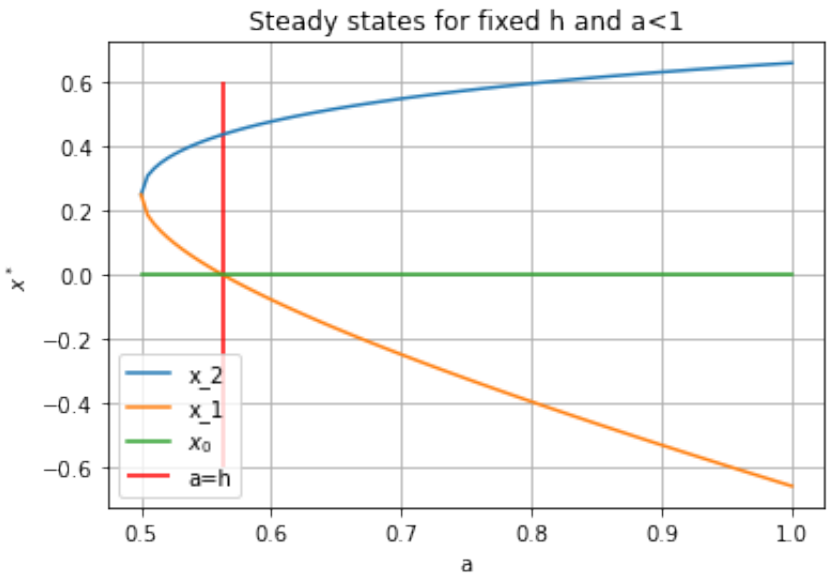
In [133]:
```
import numpy as np
import matplotlib.pyplot as plt
from scipy import optimize
```

Q1

Let's plot the bifurcation diagram (x^* vs a) for two fixed values of $h = \frac{9}{16}, \frac{36}{16}$

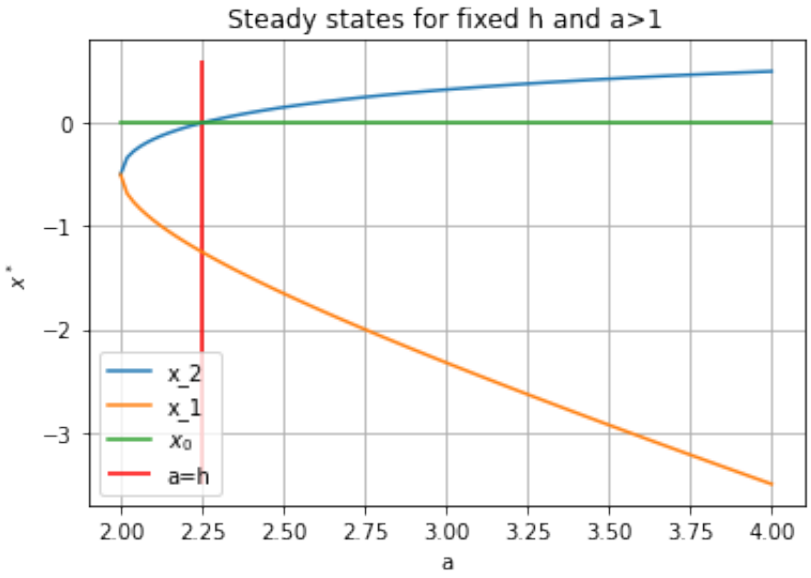
In [446]:
```
def x_plus(a,h):
    return 0.5*(1-a+np.sqrt((a+1)**2-4*h))
def x_minus(a,h):
    return 0.5*(1-a-np.sqrt((a+1)**2-4*h))

h=9/16
a_min=2*np.sqrt(h)-1
print("a_min=",a_min)

mas_a=[a for a in np.linspace(a_min, 2*a_min,100)]
mas_x_plus=[x_plus(a,h) for a in mas_a]
mas_x_minus=[x_minus(a,h) for a in mas_a]
mas_x_zero=[0 for a in mas_a]

plt.plot(mas_a,mas_x_plus,label='x_2')
plt.plot(mas_a,mas_x_minus,label='x_1')
plt.plot(mas_a,mas_x_zero,label='$x_0$')
plt.xlabel('a')
plt.ylabel('$x^*$')
plt.title('Steady states for fixed h and a<1')
plt.vlines(h,ymin=-0.6, ymax=0.6,color='red',label='a=h')
plt.grid()
plt.legend()
plt.show()
```

```
a_min= 0.5
```



In [449]:
```
def x_plus(a,h):
    return 0.5*(1-a+np.sqrt((a+1)**2-4*h))
def x_minus(a,h):
    return 0.5*(1-a-np.sqrt((a+1)**2-4*h))

h=36/16
a_min=2*np.sqrt(h)-1
print("a_min=",a_min)

mas_a=[a for a in np.linspace(a_min, 2*a_min,100)]
mas_x_plus=[x_plus(a,h) for a in mas_a]
mas_x_minus=[x_minus(a,h) for a in mas_a]
mas_x_zero=[0 for a in mas_a]

plt.plot(mas_a,mas_x_plus,label='x_2')
plt.plot(mas_a,mas_x_minus,label='x_1')
plt.plot(mas_a,mas_x_zero,label='$x_0$')
plt.xlabel('a')
plt.ylabel('$x^*$')
plt.title('Steady states for fixed h and a>1')
plt.vlines(h,ymin=-3.5, ymax=0.6,color='red',label='a=h')
plt.grid()
plt.legend()
plt.show()
```

