University of Warwick

April 2019

Topics in Mathematical Finance

Instructions

This is a CLOSED book examination.

Time allowed: 2 hours

Only silent calculators that are provided by the Programme Team are permitted. Electronic devices such as, for example, a mobile phone, tablet, smart watch, fitbit or similar device are not permitted.

Answer **ALL Three** questions from Section 1 and **One** question from Section 2. Full marks may be obtained by correctly answering three complete questions from Section 1 and one complete question from Section 2. Candidates may attempt all questions. Marks will be awarded for the best answer from section 2 only.

Read carefully the instructions on the answer book and make sure that the particulars required are entered on each answer book.

PLEASE TURN OVER

SECTION 1

[Question 1]

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space, $\mathcal{X} := L^1(\Omega, \mathcal{F}, \mathbb{P})$ and $\mathcal{M}^\infty(\mathbb{P})$ the set of all probability measures $\mathbb{Q}$ on $(\Omega, \mathcal{F})$ that are absolutely continuous with respect to $\mathbb{P}$ and whose Radon-Nikodým derivative is $\mathbb{P}$-a.s. bounded.

A. Define the map $\rho : \mathcal{X} \to \mathbb{R}$ by $\rho(X) = \mathbb{E}[-X + 2019]$.

(i) Show that ρ is a monetary measure of risk by stating and checking the defining properties of a monetary measure of risk. [2]

(ii) Is ρ convex? Either give a proof or provide a counterexample. [2]

B. State a dual representation result for a convex risk measure $\rho : \mathcal{X} \to \mathbb{R}$ and define all objects involved. [2]

C. Calculate $\mathrm{VaR}_\alpha(-X)$ and $\mathrm{ES}_\alpha(-X)$ for $\alpha = 0.001$, where X is exponentially distributed with rate parameter $\lambda = 2$. [4]

[Question 2]

Let $L \in \mathbb{R}^{n \times n}$ be a liability matrix, where the ij^{th} entry L_{ij} represents the nominal liability of bank i to j. Assume that $L_{ij} \geq 0$ and $L_{ii} = 0$ for any i, j. Let $e \in \mathbb{R}^n$ be an external cash flow vector, where the i^{th} entry e_i represents the external assets of bank i minus its external liabilities.

A. Write down the corresponding relative liability matrix Π. [2]

B. Write down the definition of a clearing payment vector, and the corresponding clearing payment equation. [3]

C. For any two clearing payment vectors p^* and $\hat{p}^*$, prove that their corresponding net value vectors are the same, i.e.

$$\Pi^{tr} p^* + e - p^* = \Pi^{tr} \hat{p}^* + e - \hat{p}^*.$$

[Hint: you may use without proof the existence of the greatest clearing payment vector.] [5]

Please turn over

[Question 3]

A. Let $X = (X_t)_{t\geq 0}$ be a semi-martingale and let $f : \mathbb{R} \mapsto \mathbb{R}$ be a twice continuously differentiable function. State Itô's formula for $f(X_t)$. [3]

B. Let $Y = (Y_t)_{t\geq 0}$ be a time homogeneous diffusion which satisfies the stochastic differential equation (SDE)

$$dY_t = a(Y_t)dB_t + b(Y_t)dt, \qquad Y_0 = y,$$

where $B = (B_t)_{t\geq 0}$ is a Brownian motion. Suppose $S = (S_t)_{t\geq 0}$ solves

$$dS_t = S_t(\sigma(Y_t)dW_t + \mu(Y_t)dt), \qquad S_0 = s,$$

where σ is a twice continuously differentiable, invertible function and W is a Brownian motion. Suppose B and W have correlation $\rho \in (-1, 1)$.

Let $\Sigma = (\Sigma_t)_{t\geq 0}$ be given by $\Sigma_t = \sigma(Y_t)$. Derive an autonomous SDE for Σ and write down an equation for S in terms of Σ. [4]

C. In the Hull-White model the price process is modelled as

$$dS_t = S_t(\sqrt{V_t}dW_t + rdt), \qquad S_0 = s,$$

where $V = (V_t)_{t\geq 0}$ solves

$$dV_t = \theta V_t dB_t + \kappa V_t dt$$

subject to $V_0 = v$. (Here, as above, the Brownian motions B and W have correlation $\rho \in (-1, 1)$, and r, θ, κ and ρ are all constants.)

