## 2 Erdos-Renyi graphs

### 2a Plot the average size of the two largest components in each realisation divided by N, against z for both values of N in a single plot (4 data series in total, use different colours).

### Use all 20 (or more) realisations and include error bars indicating the standard deviation.

In [36]:

```
import networkx as nx
import numpy as np
import matplotlib.pyplot as plt
from tqdm import tqdm

mas_N=np.arange(100,1100,step=100)
amt_samples=20
mas_z= np.linspace(0.1,3,30)
mas_graphs=[[] for i in range(len(mas_N))]
for i in range(len(mas_N)):
    N=mas_N[i]
    pN=mas_z/N
    for j in range(len(mas_z)):
        sample_of_graphs=[]
        for k in range(amt_samples):
            sample_of_graphs.append(nx.erdos_renyi_graph(N,pN[j
        mas_graphs[i].append(sample_of_graphs)

matr_largest_component_sizes_mean=np.zeros((len(mas_N),2,len(ma
matr_largest_component_sizes_std=np.zeros((len(mas_N),2,len(mas_
for i in tqdm(range(len(mas_N))):
    for j in range(len(mas_z)):
        #print(i,j)
        mas0=[]
        mas1=[]
        std0=[]
        std1=[]
        for k in range(amt_samples):
            G=mas_graphs[i][j][k]
            Gcc=sorted([G.subgraph(c) for c in nx.connected_com
            #print(len(Gcc))
            mas0.append(Gcc[0].number_of_nodes())
            mas1.append(Gcc[1].number_of_nodes())
        #print(mas0,np.mean(mas0))
        #print(mas1,np.mean(mas1))
        matr_largest_component_sizes_mean[i][0][j]=np.mean(mas0
        matr_largest_component_sizes_mean[i][1][j]=np.mean(mas1
        matr_largest_component_sizes_std[i][0][j]=np.std(mas0)
        matr_largest_component_sizes_std[i][1][j]=np.std(mas1)


plt.errorbar(x=mas_z, y=matr_largest_component_sizes_mean[0][0]
plt.errorbar(x=mas_z, y=matr_largest_component_sizes_mean[0][1]