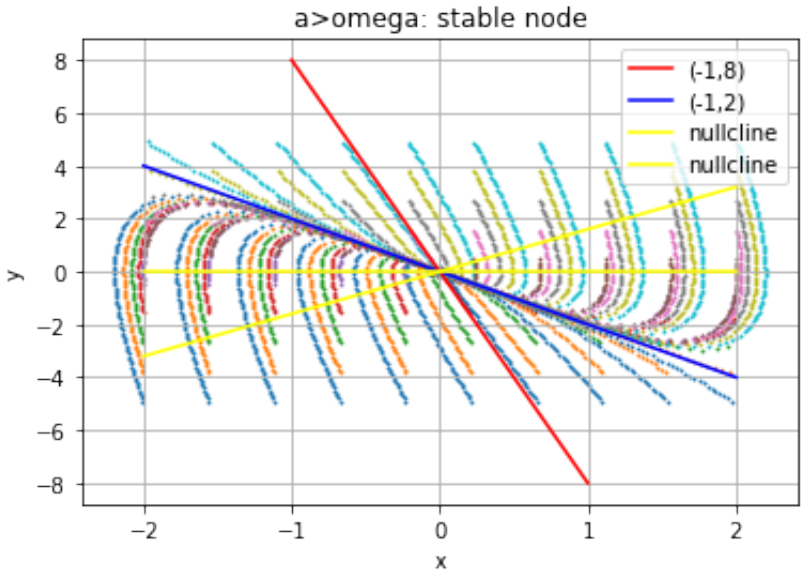
```
a_min= 2.0
```



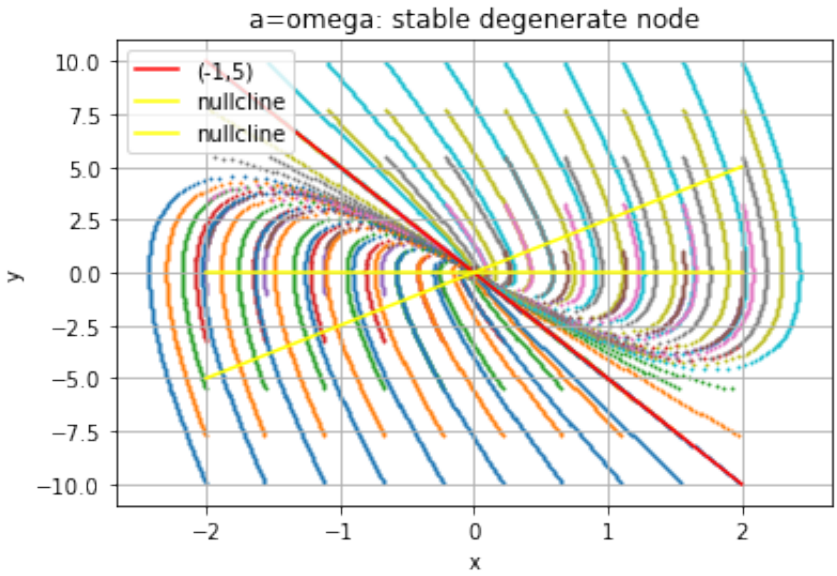
Q2

Let's draw sample phase portraits for cases (1),(2) and (3) from the handwritten solutions and validate that they are really look like stable node, stable degenerate node and stable spiral.

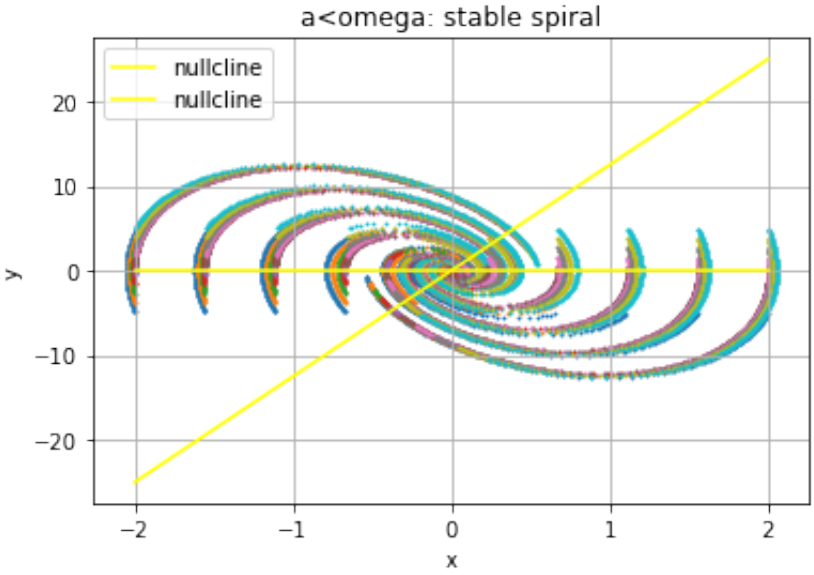
In [432]:
```
#  case 1: a > omega => stable node - yes
a=5
omega=4
def f(x,y):
    return y
def g(x,y):
    return -omega**2*x -2*a*y

k=0.1

for x0 in np.linspace(-2,2,10):
    for y0 in np.linspace(-5,5,10):
        masx=[x0]
        masy=[y0]
        #plt.scatter(x0,y0,color='red')
        #print(x,y)
        for i in range(200):
            x,y=masx[i],masy[i]
            u,v=f(x,y),g(x,y)
            dx,dy=k*u/np.sqrt(u**2 + v**2),k*v/np.sqrt(u**2 + v**2)
            #print(dx,dy)
            masx.append(x+dx)
            masy.append(y+dy)
        #plt.arrow(x=x,y=y,dx=u/np.sqrt(u**2 + v**2),dy=v/(u**2 + v
        plt.scatter(masx[1:],masy[1:],s=0.8)
plt.plot([-t for t in np.linspace(-1,1)],[8*t for t in np.linspace(
plt.plot([-t for t in np.linspace(-2,2)],[2*t for t in np.linspace(
plt.plot([-t for t in np.linspace(-2,2)],[0 for t in np.linspace(-2
plt.plot([-t for t in np.linspace(-2,2)],[-omega**2/(2*a)*t for t i
plt.xlabel('x')
plt.ylabel('y')
plt.title('a>omega: stable node')
plt.grid()
plt.legend()
plt.show()
```



In [435]:
```
# case 2: a =omega => stable degenerate node - yes
a=5
omega=5
def f(x,y):
    return y
def g(x,y):
    return -omega**2*x -2*a*y

k=0.1

for x0 in np.linspace(-2,2,10):
    for y0 in np.linspace(-10,10,10):
        masx=[x0]
        masy=[y0]
        #plt.scatter(x0,y0,color='red')
        #print(x,y)
        for i in range(200):
            x,y=masx[i],masy[i]
            u,v=f(x,y),g(x,y)
            dx,dy=k*u/np.sqrt(u**2 + v**2),k*v/np.sqrt(u**2 + v**2)
            #print(dx,dy)
            masx.append(x+dx)
            masy.append(y+dy)
        #plt.arrow(x=x,y=y,dx=u/np.sqrt(u**2 + v**2),dy=v/(u**2 + v
        plt.scatter(masx[1:],masy[1:],s=0.8)

plt.plot([-t for t in np.linspace(-2,2)],[5*t for t in np.linspace(
plt.plot([-t for t in np.linspace(-2,2)],[0 for t in np.linspace(-2
plt.plot([-t for t in np.linspace(-2,2)],[-omega**2/(2*a)*t for t i

plt.xlabel('x')
plt.ylabel('y')
plt.title('a=omega: stable degenerate node')
plt.grid()
plt.legend()
plt.show()
```



