Stochastic Modelling and Random Processes

Assignment 1

This assignment counts for $1/3$ of your homework marks. It is marked out of 100, and [x] indicates the weight of each question.

You need to **justify all your answers** unless it is clear you do not need to. All plots must contain axis labels and a legend (use your own judgement to find reasonable and relevant plot ranges), and corresponding comments on the text of your answers.

The written part of your assignment should be submitted as a pdf file (either latex, scan of handwritten answers, or extract a pdf version of a Jupyter notebook or similar) and you should also include your code (Jupyter notebook, .m file, or any other format you choose to use). Your files should be named MA933_assignment1_1234567 (where 1234567 is replaced by your university ID number).

You should submit your solutions via this link by **Thursday, 27.10.2022, 5pm UK time**.

1 Simple Random Walk [40]

Consider a Simple Random Walk (SRW) with state space $S = \{1, \dots, L\}$ with probabilities $p \in [0, 1]$ and $q = 1 - p$ to jump right and left:

$$p(x, y) = p\delta_{y,x+1} + q\delta_{y,x-1}.$$

(a) Consider the following boundary conditions:

- periodic boundary conditions, i.e. $p_{L,1} = p$, $p_{1,L} = q$,
- closed boundary conditions, i.e. $p_{1,1} = q$, $p_{L,L} = p$.

For each case, write down the transition matrix P of the process. Is the corresponding Markov chain irreducible? Give (all) the stationary distribution(s) $\boldsymbol{\pi}$ and state whether they are reversible. Justify your answers.
Hint: Where necessary, discuss the cases $p = 1$ and $p = q = 1/2$ separately from the general case $p \in (0, 1)$.

(b) Now consider absorbing boundary conditions ($p_{1,1} = p_{L,L} = 1$). Sketch the transition matrix P, decide whether the process is irreducible, and give all stationary distributions $\boldsymbol{\pi}$, stating whether they are reversible.
Let $h_k^L = \mathbb{P}[X_n = L \text{ for some } n \geq 0 | X_0 = k]$ be the absorption probability in site L. Give a recursion formula for h_k^L and solve it for $p \neq q$ and $p = q$.

(c) Simulate 500 realizations of a SRW with $L = 10$, closed boundary conditions and with a value for $p = 1 - q \in (0.6, 0.9)$ of your choice. For all simulations use $X_0 = 1$.
Plot the empirical distribution after 10 and 100 time steps in form of a histogram, and compare it with the theoretical stationary distribution from (a). Comment on your results.

(d) Repeat a single realization of the same simulation up to 50 and 500 time steps and plot the fraction of time spent in each state as a histogram, comparing to the stationary distribution and commenting on your results.

2 Generators and eigenvalues [30]

Consider the continuous-time Markov chain $(X_t : t \geq 0)$ with generator $G = \begin{pmatrix} -2 & 1 & 1 \\ 1 & -4 & 3 \\ 0 & 1 & -1 \end{pmatrix}$.

(a) Draw a graph representation for the chain (i.e. connect the three states by their **jump rates**), and give the transition matrix P^Y of the corresponding jump chain $(Y_n : n \in \mathbb{N}_0)$.

(b) Consider the Taylor series of the matrix P_t and confirm that $\frac{d}{dt} P_t|_{t=0} = G$, $\frac{d^2}{dt^2} P_t|_{t=0} = G^2$ etc..
Assume that $G = Q^{-1} \Lambda Q$, with diagonal matrix $\Lambda \in \mathbb{C}^{3\times 3}$ and eigenvalues λ_i of G on the diagonal, and with some matrix $Q \in \mathbb{C}^{3\times 3}$. Show that

$$P(t) = \exp(tG) = Q^{-1} \begin{pmatrix} 1 & 0 & 0 \\ 0 & e^{\lambda_2 t} & 0 \\ 0 & 0 & e^{\lambda_3 t} \end{pmatrix} Q \, .$$

(You do not need to compute entries of the matrix Q)

(c) Compute λ_2 and λ_3. Use this to compute $p_{11}(t)$, i.e. determine the coefficients in

$$p_{11}(t) = a + b\, e^{\lambda_2 t} + c\, e^{\lambda_3 t} \, .$$

(Again, it is not necessary to compute the matrix Q, instead use what you know about $p_{11}(0)$ and $\frac{d}{dt} p_{11}(t)|_{t=0}$ etc.)

(d) What is the stationary distribution $\boldsymbol{\pi}$ of X?

3 Pólya urn models [30]

Consider the following experiment: Place k balls each of distinct color indexed by $i = 1, \ldots, k$ in an urn. Draw one ball uniformly at random, then replace *two* balls of the colour just drawn in the urn. Iterate.

(a) Suppose we want to keep track of the contents using a stochastic process $\underline{X}(n)$ of the urn as a function of discrete time n. Give the state space S, the initial condition $\underline{X}(0)$ and the transition probabilities $p(\underline{x}, \underline{y})$, $\underline{x}, \underline{y} \in S$.

(b) For $k = 2$ sketch the state space and the transition probabilities between states and show that for all $(x_1, x_2) \in S$ and all $n \geq 1$

$$\mathbb{P}\big[\underline{X}(n) = (x_1, x_2)\big] = \frac{1}{n+1}\, \delta_{n+2, x_1+x_2} \ ,$$

i.e., the distribution at time n is uniform. Then use this to show that

$$\frac{1}{n+2}\underline{X}(n) \to (U, 1-U) \quad \text{as } n \to \infty,$$

where $U \sim U[0,1]$ is a uniform random variable on $[0,1]$.

Consider a **generalized Pólya urn model** with k types or colours on the same state space S as in (a) above. In this case, the transition probabilities are

$$p(\underline{x}, \underline{x} + \underline{e}_i) = \frac{f_i x_i^\gamma}{\sum_{j=1}^k f_j x_j^\gamma} \qquad \big(\text{and } p(\underline{x}, \underline{y}) = 0 \text{ if } \underline{y} \neq \underline{x} + \underline{e}_i\big) \ ,$$

where the $f_i > 0$ denote the **fitness** of type i and $\gamma \geq 0$ is a **reinforcement parameter**.

(c) Simulate the model for $k = 500$ types with equal fitness $f_i \equiv 1$ for $\gamma = 0,\ 0.5,\ 1$ and 1.5. For each γ, show the **empirical cumulative distribution functions** of $\underline{X}(n)$ for $n = 5000,\ 20000$ and 80000 in one plot, and do the same for the normalized data $\frac{1}{n+k}\underline{X}(n)$ (8 plots in total). Choose the plot ranges reasonably and explain what you observe.

Background info: For $\gamma > 1$ it is known that the system exhibits **monopoly**, i.e. as $n \to \infty$ almost all balls in the urn will be of a single type.

(d) Play around with the fitness parameters f_i, find something 'interesting', and show one plot and write a few sentences to explain it.