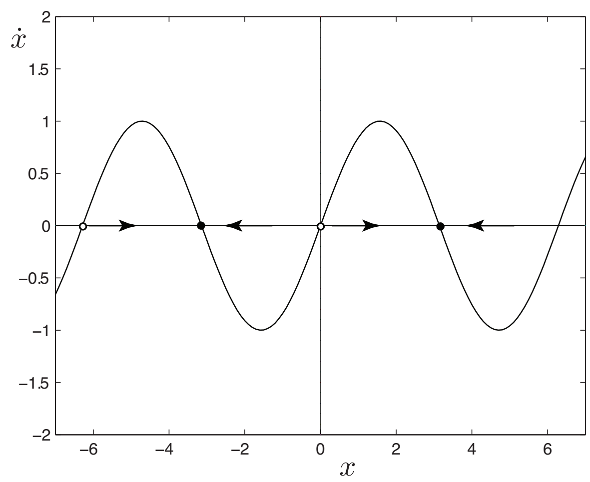
Reminder: Linear ODE

$$\dot{x} = -\tau x, \qquad x(0) = x_0.$$

$$\text{Solution:} \qquad x(t) = x_0 \mathrm{e}^{-\tau t}$$

One of the most basic techniques of dynamics is to interpret a differential equation as a vector field.

Example 1. $\dot{x} = \sin x$



Fixed points and stability

A solution $x(t)$ of differential equation starting from the initial condition x_0 is also called a *trajectory*. The fixed points (equilibrium solutions, steady-states) are defined by $f(x^*) = 0$.

- A fixed point is defined to be *stable* if all sufficiently small disturbances away from it damp out in time.
  - If all solutions of the dynamical system that start out near an equilibrium point x^* stay near x^* forever, then x^* is *Lyapunov stable*.
  - More strongly, if all solutions that start out near x^* converge to x^*, then x^* is *asymptotically stable*.
  - The notion of *exponential stability* guarantees a minimal rate of convergence.
- If disturbances grow in time a fixed point is defined to be *unstable*.

Linear stability analysis

Let x^* - a fixed point and $\eta(t) = x(t) - x^*$ - a small perturbation.

$$\dot{\eta} = f(x^* + \eta)$$

Reminder: Taylor series of f about the point x^*

$$f(x) = \sum_{n=0}^{\infty} f^{(n)}(x^*) \frac{(x - x^*)^n}{n!},$$

where $f^{(k)}(x^*)$ is the kth derivative of the function f evaluated at x^*.

Then $f(\eta + x^*) = f(x^*) + f'(x^*)\eta + O(\eta^2) \Rightarrow$

$$\dot{\eta} = f'(x^*)\eta$$

- $\eta(t)$ grows exponentially if $f'(x^*) > 0 \Rightarrow x^*$ - unstable
- $\eta(t)$ decays exponentially if $f'(x^*) < 0 \Rightarrow x^*$ - stable

Local existence and uniqueness (in $\mathbb{R}$)

Example 2.

$$\dot{x} = x^{1/3} \qquad x(0) = 0$$

The point $x = 0$ is a fixed point, so one solution is $x(t) = 0$ for all t. Also

$$\int \frac{\mathrm{d}x}{x^{1/3}} = \int \mathrm{d}t \Rightarrow \frac{3}{2}x^{2/3} = t + C$$

Initial data ensures $C = 0$ so $x(t) = (2t/3)^{3/2}$. The co-existence of solutions (non-uniqueness) is due to the fact that $x^{1/3}$ is not continuously differentiable.

Theorem: Consider the initial value problem (IVP)

$$\dot{x} = f(x), \qquad x(0) = x_0$$

Suppose that f is continuously differentiable on an open interval $I \subset \mathbb{R}$ with $x_0 \in I$. Then the IVP has a solution $x(t)$ on some interval $(-\tau, \tau)$ about $t = 0$ and the solution is unique.

