UNIVERSITY OF WARWICK

Paper Details

Paper code: ST9090_A

Paper Title: Applications of Stochastic Calculus in Finance

Exam Period: May 2023

Exam Rubric

Time allowed: **2 hours**

Calculators are not permitted.

Instructions

Full marks may be obtained by correctly answering ALL FOUR questions.

All questions carry an equal weight of 20 marks. There are a total of **80** marks available.

A guideline to the number of marks usually available is shown for each question section.

Question 1

Let $M = (M_t)_{t\geq 0}$ be a continuous local martingale with $M_0 = 0$. Its stochastic exponential is defined as $\mathcal{E}(M)_t = e^{M_t - \frac{1}{2}\langle M \rangle_t}$ for $t \geq 0$.

(a) Prove that $\mathcal{E}(M)$ is a nonnegative continuous local martingale.
[Hint: you may apply Itô's formula to $\mathcal{E}(M)$.] [5 marks]

(b) Prove that $\mathcal{E}(M)$ is a supermartingale, and state the Novikov's condition under which $\mathcal{E}(M)$ is a martingale. [5 marks]

(c) Prove that $\mathcal{E}(M)_t^{-1} = \mathcal{E}(-M)_t e^{\langle M \rangle_t}$. [5 marks]

(d) Let N be another continuous local martingale with $N_0 = 0$. Prove that

$$\mathcal{E}(M)_t \mathcal{E}(N)_t = \mathcal{E}(M+N)_t e^{\langle M,N \rangle_t}.$$

[5 marks]

Continued...

Question 2

Let $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, \mathbf{P})$ be a filtered probability space supporting a d-dimensional Brownian motion $W = (W^1, \dots, W^d)$ with its augmented filtration $\{\mathcal{F}_t\}_{t\geq 0}$. Suppose that the forward rate $f(t,T)$ follows the stochastic differential equation (SDE)

$$df(t,T) = \alpha(t,T)dt + e^{-b(T-t)} \sum_{j=1}^{d} \sigma^j dW_t^j,$$

where the drift $\alpha(t,T)$ is a real valued bounded deterministic function, and the volatility parameters $\sigma = (\sigma^1, \dots, \sigma^d) \in \mathbb{R}^d$ and $b \in \mathbb{R}_+$ are constants.

(a) State the definition of an equivalent local martingale measure (ELMM) $\mathbf{Q}$ for such a forward rate model. [2 marks]

(b) Prove that the ELMM $\mathbf{Q}$ exists if the following HJM drift condition holds

$$-e^{-b(T-t)} \sum_{j=1}^{d} \sigma^j \Theta_t^j = -\alpha(t,T) + \frac{1}{b}(e^{-b(T-t)} - e^{-2b(T-t)}) \sum_{j=1}^{d} |\sigma^j|^2$$

for all $0 \leq t < T < \infty$, where $\Theta_t = (\Theta_t^1, \dots, \Theta_t^d)$ is the market price of risk. [8 marks]

(c) Under the ELMM $\mathbf{Q}$, derive the SDE for the forward rate $f(t,T)$, and identify the relationship between its drift and volatility. [3 marks]

(d) Prove that the corresponding short rate follows the extended Vasicek model:

$$dr_t = (a(t) - br_t)dt + \sum_{j=1}^{d} \sigma^j W_t^{\mathbf{Q},j},$$

where $W^{\mathbf{Q}} = (W^{\mathbf{Q},1}, \dots, W^{\mathbf{Q},d})$ is the d-dimensional Brownian motion under the ELMM $\mathbf{Q}$, and $a(t)$ is given as

$$a(t) = \partial_t f(0,t) + bf(0,t) + \frac{\sum_{j=1}^{d} |\sigma^j|^2}{2b}(1 - e^{-2bt}).$$

[7 marks]

Continued...

Question 3

Let $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, \mathbf{Q})$ be a filtered probability space supporting a one-dimensional Brownian motion $W^{\mathbf{Q}}$ with its augmented filtration $\{\mathcal{F}_t\}_{t\geq 0}$, where $\mathbf{Q}$ is an equivalent martingale measure (EMM). Assume that the volatility $\sigma(t,T)$ of the forward rate is some bounded and deterministic function in the HJM framework.

