An Efficient SNARK for Field-Programmable and RAM Circuits

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Abstract – The advancement of succinct non-interactive argument of knowledge (SNARK) with constant proof size has significantly enhanced the efficiency and privacy of verifiable computation. Verifiable computation finds applications in distributed computing networks, particularly in scenarios where nodes cannot be generally trusted, such as blockchains. However, fully harnessing the efficiency of SNARK becomes challenging when the computing targets in the network change frequently, as the SNARK verification can involve some untrusted preprocess of the target, which is expected to be reproduced by other nodes. This problem can be addressed with two approaches: One relieves the reproduction overhead by reducing the dimensionality of preprocessing data; The other utilizes *verifiable machine computation*, which eliminates the dependency on preprocess at the cost of increased overhead to SNARK proving and verification. In this paper, we propose a new SNARK with constant proof size applicable to both approaches. The proposed SNARK combines the efficiency of Groth16 protocol, albeit lacking universality for new problems, and the universality of PlonK protocol, albeit with significantly larger preprocessing data dimensions. Consequently, we demonstrate that our proposed SNARK maintains efficiency and universality while significantly reducing the dimensionality of preprocessing data. Furthermore, our SNARK can be seamlessly applied to the verifiable machine computation, requiring a proof size smaller about four to ten times than other related works.

1 Introduction

Succinct non-interactive arguments of knowledge (SNARK) are protocols for practical verifiable computation [1] of general programs, translated into statements in NP languages [2]. The protocols include at least two parties including a prover and a verifier. A verifier outsources the execution of a computationally intensive program to a prover and gets back the execution result along with a proof of its correctness. A SNARK verifier should be able to decide the validity of proof in a way more efficient than reproducing the execution result. A SNARK is said to be a zero-knowledge (zk-) SNARK, if a prover cannot obtain any information from a valid proof other than the correctness of computation. For the recent decade, zk-SNARKs with constant-length proofs of which length is independent of the program size have been proposed [3–8].

SNARKs, for instance, can find applications in blockchain. A blockchain operates as a chain of signatures for correct computation of user transactions, which transcribe programs with specific input and output. Trustless validators, referred to as full nodes, are randomly selected to sign the correctness of transactions accumulated to date. Blockchain security relies on the number of validations to each transaction, indicating the probability of involving at least one honest validator [9]. While traditional validation entails reproducing the same transaction results, verifiable computation with SNARKs may provide more efficient validation [10–12] of a large accumulation of signatures [13]. Furthermore, zk-SNARKs contributes to preserving privacy [14].

As instantiation of a SNARK with constant proof size, Parno and Gentry in [3] have proposed *Pinocchio*, which is the first practical SNARK for general programs. The efficiency of Pinocchio in terms of the succinctness of verifying a computation comes from the use of an offline compiler that translates a program along with specific input and output into an NP statement, expressed in a deterministic *circuit*. SNARKs after Pinocchio have also

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inherited the use of such compilers. For example, in [4], Groth has proposed a nearly-optimal SNARK called *Groth16*, whose proof length and proving overhead are the smallest to date.

Security of the constant-length proof SNARKs usually depends on a *trusted setup*. For example, trusted setups in Pinocchio and Groth16 encodes and compresses a circuit into a *common reference string* (CRS) in a cryptographic way. Provers and verifiers are enforced to use a CRS, which ensures they argue a common circuit. Although trust in the setup can be altered by a multi-party computation (MPC) manner [15–17], researchers have pointed out that in this model, for every new circuit, the problem of composing a multi-party and renewed opportunity of adversarial subversion remain unresolved [18]. For example, MPC takes a long time more than a month to organize a multi-party [19–21].

More recently, SNARKs in [5–8] have incorporated *updatable and universal setups* [18], which produce *structured reference strings* (SRS). These setups reduce dependence on trust or MPC, as SRS can be updated by variable participant group to thwart adversarial subversion and are applicable universally across all circuits. Notably, Groth *et al.* in [18] have shown that the setups for CRS, such as those used in Pinocchio and Groth16, are not updatable. Maller *et al*. proposed *Sonic* [5], which stands out as the first practical SNARK featuring updatable and universal setup. *Marlin* [6] and *PlonK* [7], proposed by Chiesa *et al.* and Gabizon *et al.*, respectively, have improved communication and computation efficiency. Lipmaa et. al. [8] have proposed *Vampire*, which further optimized Marlin by adopting a simplified system of constraints referred to as R1CSlite that only has linear constraints to variables. Despite the reduced reliance on the setup, the security of these works relies on an assumption of *preprocessed verifiers*, where the online verifier is required to preprocess encoding of some or the entire circuit.

Applying the above SNARKs to a blockchain would be crucially tackled, if the blockchain operates on a random-access machine (RAM), such as Ethereum [22]. In RAM, an initial sequence of instructions that defines a program can be modified into *unrolled* instructions by input during execution. Thus, with a CRS-based SNARK, for every transaction, which describes a program with input, issued, a new circuit must be generated, requiring a trusted setup or an MPC for a fresh CRS. Even with an SRS-based SNARK, a new transaction invokes a new verifier preprocess, which must be verified or reproduced by other verifiers afterward, as blockchain verifiers are not generally trustworthy[1,2]. Based on this observation, we claim that the more SNARK security relies on verifier preprocess, the higher overhead to communication is imposed for the blockchain security. This problem is not limited to blockchain applications but potentially affects any verifiable computation of RAM through the distributed computing networks.

Works in [24–31] have addressed this problem by utilizing a *universal circuit*, which encompasses a portion or the entirety of computation in a RAM, rather than program-specific circuits. Subsequently, a setup then processes a universal circuit into a CRS that is reusable for general programs, unless there is a change in the machine specification. There exists a trade-off between the portion of RAM computation covered by a universal circuit and the complexity of a SNARK [29]. For instance, to emulate the entire computation of a RAM using a universal circuit, the circuit size must be proportional to the register size of the RAM, affecting both the overhead to the SNARK prover and the communication complexity with the verifier. Conversely, allowing the SNARK verifier to take over a portion of RAM's computation, specifically unrolling instructions, and preprocess the corresponding data, we can maintain the SNARK complexity closely proportional to that of the program-specific circuits.

In this paper, we propose a SNARK that effectively manages the dependency on verifier preprocess. We utilize *field-programmable* circuit derivation: Starting from our universal circuit, defined as a set of subcircuits, a

[1] The examples can be found from zk-rollups [23]. In a rollup manner, each transaction in a second layer (as an execution layer) of a blockchain network can be verified in a succinct way based on the trust in each verifier's preprocess, but to upload to the first layer (as a validation layer) a rollup of those for a specific period, the preprocesses must be validated.

[2] Fortunately, CRS or SRS generated by a trusted setup are not the case, as multi-party computation can replace the trust.

program-specific circuit can be derived by placing copies of subcircuits and establishing wiring between them. While our setup does not support updatability, the field-programmable derivation reduces the data dimensionality to be preprocessed by the verifier, focusing primarily on the wiring of subcircuits. As discussed above, we expect that this reduced dependency on the verifier preprocess would address the high communication complexity encountered when implementing verifiable RAM computation in a distributed computing network of untrusted nodes. Additionally, the verifier preprocess can be eliminated without modifying the SNARK itself; rather, by enhancing the complexity of subcircuit designs to handle unrolling instructions.

Our contributions are summarized as follows:

- We demonstrate a method to transform a SNARK with a common reference string, like Groth16, into one with a universal setup by integrating a permutation argument [32, 33, 5, 7]. This integration divides the responsibility of configuring a circuit into two algorithms: the setup and verifier, unlike the previous SNARKs where one algorithm handled both tasks. This integration allows for adjusting the dependency of both parties on maintaining security. An overview of our methodology is provided in subsection 1.2.

- The proposed SNARK is efficient. When used with verifier preprocessing, its communication and computation efficiency is asymptotically comparable to the state-of-the-art universal SNARKs. Moreover, when the verifier preprocessing is eliminated, the communication efficiency of our result surpasses that of other related works. A detailed comparison is provided in subsection 1.1.

- We provide a rigorous security analysis of the proposed SNARK.

The remainder of this paper is organized as follows. Section 2 defines preliminaries. In Section 3, we define a system of constraints, followed by construction of the proposed SNARK in Section 4. In Section 5, we provide security analysis of our SNARK, including adding zero-knowledge. Section 6 illustrates the elimination of verifier preprocess. Section 7 concludes the paper.

1.1 Comparison with related works

For the comparison, we informally use two models of circuits. One represents the size of a circuit in terms of the maximum numbers of addition and multiplication gates and wiring between them, denoted by N_+, $N_\times$, and $N_=$, respectively. In the other model, a circuit is placement and wiring of at most $s_{\max}$ copies of s_D subcircuits, where each subcircuit has at most n_+ addition and $n_\times$ multiplication gates and $n_=$ wires connecting them. Assuming that the subcircuits in the latter model are optimized, we can write $(n_+ + n_\times + n_=)s_{\max} = N_+ + N_\times + N_=$.

As a SNARK with a universal setup. SNARKs with universal setups in common have an intermediate process of deriving a circuit specific to a program and input from URS. We refer to the intermediate outputs as *feature polynomials*, which involve data about the entirety or a part of circuit description. Given the feature polynomials, the SNARKs convince the verifier of the correctness of a circuit evaluation, whereas correctness of the feature polynomials is left to be preprocessed by the verifier. In Figure 1, we summarize a comparison of communication and computation efficiency, including the dimensionality of data that must be preprocessed by the verifier, denoted by $\dim(\text{pre-input})$. Technically, our work is differentiated by the other works in the choice of feature polynomials.

Protocol	Setup			Prove	Verify
	Universality	Updatability	| reference string |	| proof |	dim(pre-input)
Groth16 [3]	no	no	$O(m_{\times,+})\ \mathbb{G}$	$3\ \mathbb{G}$	-
Sonic [4]	yes	yes	$O(N_{\times})\ \mathbb{G}$	$20\ \mathbb{G}\ +16\ \mathbb{F}$	$O(N_{\times}N_{+,=})\ \mathbb{F}$
PlonK [6]	yes	yes	$O(N_{\times,+})\ \mathbb{G}$	$9\ \mathbb{G}\ +6\ \mathbb{F}$	$O(N_{\times,+})\ \mathbb{F}$
Marlin [5]	yes	yes	$O(N_{\times,+})\ \mathbb{G}$	$13\ \mathbb{G}\ +8\ \mathbb{F}$	$O(N_{\times,+})\ \mathbb{F}$
This work	yes	no	$O\left((s_D+n_{\times,+})s_{\max}\right)\ \mathbb{G}$	$19\ \mathbb{G}\ +4\ \mathbb{F}$	$O(s_D s_{\max})\ \mathbb{F}$

Protocol	Prove time	Verify time
Groth16 [3]	$O(m_{\times,+})\ \mathbb{G}_{Ex.}\ +\ O(m_{\times}\log m_{\times})\ \mathbb{F}_{Arit.}$	$l\ \mathbb{G}_{Ex.}\ +3\ \mathrm{P}$
Sonic [4]	$O(N_{\times})\ \mathbb{G}_{Ex.}\ +\ O(N_{\times}N_{+,=}\log N_{\times})\ \mathbb{F}_{Arit.}$	$O(l)\ \mathbb{F}_{Arith.}\ +13\ \mathrm{P}$
PlonK [6]	$O(N_{\times,+})\ \mathbb{G}_{Ex.}\ +\ O(N_{\times,+}\log N_{\times,+})\ \mathbb{F}_{Arit.}$	$O(l)\ \mathbb{F}_{Arith.}\ +2\ \mathrm{P}$
Marlin [5]	$O(N_{\times,+})\ \mathbb{G}_{Ex.}\ +\ O(N_{\times,+}\log N_{\times,+})\ \mathbb{F}_{Arit.}$	$O(l+\log N_{\times,+})\ \mathbb{F}_{Arith.}\ +2\ \mathrm{P}$
This work	$O\left((s_D+n_{\times,+})s_{\max}\right)\ \mathbb{G}_{Ex.}\ +\ O\left((s_D+n_{\times})s_{\max}\log s_D s_{\max} n_{\times}\right)\ \mathbb{F}_{Arit.}$	$O(l)\ \mathbb{F}_{Arith.}\ +10\ \mathrm{P}$

Figure 1. Efficiency comparison of SNARKs with universal (and updatable) setups, including Groth16 as a reference state-of-the-art SNARK with a circuit-specific setup. Tables depict the numbers of elements in $\mathbb{G}$ and $\mathbb{F}$, denoted as $\mathbb{G}_1$ or $\mathbb{G}_2$ and $\mathbb{F}$, respectively. $\mathbb{G}_{Ex.}$ and $\mathbb{F}_{Arith.}$ represent the numbers of exponentiations in $\mathbb{G}_1$ or $\mathbb{G}_2$ and arithmetic operations in $\mathbb{F}$, respectively. Parameters $M_{\times}$, M_{+}, and $M_{=}$ for $M \in \{m, N, n\}$ denote the numbers of multiplication constraints, linear constraints, and equal constraints, respectively. Additionally, $M_{\times,+}=M_{\times}+M_{+}$ and $M_{+,=}=M_{+}+M_{=}$. When $M=m$, constraints are counted from an optimized circuit; when $M=N$, it represents the maximum number of constraints a circuit can contain; when $M=n$, it indicates the maximum number of constraints in a subcircuit. In our work, a circuit is placement of at most $s_{\max}$ copies of s_D subcircuits.

In [18], Groth *et al.* proposed a SNARK with updatable and universal setups. The protocol works with a circuit representation referred to as rank 1 constraint system (R1CS), which is of three matrices of size $N_{\times}\times(N_{\times}+N_{+})$ to represent the relationship of how each wire contributes to each gate and the wiring. The authors encoded a kernel vector that is orthogonal to all columns of the R1CS matrices into a polynomial feature. This naturally leads that $\dim(\text{pre-input}) = O(N_{\times}(N_{\times}+N_{+}))$. In addition, computing the kernel costs $O(N_{\times}^3)$ field operations.

Maller *et al.* in [5] proposed Sonic, which is known as the first practical SNARK with updatable and universal setups. Sonic is combination of a polynomial commitment scheme referred to as a KZG scheme [34] and a permutation argument [32, 33]. Sonic represents a circuit with three vectors of wires and three matrices of gate configuration. The wire vectors contain values to be assigned to the wires, and therefore an equation of Hadamard product between them specifies nonlinear (multiplication) gates in a circuit. The gate configuration matrices specify linear relationship between the wires, such as addition and wiring. The authors chose to encode the gate configuration matrices into a single feature polynomial, leading that $\dim(\text{pre-input}) = O(N_{\times}(N_{+}+N_{=}))$.

PlonK proposed by Gabizon *et al.* in [7] further optimized the circuit representation of Sonic. PlonK is known as one of the most efficient protocols in terms of communication and computation efficiency. Besides theory, the protocol has been implemented by Iden3 in practice [35] with applying computing acceleration techniques in [36]. Circuits of PlonK, referred to as Plonkish circuits, are restricted to have at most two input wires and an output wire[3]. Gates are configured by five selector vectors, which enable or disable the input and output wires and set the gate operation as either addition or multiplication. What comparable to the circuits of Sonic is that the multiplication between wires is not constrained by default. Also, wiring between gates is described by

[3] This restriction does not compromise generality, as Plonkish circuits can encompass any circuit by increasing the number of gates.

separated three vectors that form a permutation map. The authors encoded the eight vectors into eight feature polynomials, respectively, leading that $\dim(\text{pre-input}) = O(N_{\times} + N_{+})$.

Chiesa *et al.* in [6] proposed Marlin by combining holographic proof system [37] with an optimized KZG scheme. Marlin works with the R1CS circuit representation. The authors utilized the fact that R1CS matrices are sparse, if there are small number of addition gates in a circuit. Sparsity can be preserved even for additionally dense circuits by further splitting constraints. As a result, sparsity equals the number of wires involved in multiplication and addition gates. The verifier of Marlin encodes the sparse representation of R1CS matrices into nine feature polynomials, leading that $\dim(\text{pre-input}) = O(N_{\times} + N_{+})$.

Our SNARK combines the R1CS and the Plonkish circuit representation. Circuits of our interest are placement of at most $s_{\max}$ copies of predefined s_D subcircuits. Each subcircuit is represented by three R1CS matrices. Wiring between subcircuits is not established by R1CS but by a permutation map. As the subcircuits are predefined and committed by the setup, the permutation is the only feature that uniquely specifies a derived circuit *on-the-fly*. Letting the number of input and output wires in each subcircuit be less than a constant c, the data dimensionality to define a permutation map is $\dim(\text{pre-input}) = O(s_D s_{\max})$. For applications where $n_{\times} + n_{+} \gg s_D$, our SNARK has the reduced $\dim(\text{pre-input})$ compared to that of PlonK or Marlin. This advantage costs that our setup is not updatable, as it outputs a CRS for the R1CS [18]. Though, the subversion of CRS still can be prevented by MPC in weaker sense [15, 16].

As a machine computation eliminating verifier preprocess. In [24–28], various universal circuits for RAM computation have been introduced to eliminate the need for the verifier preprocess. These circuits can be structured into $s_{\max}$ layers, each comprising s_D subcircuits corresponding to instructions of a RAM. Each layer disputes instruction execution at each machine step, including unrolling the next step instruction. Data transfer is restricted to occur only between adjacent layers, ensuring the circuit's layered structure remains independent of RAM programs and input. We refer to SNARKs applications as *machine computation*. While the deterministic nature of universal circuits frees the verifier from reproducing feature polynomials, the size of subcircuits remains proportional to the number of slots in the register and memory of a RAM, which is necessary for tracking the machine's internal state changes. This requisite poses implementation challenges, especially for large-scale machines like the Ethereum virtual machine [22].