Example 3.

$$\dot{x} = 1 + x^2, \qquad x(0) = 0$$

$$\int \frac{\mathrm{d}x}{1+x^2} = \int \mathrm{d}t \Rightarrow \tan^{-1} x = t + C$$

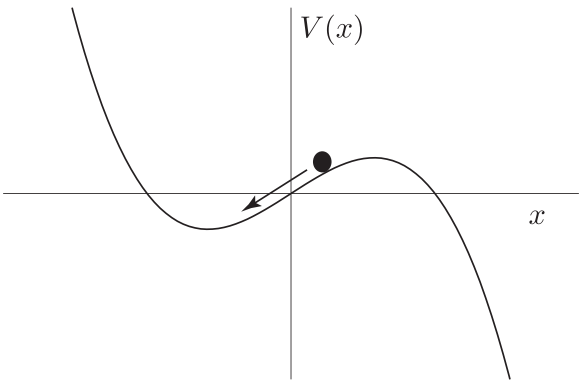
Initial data ensures $C = 0$ so $x = \tan t$ and only exists for $-\pi/2 < t < \pi/2$. This is an example of *blow-up* where $x(t)$ reaches ∞ in finite time.

As we have seen, all flows on a line either approach a fixed point or diverge to $\pm\infty$. In essence overshoot and damped oscillations can never occur in a first order system. It is never possible to have periodic motion for the first order system $\dot{x} = f(x)$, $x \in \mathbb{R}$, $f : \mathbb{R} \to \mathbb{R}$.

Potentials

For a dynamical system $\dot{x} = f(x)$, the potential $V(x)$ is defined by

$$f(x) = -\frac{\mathrm{d}V}{\mathrm{d}x}.$$



Numerical simulation of dynamical systems

This is a vast subject in its own right. It is a powerful means of obtaining insight into the behaviour of nonlinear dynamical systems (thanks to cheap high speed computers) by obtaining approximate solutions to problems that are analytically intractable.

Euler's method. Given a dynamical systems $\dot{x} = f(x)$ and a known solution at time t we approximate the solution at time $t + \Delta t$ by $x(t + \Delta t)$ where

$$x(t + \Delta t) = x(t) + f(x(t))\Delta t.$$

Introducing the notation $x_n = x(t_n)$ where $t_n = n\Delta t$ we have the following iterative scheme

$$x_{n+1} = x_n + f(x_n)\Delta t.$$

Improved Euler method. One problem with the Euler method is that it estimates the derivative $\dot{x}$ only at the left end of the time-interval between t_n and t_{n+1}. A better approach is to use the average derivative across this interval. First construct the estimate $\widetilde{x}_{n+1} \approx x_n + f(x_n)\Delta t$ and then take the average of $f(x_n)$ and $f(\widetilde{x}_{n+1})$ and use that to construct the next step in the iterative scheme. The improved Euler method is then given by

$$\begin{aligned} \widetilde{x}_{n+1} &= x_n + f(x_n)\Delta t && \text{the trial step} \\ x_{n+1} &= x_n + \frac{1}{2}[f(x_n) + f(\widetilde{x}_{n+1})]\Delta t \end{aligned}$$

This scheme tends to a smaller *error* $E = |x(t_n) - x_n|$ for a given step-size Δt. For the (first order) Euler method $E \propto \Delta t$, whilst for the (second order) modified Euler $E \propto (\Delta t)^2$.

An accurate and commonly used scheme is the so-called fourth-order **Runge-Kutta scheme**:

$$x_{n+1} = x_n + \Delta t \left[\frac{1}{6}k_1(n) + \frac{1}{3}k_2(n) + \frac{1}{3}k_3(n) + \frac{1}{6}k_4(n)\right]$$

where

$$\begin{aligned} k_1(n) &= f(x_n) \\ k_2(n) &= f(x_n + \frac{1}{2}\Delta t k_1(n)) \\ k_3(n) &= f(x_n + \frac{1}{2}\Delta t k_2(n)) \\ k_4(n) &= f(x_n + \Delta t k_3(n)) \end{aligned}$$

