Birth-death processes

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Outline

Outline

Pure Birth Process (Yule-Furry Process)

Example: Consider cells which reproduce according to the following rules:

- A cell present at time t has probability $\lambda h + o(h)$ of splitting in two in the interval $(t, t+h)$
- This probability is independent of age
- Events betweeen different cells are independent

Pure Birth Process (Yule-Furry Process)

Example: Consider cells which reproduce according to the following rules:

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- This probability is independent of age
- Events betweeen different cells are independent

What is the time evolution of the system?

Pure Birth Process (Yule-Furry Process)

Non-Probabilistic Analysis

- Let $n(t)$ =number of cells at time t
- Let λ be the birth rate per single cell

Thus $\approx \lambda n(t)\Delta(t)$ births occur in $(t, t+\Delta t)$
Then:

$$n(t+\Delta t) = n(t) + n(t)\lambda\Delta t$$

$$\frac{n(t+\Delta t) - n(t)}{\Delta t} = n(t)\lambda \rightarrow \frac{dn}{dt} = n'(t) = n(t)\lambda$$

- The solution of this differential equation is: $n(t) = Ke^{\lambda t}$
- If $n(0) = n_0$ then

$$n(t) = n_0 e^{\lambda t}$$

Pure Birth Process (Yule-Furry Process)

Probabilistic Analysis

Notation:

- $N(t) =$ number of cells at time t
- $P\{N(t) = n\} = P_n(t)$

Assumptions:

- A cell present at time t has probability $\lambda h + o(h)$ of splitting in two in the interval $(t, t + h)$
- The probability of more than one birth occurring in time interval $(t, t + h)$ is $o(h)$

All states are transient

Pure Birth Process (Yule-Furry Process)

Assumptions:

- Probability of splitting in $(t, t+h)$: $\lambda h + o(h)$
- Probability of more than one split in $(t, t+h)$: $o(h)$

The probability of birth in $(t, t+h)$ if $N(t) = n$ is $n\lambda h + o(h)$. Then,

$$P_n(t+h) = P_n(t)(1 - n\lambda h - o(h)) + P_{n-1}(t)((n-1)\lambda h + o(h))$$

Pure Birth Process (Yule-Furry Process)

Assumptions:

- Probability of splitting in $(t, t+h)$: $\lambda h + o(h)$
- Probability of more than one split in $(t, t+h)$: $o(h)$

The probability of birth in $(t, t+h)$ if $N(t) = n$ is $n\lambda h + o(h)$.
Then,

$$P_n(t+h) = P_n(t)(1 - n\lambda h - o(h)) + P_{n-1}(t)((n-1)\lambda h + o(h))$$

$$P_n(t+h) - P_n(t) = -n\lambda h P_n(t) + P_{n-1}(t)(n-1)\lambda h + f(h), \text{ with } f(h) \in o(h)$$

$$\frac{P_n(t+h) - P_n(t)}{h} = -n\lambda P_n(t) + P_{n-1}(t)(n-1)\lambda + \frac{f(h)}{h}$$

Let $h \to 0$,

$$P_n'(t) = -n\lambda P_n(t) + (n-1)\lambda P_{n-1}(t)$$

Initial condition $P_{n_0}(0) = P\{N(0) = n_0\} = 1$

Pure Birth Process (Yule-Furry Process)

Probabilities are given by a set of *ordinary differential equations*.

$$P'_n(t) = -n\lambda P_n(t) + (n-1)\lambda P_{n-1}(t)$$
$$P_{n_0}(0) = P\{N(0) = n_0\} = 1$$

Solution

$$P_n(t) = \binom{n-1}{n-n_0} e^{-\lambda n_0 t}(1-e^{-\lambda t})^{n-n_0} \quad n = n_0, n_0+1, \ldots$$

where $\binom{n}{k} = \dfrac{n!}{k!(n-k)!}$.