In the earlier design of the universal circuit in [24], multiplexer (MUX) components were used within each layer to choose a single output from all subcircuit outputs, depending on the input instruction provided to the layer. This design inherently resulted in an asymptotic prover overhead of $O((n_{\times} + n_{+}) s_D s_{\max})$. Subsequent works [25, 26] addressed the redundant structure within the universal circuit, which involved replicating identical layers. Instead, the data transfer between adjacent machine steps was argued externally to the circuit, employing arguments such as recursive proof composition [38] or folding schemes [39].

Another simplifications were made in [27, 28] where the universal circuit was presented as a set of s_D subcircuits. The authors proposed a protocol where the prover initially derives a program-specific circuit based on unrolled instructions using a lookup argument [40], followed by disputing the program execution. Specifically, *MUX-Marlin* proposed by Di *et al.* in [27] combined Marlin for program execution verification, a permutation argument for verifying the layered structure in a derived circuit, and a variant of the lookup argument for ensuring the correct copying of subcircuits from the predefined universal circuit. This design can be seen as replacing the role of MUX components with Marlin and the lookup argument compared to the design in [24]. Independently, *SublonK*, proposed by Choudhuri *et al.* in [28], adopted a similar approach but with PlonK instead of Marlin. These works resulted in reducing the size of universal circuits and the asymptotic prover overhead. Figure 2 provides further details.

Protocol	RAM-specific setup	Prove		Verify
	| reference string |	| proof |	Time	Time
MUX-Marlin [27]	$O\big((s_D + s_{\max})n\big)$ $\mathbb{G}$	125 $\mathbb{G}$ + 116 $\mathbb{F}$	$O((n+s_D)s_{\max} + s_D n)$ $\mathbb{G}_{Ex.}$ $+ O\big((s_{\max}\log^2 s_{\max} n + s_D \log s_D n)n\big)$ $\mathbb{F}_{Arith.}$	$O(l + \log s_{\max})$ $\mathbb{F}_{Arith.}$ + 2 P
SublonK [28]	$O\big((s_D + s_{\max}\log s_D)n\big)$ $\mathbb{G}$	42 $\mathbb{G}$ + 12 $\mathbb{F}$	$O\big((s_D + s_{\max}\log s_D)n\big)$ $\mathbb{G}_{Ex.}$ $+ O\big((s_D \log s_D n + s_{\max}\log s_{\max} n)n\big)$ $\mathbb{F}_{Arith.}$	$O(l + \log s_{\max})$ $\mathbb{F}_{Arith.}$ + 23 P
This work	$O((s_D + s_{\max})n)$ $\mathbb{G}$	17 $\mathbb{G}$ + 4 $\mathbb{F}$	$O((s_D + n)s_{\max})$ $\mathbb{G}_{Ex.}$ $+ O(s_{\max}(s_D \log s_D s_{\max} + n\log s_{\max} n))$ $\mathbb{F}_{Arith.}$	$O(l + \log s_D)$ $\mathbb{F}_{Arith.}$ + 10 P

Figure 2. Efficiency comparison of machine computation protocols. Tables depict the numbers of elements in $\mathbb{G}$ and $\mathbb{F}$, denoted as $\mathbb{G}_1$ or $\mathbb{G}_2$ and $\mathbb{F}$, respectively. $\mathbb{G}_{Ex.}$ and $\mathbb{F}_{Arith.}$ represent the numbers of exponentiations in $\mathbb{G}_1$ or $\mathbb{G}_2$ and arithmetic operations in $\mathbb{F}$, respectively. Setup builds s_D library subcircuits, each can have at most n constraints. A RAM circuit can consist of at most $s_{\max}$ copies of the library subcircuits.

Our SNARK also supports the universal circuits of MUX-Marlin [27] and SublonK [28]. In terms of argument composition, we also utilize a permutation argument to verify the layered structure in a derived circuit, similar to these works. However, we use Groth16 instead of Marlin or PlonK for program execution verification and introduce a polynomial binding argument for connecting the other two arguments. As shown in Figure 2, our SNARK achieves remarkable efficiency in proof size.

As a RAM-to-circuit reduction. To alleviate the high prover complexity arising from machine computation of large-scaled RAMs, the works in [29–31] have explored *RAM-to-circuit reduction* [30]. This approach allows the verifier to preprocess feature polynomials containing essential features for unrolling instructions. Consequently, the prover can focus on their program-specific computation rather than the entire machine. Their universal circuits no longer need to represent all RAM states but capture the minimal structure of a machine that can be efficiently derived into an unrolled program-specific circuit. However, this reduction necessitates the verifier's knowledge of non-deterministic behavior of a RAM during unrolling program instructions, which includes the entire [29] or the multiplicity [30] of unrolled program instructions, or input and output values at intermediate machine steps [31].

Our SNARK can be seen as a form of RAM-to-circuit reduction, since the feature polynomial requires the knowledge of how instructions are unrolled. *Mirage*, proposed by Kosba *et al.* in [31], shares a similar approach with ours, where a derived circuit is represented by the placement of predefined subcircuits and wiring between them. Thus, the circuit derivation can be uniquely specified by data of the dimensionality $O(s_D s_{\max})$. The key technical distinction lies in the verification of the wiring between subcircuits: In Mirage, it is verified by a predefined permutation subcircuit, whereas in our approach, it is done by the permutation and polynomial binding arguments, externally to the circuit. As a result, Mirage's verifier requires preprocessed data containing the input and output values to each subcircuit, while our approach does not. This feature gives our SNARK the option to eliminate the verifier preprocess through machine computation, depending on the application.

1.2 Technical overview

A universal circuit we consider is defined as a library $\mathcal{L}$ of s_D subcircuits. A circuit can be derived from the universal circuit as placement of at most $s_{\max}$ copies of the subcircuits and specified by a wire map that describes wiring between the copies. Figure 3 and Figure 4 illustrate the derivation. Analogously, a system of constraints for a circuit derivation can be split into two subsystems, one argues arithmetic constraints inside each subcircuit copy, and the other argues copy constraints that the wires connecting two or more subcircuit copies must share the same values. We represent variables of our constraint system as a triple of vectors $(\mathbf{a}, \mathbf{b}, \mathbf{c})$, where $\mathbf{a}$ is public instance to a derived circuit, $\mathbf{b}$ is private and of the values shared by the connecting wires, and $\mathbf{c}$ is of the values assigned to the internal wires inside each subcircuit copy. The arithmetic constraints check whether all the variables $(\mathbf{a},\mathbf{b},\mathbf{c})$ satisfy the constraints of all subcircuit copies placed in the circuit. The copy constraints checks whether $\mathbf{b}$ satisfies a permutation, which is defined by the wire map.

Our SNARK shown in Figure 5 consists of three arguments: an arithmetic argument based on Groth16 [4] for the arithmetic constraints, a permutation argument based on [32, 33, 5, 7] for the copy constraints, and an polynomial binding argument that connects the two arguments. More specifically, the polynomial binding argument convinces a verifier that proofs of the other two arguments are generated from the same witness $\mathbf{b}$.

The arithmetic argument is a modification of Groth16. Originally, Groth16 worked with a univariate polynomial $p_A(X)$ and a finite range $\mathcal{X}$ such that $p_A(x_i)=0$ for $x_i \in \mathcal{X}$, if and only if the i-th constraint in a circuit is satisfied. On the contrary, as we consider $s_{\max}$ copies of subcircuits, we extend the protocol to work with a bivariate polynomial $p_A(X,Y)$ and two finite ranges $\mathcal{X}$ and $\mathcal{Y}$, where $\mathcal{X}$ indicates the constraint indices, and $\mathcal{Y}$ indicates the copy indices. In other words, $p_A(x_h, y_i)=0$ for $(x_h, y_i)\in\mathcal{X}\times\mathcal{Y}$, if and only if the h-th constraint of the i-th subcircuit copy in a derived circuit is satisfied.

The permutation argument is a modification of that used in PlonK [7]. The original argument worked with a univariate permutation polynomial $p_C(X)$ and a finite range $\mathcal{X}$ such that $p_C(x_i)=x_k$ for $x_i, x_k \in \mathcal{X}$ holds if and only if the two values to be assigned respectively to the i-th wire and the k-th wire in a circuit are the same. On the contrary, we introduce a bivariate permutation polynomial $p_C(X,Y)$ over finite range $\mathcal{Z}\times\mathcal{Y}$, where $\mathcal{Y}$ indicates the copy indices, and $\mathcal{Z}$ indicates the wire indices. For example, $p_C(z_i, y_h)=y_j\theta+z_k$ for $(z_i, y_h),(z_k, y_j)\in\mathcal{Z}\times\mathcal{Y}$ holds true for any θ, if and only if the two values to be assigned respectively to the i-th wire of the h-th subcircuit copy and the k-th wire of the j-th subcircuit copy must be the same.

As the last argument, the polynomial binding argument encloses the two other arguments. Let $b_i(Y)$ and $b_i'(Y)$ denote polynomial encodings of the witness $\mathbf{b}$ and $\mathbf{b}'$ to the arithmetic and permutation arguments, respectively, over $\mathcal{Y}$. The arithmetic argument produces a proof polynomial involving $\sum_i b_i(Y)o_i(X)$, which is a part of $p_A(X,Y)$, where basis polynomials $o_i(X)$ represent the feature of wires defined over $\mathcal{X}$. The permutation argument produces a proof polynomial involving $\sum_i b_i'(Y)K_i(X)$, which will be combined with $p_C(X,Y)$, where $K_i(X)$ are Lagrange bases over $\mathcal{Z}$. The goal of polynomial binding argument is to identify $b_i(Y)=b_i'(Y)$.

The remaining aspect involves configuring the setup to efficiently compress the precomputable polynomials forming the constraint system, while maintaining universality. Specifically, the setup output, a CRS, must remain reusable for new circuits and instances $\mathbf{a}$. To achieve this, the setup publishes commitments to randomized monomials and polynomials, which consistently include the wire polynomials of each subcircuit in the library to form $p_A(X,Y)$. However, including a permutation polynomial $p_C(X,Y)$ into the CRS could be done cautiously. Since wire maps are typically circuit-specific, including it in the CRS might compromise the universality of the setup. Nevertheless, for specific applications such as machine computation, which have fixed wiring structures, the permutation can also be committed by the setup (will be further discussed in Section 6).

2 Preliminaries

2.1 Notations

Unless otherwise stated, all sets appearing in this paper are regarded as multisets. We use the notation $\{x_{i_1,i_2,\dots,i_n}\}_{i_1=m_1,i_2=m_2,\dots,i_n=m_n}^{l_1,l_2,\dots,l_n}$ as a compact set-builder notation representing the collection of indexed elements $x_{i_1,i_2,\dots,i_n}$ for all $(i_1,i_2,\dots,i_n)\in\prod_{k=1}^{n}\{m_k, m_k+1,\dots,l_k\}$. Given a field $\mathbb{F}$, we denote by $\mathbb{F}_{d_1,d_2,\dots,d_n}[X_1,X_2,\dots,X_n]$ the set of multivariate polynomials over $\mathbb{F}$ in variables $X_1,X_2,\dots,X_n$ where each X_i has degree less than d_i for all $i\in\{1,\dots,n\}$. For any positive integer n, we define the polynomial $t_n\in\mathbb{F}[X]\cup\mathbb{F}[Y]$ by $t_n(X):=X^n-1$

or $t_n(Y) := Y^n - 1$. The primitive n-th root of unity is denoted by ω_n. Finally, for vectors $\mathbf{x}$ and $\mathbf{y}$, we denote their concatenation by $\mathbf{z} = (\mathbf{x}, \mathbf{y})$.

2.2 Cryptographic definitions

Let $pp_\lambda = (\mathbb{F}, \mathbb{G}_1, \mathbb{G}_2, \mathbb{G}_T, e, G, H)$ be a bilinear group generated from a security parameter $\lambda \in \mathbb{N}$. $\mathbb{G}_1, \mathbb{G}_2$ are additive groups and $\mathbb{G}_T$ is a multiplicative group defined over a field $\mathbb{F}$. Letting $G_i \in \mathbb{G}_i$ for $i \in \{1,2\}$ denote the generators, we write group encodings $[\mathbf{x}]_i := (x_0 G_i, \cdots, x_{n-1} G_i)$ for $\mathbf{x} \in \mathbb{F}^n$. We define $e : \mathbb{G}_1 \times \mathbb{G}_2 \to \mathbb{G}_T$ as a non-degenerative bilinear map that holds $e([x]_1, [y]_2) = e([1]_1, [1]_2)^{xy}$ for $x, y \in \mathbb{F}$. It is deduced that $e([1]_1, [1]_2)$ is the generator of $\mathbb{G}_T$. We say a function $\epsilon : \mathbb{N} \to [0,1]$ is *negligible* in λ, shortly $negl(\lambda)$, if there exists a constant c such that for all λ, $\epsilon(\lambda) < \lambda^{-c}$.

We consider generic group model (GGM) with affine prover strategy, where generic polynomial-time adversaries $\mathcal{A}$ have no direct access to the group operations in pp_λ. General group model has been defined in [41] by a random injective encoding $[\cdot]_i$ from a field $\mathbb{F}$ to a group $\mathbb{G}_i$ for $i \in \{1,2\}$. As $\mathcal{A}$ has no access to the randomness of $[\cdot]_i$, group operations can be handled only through an oracle. This implies that every $\mathbf{y} \in \mathbb{G}_i^k$ produced by $\mathcal{A}$ there is an affine strategy $\mathbf{P} \in \mathbb{F}^{k \times l}$ such that $\mathbf{y} = \mathbf{P}\mathbf{x}$ for given random encoding $\mathbf{x} \in \mathbb{G}_i^l$.

2.3 Useful lemmas for polynomials

Lemma 1 (Schwartz-Zippel (SZ) Lemma). *Let $p \in \mathbb{F}[X_1, \cdots, X_n]$ be a non-zero n-variate polynomial with total degree not greater than d. Let $\mathcal{X} \subseteq \mathbb{F}$ and $(x_1, \cdots, x_n)$ be picked at random independently and uniformly from $\mathcal{X}^n$. Then,* $\Pr[f(x_1, \cdots, x_n) = 0] \le d\,|\mathcal{X}|^{-1}$.

Lemma 2 (Ben-Sasson and Sudan [42]**).** *Let $\mathcal{X} \subseteq \mathbb{F}$ with $|\mathcal{X}| = n$. A polynomial $p \in \mathbb{F}_N[X]$ with degree $N - 1 \ge n$ vanishes on $\mathcal{X}$ if and only if the vanishing polynomial $t_{\mathcal{X}}(X) := \prod_{x \in \mathcal{X}} (X - x)$ divides p, i.e., if and only if there exists a quotient polynomial $h \in \mathbb{F}[X]$ (i.e., $h(X)$ does not involve negative powers of X) such that $p(X) = t_{\mathcal{X}}(X) h(X)$.*

Proof. See Appendix A.

Corollary 1 (An extension of Lemma 2 to bivariate polynomials). *Let $\mathcal{X}, \mathcal{Y} \subseteq \mathbb{F}$ with $|\mathcal{X}| = n$ and $|\mathcal{Y}| = s$. A polynomial $p \in \mathbb{F}_{N,S}[X, Y]$ with degree $N \ge n$ in X and $S \ge s$ in Y vanishes on $\mathcal{X} \times \mathcal{Y}$ if and only if, given two vanishing polynomials $t_{\mathcal{X}}(X) := \prod_{x \in \mathcal{X}} (X - x)$ and $t_{\mathcal{Y}}(Y) := \prod_{y \in \mathcal{Y}} (Y - y)$, there exist quotient polynomials $h_{\mathcal{X}}, h_{\mathcal{Y}} \in \mathbb{F}[X, Y]$ such that $p(X, Y) = t_{\mathcal{X}}(X) h_{\mathcal{X}}(X, Y) + t_{\mathcal{Y}}(Y) h_{\mathcal{Y}}(X, Y)$.*

Proof. See Appendix B.

2.4 Circuit, rank-1 constraint system, and quadratic arithmetic program

An arithmetic circuit comprises multiplication gates, addition gates, and wires. Each gate has two input wires and one output wire, with some gates possibly have wiring connections. For a given list of values assigned to the wires, the circuit is considered satisfied if these values satisfy the input-output relationships of all gates.

A rank-1 constraint system (R1CS) is a matrix representation of a circuit. R1CS is a set of matrices

$\mathbf{U},\mathbf{V},\mathbf{W}\in\mathbb{F}^{n\times m}$, where the triple of column vectors $(\mathbf{u}_k=(u_{i,k})_{i=0}^{n-1},\mathbf{v}_k=(v_{i,k})_{i=0}^{n-1},\mathbf{w}_k=(w_{i,k})_{i=0}^{n-1})$ for each $k\in\{0,\cdots,m-1\}$ represents a wire. Given assignments to the wires, denoted by a vector $\mathbf{d}=(d_0,\cdots,d_{m-1})$, a circuit is satisfied if and only if the following n equations for $i=0,\cdots,n$ holds simultaneously,

$$\left(\sum_{k=0}^{m-1} d_k u_{i,k}\right)\cdot\left(\sum_{k=0}^{m-1} d_k v_{i,k}\right)-\left(\sum_{k=0}^{m-1} d_k w_{i,k}\right)=0. \tag{1}$$

Each equation is referred to as a *constraint*, which represents the relationship between a left input, a right input, and an output within a multiplication gate. In R1CS, addition gates are typically not considered independent constraints but instead merged into the n multiplication gates as summations of the inputs or outputs, except for special uses such as Marlin [6], where n is expanded as increase of the addition gates to preserve sparsity of the R1CS matrices. It is also notable that R1CS does not count equalities between the wires as constraints, unlike constraint systems of PlonK [7] or Sonic [5].