plt.errorbar(x=mas_z, y=matr_largest_component_sizes_mean[-1][0
plt.errorbar(x=mas_z, y=matr_largest_component_sizes_mean[-1][1
plt.legend()
plt.title('Largest and second largest components sizes of a E-R
plt.xlabel('z')
plt.ylabel('Component size')
plt.show()
```

```
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```
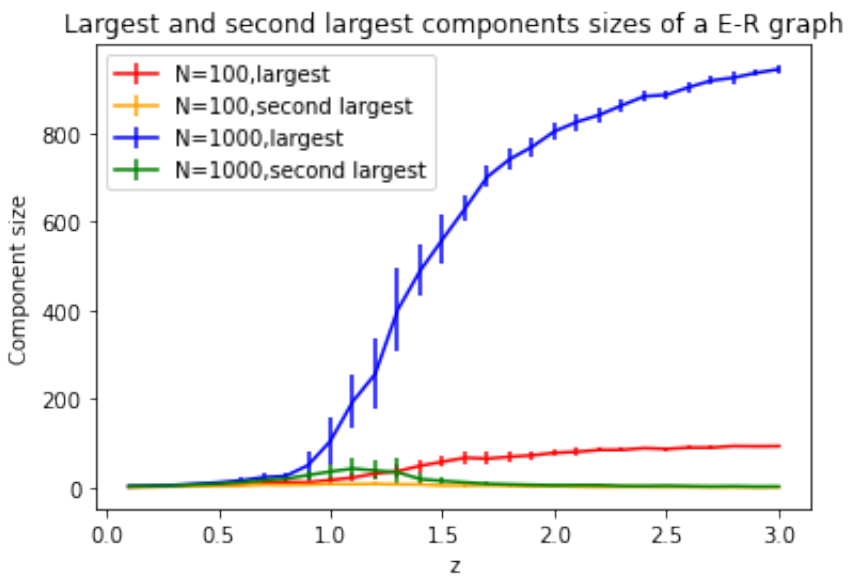



We see two things:

1) the largest component size (denote it by $C_1(N)$) grows significantly as $N$ grows, while the second component size (denote it by $C_2(N)$) does not show significant growth 2) Behaviour changes at $p = \frac{1}{N}$, that is, for $p = \frac{c}{N}$ with $c < 1$ we have less significant growth, while for $c > 1$ we have very fast growth of largest component size.

In fact, I found in the literature, that:

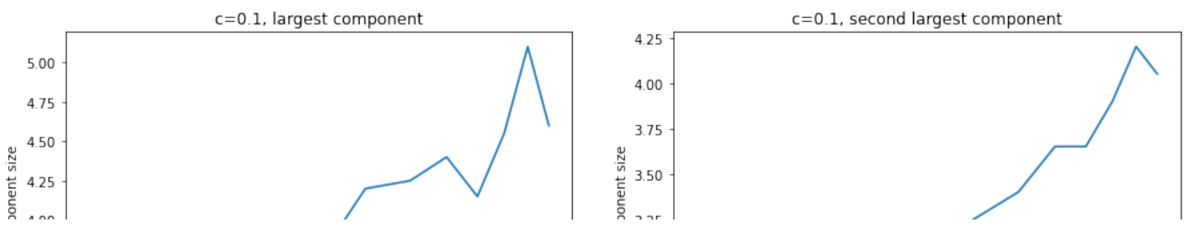
for $c < 1$: $C_1(N) \to \frac{\ln N}{c - \ln c - 1}$ as $N \to \infty$

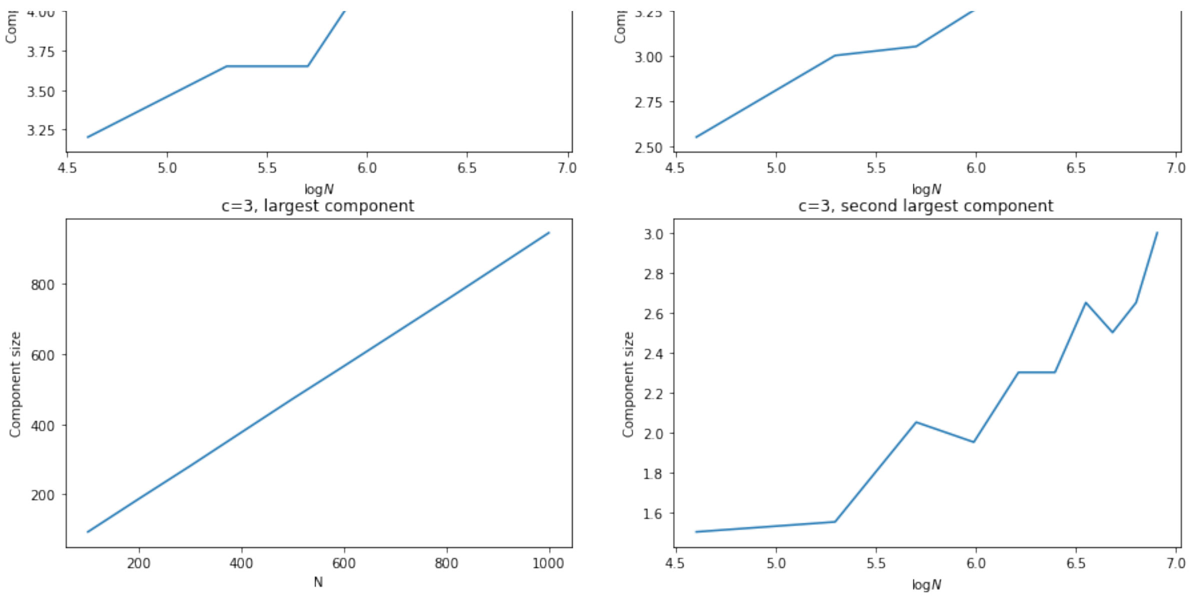
for $c > 1$: $C_1(N) \to n\beta(c)$, where $\beta$ is the unique solution of equation $\beta + e^{-\beta c} = 1$ on interval $(0, 1)$,

and $C_2(N) \to \frac{\ln N}{c - \ln c - 1}$ as $N \to \infty$. Lets check it for $c = 0.1$ and $c = 3$

We see that for $c = 3$ $C_1(N)$ really grows proportional to $N$; other three graphs are not looking like lines, may be we need more amt_samples to see the desired behaviour

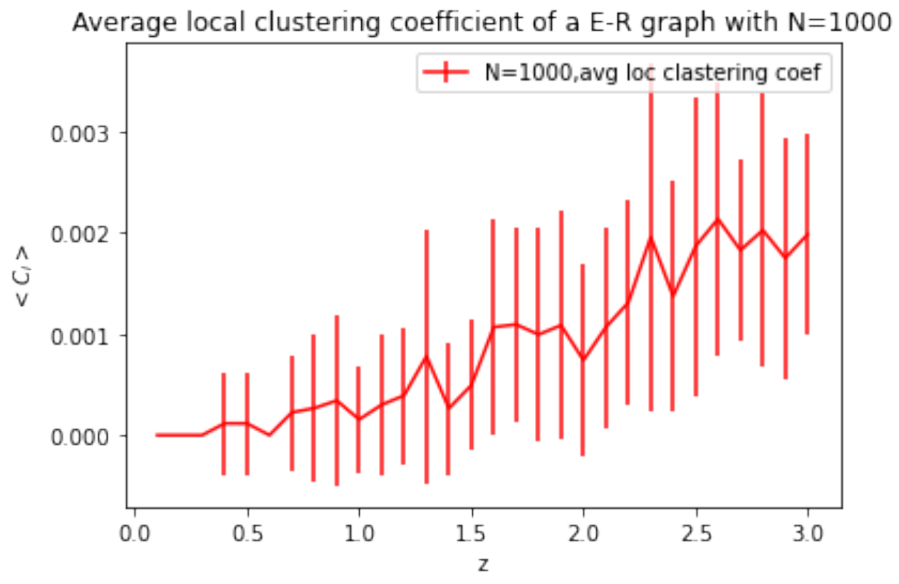
In [37]:
```python
m01_largest=[]
m01_second_largest=[]
m3_largest=[]
m3_second_largest=[]
for i in range(len(mas_N)):
    m01_largest.append(matr_largest_component_sizes_mean[i][0][
    m01_second_largest.append(matr_largest_component_sizes_mean
    m3_largest.append(matr_largest_component_sizes_mean[i][0][-
    m3_second_largest.append(matr_largest_component_sizes_mean[