Pure Birth Process (Yule-Furry Process)

Solution

$$P_n(t) = \binom{n-1}{n-n_0} e^{-\lambda n_0 t}(1-e^{-\lambda t})^{n-n_0} \quad n = n_0, n_0+1, \ldots$$

Observation: The solution can be seen as a negative binomial distribution, i.e., probability of obtaining n_0 successes in n trials. Suppose $p =$prob. of success and $q = 1 - p =$prob. of failure. Then, the probability that the first $(n-1)$ trials result in $(n_0 - 1)$ successes and $(n - n_0)$ failures followed by success on the n^{th} trial is:

$$\binom{n-1}{n-n_0} p^{n_0-1} q^{n-n_0} p = \binom{n-1}{n-n_0} p^{n_0} q^{n-n_0}; \quad n = n_0, n_0+1, \ldots$$

If $p = e^{-\lambda t}$ and $q = 1 - e^{-\lambda t}$, both equations are the same.

Pure Birth Process (Yule-Furry Process)

- Yule studied this process in connection with the theory of evolution, i.e., population consists of the species within a genus and creation of a new element is due to mutations.
- This approach neglects the probability of species dying out and size of species.
- Furry used the same model for radioactive transmutations.

Pure Birth Processes. Generalization

- In a Yule-Furry process, for $N(t) = n$ the probability of a change during $(t, t+h)$ depends on n.
- In a Poisson process, the probability of a change during $(t, t+h)$ is independent of $N(t)$.

Generalization

- Assume that for $N(t) = n$ the probability of a new change to $n+1$ in $(t, t+h)$ is $\lambda_n h + o(h)$.
- The probability of more than one change is $o(h)$.

Pure Birth Processes. Generalization

Generalization

- Assume that for $N(t) = n$ the probability of a new change to $n+1$ in $(t, t+h)$ is $\lambda_n h + o(h)$.
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Then,

$$
\begin{aligned}
&P_n(t+h) = P_n(t)(1-\lambda_n h) + P_{n-1}(t)\lambda_{n-1}h + o(h), \quad n \neq 0\\
&P_0(t+h) = P_0(t)(1-\lambda_0 h) + o(h)\\
\Rightarrow &P_n'(t) = -\lambda_n P_n(t) + \lambda_{n-1}P_{n-1}(t)\\
&P_0'(t) = -\lambda_0 P_n(t)
\end{aligned}
$$

Equations can be solved recursively with $P_0(t) = P_0(0)e^{-\lambda_0 t}$

Pure Birth Process. Generalization

Let the initial condition be $P_{n_0}(0) = 1$.

The resulting equations are:

$$P_n'(t) = -\lambda_n P_n(t) + \lambda_{n-1} P_{n-1}(t), \quad n > n_0$$
$$P_{n_0}'(t) = -\lambda_{n_0} P_{n_0}(t)$$

Yule-Furry processes assumed $\lambda_n = n\lambda$

Outline

Birth-Death Processes

Notation

- Pure Birth process: If n transitions take place during $(0, t)$, we may refer to the process as being in state E_n.
- Changes in the pure birth process: $E_n \to E_{n+1} \to E_{n+2} \to \ldots$
- Birth-Death Processes consider transitions $E_n \to E_{n-1}$ as well as $E_n \to E_{n+1}$ if $n \geq 1$. If $n = 0$, only $E_0 \to E_1$ is allowed.

Birth-Death Processes

Birth-Death Processes

Assumptions

If the process at time t is in E_n, then during $(t, t+h)$:

- Transition $E_n \to E_{n+1}$ has probability $\lambda_n h + o(h)$
- Transition $E_n \to E_{n-1}$ has probability $\mu_n h + o(h)$
- Probability that more than 1 change occurs $= o(h)$.

$$\begin{aligned} P_n(t+h) = & P_n(t)(1 - \lambda_n h - \mu_n h) \\ & + P_{n-1}(t)(\lambda_{n-1} h) + P_{n+1}(t)(\mu_{n+1} h) + o(h) \end{aligned}$$