(a) Denote the price of a zero-coupon bond by $P(t,T)$, and the bank account by $B(t)$. Write down the stochastic differential equation (SDE) for the discounted price $\frac{P(t,T)}{B(t)}$ under the EMM $\mathbf{Q}$. [3 marks]

(b) State the definition of the T-forward measure $\tilde{\mathbf{Q}}^T$. [3 marks]

(c) Let $S > T$ be some future date. Derive the SDE for $\frac{P(t,S)}{P(t,T)}$ under the T-forward measure $\tilde{\mathbf{Q}}^T$, and solve this SDE. [7 marks]

(d) Consider a binary call option with maturity T and strike price K, written on a zero-coupon bond with maturity $S > T$, so its payoff at maturity T is

$$\mathbf{1}_{\{P(T,S)\geq K\}}.$$

Write down the arbitrage price for this binary option, and prove that the price is given by

$$P(0,T)\Phi(d),$$

where Φ is the standard normal cumulative distribution function, and

$$d = \frac{\ln \frac{P(0,S)}{KP(0,T)} - \frac{1}{2}\int_0^T |\sigma^*(u,S) - \sigma^*(u,T)|^2 du}{\sqrt{\int_0^T |\sigma^*(u,S) - \sigma^*(u,T)|^2 du}},$$

with $\sigma^*(t,T) = -\int_t^T \sigma(t,u)du$.

[Hint: Use the zero-coupon bond price $P(t,T)$ as the numeraire.] [7 marks]

Continued...

Question 4

Let τ be a non-negative random variable defined on a complete probability space $(\Omega, \mathcal{F}, \mathbf{P})$, and $\mathbb{F} = \{\mathcal{F}_t\}_{t\geq 0}$ be the filtration given by $\mathcal{F}_t = \sigma(\{\tau \leq u\} : u \leq t)$.

(a) For any $A \in \mathcal{F}_t$, write down two possibilities of $A \cap \{\tau > t\}$. [2 marks]

(b) Let Y be an $\mathcal{F}_\infty$-measurable and bounded random variable, where $\mathcal{F}_\infty = \sigma\left(\cup_{t\geq 0}\mathcal{F}_t\right)$. Prove that

$$\mathbb{E}[\mathbf{1}_{\{\tau>t\}}Y|\mathcal{F}_t] = \mathbf{1}_{\{\tau>t\}}\frac{\mathbb{E}[\mathbf{1}_{\{\tau>t\}}Y]}{\mathbf{P}(\tau > t)}.$$

[3 marks]

(c) Prove that τ follows an exponential distribution with a constant intensity $\lambda > 0$ if and only if the process $M = (M_t)_{t\geq 0}$, where

$$M_t = 1_{\{\tau\leq t\}} - \int_0^t 1_{\{\tau>s\}}\lambda ds,$$

is an $(\mathbb{F}, \mathbf{P})$-martingale and $\mathbf{P}(\tau > 0) = 1$. [5 marks]

(d) Let $T > 0$. Under the assumption in part (c), prove that the process $Z = (Z_t)_{t\in[0,T]}$, where

$$Z_t = 1_{\{\tau>t\}}e^{\int_0^t 1_{\{\tau>s\}}\lambda ds},$$

is an $(\mathbb{F}, \mathbf{P})$-martingale.
[Hint: Let $H_t := 1_{\{\tau\leq t\}}$ and $V_t := 1 - H_t$. You may first prove that $\Delta V_s = -V_{s-}\Delta H_s$.]
[5 marks]

(e) Let Z_T be given as in part (d) and let $\mathbf{Q}$ be the probability measure on $(\Omega, \mathcal{F}_T)$ with the Radon-Nikodym density $\frac{d\mathbf{Q}}{d\mathbf{P}}\Big|_{\mathcal{F}_T} = Z_T$. Prove that the process $M = (M_t)_{t\in[0,T]}$, where $M_t = 1_{\{\tau\leq t\}}$, is an $(\mathbb{F}, \mathbf{Q})$-martingale.
[Hint: you may use without proof the fact that M is an $(\mathbb{F}, \mathbf{Q})$-martingale if and only if MZ is an $(\mathbb{F}, \mathbf{P})$-martingale.] [5 marks]

End.