fig,ax=plt.subplots(2,2)
fig.set_figheight(10)
fig.set_figwidth(15)
ax[0][0].plot(np.log(mas_N), m01_largest,label='m01_largest')
ax[0][0].set_xlabel(r'$\logN$')
ax[0][0].set_ylabel('Component size')
ax[0][0].set_title(r'c=0.1, largest component')

ax[0][1].plot(np.log(mas_N), m01_second_largest,label='m01_seco
ax[0][1].set_xlabel(r'$\logN$')
ax[0][1].set_ylabel('Component size')
ax[0][1].set_title(r'c=0.1, second largest component')

ax[1][0].plot(mas_N, m3_largest,label='m3_largest')
ax[1][0].set_xlabel('N')
ax[1][0].set_ylabel('Component size')
ax[1][0].set_title(r'c=3, largest component')

ax[1][1].plot(np.log(mas_N), m3_second_largest,label='m3_second_
ax[1][1].set_xlabel(r'$\logN$')
ax[1][1].set_ylabel('Component size')
ax[1][1].set_title(r'c=3, second largest component')
plt.show()
```





## 2b For N = 1000, plot the average local clustering coefficient ⟨Ci⟩ against z using all 20 re- alisations and i = 1, . . . , N for averaging, and including error bars indicating the standard deviation for all 20N data points.

In [48]:
```python
local_clustering_coef_mean=np.zeros(len(mas_z))
local_clustering_coef_std=np.zeros(len(mas_z))

for j in range(len(mas_z)):
    mas_mean=[]
    mas_std=[]
    for k in range(amt_samples):
        G=mas_graphs[-1][j][k]
        clastering_coefs=nx.clustering(G)
        clastering_coefs=list(map(lambda x: clastering_coefs[x]
        mas_mean.append(np.mean(clastering_coefs))
        mas_std.append(np.std(clastering_coefs))
    local_clustering_coef_mean[j]=np.mean(mas_mean)
    local_clustering_coef_std[j]=np.std(mas_mean)

plt.errorbar(x=mas_z, y=local_clustering_coef_mean,yerr=local_c
plt.legend()
plt.title('Average local clustering coefficient of a E-R graph 
plt.xlabel('z')
plt.ylabel(r'$<C_i>$')
plt.show()
```



We see that average local clustering coefficient grows, but very slowly with growth of $z$, so E-R graphs look more like trees than as clustering nodes, and local clustering coefficient has big standard deviation that does not seem to decrease with the growth of $N$.

## 2c Use results in lectures to state what is the expected number of edges, as well as the ex- pected average degree as a function of z and N and plot these as a function of z for the two values of N, comparing your results with the expected ones. Include error bars to indicate the standard deviation for your data points.

In [92]:
```python
matr_nb_edges_mean=np.zeros((len(mas_N),len(mas_z)))
matr_nb_edges_std=np.zeros((len(mas_N),len(mas_z)))
matr_degree_mean=np.zeros((len(mas_N),len(mas_z)))
matr_degree_std=np.zeros((len(mas_N),len(mas_z)))
for i in tqdm(range(len(mas_N))):
    for j in range(len(mas_z)):
        #print(i,j)
        mas_nb_edges=[]
        mas_degree=[]
        for k in range(amt_samples):
            G=mas_graphs[i][j][k]
            nb_edges=G.number_of_edges()
            degree=np.mean(list(map(lambda x: x[1],G.degree)))
            mas_nb_edges.append(nb_edges)
            mas_degree.append(degree)
        matr_nb_edges_mean[i][j]=np.mean(mas_nb_edges)
        matr_nb_edges_std[i][j]=np.std(mas_nb_edges)
        matr_degree_mean[i][j]=np.mean(mas_degree)
        matr_degree_std[i][j]=np.std(mas_degree)

fig,ax=plt.subplots(1,2)
fig.set_figheight(5)
fig.set_figwidth(15)

ax[0].errorbar(x=mas_z, y=matr_nb_edges_mean[0],yerr=matr_nb_ed
ax[0].errorbar(x=mas_z, y=matr_nb_edges_mean[-1],yerr=matr_nb_e
ax[0].scatter(mas_z,(100-1)/2*mas_z,label=r'$\frac{z(100-1)}{2}
ax[0].scatter(mas_z,(1000-1)/2*mas_z,label=r'$\frac{z(1000-1)}{
ax[0].set_xlabel('z')
ax[0].set_ylabel('Nb of edges')
ax[0].legend()
ax[0].set_title('Avg number of edges in E-R graph')

ax[1].errorbar(x=mas_z, y=matr_degree_mean[0],yerr=matr_degree_
ax[1].errorbar(x=mas_z, y=matr_degree_mean[-1],yerr=matr_degree
ax[1].scatter(mas_z,(1-1/100)*mas_z,linewidths=5,label=r'$(1-\f
ax[1].set_xlabel('z')
ax[1].set_ylabel('Degree')
ax[1].legend()
```

```python
ax[1].set_title('Avg degree in E-R graph')

plt.show()
```

```
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```
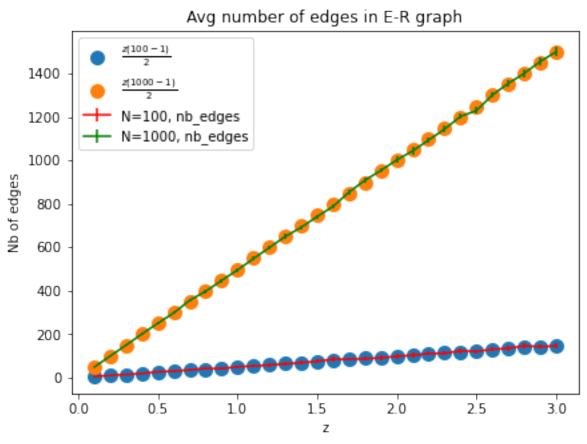
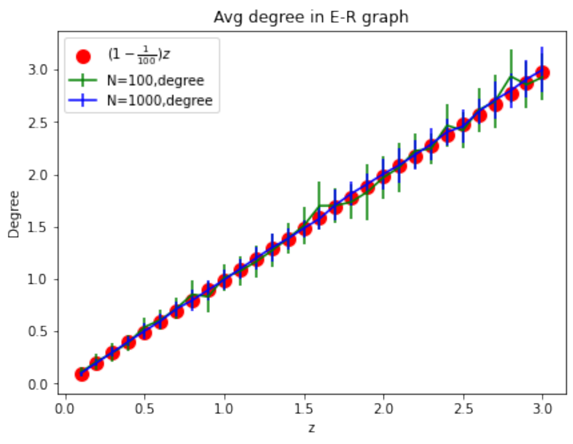