Quadratic arithmetic program (QAP) is a polynomial representation of rank-1 constraint systems (R1CS) [43]. We first define a vanishing set $\mathcal{X}\subseteq\mathbb{F}$ of $|\mathcal{X}|=n$. It is preferred to pick $\mathcal{X}$ as a group of order n generated by an n-th root of unity ω_X (i.e., $\omega_X^n=1$) for computation efficiency. A vanishing polynomial is then defined as $t_{\mathcal{X}}(X):=\prod_{x\in\mathcal{X}}(X-x)=\omega_X^n-1$. A QAP is defined as a set of polynomials, $\mathcal{Q}\subseteq\mathbb{F}_n[X]$ such that

$$\mathcal{Q}:=\left\{\left(u_k(X),v_k(X),w_k(X)\right)\right\}_{k=0}^{m-1},$$

where $u_k(X)$, $v_k(X)$, and $w_k(X)$ encodes the column vectors $\mathbf{u}_k$, $\mathbf{v}_k$, $\mathbf{w}_k$ over $\mathcal{X}$, respectively. In other words, $u_k(\omega_X^i)=u_{k,i}$, $v_k(\omega_X^i)=v_{k,i}$, $w_k(\omega_X^i)=w_{k,i}$ for $i=0,\cdots,n-1$ and $k=0,\cdots,m-1$. The left-hand side of equation (1) can be equivalently represented a circuit polynomial $p(X)$ as defined by

$$p(X):=\left(\sum_{k=0}^{m-1} d_k u_k(X)\right)\cdot\left(\sum_{k=0}^{m-1} d_k v_k(X)\right)-\left(\sum_{k=0}^{m-1} d_k w_k(X)\right).$$

By Lemma 2, a circuit is satisfied if and only if there exists $q\in\mathbb{F}_{n-1}[X]$ such that

$$p(X)=q(X)t_{\mathcal{X}}(X). \tag{2}$$

QAP is useful to define NP language statements. A statement can be a tuple $(\mathbf{a},\mathbf{c})$ of instance and witness, where the instance $\mathbf{a}=(d_0,\cdots,d_{l-1})$ would incorporate public input and output of a circuit, and the witness $\mathbf{c}=(d_l,\cdots,d_{m-1})$ would involve the intermediate outputs of the gates. We specify a relation generator $\mathcal{R}$ for a security parameter λ that outputs a polynomial-time decidable binary relation R_λ, which is a set of the instance and witness tuples that satisfy a given QAP. For notational simplicity, we elide λ from R_λ and denote it by R. Formally, we can define

$$R:=\left\{(\mathbf{a},\mathbf{c})\middle|\begin{array}{l}\mathbf{a}=\left(d_0,\cdots,d_{l-1}\right)\in\mathbb{F}^l,\mathbf{c}=\left(d_l,\cdots,d_{m-1}\right)\in\mathbb{F}^{m-l},\\ \exists q\in\mathbb{F}_{n-1}[X]:p(X)=q(X)t_{\mathcal{X}}(X)\end{array}\right\}.$$

2.5 Groth16: Non-interactive linear proof system for QAP-based R

Groth16, proposed by J. Groth [4], is a non-interactive system to argue a statement in R, known as the most succinct protocol under generic group model. The system is a quadruple of polynomial-time algorithms $(\texttt{Setup}, \texttt{Prove}, \texttt{Verify}, \texttt{Sim})$. In detail, given a relation R based on a QAP $\mathcal{Q}$ with parameters l, m, n, each algorithm is defined as follows:

- $\texttt{Setup}: (pp_\lambda, \mathcal{Q}) \mapsto (\boldsymbol{\tau}, \boldsymbol{\sigma})$ is a probabilistic polynomial-time (PPT) algorithm that takes as input the bilinear group pp_λ and a QAP $\mathcal{Q}$ and outputs a simulation trapdoor $\boldsymbol{\tau} \in \mathbb{F}^5$ and a common reference string (CRS) $\boldsymbol{\sigma} \in \mathbb{G}_1^{m+2n+3} \times \mathbb{G}_2^{n+5}$,
- $\texttt{Prove}: (\mathcal{Q}, \boldsymbol{\sigma}, \mathbf{a}, \mathbf{c}) \mapsto \boldsymbol{\pi}$ is a deterministic polynomial-time (DPT) algorithm that takes as input statement $\mathbf{a} \in \mathbb{F}^l$ and witness $\mathbf{c} \in \mathbb{F}^{m-l}$ and outputs a proof $\boldsymbol{\pi} \in \mathbb{G}_1{}^2 \times \mathbb{G}_2$,
- $\texttt{Verify}: (\boldsymbol{\sigma}, \mathbf{a}, \boldsymbol{\pi}) \mapsto 0/1$ is a DPT algorithm that takes as input a proof $\boldsymbol{\pi}$ and returns 0 (reject) or 1 (accept),
- $\texttt{Sim}: (\mathcal{Q}, \boldsymbol{\tau}, \mathbf{a}) \mapsto \boldsymbol{\pi}^\star$ is a PPT algorithm that outputs a simulated proof $\boldsymbol{\pi}^\star$.

Definition 1 (Perfect completeness). *A proof system* $(\texttt{Setup}, \texttt{Prove}, \texttt{Verify})$ *for* R *is perfect complete, if, for all* $(\mathbf{a}, \mathbf{c}) \in R$,

$$\Pr\left[(\boldsymbol{\sigma}, \boldsymbol{\tau}) \leftarrow \texttt{Setup}(pp_\lambda, \mathcal{Q}); \boldsymbol{\pi} \leftarrow \texttt{Prove}(\mathcal{Q}, \boldsymbol{\sigma}, \mathbf{a}, \mathbf{c}) \quad : \texttt{Verify}(\boldsymbol{\sigma}, \mathbf{a}, \boldsymbol{\pi}) = 1\right] = 1.$$

Definition 2 (Statistical knowledge soundness). *A proof system* $(\texttt{Setup}, \texttt{Prove}, \texttt{Verify})$ *for* R *is statistical knowledge sound, if for all polynomial-time generic adversaries* $\mathcal{A}$, *there exists a polynomial-time extractor* $\mathcal{X}_\mathcal{A}$ *such that*

$$\Pr\left[\begin{array}{c} (\boldsymbol{\sigma}, \boldsymbol{\tau}) \leftarrow \texttt{Setup}(pp_\lambda, \mathcal{Q}); \\ (\mathbf{a}, \boldsymbol{\pi}) \leftarrow \mathcal{A}(\mathcal{Q}, \boldsymbol{\sigma}); \mathbf{c} \leftarrow \mathcal{X}_\mathcal{A}(R, \boldsymbol{\sigma}, \mathbf{a}, \boldsymbol{\pi}) \end{array} : \begin{array}{c} \texttt{Verify}(\boldsymbol{\sigma}, \mathbf{a}, \boldsymbol{\pi}) = 1 \wedge \\ (\mathbf{a}, \mathbf{c}) \notin R \end{array}\right] = negl(\lambda).$$

We introduce the Groth16 algorithms that satisfy the above definitions.

$\texttt{Setup}(pp_\lambda, \mathcal{Q})$ picks $\boldsymbol{\tau} = (x, \alpha, \beta, \gamma, \delta)$ uniformly from $(\mathbb{F}^*)^5$ at random, defines

$$o_i(X) := \beta u_i(X) + \alpha v_i(X) + w_i(X),$$

computes

$$\begin{aligned} \boldsymbol{\sigma}_1 &= \left(\alpha, \beta, \delta, \left\{x^h\right\}_{h=0}^{n-1}, \left\{\gamma^{-1} o_i(x)\right\}_{i=0}^{l-1}, \left\{\delta^{-1} o_i(x)\right\}_{i=l}^{m-1}, \left\{\delta^{-1} x^h t_\mathcal{X}(x)\right\}_{h=0}^{n-2}\right), \\ \boldsymbol{\sigma}_2 &= \left(\beta, \gamma, \delta, \left\{x^h\right\}_{h=0}^{n-1}\right), \end{aligned}$$

and returns $(\boldsymbol{\tau}, \boldsymbol{\sigma})$, where $\boldsymbol{\sigma} = ([\boldsymbol{\sigma}_1]_1, [\boldsymbol{\sigma}_2]_2)$.

$\texttt{Prove}(\mathcal{Q}, \boldsymbol{\sigma}, \mathbf{a}, \mathbf{c})$ parses $\mathbf{a} = (d_0, \cdots, d_{l-1})$ and $\mathbf{c} = (d_l, \cdots, d_{m-1})$, computes $q \in \mathbb{F}_{n-1}[X]$ defined in (2), computes

$$
\begin{aligned}
[A]_1 &= [\alpha]_1 + \left[\sum_{i=0}^{m-1} d_i u_i(x)\right]_1, \\
[B]_k &= [\beta]_k + \left[\sum_{i=0}^{m-1} d_i v_i(x)\right]_k \text{ for } k \in \{0,1\}, \\
[C]_1 &= \sum_{i=l}^{m-1} d_i \left[\delta^{-1} o_i(x)\right]_1 + \left[\delta^{-1} q(x) t_{\mathcal{X}}(x)\right]_1,
\end{aligned} \tag{3}
$$

and returns $\boldsymbol{\pi} = ([A]_1, [B]_2, [C]_1)$.

$\texttt{Verify}(\boldsymbol{\sigma}, \mathbf{a}, \boldsymbol{\pi})$ parses $\mathbf{a} = (d_0, \cdots, d_{l-1})$, $\boldsymbol{\pi} = ([A]_1, [B]_2, [C]_1)$ and returns $\texttt{accept}$ if and only if the following equation holds,

$$
e([A]_1, [B]_2) = e([\alpha]_1, [\beta]_2) e\left(\sum_{i=0}^{l-1} d_i \left[\gamma^{-1} o_i(x)\right]_1, [\gamma]_2\right) e([C]_1, [\delta]_2).
$$

Theorem 1 (Groth [4]**).** *The Groth16 system* $(\texttt{Setup}, \texttt{Prove}, \texttt{Verify})$ *for* R *constructed in Section 2.5 is perfect complete and statistical knowledge soundness in generic group model.*

We omit the proof of Theorem 1, as further discussion on completeness and knowledge soundness will be provided in Section 5.

The above construction of $\texttt{Prove}$ as a DPT algorithm can be converted into a PPT algorithm $\texttt{Prove}^\star$ to have one additional security property, zero-knowledge as defined below.

Definition 3 (Perfect zero-knowledge). A proof system $(\texttt{Setup}, \texttt{Prove}^\star, \texttt{Verify}, \texttt{Sim})$ for R has perfect zero-knowledge if for all $(\mathbf{a}, \mathbf{c}) \in R$ and all adversaries $\mathcal{A}$

$$
\begin{aligned}
&\Pr\left[(\boldsymbol{\sigma}, \boldsymbol{\tau}) \leftarrow \texttt{Setup}(pp_\lambda, \mathcal{Q}); \boldsymbol{\pi} \leftarrow \texttt{Prove}(\mathcal{Q}, \boldsymbol{\sigma}, \mathbf{a}, \mathbf{c}) \quad : \mathcal{A}(R, \boldsymbol{\sigma}, \boldsymbol{\tau}, \mathbf{a}, \boldsymbol{\pi}) = 1\right] = \\
&\Pr\left[(\boldsymbol{\sigma}, \boldsymbol{\tau}) \leftarrow \texttt{Setup}(pp_\lambda, \mathcal{Q}); \boldsymbol{\pi} \leftarrow \texttt{Sim}(\mathcal{Q}, \boldsymbol{\tau}, \mathbf{a}) \quad : \mathcal{A}(R, \boldsymbol{\sigma}, \boldsymbol{\tau}, \mathbf{a}, \boldsymbol{\pi}) = 1\right].
\end{aligned}
$$

The perfect zero-knowledge means that the proof generated from a valid statement $(\mathbf{a}, \mathbf{c}) \in R$ is probabilistically indistinguishable from a simulated proof $\boldsymbol{\pi}^\star$. For the Groth16 system to have the perfect zero-knowledge, a simulation algorithm has been constructed as

$\texttt{Sim}(\mathcal{Q}, \boldsymbol{\tau}, \mathbf{a})$ parses $\mathbf{a} = (d_0, \cdots, d_{l-1})$, picks $(A^\star, B^\star)$ uniformly from $(\mathbb{F}^*)^2$ at random, computes

$$
[C^\star]_1 = \left[\delta^{-1}\left(A^\star B^\star - \alpha\beta - \sum_{i=0}^{l-1} d_i \gamma^{-1} o_i(x)\right)\right]_1,
$$

and returns $\boldsymbol{\pi}^\star = ([A^\star]_1, [B^\star]_2, [C^\star]_1)$.

It is straightforward to see that the simulated proof $\boldsymbol{\pi}^\star$ is always accepted by $\texttt{Verify}$. All that remains is to construct a PPT algorithm $\texttt{Prove}^\star$ that outputs a proof $\boldsymbol{\pi}$ with the identical probability distribution as the simulated proof $\boldsymbol{\pi}^\star$:

$\texttt{Prove}^\star(\mathcal{Q}, \boldsymbol{\sigma}, \mathbf{a}, \mathbf{c})$ is a modification of $\texttt{Prove}$, where it additionally picks (r, s) uniformly from $(\mathbb{F}^*)^2$ at random and modifies the computation (3) as $[\tilde{A}]_1 = [A]_1 + r[\delta]_1$, $[\tilde{B}]_2 = [B]_2 + s[\delta]_2$, $[\tilde{C}]_1 = [C]_1 + s[A]_1 + r[B]_1$

$+rs[\delta]_1$. The output is $\boldsymbol{\pi} = ([\tilde{A}]_1, [\tilde{B}]_2, [\tilde{C}]_1)$.

Theorem 2 (Groth [4]). *The Groth16 system* $(\texttt{Setup}, \texttt{Prove}^{\star}, \texttt{Verify}, \texttt{Sim})$ *for* R *constructed in Section 2.5 has perfect zero-knowledge.*

Circuit-specific setup of the Groth16 system. The setup algorithm allows the prover to compress a statement into three group elements. However, the security relies on the assumption that any intermediate output, such as the trapdoor $\boldsymbol{\tau}$ or the randomized monomials $\{[\delta^{-1}x^h]_1, [\gamma^{-1}x^h]_1\}_h$, other than the final output $\boldsymbol{\sigma}$, is kept secret. Some real-world applications can realize this assumption based on trust. In this case, since the setup algorithm requires a QAP as input, each new circuit requires new trust.

2.6 KZG polynomial commitment scheme and witness-extended emulation

To reduce dependence on the trusted setup of succinct proof systems like Groth16, interactive public-coin protocols for R such as Sonic [5], PlonK [7], and Marlin [6] have introduced universal setups that are independent of specific circuits. Instead, these protocols have incorporated polynomial commitment schemes as sub-protocols to ensure that the parties—the prover and verifier—agree on the same circuit or polynomials before commencing larger protocols. Specifically, in a larger protocol, the verifier is convinced of the prover's knowledge given circuit polynomials that were previously committed to.

KZG polynomial commitment scheme, proposed by Kate *et al.* in [34], is an efficient polynomial scheme that utilizes the following algebraic behavior: Let $p \in \mathbb{F}_n[X]$ be a polynomial to be committed. By Lemma 2, for every $\xi \in \mathbb{F}$, there exists a pair of a quotient polynomial $q \in \mathbb{F}[X]$ and the evaluation $p(\xi)$ such that $p(X) - p(\xi) = q(X)(X - \xi)$. Suppose there is a structured reference string (SRS) $\boldsymbol{\sigma} = ([(x^i)_{i=0}^{n-1}]_1, [1]_2, [x]_2)$ with a random trapdoor $x \in \mathbb{F}^*$. We let $[f(x)]_1$ denote a commitment to a polynomial $f \in \mathbb{F}_n[X]$. The protocol proceeds as follows: *1)* prover commits to $p(X)$; *2)* verifier picks ξ at random from $\mathbb{F}$; *3)* prover evaluates $p(\xi)$ and commits to $q(X)$ that satisfies $p(X) - p(\xi) = q(X)(X - \xi)$; *4)* verifier accepts the transcript $\mathbf{t} = ([p(x)]_1, \xi, p(\xi), [q(x)]_1)$, if and only if

$$e\big([p(x)]_1 - p(\xi)[1]_1, [1]_2\big) = e\big([q(x)]_1, [x]_2 - \xi[1]_2\big).$$

As a sub-protocol, an advantage of the KZG commitment scheme is that a larger protocol can manipulate the evaluation $p(\xi)$ in place of the commitment $[p(x)]_1$, which is referred to as *evaluation binding*. Manipulating the additive group elements is costly and inefficient, and is only allowed with limited operations, such as linear operations or bilinear pairings. Technically, the verifier can be convinced of the prover's knowledge of the coefficients of $p(x)$, once an acceptable evaluation $p(\xi)$ is provided, through *witness-extended emulation* [44] (refer to Definition 5 for a formal definition) with high probability in λ. Below, we outline how the extraction works.