Dynamical Systems Software

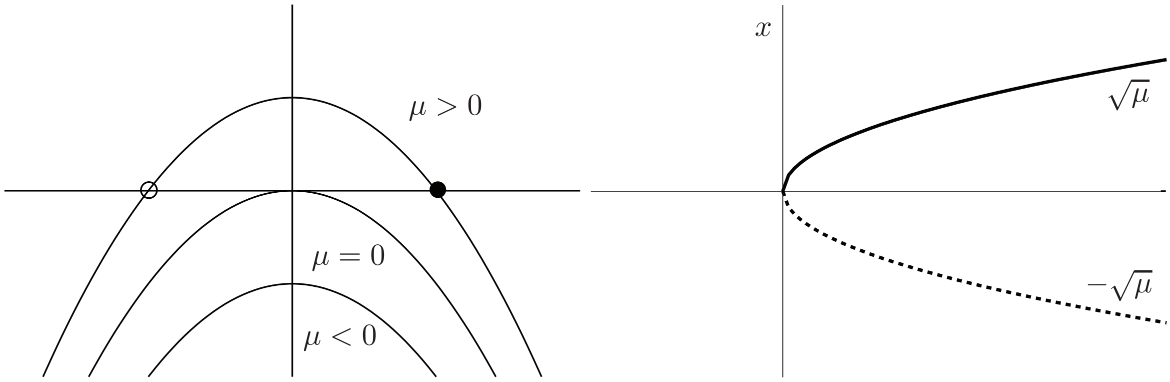
See an extensive list for more information http://www.dynamicalsystems.org/sw/sw/ .

Bifurcations

The qualitative structure of a flow can change as a parameter is varied. These qualitative changes are called *bifurcations* and the parameter values at which they occur are called *bifurcation* points.

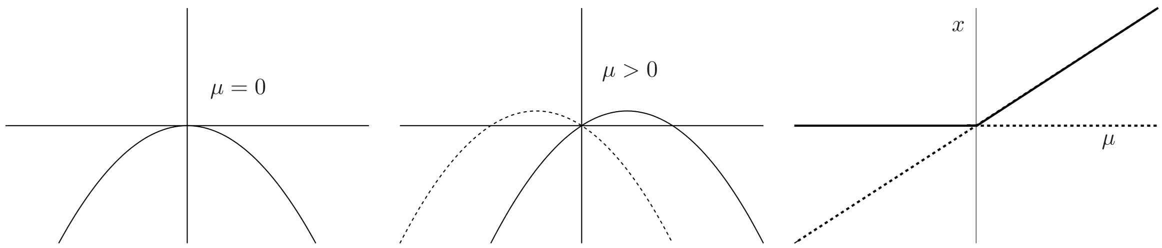
Saddle-Node bifurcation

$$\dot{x} = \mu - x^2$$



Transcritical bifurcation

$$\dot{x} = \mu x - x^2$$



Example 4.

$$\dot{x} = r \ln x + x - 1$$

Fixed point at $x = 1$. Let $u = x - 1$, then

$$\begin{aligned}\dot{u} &= r \ln(1+u) + u \approx r\left(u - \frac{u^2}{2} + \ldots\right) + u \\ &= (r+1)u - \frac{1}{2} r u^2\end{aligned}$$

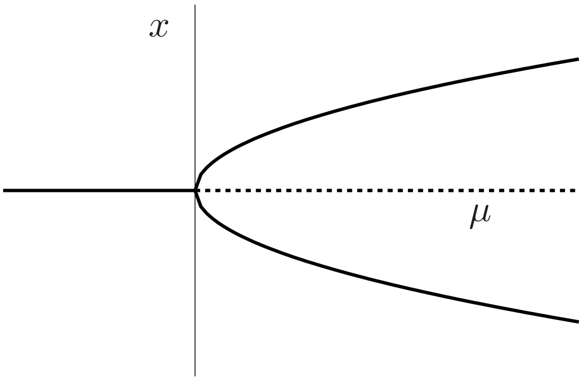
Rescale ($v = (r/2)u$):

$$\dot{v} = (r+1)v - v^2$$

By a near identity change of co-ords we have found the *normal form* for the dynamics (valid close to the bifurcation point).