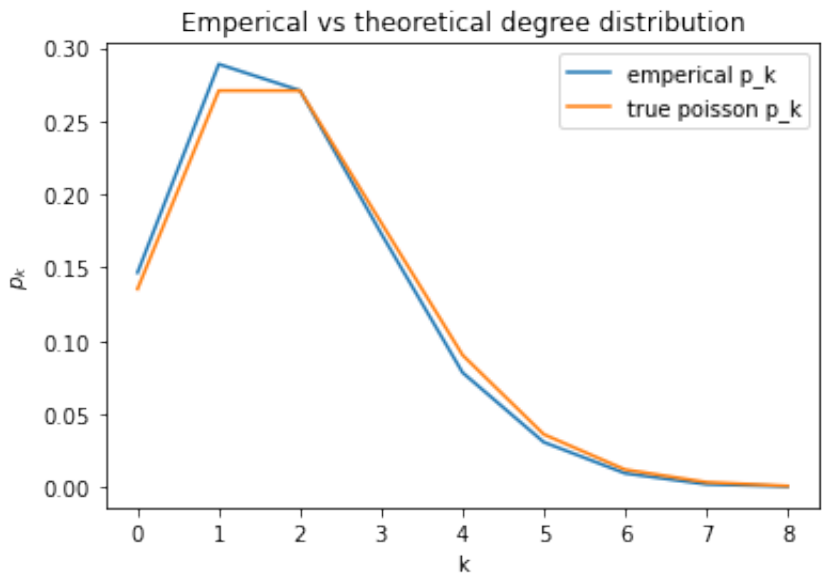


From lectures we know that expected number of edges is $E(K) = p\frac{N(N-1)}{2}$ and average degree is $E[<k>] = (N-1)p$; We have $P_N = \frac{z}{N}$, so expected number of edges is $E(K) = \frac{z(N-1)}{2}$ and and average degree is $E[<k>] = z(1-\frac{1}{N})$;

Avg degree is clearly right - the graph shows as a line with slope nearly 1; And number of edges also goes in line with theoretical (scattered) average number of edges.

## 2d For N = 1000 and your favourite value of z ∈ [0.5, 2], plot the degree distribution p(k) against k = 0, 1, . . . using all 20 realisations, and compare it to the mass function of the Poi(z) Poisson distribution in a single plot.

In [121]:
```python
import scipy
from scipy import stats

def degree_distribution(G):
    vk = dict(G.degree())
    vk = list(vk.values())
    vk = np.array(vk)          # store degree values in array

    maxk = np.max(vk)
    k = np.arange(0,maxk+1) # possible values of k

    pk = np.zeros(maxk+1) # degree distribution p(k)
    for i in vk:
        pk[i] = pk[i] + 1
    pk = pk/sum(pk) # the sum of the elements of P(k) must to b

    return k,pk

mas_pks=[]
z=mas_z[19] #z=2
dict_ks=dict()
for s in range(amt_samples):
    G=mas_graphs[1][19][s]
    k,pk=degree_distribution(G)
    for a,b in zip(k,pk):
        if a not in dict_ks:
            dict_ks[a]=b
        else:
            dict_ks[a]+=b
mas_k=[]
mas_pk=[]
for k in dict_ks.keys():
    mas_k.append(k)
    mas_pk.append(dict_ks[k]/amt_samples)

mas_pk=np.array(mas_pk)
mas_poisson=[scipy.stats.poisson.pmf(k=k,mu=z) for k in mas_k]
plt.plot(mas_k,mas_pk,label='emperical p_k')
plt.plot(mas_k,mas_poisson, label='true poisson p_k')
plt.xlabel('k')
plt.ylabel(r'$p_k$')
plt.legend()
plt.title('Emperical vs theoretical degree distribution')
plt.show()
```

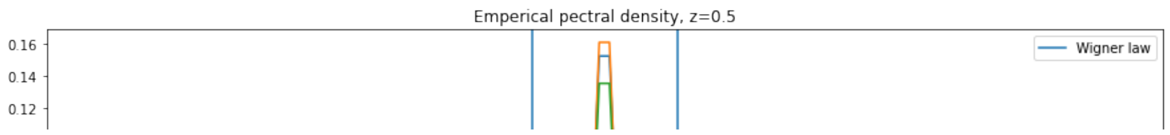


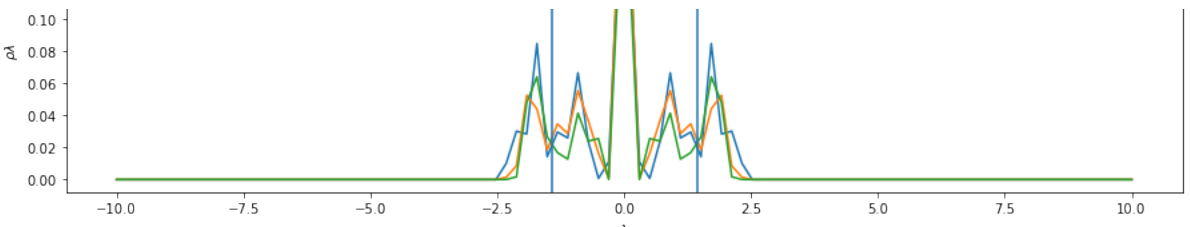
We see that for chosen $z = 2$ emperical and theoretical degree distributions are very close!