Using Part B or otherwise, derive an SDE for Σ where $\Sigma_t = \sqrt{V_t}$, and an equation for S in terms of Σ. [3]

Please turn over

SECTION 2

[Question 4]

Let $\mathcal{X}$ be a linear subspace of bounded real-valued random variables on a measurable space $(\Omega, \mathcal{F})$ containing the constants.

A. Let $\mathcal{A}$ be a nonempty convex subset of $\mathcal{X}$ such that $\inf\{m \in \mathbb{R} : m \in \mathcal{A}\} > -\infty$ and $X \in \mathcal{A}, Y \in \mathcal{X}, Y \geq X \implies Y \in \mathcal{A}$. Let $\rho_{\mathcal{A}} = \inf\{m \in \mathbb{R} : m + X \in \mathcal{A}\}$ be the associated risk measure. Show that $\rho_{\mathcal{A}}$ is convex. [4]
Hint: In your answer, you may use without proof that $\rho_{\mathcal{A}}$ is monotone and cash-invariant.

B. Let $S \in \mathcal{F}$ be a nonempty set of stress scenarios and define the map $\rho_S : \mathcal{X} \to \mathbb{R}$ by

$$\rho_S(X) = -\inf_{\omega \in S} X(\omega).$$

(i) Show that ρ_S is a coherent risk measure. [4]

(ii) Show that ρ_S is continuous from above. [4]

(iii) Show that ρ_S satisfies the dual representation

$$\rho_S(X) = \sup_{\mathbb{Q} \in \mathcal{Q}_S} \mathbb{E}^{\mathbb{Q}}[-X], \qquad (*)$$

where $\mathcal{Q}_S$ denotes the set of all probability measures on $(\Omega, \mathcal{F})$ satisfying $\mathbb{Q}[S] = 1$. [5]
Hint: Consider the probability measures $\delta_{\{\omega\}}$ for $\omega \in S$.

(iv) Show that the supremum in $(*)$ can be replaced by a maximum if S is a finite set. [3]

Please turn over

[Question 5]

Let τ be a non-negative random variable defined on a complete probability space $(\Omega, \mathcal{F}, \mathbb{P})$, and $\mathbb{F} = \{\mathcal{F}_t\}_{t\geq 0}$ be the filtration given by $\mathcal{F}_t = \sigma(\{\tau \leq u\} : u \leq t)$.

A. For any $A \in \mathcal{F}_t$, write down the two possibilities of $A \cap \{\tau > t\}$. [2]

B. Let Y be an $\mathcal{F}_\infty$-measurable and bounded random variable, where $\mathcal{F}_\infty = \sigma\left(\cup_{t\geq 0}\mathcal{F}_t\right)$. Prove that

$$\mathbb{E}[1_{\{\tau>t\}}Y|\mathcal{F}_t] = 1_{\{\tau>t\}}\frac{\mathbb{E}[1_{\{\tau>t\}}Y]}{\mathbb{P}(\tau > t)}.$$

[3]

C. Prove that τ follows exponential distribution with a constant intensity $\lambda > 0$ if and only if the process $M = (M_t)_{t\geq 0}$, where

$$M_t = 1_{\{\tau\leq t\}} - \int_0^t 1_{\{\tau>s\}}\lambda ds,$$

is an $(\mathbb{F}, \mathbb{P})$-martingale and $\mathbb{P}(\tau > 0) = 1$. [5]

D. Let $T > 0$ and $\mu \in [0, 1]$ be fixed numbers. Under the assumption in part C, prove that the process $Z^\mu = (Z_t^\mu)_{t\in[0,T]}$, where

$$Z_t^\mu = \left(1_{\{\tau>t\}} + (1-\mu)1_{\{\tau\leq t\}}\right) e^{\int_0^t \mu 1_{\{\tau>s\}}\lambda ds},$$

is an $(\mathbb{F}, \mathbb{P})$-martingale.

