THE UNIVERSITY OF WARWICK

FOURTH YEAR EXAMINATION: SUMMER 2018

QUANTUM MECHANICS: BASIC PRINCIPLES AND PROBABILISTIC METHODS

Time Allowed: **3 hours**

Read carefully the instructions on the answer book and make sure that the particulars required are entered on each answer book.

Calculators are not needed and are not permitted in this examination.

Candidates should answer COMPULSORY QUESTION 1 and THREE QUESTIONS out of the four optional questions 2, 3, 4 and 5.

The compulsory question is worth 40% of the available marks. Each optional question is worth 20%.

If you have answered more than the compulsory Question 1 and three optional questions, you will only be given credit for your QUESTION 1 and THREE OTHER best answers.

The numbers in the margin indicate approximately how many marks are available for each part of a question.

COMPULSORY QUESTION

1. a) Suppose that the Hamilton operator of a quantum system is described by the matrix

$$H = \begin{pmatrix} -3 & 4 \\ 4 & 3 \end{pmatrix}.$$

and that the system is in the state ω which corresponds to the orthogonal projection on the direction of the vector

$$\psi = \begin{pmatrix} 2 \\ 1 \end{pmatrix}.$$

(i) Find the expectation $\langle H \rangle_\omega$. **[2]**

(ii) Find possible values for the energy of the quantum system and determine the probabilities for each one. **[4]**

(iii) Find the density matrix for the ground state of the quantum system. **[4]**

b) Consider a quantum system described by a Hamilton operator $H : \mathbb{C}^n \to \mathbb{C}^n$ where $n \in \mathbb{N}$. Suppose the quantum system is in a pure state ω. State and prove the Heisenberg uncertainty relation for two observables A and B. **[8]**

c) Let $P\psi = -i\hbar\dfrac{d\psi}{dx}$ and $Q\psi = x\psi$ be momentum and position operators on $L^2(\mathbb{R})$. Here $\hbar$ is the Planck constant.

(i) State without proof the Heisenberg uncertainty relation for the operators Q and P. **[2]**

(ii) Show that the point spectrum of P is empty. **[4]**

(iii) Suppose that $H = P$ is a Hamilton operator of a quantum system. For every $\psi_0 \in L^2(\mathbb{R})$ and every $t \in \mathbb{R}$ find an expression for $U_t\psi_0$, where U_t is the evolution operator of the quantum system. Hint: state and solve the corresponding Schrödinger equation. **[4]**

d) Suppose that a quantum particle is placed in a one-dimensional box and its Hamilton operator acts by the formula $H\psi = -\dfrac{1}{2}\dfrac{d^2\psi(x)}{dx^2}$ on functions $\psi \in D(H)$ where

$$D(H) = \Big\{\psi \in C^2([0, 2\pi]) : \psi(0) = \psi(2\pi) = 0\Big\} \subset L^2([0, 2\pi])\,.$$

(i) Show that H is symmetric on $D(H)$. **[3]**

(ii) For every stationary state of the particle find a wave function. **[5]**

e) Let a potential $V \in C^0(\mathbb{R}^d)$ be continuous and bounded below and $H = -\frac{1}{2}\Delta + V$ be a self-adjoint operator in $L^2(\mathbb{R}^d)$. State without proof the Feynman-Kac formula. **[4]**

OPTIONAL QUESTIONS

2. a) In the momentum representation, the position and momentum operators of a particle are given by $(\hat{Q}\hat{\psi})(k) = i\dfrac{\partial\hat{\psi}(k)}{\partial k}$ and $(\hat{P}\hat{\psi})(k) = \hbar k\hat{\psi}(k)$.

(i) Write down the Schrödinger equation for a free particle in the momentum representation. You may assume that the particle's mass $m = 1$. **[2]**

(ii) Find a solution for the corresponding initial value problem assuming $\hat{\psi}(k, 0) = \hat{\psi}_0(k)$ where $\hat{\psi}_0$ is a smooth function with compact support. **[2]**

(iii) Deduce that the expectation $\langle P\rangle_{\psi(t)}$ is independent of t and the expectation $\langle Q\rangle_{\psi(t)}$ has ballistic motion, i.e. $\langle Q\rangle_{\psi(t)} = q_0 + v_0 t$ for some $q_0, v_0 \in \mathbb{R}$. **[4]**

b) Consider a quantum particle of mass $m > 0$ which moves in a smooth one-dimensional potential V. Suppose that the state $\omega(t)$ of the quantum particle is described by a classical solution of the Schrödinger equation

$$i\hbar\frac{\partial\psi(x,t)}{\partial t} = -\frac{\hbar^2}{2m}\frac{\partial^2\psi(x,t)}{\partial x^2} + V(x)\psi(x,t).$$