Time evolution of the probabilities

$\Rightarrow P_n'(t) = -(\lambda_n + \mu_n) P_n(t) + \lambda_{n-1} P_{n-1}(t) + \mu_{n+1} P_{n+1}(t)$

Birth-Death Processes

For $n = 0$

$$P_0(t+h) = P_0(t)(1-\lambda_0 h) + P_1(t)\mu_1 h + o(h)$$

$$\Rightarrow P_0'(t) = -\lambda_0 P_0(t) + \mu_1 P_1(t)$$

- If $\lambda_0 = 0$, then $E_0 \to E_1$ is impossible and E_0 is an absorbing state.
- If $\lambda_0 = 0$, then $P_0'(t) = \mu_1 P_1(t) \geq 0$ and hence $P_0(t)$ increases monotonically.

Note:

$\lim_{t\to\infty} P_0(t) = P_0(\infty) =$ Probability of being absorbed.

Steady-state distribution

$P_0'(t) = -\lambda_0 P_0(t) + \mu_1 P_1(t)$
$P_n'(t) = -(\lambda_n + \mu_n) P_n(t) + \lambda_{n-1} P_{n-1}(t) + \mu_{n+1} P_{n+1}(t)$

As $t \to \infty$, $P_n(t) \to P_n(limit)$.
Hence, $P_0'(t) \to 0$ and $P_n'(t) \to 0$.
Therefore,

$$
\begin{aligned}
0 &= -\lambda_0 P_0 + \mu_1 P_1 \\
\Rightarrow P_1 &= \frac{\lambda_0}{\mu_1} P_0 \\
0 &= -(\lambda_1 + \mu_1) P_1 + \lambda_0 P_0 + \mu_2 P_2 \\
\Rightarrow P_2 &= \frac{\lambda_0 \lambda_1}{\mu_1 \mu_2} P_0 \\
\Rightarrow P_3 &= \frac{\lambda_0 \lambda_1 \lambda_2}{\mu_1 \mu_2 \mu_2} P_0 \qquad \qquad etc
\end{aligned}
$$

Steady-state distribution

$$P_1 = \frac{\lambda_0}{\mu_1} P_0; \quad P_2 = \frac{\lambda_0 \lambda_1}{\mu_1 \mu_2} P_0; \quad P_3 = \frac{\lambda_0 \lambda_1 \lambda_2}{\mu_1 \mu_2 \mu_2} P_0; \quad P_4 = \dots$$

The dependence on the initial conditions has disappeared.

After normalizing, i.e., $\sum_{n=1}^{\infty} P_n = 1$:

$$P_0 = \frac{1}{1 + \sum_{n=1}^{\infty} \prod_{i=0}^{n-1} \frac{\lambda_i}{\mu_{i+1}}}; \quad P_n = \frac{\prod_{i=0}^{n-1} \frac{\lambda_i}{\mu_{i+1}}}{1 + \sum_{n=1}^{\infty} \prod_{i=0}^{n-1} \frac{\lambda_i}{\mu_{i+1}}}, \; n \geq 1$$

Steady-state distribution

$$P_0 = \frac{1}{1 + \sum_{n=1}^{\infty} \prod_{i=0}^{n-1} \frac{\lambda_i}{\mu_{i+1}}}; \quad P_n = \frac{\prod_{i=0}^{n-1} \frac{\lambda_i}{\mu_{i+1}}}{1 + \sum_{n=1}^{\infty} \prod_{i=0}^{n-1} \frac{\lambda_i}{\mu_{i+1}}}, \; n \geq 1$$

Ergodicity condition

$P_n > 0$, for all $n \geq 0$, i.e.,:

$$\sum_{n=1}^{\infty} \prod_{i=0}^{n-1} \frac{\lambda_i}{\mu_{i+1}} < \infty$$

Example. A single server system

- constant arrival rate λ (Poisson arrivals)
- stopping rate of service μ (exponential distribution)
- states of the system: 0 (server free), 1 (server busy)

$$P_0'(t) = -\lambda P_0(t) + \mu P_1(t)$$
$$P_1'(t) = \lambda P_0(t) - \mu P_1(t)$$