In [438]:
```
# case 3:  a<omega => stable spiral
a=4
omega=10
def f(x,y):
    return y
def g(x,y):
    return -omega**2*x -2*a*y

k=0.1

for x0 in np.linspace(-2,2,10):
    for y0 in np.linspace(-5,5,10):
        masx=[x0]
        masy=[y0]
        #plt.scatter(x0,y0,color='red')
        #print(x,y)
        for i in range(200):
            x,y=masx[i],masy[i]
            u,v=f(x,y),g(x,y)
            dx,dy=k*u/np.sqrt(u**2 + v**2),k*v/np.sqrt(u**2 + v**2)
            #print(dx,dy)
            masx.append(x+dx)
            masy.append(y+dy)
        #plt.arrow(x=x,y=y,dx=u/np.sqrt(u**2 + v**2),dy=v/(u**2 + v
        plt.scatter(masx[1:],masy[1:],s=0.8)
plt.plot([-t for t in np.linspace(-2,2)],[0 for t in np.linspace(-2
plt.plot([-t for t in np.linspace(-2,2)],[-omega**2/(2*a)*t for t i

plt.xlabel('x')
plt.ylabel('y')
plt.title('a<omega: stable spiral')
plt.grid()
plt.legend()
plt.show()
```



And we see that yes, for case (3) point $(0, 0)$ really looks like a stable spiral.

Q3

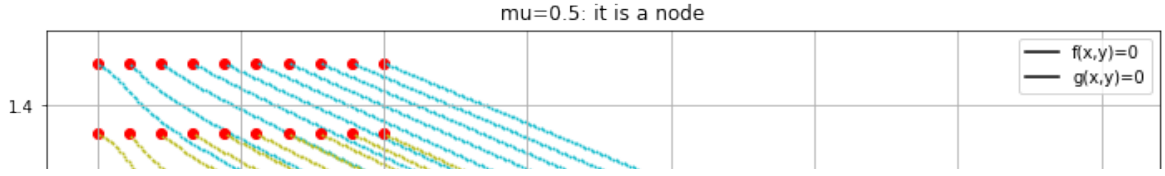
3a) Plot phase portraits for $\mu = 0.5, 0.8, 1.4$

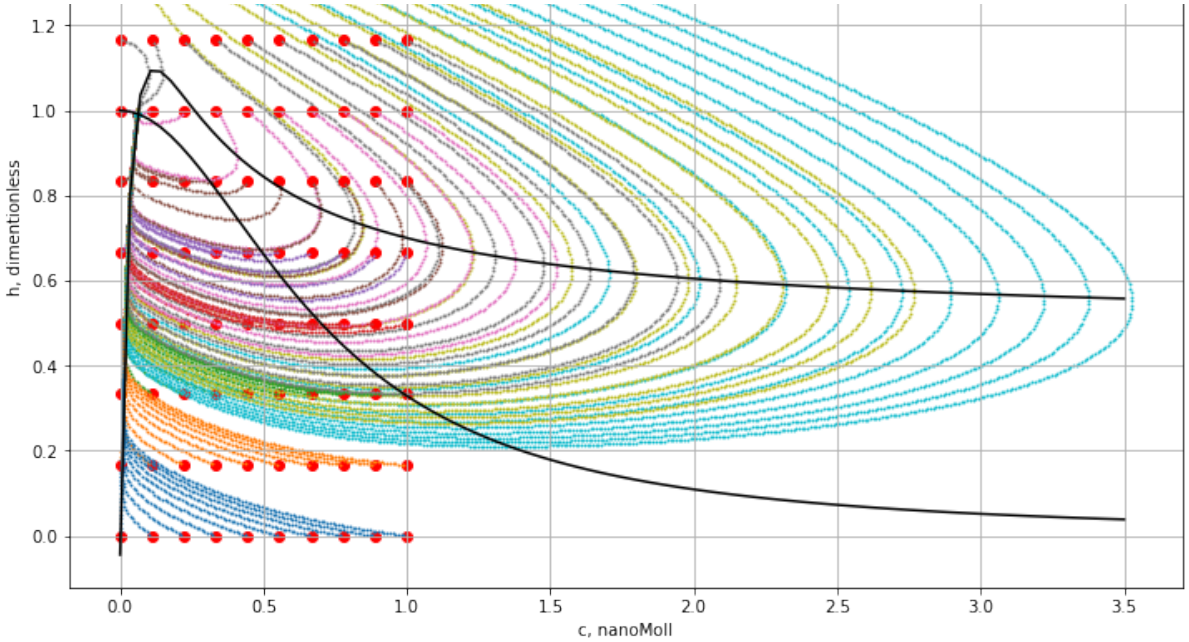
Let's denote $x := c$, $y := h$, plug in the needee variables (and make sure that the dimensionalities for both sides really coincide - I checked this); we will get the following system: $\dot{x} = f(x, y, \mu)$, $\dot{y} = g(x, y, \mu)$, where $f(x, y, \mu) = \mu 8.1 y \frac{0.111+0.889x}{0.7+x} - \frac{2x}{0.1+x} = 0.02$, $g(x, y, \mu) = 0.5 \frac{1}{1+\left(\frac{x}{0.7}\right)^2}$. So nullclines are given by equations $f(x, y, \mu) = 0$, $g(x, y, \mu) = 0$, where in both cases we reorganize the idedntity to express y as a function of x to draw the curve in xOy plane.

Now let's plot the nullclines and numerically integrated trajectories for 1000 steps (using Euler's scheme) to identify the type of fixed point obtained in the point of intersection of nullclines for three different parametrs of μ, where big red dots denote the points where trajectories started.

