Ex 1 (1) Apply Itô's formula to $\Phi(t)\cdot\left[x+\int_0^t \Phi^{-1}(s)a(s)ds+\int_0^t \Phi^{-1}(s)\sigma(s)dW_s\right]$

$$dX_t = d\Phi(t)\left[x+\int_0^t \Phi^{-1}(s)a(s)\,ds+\int_0^t \Phi^{-1}(s)\sigma(s)dW_s\right]$$
$$+\Phi(t)\left[\Phi^{-1}(t)a(t)dt+\Phi^{-1}(t)\sigma(t)dW_t\right]$$

$$= b(t)\Phi(t)\left[x+\int_0^t \Phi^{-1}(s)a(s)ds+\int_0^t \Phi^{-1}(s)\sigma(s)dW_s\right]+a(t)dt+\sigma(t)dW_t$$
$$=(a(t)+b(t)X_t)\,dt+\sigma(t)dW_t$$

It is clear that $X_0=x$ #

(2) Take $a(t)=a$, $b(t)=-b$ and $\sigma(t)=\sigma$. Then, $\Phi(t)=e^{-bt}$ and

$$X_t=e^{-bt}\left[x_0+\int_0^t e^{bs}a\,ds+\int_0^t e^{bs}\sigma dW_s\right]$$
$$=x_0e^{-bt}+\frac{a}{b}(1-e^{-bt})+\sigma\int_0^t e^{b(s-t)}dW_s \qquad \#$$

(3) Consider stochastic integral $\int_0^t e^{b(s-t)}dW_s$ which is a mart.

We have $E\left[\int_0^t e^{b(s-t)}dW_s\right]=0$

$$Var\left[\int_0^t e^{b(s-t)}dW_s\right]=E\left[\left(\int_0^t e^{b(s-t)}dW_s\right)^2\right]\overset{\text{Itô's isometry}}{=\!=\!=}\int_0^t e^{2b(s-t)}ds$$
$$=\frac{1}{2b}(1-e^{-2bt})$$

Therefore. $E[X_t]=x_0e^{-bt}+\frac{a}{b}(1-e^{-bt})\to\frac{a}{b}$ as $t\to\infty$

$$Var[X_t]=Var\left[\sigma\int_0^t e^{b(s-t)}dW_s\right]=\frac{\sigma^2}{2b}(1-e^{-2bt})\to\frac{\sigma^2}{2b} \text{ as } t\to\infty$$
#

(4) To show $B_t=\sqrt{2b}\,Y_{\frac{\ln(1+t)}{2b}}$ is a BM, we first observe that $B$ is Gaussian since $Y_t=\int_0^t e^{bs}dW_s$ is Gaussian. Hence, we only need to show that $B$ has zero mean and covariance $E[B_tB_s]=t\wedge s$. Indeed,

$$E[B_t]=E\left[\sqrt{2b}\int_0^{\frac{\ln(1+t)}{2b}}e^{bs}dW_s\right]=0$$

For $t\geq s\geq 0$,

$$E[B_tB_s] = E[(B_t-B_s)B_s + B_s^2]$$

$$= E\left[\sqrt{2b}\int_{\frac{\ln(s+1)}{2b}}^{\frac{\ln(t+1)}{2b}} e^{bs}dW_s \cdot \sqrt{2b}\int_0^{\frac{\ln(s+1)}{2b}} e^{bs}dW_s + 2b\left(\int_0^{\frac{\ln(s+1)}{2b}} e^{bs}dW_s\right)^2\right]$$

$$= 2b\int_0^{\frac{\ln(s+1)}{2b}} e^{2bs}ds = e^{2b\left(\frac{\ln(s+1)}{2b}\right)}-1 = s$$