Example. A single server system

$$P_0'(t) = -\lambda P_0(t) + \mu P_1(t)$$
$$P_1'(t) = \lambda P_0(t) - \mu P_1(t)$$

Given that: $P_0(t) + P_1(t) = 1$, $P_0'(t) + (\lambda + \mu)P_0(t) = \mu$.

$$P_0(t) = \frac{\mu}{\lambda + \mu} + \left(P_0(0) - \frac{\mu}{\lambda + \mu}\right) e^{-(\lambda+\mu)t}$$
$$P_1(t) = \frac{\lambda}{\lambda + \mu} + \left(P_1(0) - \frac{\lambda}{\lambda + \mu}\right) e^{-(\lambda+\mu)t}$$

Solution = Equilibrum distribution + Deviation from the equilibrium with exponential decay.

Poisson Process. Probabilities

Poisson Process

- Birth probability per time unit is constant λ
- The population size is initially 0

All states are transient

Equations

$$P_i'(t) = -\lambda P_i(t) + \lambda P_{i-1}(t), \quad i > 0$$
$$P_0'(t) = -\lambda P_0(t)$$

Poisson Process. Probabilities

Equations

$$P_i'(t) = -\lambda P_i(t) + \lambda P_{i-1}(t), \quad i > 0$$
$$P_0'(t) = -\lambda P_0(t)$$

$$\Rightarrow P_0(t) = e^{-\lambda t}$$

$$\frac{d}{dt}[e^{\lambda t} P_i(t)] = \lambda P_{i-1}(t) e^{\lambda t} \Rightarrow P_i(t) = e^{-\lambda t} \lambda \int_0^t P_{i-1}(t') e^{\lambda t'} dt'$$

$$P_1(t) = e^{-\lambda t} \lambda \int_0^t e^{-\lambda t'} e^{\lambda t'} dt' = e^{-\lambda t} (\lambda t)$$

Recursively: $P_i(t) = \frac{(\lambda t)^i}{i!} e^{-\lambda t}$
Number of births in interval $(0, t) \sim$ Poisson(λt).

Pure Death Process. Probabilities

Pure Death Process

- All the individuals have the same mortality rate μ
- The population size is initially n

State 0 is an absorbing state. The rest are transient.

Equations

$$P'_n(t) = -n\mu P_n(t)$$
$$P'_i(t) = (i+1)\mu P_{i+1}(t) - i\mu P_i(t), \quad i = 0, \ldots, n-1$$

Pure Death Process. Probabilities

Equations

$$P_n'(t) = -n\mu P_n(t)$$
$$P_i'(t) = (i+1)\mu P_{i+1}(t) - i\mu P_i(t), \quad i = 0, \dots, n-1$$

$\Rightarrow P_n(t) = e^{-n\mu t}$

$$\frac{d}{dt}[e^{i\mu t} P_i(t)] = (i+1)\mu P_{i+1}(t) e^{i\mu t} \Rightarrow P_i(t) = (i+1)e^{-i\mu t}\mu \int_0^t P_{i+1}(t') e^{i\mu t'} dt'$$

$$P_{n-1}(t) = ne^{-(n-1)\mu t}\mu \int_0^t e^{-n\mu t'} e^{(n-1)\mu t'} dt' = ne^{-(n-1)\mu t}(1 - e^{-\mu t})$$

Recursively: $P_i(t) = \binom{n}{i}(e^{-\mu t})^i (1 - e^{-\mu t})^{n-i}$

Binomial distribution: The survival probability at time t is $e^{-\mu t}$ independent of others.