Consider a PPT adversary $\mathcal{A}$. Suppose that a transcript $\mathbf{t} = ([a]_1, \xi, b, [c]_1)$, generated between $\mathcal{A}$ and the verifier, is acceptable. In generic group model, the verification equation can be rewritten into $a - b = c(x - \xi)$, where the transcript components can be rewritten into $a = \sum_{i=0}^{n-1} a_i x^i$ and $c = \sum_{i=0}^{n-1} c_i x^i$. Lemma 1 allows to rewrite the verification equation into a polynomial equation $a(X) - b = c(X)(X - \xi)$, with high probability greater than $1 - n|\mathbb{F}|^{-1}$. Suppose this case has happened. Then, by Lemma 2, we can imply that $a(\xi_i) = \sum_{i=0}^{n-1} a_i \xi^i = b$. Suppose there exists a DPT emulator that accesses all the internal states of $\mathcal{A}$, including coin-tossing, used to generate $\mathbf{t}$. This emulator thus can interact with the verifier by mimicking the behavior of $\mathcal{A}$. By emulating the protocol to collect n acceptable transcripts with fresh (distinct) randomness $\xi_1, \cdots \xi_n$, the verifier can obtain n simultaneous linear equations $a(\xi_i) = b$, where the linear map is a Vandermonde matrix, so it is invertible. As a result, the coefficients $(a_0, \cdots, a_{n-1})$ can be extractable from the emulated transcripts, with probability greater than $1 - n^2 \,|\, \mathbb{F} \,|$. Lindel in [44] has shown that the emulation can be done expectedly in n runs.

2.7 PlonK's permutation argument

We revisit a permutation argument based on the KZG commitment scheme, refined by Gabizon *et al.* in [7]. Recall a circuit represented by a constraint system R. Suppose there are wiring constraints in the circuit that necessitate repetitions in some of the wire assignments within a statement $\mathbf{d} = (d_0, \cdots, d_{m-1})$. The wiring constraints, referred to as *copy constraints* throughout the rest of this paper, indicate that two wires attached to different gates share the same assignment. As an interactive public-coin protocol, the objective of a permutation argument is to convince the verifier of the prover's knowledge of $\mathbf{d}$ that satisfies the copy constraints.

Define a vanishing set $\mathcal{Z} := \{\omega_Z^i\}_{i=0}^{m-1}$ of the m-th root of unity and the corresponding vanishing polynomial $t_{\mathcal{Z}}(Z)$. Formally, copy constraints in a statement $\mathbf{d} = (d_0, \cdots, d_{m-1})$ can be defined using a permutation polynomial $s : \mathcal{Z} \to \mathcal{Z}$ such that $d_i = d_k$, if $s(\omega_Z^i) = \omega_Z^k$. Encoding the wire assignments into a polynomial $b : \mathcal{Z} \to \mathbb{F}$ such that $b(\omega_Z^i) = d_i$, the following claim can be used to equivalently check the copy constraints $s(Z)$ with respect to $b(Z)$.

Claim. *Denoting indeterminates* $\boldsymbol{\theta} = (\theta_0, \theta_1)$, *let* $f(Z, \boldsymbol{\theta}) := b(Z) + \theta_0 s(Z) + \theta_1$ *and* $g(Z, \boldsymbol{\theta}) := b(Z) + \theta_0 Z + \theta_1$. *The wire assignments* $b(Z)$ *satisfies copy constraints of* $s(Z)$, *if and only if the equation* $\prod_{z \in \mathcal{Z}} f(z, \boldsymbol{\theta}) = \prod_{z \in \mathcal{Z}} g(z, \boldsymbol{\theta})$ *holds.*

We omit the proof of this claim. Instead, in Lemma 3, we will provide an extended version of this claim for bivariate polynomials with formal proof.

The sufficient and necessary condition of the above claim can be equivalently satisfied by showing a recursion polynomial $r \in \mathbb{F}_{m,2,2}[Z, \boldsymbol{\theta}]$, constructed as

$$\begin{cases} r(\omega_Z^0, \boldsymbol{\theta}) = 1, \\ r(\omega_Z^{i+1}, \boldsymbol{\theta}) g(\omega_Z^i, \boldsymbol{\theta}) = r(\omega_Z^i, \boldsymbol{\theta}) f(\omega_Z^i, \boldsymbol{\theta}), \quad \text{for } 0 \le i \le m-1. \end{cases} \tag{4}$$

In a permutation argument, the prover is requested to show the knowledge of $r(Z, \boldsymbol{\theta})$. Suppose the verifier can access $f(Z, \boldsymbol{\theta})$ and $g(Z, \boldsymbol{\theta})$. The verifier accepts a claim polynomial $r(Z, \boldsymbol{\theta})$ if and only if two polynomial $p_1(Z, \boldsymbol{\theta}) := (r(Z, \boldsymbol{\theta}) - 1) K_0(Z)$ and $p_2(Z, \boldsymbol{\theta}) := r(\omega_Z Z, \boldsymbol{\theta}) g(Z, \boldsymbol{\theta}) - r(Z, \boldsymbol{\theta}) f(Z, \boldsymbol{\theta})$ vanishes on $\mathcal{Z}$, where $K_0(Z)$ vanishes on $\mathcal{Z} - \{\omega_Z^0\}$ and $K_0(\omega_Z^0) = 1$. It is easy to see that the two polynomials p_1 and p_2 are to check the first and second conditions of the above claim, respectively.

By Lemma 2, the constraints on $r(Z, \boldsymbol{\theta})$ are satisfied if and only if there exists quotient polynomials $q_1, q_2 \in \mathbb{F}[Z, \boldsymbol{\theta}]$ of degree less than $m-1$ in Z such that, for each $i \in \{0,1\}$,

$$p_i(Z, \boldsymbol{\theta}) = q_i\left(Z, \boldsymbol{\theta}\right) t_{\mathcal{Z}}\left(Z\right). \tag{5}$$

To argue the copy constraints in (5) efficiently, Gabizon *et al.* in [7] have introduced a highly optimized protocol. Suppose there is an SRS $\boldsymbol{\sigma} = ([(z^i)_{i=0}^{m-1}]_1, [1]_2, [z]_2)$ with a random trapdoor $z \in \mathbb{F}^*$. The protocol proceeds as follows: *1)* prover commits to $b(Z)$; *2)* verifier picks $\boldsymbol{\theta} \in (\mathbb{F}^*)^2$ at random; *3)* prover commits to $r(Z,\boldsymbol{\theta})$; *4)* verifier picks $\kappa_0 \in \mathbb{F}^*$ at random; *5)* prover commits to a quotient polynomial $q(Z,\boldsymbol{\theta}) := q_1(Z,\boldsymbol{\theta}) + \kappa_0 q_2(Z,\boldsymbol{\theta})$; *6)* verifier picks $\xi \in \mathbb{F}$ at random; *7)* prover computes the evaluations $r(\xi,\boldsymbol{\theta})$ and $r(\omega_Z\xi,\boldsymbol{\theta})$; *8)* verifier picks $\kappa_1 \in \mathbb{F}^*$ at random; *9)* prover commits to two quotient polynomials $h_0, h_1 \in \mathbb{F}[Z]$ that satisfies, respectively,

$$\begin{pmatrix} \left(r(\xi,\boldsymbol{\theta})-1\right)K_0(Z) + \kappa_0\left(r(\omega_Z\xi,\boldsymbol{\theta})g(Z,\boldsymbol{\theta}) - r(\xi,\boldsymbol{\theta})f(Z,\boldsymbol{\theta})\right) \\ -q(Z)t_{\mathcal{Z}}(\xi) + \kappa_1\left(r(Z,\boldsymbol{\theta}) - r(\xi,\boldsymbol{\theta})\right) \end{pmatrix} = h_0(Z)(Z-\xi),$$
$$r(Z,\boldsymbol{\theta}) - r(\omega_Z\xi,\boldsymbol{\theta}) = h_1(Z)(Z-\omega_Z\xi);$$

10) verifier finally accepts the transcript $\mathbf{t} = ([b(z)]_1,\ \boldsymbol{\theta},\ [r(z)]_1,\ \kappa_0,\ [q(z)]_1,\ \xi,\ r(\xi),\ r(\omega_Z\xi),\ \kappa_1,\ [h_0(z)]_1,\ [h_1(z)]_1)$, if and only if the following two equations hold:

$$e\begin{pmatrix} \left(r(\xi,\boldsymbol{\theta})-1\right)[K_0(z)]_1 + \kappa_0\left(r(\omega_Z\xi,\boldsymbol{\theta})[g(z,\boldsymbol{\theta})]_1 - r(\xi,\boldsymbol{\theta})[f(z,\boldsymbol{\theta})]_1\right) & ,[1]_2 \\ -t_{\mathcal{Z}}(\xi)[h(z)]_1 + \kappa_1\left([r(z,\boldsymbol{\theta})]_1 - r(\xi,\boldsymbol{\theta})[1]_1\right) & \end{pmatrix} = e\left([h_0(z)]_1, [z]_2 - \xi[1]_2\right),$$
$$e\left([r(z,\boldsymbol{\theta})]_1 - r(\omega_Z\xi,\boldsymbol{\theta})[1]_1 \quad ,[1]_2\right) = e\left([h_1(z)]_1, [z]_2 - \omega_Z\xi[1]_2\right), \tag{6}$$

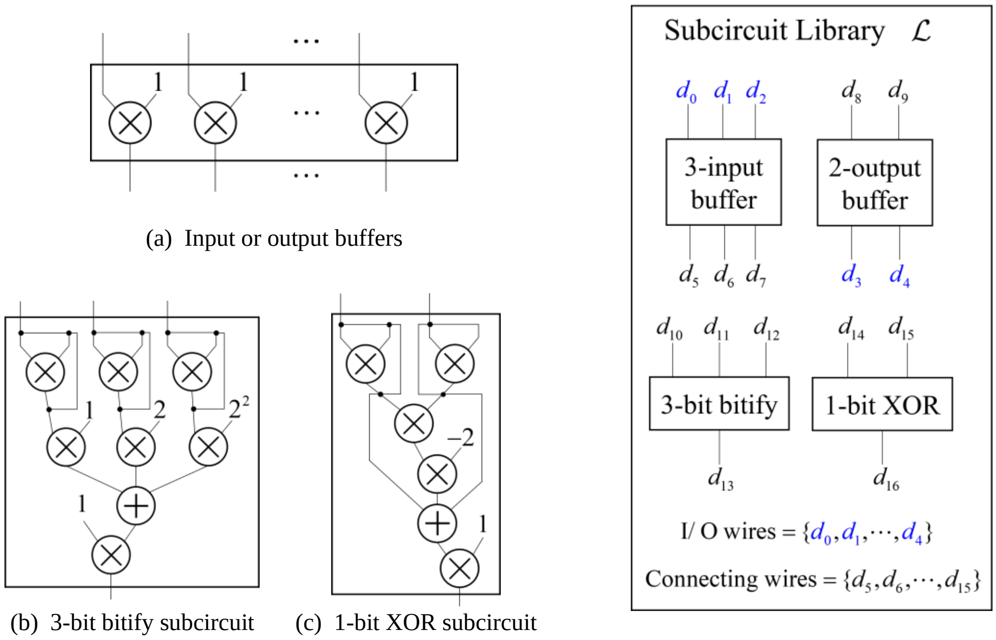
where $[f(z,\boldsymbol{\theta})]_1 = [b(z)]_1 + \theta_0[s(z)]_1 + \theta_1[1]_1$ and $[g(z,\boldsymbol{\theta})]_1 = [b(z)]_1 + \theta_0[z]_1 + \theta_1[1]_1$. It was assumed that the verifier can access $[s(z)]_1$, which is an honest commitment to $s(Z)$.

It is straightforward to see the completeness that if the prover knows the recursion polynomial r that satisfies (4), the verifier accepts the transcript (with assuming $\boldsymbol{\theta}$ is of indeterminates). Conversely, by recalling the evaluation binding of KZG commitment scheme, we can see this protocol is knowledge sound (with assuming that κ_0 and κ_1 are indeterminates): By the evaluation binding, satisfying (6) implies satisfying (5). We will provide formal proof for completeness, knowledge soundness, and zero-knowledge of an extended protocol in Section 5.

3 Front-end preprocess: System of constraints and setup algorithm

We define a system of constraints based on a *subcircuit library* $\mathcal{L}$. Our system is parameterized by the number of subcircuits in $\mathcal{L}$, denoted by s_D, the maximum number of constraints that a subcircuit can contain, denoted by n, and the maximum number of subcircuit copies that can be placed in a circuit, denoted by $s_{\max}$.

Subcircuit library. A subcircuit library is defined as $\mathcal{L} =:= \bigcup_{k=0}^{s_D-1} \mathrm{c}_k$, where c_0 is an input buffer circuit, c_{s_D-1} is an output buffer circuit, and $\mathrm{c}_1,\cdots,\mathrm{c}_{s_D-2}$ are custom circuits. All the circuits in $\mathfrak{L}$ are referred to as *subcircuits* and defined in the QAP representation. We let $m^{(k)}$ denote the total number of wires that compose the subcircuit c_k, and out of those we let $l^{(k)}$ denote the number of input and output wires. The input buffer c_0 can be placed in a circuit to take as input instance of length l_{in}, and the output buffer c_{s_D-1} returns output instance of length l_{out}. In other words, $l^{(0)} = 2l_{in}$ and $l^{(s_D-1)} = 2l_{out}$. Figure 3 depicts an example of the subcircuit library $\mathcal{L}$.



(a) Input or output buffers

(b) 3-bit bitify subcircuit

(c) 1-bit XOR subcircuit

(d) An example of subcircuit library

Figure 3. An example of subcircuit library $\mathcal{L}$, which consists of I/O buffers, 3-bit bitify subcircuit, 1-bit XOR subcircuit. (a) I/O buffers check consistency between the input and output. (b) Bitify subcircuit verifies first whether each input is binary and then the binary representation of the output. (c) XOR subcircuit verifiers first whether each input is binary and then XOR of the input. (d) In $\mathcal{L}$, I/O wires and connecting wires are separately listed.

Circuit derivation. A system of constraints constraint we consider are a class of field-programmable circuits. In this class, a circuit $\mathcal{C}$ is programed by a *placement* of at most $s_{\max}$ copies of the subcircuits in $\mathcal{L}$, denoted by $\mathbf{Q} = (\mathrm{d}^{(0)}, \cdots, \mathrm{d}^{(s_{\max}-1)})$, where $\mathrm{d}^{(i)} \in \{\mathrm{c}_k\}_{k=0}^{s_D-1}$ denotes an *active subcircuit*, and wiring between the copies. The wiring is encoded into a permutation ρ, called a *wire map*, which entails data transfer only between subcircuits but does not involve data transfer between internal gates within each subcircuit. In other words, given a fixed $\mathcal{L}$, $\mathcal{C} := (\mathbf{Q}, \rho)$. The activation of subcircuits is determined by ρ. Figure 4 exemplifies how a wire map programs a circuit.

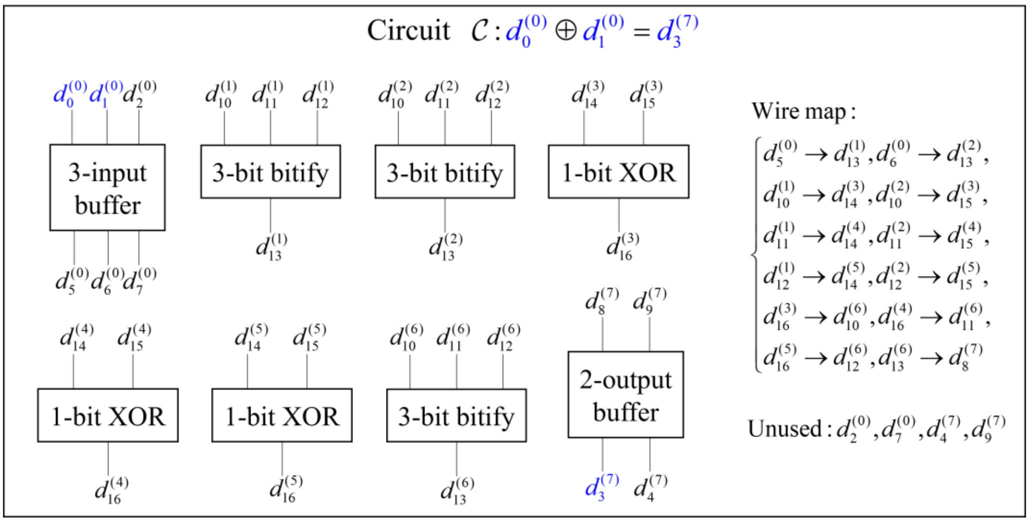


Figure 4. An example circuit $\mathcal{C}$, where the program takes two inputs of 3-bits and returns the XOR of them. Subcircuits that make up this circuit are copied from $\mathcal{L}$ in Figure 3. Each wire is represented as $d_j^{(i)}$, which denotes the i-th copy of $\mathcal{L}$'s j-th wire. In the wire map, $a \to b$ denotes that an input b is driven by an output a.

