Terse hints for some of the exercise sheet questions

Please email me corrections for these hints, or demands for better hints.

1.1 Use $|W_t|/t \to 0$ as $t \to \infty$ to help you see what sample paths do.

1.2 If Z and $2Z$ have the same distribution then $P[Z > a] = P[Z > 2a]$ for all a, implying that $P[Z \in (a, 2a]] = 0$ for all a. Deduce that $P[Z \in (0, \infty)] = 0$. Now try and show that $P[Z = 0] = 0$.

1.3 (a) could be wrong? (b) $\sigma(I_A) = \{\emptyset, A, A^c, \Omega\}$.

1.4 Quote the result from transforms that $Z_1, \ldots, Z_N$ are independent if and only if $E[\exp(i\theta_1 Z_1 + \ldots + i\theta_N Z_N)] = E[\exp(i\theta_1 Z_1)] \ldots E[\exp(i\theta_N Z_N)]$ for all $\theta_1, \ldots, \theta_N \in \mathbf{R}$. Or just prove fact [G6].

1.5 $E[X_s X_t] = a^2 e^{-cs}$ when $c > 0$. You might want to calculate $E[|X_t - X_s|^2]$ to help you draw a sample path.

1.7 Yes it does!

1.8 Dull - but we will use this. Shall I just do it in a video?

2.1 You might state the result as $\int_0^t Z G_r I(r \geq s) dW_r = Z \int_0^t G_r I(r \geq s) dW_r$. I was thinking that this could be checked directly for for simple integrands (G_r), then for bounded continuous adapted integrands, and finally for continuous adapted integrands.

2.2 Use the isometry.

2.3 (b) I don't believe the identity $\{\tau \leq T\} = \cup_{t_i \leq T} \{X_{t_i} \in U\}$ form Evans.

2.4 For part (a) you might use [G7] from the Gaussian primer - 'the limit of Gaussians must be Gaussian'.

2.5 Check your answers - I make them: (a) $dX_t = 2tW_t dt + t^2 dW_t$, (b) $dX_t = 3(W_t^2 - t)dW_t$, (c) $dX = (1 + (1/2)e^W)dt + e^W dW$. (If the Ito diffusion is time-homogeneous I omit the subscript t's.)

2.6 I needed $\sigma^2 = 2ca^2$ to make the covariances match.

2.7 Check your answers: (a) $X_t = x \exp\left(\int_0^t (\mu(s) - \frac{1}{2}\sigma^2(s))ds + \int_0^t \sigma(s) dW_s\right)$,

(b) $X_t = \frac{at}{T} + \sigma(T - t) \int_0^t \frac{1}{T-s} dW_s$.

2.8 (d) This is an example of the 'moment problem' but here it is for a bounded random variable.

2.9 In part (c), what does it mean that Z_t converges conditionally on events A_t? This means that the probabilities defined by $P[Z_t \in \cdot | A_t]$ converge in distribution as $t \to \infty$. You should use transforms, and it would is enough to check that $E[\exp(-\theta Z_t)|A_t] \to \phi(\theta)$ for all $\theta \geq 0$. The limit $\phi(\theta)$ will describe the limiting probability, in this case it should be the transform of an exponential variable.

2.10 Check your answer: I made it $\lim_{t\to\infty} P[X_t = 0] = e^{-2\mu x/\sigma^2}$. I needed to solve $\dot{\lambda}_t = -\mu\lambda_t + \frac{\sigma^2}{2}\lambda_t^2 \ldots$.

2.11 No hint.

2.12 I think the key part when letting $n \to \infty$ is to pass to the limit for the integral

$$\begin{aligned} \int_0^t f_n''(X_s - Y_s)(d(X_s - Y_s))^2 &= \int_0^t \phi_n(X_s - Y_s)(\sigma(X_s) - \sigma(Y_s))^2 ds \\ &\leq K^2 \int_0^t \phi_n(X_s - Y_s)|X_s - Y_s| ds. \end{aligned}$$

I then used the bounds $\phi_n(z)|z| \leq 2n|z|I(|z| \leq 1/n) \leq 2I(0 < |z| < 1/n)$ (why is this true?) to show this term will converge to zero as $n \to \infty$.

3.1 For $f : \mathbf{R}^n \to \mathbf{R}^m$ you can just use Ito's formula on each component of $f = (f_1, \ldots, f_m)$.