Outline

Relation to CTMC

Infinitesimal generator matrix:

$$Q = \begin{pmatrix} -\lambda_0 & \lambda_0 & 0 & \dots & \dots & \dots & \dots \\ \mu_1 & -(\lambda_1 + \mu_1) & \lambda_1 & 0 & \dots & \dots & \dots \\ 0 & \mu_2 & -(\lambda_2 + \mu_2) & \lambda_2 & 0 & \dots & \dots \\ \vdots & 0 & \mu_3 & -(\lambda_3 + \mu_3) & \lambda_3 & 0 & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \end{pmatrix}$$

Relation to DTMC

Embedded Markov chain of the process.
For $t \to \infty$, define:

$$\begin{aligned} P(E_{n+1}|E_n) &= \text{Prob. of transition } E_n \to E_{n+1} \\ &= \text{Prob. of going to } E_{n+1} \text{conditional on being in} E_n \end{aligned}$$

Define $P(E_{n-1}|E_n)$ similarly. Then

$$P(E_{n+1}|E_n) \backsim \lambda_n, P(E_{n-1}|E_n) \backsim \mu_n$$

$$P(E_{n+1}|E_n) = \frac{\lambda_n}{\lambda_n + \mu_n}, P(E_{n-1}|E_n) = \frac{\mu_n}{\lambda_n + \mu_n}$$

The same conditional probabilities hold if it is given that a transition will take place in $(t, t+h)$ conditional on being in E_n.

Outline

Linear Birth-Death Processes

Linear Birth-Death Process

- $\lambda_n = n\lambda$
- $\mu_n = n\mu$

$$\Rightarrow P_0'(t) = \mu P_1(t)$$
$$P_n'(t) = -(\lambda + \mu)nP_n(t) + \lambda(n-1)P_{n-1}(t) + \mu(n+1)P_{n+1}(t)$$

Steady state behavior is characterized by:

$$\lim_{t\to\infty} P_0'(t) = 0 \quad \Rightarrow \quad P_1(\infty) = 0$$

Similarly as $t \to \infty$ $P_n'(\infty) = 0$

Linear Birth-Death Processes

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Two cases can happen:

- If $P_0(\infty) = 1$ $\Rightarrow$ the probability of ultimate extinction is 1.
- If $P_0(\infty) = P_0 < 1$, the relations $P_1 = P_2 = P_3 \ldots = 0$ imply with probability $1 - P_0$ that the population can increase without bounds.

The population must either die out or increase indefinitely.

Mean of a Linear Birth-Death Process

$$P_n'(t) = -(\lambda+\mu)nP_n(t) + \lambda(n-1)P_{n-1}(t) + \mu(n+1)P_{n+1}(t)$$

Define Mean by $M(t) = \sum_{n=1}^{\infty} nP_n(t)$

and consider $M'(t) = \sum_{n=1}^{\infty} nP_n'(t)$, then:

$$M'(t) = -(\lambda+\mu)\sum_{n=1}^{\infty} n^2 P_n(t) + \lambda \sum_{n=1}^{\infty}(n-1)nP_{n-1}(t) + \mu \sum_{n=1}^{\infty}(n+1)nP_{n+1}(t)$$

Write $(n-1)n = (n-1)^2 + (n-1)$, $(n+1)n = (n+1)^2 - (n+1)$

Mean of a Linear Birth-Death Process

$$
\begin{aligned}
M'(t) &= -(\lambda+\mu)\sum_{n=1}^{\infty} n^2 P_n(t) \\
&\quad + \lambda \sum_{n=1}^{\infty}(n-1)^2 P_{n-1}(t) + \mu\left(\sum_{n=1}^{\infty}(n+1)^2 P_{n+1}(t) + P_1(t)\right) \\
&\quad + \lambda \sum_{n=1}^{\infty}(n-1) P_{n-1}(t) - \mu\left(\sum_{n=1}^{\infty}(n+1) P_{n+1}(t) + P_1(t)\right) \\
\Rightarrow\ M'(t) &= \lambda \sum_{n=1}^{\infty} nP_n(t) - \mu \sum_{n=1}^{\infty} nP_n(t) = (\lambda-\mu)M(t)
\end{aligned}
$$

$$M(t) = n_0 e^{(\lambda-\mu)t} \text{ if } P_{n_0}(0) = 1$$

Mean of a Linear Birth-Death Process

$M(t) = n_0 e^{(\lambda-\mu)t}$

- If $\lambda > \mu$ then $M(t) \to \infty$
- If $\lambda < \mu$ then $M(t) \to 0$