In [227]:

```
mu=0.5

def f(x,y,mu):
    return mu*8.1*y*(0.111+0.889*x/(0.7+x))-2*x/(0.1+x) + 0.02
def g(x,y,mu):
    return 1/(1+(x/0.7)**2)/2 - y/2

def nullcline1(x,mu):
    return (2*x/(0.1+x)-0.02)/(8.1*mu*(0.111+0.889*x/(0.7+x)))
def nullcline2(x,mu):
    return 1/(1+(x/0.7)**2)

k=0.01
xrange=np.linspace(0,1,10)
yrange=np.linspace(0,1.5,10)
nullclinerange=np.linspace(0,3.5,100)
plt.figure(figsize=(12, 8))

for x0 in xrange:
    for y0 in yrange:
        masx=[x0]
        masy=[y0]
        plt.scatter(x0,y0,color='red')
        #print(x,y)
        for i in range(1000):
            x,y=masx[i],masy[i]
            u,v=f(x,y,mu),g(x,y,mu)
            dx,dy=k*u/np.sqrt(u**2 + v**2),k*v/np.sqrt(u**2 + v**2)
            #print(dx,dy)
            masx.append(x+dx)
            masy.append(y+dy)
        #plt.arrow(x=x,y=y,dx=u/np.sqrt(u**2 + v**2),dy=v/(u**2 + v
        plt.scatter(masx[1:],masy[1:],s=0.8)

plt.plot([x for x in nullclinerange], [nullcline1(x,mu) for x in nu
plt.plot([x for x in nullclinerange],[nullcline2(x,mu) for x in nul

# opt_func = lambda x: nullcline1(x,mu)-nullcline2(x,mu)
# fp=optimize.root_scalar(opt_func, bracket=[0, 5]).root
# print("fp=",fp)

plt.grid()
plt.xlabel('c, nanoMoll')
plt.ylabel('h, dimentionless')
plt.title('mu='+str(mu)+": it is a node")
plt.legend()
plt.show()
```

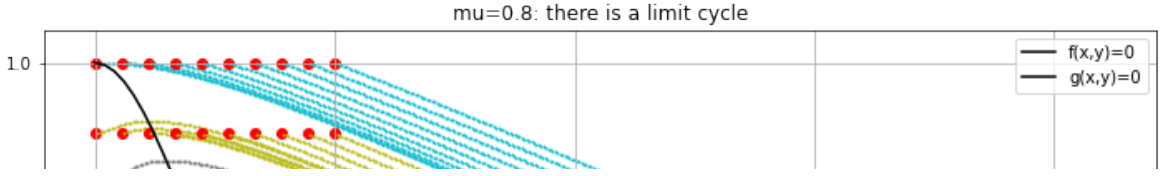


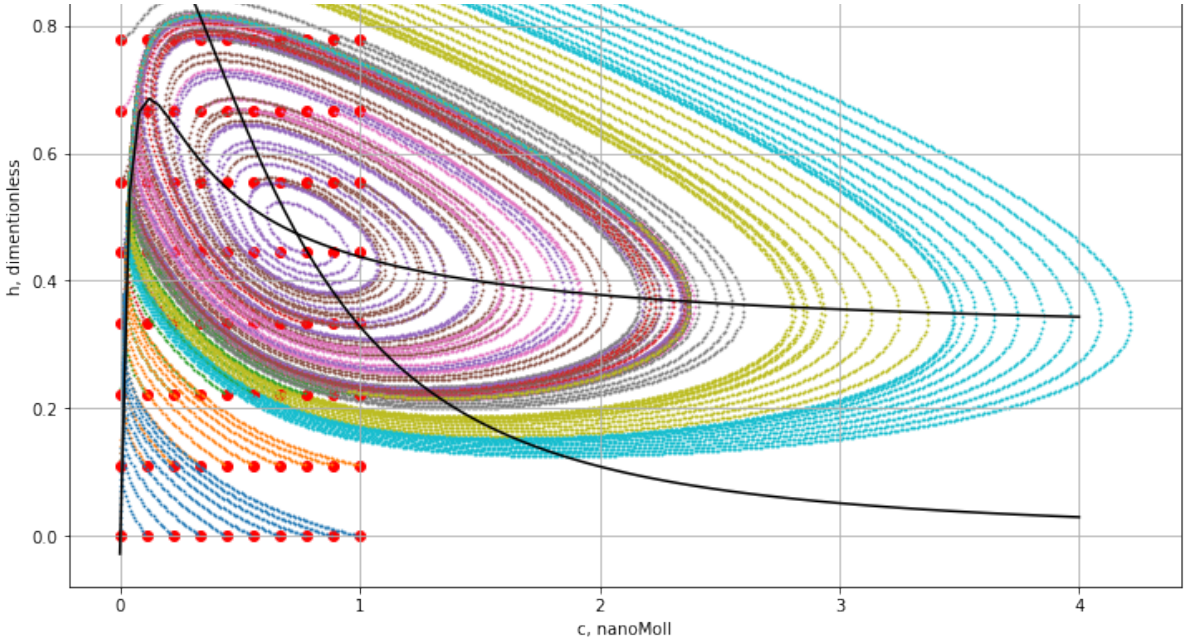


We see that for $\mu = 0.5$, we have a stable node close to $(1, 0)$ on (c, h) plane. All trajectories tend to this point for fixed μ, so there will be the only fixed point for μ from regime 1.

In [226]:

```
mu=0.8

def f(x,y,mu):
    return mu*8.1*y*(0.111+0.889*x/(0.7+x))-2*x/(0.1+x) + 0.02
def g(x,y,mu):
    return 1/(1+(x/0.7)**2)/2 - y/2

def nullcline1(x,mu):
    return (2*x/(0.1+x)-0.02)/(8.1*mu*(0.111+0.889*x/(0.7+x)))
def nullcline2(x,mu):
    return 1/(1+(x/0.7)**2)

k=0.01
xrange=np.linspace(0,1,10)
yrange=np.linspace(0,1,10)
nullclinerange=np.linspace(0,4,100)
plt.figure(figsize=(12, 8))

for x0 in xrange:
    for y0 in yrange:
        masx=[x0]
        masy=[y0]
        plt.scatter(x0,y0,color='red')
        #print(x,y)
        for i in range(1000):
            x,y=masx[i],masy[i]
            u,v=f(x,y,mu),g(x,y,mu)
            dx,dy=k*u/np.sqrt(u**2 + v**2),k*v/np.sqrt(u**2 + v**2)
            #print(dx,dy)
            masx.append(x+dx)
            masy.append(y+dy)
        #plt.arrow(x=x,y=y,dx=u/np.sqrt(u**2 + v**2),dy=v/(u**2 + v
        plt.scatter(masx[1:],masy[1:],s=0.8)

plt.plot([x for x in nullclinerange], [nullcline1(x,mu) for x in nu
plt.plot([x for x in nullclinerange],[nullcline2(x,mu) for x in nul

# opt_func = lambda x: nullcline1(x,mu)-nullcline2(x,mu)
# fp=optimize.root_scalar(opt_func, bracket=[0, 5]).root
# print("fp=",fp)

plt.grid()
plt.xlabel('c, nanoMoll')
plt.ylabel('h, dimentionless')
plt.title('mu='+str(mu)+': there is a limit cycle')
plt.legend()
plt.show()
```



