29.02.2022 Brownian Motion HW3.

① Martingales

Show that the following are martingales:

a) $W_t^2 - t$

$f(x,t) = x^2 - t \Rightarrow f'_t = -1;\ f'_x = 2x;\ f''_{xx} = 2.$

Ito's formula: $df(x,t) = \cancel{-1\,dt} + 2W_t dW_t + \cancel{\frac{1}{2}\cdot 2\,dt} = 2W_t dW_t$ ← has only dW_t part with no dt part ⇒ local martingale and by Novikov condition, since $E\left[e^{\frac{1}{2}\int_0^T |W_t|^2 dt}\right] < \infty$, it is a martingale.

б) $W_t^3 - 3tW_t$

$f(x,t) = x^3 - 3t\cdot x \Rightarrow f'_t = -3x;\ f'_x = 3x^2 - 3t;\ f''_{xx} = 6x$

$\Rightarrow df(x,t) = \cancel{-3W_t dt} + 3(W_t^2 - t)dW_t + \cancel{\frac{1}{2}\cdot 6W_t dt} = 3(W_t^2 - t)dW_t$ ⇒ only dW_t part ⇒ mart.

в) $W_t^4 - 6tW_t^2 + 3t^2$

$f(x,t) = x^4 - 6tx^2 + 3t^2;\ \Rightarrow f'_t = -6x^2 + 6t;$

$f'_x = 4x^3 - 12tx$

$f''_{xx} = 12x^2 - 12t$

$\Rightarrow df(x,t) = (\cancel{-6W_t^2} + 6t)dt + (4W_t^3 - 12t)dW_t + \frac{1}{2}(\cancel{12W_t^2} - 12t)dt = 4(W_t^3 - 3t)dW_t$ ⇒ only dW_t part ⇒ mart.

② Laplacian in generalized polar coordinates

a) Show that for $|x| > 0$: $\Delta u(x) = \frac{1}{r^{d-1}}\partial_r\left(r^{d-1}\partial_r \tilde{u}\right)(|x|)$

▶ $\frac{\partial u(x_1 \dots x_d)}{\partial x_i} = \frac{\partial u}{\partial r}\cdot\frac{\partial r}{\partial x_i} = \frac{\partial u}{\partial r}\cdot\frac{x_i}{r}$

$u(x_1, \dots x_d) = u(r)$

$\frac{\partial r}{\partial x_i} = \left(\sqrt{x_1^2 + \dots + x_d^2}\right)'_{x_i} = \frac{2x_i}{2\sqrt{x_1^2 + \dots + x_d^2}} = \frac{x_i}{\sqrt{x_1^2 + \dots + x_d^2}} = \frac{x_i}{r}$

$\Rightarrow \frac{\partial^2 u}{\partial x_i^2} = \frac{\partial}{\partial x_i}\left(\frac{\partial u}{\partial x_i}\right) = \frac{\partial}{\partial x_i}\left(\frac{\partial u}{\partial r}\cdot\frac{x_i}{r}\right) = \frac{\partial^2 u}{\partial r^2}\cdot\frac{x_i^2}{r^2} + \frac{\partial u}{\partial r}\cdot\underbrace{\frac{\partial}{\partial x_i}\left(\frac{x_i}{r}\right)}$

$\frac{\partial}{\partial x_i}\left(\frac{x_i}{r}\right) = \frac{1\cdot r - x_i\cdot\frac{\partial r}{\partial x_i}}{r^2} = \frac{r - x_i\cdot\frac{x_i}{r}}{r^2} = \frac{r^2 - x_i^2}{r^3}$

$\Rightarrow \Delta u = \sum_{i=1}^d \frac{\partial^2 u}{\partial x_i^2} = \sum_{i=1}^d\left(\frac{\partial^2 u}{\partial r^2}\cdot\frac{x_i^2}{r^2} + \frac{r^2 - x_i^2}{r^3}\cdot\frac{\partial u}{\partial r}\right)$

$\Rightarrow \Delta u = \frac{\partial^2 u}{\partial r^2}\cdot\underbrace{\frac{\sum x_i^2}{r^2}}_{=1} + \frac{d\cdot r^2 - \overbrace{\sum_{i=1}^d x_i^2}^{=r^2}}{r^3}\cdot\frac{\partial u}{\partial r} = \frac{\partial^2 u}{\partial r^2} + \frac{(d-1)}{r}\cdot\frac{\partial u}{\partial r} = \frac{1}{r^{d-1}}\cdot\partial_r\left(r^{d-1}\cdot\frac{\partial u}{\partial r}\right).$ ◀

б) Use the formula above to prove Stokes' theorem on the disc for radial functions.