## 2e Consider z = 0.5, 1.5, 5 and 10. Plot the spectrum of the adjacency matrix A using all 20 realisations with a kernel density estimate, and compare it to the Wigner semi-circle law. Comment on your results based on what you expect from the lectures.

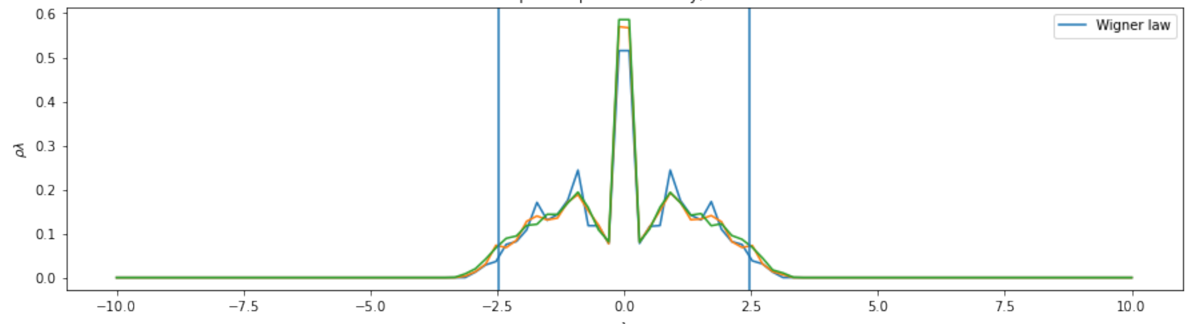
In [157]:
```python
N=1000
amt_samples=3
mas_z= [0.5,1.5,5,10]
mas_kdes=[[] for i in range(len(mas_z))]

lambd = np.linspace(-10,10,100)
fig,ax=plt.subplots(4,1)
fig.set_figheight(18)
fig.set_figwidth(15)


for i in range(len(mas_z)):
    pN=mas_z[i]/N
    sample_of_graphs=[]
    for j in range(amt_samples):
        G=nx.erdos_renyi_graph(N,pN)
        A = nx.to_numpy_matrix(G)    #find adjacency matrix
        sigm=np.std(A.flatten())
        r=2*sigm*np.sqrt(N)
        if (j==0):
            print("Sigma=",sigm, "2*sigm*sqrt(N)=",r)
        evals, evecs = np.linalg.eig(A)  #calculate e'vales
        # find the spectral density using a kernel density esti
        evals=list(map(lambda x: x.real,evals))
        spectral_density = stats.gaussian_kde(evals, bw_method 
        ax[i].plot(lambd, spectral_density(lambd))
    ax[i].axvline(x=r,label='Wigner law')
    ax[i].axvline(x=-r)
    ax[i].set_title('Emperical pectral density, z='+str(mas_z[i
    ax[i].set_xlabel(r'$\lambda$')
    ax[i].set_ylabel(r'$\rho\lambda$')
    ax[i].legend()
plt.show()
```

```
Sigma= 0.023058989049826093 2*sigm*sqrt(N)= 1.4583785187666467
Sigma= 0.037363693554037194 2*sigm*sqrt(N)= 2.363087468546182
Sigma= 0.07046346500137501 2*sigm*sqrt(N)= 4.456500824638093
Sigma= 0.1001563879340704 2*sigm*sqrt(N)= 6.334446161741373
```