Let $\langle P\rangle_{\omega(t)}$ and $\langle Q\rangle_{\omega(t)}$ be expectations for the particle's momentum and position respectively. Show that

$$\frac{d\langle Q\rangle_{\omega(t)}}{dt} = \frac{1}{m}\langle P\rangle_{\omega(t)} \qquad \text{and} \qquad \frac{d\langle P\rangle_{\omega(t)}}{dt} = -\langle V'(Q)\rangle_{\omega(t)}.$$

State all assumptions on the function ψ needed for your proof. **[12]**

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3. Consider the eigenvalue problem $H\psi_n = E_n\psi_n$ for the Hamilton operator H which acts on smooth functions from $L^2(\mathbb{R})$ by

$$H\psi = -\frac{d^2\psi}{dx^2} + x^2\psi.$$

This problem can be solved with the help of an auxiliary operator a and its adjoint

$$a\psi = \frac{d\psi}{dx} + x\psi, \qquad a^*\psi = -\frac{d\psi}{dx} + x\psi.$$

a) Find a function $\psi_0 \in L^2(\mathbb{R})$, $\psi_0 \neq 0$, such that $a\psi_0 = 0$. **[2]**

b) Find an expression for the commutator $[a, a^*]$. **[3]**

c) Show that eigenvalues of the operator $N = a^*a$ are not negative. **[3]**

d) Let $a\psi_0 = 0$, $\|\psi_0\| = 1$ and $\psi_n = \dfrac{1}{\sqrt{n!2^n}}(a^*)^n\psi_0$.

(i) Show that $N\psi_n = 2n\psi_n$ for $n \in \mathbb{N}$. **[4]**

(ii) Show that $H\psi_n = E_n\psi_n$ with $E_n = 2n + 1$. **[4]**

(iii) Show that $(\psi_n, \psi_m) = \delta_{nm}$ where δ_{nm} is the Kronecker δ. **[4]**

You may ignore issues related to the domains of the operators.

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4. Let $H : D(H) \to \mathcal{H}$ be a self-adjoint Hamilton operator in a Hilbert space $\mathcal{H}$.

a) Let $\lambda \in \mathbb{R}$. State the definition of a Weyl sequence for H at λ. **[3]**

b) Show that the continuous spectrum of the momentum operator $P\psi = -i\hbar\dfrac{d\psi}{dx}$ coincides with $\mathbb{R}$.

Hint: You may use without proof Weyl's criterion and the fact that the operator P has no eigenvalues. **[6]**

c) Show that for every $\lambda > 0$ the resolvent $R_\lambda = (H - i\lambda)^{-1}$ is a bounded operator on $\mathcal{H}$. You may use without proof that $i\lambda$ belongs to the resolvent set of H. **[3]**

d) Show that for every $\lambda > 0$ the operator $H_\lambda = -i\lambda H R_\lambda$ is a bounded operator on $\mathcal{H}$. **[2]**

e) Let $f \in D(H)$. Show that $\|H_\lambda f - Hf\| \to 0$ when $\lambda \to +\infty$. **[6]**

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5. a) Let $H : \mathcal{H} \to \mathcal{H}$ be a bounded self-adjoint operator on a Hilbert space $\mathcal{H}$.

(i) Show that the following series converges in the operator norm

$$U_t = \sum_{k=0}^{\infty} \frac{(-i\frac{t}{\hbar})^k}{k!} H^k.$$

You may use without proof that the space of bounded operators is complete. **[4]**

(ii) Show that the operator $U_{t+s} = U_t U_s$ for every $t, s \in \mathbb{R}$. **[4]**

(iii) Show that $\psi(t) = U_t \psi$ satisfies the Schrödinger equation with the Hamilton operator H. **[4]**

b) Let $A, B : \mathcal{H} \to \mathcal{H}$ be bounded operators on a Hilbert space $\mathcal{H}$. Prove the Trotter product formula

$$\mathrm{e}^{A+B} = \lim_{n\to\infty} \left(\mathrm{e}^{\frac{1}{n}A}\mathrm{e}^{\frac{1}{n}B}\right)^n.$$

Explain your arguments carefully. **[8]**
