Chapter 5 Solutions

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Ex 1. (1) See Ex 2 in Chapter 1

(2) The arbitrage / risk-neutral price is

$$N\delta\, E^{Q}\left[\frac{(R_{swap}(T_0)-K)^+ \sum_{i=1}^{n} P(T_0,T_i)}{B(T_0)}\right]$$

$$= N\delta\, P(0,T_0)\, E^{Q}\left[\frac{(R_{swap}(T_0)-K)^+ \sum_{i=1}^{n} P(T_0,T_i)}{P(0,T_0)\, B(T_0)}\right]$$

$$= N\delta\, P(0,T_0)\, E^{\tilde{Q}^{T_0}}\left[(R_{swap}(T_0)-K)^+ \sum_{i=1}^{n} P(T_0,T_i)\right] \quad \text{with } \left.\frac{d\tilde{Q}^{T_0}}{dQ}\right|_{\mathcal{F}_t} = \frac{P(t,T_0)}{B(t)\, P(0,T_0)}$$

#

(3) Since $d\frac{P(t,T_i)}{P(t,T_0)} = \frac{P(t,T_i)}{P(t,T_0)} \underbrace{[\sigma^*(t,T_i) - \sigma^*(t,T_0)]}_{\text{deterministic}} dW_t^{\tilde{Q}^{T_0}}$

it follows from Novikov's condition that $\frac{P(t,T_i)}{P(t,T_0)}$ is a martingale under $\tilde{Q}^{T_0}$, so is $D(t) = \sum_{i=1}^{n} \frac{P(t,T_i)}{P(t,T_0)}$

Define an equivalent probability measure $\tilde{Q}^{swap}$ by its RN density

$$\left.\frac{d\tilde{Q}^{swap}}{d\tilde{Q}^{T_0}}\right|_{\mathcal{F}_t} = \frac{D(t)}{D(0)} \qquad t\in[0,T_0]$$

Bayes' rule then implies

$$N\delta\, P(0,T_0)\, E^{\tilde{Q}^{T_0}}\left[(R_{swap}(T_0)-K)^+ \sum_{i=1}^{n} P(T_0,T_i)\right]$$

$$= N\delta\, P(0,T_0)\, D(0)\, E^{\tilde{Q}^{T_0}}\left[(R_{swap}(T_0)-K)^+ \frac{D(T)}{D(0)}\right]$$

$$= N\delta\, P(0,T_0)\, D(0)\, E^{\tilde{Q}^{swap}}\left[(R_{swap}(T_0)-K)^+\right]$$

$$= N\delta \sum_{i=1}^{n} P(0,T_i)\, E^{\tilde{Q}^{swap}}\left[(R_{swap}(T_0)-K)^+\right] \qquad \#$$

(4) Since $R_{swap}(t) = \frac{P(t,T_0) - P(t,T_n)}{\delta \sum_{i=1}^{n} P(t,T_i)} = \frac{1 - \frac{P(t,T_n)}{P(t,T_0)}}{\delta D(t)}$

For any $S \geq t \geq 0$,

$$E^{\tilde{Q}^{swap}}\left[\frac{1}{D(S)} \middle| \mathcal{F}_t\right] = E^{\tilde{Q}^{swap}}\left[\frac{D(0)/D(S)}{D(0)/D(t)} \cdot \frac{1}{D(t)} \middle| \mathcal{F}_t\right]$$

$$= \frac{1}{D(t)} E^{\tilde{Q}^{swap}}\left[\frac{\frac{d\tilde{Q}^{T_0}}{d\tilde{Q}^{swap}}\Big|_{\mathcal{F}_S}}{\frac{d\tilde{Q}^{T_0}}{d\tilde{Q}^{swap}}\Big|_{\mathcal{F}_t}} \middle| \mathcal{F}_t\right]$$

$$= \frac{1}{D(t)} E^{\tilde{Q}^{T_0}}[1 \mid \mathcal{F}_t] = \frac{1}{D(t)}$$

Similarly, $E^{\tilde{Q}^{swap}}\left[\frac{P(S,T_n)/P(S,T_0)}{D(S)} \middle| \mathcal{F}_t\right] = E^{\tilde{Q}^{swap}}\left[\frac{P(S,T_n)}{P(S,T_0)\,D(t)} \cdot \frac{D(0)/D(S)}{D(0)/D(t)} \middle| \mathcal{F}_t\right]$

$$= \frac{1}{D(t)} E^{\tilde{Q}^{T_0}}\left[\frac{P(S,T_n)}{P(S,T_0)} \middle| \mathcal{F}_t\right]$$

$$= \frac{1}{D(t)} \cdot \frac{P(t,T_n)}{P(t,T_0)}$$

by using the fact that $\frac{P(t,T_n)}{P(t,T_0)}$ is a martingale under $\tilde{Q}^{T_0}$.

