Tommaso Rosati

Brownian Motion

Problem sheet 1

1. **Gaussians**

   (a) Compute the characteristic function $t \mapsto \mathbb{E}[e^{itX}]$ of a (univariate) Gaussian $X \sim \mathcal{N}(\mu, \sigma^2)$, and its moment generating function $t \mapsto \mathbb{E}[e^{tX}]$.

   Now let $\mathbf{X} = (X_1, \dots, X_n)$ be a **multivariate Gaussian**.

   (b) Write $\mu_i = \mathbb{E}[X_i]$ for the vector of means and $\Sigma = \big(\mathrm{Cov}[X_i, X_j] : i, j = 1, .., n\big)$ for the covariance matrix. Compute the characteristic function of $\mathbf{X}$.

   (c) Show that $\{X_1, \dots, X_n\}$ are mutually independent iff they are pairwise uncorrelated.

   (d) Let $\mathbf{X} = (X_1, \dots, X_n)$ be i.i.d. $\mathcal{N}(0, 1)$. Show that then also $O\mathbf{X}^T$ are i.i.d. $\mathcal{N}(0, 1)$ for any orthogonal matrix $O \in \mathbb{R}^{n \times n}$.
   Show that for X_1, X_2 i.i.d. $\mathcal{N}(0, \sigma^2)$, $X_1 + X_2$ and $X_1 - X_2$ are i.i.d. $\mathcal{N}(0, 2\sigma^2)$.

2. **Continuous modifications**

   Let $\big(X(t) : t \in [0, \infty)\big)$ be a family of i.i.d. (independent, identically distributed) standard Gaussians $X(t) \sim \mathcal{N}(0, 1)$. Then $\big(X(t) : t \in \mathbb{N}\big)$ is a Gaussian process with mean $\mu(t) \equiv 0$ and covariance

   $$\sigma(s, t) = \begin{cases} 1\,, & \text{if } s = t \\ 0\,, & \text{if } s \neq t \end{cases} .$$

   Does there exists a modification $\widetilde{X}$ of $t \mapsto X_t$ such that almost surely $\widetilde{X}$ is continuous at a given point $t_0 \in [0, \infty)$ (meaning that $\mathbb{P}(\widetilde{X} \text{ is continuous at } t_0) = 1$)?

3. **Continuity**

   The set $C = \{t \mapsto \omega(t) : \omega \text{ is continuous}\}$ is not an element of $\mathcal{B}(\mathbb{R}^{[0,\infty)})$ (the infinite product σ-algebra).

   Hint: Use (and prove that) if $A \in \mathcal{B}(\mathbb{R}^{[0,\infty)})$, then there exists a sequence of time $\{t_i\}_{i \in \mathbb{N}} \subseteq [0, \infty)$ and a set $B \in \mathcal{B}(\mathbb{R}^{\mathbb{N}})$ such that

   $$A = \{\omega \in \mathbb{R}^{[0,\infty)} \ : \ (\omega_{t_i})_{i \in \mathbb{N}} \in B\} .$$

4. **Consistency**

   Consider a mean process $\mu \colon [0, \infty) \to \mathbb{R}^d$ and a covariance process $(s, t) \mapsto \Sigma_{s,t}$ such that for any $\mathbf{t} = (t_1, \dots, t_n)$, the matrix $(\Sigma_{t_i, t_j})_{i,j}$ is positive semi-definite and symmetric. Then let $\sigma_{\mathbf{t}}$ be the probability distribution of a Gaussian on $\mathbb{R}^d$ with mean $(\mu(t_1), \dots, \mu(t_n))$

and covariance $(\Sigma_{t_i,t_j})_{i,j}$.

Prove that $\{\sigma_{\mathbf{t}}\}_{\mathbf{t}}$ forms a consistent family of finite-dimensional distributions.