[Hint: Let $H_t := 1_{\{\tau\leq t\}}$ and $V_t := 1 - H_t + (1-\mu)H_t$. You may first prove that $\Delta V_s = -\mu V_{s-}\Delta H_s$.] [5]

E. Let Z_T^μ be given as in part D and define $\mathbb{Q}$ on $\mathcal{F}_T$ by $\frac{d\mathbb{Q}}{d\mathbb{P}} = Z_T^\mu$. Prove that the process $M^\mu = (M_t^\mu)_{t\in[0,T]}$, where

$$M_t^\mu = 1_{\{\tau\leq t\}} - \int_0^t (1-\mu)1_{\{\tau>s\}}\lambda ds,$$

is an $(\mathbb{F}, \mathbb{Q})$-martingale.

[Hint: you may use without proof the fact that M^μ is an $(\mathbb{F}, \mathbb{Q})$-martingale if and only if $M^\mu Z^\mu$ is an $(\mathbb{F}, \mathbb{P})$-martingale.] [5]

Please turn over

[Question 6]

A. Let $X = (X_t)_{t\geq 0}$ be a diffusion process with state space $\mathbb{R}$. Suppose that X is sufficiently regular so that at time t the law of X has a density $p_t^X(\cdot)$, and that $p_t^X(x)$ is continuous in both x and t. The functions h and a below may also be assumed to be continuous.

(i) Explain briefly why $\mathbb{E}[h(X_T, T)] = \int_{\mathbb{R}} h(s,T)p_T^X(s)ds$

(ii) Explain briefly why $p_T^X(x) = \frac{\partial^2}{\partial x^2}\mathbb{E}[(X_T - x)^+]$.

(iii) Explain briefly why $\lim_{\Delta\downarrow 0}\mathbb{E}[\frac{1}{\Delta}\int_T^{T+\Delta} h(X_u, u)du] = \mathbb{E}[h(X_T, T)]$.

(iv) Explain briefly why if $f(y) = (y-\kappa)^+$ we have the heuristic

$$\mathbb{E}\left[X_T^2 a(X_T,T)^2 f''(X_T)\right] = \kappa^2 a(\kappa, T)^2 p_T^X(\kappa)$$

[7]

B. Let $S = (S_t)_{t\geq 0}$ be a martingale diffusion process with state space $\mathbb{R}^+$ and dynamics $dS_t = S_t\sigma(S_t,t)dW_t$ where W is a Brownian motion. Suppose that at time T the law of S has a density $p_T^S(\cdot)$. [You may assume sufficient regularity for S that the results of Part A of the question apply, that changing orders of taking limits and integrating is possible, and that any local martingales are martingales.]

(i) Let $C(K,T) = \mathbb{E}[(S_T - K)^+]$. Show that $\frac{\partial^2}{\partial K^2}C(K,T) = p_T^S(K)$.

(ii) For h twice differentiable, write down an expression for $h(S_{T+\Delta}) - h(S_T)$ using Itô's formula. Hence deduce an expression for $\mathbb{E}[\frac{1}{\Delta}\{h(S_{T+\Delta}) - h(S_T)\}]$.

(iii) Setting $h(s) = (s-K)^+$ explain why we expect

$$\frac{\partial}{\partial T}C(K,T) = \frac{1}{2}K^2\sigma(K,T)^2 p_T^S(K).$$

(iv) Deduce Dupire's formula for the squared local volatility

$$\sigma(K,T)^2 = \frac{2\frac{\partial}{\partial T}C(K,T)}{K^2\frac{\partial^2}{\partial K^2}C(K,T)}.$$

[8]

C. Suppose interest rates are zero and call prices are given by

$$C(K,T) = S_0\frac{\Phi(d_{1,+}) + \Phi(d_{2,+})}{2} - K\frac{\Phi(d_{1,-}) + \Phi(d_{2,-})}{2},$$

where $\Phi(\cdot)$ is the cumulative normal distribution,

$$d_{i,\pm} = d_{i,\pm}(K,T) = \frac{\ln(S_0/K) \pm \frac{1}{2}\sigma_i^2 T}{\sigma_i\sqrt{T}},$$

and $0 < \sigma_1 < \sigma_2$.

(i) Find an expression for the local volatility $\sigma(K,T)$ in this model.

(ii) Explain why $\sigma_1 < \sigma(K,T) < \sigma_2$.

Hint: you may use known properties of the Black-Scholes model. [5]

End of Paper