ST9090

UNIVERSITY OF WARWICK

April 2023

Applications of Stochastic Calculus in Finance: Sample Solutions

Question 1 The question is taken from Exercise 1 of Chapter 0 Review of Stochastic Calculus. Since stochastic exponential is ubiquitous in this course, it deserves to be tested in its own right.

(a) Applying Ito's formula to $\mathcal{E}(M)_t = \exp(M_t - \frac{1}{2} < M >_t)$ gives

$$\begin{aligned} d\mathcal{E}(M)_t &= \mathcal{E}(M)_t[dM_t - \frac{1}{2}d < M >_t] + \frac{1}{2}\mathcal{E}(M)_t d < M >_t \\ &= \mathcal{E}(M)_t dM_t. \end{aligned}$$

Hence, $\mathcal{E}(M)$ is a local martingale. [5]

(b) Since $\mathcal{E}(M)$ is a local martingale, there exists a stopping sequence $T_n \uparrow \infty$ such that

$$\mathbf{E}[\mathcal{E}(M)_t^{T_n}|\mathcal{F}_s] = \mathcal{E}(M)_s^{T_n}$$

for $t \geq s \geq 0$. Since $\mathcal{E}(M)$ is nonnegative, Fatou's lemma further implies that

$$\liminf_n \mathbf{E}[\mathcal{E}(M)_t^{T_n}|\mathcal{F}_s] \geq \mathbf{E}[\liminf_n \mathcal{E}(M)_t^{T_n}|\mathcal{F}_s] = \mathbf{E}[\mathcal{E}(M)_t|\mathcal{F}_s].$$

On other hand, $\liminf_n \mathcal{E}(M)_s^{T_n} = \mathcal{E}(M)_s$ from which we conclude the supermartingale property of $\mathcal{E}(M)$. [4]

Furthermore, $\mathcal{E}(M)$ is a martingale (up to $T > 0$) if Novikov's condition holds: $\mathbf{E}[e^{\frac{1}{2}<M>_T}] < \infty$. [1]

(c) Note that $< M >_t = < -M >_t$. Indeed, by Ito's formula,

$$\begin{aligned} dM_t^2 &= 2M_t dM_t + d < M >_t \\ d(-M_t)^2 &= 2(-M_t)d(-M_t) + d < -M >_t \ . \end{aligned}$$

[4]

In turn, we have

$$\mathcal{E}(M)_t\mathcal{E}(-M)_t = e^{M_t - \frac{1}{2}<M>_t} e^{-M_t - \frac{1}{2}<-M>_t} = e^{-<M>_t}.$$

[1]

(d) Apply Ito's formula to $(M_t + N_t)^2$,

$$(M_t + N_t)^2 = 2\int_0^t (M_s + N_s)d(M_s + N_s) + < M + N >_t \ .$$

[2]

Hence,

$$
\begin{aligned}
&< M + N >_t \\
&= M_t^2 - 2\int_0^t M_s dM_s + N_t^2 - 2\int_0^t N_s dN_s + 2(M_t N_t - \int_0^t M_s dN_s - \int_0^t N_s dM_s) \\
&=< M >_t + < N >_t + 2 < M, N >_t \ .
\end{aligned}
$$

[2]

In turn, we have

$$
\begin{aligned}
\mathcal{E}(M+N)_t e^{<M,N>_t} &= e^{M_t + N_t - \frac{1}{2}<M+N>_t + <M,N>_t} \\
&= e^{M_t - \frac{1}{2}<M>_t} e^{N_t - \frac{1}{2}<N_t>} = \mathcal{E}(M)_t \mathcal{E}(N)_t.
\end{aligned}
$$

[1]

Question 2 The question is taken from Chapter 3 HJM methodology. It is modified from Exercise 2 with a generalisation to a multi-dimensional case.