Similarly if $M_2(t) = \sum_{n=1}^{\infty} n^2 P_n(t)$ one can show that:

$$M_2'(t) = 2(\lambda - \mu)M_2(t) + (\lambda + \mu)M(t)$$

and when $\lambda > \mu$, the variance is:

$$n_0 e^{2(\lambda-\mu)t}\left(1 - e^{(\mu-\lambda)t}\right)\frac{\lambda+\mu}{\lambda-\mu}$$

Outline

Linear Birth-Death Process. Example

Let $X(t)$ be the number of bacteria in a colony at instant t.
Evolution of the population is described by:

- the time that each of the individuals takes for division in two (binary fission), independently of the other bacteria
- the life time of each bacterium (also independent)

Assume that:

- Time for division is exponentially dist. (rate λ)
- Life time is also exponentially dist. (rate μ)

$M(t) = n_0 e^{(\lambda-\mu)t}$

- If $\lambda > \mu$ then the population tends to infinity
- If $\lambda < \mu$ then the population tends to 0

A queueing system

- s servers
- K waiting places
- λ arrival rate (Poisson)
- μ Exp(μ) holding time (expectation $1/\mu$)

Is it a birth-death process?

A queueing system

- s servers
- K waiting places
- λ arrival rate (Poisson)
- μ Exp(μ) holding time (expectation $1/\mu$)

Let "$N =$number of customers in the system" be the state variable.

- N determines uniquely the number of customers in service and waiting room.
- After each arrival and departure the remaining service times of the customers in service are Exp(μ) distributed (memoryless).

Call blocking in an ATM network

An ATM network offers calls of two different types.

$$\begin{cases} R_1 = 1\,Mbps \\ \lambda_1 = \text{arrival rate} \\ \mu_1 = \text{mean holding time} \end{cases} \qquad \begin{cases} R_2 = 2\,Mbps \\ \lambda_2 = \text{arrival rate} \\ \mu_2 = \text{mean holding time} \end{cases}$$

Assume that the capacity of the link is infinite:

Is it a birth-death process?

Call blocking in an ATM network

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Assume that the capacity of the link is infinite:

The state variable is the pair (N_1, N_2) where N_i defines the number of class-i connections in progress.

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Assume that the capacity of the link is limited to 4.5 Mbps

Is it a birth-death process?

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Assume that the capacity of the link is limited to 4.5 Mbps

Exercise 1

Process definition

- There are two transatlantic cables each of which handle one telegraph message at a time.
- The time-to-breakdown for each has the same exponential random distribution with parameter λ.
- The time to repair for each cable has the same exponential random distribution with parameter μ.

Tasks:

- Draw the corresponding birth-death process.
- Write its infinitesimal generator.
- Write differential equations for the probabilities.
- Compute the steady state distribution

Exercise 2

Birth-disaster process

Consider that X_t is a continuous-time Markov process defined as follows:

- Each individual gives a birth after an exponential random time of parameter λ, independent of each other.
- A disaster occurs randomly at exponential random time of parameter δ.
- Once a disaster occurs, it wipes out all the entire population.

Tasks:

- What is the infinitesimal generator matrix of the process?
- What is the time evolution of $M(t) = \mathbb{E}[X_t]$?

Acknowledgments

Much of the material in the course is based on the following courses:

- Queueing Theory / Birth-death processes.
  J. Vitano
- Birth and Death Processes
  http://www.bibalex.org/supercourse/
- Performance modelling and evaluation. Birth-death processes.
  J. Campos
- Discrete State Stochastic Processes
  J. Baik