THE UNIVERSITY OF WARWICK

FOURTH YEAR EXAMINATION: APRIL 2018

STOCHASTIC ANALYSIS

Time Allowed: **3 hours**

Read carefully the instructions on the answer book and make sure that the particulars required are entered on each answer book.

Calculators are not needed and are not permitted in this examination.

Candidates should answer COMPULSORY QUESTION 1 and THREE QUESTIONS out of the four optional questions 2, 3, 4 and 5.

The compulsory question is worth 40% of the available marks. Each optional question is worth 20%.

If you have answered more than the compulsory Question 1 and three optional questions, you will only be given credit for your QUESTION 1 and THREE OTHER best answers.

The numbers in the margin indicate approximately how many marks are available for each part of a question.

Notation: Throughout the exam we fix a filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t), \mathbb{P})$. We assume that the filtration $(\mathcal{F}_t)$ is complete and right-continuous. In some questions we assume that $(\mathcal{F}_t)$ is the augmented filtration generated by some explicitly given processes. All random variables are defined on this probability space.

$\mathbb{H}^2$ is the Hilbert space of L^2 bounded continuous martingales on $(\Omega, \mathcal{F}, (\mathcal{F}_t), \mathbb{P})$. Given a continuous local martingale M denote by $L^2(M)$ the space of progressively measurable processes H_t with $\mathbb{E}\int_0^\infty H_t^2 d\langle M, M\rangle_t < \infty$. A process (H_t) is called an *elementary process* if there exists an $n \geq 1$ and $0 \leq t_0 < \ldots < t_n$ such that $H_t = \sum_{j=1}^n H^{(j)} \mathbf{1}_{(t_{j-1}, t_j]}$ and

- for each $j \in \{0, \ldots, n-1\}$ the random variable $H^{(j)}$ is $\mathcal{F}_{t_j}$ measurable.
- There exists a $C \in (0, \infty)$ such that for all $j \in \{1, \ldots, n\}$ we have $|H_j| \leq C$ almost surely.

COMPULSORY QUESTION

1. a) Let (B_t) be a standard Brownian motion.

(i) Show that for $\sigma > 0$ the process (Y_t) with $Y_t = \exp(\sigma B_t - \frac{\sigma^2}{2}t)$ is a continuous martingale. **[4]**

(ii) For $t \geq 0$ let $B_t^* = \sup_{0 \leq s \leq t} B_s$. Show that for all $a > 0$ we have $\mathbb{P}\big(B_t^* \geq a\big) \leq \exp(-\frac{a^2}{2t})$. **[4]**

b) (i) Let $M \in \mathbb{H}^2$ and $H \in L^2(M)$. State the Itô isometry for $\int_0^t H_s dM_s$. **[3]**

(ii) Prove Itô's isometry for elementary processes (H_t). **[7]**
Hint: The necessary definitions from the course are recalled above.

c) (i) Let $F\colon \mathbb{R}^d \to \mathbb{R}$ be a C^2 function and let $X_t^1, \ldots, X_t^d$ be continuous semimartingales. State Itô's formula for $F(X_t^1, \ldots, X_t^d)$. **[3]**

(ii) Explain briefly the main idea of the proof of Itô's formula. **[7]**

d) (i) Let (B_t) be a Brownian motion and set $Z_t = \exp(2B_t - 16t^2)$. Calculate $\langle Z, Z\rangle_t$. **[4]**

(ii) Let (B_t^1, B_t^2) be a two-dimensional Brownian motion and for $(x, y) \in \mathbb{R}^2$ set $f(x, y) = x^3y - y^3x$. Show that $f(B_t^1, B_t^2)$ is a local martingale. **[4]**

e) Let (H_t) be a progressively measurable process and let (B_t^1) and (B_t^2) be two independent Brownian motions. Set $Y_t = \int_0^t \sin(H_s)dB_s^{(1)} + \int_0^t \cos(H_s)dB_s^{(2)}$. Show that (Y_t) is a Brownian motion. **[4]**

OPTIONAL QUESTIONS

2. a) Let (B_t) be a standard one-dimensional Brownian motion. For $a > 0$ let $U_a = \inf\{t \geq 0\colon |B_t| = a\}$. Calculate $\mathbb{E}U_a$. **[6]**

b) (i) Let (M_t) be a martingale with $\mathbb{E}M_t^2 < \infty$ for all t. Show that for $t > s \geq r$ we have $\mathbb{E}(M_t - M_s)M_r = 0$. **[2]**

(ii) Assume that the martingale (M_t) is additionally a Gaussian process and that $(\mathcal{F}_t)$ is the (augmented) filtration generated by (M_t). Show that for $t > s \geq 0$ the random variable $(M_t - M_s)$ is independent from $\mathcal{F}_s$. **[3]**

(iii) Now, let (M_t) be a Gaussian martingale with continuous sample paths. Assume again that $(\mathcal{F}_t)$ is the (augmented) filtration generated by (M_t). Show that the quadratic variation of M_t is deterministic and given by $\langle M, M\rangle_t = \mathbb{E}(M_t - M_0)^2$. **[4]**

(iv) Conversely, let (M_t) be a continuous martingale with deterministic quadratic variation. Use the Dambis-Dubins-Schwarz Theorem (method of time-change) to show that (M_t) is a Gaussian process. **[5]**