(a) An ELMM $\mathbf{Q} \sim \mathbf{P}$ such that the discounted T-bond price $\frac{P(t,T)}{B_t}$, $0 \le t \le T$, is local martingales for any $T > 0$. [2]

(b) Denote $\sigma^j(t,T) = e^{-b(T-t)}\sigma^j$. Applying Ito's formula to $P(t,T) = \exp\{-\int_t^T f(t,u)du\}$ and using stochastic Fubini theorem give

$$\frac{dP(t,T)}{P(t,T)} = \left[r_t - \int_t^T \alpha(t,u)du + \frac{1}{2}\sum_j (\int_t^T \sigma^j(t,u)du)^2\right]dt - \sum_j (\int_t^T \sigma^j(t,u)du)dW_t^j$$

In turn,

$$\begin{aligned} &d(\frac{P(t,T)}{B_t})/\frac{P(t,T)}{B_t} \\ &= \left[-\int_t^T \alpha(t,u)du + \frac{1}{2}\sum_j (\int_t^T \sigma^j(t,u)du)^2\right]dt - \sum_j (\int_t^T \sigma^j(t,u)du)dW_t^j \\ &= -\sum_j (\int_t^T \sigma^j(t,u)du)(dW_t^j + \Theta_t^j dt) \end{aligned}$$

where $\Theta_t = (\Theta_t^1, \dots, \Theta_t^d)$ are such that

$$-\sum_j (\int_t^T \sigma^j(t,u)du)\Theta_t^j = -\int_t^T \alpha(t,u)du + \frac{1}{2}\sum_j (\int_t^T \sigma^j(t,u)du)^2$$

for any $T > 0$. By differentiating against T, we obtain

$$-\sum_j \sigma^j(t,T)\Theta_t^j = -\alpha(t,T) + \sum_j \sigma^j(t,T)\int_t^T \sigma^j(t,u)du,$$

i.e.

$$-e^{-b(T-t)}\sum_{j=1}^d \sigma^j\Theta_t^j = -\alpha(t,T) + \frac{1}{b}(e^{-b(T-t)} - e^{-2b(T-t)})\sum_{j=1}^d |\sigma^j|^2.$$

[4]

If the above HJM drift condition holds, then we may introduce a new probability measure via RN density

$$\left.\frac{d\mathbf{Q}}{d\mathbf{P}}\right|_{\mathcal{F}_t} = \mathcal{E}(-\sum_j \int_0 \Theta_s^j dW_s^j)_t,$$

where the stochastic exponential $\mathcal{E}(-\sum_j \int_0 \Theta_s^j dW_s^j)$ is a martingale by Novikov's condition. Furthermore, by Girsanov's theorem, $W_t^{\mathbf{Q},j} := W_t^j + \int_0^t \Theta_s^j ds$, $1 \le j \le d$, is a BM under $\mathbb{Q}$ and, therefore, $\frac{P(t,T)}{B_t}$, $0 \le t \le T$, is a local martingale. [4]

(c) Under the ELMM $\mathbf{Q}$, we have

$$\begin{aligned} df(t,T) &= \alpha(t,T)dt + e^{-b(T-t)}\sum_j \sigma^j (dW_t^{\mathbf{Q},\mathbf{j}} - \Theta_t^j dt) \\ &= \left[\alpha(t,T) - e^{-b(T-t)}\sum_j \sigma^j \Theta_t^j\right] dt + e^{-b(T-t)}\sum_j \sigma^j dW_t^{\mathbf{Q},j} \\ &= \left[\frac{1}{b}(e^{-b(T-t)} - e^{-2b(T-t)})\sum_j |\sigma^j|^2\right] dt + e^{-b(T-t)}\sum_j \sigma^j dW_t^{\mathbf{Q},j}. \end{aligned}$$

where we used the HJM drift condition in the last equality. The drift and volatility are related by

$$\frac{1}{b}(e^{-b(T-t)} - e^{-2b(T-t)})\sum_j |\sigma^j|^2 = \sum_j e^{-b(T-t)}\sigma^j \int_t^T e^{-b(s-t)}\sigma^j ds.$$

[3]