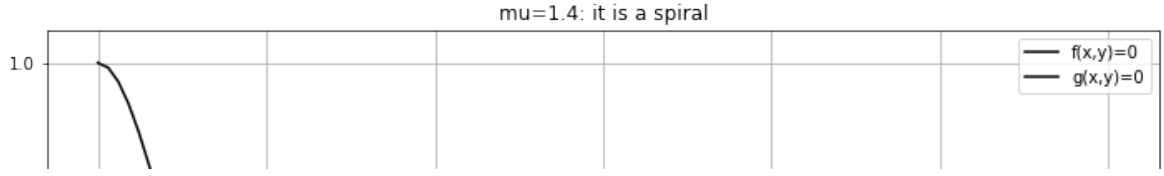


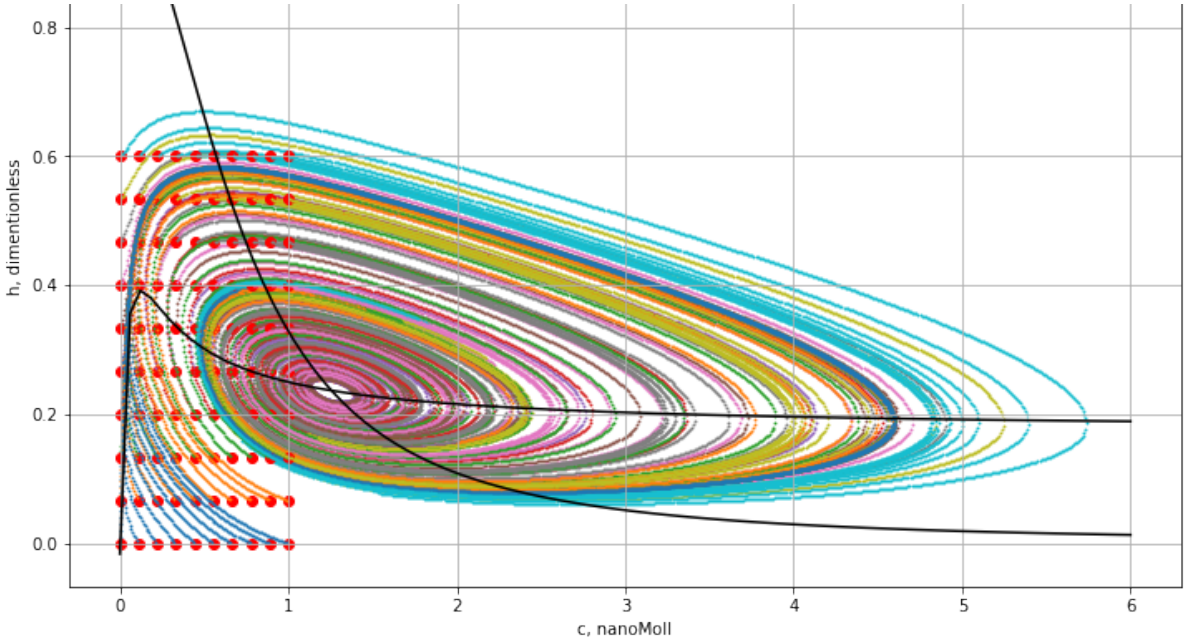
We see that for $\mu = 0.8$, we have a stable limit cycle close to $(0.8, 0.5)$ on (c, h) plane.

We see that for this regime there will be a limit cycle and hence oscillations, so we will need to compute max and min aplitudes of trajectories.

In [269]:

```
mu=1.4

def f(x,y,mu):
    return mu*8.1*y*(0.111+0.889*x/(0.7+x))-2*x/(0.1+x) + 0.02
def g(x,y,mu):
    return 1/(1+(x/0.7)**2)/2 - y/2

def nullcline1(x,mu):
    return (2*x/(0.1+x)-0.02)/(8.1*mu*(0.111+0.889*x/(0.7+x)))
def nullcline2(x,mu):
    return 1/(1+(x/0.7)**2)

k=0.01
xrange=np.linspace(0,1,10)
yrange=np.linspace(0,0.6,10)
nullclinerange=np.linspace(0,6,100)
plt.figure(figsize=(12, 8))

for x0 in xrange:
    for y0 in yrange:
        masx=[x0]
        masy=[y0]
        plt.scatter(x0,y0,color='red')
        #print(x,y)
        for i in range(1000):
            x,y=masx[i],masy[i]
            u,v=f(x,y,mu),g(x,y,mu)
            dx,dy=k*u/np.sqrt(u**2 + v**2),k*v/np.sqrt(u**2 + v**2)
            #print(dx,dy)
            masx.append(x+dx)
            masy.append(y+dy)
        #plt.arrow(x=x,y=y,dx=u/np.sqrt(u**2 + v**2),dy=v/(u**2 + v
        plt.scatter(masx[1:],masy[1:],s=0.8)

plt.plot([x for x in nullclinerange], [nullcline1(x,mu) for x in nu
plt.plot([x for x in nullclinerange],[nullcline2(x,mu) for x in nul

# opt_func = lambda x: nullcline1(x,mu)-nullcline2(x,mu)
# fp=optimize.root_scalar(opt_func, bracket=[0, 5]).root
# print("fp=",fp)

plt.grid()
plt.xlabel('c, nanoMoll')
plt.ylabel('h, dimentionless')
plt.title('mu='+str(mu)+': it is a spiral')
plt.legend()
plt.show()
```