Pitchfork bifurcation: supercritical

$$\dot{x} = \mu x - x^3$$



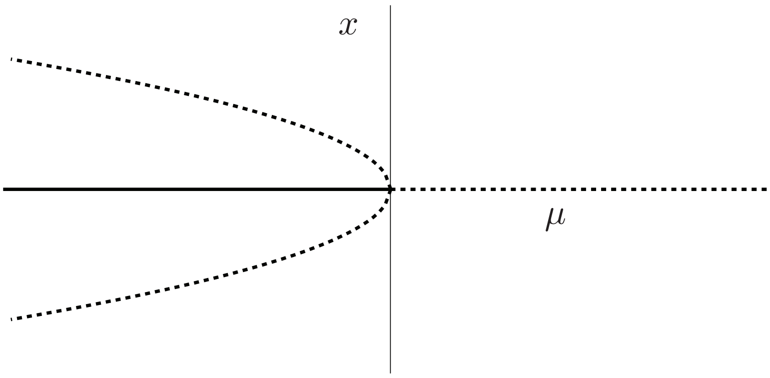
Shows critical slowing down at $\mu = 0$:

$$\int \frac{\mathrm{d}x}{x^3} = -\int \mathrm{d}t \Rightarrow x = \sqrt{\frac{1}{2(t+C)}}, \quad C = \frac{1}{2x_0^2}(x_0 \neq 0)$$

For large t, $x \sim t^{-1/2}$: power law decay rather than exponential $\mathrm{e}^{\mu t}$.

Pitchfork bifurcation: subcritical

$$\dot{x} = \mu x + x^3$$



Example 5.

$$\dot{x} = \mu x + x^3 - x^5$$

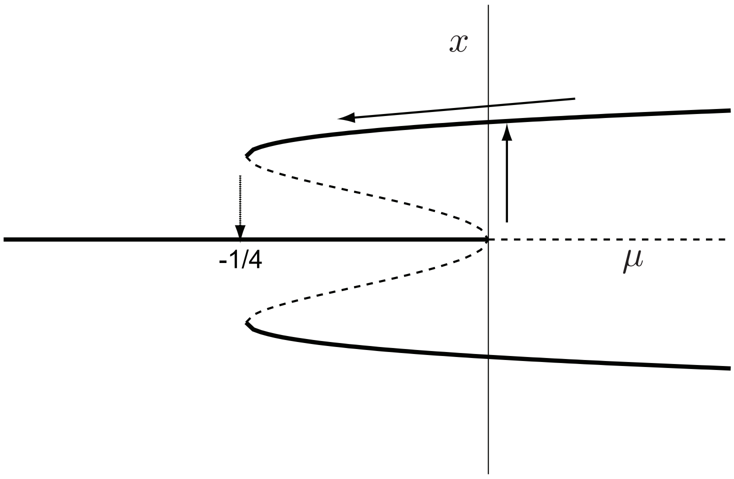
Fixed points

$$-(\mu + x^2) + x^4 = 0 \qquad \text{and} \qquad x = 0$$

roots:

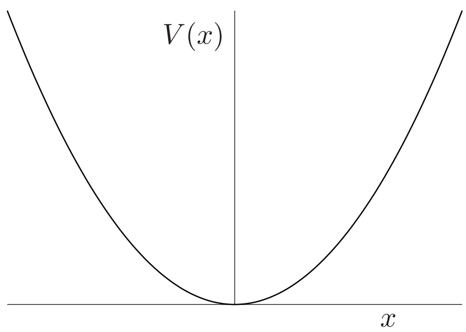
$$x^2 = \frac{1 \pm \sqrt{1+4\mu}}{2}$$

If $\mu > 0$ then $x^2 = (1 + \sqrt{1+4\mu})/2$: total of three fixed points. If $-1/4 < \mu < 0$, $x^2 = (1 \pm \sqrt{1+4\mu})/2$: total of five fixed points. Define $\mu_c = -1/4$.