(d) Since $r_t = f(t,t)$, by using part (c), we have

$$r_t = f(0,t) + \int_0^t \frac{1}{b}(e^{-b(t-s)} - e^{-2b(t-s)})\sum_j |\sigma^j|^2 ds + \int_0^t e^{-b(t-s)}\sum_j \sigma^j dW_s^{\mathbf{Q},j}. \quad (1)$$

[2]

On the other hand, if $dr_t = (a(t) - br_t)dt + \sum_{j=1}^d \sigma^j W_t^{\mathbf{Q},j}$ for $a(t)$ to be determined, then

$$r_t = r_0 e^{-bt} + \int_0^t a(s)e^{-b(t-s)}ds + \int_0^t e^{-b(t-s)}\sum_j \sigma^j dW_s^{\mathbf{Q},j}. \quad (2)$$

[2]

By comparing (1) and (2), we must have $a(s) = a_1(s) + a_2(s)$ where

$$\begin{aligned} &f(0,t) = r_0 e^{-bt} + \int_0^t a_1(s)e^{-b(t-s)}ds; \\ &\int_0^t \frac{1}{b}(e^{-b(t-s)} - e^{-2b(t-s)})\sum_j |\sigma^j|^2 ds = \int_0^t a_2(s)e^{-b(t-s)}ds. \end{aligned}$$

Solving the above two equations gives

$$\begin{aligned} a_1(t) &= bf(0,t) + \partial_t f(0,t) \\ a_2(t) &= \frac{1}{2b}\sum_j |\sigma^j|^2(1 - e^{-2bt}). \end{aligned}$$

[3]

Question 3 This question is taken from Chapter 4 change of numeraire. It is modified from Exercise 2 with a simplified payoff function.

(a) Under the EMM $\mathbf{Q}$, the discounted zero-coupon bond price follows

$$d(\frac{P(t,T)}{B(t)}) = \frac{P(t,T)}{B(t)}\sigma^*(t,T)dW_t^{\mathbf{Q}}.$$

[3]

(b) The T-forward measure $\tilde{\mathbf{Q}}^T$ is defined by the RN density

$$\left.\frac{d\tilde{\mathbf{Q}}^T}{d\mathbf{P}}\right|_{\mathcal{F}_t} = \mathcal{E}(\int_0 \sigma^*(s,T)dW_s^{\mathbf{Q}})_t = \frac{P(t,T)}{P(0,T)B(t)},$$

which is a martingale by Novikov's condition. [3]

(c) Applying Ito's formula to $(\frac{P(t,T)}{B(t)})^{-1}$ yields

$$d(\frac{P(t,T)}{B(t)})^{-1} = (\frac{P(t,T)}{B(t)})^{-1}\left[-\sigma^*(t,T)dW_t^{\mathbf{Q}} + |\sigma^*(t,T)|^2 dt\right].$$

In turn, for $S > T$,

$$\begin{aligned} d(\frac{P(t,S)}{P(t,T)}) &= d(\frac{P(t,S)}{B(t)}(\frac{P(t,T)}{B(t)})^{-1}) \\ &= \frac{P(t,S)}{P(t,T)}[\sigma^*(t,S) - \sigma^*(t,T)][dW_t^{\mathbf{Q}} - \sigma^*(t,T)dt]. \end{aligned}$$

[4]

By Girsanov's theorem, $W_t^{\tilde{\mathbf{Q}}^T} = W_t^{\mathbf{Q}} - \int_0^t \sigma^*(u,T)du$ is a BM under the T-forward measure $\tilde{\mathbf{Q}}^T$, and

$$d(\frac{P(t,S)}{P(t,T)}) = \frac{P(t,S)}{P(t,T)}[\sigma^*(t,S) - \sigma^*(t,T)]dW_t^{\tilde{\mathbf{Q}}^T},$$

which admits the explicit solution

$$\begin{aligned} \frac{P(t,S)}{P(t,T)} &= \frac{P(0,S)}{P(0,T)}\mathcal{E}\left(\int_0 [\sigma^*(u,S) - \sigma^*(u,T)]dW_u^{\tilde{\mathbf{Q}}^T}\right)_t \\ &= \frac{P(0,S)}{P(0,T)}\exp\left(\int_0^t [\sigma^*(u,S) - \sigma^*(u,T)]dW_u^{\tilde{\mathbf{Q}}^T} - \frac{1}{2}\int_0^t |\sigma^*(u,S) - \sigma^*(u,T)|^2 du\right). \end{aligned}$$