Similarly, for $s \geq t \geq 0$,

$$E[B_tB_s] = t$$

Finally, it is clear that $Y_t = \frac{1}{\sqrt{2b}} B_{e^{2bt}-1}$. Plugging this expression into the formula for $X_t$, we get

$$X_t = X_0e^{-bt} + \frac{a}{b}(1-e^{-bt}) + \frac{\sigma e^{-bt}}{\sqrt{2b}} B_{e^{2bt}-1} \qquad \#$$

Ex 2 (1) Applying Itô's formula to $e^{M_t - \frac{1}{2}\langle M\rangle_t}$

$$d\mathcal{E}(M)_t = \mathcal{E}(M)_t\left[dM_t - \tfrac{1}{2}d\langle M\rangle_t\right] + \tfrac{1}{2}\mathcal{E}(M)_t\, d\langle M\rangle_t = \mathcal{E}(M)_t\, dM_t$$

It is clear that $\mathcal{E}(M)_0 = 1$ #

(2) Since $\mathcal{E}(M)$ is a local mart. there exists a sequence of stopping times $T_n \uparrow \infty$ s.th. for $t \geq s \geq 0$,

$$E[\mathcal{E}(M)_t^{T_n} \mid F_s] = \mathcal{E}(M)_s^{T_n}$$

Since $\mathcal{E}(M)$ is non-negative, by Fatou's lemma,

$$\lim_{n\to\infty} E[\mathcal{E}(M)_t^{T_n} \mid F_s] \geq E[\lim_{n\to\infty}\mathcal{E}(M)_t^{T_n} \mid F_s] = E[\mathcal{E}(M)_t \mid F_s]$$

On the other hand, $\lim_{n\to\infty} \mathcal{E}(M)_s^{T_n} = \mathcal{E}(M)_s$, from which we conclude

$$E[\mathcal{E}(M)_t \mid F_s] \leq \mathcal{E}(M)_s \qquad \#$$

(3) $\mathcal{E}(M)_t\,\mathcal{E}(-M)_t = e^{M_t - \frac{1}{2}\langle M\rangle_t}\; e^{-M_t - \frac{1}{2}\langle -M\rangle_t}$

Note that $\langle M\rangle_t = \langle -M\rangle_t$. Indeed, $dM_t^2 = 2M_t dM_t + d\langle M\rangle_t$

$$d(-M_t)^2 = 2(-M_t)\,d(-M)_t + d\langle -M\rangle_t$$

Hence, $\mathcal{E}(M)_t\mathcal{E}(-M)_t = e^{-\langle M\rangle_t}$

..

Hence, $\mathcal{E}(M)_t\,\mathcal{E}(-M)_t = e^{-\langle M\rangle_t}$

#

(4) We first show that $\langle M, N\rangle_t = \frac{1}{2}\left(\langle M+N\rangle_t - \langle M\rangle_t - \langle N\rangle_t\right)$ (★)

Indeed, applying Itô's formula to $(M+N)^2$:

$$(M_t+N_t)^2 = 2\int_0^t (M_s+N_s)\,d(M_s+N_s) + \langle M+N\rangle_t$$

Hence, $$\langle M+N\rangle_t = \underbrace{\left(M_t^2 - 2\int_0^t M_s\,dM_s\right)}_{\langle M\rangle_t} + \underbrace{\left(N_t^2 - 2\int_0^t N_s\,dN_s\right)}_{\langle N\rangle_t} + 2\underbrace{\left(M_tN_t - \int_0^t M_s\,dN_s - \int_0^t N_s\,dM_s\right)}_{\langle M,N\rangle_t}$$

In turn, $$\mathcal{E}(M+N)_t\, e^{\langle M,N\rangle_t}$$
$$= e^{M_t+N_t-\frac{1}{2}\langle M+N\rangle_t + \langle M,N\rangle_t}$$
$$= e^{M_t-\frac{1}{2}\langle M\rangle_t}\, e^{N_t-\frac{1}{2}\langle N\rangle_t} = \mathcal{E}(M)_t\,\mathcal{E}(N)_t$$

#