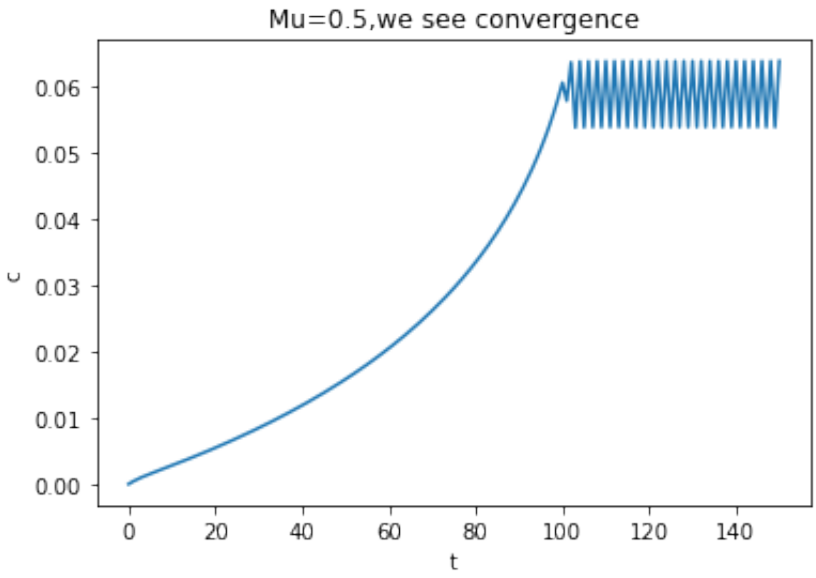


We see that for $\mu = 1.4$, we have a stable spiral close to $(1.2, 0.25)$ on (c, h) plane. All trajectories tend to this point for fixed μ, so there will be the only fixed point for μ from regime 1.

3b) find out in which of three regimes we see significanr oscillations

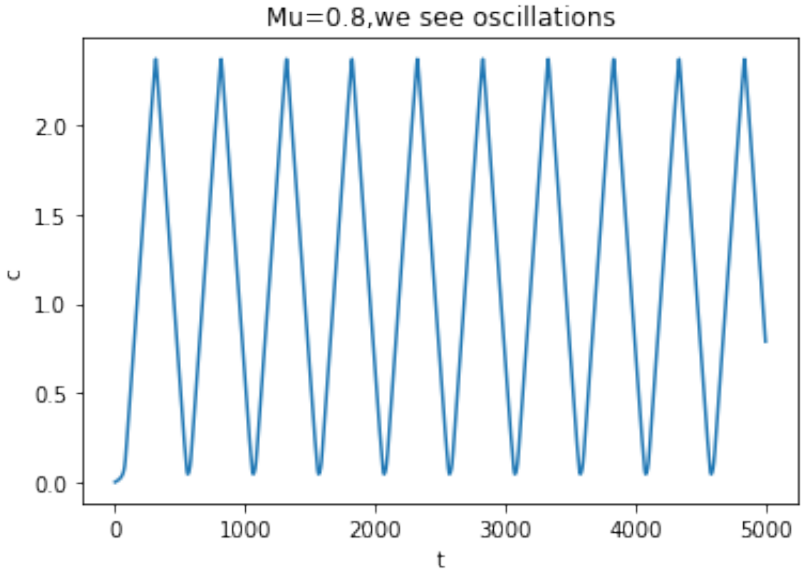
In [455]:
```
mu=0.5 # we see convergence
k=0.01
masx=[0.0]
masy=[0.0]
t_range=range(150)
for t in t_range:
    x,y=masx[t],masy[t]
    u,v=f(x,y,mu),g(x,y,mu)
    dx,dy=k*u/np.sqrt(u**2 + v**2),k*v/np.sqrt(u**2 + v**2)
    masx.append(x+dx)
    masy.append(y+dy)
print("c_min=",np.min(masx[-50:-1]),"c_max=",np.max(masx[-50:-1]),"
plt.plot(np.arange(len(masx)),masx)
plt.xlabel('t')
plt.ylabel('c')
plt.title('Mu=0.5,we see convergence')
plt.show()
```

```
c_min= 0.0538990442318773 c_max= 0.0639576723369066 diff= 0.010058
628105029305
```



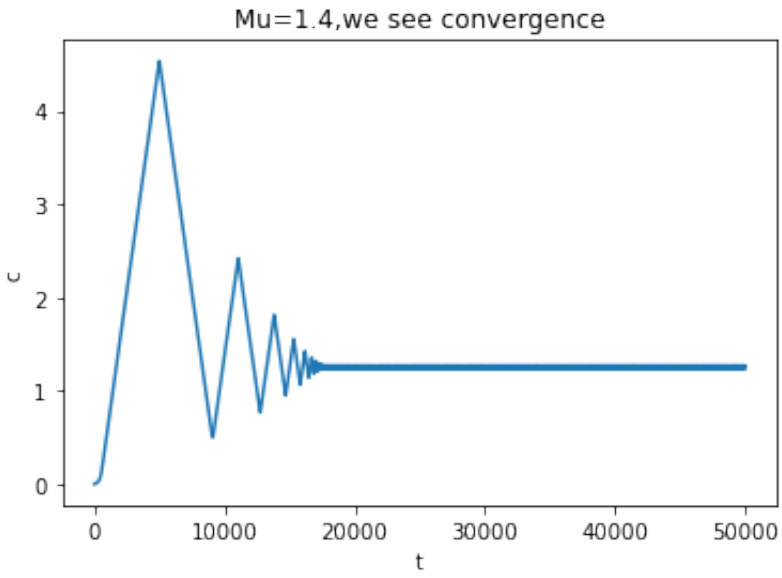
In [456]:
```
mu=0.8 # we see big amplitude oscillations
k=0.01
masx=[0.0]
masy=[0.0]
t_range=range(5000)
for t in t_range:
    x,y=masx[t],masy[t]
    u,v=f(x,y,mu),g(x,y,mu)
    dx,dy=k*u/np.sqrt(u**2 + v**2),k*v/np.sqrt(u**2 + v**2)
    masx.append(x+dx)
    masy.append(y+dy)

print("c_min=",np.min(masx[-50:-1]),"c_max=",np.max(masx[-50:-1]),"

plt.plot(np.arange(len(masx)),masx)
plt.xlabel('t')
plt.ylabel('c')
plt.title('Mu=0.8,we see oscillations')
plt.show()
```

```
c_min= 0.7985847927750621 c_max= 1.2776893828268887 diff= 0.479104
5900518266
```