3.1 Compilers

QAP compiler. QAP compiler outputs the QAP of $\mathfrak{L}$. According to the QAP representation, we define $\mathrm{c}_k \subset \mathcal{L}$ for each $k = 0, \cdots, s_D - 1$ as

$$\mathrm{c}_k := \left\{ u_0^{(k)}, \cdots, u_{m^{(k)}-1}^{(k)}, v_0^{(k)}, \cdots, v_{m^{(k)}-1}^{(k)}, w_0^{(k)}, \cdots, w_{m^{(k)}-1}^{(k)} \right\} \subset \mathbb{F}_n[X].$$

A library $\mathcal{L}$ is defined as $\mathcal{L} := \bigcup_{k=0}^{s_D-1} \mathrm{c}_k$. We denote the total number of wires by $m_D := \sum_{k=0}^{s_D-1} m^{(k)}$, the number of public wires by $l := 2^{-1}(l^{(0)} + l^{(s_D-1)}) = l_{in} + l_{out}$, and the number of (private) interface wires by $m_I := \sum_{k=0}^{s_D-1} l^{(k)} - l$. The library contains $3m_D$ polynomials in total, which can be rearranged as

$$\mathcal{L} = \left\{ \begin{matrix} u_0, \cdots, u_{l_{in}-1}, u_{l_{in}}, \cdots, u_{l-1}, u_l, \cdots, u_{l_D-1}, u_{l_D}, \cdots, u_{m_D-1}, \\ v_0, \cdots, v_{l_{in}-1}, v_{l_{in}}, \cdots, v_{l-1}, v_l, \cdots, v_{l_D-1}, v_{l_D}, \cdots, v_{m_D-1}, \\ w_0, \cdots, w_{l_{in}-1}, w_{l_{in}}, \cdots, w_{l-1}, w_l, \cdots, w_{l_D-1}, w_{l_D}, \cdots, w_{m_D-1} \end{matrix} \right\} \subset \mathbb{F}_n[X].$$

Each polynomial $o_j \in \{u_j, v_j, w_j\} \subset \mathcal{L}$ is picked from $o_j^{(k)} \in \{u_j^{(k)}, v_j^{(k)}, w_j^{(k)}\} \subset \mathrm{c}_k$ by the following rules:

$$\begin{aligned} &(o_0, \cdots, o_{l-1}) = (o_0^{(0)}, \cdots, o_{l_{in}-1}^{(0)}, o_{l_{out}}^{(s_D-1)}, \cdots, o_{2l_{out}-1}^{(s_D-1)}), \\ &(o_l, \cdots, o_{l_D-1}) = (o_{l_{in}}^{(0)}, \cdots, o_{l^{(0)}-1}^{(0)}, o_0^{(s_D-1)}, \cdots, o_{l_{out}-1}^{(s_D-1)}, o_0^{(1)}, \cdots, o_{l^{(1)}-1}^{(1)}, o_0^{(2)}, \cdots, o_{l^{(2)}-1}^{(2)}, \cdots, o_0^{(s_D-2)}, \cdots, o_{l^{(s_D-2)}-1}^{(s_D-2)}), \\ &(o_{l_D}, \cdots, o_{m_D-1}) = (o_{l^{(0)}}^{(0)}, \cdots, o_{m^{(0)}}^{(0)}, o_{l^{(1)}}^{(1)}, \cdots, o_{m^{(1)}}^{(1)}, \cdots, o_{l^{(s_D-1)}}^{(s_D-1)}, \cdots, o_{m^{(s_D-1)}}^{(s_D-1)}). \end{aligned} \tag{7}$$

The polynomials are arranged by the role of the corresponding wires. The first polynomials o_j for $j \in \{0, \cdots, l-1\}$ represent public *I/O wires*, which are used only for taking and returning the circuit input and output. We put a restriction on $\mathcal{C}$ that the first and last slots in a placement must be occupied by the input and output buffers c_0 and c_{s_D-1}, respectively. The next polynomials o_j for $j \in \{l, \cdots, l + m_I - 1\}$ represent *interface wires*, which are used only for transferring data from one subcircuit to other subcircuits. The rest polynomials

o_j for $j \in \{l+m_I, \cdots, m_D - 1\}$ represent *internal wires* inside subcircuits.

Synthesizer. Given $\mathcal{L}$ as input, synthesizer outputs $\mathbf{Q}$ and ρ, those together form a circuit $\mathcal{C} = (\mathbf{Q}, \rho)$. By setting the wire assignments in all inactive subcircuits to zero, we can rewrite $\mathbf{Q} = (\mathcal{L}^{(0)}, \cdots, \mathcal{L}^{(s_{\max}-1)})$, where $\mathcal{L}^{(i)}$ is the i-th copy of $\mathcal{L}$. Synthesis draws a wire map that describes how the interface wires in each copy are connected to each other. The connected wires must share the same value assignment.

Formally, motivated by [5, 7], we define a wire map as a permutation ρ. Let $d_j^{(i)} \in \mathbb{F}$ denote a *value assignment* to the wire $u_j, v_j, w_j \in \mathcal{L}^{(i)}$ for $i \in \{0, \cdots, s_{\max} - 1\}$. We collect and write $b_{j-l}^{(i)} = d_j^{(i)}$ for $j \in \{l, \cdots, l_D - 1\}$, which are the value assignments to the connecting wires of $\mathcal{L}^{(i)}$. As an index set for $b_j^{(i)}$, we define $\mathcal{N} := \{0, \cdots, s_{\max} - 1\} \times \{0, \cdots, m_I - 1\}$. Then, a wire map is defined as a permutation $\rho : \mathcal{N} \to \mathcal{N}$. To construct the mapping rule of ρ, we divide $\mathcal{N}$ into M disjoint subsets $\mathcal{N}_k$ for $k = 0, \cdots, M-1$ so that all $b_j^{(i)}$ indexed by $(i, j) \in \mathcal{N}_k$ share the same value. We define cycles $\rho_k : \mathcal{N}_k \to \mathcal{N}_k$ for $k = 0, \cdots, M-1$. The permutation ρ is finally constructed as an integration of the cycles

$$\rho = (\rho_0) \cdots (\rho_{M-1}). \tag{8}$$

In the example circuit in Figure 4, a permutation ρ can be defined over $M = 12$ disjoint subsets, each with $|\mathcal{N}_k| = 2$.

3.2 System of constraints

Given a circuit $\mathcal{C} = (\mathbf{Q}, \rho)$, a system of constraints contains two subsystems: arithmetic constraints and copy constraints. Arithmetic constraints checks whether all wire assignments $d_j^{(i)}$ for $h \in \{0, \cdots, s_{\max} - 1\}$ and $j \in \{0, \cdots, m_D\}$ satisfy the QAP of $\mathbf{Q}$. Copy constraints checks the correctness of the connection between subcircuits in the placement by checking whether the wire assignments to the connecting wires, i.e., $b_j^{(i)}$ for $(i, j) \in \mathcal{N}$ satisfy a permutation ρ.

For the construction of a constraint system, we need Lagrange bases $K_i \in \mathbb{F}[Z]$ for $i \in \{0, \cdots, m_I - 1\}$ and $L_i \in \mathbb{F}[Y]$ for $i \in \{0, \cdots, s_{\max} - 1\}$ such that $K_i(\omega_{m_I}^i) = L_i(\omega_{s_{\max}}^i) = 1$ and $K_i(\omega_{m_I}^k) = L_i(\omega_{s_{\max}}^k) = 0$ for every $k \neq i$. We encode some of the wire assignments $\{d_j^{(i)}\}_{i=0}^{s_{\max}-1}$ for each $j \in \{0, \cdots, m_D - 1\}$ into polynomials according to the roles of wires as discussed in (7):

$$\sum_{i=0}^{s_{\max}-1} d_j^{(i)} L_i(Y) =: \begin{cases} b_{j-l}(Y), & \text{for } j = l, \cdots, l + m_I - 1, \\ c_{j-(l+m_I)}(Y), & \text{for } j = l + m_I, \cdots, m_D - 1. \end{cases} \tag{9}$$

1) *Arithmetic constraints*

The arithmetic constraints can be represented by a set of equations: for all $(x, y) \in \{\omega_n^i\}_{i=0}^{n-1} \times \{\omega_{s_{\max}}^i\}_{i=0}^{s_{\max}-1}$,

$$p_0(x, y) = 0, \tag{10}$$

where

$$p_0(X,Y) := U(X,Y)V(X,Y) - W(X,Y),$$

and for $O \in \{U,V,W\}$,

$$O(X,Y) := \sum_{j=0}^{l_{in}-1} d_j^{(0)} L_0(Y) o_j(X) + \sum_{j=l_{in}}^{l-1} d_j^{(s_D-1)} L_{-1}(Y) o_j(X) + \sum_{j=l}^{l_D-1} b_{j-l}(Y) o_j(X) + \sum_{j=l_D}^{m_D-1} c_{j-l_D}(Y) o_j(X), \tag{11}$$

where $o_i = u_i$, if $O = U$; $o_i = v_i$, if $O = V$; and $o_i = w_i$, if $O = W$.

Applying Corollary 1 to the $ns_{\max}$ equations of (10), the arithmetic constraints are satisfied if and only if

$$\exists q_0, q_1 \in \mathbb{F}[X,Y]: p_0(X,Y) = q_0(X,Y) t_n(X) + q_1(X,Y) t_{s_{\max}}(Y). \tag{12}$$

2) Copy constraints

The copy constraints check whether $b(X,Y)$ satisfies a permutation ρ, where

$$b(X,Y) := \sum_{j=0}^{l_D-l-1} b_j(Y) K_j(X). \tag{13}$$

For $\rho(i,j) \mapsto (h,k)$, we write $\rho(i,j)_1 = h$ and $\rho(i,j)_2 = k$. We say the copy constraints are satisfied if and only if $B(\omega_{m_I}^j, \omega_{s_{\max}}^i) = B(\omega_{m_I}^{\rho(i,j)_2}, \omega_{s_{\max}}^{\rho(i,j)_1})$, i.e., $b_j^{(i)} = b_{\rho(i,j)_2}^{(\rho(i,j)_1)}$ for every connecting wire index $(i,j) \in \mathcal{N}$.

Motivated by [5, 7, 33], we construct a permutation check algorithm for the copy constraints. We first encode ρ into permutation polynomials $s^{(0)}, s^{(1)}, s^{(2)} \in \mathbb{F}[X,Y]$ such that for $(i,j) \in \mathcal{N}$,

$$\begin{aligned} s^{(0)}(\omega_{m_I}^j, \omega_{s_{\max}}^i) &= \omega_{m_I}^{\rho(i,j)_2}, \\ s^{(1)}(\omega_{m_I}^j, \omega_{s_{\max}}^i) &= \omega_{s_{\max}}^{\rho(i,j)_1}, \\ s^{(2)}(\omega_{m_I}^j, \omega_{s_{\max}}^i) &= \omega_{m_I}^j \Leftrightarrow s^{(2)}(X,Y) := X. \end{aligned} \tag{14}$$

With introducing indeterminates $\boldsymbol{\theta} = (\theta_0, \theta_1, \theta_2)$, we also define

$$\begin{aligned} f(X,Y,\boldsymbol{\theta}) &:= b(X,Y) + \theta_0 s^{(0)}(X,Y) + \theta_1 s^{(1)}(X,Y) + \theta_2, \\ g(X,Y,\boldsymbol{\theta}) &:= b(X,Y) + \theta_0 s^{(2)}(X,Y) + \theta_1 Y + \theta_2. \end{aligned} \tag{15}$$

Lemma 3 below is useful for checking the copy constraints.

Lemma 3. *Given polynomials f, g defined in (15), $b(X,Y)$ satisfies copy constraints with ρ, if and only if the following equation holds*

$$\prod_{y \in \mathcal{X}, z \in \mathcal{Y}} f(x,y,\boldsymbol{\theta}) = \prod_{y \in \mathcal{X}, z \in \mathcal{Y}} g(x,y,\boldsymbol{\theta}), \tag{16}$$

where $\mathcal{X} := \{\omega_{m_I}^i\}_{i=0}^{m_I-1}$ and $\mathcal{Y} := \{\omega_{s_{\max}}^i\}_{i=0}^{s_{\max}-1}$.

Proof. For the simplicity of expression, we denote $b_{i,j}=b(\omega_{m_I}^{j},\omega_{s_{\max}}^{i})$ for $(i,j)\in\mathcal{N}$. If $b_{i,j}=b_{\rho(i,j)}$ then the factors on both sides are the same, just in a different order, so the equation holds. Conversely, suppose the equation (16) holds. Consider the roots of θ_0 on both sides given by

$$\left\{-\frac{b_{i,j}+\omega_{m_I}^{\rho(i,j)_2}\theta_1+\theta_2}{\omega_{s_{\max}}^{\rho(i,j)_1}}:(i,j)\in\mathcal{N}\right\},\left\{-\frac{b_{h,k}+\omega_{m_I}^{k}\theta_1+\theta_2}{\omega_{s_{\max}}^{h}}:(h,k)\in\mathcal{N}\right\}.$$

The equation implies that two sets of roots must be the same, i.e., for every $(i,j)\in\mathcal{N}$, there must exist $(h,k)\in\mathcal{N}$ such that

$$\omega_{s_{\max}}^{h}\left(b_{i,j}+\omega_{m_I}^{\rho(i,j)_2}\theta_2+\theta_3\right)=\omega_{s_{\max}}^{\rho(i,j)_1}\left(b_{h,k}+\omega_{m_I}^{k}\theta_2+\theta_3\right).$$

Since there is the unique pair $((h,i),(j,k))\in\mathcal{N}$ such that $\omega_Y^{j}=\omega_Y^{\rho(h,i)_1}$ and $\omega_Z^{k}=\omega_Z^{\rho(h,i)_2}$ by the definition of ρ, we conclude that on those indices it holds $b_{h,i}=b_{j,k}$. In other words, $b_{i,j}=b_{\rho(i,j)}$. □

For an efficient utilization of Lemma 3, we define a recursion polynomial $r\in\mathbb{F}[X,Y,\boldsymbol{\theta}]$ such that

$$\begin{cases} r(\omega_{m_I}^{m_I-1},\omega_{s_{\max}}^{s_{\max}-1},\boldsymbol{\theta})=1, \\ r(\omega_{m_I}^{j},\omega_{s_{\max}}^{i},\boldsymbol{\theta})g(\omega_{m_I}^{j},\omega_{s_{\max}}^{i},\boldsymbol{\theta})=r(\omega_{m_I}^{j-1},\omega_{s_{\max}}^{i},\boldsymbol{\theta})f(\omega_{m_I}^{j},\omega_{s_{\max}}^{i},\boldsymbol{\theta}), \quad \text{for } 0\le i\le s_{\max}-1, 0<j\le m_I-1, \\ r(\omega_{m_I}^{0},\omega_{s_{\max}}^{i},\boldsymbol{\theta})g(\omega_{m_I}^{0},\omega_{s_{\max}}^{i},\boldsymbol{\theta})=r(\omega_{m_I}^{m_I-1},\omega_{s_{\max}}^{i-1},\boldsymbol{\theta})f(\omega_{m_I}^{0},\omega_{s_{\max}}^{i},\boldsymbol{\theta}), \quad \text{for } 0\le i\le s_{\max}-1. \end{cases} \tag{17}$$

It is straightforward to see that there exists $r(X,Y,\boldsymbol{\theta})$ that holds (17), if and only if the equation (16) holds.

By Corollary 1, the polynomial $r(X,Y,\boldsymbol{\theta})$ satisfies (17), if and only if there exist $q_i\in\mathbb{F}[X,Y,\boldsymbol{\theta}]$ for $i\in\{2,\cdots,7\}$ such that

$$\begin{aligned} p_1(X,Y,\boldsymbol{\theta})&=q_2(X,Y,\boldsymbol{\theta})t_{m_I}(X)+q_3(X,Y,\boldsymbol{\theta})t_{s_{\max}}(Y), \\ p_2(X,Y,\boldsymbol{\theta})&=q_4(X,Y,\boldsymbol{\theta})t_{m_I}(X)+q_5(X,Y,\boldsymbol{\theta})t_{s_{\max}}(Y), \\ p_3(X,Y,\boldsymbol{\theta})&=q_6(X,Y,\boldsymbol{\theta})t_{m_I}(X)+q_7(X,Y,\boldsymbol{\theta})t_{s_{\max}}(Y), \end{aligned} \tag{18}$$

where

$$\begin{aligned} p_1(X,Y,\boldsymbol{\theta})&:=\left(r(X,Y,\boldsymbol{\theta})-1\right)K_{-1}(X)L_{-1}(Y), \\ p_2(X,Y,\boldsymbol{\theta})&:=\left(X-1\right)\left(r(X,Y,\boldsymbol{\theta})g(X,Y,\boldsymbol{\theta})-r(\omega_{m_I}^{-1}X,Y,\boldsymbol{\theta})f(X,Y,\boldsymbol{\theta})\right), \\ p_3(X,Y,\boldsymbol{\theta})&:=K_0(X)\left(r(X,Y,\boldsymbol{\theta})g(X,Y,\boldsymbol{\theta})-r(\omega_{m_I}^{-1}X,\omega_{s_{\max}}^{-1}Y,\boldsymbol{\theta})f(X,Y,\boldsymbol{\theta})\right). \end{aligned} \tag{19}$$

3) *Integrating all the constraints*

We finally define the constraint system as a relation generator $\mathcal{R}$. The relation generator takes as input a security parameter λ, a subcircuit library $\mathcal{L}\subset\mathbb{F}[X]$, and permutation polynomials $s^{(0)},s^{(1)},s^{(2)}\in\mathbb{F}[X,Y]$ and generates a polynomial-time decidable binary relation R, which is a set of pairs of instance and witness $(\mathbf{a},(\mathbf{b}(Y),\mathbf{c}(Y)))$, where

$$\mathbf{a} := (d_0^{(0)}, \cdots, d_{l_{in}-1}^{(0)}, d_{l_{in}}^{(s_{\max}-1)}, \cdots, d_{l-1}^{(s_{\max}-1)}),$$
$$\mathbf{b}(Y) := (b_0(Y), \cdots, b_{l_D-l-1}(Y)),$$
$$\mathbf{c}(Y) := (c_0(Y), \cdots, c_{m_D-l_D-1}(Y)),$$

such that (12) and (18) hold. In other words, $R \subseteq \mathbb{F}^l \times \mathbb{F}^{m_D-l}[Y]$ is constructed as

$$R = \left\{ \left(\mathbf{a}, \left(\mathbf{b}(Y), \mathbf{c}(Y)\right)\right) \middle| \begin{bmatrix} \forall x \in \{\omega_n^i\}_{i=0}^{n-1}, \forall y \in \{\omega_{s_{\max}}^i\}_{i=0}^{s_{\max}-1}, \forall z \in \{\omega_{m_I}^i\}_{i=0}^{m_I-1}, \\ p_0(x,y)=0,\, p_i(z,y)=0 \text{ for } i=1,2,3 \end{bmatrix} \right\}. \tag{20}$$

3.3 Setup of subcircuit library

Our back-end protocol that will be defined in the next section relies on a probabilistic algorithm `Setup` for R that produces an encoded reference string $\boldsymbol{\sigma}$ of the library subcircuit polynomials in $\mathcal{L}$. Parties of the back-end protocol will be enforced to use $\boldsymbol{\sigma}$, by which a prover can compress a claim statement $(\mathbf{a}, (\mathbf{b}(Y), \mathbf{c}(Y))$ for R into proof of a small size. A verifier can be convinced by $\boldsymbol{\sigma}$ that the counterparty is disputing a circuit derived from the same library $\mathcal{L}$. Also, randomness in $\boldsymbol{\sigma}$ keeps $(\mathbf{a}, (\mathbf{b}(Y), \mathbf{c}(Y))$ extractable from the compression.

