Stochastic Modelling and Random Processes

Example sheet 6

1. **Getting used to the definitions on graphs**

   For this question, you should find all the definitions you need in Slide set 7. The goal of this question is for you to get used to all the definitions and computations that are usually done with graphs, and this graph is big (and time consuming!) so that you can see some things more easily. That being said, I drew it randomly, so it might not work out well :) And so you are free to draw our favourite graph and repeat the exercise. You can use python or matlab or any other language to do this question, but not networkx yet please (see below)

   Consider an undirected graph with $N = 11$ nodes and the following adjacency matrix:

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 & 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 \end{bmatrix}$$

   (a) Draw the graph, and find all its cliques. Which of the cliques you found are maximal?

   (b) What is the number of edges $K = \frac{|E|}{2}$? And what is the degree of each node?

   (c) Find the length of the shortest paths between a few pairs of nodes (e.g. 1 and 10, 3 and 5, 6 and 10 and many others) and find one of these shortest paths.

   (d) How many walks (remember, walks *can repeat* nodes) are there between 2 and 10 with $n = 10$ steps? List a few of them.

   (e) What is the diameter and the characteristic path length of this graph?

   (f) Write down the degree distribution of this graph and its average degree and the corresponding variance. Is this network scale-free?

   (g) Write down a few joint degree distributions for pairs of degrees of your choosing. What is the marginal distribution for $k = 3$ and $k = 4$? And what are the average nearest neighbouts of nodes with degrees 3 and 4?

   (h) (if there is time) Is this network uncorrelated? Assortative? Disassortative? Compute its correlation coefficient.

   (i) How many triangles does the graph have? (You already found them, but feel free to use either of the formulae from lectures). Compute the clustering coefficient (you can use both the global clustering coefficient, or the local definition and take the average).

   (j) What is the spectral density of this graph?

(k) Write down the closeness and Katz centrality for all the nodes in this graph, and the betweenness centrality for a few of them (e.g. see what happens with 4, 5 and 6, as well 8 and 9).

(l) Define the graph Laplacian and compute the spectral gap and the Fiedler vector for this graph. Can you already say anything about graph partitioning / communities in this graph?

(m) Write down the normalised graph Laplacian and compute the PageRank for this graph.

2. **Simulation of random graphs**

The goal of this question is for you to get used to the package networkx and to simulate some of the random graphs we have seen (or will see in week 10) in this module, so that you can use this for the assignment.

(a) Read the documentation of the networkx package to get familiar with it

(b) Generate the graph from Q1 and compute some of the measures from Q1.

(c) Generate a few realisations of the random graphs we saw over the last couple of weeks: Erdős-Rényi, configuration model, Barabási-Albert and Watts-Strogatz. You can simulate a random lobster one as well, but I don't know what this one is :) Do this for increasing N and various values of p (for E-R), degree distributions (for configuration) and other values of parameters, so that you can observe the behaviours we discussed.

(d) Compute some of the measures we discussed in Q1 (especially the ones we looked at on our lectures for the random ones) for these graphs so that you see the typical behaviours discussed above