In [457]:
```
mu=1.4 #we see convergence
k=0.001
masx=[0]
masy=[0]
t_range=range(50000)
for t in t_range:
    x,y=masx[t],masy[t]
    u,v=f(x,y,mu),g(x,y,mu)
    dx,dy=k*u/np.sqrt(u**2 + v**2),k*v/np.sqrt(u**2 + v**2)
    masx.append(x+dx)
    masy.append(y+dy)
print("c_min=",np.min(masx[-50:-1]),"c_ma=x",np.max(masx[-2000:-1])
plt.plot(np.arange(len(masx)),masx)
plt.xlabel('t')
plt.ylabel('c')
plt.title('Mu=1.4,we see convergence')
plt.show()
```

```
c_min= 1.2438772658077775 c_ma=x 1.2701942653251108 diff= 0.026316
99951733335
```

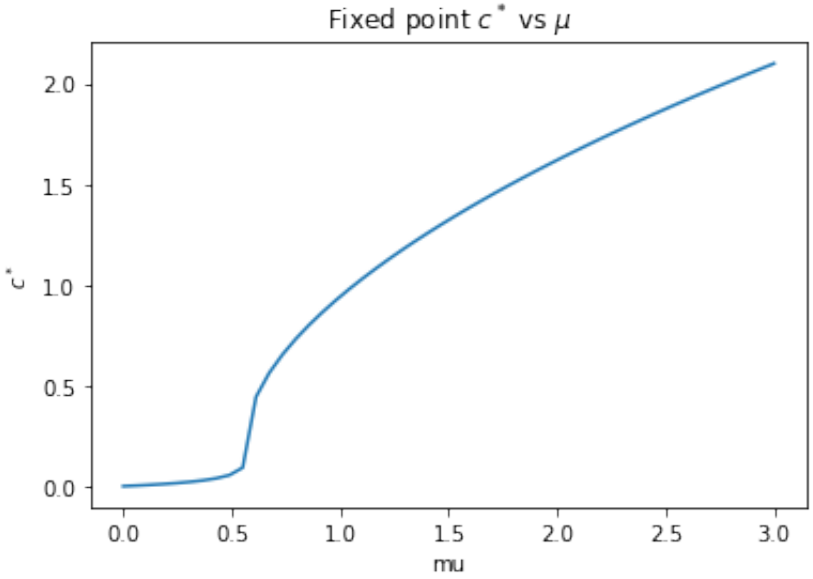


3c) Plot bifurcation diagram for μ

First print the line that is given by the intersection of nullclines.

In [458]:
```
#fixed point
mas_mu=[]
mas_fp=[]

for mu in np.linspace(0,3,50):
    opt_func = lambda x: nullcline1(x,mu)-nullcline2(x,mu)
    fp=optimize.root_scalar(opt_func, bracket=[0, 5]).root
    mas_mu.append(mu)
    mas_fp.append(fp)
plt.plot(mas_mu,mas_fp)
plt.xlabel('mu')
plt.ylabel('$c^*$')
plt.title('Fixed point $c^*$ vs $\mu$')
plt.show()
```

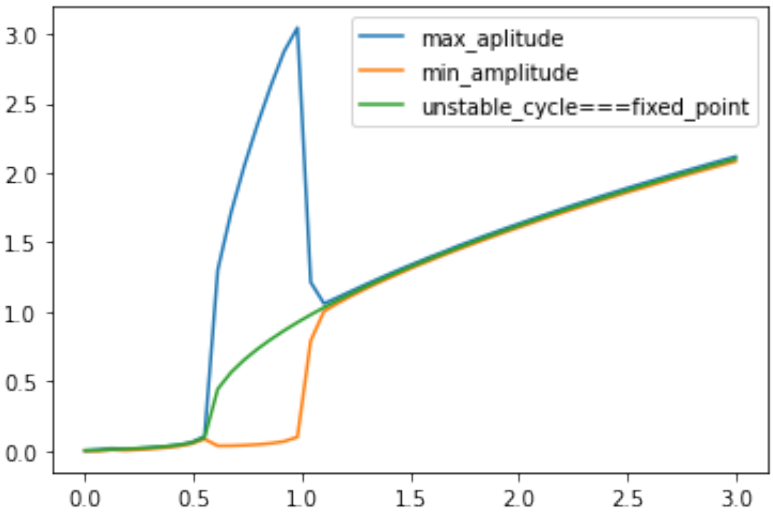
```
<ipython-input-269-a13c8b766be4>:9: RuntimeWarning: divide by zero
encountered in double_scalars
  return (2*x/(0.1+x)-0.02)/(8.1*mu*(0.111+0.889*x/(0.7+x)))
```



Now plot the bifurcation diagram: to do this, we take different μ, and for each μ we run many simulations of trajectories of the system and measure the min and max amlitude of them in a long-time runned (converged) mode. Note that for μ in regime 2 green line (that is given) is unstable limit cycle (we can check this by reversing the time and running simulations -
now all trajectories with μ from regime 2 will converge to green curve).
While blue and orange lines are stable limit cycles.

In [287]:
```
from tqdm import tqdm
mas_mu=[]
mas_max_amplitude=[]
mas_min_amplitude=[]
for mu in tqdm(np.linspace(0,3,50)):
    mas_mu.append(mu)
    masx=[0.0]
    masy=[0.0]
    k=0.01
    t_range=range(5000)
    if (mu >1):
        k=0.001
        t_range=range(50000)
    for t in t_range:
        x,y=masx[t],masy[t]
        u,v=f(x,y,mu),g(x,y,mu)
        dx,dy=k*u/np.sqrt(u**2 + v**2),k*v/np.sqrt(u**2 + v**2)
        masx.append(x+dx)
        masy.append(y+dy)
    mas_max_amplitude.append(np.max(masx[-2000:-1]))
    mas_min_amplitude.append(np.min(masx[-2000:-1]))

plt.plot(mas_mu,mas_max_amplitude,label='max_aplitude')
plt.plot(mas_mu,mas_min_amplitude,label='min_amplitude')
plt.plot(mas_mu,mas_fp,label='unstable_cycle===fixed_point')
plt.legend()
plt.show()
```

```
100%|██████████| 50/50 [00:11<00:00,  4.20it/s]
```