However, `Setup` may not include permutation polynomials $\{s^{(0)}, s^{(1)}, s^{(2)}\}$, when leaving the parties to commit to them by themselves grants universality to the back-end protocol. In special cases where universality is guaranteed even if the permutation polynomials are fixed, we can consider appending them to the reference string. Section 6 will illustrate one of these cases, verifiable machine computation.

In addition to the previously defined Lagrange bases $\{K_i\}_{i=0}^{m_I-1}$ and $\{L_i\}_{i=0}^{s_{\max}}$, we define another Lagrange bases $\{M_i\}_{i=0}^{l-1} \subset \mathbb{F}[X]$ such that $M_i(\omega_n^i) = 1$ and $M_k(\omega_n^i) = 0$ for every $i \neq k$.

`Setup`$(pp_\lambda, \mathcal{L})$ takes as input the bilinear pairing group pp_λ and the subcircuit library $\mathcal{L} = \{u_j(X), v_j(X), w_j(X)\}_{j=0}^{m_D-1}$, randomly picks trapdoors

$$\boldsymbol{\tau} := \left(\alpha, \gamma, \delta, \eta, x, y\right) \in (\mathbb{F}^*)^6, \tag{21}$$

computes, for every $j \in \{0, \cdots, m_D - 1\}$,

$$o_j(X) := \alpha u_j(X) + \alpha^2 v_j(X) + \alpha^3 w_j(X),$$

and returns $\boldsymbol{\sigma} = ([\boldsymbol{\sigma}_{A,C}]_1, [\boldsymbol{\sigma}_B]_1, [\boldsymbol{\sigma}_V]_2)$, where

$$\boldsymbol{\sigma}_{A,C} := \left(\left\{ x^h y^i \right\}_{h=0,i=0}^{\max(2n-2, 3m_I-3), 2s_{\max}-2} \right),$$

$$\boldsymbol{\sigma}_B := \begin{pmatrix} \delta, \eta, \\ \left\{\gamma^{-1}\left(L_0(y)o_j(x)+M_j(x)\right)\right\}_{j=0}^{l_{in}-1}, \left\{\gamma^{-1}\left(L_{-1}(y)o_j(x)+M_j(x)\right)\right\}_{j=l_{in}}^{l-1}, \\ \left\{\eta^{-1}L_i(y)\left(o_{j+l}(x)+\alpha^4 K_j(x)\right)\right\}_{i=0,j=0}^{s_{\max}-1,m_I-1}, \left\{\delta^{-1}L_i(y)o_j(x)\right\}_{i=0,j=l+m_I}^{s_{\max}-1,m_D-1}, \\ \left\{\delta^{-1}\alpha^k x^h t_n(x)\right\}_{h=0,k=1}^{2,3}, \left\{\delta^{-1}\alpha^4 x^j t_{m_I}(x)\right\}_{j=0}^{1}, \left\{\delta^{-1}\alpha^k y^i t_{s_{\max}}(y)\right\}_{i=0,k=1}^{2,4} \end{pmatrix},$$

$$\boldsymbol{\sigma}_V := \left(\alpha, \alpha^2, \alpha^3, \alpha^4, \gamma, \delta, \eta, x, y\right).$$

4 Back-end Protocol: A SNARK for R

We construct an interactive protocol $\mathrm{IP}_{\mathcal{P},\mathcal{V}}$ for the relation R, as shown in Figure 5. The protocol consists of a tuple of prover algorithms

$$\mathcal{P} := (\texttt{Prove}_0, \texttt{Prove}_1, \texttt{Prove}_2, \texttt{Prove}_3, \texttt{Prove}_4)$$

and a tuple of verifier algorithms

$$\mathcal{V} := (\texttt{Verify}_0, \texttt{Verify}_1, \texttt{Verify}_2, \texttt{Verify}_3, \texttt{Verify}_4).$$

We sometimes write $\mathrm{IP}_{\mathcal{P},\mathcal{V}} = \langle \mathcal{P}, \mathcal{V} \rangle$. We assume that $\mathcal{V}$ is given preprocessed commitments $[s^{(i)}(x,y)]_1$ for $i \in \{0,1,2\}$ to the permutation polynomials $s^{(i)}(X,Y)$ in (14). We write $\mathbf{s}(X,Y) = (s^{(i)}(X,Y))_{i=0}^{2}$. Let $(\mathbf{a}, (\mathbf{b}(Y), \mathbf{c}(Y)))$ denote a claimed pair of an instance and a witness. We write a concatenation $\mathbf{d}(Y) := \mathbf{a}^{(in)}(Y) \,|\, \mathbf{a}^{(out)}(Y) \,|\, \mathbf{b}(Y) \,|\, \mathbf{c}(Y)$, where $a_i^{(in)}(Y) = a_i L_0(Y)$ for $i = 0, \cdots, l_{in}-1$ and $a_j^{(out)}(Y) = a_{j+l_{in}} L_{-1}(Y)$ for $j = 0, \cdots, l_{out}-1$.

It is assumed that $\mathcal{P}$ has access to the subcircuit library $\mathcal{L}$, the permutation polynomials $s^{(i)}(X,Y)$ for $i \in \{0,1,2\}$, the reference string $\boldsymbol{\sigma}$, and the claimed pair $\mathbf{d}(Y)$, and that each algorithm in $\mathcal{P}$ shares interim calculations, as well as random coin tosses, with all other algorithms in $\mathcal{P}$. Also, it is assumed that $\mathcal{V}$ can only access the reference string $\boldsymbol{\sigma}$, the preprocessed commitments $[s^{(i)}(x,y)]_1$ for $i \in \{0,1,2\}$, and the public input $\mathbf{a}$.

$\texttt{Prove}_0(\ )$ is a probabilistic algorithm that returns

$$\left([U]_1, [V]_1, [W]_1, [O_{mid}]_1, [O_{prv}]_1, [B]_1, [Q_{A,X}]_1, [Q_{C,X}]_1\right) \in \mathbb{G}_1^8.$$

$\texttt{Verify}_0([U]_1, [V]_1, [W]_1, [O_{mid}]_1, [O_{prv}]_1, [B]_1, [Q_{A,X}]_1, [Q_{C,X}]_1)$ is a probabilistic algorithm that takes 8 elements in $\mathbb{G}_1$ as input and returns randomly picked challenges $\boldsymbol{\theta} := (\theta_0, \theta_1, \theta_2) \in (\mathbb{F}^*)^3$.

$\texttt{Prove}_1(\boldsymbol{\theta})$ is a probabilistic algorithm that takes challenges $\boldsymbol{\theta} \in (\mathbb{F}^*)^3$ as input and returns $[R]_1 \in \mathbb{G}_1$.

$\texttt{Verify}_1([R]_1)$ is a probabilistic algorithm that takes one element in $\mathbb{G}_1$ as input and returns a randomly picked challenge $\kappa_0 \in \mathbb{F}^*$.

$\texttt{Prove}_2(\kappa_0)$ is a deterministic algorithm that takes a challenge $\kappa_0 \in \mathbb{F}^*$ as input and returns

$([Q_{C,X}]_1,[Q_{C,Y}]_1)\in\mathbb{G}_1^2$.

$\texttt{Verify}_2([Q_{C,X}]_1,[Q_{C,Y}]_1)$ is a probabilistic algorithm that takes 2 elements in $\mathbb{G}_1$ as input and returns randomly picked challenges $(\chi,\zeta)\in(\mathbb{F}^*)^2$.

$\texttt{Prove}_3(\chi,\zeta)$ is a deterministic algorithm that takes challenges $(\chi,\zeta)\in(\mathbb{F}^*)^2$ as input and returns $(V_{x,y},R_{x,y},R'_{x,y},R''_{x,y})\in\mathbb{F}^4$.

$\texttt{Verify}_3(V_{x,y},R_{x,y},R'_{x,y},R''_{x,y})$ is a probabilistic algorithm that takes as input four elements in $\mathbb{F}$ and returns a randomly picked challenge $\kappa_1\in\mathbb{F}^*$.

$\texttt{Prove}_4(\kappa_1)$ is a deterministic algorithm that takes as an input challenge $\kappa_1\in\mathbb{F}^*$ and returns $([\Pi_\chi]_1,[\Pi_\zeta]_1,[M_\chi]_1,[M_\zeta]_1,[N_\chi]_1,[N_\zeta]_1)\in\mathbb{G}_1^6$.

We write a transcript as

$$\mathbf{t}:=\begin{pmatrix}[U]_1,[V]_1,[W]_1,[O_{mid}]_1,[O_{prv}]_1,[B]_1,[Q_{A,X}]_1,[Q_{A,Y}]_1,\theta_0,\theta_1,\theta_2,\\ [R]_1,\kappa_0,\\ [Q_{C,X}]_1,[Q_{C,Y}]_1,\chi,\zeta,\\ V_{x,y},R_{x,y},R'_{x,y},R''_{x,y},\kappa_1,\\ [\Pi_\chi]_1,[\Pi_\zeta]_1,[M_\chi]_1,[M_\zeta]_1,[N_\chi]_1,[N_\zeta]_1\end{pmatrix}. \tag{22}$$

$\texttt{Verify}_4(pp_\lambda,\mathbf{t})$ is a probabilistic algorithm that takes as input a bilinear group pp_λ and the transcript $\mathbf{t}$ and returns true or false.

The protocol $\texttt{IP}_{\mathcal{P},\mathcal{V}}$ can be described by dividing it into three arguments: the arithmetic constraint argument, the copy constraint argument, and the polynomial binding argument. The arithmetic constraint argument addresses the arithmetic constraints, while the copy constraint argument addresses the copy constraints. The polynomial binding argument connects these two by ensuring that the witnesses used in both arguments are identical. We explain each argument and then present the integrated protocol $\texttt{IP}_{\mathcal{P},\mathcal{V}}$.

	Protocol $\mathrm{IP}_{\mathcal{P},\mathcal{V}}$ for R	
	$\mathcal{P}$	$\mathcal{V}$
1:	$\mathrm{Prove}_0(\ ) \mapsto \begin{pmatrix} [U]_1,[V]_1,[W]_1,[O_{mid}]_1,[O_{prv}]_1, \\ [Q_{A,X}]_1,[Q_{A,Y}]_1,[B]_1 \end{pmatrix}$	
2:		$\mathrm{Verify}_0\begin{pmatrix} [U]_1,[V]_1,[W]_1,[O_{mid}]_1,[O_{prv}]_1, \\ [Q_{A,X}]_1,[Q_{A,Y}]_1,[B]_1 \end{pmatrix} \mapsto \mathbf{\theta}$
3:	$\mathrm{Prove}_1(\mathbf{\theta}) \mapsto [R]_1$	
4:		$\mathrm{Verify}_1([R]_1) \mapsto \kappa_0$
	$\mathrm{Prove}_2(\kappa_0) \mapsto ([Q_{C,X}]_1,[Q_{C,Y}]_1)$	
6:		$\mathrm{Verify}_2([Q_{C,X}]_1,[Q_{C,Y}]_1) \mapsto (\chi,\zeta)$
7:	$\mathrm{Prove}_3(\chi,\zeta) \mapsto (V_{x,y},R_{x,y},R'_{x,y},R''_{x,y})$	
8:		$\mathrm{Verify}_3(V_{x,y},R_{x,y},R'_{x,y},R''_{x,y}) \mapsto \kappa_1$
9:	$\mathrm{Prove}_4(\kappa_1) \mapsto ([\Pi_\chi]_1,[\Pi_\zeta]_1,[M_\chi]_1,[M_\zeta]_1,[N_\chi]_1,[N_\zeta]_1)$	
10:		$\mathrm{Verify}_4(pp_\lambda,\mathbf{t}) \mapsto 0/1$

Figure 5. An interactive protocol $\mathrm{IP}_{\mathcal{P},\mathcal{V}}$ for R. In the protocol, every output produced by $\mathcal{P}$ or $\mathcal{V}$ is sent immediately to the other party.

4.1 Arithmetic constraint argument

The arithmetic constraint argument is based on Groth16 [4], with modifications for bivariate polynomials and linearization. In this argument, the prover claims instance and witness that satisfies the arithmetic constraints (12).

For this argument, $\langle \mathcal{P},\mathcal{V} \rangle$ produces a transcript as follows,

$$\left([U]_1,[V]_1,[W]_1,[Q_{A,X}]_1,[Q_{A,Y}]_1,\chi,\zeta,V_{x,y},\kappa_1,[\Pi_{A,\chi}]_1,[\Pi_{A,\zeta}]_1\right).$$

We explain how each component in the transcript is computed.

`Prove`$_0$ randomly picks $(r_{U_X},r_{U_Y},r_{V_X},r_{V_Y}) \in (\mathbb{F}^*)^4$, generate two random polynomials $r_{W_X} \in \mathbb{F}[X]$ and $r_{W_Y} \in \mathbb{F}[Y]$ with nonzero coefficients, and computes the quotient polynomials q_0 and q_1 as defined in (12). It also computes

$$\begin{aligned} U(X,Y) &:= \sum_{j=0}^{m_D-1} d_j(Y)u_j(X) + r_{U_X}t_n(X) + r_{U_Y}t_{s_{\max}}(Y), \\ V(X,Y) &:= \sum_{j=0}^{m_D-1} d_j(Y)v_j(X) + r_{V_X}t_n(X) + r_{V_Y}t_{s_{\max}}(Y), \\ W(X,Y) &:= \sum_{j=0}^{m_D-1} d_j(Y)w_j(X) + r_{W_X}(X)t_n(X) + r_{W_Y}(Y)t_{s_{\max}}(Y). \end{aligned} \tag{23}$$

It then returns

$$[U]_1 := [U(x,y)]_1,\quad [V]_1 := [V(x,y)]_1,\quad [W]_1 := [W(x,y)]_1, \tag{24}$$

$$
\begin{aligned}
[Q_{A,X}]_1 &:= [q_0(x,y)]_1 + r_{U_X}\left[v(x,y)\right]_1 + r_{V_X}\left[u(x,y)\right]_1 - [r_{W_X}(x)]_1 \\
&\quad + r_{U_X} r_{V_X}[t_n(x)]_1 + r_{U_Y} r_{V_X}[t_{s_{\max}}(y)]_1, \\
[Q_{A,Y}]_1 &:= [q_1(x,y)]_1 + r_{U_Y}\left[v(x,y)\right]_1 + r_{V_Y}\left[u(x,y)\right]_1 - [r_{W_Y}(y)]_1 \\
&\quad + r_{U_X} r_{V_Y}[t_n(x)]_1 + r_{U_Y} r_{V_Y}[t_{s_{\max}}(y)]_1.
\end{aligned} \tag{25}
$$

$\texttt{Prove}_1$ and $\texttt{Prove}_2$ have no role in this argument.

$\texttt{Prove}_3$, given the challenges χ and ζ, returns

$$
V_{x,y} := V(\chi,\zeta). \tag{26}
$$

$\texttt{Prove}_4$ first computes the main quotient polynomials $\pi_{A,\chi} \in \mathbb{F}[X]$ and $\pi_{A,\zeta} \in \mathbb{F}[X,Y]$ for the arithmetic constraint such that

$$
\begin{pmatrix} u(X,Y)v(\chi,\zeta) - w(X,Y) \\ -q_0(X,Y)t_{\mathcal{X}}(\chi) - q_1(X,Y)t_{\mathcal{Y}}(\zeta) \end{pmatrix} = \pi_{A,\chi}(X)\left(X-\chi\right) + \pi_{A,\zeta}(X,Y)\left(Y-\zeta\right). \tag{27}
$$

It then computes quotient polynomials $\pi_{KZG,\chi} \in \mathbb{F}[X]$ and $\pi_{KZG,\zeta} \in \mathbb{F}[X,Y]$ such that

$$
V(X,Y) - V(\chi,\zeta) = \pi_{KZG,\chi}(X)\left(X-\chi\right) + \pi_{KZG,\zeta}(X,Y)\left(Y-\zeta\right). \tag{28}
$$

The last quotient polynomials that $\texttt{Prove}_4$ computes for this argument are utility quotient polynomials $\pi_{ZK,\chi} \in \mathbb{F}[X]$ and $\pi_{ZK,\zeta} \in \mathbb{F}[X,Y]$, which cancel out the randomized cross terms from the products of (23), such that

$$
\begin{pmatrix} V(\chi,\zeta)(r_{U_X} t_n(X) + r_{U_Y} t_{s_{\max}}(Y)) \\ -V(X,Y)(r_{U_X} t_n(\chi) + r_{U_Y} t_{s_{\max}}(\zeta)) \\ + r_{W_X}(X)\left(t_n(\chi) - t_n(X)\right) \\ + r_{W_Y}(Y)\left(t_{s_{\max}}(\zeta) - t_{s_{\max}}(Y)\right) \end{pmatrix} = \pi_{ZK,\chi}(X)(X-\chi) + \pi_{ZK,\zeta}(X,Y)(Y-\zeta). \tag{29}
$$

It finally returns, given the challenge κ_1,

$$
\begin{aligned}
[\Pi_{A,\chi}]_1 &:= \left[\pi_{A,\chi}(x) + \pi_{ZK,\chi}(x) + \kappa_1 \pi_{KZG,\chi}(x)\right]_1, \\
[\Pi_{A,\zeta}]_1 &:= \left[\pi_{A,\zeta}(x,y) + \pi_{ZK,\zeta}(x,y) + \kappa_1 \pi_{KZG,\zeta}(x,y)\right]_1.
\end{aligned} \tag{30}
$$

The verifier algorithm $\texttt{Verify}_4$ computes

$$
[\mathrm{LHS}_A]_1 := \begin{pmatrix} V_{x,y}[U]_1 - [W]_1 + \kappa_1\left([V]_1 - V_{x,y}[1]_1\right) \\ -t_n(\chi)[Q_{A,X}]_1 - t_{s_{\max}}(\zeta)[Q_{A,Y}]_1 \end{pmatrix}, \tag{31}
$$

$$
[\mathrm{AUX}_A]_1 := \chi[\Pi_{A,\chi}]_1 + \zeta[\Pi_{A,\zeta}]_1, \tag{32}
$$

and accepts the transcript, given the public input $\mathbf{a}(Y)$, only if the following equation holds:

$$e([\mathrm{LHS}_A]_1+[\mathrm{AUX}_A]_1,[1]_2)=e([\Pi_{A,\chi}]_1,[x]_2)e([\Pi_{A,\zeta}]_1,[y]_2). \tag{33}$$

4.2 Copy constraint argument

The copy constraint argument is based on the permutation argument in [33] that was also used in Sonic [5] and PlonK [7]. We modify the permutation argument to work with bivariate polynomials. In the argument, the prover claims a polynomial $b(X,Y):=\sum_{j=0}^{l_D-l-1} b_j(Y)K_j(X)$ that satisfies the copy constraints in Lemma 3.