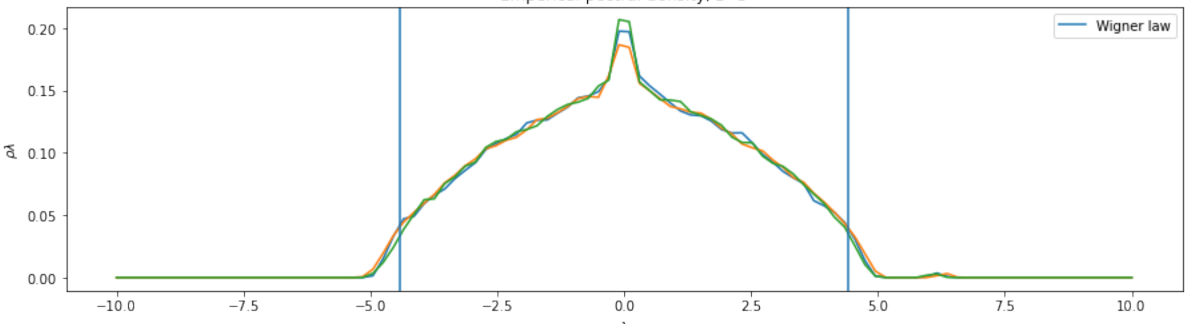




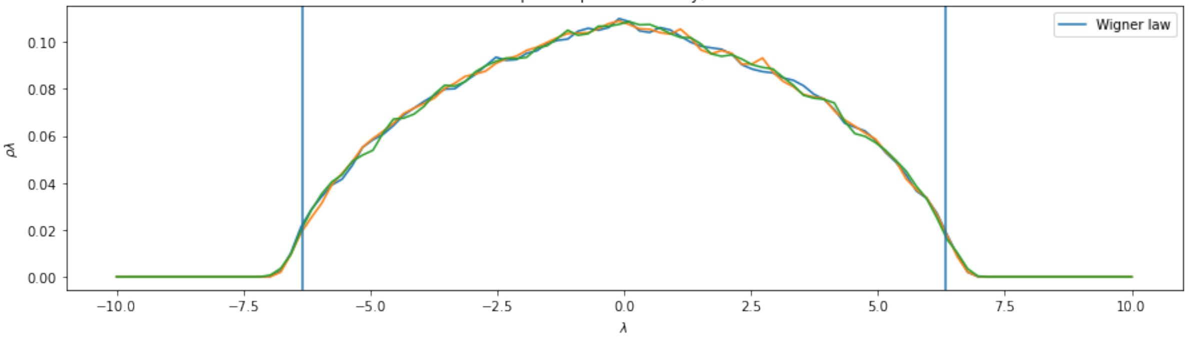
Emperical pectral density, z=1.5



Emperical pectral density, z=5



Emperical pectral density, z=10



We see that:

for $z > 1$ Wigner semi-circle law holds with support $4\sqrt{N}\sigma_N$, where $\sigma_N$ is the standad deviation of adjacency matric coefficients.

for $z < 1$ the asymptotic spectra; density deviates from Wigner semi-circle spectral density.

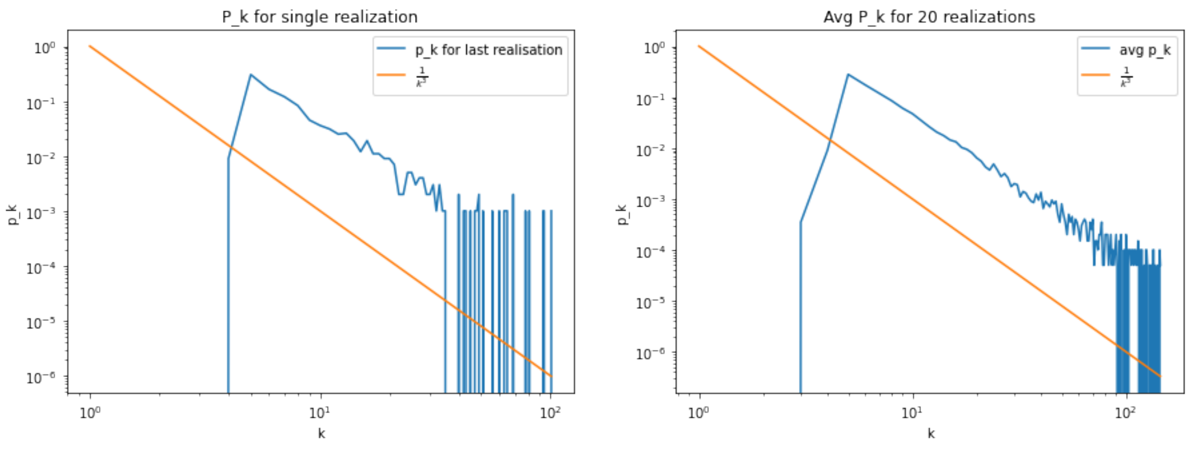
## 3 Baraba´si-Albert model

In [339]:
```python
import numpy as np
from numpy.random import choice

ba_graphs=[]
```

```

def generate_barabasi_albert(m0,m,N):
    A=np.zeros((N,N))
    for i in range(m0):
        for j in range(m0):
            if i!=j:
                A[i][j]=1
    for t in range(1,N-m0+1):
        mas_ks=A[0:m0+t-1,0:m0+t-1].sum(axis=1)
        #print(mas_ks)
        mas_ks/=sum(mas_ks)
        possible_nodes=np.arange(len(mas_ks))
        realized_nodes = np.random.choice(possible_nodes, m, p=
        #print(realized_nodes)
        for i in realized_nodes:
            A[m0+t-1,i]=1
            A[i,m0+t-1]=1
    return nx.from_numpy_array(A)

m0=5
m=5
N=1000
amt_samples=20
for i in range(amt_samples):
    G=generate_barabasi_albert(m0,m,N)
    degree=np.mean(list(map(lambda x: x[1],G.degree)))
    #print("Avg degree=", degree) #is is 2m=10
    ba_graphs.append(G)

mas_pks=[]
dict_ks=dict()
for G in ba_graphs:
    k,pk=degree_distribution(G)
    for a,b in zip(k,pk):
        if a not in dict_ks:
            dict_ks[a]=b
        else:
            dict_ks[a]+=b
mas_k=[]
mas_pk=[]
for kk in dict_ks.keys():
    mas_k.append(kk)
    mas_pk.append(dict_ks[kk]/amt_samples)

fig,ax=plt.subplots(1,2)
fig.set_figheight(5)
fig.set_figwidth(15)

ax[0].loglog(k,pk,label='p_k for last realisation')
ax[0].loglog(k,1/k**3,label=r'$\frac{1}{k^3}$')
ax[0].set_xlabel('k')
ax[0].set_ylabel('p_k')
ax[0].legend()
ax[0].set_title('P_k for single realization')
```

```


ax[1].loglog(mas_k,mas_pk,label="avg p_k")
ax[1].loglog(mas_k,1/(np.array(mas_k)**3),label=r'$\frac{1}{k^3
ax[1].set_xlabel('k')
ax[1].set_ylabel('p_k')
ax[1].legend()
ax[1].set_title('Avg P_k for 20 realizations')

plt.show()
```