$\int_{B_r(0)} \Delta u(x)\,dx = \int_{\partial B_r(0)} \partial_n u(z)\,d\sigma(z)$

▶ Use disintegration formula: integral over the ball is integral over r of spherical integrals:

$\int_{B_r(0)} \Delta u(\bar{x})\,d\bar{x} = \int_{B_r(0)} \frac{1}{r^{d-1}}\cdot\partial_r u\left(r^{d-1}\frac{\partial u}{\partial r}\right)dx_1\dots dx_n = \int_0^r \frac{1}{r^{d-1}}\int_{\partial B_r(0)}\partial_r u\left(r^{d-1}\frac{\partial u}{\partial r}\right) r^{d-1}\,d\sigma\,dr$

$= \int_{\partial B_r(0)}\left(\int_0^r \partial_r\left(r^{d-1}\frac{\partial u}{\partial r}\right)dr\right)d\sigma = \int_{\partial B_r(0)}\left(r^{d-1}\cdot\frac{\partial u}{\partial r}\Big|_0^r\right)d\sigma = r^{d-1}\int_{\partial B_r(0)}\frac{\partial u}{\partial r}(z)\,d\sigma(z) = r^{d-1}\int_{\partial B_r(0)}\frac{\partial u}{\partial n}(z)\,d\sigma(z)$

↑ this coefficient is superfluous?

For example, for $d=2$: $\begin{cases} x = r\cos\varphi \\ y = r\sin\varphi \end{cases} \Rightarrow \begin{cases} dx = -r\sin\varphi\,d\varphi + \cos\varphi\,dr \\ dy = r\cos\varphi\,d\varphi + \sin\varphi\,dr \end{cases} \Rightarrow |J| = r\,dr\,d\varphi$

$\Rightarrow \int_{B_r(0)} \underbrace{\frac{1}{r}\partial_r\left(r\frac{\partial u}{\partial r}\right)}_{=\Delta u(x,y)}dx\,dy = \int_0^r\int_0^{2\pi}\frac{1}{\cancel{r}}\cdot\partial_r\left(r\frac{\partial u}{\partial r}\right)\cancel{r}\,dr\,d\varphi = \int_0^{2\pi}\left(r\cdot\frac{\partial u}{\partial r}\Big|_0^r\right)d\varphi = r\cdot\int_0^{2\pi}\underbrace{\frac{\partial u}{\partial r}}_{=\frac{\partial u}{\partial n}}d\varphi$ ◀

3. Conformal invariance

a) $C_t = B_t^1 + iB_t^2$

$f: D \to \mathbb{C}$ - analytic

prove that $f(C_t) = f(W_t^1 + iW_t^2)$ is a martingale

⇒ If f is analytic, then its Re and Im parts are harmonic functions: $\Delta u = 0$; $\Delta v = 0$.

Indeed: f - analytic ⇒ Cauchy-Riemann equations: $\begin{cases} u'_x = v'_y \\ u'_y = -v'_x \end{cases}$

$$\Rightarrow \frac{\partial^2 u}{\partial x^2} = (v'_y)'_x = \frac{\partial^2 v}{\partial y \partial x}$$

$$\frac{\partial^2 u}{\partial y^2} = (-v'_x)'_y = -\frac{\partial^2 v}{\partial x \partial y}$$

$$\Rightarrow \Delta u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = \frac{\partial^2 v}{\partial y \partial x} - \frac{\partial^2 v}{\partial x \partial y} = 0.$$

since v has continuous derivatives

And on lecture we proved, that $M_t^g = g(B_t) - \int_0^t \frac{1}{2} \Delta g(B_s) ds$

is a martingale, for $g \in C^2(\mathbb{R}^d; \mathbb{R})$.

Taking $g := u$ and then $g := v$, and using that $\Delta u \equiv 0$ and $\Delta v \equiv 0$,

we obtain that $f = (u; v)$ is a 2d martingale ◀

б) Show that if f is harmonic, then $f(z_0) = \int_0^{2\pi} f(z_0 + \rho e^{i\theta}) d\theta$; $\forall z_0 \in \mathbb{C}$; $\rho > 0$.