[3]

(d) The no-arbitrage price is

$$\begin{aligned}
&\mathbf{E}^{\mathbf{Q}}[\frac{1}{B(T)}\mathbf{1}_{\{P(T,S)\geq K\}}] \\
&= P(0,T)\mathbf{E}^{\mathbf{Q}}[\frac{P(T,T)}{P(0,T)B(T)}\mathbf{1}_{\{P(T,S)\geq K\}}] \\
&= P(0,T)\mathbf{E}^{\tilde{\mathbf{Q}}^T}[\mathbf{1}_{\{P(T,S)\geq K\}}] \\
&= P(0,T)\tilde{\mathbf{Q}}^T(\frac{P(T,S)}{P(T,T)} \geq K) \\
&= P(0,T)\tilde{\mathbf{Q}}^T(-\int_0^T [\sigma^*(u,S)-\sigma^*(u,T)]dW_u^{\tilde{\mathbf{Q}}^T} \leq d\sqrt{\int_0^T |\sigma^*(u,S)-\sigma^*(u,T)|^2 du})
\end{aligned}$$

[5]

Since $\int_0^T[\sigma^*(u,S)-\sigma^*(u,T)]dW_u^{\tilde{\mathbf{Q}}^T}$ is Gaussian $N(0,\int_0^T|\sigma^*(u,S)-\sigma^*(u,T)|^2du)$, it follows that the no-arbitrage price is given by $P(0,T)\Phi(d)$. [2]

Question 4 This question is based on Question 4 of the 2022 exam paper. Only 8 out of 30 students attempted to answer the question with average 9/20, so this question deserves to be tested again but with a special case of $\mu = 1$.

(a) For any $A \in \mathcal{F}_t$, we have $A \cap \{\tau > t\} = \{\tau > t\}$ or $\emptyset$. [2]

(b) We show that $\mathbf{1}_{\{\tau>t\}}\mathbf{E}[\mathbf{1}_{\{\tau>t\}}Y]$ is the conditional expectation of $\mathbf{1}_{\{\tau>t\}}Y\mathbb{P}(\tau > t)$ w.r.t $\mathcal{F}_t$. Let $A \in \mathcal{F}_t$.

If $A \cap \{\tau > t\} = \emptyset$, obviously,

$$\mathbf{E}\left[\mathbf{1}_A\mathbf{1}_{\{\tau>t\}}\mathbf{E}[\mathbf{1}_{\{\tau>t\}}Y]\right] = \mathbf{E}\left[\mathbf{1}_A\mathbf{1}_{\{\tau>t\}}Y\mathbf{P}(\tau > t)\right] = 0.$$

If $A \cap \{\tau > t\} = \{\tau > t\}$, then,

$$\mathbf{E}\left[\mathbf{1}_A\mathbf{1}_{\{\tau>t\}}\mathbf{E}[\mathbf{1}_{\{\tau>t\}}Y]\right] = \mathbf{P}(\tau > t)\mathbf{E}[\mathbf{1}_{\{\tau>t\}}Y],$$

and

$$\mathbf{E}\left[\mathbf{1}_A\mathbf{1}_{\{\tau>t\}}Y\mathbf{P}(\tau > t)\right] = \mathbf{E}[\mathbf{1}_{\{\tau>t\}}Y]\mathbf{P}(\tau > t),$$

from which we conclude. [3]

(c) Only if part: For any $T \geq t \geq 0$, we have

$$\begin{aligned}\mathbf{E}[M_T|\mathcal{F}_t] =&1 - \mathbf{E}[1_{\{\tau>T\}}|\mathcal{F}_t] - \int_0^t \mathbf{1}_{\{\tau>s\}}\lambda ds - \int_t^T \mathbf{E}[\mathbf{1}_{\{\tau>s\}}\lambda|\mathcal{F}_t]ds\\ =&1 - \mathbf{1}_{\{\tau>t\}}e^{-\lambda(T-t)} - \int_0^t \mathbf{1}_{\{\tau>s\}}\lambda ds - \mathbf{1}_{\{\tau>t\}}\int_t^T \lambda e^{-\lambda(s-t)}ds\\ =&1_{\{\tau\leq t\}} - \int_0^t \mathbf{1}_{\{\tau>s\}}\lambda ds = M_t.\end{aligned}$$