For this argument, $\langle \mathcal{P},\mathcal{V}\rangle$ produces a transcript as follows,

$$\begin{pmatrix} [B]_1,\theta_0,\theta_1,\theta_2,[R]_1,\kappa_0,[Q_{C,X}]_1,[Q_{C,Y}]_1,\chi,\zeta,R_{x,y},R'_{x,y},R''_{x,y}, \\ \kappa_1,[\Pi_{C,\chi}]_1,[\Pi_{C,\zeta}]_1,[M_\chi]_1,[M_\zeta]_1,[N_\chi]_1,[N_\zeta]_1 \end{pmatrix}.$$

We explain how each transcript component is computed in sequence.

For this argument, $\texttt{Prove}_0$ generates two random polynomials $r_{B_X}\in\mathbb{F}[X]$ and $r_{B_Y}\in\mathbb{F}[Y]$ with nonzero coefficients, computes

$$B(X,Y):=b(X,Y)+r_{B_X}(X)t_{m_I}(X)+r_{B_Y}(Y)t_{s_{\max}}(Y), \tag{34}$$

and returns

$$[B]_1:=\left[B(x,y)\right]_1. \tag{35}$$

$\texttt{Prove}_1$ randomly picks $(r_{R_X},r_{R_Y})\in(\mathbb{F}^*)^2$, and given the challenges θ_0, θ_1, and θ_2, it computes polynomials f,g as defined in (15) and the recursion polynomial r as defined in (17). It then computes

$$R(X,Y):=r(X,Y)+r_{R_X}t_{m_I}(X)+r_{R_Y}t_{s_{\max}}(Y) \tag{36}$$

and returns

$$[R]_1:=\left[R(x,y)\right]_1. \tag{37}$$

$\texttt{Prove}_2$, given the challenge κ_0, computes

$$\begin{aligned} r_{D_1}(X,Y)&:=r(X,Y)-r(\omega_{m_I}^{-1}X,Y),\\ r_{D_2}(X,Y)&:=r(X,Y)-r(\omega_{m_I}^{-1}X,\omega_{s_{\max}}^{-1}Y),\\ g_D(X,Y)&:=g(X,Y)-f(X,Y). \end{aligned} \tag{38}$$

It then computes the quotient polynomials $q_2,\cdots,q_7$ as defined in (18). It and returns

$$\begin{aligned} [Q_{C,X}]_1 &:= [q_2(x,y)]_1+\kappa_0[q_4(x,y)]_1+\kappa_0^2[q_6(x,y)]_1+r_{R_X}\left[K_{-1}(x)L_{-1}(y)\right]_1\\ &+\kappa_0\left(\left[r_{B_X}(x)(x-1)r_{D_1}(x,y)\right]_1+r_{R_X}\left[(x-1)g_D(x,y)\right]_1\right)\\ &+\kappa_0^2\left(\left[r_{B_X}(x)K_0(x)r_{D_2}(x,y)\right]_1+r_{R_X}\left[K_0(x)g_D(x,y)\right]_1\right) \end{aligned} \tag{39}$$

and

$$\begin{aligned}[Q_{C,Y}]_1 &:= [q_3(x,y)]_1 + \kappa_0[q_5(x,y)]_1 + \kappa_0^2[q_7(x,y)]_1 + r_{R_Y}\left[K_{-1}(x)L_{-1}(y)\right]_1 \\ &+\kappa_0\left(\left[r_{B_Y}(y)(x-1)r_{D_1}(x,y)\right]_1 + r_{R_Y}\left[(x-1)g_D(x,y)\right]_1\right) \\ &+\kappa_0^2\left(\left[r_{B_Y}(y)K_0(x)r_{D_2}(x,y)\right]_1 + r_{R_Y}\left[K_0(x)g_D(x,y)\right]_1\right).\end{aligned} \tag{40}$$

$\texttt{Prove}_3$, given the challenges χ and ζ, returns

$$R_{x,y} := R(\chi,\zeta),\ R'_{x,y} := R(\omega_{m_I}^{-1}\chi,\zeta),\ R''_{x,y} := R(\omega_{m_I}^{-1}\chi,\omega_{s_{\max}}^{-1}\zeta). \tag{41}$$

$\texttt{Prove}_4$ first computes quotient polynomials $\pi_{C,\chi} \in \mathbb{F}[X]$ and $\pi_{C,\zeta} \in \mathbb{F}[X,Y]$ for the copy constraint such that

$$p_C(X,Y) = \pi_{C,\chi}(X)(X-\chi) + \pi_{C,\zeta}(X,Y)(Y-\zeta), \tag{42}$$

where

$$\begin{aligned}p_C(X,Y) &:= \left(r(\chi,\zeta)-1\right)K_{-1}(X)L_{-1}(Y) \\ &+\kappa_0(\chi-1)\left(r(\chi,\zeta)g(X,Y) - r(\omega_{m_I}^{-1}\chi,\zeta)f(X,Y)\right) \\ &+\kappa_0^2K_0(\chi)\left(r(\chi,\zeta)g(X,Y) - r(\omega_{m_I}^{-1}\chi,\omega_{s_{\max}}^{-1}\zeta)f(X,Y)\right) \\ &-\left(q_2(X,Y)+\kappa_0q_4(X,Y)+\kappa_0^2q_6(X,Y)\right)t_{m_I}(\chi) \\ &-\left(q_3(X,Y)+\kappa_0q_5(X,Y)+\kappa_0^2q_7(X,Y)\right)t_{s_{\max}}(\zeta).\end{aligned} \tag{43}$$

It then computes quotient polynomials $\pi_{1,\chi},\pi_{2,\chi},\pi_{3,\chi} \in \mathbb{F}[X]$ and $\pi_{1,\zeta},\pi_{2,\zeta},\pi_{3,\zeta} \in \mathbb{F}[X,Y]$ for KZG openings such that

$$\begin{aligned}R(X,Y) - R_{x,y} &= \pi_{1,\chi}(X)(X-\chi) + \pi_{1,\zeta}(X,Y)(Y-\zeta), \\ R(X,Y) - R'_{x,y} &= \pi_{2,\chi}(X)(X-\omega_{m_I}^{-1}\chi) + \pi_{2,\zeta}(X,Y)(Y-\zeta), \\ R(X,Y) - R''_{x,y} &= \pi_{3,\chi}(X)(X-\omega_{m_I}^{-1}\chi) + \pi_{3,\zeta}(X,Y)(Y-\omega_{s_{\max}}^{-1}\zeta).\end{aligned} \tag{44}$$

The last quotient polynomials that $\texttt{Prove}_4$ computes for this argument are utility quotient polynomials $\pi_{i,\chi} \in \mathbb{F}[X]$ and $\pi_{i,\zeta} \in \mathbb{F}[X,Y]$ for $i \in \{4,5\}$, which cancel out the randomized cross terms from the products of (34) and (36), such that

$$\begin{aligned}&\begin{pmatrix}(\chi-1)r_{D_1}(\chi,\zeta)\left(r_{B_X}(X)t_{m_I}(X) + r_{B_Y}(Y)t_{s_{\max}}(Y)\right) \\ -(X-1)r_{D_1}(X,Y)\left(r_{B_X}(X)t_{m_I}(\chi) + r_{B_Y}(Y)t_{s_{\max}}(\zeta)\right) \\ +\left(r_{R_X}t_{m_I}(\chi) + r_{R_Y}t_{s_{\max}}(\zeta)\right)g_D(X,Y)(\chi - X)\end{pmatrix} = \pi_{4,\chi}(X)(X-\chi) + \pi_{4,\zeta}(X,Y)(Y-\zeta), \\ &\begin{pmatrix}K_0(\chi)r_{D_2}(\chi,\zeta)\left(r_{B_X}(X)t_{m_I}(X) + r_{B_Y}(Y)t_{s_{\max}}(Y)\right) \\ -K_0(X)r_{D_2}(X,Y)\left(r_{B_X}(X)t_{m_I}(\chi) + r_{B_Y}(Y)t_{s_{\max}}(\zeta)\right) \\ +\left(r_{R_X}t_{m_I}(\chi) + r_{R_Y}t_{s_{\max}}(\zeta)\right)g_D(X,Y)\left(K_0(\chi) - K_0(X)\right)\end{pmatrix} = \pi_{5,\chi}(X)(X-\chi) + \pi_{5,\zeta}(X,Y)(Y-\zeta).\end{aligned} \tag{45}$$

Given the challenge κ_0, we define aggregated quotient polynomials as

$$\begin{aligned}\pi_{ZK,\chi}(X) &:= \kappa_0 \pi_{4,\chi}(X) + \kappa_0^2 \pi_{5,\chi}(X),\\ \pi_{ZK,\zeta}(X,Y) &:= \kappa_0 \pi_{4,\zeta}(X,Y) + \kappa_0^2 \pi_{5,\zeta}(X,Y).\end{aligned} \tag{46}$$

It finally returns, given the challenge κ_1,

$$\begin{aligned}
[\Pi_{C,\chi}]_1 &:= \left[\kappa_1^2\left(\pi_{C,\chi}(x) + \pi_{ZK,\chi}(x)\right) + \kappa_1^3 \pi_{1,\chi}(x)\right]_1,\\
[\Pi_{C,\zeta}]_1 &:= \left[\kappa_1^2\left(\pi_{C,\zeta}(x,y) + \pi_{ZK,\zeta}(x,y)\right) + \kappa_1^3 \pi_{1,\zeta}(x,y)\right]_1,\\
[M_\chi]_1 &:= \left[\pi_{2,\chi}(x)\right]_1,\\
[M_\zeta] &:= [\pi_{2,\zeta}(x,y)]_1,\\
[N_\chi]_1 &:= \left[\pi_{3,\chi}(x)\right]_1,\\
[N_\zeta]_1 &:= \left[\pi_{3,\zeta}(x,y)\right]_1.
\end{aligned} \tag{47}$$

The verifier algorithm Verify_4 picks a random value $\kappa_2 \in \mathbb{F}^*$, computes

$$\begin{aligned}
[F]_1 &:= [B]_1 + \theta_0[s^{(0)}(x,y)]_1 + \theta_1[s^{(1)}(x,y)]_1 + \theta_2[1]_1,\\
[G]_1 &:= [B]_1 + \theta_0[s^{(2)}(x,y)]_1 + \theta_1[y]_1 + \theta_2[1]_1,
\end{aligned} \tag{48}$$

$$[\mathrm{LHS}_C]_1 := \left(\begin{aligned}
&\kappa_1^2 \left(\begin{aligned}
&\left(R_{x,y} - 1\right)\left[K_{-1}(x)L_{-1}(y)\right]_1\\
&+\kappa_0\left(\chi - 1\right)\left(R_{x,y}[G]_1 - R'_{x,y}[F]_1\right)\\
&+\kappa_0^2 K_0(\chi)\left(R_{x,y}[G]_1 - R''_{x,y}[F]_1\right)\\
&-t_{m_I}(\chi)[Q_{C,X}]_1 - t_{s_{\max}}(\zeta)[Q_{C,Y}]_1
\end{aligned}\right)\\
&+\kappa_1^3\left([R]_1 - R_{x,y}[1]_1\right) + \kappa_2\left([R]_1 - R'_{x,y}[1]_1\right) + \kappa_2^2\left([R]_1 - R''_{x,y}[1]_1\right)
\end{aligned}\right), \tag{49}$$

and

$$[\mathrm{AUX}_C]_1 := \left(\begin{aligned}
&\chi[\Pi_{C,\chi}]_1 + \kappa_2 \omega_{m_I}^{-1}\chi[M_\chi]_1 + \kappa_2^2 \omega_{m_I}^{-1}\chi[N_\chi]_1\\
&+\zeta[\Pi_{C,\zeta}]_1 + \kappa_2 \zeta[M_\zeta]_1 + \kappa_2^2 \omega_{s_{\max}}^{-1}\zeta[N_\zeta]_1
\end{aligned}\right), \tag{50}$$

and accepts the transcript, only if the following equation holds:

$$e\left([\mathrm{LHS}_C]_1 + [\mathrm{AUX}_C]_1, [1]_2\right) = \left(\begin{aligned}
&e\left([\Pi_{C,\chi}]_1 + \kappa_2[M_\chi]_1 + \kappa_2^2[N_\chi]_1, [x]_2\right)\\
&\cdot e\left([\Pi_{C,\zeta}]_1 + \kappa_2[M_\zeta]_1 + \kappa_2^2[N_\zeta]_1, [y]_2\right)
\end{aligned}\right). \tag{51}$$

4.3 Polynomial binding argument

We construct the *polynomial binding argument* that forces the prover $\mathcal{P}$ to use predefined polynomials when constructing the arithmetic and copy constraint arguments. This convinces the verifier $\mathcal{V}$ of three statements: 1) the public input $\mathbf{a}^{(in)}(Y)$, $\mathbf{a}^{(out)}(Y)$ is included in the arithmetic argument and is never cancelled with malicious intent; 2) the construction of the arithmetic argument is based on linear combinations of the predefined circuit polynomials in $\mathcal{L}$; 3) the arithmetic and copy constraint arguments share the same witness $\mathbf{b}(Y)$.

For this argument, $\langle \mathcal{P}, \mathcal{V} \rangle$ produces a transcript as follows,

$$\left([U]_1,[V]_1,[W]_1,[O_{mid}]_1,[O_{prv}]_1,[B]_1,\chi,\zeta,\kappa_1,[\Pi_{B,\chi}]_1\right).$$

We explain how each transcript component is computed in sequence.

For this argument, $\texttt{Prove}_0$ randomly picks $(r_{U_X}, r_{U_Y}, r_{V_X}, r_{V_Y}, r_{O_{pub}}, r_{O_{mid}}) \in (\mathbb{F}^*)^6$, generates random polynomials $r_{W_X}, r_{B_X} \in \mathbb{F}[X]$ and $r_{W_Y}, r_{B_Y} \in \mathbb{F}[Y]$ with nonzero coefficients, and computes $[U]_1$, $[V]_1$, $[W]_1$, $[B]_1$, as defined in (24) and (35). It then returns $[O_{mid}]_1$ and $[O_{prv}]_1$, where

$$\begin{aligned}
O_{mid} &:= \sum_{j=l}^{l_D-1}\left[b_{j-l}(y)\eta^{-1}\left(o_j(x)+\alpha^4 K_j(x)\right)\right]_1 + r_{O_{mid}}[\delta]_1,\\
O_{prv} &:= \begin{pmatrix}
\sum_{j=l_D}^{m_D-1}\left[d_j(y)\delta^{-1}o_j(x)\right]_1 - r_{O_{mid}}[\eta]_1\\
+r_{U_X}\left[\delta^{-1}\alpha t_n(x)\right]_1 + r_{V_X}\left[\delta^{-1}\alpha^2 t_n(x)\right]_1 + \left[\delta^{-1}\alpha^3 r_{W_X}(x)t_n(x)\right]_1 + \left[\delta^{-1}\alpha^4 r_{B_X}(x)t_{m_I}(x)\right]_1\\
+r_{U_Y}\left[\delta^{-1}\alpha t_{s_{\max}}(y)\right]_1 + r_{V_Y}\left[\delta^{-1}\alpha^2 t_{s_{\max}}(y)\right]_1 + \left[\delta^{-1}\alpha^3 r_{W_Y}(y)t_{s_{\max}}(y)\right]_1 + \left[\delta^{-1}\alpha^4 r_{B_Y}(y)t_{s_{\max}}(y)\right]_1
\end{pmatrix}.
\end{aligned} \tag{52}$$

$\texttt{Prove}_4$, given the challenge κ_1, computes quotient polynomials $\pi_{B,\chi} \in \mathbb{F}[X]$ for KZG openings such that

$$A(X) - A(\chi) = \pi_{B,\chi}(X)\left(X-\chi\right)$$

and returns

$$[\Pi_{B,\chi}]_1 := [\kappa_1^4 \pi_{B,\chi}(x)]_1. \tag{53}$$

The verifier algorithm $\texttt{Verify}_4$, given instance $\mathbf{a}$, picks a random value $\kappa_2 \in \mathbb{F}^*$, computes,

$$\begin{aligned}
[O_{pub}]_1 &:= \sum_{j=0}^{l_{in}-1} a_j\left[\gamma^{-1}\left(L_0(y)o_j(x)+M_j(x)\right)\right]_1 + \sum_{j=l_{in}}^{l-1} a_j\left[\gamma^{-1}\left(L_{-1}(y)o_j(x)+M_j(x)\right)\right]_1,\\
[A]_1 &:= [A(x)]_1,\\
A_{pub} &:= A(\chi),
\end{aligned} \tag{54}$$

where $A(X) := \sum_{j=0}^{l-1} a_j M_j(X)$, and

$$[\mathrm{LHS}_B]_1 := \left(1+\kappa_2\kappa_1^4\right)[A]_1 - \kappa_2\kappa_1^4 A_{pub}[1]_1, \tag{55}$$

$$[\mathrm{AUX}_B]_1 := \kappa_2\chi[\Pi_{B,\chi}]_1$$

and accepts the transcript, only if the following equation holds:

$$\begin{pmatrix} e\left([\mathrm{LHS}_B]_1+[\mathrm{AUX}_B]_1,[1]_2\right)e\left([B]_1,[\alpha^4]_2\right) \\ \cdot e\left([U]_1,[\alpha]_2\right)e\left([V]_1,[\alpha^2]_2\right)e\left([W]_1,[\alpha^3]_2\right) \end{pmatrix} = \begin{pmatrix} e\left([O_{pub}]_1,[\gamma]_2\right)e\left([O_{mid}]_1,[\eta]_2\right)e\left([O_{prv}]_1,[\delta]_2\right) \\ \cdot e\left(\kappa_2[\Pi_{B,\chi}]_1,[x]_2\right) \end{pmatrix}. \tag{56}$$

We put an assumption on the subcircuit library polynomials $\{u_j, v_{j,} w_j\}_{j=0}^{m_D-1}$ that will be useful for knowledge soundness of the protocol $\langle \mathcal{P}, \mathcal{V} \rangle$.