Q4

4a) Fixed points

Fixed points are such x that $f(x) = x$, this means $rx(1 - x^{\theta}) = 0$ so $x = 0$ or $x = 1$;

To analyse stability use $f'(x) = 1 + r(1 - x^{\theta}) - r\theta x^{\theta}$;

$f'(0) = 1 + r > 1$ so $x = 0$ is unsatable;

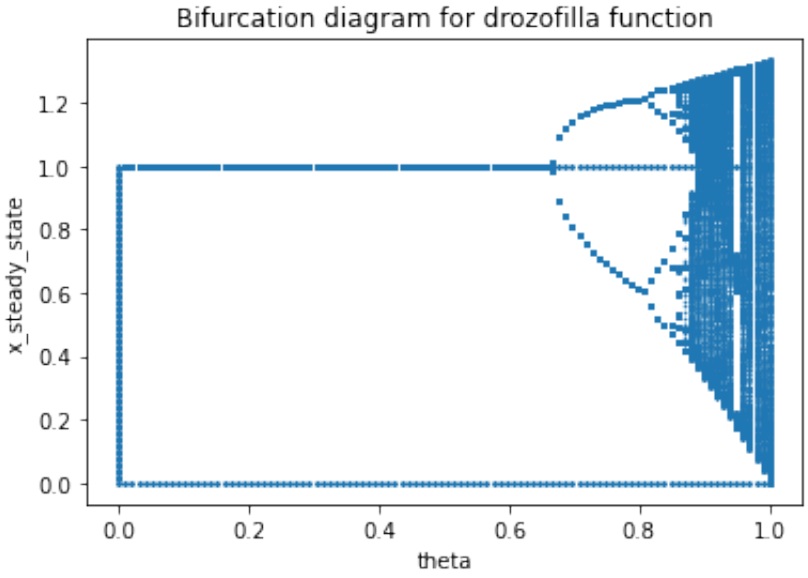
$f'(1) = 1 - r\theta$ is < 1 in modulo when $\theta \in (0, \frac{2}{r})$, so for these θ $x = 1$ is stable;
Stable 2-cycle occurs for the firs time, when the fixed point loses stability for the first time, and it happens at $\frac{2}{r}$.

4b) Plot bifurcation diagram

The algorithm is the follows: for all theta run N-lengs simulations each starting at a different $x0$, take the steady state of trajectory by throwing away the beginning of each tragectory, and plot the (θ, x_i) pairs on the plane.

In [365]:

```
r=3

def fdroz(x,r,theta):
    if np.abs(x) < 1e-15:
        return 0.0
    if np.abs(1-x) < 1e-15:
        return 1.0
    return x + r*x*(1-x**theta)
r=3

otvtheta=[]
otvx=[]
for theta in np.linspace(0,3/3,100):
    for x0 in np.linspace(0,1,100):
        masx=[x0]
        N=1000
        for i in range(N):
            x=masx[i]
            masx.append(fdroz(x,3,theta))
        masx=masx[-20:-1]
        for el in masx:
            otvtheta.append(theta)
            otvx.append(el)
plt.scatter(otvtheta,otvx,s=0.5)
plt.xlabel('theta')
plt.ylabel('x_steady_state')
plt.title('Bifurcation diagram for drozofilla function')
plt.show()
```



We see that the first point when $x = 1$ looses stability and transforms into two points is exactly at point $\theta = \frac{2}{r} = \frac{2}{3}$ that we found analytically earlier.

4c calculate Lyapunov exponent for different thetas

We just take the formula from the lectures that say we need to take any starting point x_0, calculate $x_n := f^n(x_0)$, take logarithms of the absolute value of each $\frac{df(x_i)}{dx}$, where $\frac{df}{dx}$ is the derivative of drozofilla function, and take the limit of mean values of these logarithms. This limit is called the Lyapunov exponent. And we just plot these limit values for different θs.

In [411]:
```
def fdroz_prime(x,r,theta):
    return 1 +r*(1-x**theta) -r*theta*x**theta
r=3
mas_lyapunovs=[]
mas_theta=np.linspace(0,3/3,2000)
for theta in tqdm(mas_theta):
    N=1000
    x0=np.pi/5 #take any initial condition
    s=0
    masx=[x0]
    for i in range(N):
        xi=masx[-1]
        s+=np.log(np.abs(fdroz_prime(xi,r,theta)))
        masx.append(fdroz(xi,r,theta))
    mas_lyapunovs.append(s/N)
plt.scatter(mas_theta,mas_lyapunovs,s=0.5)
plt.xlabel('theta')
plt.ylabel('Lyapunov exponent')
plt.grid()
plt.title('Lyapunov exponent for drozofilla map')
plt.show()
```

```
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```

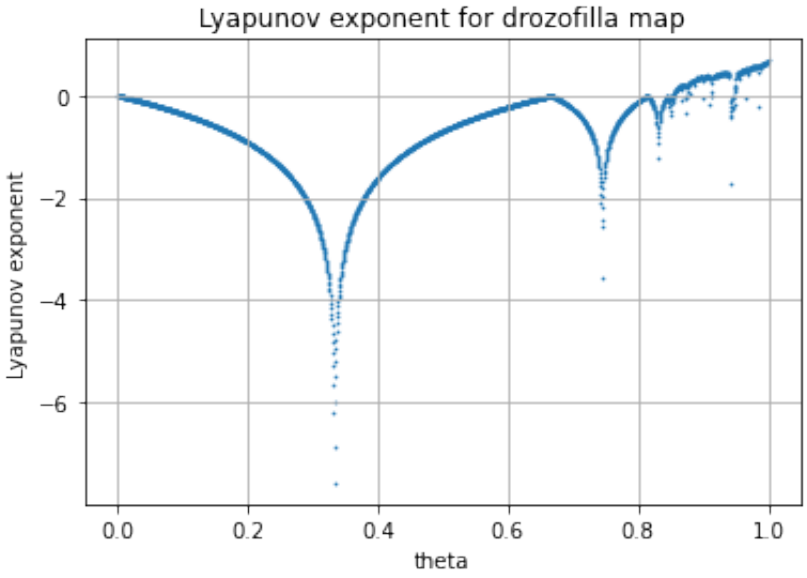

Note that for negative Lyapunov exponents the regime is stable.

We also see that both the bifurcation diagram and Lyapunov exponent graphs look similar to what we had in the lecture with logistic map.

In [ ]: