Mutual Induction Infinite Wire

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1 Introduction

We define $A(x, y)$ as the z-component of the magnetic vector potential which satisfies

$$\frac{\partial A}{\partial y} = B_x(x, y),$$

$$\frac{\partial A}{\partial x} = -B_y(x, y),$$

and we choose our integration constant to ensure

$$A(0, 0) = 0.$$

The mutual induction $M_i(x, y, x_i, y_i)$ enables us to calculate the A_i generated at (x, y) from an infinitely long vertical wire with current, I_i, at (x_i, y_i), i.e.

$$A_i(x, y) = M(x, y, x_i, y_i)I_i.$$

The total A generated by multiple filaments is given by

$$A(x, y) = \sum_i M(x, y, x_i, y_i)I_i.$$

2 Mutual induction formula

$$M(x, y, x_i, y_i) = -\frac{\mu_0}{2\pi} \ln\left(\frac{\rho_i}{d_i}\right)$$

where

$$\rho_i = \sqrt{(x - x_i)^2 + (y - y_i)^2},$$

$$d_i = \sqrt{x_i^2 + y_i^2}.$$

Note that we divide by d_i to ensure $A(0, 0) = 0$.

3 Check Ampere's law is satisfied

We can check Ampere's law is satisfied by switching to cylindrical coordinates with our origin at (x_i, y_i). Hence,

$$A_i(x, y) = A_i(\rho_i) = -\frac{\mu_0 I_i}{2\pi} \ln\left(\frac{\rho_i}{d_i}\right)$$

$$\begin{aligned} B_{\phi,i}(\rho_i) &= -\frac{\partial A_i}{\partial \rho_i}, \\ &= \frac{\mu_0 I_i}{2\pi\rho_i}. \end{aligned}$$

Hence, we can clearly see that Ampere's law is satisfied.

4 Resolving the singularity

$M(x, y, x_i, y_i)$ has a singularity at $(x, y) = (x_i, y_i)$. Strictly speaking, there is no way to resolve this singularity. However, if we wish to know A at (x_i, y_i), we can instead calculate the average value of A over a box with vertices $\left(x_i \pm \frac{\Delta x}{2}, y_i \pm \frac{\Delta y}{2}\right)$, which is given by

$$\begin{aligned} \langle A_i \rangle &= \int_{y_i-\Delta y/2}^{y_i+\Delta y/2} \int_{x_i-\Delta x/2}^{x_i+\Delta x/2} \frac{M(x, y, x_i, y_i) I_i}{\Delta x \Delta y} dx dy \\ &= I_i \langle M(x_i, y_i) \rangle \end{aligned}$$

4.1 Calculating average value of the self-inductance

Let

$$\begin{aligned} x' &= (x - x_i)/d_i, \\ y' &= (y - y_i)/d_i, \\ \Delta x' &= \frac{\Delta x}{d_i}, \\ \Delta y' &= \frac{\Delta y}{d_i}. \end{aligned}$$

Hence,

$$\langle M(x_i, y_i) \rangle = -\frac{\mu_0 d_i^2}{4\pi \Delta x \Delta y} \int_{-\Delta y/(2d_i)}^{\Delta y/(2d_i)} \int_{-\Delta x/(2d_i)}^{\Delta x/(2d_i)} \ln(x'^2 + y'^2) dx' dy'.$$

Note that

$$\int_{-\Delta x/(2d_i)}^{\Delta x/(2d_i)} \ln(x'^2 + y'^2) dx' = \frac{\Delta x}{d_i} \ln\left(\frac{\Delta x^2}{4d_i^2} + y'^2\right) + 4y' \tan^{-1}\left(\frac{\Delta x/(2d_i)}{y'}\right) - \frac{2\Delta x}{d_i}.$$

Now we will integrate each term of the above equation one by one

$$\int_{-\Delta y/(2d_i)}^{\Delta y/(2d_i)} \frac{\Delta x}{d_i} \ln\left(\frac{\Delta x^2}{4d_i^2} + y'^2\right) dy' = \frac{\Delta x \Delta y}{d_i^2} \ln\left(\frac{\Delta x^2}{4d_i^2} + \frac{\Delta y^2}{4d_i^2}\right) + 2\frac{\Delta x^2}{d_i^2} \tan^{-1}\left(\frac{\Delta y}{\Delta x}\right) - 2\frac{\Delta x \Delta y}{d_i^2},$$

$$\int_{-\Delta y/(2d_i)}^{\Delta y/(2d_i)} 4y' \tan^{-1}\left(\frac{\Delta x/(2d_i)}{y'}\right) dy' = \frac{\Delta y^2}{d_i^2} \tan^{-1}\left(\frac{\Delta x}{\Delta y}\right) - \frac{\Delta x^2}{d_i^2} \tan^{-1}\left(\frac{\Delta y}{\Delta x}\right) + \frac{\Delta x \Delta y}{d_i^2},$$

$$\int_{-\Delta y/(2d_i)}^{\Delta y/(2d_i)} -2\frac{\Delta x}{d_i^2} dy' = -2\frac{\Delta x \Delta y}{d_i^2}.$$

Hence,

$$\begin{aligned} \int_{-\Delta y/(2d_i)}^{\Delta y/(2d_i)} \int_{-\Delta x/(2d_i)}^{\Delta x/(2d_i)} \ln(x'^2 + y'^2) dx' dy' = &\frac{\Delta x \Delta y}{d_i^2} \ln\left(\frac{\Delta x^2}{4d_i^2} + \frac{\Delta y^2}{4d_i^2}\right) + \frac{\Delta x^2}{d_i^2} \tan^{-1}\left(\frac{\Delta y}{\Delta x}\right) \\ &+ \frac{\Delta y^2}{d_i^2} \tan^{-1}\left(\frac{\Delta x}{\Delta y}\right) - 3\frac{\Delta x \Delta y}{d_i^2} \end{aligned}$$

We verify these integrals here: `https://www.desmos.com/calculator/tiegn2skei`.

Hence,

$$\boxed{\langle M(x_i, y_i) \rangle = -\frac{\mu_0}{4\pi} \left[\ln\left(\frac{\Delta x^2 + \Delta y^2}{4d_i^2}\right) + \frac{\Delta x}{\Delta y} \tan^{-1}\left(\frac{\Delta y}{\Delta x}\right) + \frac{\Delta y}{\Delta x} \tan^{-1}\left(\frac{\Delta x}{\Delta y}\right) - 3\right]}$$

4.2 Taking limits of the self-inductance

Note that

$$\lim_{\Delta x \to 0} \left\{ \frac{\Delta y}{\Delta x} \tan^{-1} \left(\frac{\Delta x}{\Delta y} \right) \right\} = 1,$$

$$\lim_{\Delta x \to 0} \left\{ \frac{\Delta x}{\Delta y} \tan^{-1} \left(\frac{\Delta y}{\Delta x} \right) \right\} = 0.$$

Hence,

$$\begin{aligned} \lim_{\Delta x \to 0} \langle M(x_i, y_i) \rangle &= -\frac{\mu_0}{4\pi} \left[\ln \left(\frac{\Delta y^2}{4 d_i^2} \right) - 2 \right], \\ &= -\frac{\mu_0}{2\pi} \left[\ln \left(\frac{\Delta y}{2 d_i} \right) - 1 \right]. \end{aligned}$$

$$\lim_{\Delta y \to 0} \langle M(x_i, y_i) \rangle = -\frac{\mu_0}{2\pi} \left[\ln \left(\frac{\Delta x}{2 d_i} \right) - 1 \right].$$

These last two equations give the self-induction average induction along a line.