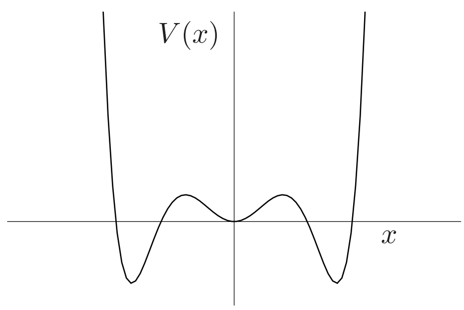


1. In range $\mu_c < \mu < 0$ there co-exist 3 stable fixed points (and 2 unstable). There is *multi-stability*. (Local not global stability). Initial conditions determine the final state.
2. Bifurcation at μ_c is a saddle-node bifurcation.
3. System exhibits hysteresis and jump phenomenon.
4. If x^5 term was absent then blow up could occur.

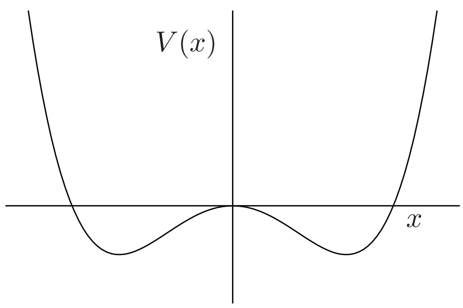
$\mu < 1/4$



$1/4 < \mu < 0$



$\mu > 0$

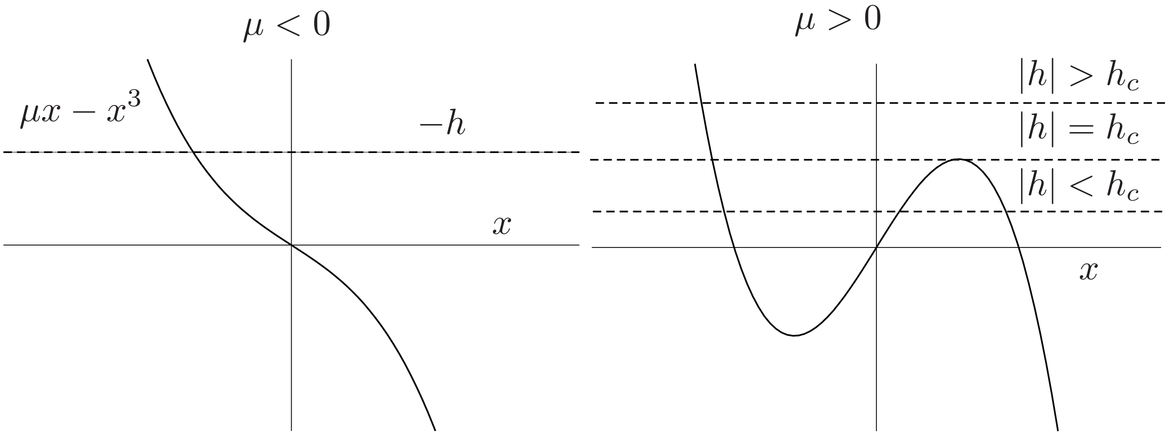


Cusp singularity

The pitchfork bifurcation is common in problems with reflection symmetry. Imperfections break this symmetry.

$$\dot{x} = h + \mu x - x^3$$

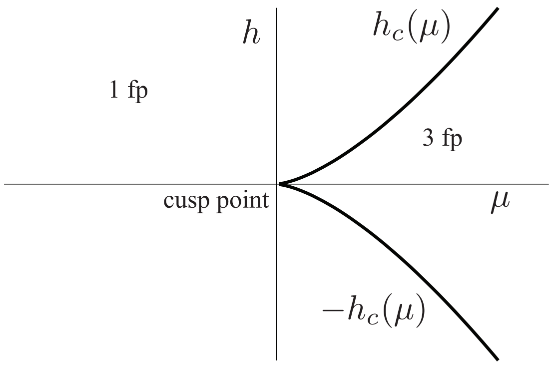
Co-dimension 2 rather than co-dimension 1.