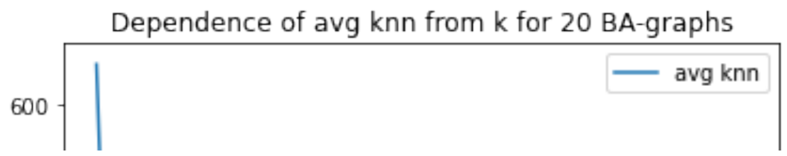
```
<ipython-input-339-6a5711e26d54>:54: RuntimeWarning: divide by zer
o encountered in true_divide
  ax[0].loglog(k,1/k**3,label=r'$\frac{1}{k^3}$')
<ipython-input-339-6a5711e26d54>:62: RuntimeWarning: divide by zer
o encountered in true_divide
  ax[1].loglog(mas_k,1/(np.array(mas_k)**3),label=r'$\frac{1}{k^3}
$')
```

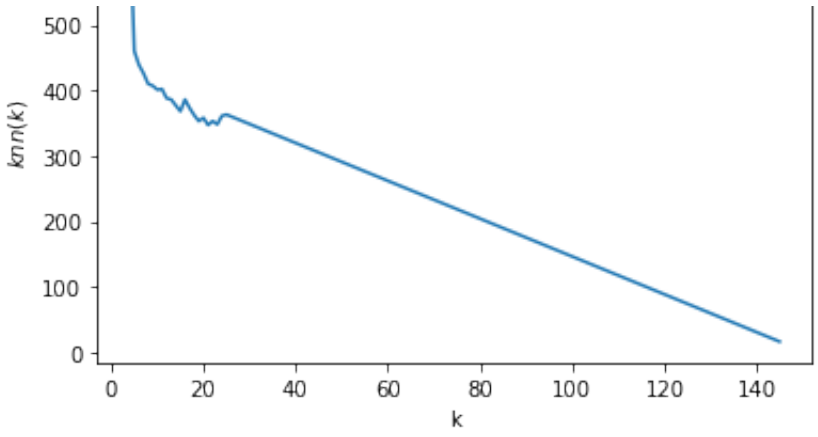


We see that in log-log plot the dependence of $p_k$ on $k$ is linear and the coefficient is -3, because the slope of the orange line (we expect $p_k$ be proportional to $k^{-3}$ from lectures ) is the same as the slope of line that could approximate blue plot.

## 3b Compute the nearest neighbour and decide whether the graphs are typically uncorrelated or (dis-)assortative.

In [340]:
```
def calculate_knn(G,k):
    A = nx.to_numpy_matrix(G)
    ki_s=A.sum(axis=1).astype(int)
    max_k=int(max(ki_s))
#     if k==0:
#         print(max_k, min(ki_s),sum(ki_s), 2*G.number_of_edges
    knn_i_s=A.dot(ki_s.reshape((-1,1)))/ki_s
    num=0
    denom=0
    for i,ki in enumerate(ki_s):
        if ki==k:
            num+=knn_i_s[i]
            denom+=1
    if denom !=0:
        return max_k, num/denom
    else:
        return max_k, np.infty


dict_knns=dict()
for G in ba_graphs:
    mas_knn=[]
    k_max, knn_tek=calculate_knn(G,0)
    mas_knn.append(knn_tek)
    if 0 not in dict_knns:
        dict_knns[0]=knn_tek
    else:
        dict_knns[0]+=knn_tek
    #print("k_max=",k_max)
    for k in range(1,k_max+1):
        _,knn_tek=calculate_knn(G,k)
        if knn_tek <np.infty:
            knn_tek=int(knn_tek)
        mas_knn.append(knn_tek)
        if k not in dict_knns:
            #print(k,knn_tek)
            dict_knns[k]=knn_tek
        else:
            dict_knns[k]+=knn_tek


mas_k=[]
mas_knns=[]
for k in dict_knns.keys():
    mas_k.append(k)
    mas_knns.append(dict_knns[k]/amt_samples)
mas_k=[]
mas_knns=[]
for k in dict_knns.keys():
    if dict_knns[k]<np.inf:
        mas_k.append(k)
        mas_knns.append(dict_knns[k])
plt.plot(mas_k,mas_knns,label='avg knn')
plt.xlabel('k')
plt.ylabel(r'$knn(k)$')
plt.legend()
plt.title(r'Dependence of avg knn from k for 20 BA-graphs')
plt.show()
```





Note that we does not plot those $(k, knn(k))$ for which average $knn(k)$ in infinity; From the graph we see that $knn(k)$ depends on $k$ - so the graphs are typycally correlated - this is strange because in the lecture it was written that BA graphs are uncorrelated - so may be we calculated not $knn$. And also $knn(k)$ is decreasing, so the graph is disaassertive.

