THE UNIVERSITY OF WARWICK

FOURTH YEAR EXAMINATION: 2019

QUANTUM MECHANICS

Time Allowed: **3 hours**

Read carefully the instructions on the answer book and make sure that the particulars required are entered on each answer book.

Calculators are not needed and are not permitted in this examination.

Candidates should answer COMPULSORY QUESTION 1 and THREE QUESTIONS out of the four optional questions 2, 3, 4 and 5.

The compulsory question is worth 40% of the available marks. Each optional question is worth 20%.

If you have answered more than the compulsory Question 1 and three optional questions, you will only be given credit for your QUESTION 1 and THREE OTHER best answers.

The numbers in the margin indicate approximately how many marks are available for each part of a question.

COMPULSORY QUESTION

1. (i) Write down the Schrödinger equation for a particle in $\mathbb{R}$ under an external potential V. [4]

(ii) Show that the Laplacian $\Delta = \frac{\mathrm{d}^2}{\mathrm{d}x^2}$ is a symmetric operator in $L^2(\mathbb{R})$ (for the domain, take the space of Schwartz functions). [4]

(iii) Show that the operator Δ in (ii) is unbounded. [4]

(iv) On the interval $[0, 1]$, consider the Laplacian $\Delta = \frac{\mathrm{d}^2}{\mathrm{d}x^2}$ with domain $\mathcal{D}(\Delta) = C^2([0, 1])$. Is Δ symmetric? Self-adjoint? [4]

(v) Consider the 1D infinite square well, that is, consider the Laplacian on $L^2([0, L])$ with Dirichlet boundary conditions. Describe its spectrum and eigenvectors. [4]

(vi) Let X be the operator $f \mapsto xf$ and P the operator $f \mapsto -\mathrm{i}f'$ on $L^2(\mathbb{R})$. Show that $[X, P] = \mathrm{i}$. [4]

(vii) Let X, P as in (vi), and Δ as in (ii). Find $[X, \Delta]$ and write it in terms of P. [4]

(viii) Find $[X^2, P]$. [4]

Question 1 continued overleaf

(ix) Give a characterisation of the Wiener measure on continuous paths. (No proofs needed.) [4]

(x) Let $H = -\frac{1}{2}\Delta + V$ in $L^2(\mathbb{R})$, where V is a bounded continuous function. Write down the Feynman-Kac formula, using the Wiener measure on continuous paths. [4]

OPTIONAL QUESTIONS

2. Consider the Hilbert space $\mathbb{C}^n$.

(a) (i) Give the definition of a quantum state. [2]

(ii) What are pure and mixed quantum states? [2]

(b) Give the definition of a density operator. [3]

Recall that if ρ is a density operator, then we can define a state ω by assigning the value $\omega(A) = \mathrm{Tr}\rho A$ to each operator A.

(c) Prove the converse, namely that to any quantum state ω there corresponds a density matrix such that $\omega(A) = \mathrm{Tr}\rho A$. [5]

(d) State the Heisenberg uncertainty principle for states. [3]

(e) Give a proof of the Heisenberg uncertainty principle for states. (You get partial credit if you give a proof for pure states only.) [5]

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3. Recall the Hamiltonian for the harmonic potential: $H = \frac{1}{2}(-\Delta + x^2)$ on $L^2(\mathbb{R})$ (with domain so that H is self-adjoint). Recall the definition of the creation and annihilation operators:

$$a^* = (X - \mathrm{i}P)/\sqrt{2}, \qquad a = (X + \mathrm{i}P)/\sqrt{2}.$$

(a) Check that $H = N + \frac{1}{2}$, where $N = a^*a$. [3]

(b) Find an eigenvector of N with eigenvalue 0. [3]

(c) Show that $\{0, 1, 2, \ldots\}$ are eigenvalues. [4]

(d) Show that $\lambda \in \mathbb{C} \setminus \{0, 1, 2, \ldots\}$ cannot be an eigenvalue. [4]

(e) Consider the operator $H = \frac{1}{2}(-\Delta - 2x + x^4)$ on $L^2(\mathbb{R})$.

(i) Show that $H \geq 0$. Hint: It is possible to write $H = b^*b$ with suitable operator b. [3]

(ii) Show that 0 is not an eigenvalue. Hint: Assume that the eigenvector exists, find it, and check that it is not in L^2, contradiction. [3]

Question 4 continued overleaf

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4. Recall the Hamiltonian for the hydrogen atom: $H = -\Delta - \frac{1}{\|x\|}$ on $L^2(\mathbb{R}^3)$ (with domain $\mathcal{D}(H) = H^2(\mathbb{R}^3)$ so that it is self-adjoint).

(a) Prove that H is bounded below. Hint: You can use the Sobolev inequality $\|\nabla f\|_2^2 \geq 3(\frac{\pi}{2})^{4/3}\|f\|_6^2$. **[6]**

(b) Show that a function of the form $f(x) = \mathrm{e}^{-a\|x\|}$ with $a > 0$ (to be found) is an eigenvector, and find the corresponding eigenvalue. **[4]**

(c) Describe the full spectrum of H, including the multiplicity of all eigenvalues (no proof required). **[5]**

(d) Prove that 0 belongs to the spectrum of H. Hint: Consider a sequence of normalised functions (g_n) of the form $g_n(x) = c_n g(\frac{x}{n})$ for some fixed g. **[5]**

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5. Let A, B be bounded operators in the Hilbert space $\mathcal{H}$.

(a) (i) Give the definition of the operator e^{sA} where $s \in \mathbb{C}$. **[3]**

(ii) Prove the group property $\mathrm{e}^{sA}\mathrm{e}^{tA} = \mathrm{e}^{(s+t)A}$ for any $s, t \in \mathbb{C}$. **[3]**

(iii) Prove that $\frac{\mathrm{d}}{\mathrm{d}t}\mathrm{e}^{tA} = A\mathrm{e}^{tA}$ for any $t \in \mathbb{R}$. **[3]**

(b) Prove the Trotter product formula:

$$\mathrm{e}^{A+B} = \lim_{n\to\infty} (\mathrm{e}^{\frac{1}{n}A}\mathrm{e}^{\frac{1}{n}B})^n.$$

[4]

Now we consider the operator $\mathrm{e}^{\frac{1}{2}t\Delta}$ in $L^2(\mathbb{R})$ where $t \geq 0$.

(c) What is the norm of $\mathrm{e}^{\frac{1}{2}t\Delta}$ for $t \geq 0$? **[3]**

(d) Find the integral kernel of $\mathrm{e}^{\frac{1}{2}t\Delta}$, that is, the function $a_t(x, y)$ such that

$$\mathrm{e}^{\frac{1}{2}t\Delta} f(x) = \int_{-\infty}^{\infty} a_t(x, y) f(y) \mathrm{d}y.$$

Give a full proof. **[4]**