Hence, we obtain that

$$E^{\tilde{Q}^{swap}}[R_{swap}(S) \mid \mathcal{F}_t] = R_{swap}(t)$$

Now if $dR_{swap}(t) = R_{swap}(t)\,\sigma_{swap}(t)\,dW_t^{\tilde{Q}^{swap}}$, the arbitrage price of the IR swaption is by (3)

$$N\delta \sum_{i=1}^{n} P(0,T_i)\, \underbrace{E^{\tilde{Q}^{swap}}\left[(R_{swap}(T_0)-K)^+\right]}_{\text{BS call price with } r\equiv 0 \text{ and } \sigma = \sigma_{swap}(t)}$$

$$= N\delta \sum_{i=1}^{n} P(0,T_i)\left[R_{swap}(0)\,\Phi(d_1^{T_0}) - K\,\Phi(d_2^{T_0})\right]$$

with $d_{1,2}^{T_0} = \dfrac{\ln \frac{R_{swap}(0)}{K} \pm \frac{1}{2}\int_0^{T_0} |\sigma_{swap}(t)|^2 dt}{\sqrt{\int_0^{T_0} |\sigma_{swap}(t)|^2 dt}}$ #

Remark: Compared to cap/floor, the payoff of swaption cannot be decomposed into more elementary payoffs. This is the fundamental difference between cap/floor more elementary payoffs. This is the fundamental difference between cap/floor and swaption

Q2 (1) Since $P(t,T_0) - P(t,T_n) = \sum_{i=1}^{n} [P(t,T_{i-1}) - P(t,T_i)]$

$$= \sum_{i=1}^{n} (T_i - T_{i-1}) P(t,T_i) F(t; T_{i-1}, T_i)$$

$$= \delta \sum_{i=1}^{n} P(t,T_i) F(t; T_{i-1}, T_i)$$

Hence, $R_{swap}(t) = \dfrac{P(t,T_0) - P(t,T_n)}{\delta \sum_{i=1}^{n} P(t,T_i)} = \sum_{i=1}^{n} \underbrace{\dfrac{P(t,T_i)}{\sum_{j=1} P(t,T_j)}}_{w_i(t)} F(t; T_{i-1}, T_i)$ #

(2) $d R_{swap}(t) \approx \sum_{i=1}^{n} w_i(0)\, d L(t, T_{i-1})$

$$= \sum_{i=1}^{n} w_i(0) L(t,T_{i-1}) \lambda(t, T_{i-1}) d W_t^{\tilde{Q}^{T_i}}$$

Note that

$$W_t^{\tilde{Q}^{T_{i-1}}} = W_t^{\tilde{Q}^{T_i}} - \int_0^t [\sigma^*(s, T_{i-1}) - \sigma^*(s, T_i)] ds$$

$$W_t^{\tilde{Q}^{T_{i-2}}} = W_t^{\tilde{Q}^{T_{i-1}}} - \int_0^t [\sigma^*(s, T_{i-2}) - \sigma^*(s, T_{i-1})] ds$$

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$$W_t^{\tilde{Q}^{T_0}} = W_t^{\tilde{Q}^{T_1}} - \int_0^t [\sigma^*(s, T_0) - \sigma^*(s, T_1)] ds$$

$$\Rightarrow \quad W_t^{\tilde{Q}^{T_0}} = W_t^{\tilde{Q}^{T_i}} - \int_0^t \sum_{l=0}^{i-1} [\sigma^*(s, T_l) - \sigma^*(s, T_{l+1})] ds$$

Hence, $d R_{swap}(t) = \sum_{i=1}^{n} w_i(0) L(t, T_{i-1}) \lambda(t, T_{i-1}) \left[d W_t^{\tilde{Q}^{T_0}} + \sum_{l=0}^{i-1} [\sigma^*(t, T_l) - \sigma^*(t, T_{l+1})] dt \right]$

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(3) The arbitrage price is the same as the Black's formular in Q1, but with $|R_{swap}(t)|^2$ replaced by $|P(t)|^2$ #