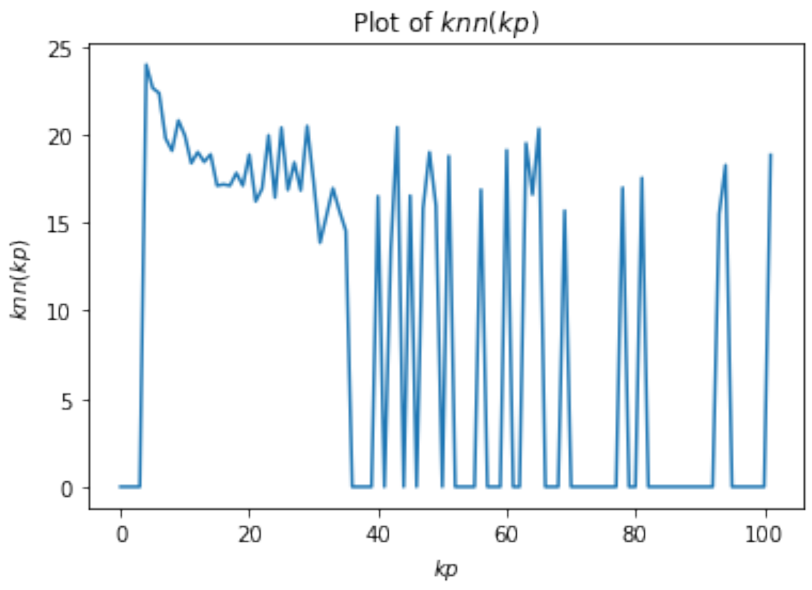
In [341]:
```
G=ba_graphs[-1]
A=nx.to_numpy_matrix(G)
ki_s=list(map(lambda x: x[1], G.degree))
k_max=max(ki_s)
matr_q=np.zeros((k_max+1,k_max+1))
for k in tqdm(range(k_max+1)):
    #print(k)
    for kp in range(k_max+1):
        q=0
        for i in range(A.shape[0]):
            for j in range(A.shape[1]):
                if (ki_s[i]==k and ki_s[j]==kp):
                    #print(A[i,j])
                    #print(k,kp,i,j,A[i,j])
                    q+=A[i,j]
        q/=A.sum()
        matr_q[k,kp]=q


```

```
100%|██████████| 102/102 [13:06<00:00,  7.71s/it]
```

In [356]:
```
matr_q.sum()
```

Out[356]:
```
1.0
```

In [381]:
```python
# compute knn(kp) for different kp
mas_knn=[]
q_kp=matr_q.sum(axis=0)

for kp in range(k_max+1):
    s=0
    for k in range(k_max+1):
        s+=k*matr_q[k,kp]
    if q_kp[kp]!=0:
        s/=q_kp[kp]
    mas_knn.append(s)
plt.plot(np.arange(k_max+1),mas_knn)
plt.title(r'Plot of $knn(kp)$')
plt.xlabel(r'$kp$')
plt.ylabel(r'$knn(kp)$')
plt.show()
```



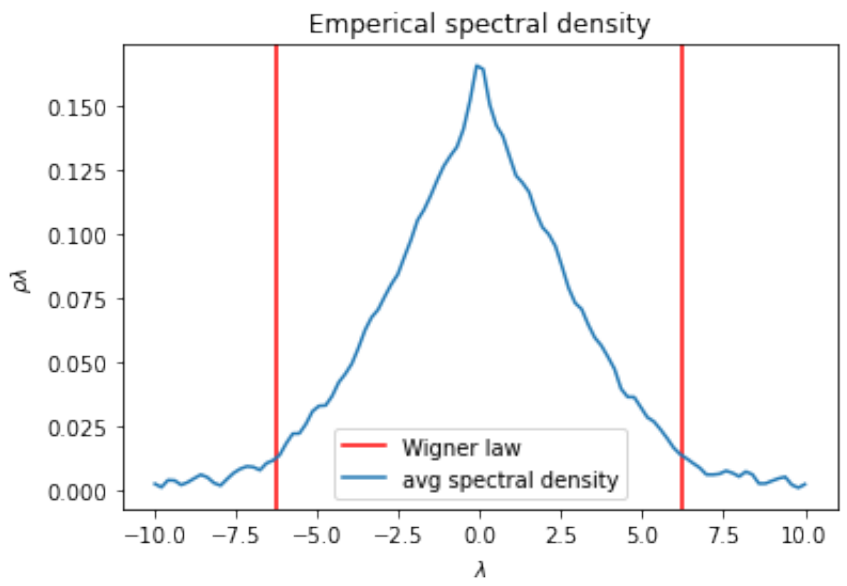
Now it is better, because knn looks like smth not depending on kp. If we averaged knn on 20 graphs, then the graph will become almost a line.

## 3c Plot the spectrum of the adjacency matrix A = (aij) using all realisations with a ker- nel density estimate, and compare it to the Wigner semi-circle law with σ2 = var[aij]. Comment on your results.

In [384]:
```
A_avg=np.zeros_like(A)
for G in ba_graphs:
    A=nx.to_numpy_matrix(G)
    for i in range(A.shape[0]):
            for j in range(A.shape[1]):
                A_avg[i,j]+=A[i,j]
A_avg/=amt_samples
```

In [397]:
```
sigm=np.std(A.flatten())
r=2*sigm*np.sqrt(N)
print("r=",r)
evals, evecs = np.linalg.eig(A)  #calculate e'vales
# find the spectral density using a kernel density estimate
evals=list(map(lambda x: x.real,evals))
spectral_density = stats.gaussian_kde(evals, bw_method = 0.05)
plt.title('Emperical spectral density')
plt.xlabel(r'$\lambda$')
plt.ylabel(r'$\rho\lambda$')
plt.axvline(x=r,label='Wigner law',color='red')
plt.axvline(x=-r,color='red')
plt.plot(lambd, spectral_density(lambd),label='avg spectral den
plt.legend()
plt.show()
```

```
r= 6.2340114391938695
```



We see that Wigner semi-circle law holds with support $4\sqrt{N}\sigma_N$, where $\sigma_N$ is the standad deviation of adjacency matric coefficients.

In [ ]:
```

```