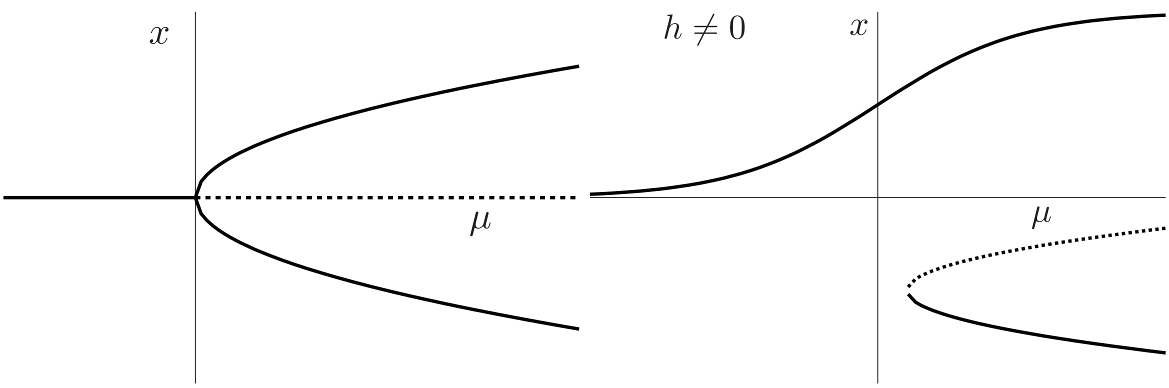


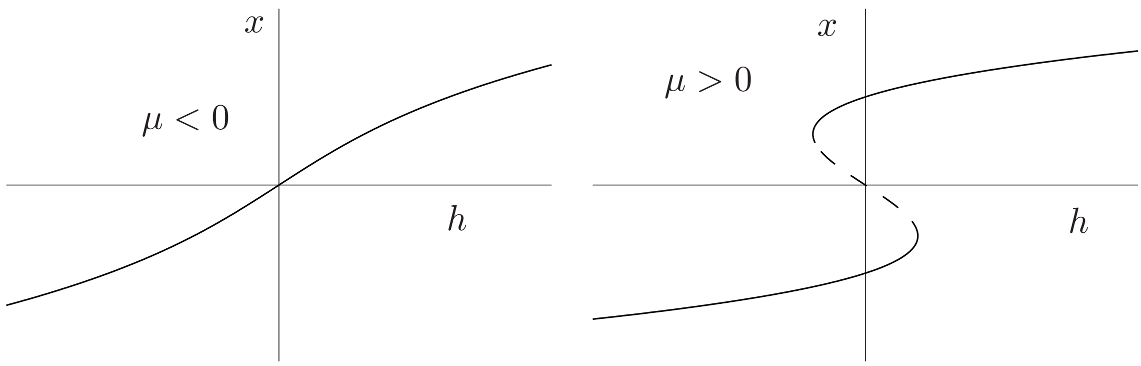
Critical case: horizontal line is tangent to min or max of $f(x) = \mu x - x^3$. Local max/min at $x = \pm\sqrt{\mu/3}$.

$$h_c(\mu) = \frac{2\mu}{3}\sqrt{\frac{\mu}{3}}$$

At $h = \pm h_c(\mu)$ there is a saddle-node bifurcation. There are two bifurcation curves $\pm h_c(\mu)$.







Jump phenomenon and catastrophe theory.

Example 6. Budworm population dynamics:

$$\dot{N} = RN\left(1 - \frac{N}{K}\right) - \frac{BN^2}{A^2 + N^2}, \qquad A, B, R > 0$$

The budworm population $N(t)$ grows logistically (first term) in the absence of predators. The second term describes mortality due to predation (mainly by birds).

