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Brownian Motion

Problem sheet 2 (Valid as an assignment)

1. **Modes of convergence [24 Points]**

   (a) Let $(X_n : n \geq 1)$ and X be r.v.s defined on the same probability space. Show that

   $$X_n \to X \ a.s. \quad \Rightarrow \quad X_n \to X \text{ in probability} \quad \Rightarrow \quad X_n \to X \text{ in distribution} .$$

   Here "a.s." stands for *almost surely*. Convergence in distribution is the same as weak convergence.

   (b) Let $\mathbf{X}_n \sim \mathcal{N}(\boldsymbol{\mu}_n, \Sigma_n)$ for $n = 1, 2, \ldots$ be a sequence of d-dimensional Gaussian variables with converging mean $\boldsymbol{\mu}_k \to \boldsymbol{\mu} \in \mathbb{R}^d$ and covariance matrix $\Sigma_k \to \Sigma \in \mathbb{R}^{d \times d}$, with both Σ_k and Σ definite positive.
   Show that then $\quad \mathbf{X}_n \to \mathbf{X} \sim \mathcal{N}(\boldsymbol{\mu}, \Sigma) \quad$ in distribution .

2. **Brownian bridge [26 Points]**
   Let $\big(B(t) : t \geq 0\big)$ be a BM and define $\quad X(t) := B(t) - tB(1)$ for $t \in [0, 1]$.

   (a) Show that $\big(X(t) : 0 \leq t \leq 1\big)$ is a Gaussian process with continuous paths, and compute its mean and covariance function.

   (b) Draw a typical sample path of $\big(X(t) : 0 \leq t \leq 1\big)$. Why do we call X a Brownian bridge? How would you define a a Brownian bridge with $X(0) = x$ and $X(T) = y$ with fixed $x, y \in \mathbb{R}$ and $T > 0$?

   (c) Define $\quad W(t) := (1 + t)X\Big(\dfrac{t}{1 + t}\Big) \quad$ for all $t \geq 0$.
   What is the mean and covariance function of $\big(W(t) : t \geq 0\big)$? Identify the process.

3. **Ornstein Uhlenbeck process [26 Points]**
   Let $\big(B(t) : t \geq 0\big)$ be a BM and define $X(t) := e^{-ct}B(e^{2ct})$ for $c > 0$ and all $t \in \mathbb{R}$.

   (a) Show that $\big(X(t) : t \in \mathbb{R}\big)$ is a Gaussian process with continuous paths, and compute its mean and covariance function. This is called the Ornstein Uhlenbeck (OU) process.

   (b) Show that the OU process is *strongly stationary*, i.e. for all $n \geq 1$, $t_1, \ldots t_n, h \in \mathbb{R}$

   $$\big(X(t_1), \ldots, X(t_n)\big) \sim \big(X(t_1 + h), \ldots, X(t_n + h)\big) ,$$

   which means that FDDs are translation invariant.

   (c) Show that the OU process is *time-reversible*, i.e. for $Y(t) := X(-t)$ the process $\big(Y(t) : t \in \mathbb{R}\big)$ has the same FDDs as $\big(X(t) : t \in \mathbb{R}\big)$.

4. **Building new Brownian Motions [24 Points]** Let B be a standard Brownian motion in $\mathbb{R}$. Show the following:

   (a) Scaling property:
   If $\lambda > 0$, then $B^\lambda := \big(\lambda^{-1/2} B_{\lambda t} : t \geq 0\big)$ is a standard Brownian motion.

   (b) Orthogonal transformations for Brownian Motion on $\mathbb{R}^d$:
   If $U \in O(d)$ is an *orthogonal* $d \times d$ matrix (i.e. $U^{-1} = U^T$), then $U\,\mathbf{B} = \big(U\,\mathbf{B}_t : t \geq 0\big)$ is a standard Brownian motion. In particular $-\mathbf{B}$ is a standard Brownian motion.

   (c) Time inversion:
   Define $\quad B' = (B'_t : t \geq 0) \quad$ by $\quad B'_t = \begin{cases} t\,B_{1/t} & ,\ t > 0 \\ 0 & ,\ t = 0 \end{cases}$,
   then B' is a standard Brownian motion.
   Hint: Use that a process $(X_t : t \geq 0)$ is a standard BM iff it has continuous paths, $X_t \sim N(0,t)$ and the right covariances, i.e. $\mathbb{E}(X_t X_s) = \min\{s,t\}$.