⇒ Since f is harmonic, since $M_t^f = f(B_t) - \underbrace{\int_0^t \frac{1}{2} \Delta f(B_s) ds}_{=0} = f(B_t)$ - is a martingale

Introduce stopping time: $\tau = \inf\{t \geq 0 : B_t \in \partial B_\rho(z_0)\}$.

Then, using optimal stopping theorem for a martingale, $X_t := f(z_0 + W_t)$

we get that ~~[illegible]~~ $\underset{f(z_0)}{X_0} = \underset{E f(z_0 + W_\tau).}{E X_\tau}$

But due to rotational symmetry of BM, $z_0 + W_\tau$ has uniform distribution over the boundary of $B_\rho(z_0)$. ⇒ $E f(z_0 + W_\tau) = \int_0^{2\pi} \frac{1}{2\pi} f(z_0 + \rho e^{i\theta}) d\theta$

⇒ we have proved that $f(z_0) = \frac{1}{2\pi} \int_0^{2\pi} f(z_0 + \rho e^{i\theta}) d\theta$

the coefficient is superfluous?

4. O-U process

$$q(t, x, y) = \frac{1}{\sqrt{2\pi(1 - e^{-2ct})}} \exp\left(\frac{-(y - xe^{-ct})^2}{2(1 - e^{-2ct})}\right)$$

a) Consider $\bar{q}(y) := \lim_{t \to \infty} q(t, x, y) = \frac{1}{\sqrt{2\pi}} e^{\frac{-y^2}{2}}$ ⇒ $X_t \to N(0; 1)$

Identify $y \to a(y)$ and $D > 0$ such that:

$$0 = -\frac{\partial}{\partial y}(a(y)\bar{q}(y)) + \frac{\partial^2}{\partial y^2}(D \cdot \bar{q}(y))$$ ← this is Fokker-Planck equation for the stationary density

$a(y) = -y$ and $D = 1$ satisfy this

- $\frac{\partial}{\partial y}(a(y)\bar{q}(y)) = \frac{\partial}{\partial y}\left(-y \cdot \frac{1}{\sqrt{2\pi}} \cdot e^{-\frac{y^2}{2}}\right)$

they are the same

- $\frac{\partial^2}{\partial y^2}(1 \cdot \bar{q}(y)) = \frac{\partial}{\partial y}\left(\left(\frac{1}{\sqrt{2\pi}} \cdot e^{-\frac{y^2}{2}}\right)'_y\right) = \frac{\partial}{\partial y}\left(\frac{1}{\sqrt{2\pi}} \cdot -y \cdot e^{-\frac{y^2}{2}}\right)$

$$\Rightarrow \frac{\partial}{\partial y}(a(y)\bar{q}(y)) = \frac{\partial^2}{\partial y^2}(D \cdot \bar{q}(y))$$

6) Derive the generator $\mathcal{L}$ for O-U process, using Dynkin's formula: $\frac{d}{dt} E_x [f(X_t)] = E_x [\mathcal{L} f(X_t)]$

We may use instead Fokker-Plank equation, because it is derived from this formula

if $dX_t = a(X_t,t)dt + \sigma(X_t,t)dW_t$,

then $\mathcal{L}(f)^{(x)} = a \cdot f'_x + \frac{\sigma^2}{2} f''_{xx}$,

and Fokker-Plank equation looks like $\frac{\partial}{\partial t} p_t(x,y) = -\frac{\partial}{\partial y}(a(y,t)p_t) + \frac{1}{2}\frac{\partial^2}{\partial y^2}(\sigma^2(y,t)p_t)$.

We have in our case $a(y,t) = -y$; $\sigma(y,t) = 1$.

$\Rightarrow$ process looks like $dX_t = -X_t dt + dW_t$,

and the generator is $\mathcal{L}(f)^{(x)} = -x \cdot f'_x + \frac{1}{2} f''_{xx}$

Now we derive Fokker-Plank equations:

$$\frac{d}{dt} E_x [f(X_t)] = E_x [\mathcal{L} f(X_t)]$$

$$\frac{d}{dt}\int f(y) p_t(x,y)dy = E_x \left[a(X_t,t) f'(X_t) + \frac{\sigma^2}{2}(X_t,t) f''(X_t) \right]$$

$$= \int f(y)\frac{\partial}{\partial t} p_t(x,y)dy \qquad = \int a(y,t) f'(y) \cdot p_t(x,y)dy + \int \frac{\sigma^2(y,t)}{2} f''(y) p_t(x,y)dy$$