Check (some) answers: I get (a) $dX = 2dt + 2W^1 dW^1 + 2W^2 dW^2$; (c) $dX^1_t = \ldots$ and $dX^2 = dt - W^1 dW^2 + (2W^1 - W^2) dW^1$.

Then $d(XY) = XdY + YdX + dXdY$ and $d(XYZ) = XYdZ + XZdY + \ldots$...

3.2 A slog writing partial derivatives carefully.

3.3 $\mathcal{E}_t = f(X_t, Y_t)$ for $f(x,y) = \exp(-x-y)$ and $dX = gdW$ and $dY = g^2 dt$. Now use Ito.

Check your answer: in the final special case I got

$$X_t = \alpha e^{\beta W_t - (\beta^2/2)t} \int_0^t e^{\beta W_s - (\beta^2/2)s} ds.$$

I can see how to use this to get $E[X_t]$ but I found getting $E[X_t^2]$ was probably easier by developing $d(X^2)$. I could not see from this formula how the sample paths evolve for large t.

3.4 (b) We know if c_1 and c_2 are positive integers then X should be the radius of a $c_1 + c_2$ dimensional Brownian motion, so it makes sense to guess and is true) that the new parameter is $c_1 + c_2$ in general.

3.5 Ito calculus should show that $d|Y|^2 = 0dt + 0dW^1 + 0dW^2 + 0dW^3$.

We know form scale analysis of Bessel processes that R can never hit zero.

Use Levy characterization of a BM on $\mathbf{R}^3$ to identify X.

3.6 The expectation $u(x) = E[\exp(-\lambda\tau_{a,b}) I(\tau_b < \tau_a)]$ solves $\frac{1}{2}\Delta u = \lambda u$ on (a,b) with boundary conditions $u(a) = 0$ and $u(b) = 1$.

Use Bayes's definition for conditioning to find $E[\exp(-\lambda\tau_{a,b})|\tau_b < \tau_a]$ (you already know the formula for $P[\tau_b < \tau_a]$). Now expand your formula as a power series in λ to read off $E[\tau_{a,b}|\tau_b < \tau_a]$. I got $E[\tau_{a,b}|\tau_b < \tau_a] = \frac{1}{3}((b-a)^2 - (x-a)^2)$.

3.7 I found that $u(0) = \frac{2\mu}{\mu + \sqrt{\mu^2 + 2\lambda}}$. Expanding in powers of λ I found $E[Z] = \frac{1}{2\mu^2}$ and $E[Z^2] = \frac{1}{\mu^4}$.

3.8 Check your answers: (a) If $c > 0$ then exit at $+\infty$ (but in infinite time by later speed analysis). (b) Oscillation between $\pm\infty$. (c) Exit at zero (in finite time by later speed analysis). (d) Oscillation between 0 and ∞. (e) For any c, exit at 0 or at π possible (always in finite time by later speed analysis).

3.9 No hint.

3.10 Exit at 0 in finite time iff $p < 1$ (and so exit at 1 in finite time iff $q < 1$.

3.11 $\Delta(1/|x|) = 0$ on $\mathbf{R}^3/\{0\}$.

When $X_0 \sim N(0, I)$ independent of B then $X_t \sim N(0, (1+t)I) \sim \sqrt{1+t}\, N(0, I)$.

3.12 Taking $u \in C^{1,2}([0,t] \times \mathbf{R}^N)$ is natural from the PDE viewpoint. To derive the probabilistic formula I started by supposing that u and the derivatives $\partial u/\partial x_i$ are bounded, and the function h is non-negative.

3.13 Check some answers: (a) $\phi(x) = (2\pi)^{-1/2} \exp(-(x-\alpha)^2/2)$ (Gaussian centered at α); (b) $\phi(x) = Cx^{2\alpha-1} e^{-2x}$ (and $C = C(\alpha)$ must normalise this density); (c) $\phi(x) = (1+x^2)^{-1} \exp(2\alpha \tan^{-1}(x))$ (and $C = C(\alpha)$ must normalise this density).

3.14 Check your answers - last year I think I reached: (a) $\alpha = \frac{3}{4}$ and $\beta = \frac{21}{4}$; (b) $\alpha = 1/2$ and $\beta = 1/12$; (c) $\alpha = \int_0^{2\pi} \cos^2(x)/(2+\sin(x))^2 dx / \int_0^{2\pi} 1/(2+\sin(x))^2 dx$.

3.15 - 3.18 I started problems 3.17 and 3.18 in the final week monday lectures. My aim is to try and make a video on problems 3.15 and 3.16.