Non-dimensionalise: $x = N/A$.

$$\frac{A}{B}\frac{\mathrm{d}x}{\mathrm{d}t} = \frac{R}{B}Ax\left(1 - \frac{Ax}{K}\right) - \frac{x^2}{1+x^2} \equiv f(x)$$

Introduce

$$\tau = \frac{Bt}{A}, \qquad r = \frac{RA}{B}, \qquad k = \frac{K}{A}$$

so that

$$\frac{\mathrm{d}x}{\mathrm{d}\tau} = rx\left(1 - \frac{x}{k}\right) - \frac{x^2}{1+x^2}$$

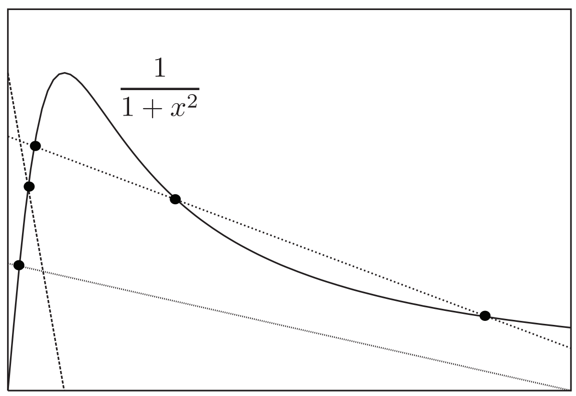
Fixed point at

$$\overline{x} = 0, \qquad r\left(1 - \frac{\overline{x}}{k}\right) - \frac{\overline{x}}{1+\overline{x}^2} = 0$$

Linearisation:

$$f'(x) = r - \frac{2rx}{k} - \frac{2x}{(1+x^2)^2}$$

so $f'(0) = r > 0$, so $\overline{x} = 0$ is unstable. Other roots may be found graphically by finding the intercepts of $x/(1+x^2)$ and $r(1 - x/k)$:



Hence there can be either 1, 2 or 3 interceptions depending upon the choice of (k, r). For example when there are three fixed points $c > b > a > 0$, then since $x = 0$ is unstable a is stable, b unstable and c stable. We compute the details of the bifurcation in the following manner:
Saddle-node occurs when $r(1-x/k)$ intersects $x/(1+x^2)$ tangentially. Thus we require $\overline{x}$ (given by $f(\overline{x}) = 0$ and

$$\frac{\mathrm{d}}{\mathrm{d}x}\left[r\left(1-\frac{x}{k}\right)\right] = \frac{\mathrm{d}}{\mathrm{d}x}\left[\frac{x}{1+x^2}\right]$$

or that

$$-\frac{r}{k} = \frac{1-x^2}{(1+x^2)^2}, \qquad x = \overline{x} \tag{1}$$

Substitution of r/k into the fixed point equation gives

$$r = \frac{2\overline{x}^3}{(1+\overline{x}^2)^2}$$

Substitution into (1) gives

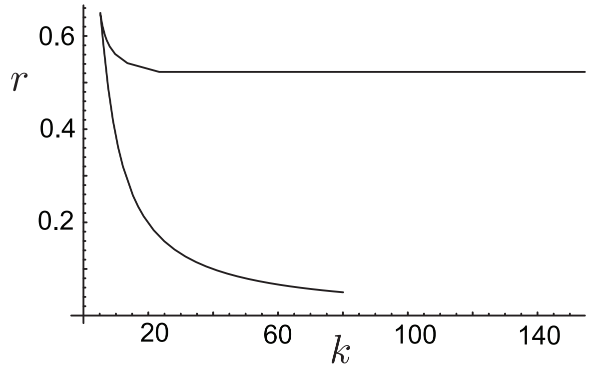
$$k = \frac{2\overline{x}^3}{\overline{x}^2 - 1}$$

Since $k > 0$, we require $x > 1$. The bifurcation curve is defined by $(k(\overline{x}), r(\overline{x}))$ Challenge: plot the bifurcation curve ($r = r(k)$). In MATLAB you could try

```
ezplot3('2*x.^3./(x.^2-1)','2*x.^3./(1+x.^2)^2','0',[1,15]);view(0,90);
```

In MATHEMATICA you could try

```
ParametricPlot[{2 x x x /(x x -1), 2 x x x/(1+x x)^2},{x,1,40}]
```



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