Since τ follows exponential distribution, it follows that $\mathbf{P}(\tau > 0) = e^{-\lambda 0} = 1$. [2]

If part: For $t \geq 0$, define $\Phi(t) = \mathbf{P}(\tau > t)$. Then, the martingale property of M yields that

$$\begin{aligned}\Phi(t) =&\mathbf{E}\left[1 - M_t - \int_0^t \mathbf{1}_{\{\tau>s\}}\lambda ds\right]\\ =&1 - M_0 - \lambda\int_0^t \Phi(s)ds.\end{aligned}$$

The condition $\tau > 0$ a.s. further yields that $M_0 = 0$, a.s. Thus,

$$\Phi(t) = 1 - \lambda\int_0^t \Phi(s)ds,$$

which implies that $\Phi(t) = e^{-\lambda t}$, i.e. τ follows exponential distribution with intensity λ. [3]

d Let $V_t = 1_{\{\tau>t\}}$ and $C_t = ^{\int_0^t 1_{\{\tau>s\}}\lambda ds}$. Note that for each ω, both $V_t(\omega)$ and $C_t(\omega)$ are BV functions. Using integration by parts formula, we obtain

$$\begin{aligned} V_tC_t &= V_0C_0 + \int_0^t V_s dC_s + \int_0^t C_{s-} dV_s \\ &= 1 + \int_0^t V_s C_s 1_{\{\tau>s\}}\lambda ds + \sum_{0<s\le t} C_{s-}\Delta V_s. \end{aligned}$$

[2]

Note that

$$\Delta V_s = V_s - V_{s-} = V_{s-}(V_s - V_{s-}) = -V_{s-}\Delta H_s.$$

[1]

We further have

$$\begin{aligned} V_tC_t &= 1 + \int_0^t V_s C_s 1_{\{\tau>s\}}\lambda ds - \sum_{0<s\le t} V_{s-}C_{s-}\Delta H_s \\ &= 1 + \int_0^t V_{s-} C_{s-} 1_{\{\tau>s\}}\lambda ds - \int_0^t V_{s-}C_{s-} dH_s \\ &= 1 - \int_0^t V_{s-}C_{s-}(dH_s - 1_{\{\tau>s\}}\lambda ds), \end{aligned}$$

from which we know that $Z = VC$ is a local martingale. Since Z is bounded, it is a martingale. [2]

e It suffices to show that HZ is a martingale under $\mathbb{P}$. Using integration by parts formula, we obtain

$$H_tZ_t = H_0Z_0 + \int_0^t H_{s-} dZ_s + \int_0^t Z_{s-} dH_s + \sum_{0<s\le t} \Delta Z_s \Delta H_s.$$

[2]

Note that

$$\int_0^t Z_{s-} dH_s = \int_0^t Z_{s-} dM_s + \int_0^t Z_{s-} 1_{\{\tau>s\}}\lambda ds,$$

and by Part D,

$$\Delta Z_s = Z_s - Z_{s-} = C_s(V_s - V_{s-}) = -C_s V_{s-}\Delta H_s = -Z_{s-}\Delta H_s.$$

Thus

$$\sum_{0<s\le t} \Delta Z_s \Delta H_s = -\sum_{0<s\le t} Z_{s-}\Delta H_s = -\int_0^t Z_{s-} dH_s,$$

[2]

and in turn,

$$
\begin{aligned}
H_t Z_t &= \int_0^t H_{s-} dZ_s + \int_0^t Z_{s-} dM_s - \int_0^t Z_{s-} (dH_s - 1_{\{\tau > s\}} \lambda ds) \\
&= \int_0^t H_{s-} dZ_s.
\end{aligned}
$$

Moreover, since HZ is bounded, it is a martingale under $\mathbb{P}$. [1]