Question 3 continued overleaf

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3. Let $x_0 \in \mathbb{R}$, $a, \sigma > 0$ and let (B_t) be a Brownian motion. Consider the SDE

$$dX_t = -aX_t dt + \sigma dB_t, \qquad X_0 = x_0. \tag{1}$$

a) (i) Show that there is at most one strong solution to (1) and that it is given by

$$X_t = e^{-at}x_0 + \sigma e^{-at}\int_0^t e^{as}dB_s.$$

[4]

(ii) Show that X_t is Gaussian and calculate its mean and variance. Show that mean and variance converge as $t \to \infty$ and determine the limit. **[5]**

b) (i) Fix a C^2 function $f\colon \mathbb{R} \to \mathbb{R}$ and $\sigma \in \mathbb{R}$. Assume that X_t solves the equation

$$dX_t = f(X_t)dt + \sigma X_t dB_t \qquad X_0 = x_0 \tag{2}$$

for a given Brownian motion (B_t) and some $x_0 \in \mathbb{R}$. Let $F_t = \exp(-\sigma B_t + \frac{\sigma^2}{2}t)$. Set $Y_t = X_t F_t$ and show that (Y_t) solves the ordinary differential equation

$$\dot{Y}_t = F_t f\Big(\frac{Y_t}{F_t}\Big). \tag{3}$$

[6]

(ii) Use formula (3) from the previous step to solve the SDE (2) for the constant function $f(x) = a$ for $a \in \mathbb{R}$. **[5]**

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4. a) Give the statement of the Kolmogorov continuity test. **[2]**

b) The aim of the next questions is to give a proof of the following special case of the Burkholder-Davis-Gundy (BDG) inequality:

Theorem 1. *Let $p \geq 2$. Then there exists a constant C_p depending only on p such that for any continuous martingale (M_t) with $M_0 = 0$ and any stopping time T we have*

$$\mathbb{E}\Big[\sup_{0\leq s\leq T}|M_s|^p\Big] \leq C_p\mathbb{E}\big[\langle M, M\rangle_T^{\frac{p}{2}}\big]. \tag{4}$$

(i) Assume that (4) is proved for bounded M and $T = \infty$. Deduce the general case. **[3]**

 Question 4 continued overleaf

(ii) Apply Itô's formula to the function $x \mapsto |x|^p$, and prove that there is a constant $C_p^{(0)}$ such that

$$E[|M_t|^p] \leq C_p^{(0)} \Big(E\Big[\sup_{0 \leq s \leq t} |M_s^*|^p \Big] \Big)^{\frac{p-2}{p}} (E[\langle M, M \rangle_t^{p/2}])^{2/p}. \qquad (5)$$

[5]

(iii) Combine the estimate (5) with the maximal inequality to get the desired estimate (4). **[4]**

c) Let (H_t) be an adapted process with continuous sample paths such that for all $0 \leq t \leq 1$ we have $|H_t| \leq 1$ almost surely. Let (B_t) be Brownian motion and set $Y_t = \int_0^t H_s dB_s$. Show that for any $\alpha < \frac{1}{2}$ and any p satisfying $\frac{1}{2} - \frac{1}{p} > \alpha$ we have

$$\mathbb{E}\Big[\Big(\sup_{0 \leq s < t \leq 1} \frac{|Y_t - Y_s|}{|t - s|^\alpha} \Big)^p \Big] < \infty.$$

[6]

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5. a) Let (X_t) solve the SDE $dX_t = b(X_t)dt + \sigma(X_t)dB_t$ for $b\colon \mathbb{R}^d \to \mathbb{R}^d$, $\sigma\colon \mathbb{R}^d \to \mathbb{R}^{d \times m}$ and an m-dimensional Brownian motion (B_t). Define the generator $\mathcal{L}$ of this SDE in terms of b and σ and show that if σ is locally bounded and f is C^2 with compact support, then $f(X_t) - f(X_0) - \int_0^t \mathcal{L}f(X_s)ds$ is a continuous martingale. **[7]**
(**Hint:** Use Itô's formula to show that it is a local martingale and then justify that the stochastic integrals appearing are true martingales.)

b) Let B_t^1, B_t^2 be two independent Brownian motions defined on some probability space $(\Omega, \mathcal{F}, \mathbb{P})$. Define

$$L_t = \int_0^t \sin(B_s^1 + 2B_s^2)e^{-s}dB_s^1 + \int_0^t \frac{\cos(B_s^1)}{1 + s^4} dB_s^2.$$

(i) Determine a bounded variation process (A_t) such that $D_t = \exp(L_t - A_t)$ is a local martingale with $D_0 = 1$. **[3]**

(ii) Show that for your choice of (A_t), the process (D_t) is a uniformly integrable martingale. **[3]**

(iii) Define a probability measure $\mathbb{Q}$ by its Radon-Nikodym density $\frac{d\mathbb{Q}}{d\mathbb{P}} = D_\infty$. Determine a bounded variation process (C_t) such that under $\mathbb{Q}$ the process $\tilde{B}_t^1 = B_t^1 - C_t$ is a local martingale. **[4]**

(iv) Let $\tilde{B}_t^1$ be as in part (iii). Show that $\tilde{B}_t^1$ is a Brownian motion under $\mathbb{Q}$. **[3]**