= (integration by parts) (integration by parts 2 times)

$$-\int f(y) \frac{\partial}{\partial y}(a(y,t)\cdot p_t(x,y))dy + \int f(y) \frac{\partial^2}{\partial y^2}\left(\frac{\sigma^2}{2}(y,t) p_t(x,y)\right)dy$$

$$\Rightarrow \frac{\partial}{\partial t} p_t(x,y) = -\frac{\partial}{\partial y}(a(y,t)p_t(x,y)) + \frac{1}{2}\frac{\partial^2}{\partial y^2}(\sigma^2(y,t)p_t(x,y))$$

5.) Poisson problem

Suppose $u \in C^2(D) \cap C(\bar{D})$

$\begin{cases} \Delta u = \psi \text{ on } D \\ \Delta u = 0 \text{ on } \partial D \end{cases}$ Show that $u(x) = -\frac{1}{2} E_x \left[\int_0^\tau \psi(B(s))ds\right]$, where $\tau = \inf\{t \geq 0 : B_t \in \partial D\}$.

Let's prove it in a more general form:

Let $Lu = \frac{1}{2} tr(A\Delta u) + \langle b; \nabla u \rangle$, and for $f, g \in C^1$ $\exists!$ solution of Dirichlet problem.

$\begin{cases} Lu = f \\ u|_{\partial D} = g. \end{cases}$

Then: 1) $E\tau^y < \infty$, where $\tau^y = \inf\{t \geq 0 : X_t^y \notin D\}$, and $dX_t^y = b(X_t^y)dt + \sigma(X_t^y)dW_t$

2) $u(y) = Eg(X^y_{\tau^y}) - E\int_0^{\tau^y} f(X_t^y)dt$

Let u be the solution of Dirichlet problem.

Let's apply Ito's formula to $u(X^y_{\tau_N})$, where $\tau_N := \min\{\tau_y; N\}$ - bounded time.

$\Rightarrow u(X^y_{\tau_N}) = u(y) + \int_0^t \left(\frac{1}{2} tr(A\Delta u) + \langle b; \nabla u\rangle\right)(X_s^y)ds + \underbrace{\int_0^t (\dots)dW_s}_{\text{has zero expectation.}}$

$\Rightarrow$ taking expectation: $Eu(X^y_{\tau_N}) = u(y) + E\int_0^{\tau_N} Lu(X_s^y)ds$. (*)

Then let u be the solution of the problem $\begin{cases} Lu = 1 \\ u/_{\partial\Omega} = 0 \end{cases}$

$\Rightarrow u$ - is a bounded function: $|u| \le c$.

$\Rightarrow$ from $(*)$: $E\tau_N \le 2C$, $N \to \infty$.

$\Rightarrow E\tau_y < \infty$, because $\tau_N(\omega) \underset{N\to\infty}{\to} \tau_y(\omega)$ $\Rightarrow (1)$ proved.

And now we have, as $N \to \infty$:

$$\underbrace{E\, u(X^y_{\tau_y})}_{=g(X^y_{\tau_y})} = u(y) + E\int_0^{\tau_y} f(X_s^y)\,ds$$

$$\Rightarrow u(y) = E g(X^y_{\tau_y}) - E\int_0^{\tau_y} f(X_s^y)\,ds. \quad (**)$$

because we want our process to be W_t.

In our case: $dX_t = 0 \cdot dt + 1 \cdot dW_t \Rightarrow b = 0; \sigma = 1$

$\Rightarrow Lu = \frac{1}{2}\Delta u.$ $\Rightarrow \begin{cases} Lu = f \\ u/_{\partial\Omega} = g \end{cases}$ looks like $\begin{cases} \frac{1}{2}\Delta u = f \\ u/_{\partial\Omega} = g. \end{cases}$

But from the problem formulation, we are given $\begin{cases} \Delta u = \psi \text{ on } D \\ u = 0 \text{ on } \partial D. \end{cases}$

$\Rightarrow \begin{cases} \psi = 2f \\ 0 = g \end{cases} \Rightarrow \begin{cases} f = \frac{\psi}{2} \\ g = 0. \end{cases}$

$\Rightarrow (**)$ reads as $u(y) = -\frac{1}{2} E_y\left[\int_0^{\tau^y} \psi(X_s^y)\,ds\right].$