4.4 Integrated protocol

We integrate the three arguments constructed above. The transcript $\mathbf{t}$ is given in (22).

$\texttt{Prove}_4$ returns aggregation of the proofs defined in (30), (47), and (53), and returns

$$\begin{aligned} &[\Pi_\chi]_1 := [\Pi_{A,\chi}]_1 + [\Pi_{C,\chi}]_1 + [\Pi_{B,\chi}]_1, \\ &[\Pi_\zeta]_1 := [\Pi_{A,\zeta}]_1 + [\Pi_{C,\zeta}]_1. \end{aligned} \tag{57}$$

Finally, the verifier algorithm $\texttt{Verify}_4$, given instance $\mathbf{a}$, picks a random value $\kappa_2 \in \mathbb{F}^*$, computes

$$\begin{aligned} &[\mathrm{LHS}]_1 := [\mathrm{LHS}_B]_1 + \kappa_2\left([\mathrm{LHS}_A]_1 + [\mathrm{LHS}_C]_1\right), \\ &[\mathrm{AUX}]_1 := \begin{pmatrix} \kappa_2\chi[\Pi_\chi]_1 + \kappa_2\zeta[\Pi_\zeta]_1 \\ +\kappa_2^2\omega_{m_I}^{-1}\chi[M_\chi]_1 + \kappa_2\zeta[M_\zeta]_1 \\ +\kappa_2^3\omega_{m_I}^{-1}\chi[N_\chi]_1 + \kappa_2\omega_{s_{\max}}^{-1}\zeta[N_\zeta]_1 \end{pmatrix}, \end{aligned} \tag{58}$$

where $[\mathrm{LHS}_A]_1$, $[\mathrm{LHS}_C]_1$, and $[\mathrm{LHS}_B]_1$ are defined in (32), (49), and (55), respectively, and accepts $\mathbf{t}$ if and only if the following equation holds:

$$\begin{pmatrix} e\left([\mathrm{LHS}]_1+[\mathrm{AUX}]_1,[1]_2\right)e\left([B]_1,[\alpha^4]_2\right) \\ e\left([U]_1,[\alpha]_2\right)e\left([V]_1,[\alpha^2]_2\right)e\left([W]_1,[\alpha^3]_2\right) \end{pmatrix} = \begin{pmatrix} e\left([O_{pub}]_1,[\gamma]_2\right)e\left([O_{mid}]_1,[\eta]_2\right)e\left([O_{prv}]_1,[\delta]_2\right) \\ \cdot e\left(\kappa_2[\Pi_\chi]_1 + \kappa_2^2[M_\chi]_1 + \kappa_2^3[N_\chi]_1,[x]_2\right) \\ \cdot e\left(\kappa_2[\Pi_\zeta]_1 + \kappa_2^2[M_\zeta]_1 + \kappa_2^3[N_\zeta]_1,[y]_2\right) \end{pmatrix}. \tag{59}$$

Overall efficiency of the protocol. The protocol $\mathrm{IP}_{\mathcal{P},\mathcal{V}}$ consists of 10 rounds of interaction. The prover sends 17 and 4 elements respectively in $\mathbb{G}_1$ and $\mathbb{F}$. The verifier picks 7 challenges from $\mathbb{F}^*$. In total, the prover performs $O(ns_{\max})$ and $O(s_{\max}m_I)$ exponentiations in $\mathbb{G}_1$ and $O((n+m_I)s_{\max}\log ns_{\max}m_I)$ operations in $\mathbb{F}$. The verifier computes $l+18$ exponentiations in $\mathbb{G}_1$ and 10 pairings.

For the computation of $[O_{mid}]_1$ and $[O_{prv}]_1$, as there are many zero coefficients, it requires only $O(ns_{\max})$ exponentiations in $\mathbb{G}_1$ instead of the seemingly given $O(m_D s_{\max})$ exponentiations in $\mathbb{G}_1$. This assessment comes from the fact that there can be at most $s_{\max}$ subcircuits placed in a circuit and each subcircuit has $O(n)$ wires.

In $\mathrm{IP}_{\mathcal{P},\mathcal{V}}$, we specify that $\mathcal{V}$ computes three elements, $[O_{pub}]_1$, $[A]_1$, and A_{pub}, using the public input $\mathbf{a}$. However, computing all three is not mandatory and may be chosen depending on the context. If $\mathcal{V}$ directly computes $[O_{pub}]_1$, it suffices to set $[A]_1$ and A_{pub} as identity elements without compromising security, although this case is omitted in the formal security analysis presented in the next section. Under this strategy, $\mathcal{V}$ performs $O(l)$ exponentiations in $\mathbb{G}_1$. Alternatively, $\mathcal{V}$ may request $\mathcal{P}$ to perform verifiable computations of $[O_{pub}]_1$ and $[A]_1$. Their integrity can be verified by locally computing A_{pub} and substituting it into (56). In this case, $\mathcal{P}$ provides two $\mathbb{G}_1$ elements in addition to $\mathbf{t}$, and $\mathcal{V}$ performs $O(l)$ operations in $\mathbb{F}$ and $O(1)$ exponentiations in $\mathbb{G}_1$.

5 Protocol Security

5.1 Completeness and knowledge-soundness

We define security properties: completeness and knowledge-soundness.

For the completeness, we modify the completeness in Definition 1 to allow statistical imperfection. We will show that the protocol $\langle \mathcal{P}, \mathcal{V} \rangle$ is imperfect in completeness due to the copy constraint argument. Noticeably, it has been discussed in [7] that the copy constraint argument can achieve perfect completeness at the cost of adding $\epsilon(\lambda)$ to knowledge soundness error discussed below, by forcing $\mathcal{V}$ to accept an incomplete transcript immediately whenever a bad challenge $\boldsymbol{\theta}$ has been chosen.

Definition 4 (Statistical completeness). *Given the* $\texttt{Setup}$ *for* R, *an interactive protocol* $\langle \mathcal{P}, \mathcal{V} \rangle$ *with preprocessed input* $\mathbf{z}$ *is statistically complete, if given a valid pair of instance and witness as input to* $\mathcal{P}$, *a transcript produced by the protocol is acceptable by* $\mathcal{V}$, *with high probability in* λ. *In other words, for every* $((\mathbf{a}^{(in)}, \mathbf{a}^{(out)}), (\mathbf{b}(Y), \mathbf{c}(Y))) \in R$, *it holds that*

$$\Pr\left[\begin{array}{l} \boldsymbol{\sigma} \leftarrow \texttt{Setup}(R); \\ \mathbf{t} \leftarrow \left\langle \mathcal{P}\left(\boldsymbol{\sigma}, \mathbf{a}^{(in)}, \mathbf{a}^{(out)}, \mathbf{b}(Y), \mathbf{c}(Y), \mathbf{z}\right), \mathcal{V}\left(\boldsymbol{\sigma}, \mathbf{a}^{(in)}, \mathbf{a}^{(out)}, \mathbf{z}\right)\right\rangle : \\ \mathbf{t} \textit{ is accepted by } \mathcal{V} \end{array}\right] \geq 1 - \epsilon(\lambda)$$

for a failure probability ϵ *negligible in* λ.

For the knowledge soundness, instead of direct extraction of witness in Definition **2**, we utilize *witness-extended emulation*. Informally, a protocol for a relation is said to be knowledge sound if a valid pair of instance and witness that reside in the relation can be extracted from an acceptable transcript with high probability. However, the relation R we consider is a collection of polynomials with a large degree $s_{\max}$. Thus, instead of knowledge of the valid polynomials, $\mathcal{V}$ in the copy constraint argument queries $\mathcal{P}$ for an evaluation of every relevant polynomial at a challenged point that cannot be predicted by $\mathcal{P}$ in advance. Since there is no efficient algorithm to extract a polynomial from a single evaluation, we need a special soundness that extracts a valid witness from two or more distinct acceptable transcripts [2]. In the same context, we follow the definition of witness-extended emulation [5, 44, 46], which is a general framework to define a special soundness for any public coin interactive protocols.

Definition 5 (Witness-extended emulation against affine prover strategy). *Consider a public coin interactive protocol* $\langle \mathcal{P}, \mathcal{V} \rangle$ *with preprocessed input* $\mathbf{z}$, *given the* $\texttt{Setup}$ *for* R. *The protocol satisfies witness-extended emulation, if, for all deterministic polynomial time* $\mathcal{P}^*$, *there exists an expected polynomial-time emulator* $\mathcal{X}$ *such that for all generic adversaries* $\mathcal{A}$ *we have*

$$\Pr\left[\begin{array}{l}\boldsymbol{\sigma} \leftarrow \texttt{Setup}(R);\\ (\mathbf{a},\mathbf{b}(Y),\mathbf{c}(Y)) \leftarrow \mathcal{A}(\boldsymbol{\sigma},\mathbf{z});\\ \mathbf{t} \overset{\$}{\leftarrow} \left\langle \mathcal{P}^*(\boldsymbol{\sigma},\mathbf{a},\mathbf{b}(Y),\mathbf{c}(Y),\mathbf{z}),\mathcal{V}(\boldsymbol{\sigma},\mathbf{a},\mathbf{z})\right\rangle :\\ \mathcal{A}(\mathbf{a},\mathbf{t},\mathbf{z}) = 1\end{array}\right] \approx \Pr\left[\begin{array}{l}\boldsymbol{\sigma} \overset{\$}{\leftarrow} \texttt{Setup}(\mathbf{b},\mathbb{Q}_A);\\ (\mathbf{a},\mathbf{b}(Y),\mathbf{c}(Y)) \leftarrow \mathcal{A}\left(\boldsymbol{\sigma},[s^{(0)}]_1,[s^{(1)}]_1,[s^{(2)}]_1\right);\\ (\mathbf{t},\mathbf{b}(Y),\mathbf{c}(Y)) \overset{\$}{\leftarrow} \mathcal{X}^{\left\langle \mathcal{P}^*(\boldsymbol{\sigma},\mathbf{a},\mathbf{b}(Y),\mathbf{c}(Y),\mathbf{z}),\mathcal{V}(\boldsymbol{\sigma},\mathbf{a},\mathbf{z})\right\rangle} :\\ \mathcal{A}(\mathbf{a},\mathbf{t},\mathbf{z}) = 1 \wedge \begin{pmatrix}\mathbf{t} \textit{ is acceptable} \Rightarrow\\ (\mathbf{a},(\mathbf{b}(Y),\mathbf{c}(Y))) \in R\end{pmatrix}\end{array}\right],$$

where $\mathcal{X}$ has access to repeatedly rewind $\langle \mathcal{P}^*,\mathcal{V}\rangle$ to a particular round for fresh randomness of $\mathcal{V}$ and produce the corresponding transcript.

Assumption 1. *For $u_j(X), v_j(X), w_j(X) \in \mathcal{L}$ and $j \in \{0,\cdots,m_D - 1\}$, define polynomials of indeterminates A and X to represent the subcircuit library wires as*

$$o_j(\mathrm{A}, X) := \mathrm{A}u_j(X) + \mathrm{A}^2 v_j(X) + \mathrm{A}^3 w_j(X).$$

We assume that the subcircuit library polynomials $\{o_j\}_{j=0}^{m_D-1}$ are linearly independent.

Regarding binding polynomials, this assumption eliminates the possibility of an adversary cancelling out $[O_{pub}]_1$ in the verification equation (56) by forging $[O_{mid}]_1$ and $[O_{prv}]_1$. Indeed, a weaker assumption than Assumption 1 has been used in the literature to show that Groth16 achieves *simulation-extractability* in a weak sense [45]. Fortunately, even if real-world circuits or subcircuits do not satisfy this assumption, they can be easily modified by adding dummy constraints to prevent linear dependencies among the wire polynomials, at the cost of increase in the degree of the polynomials in $\mathcal{L}$.

Proposition 1. *Given the* `Setup` *for R, the interactive protocol $\langle \mathcal{P},\mathcal{V}\rangle$ constructed in Section 4 is statistical complete with a failure probability of*

$$\epsilon(\lambda) \le \frac{3 s_{\max} m_I}{|\mathbb{F}^*|}.$$

Proof. See Appendix C.

Proposition 2 (Witness-extended emulation). *Given the* Setup *for R, the interactive protocol $\texttt{IP}_{\mathcal{P},\mathcal{V}} = \langle \mathcal{P},\mathcal{V}\rangle$ constructed in Section 4 satisfies witness-extended emulation with an emulator that extracts $(\mathbf{a},(\mathbf{b}(Y),\mathbf{c}(Y))) \in R$ from an acceptable transcript $\mathbf{t} \leftarrow \texttt{IP}_{\mathcal{A},\mathcal{V}}$* for all generic adversaries $\mathcal{A}$. *Specifically, there exists an emulator with a constant c that extracts a valid witness expectedly in $N = cns_{\max} m_I$ rewinds with probability bounded below by*

$$\frac{N\left(5N + 2(n + s_{\max} + m_I) + 12\right)}{|\mathbb{F}^*|}.$$

Proof. See Appendix D.

We consider GGM, which is a stronger assumption than algebraic group model (AGM). Unlike generic adversaries, an algebraic adversary has direct access to group operations. We do not provide AGM analysis for the sake of simplicity in the proof, but interested readers are referred to [47, 48]. In [47], it has been shown that Groth16, which is the original version of our arithmetic constraint argument, satisfies the knowledge soundness against computationally bounded algebraic adversaries. Also, under AGM, a general framework for online witness-extended emulation for public-coin interactive protocols, where the emulator does not rewind the protocol, has been introduced in [48].

5.2 Zero-knowledge

It would be challenging for an interactive protocol to have the perfect zero-knowledge in Definition 3. Instead, we can consider honest-verifier zero-knowledge [49]. Informally, an interactive protocol that produces a transcript $\mathbf{t}$ is said to have honest-verifier (statistical) zero-knowledge, if there exists a simulator that produces a simulated transcript $\mathbf{t}^\star$ such that the distributions of $\mathbf{t}$ and $\mathbf{t}^\star$ are identical (or statistically indistinguishable) given that the randomized verifier strictly follows the protocol. In [5, 7], the authors exemplify adding honest-verifier zero-knowledge to an interactive protocol. We can apply a similar approach to the protocol $\langle \mathcal{P}, \mathcal{V} \rangle$.

We provide the simulator that interacts with $\mathcal{V}$ to produce $\mathbf{t}^\star$ as

$\mathtt{Sim}_{\langle \mathcal{P}, \mathcal{V} \rangle}\left(pp_\lambda, \boldsymbol{\tau}, \mathbf{s}(Y,Z), \mathbf{a} \right) \mapsto \mathbf{t}^\star$: It picks

$$U^\star, V^\star, W^\star, O^\star_{mid}, B^\star, Q^\star_{A,X}, Q^\star_{A,Y}, R^\star, Q^\star_{C,X}, Q^\star_{C,Y}, V^\star_{x,y}, R^\star_{x,y}, R'^\star_{x,y}, R''^\star_{x,y}, \Pi^\star_\chi, M^\star_\chi, N^\star_\chi \in \mathbb{F}^{17}$$

uniformly at random, takes random challenges $\theta_0, \theta_1, \theta_2, \kappa_0, \chi, \zeta, \kappa_1 \in \mathbb{F}^7$ from $\mathcal{V}$, computes

$$\begin{aligned} A &:= \sum_{j=0}^{l-1} a_j M_j(x), \\ O_{pub} &:= \sum_{j=0}^{l_{in}-1} a_j \left(L_0(y) o_j(x) + M_j(x) \right) + \sum_{j=l_{in}}^{l-1} a_j \left(L_{-1}(y) o_j(x) + M_j(x) \right), \\ A_{pub} &:= \sum_{j=0}^{l-1} a_j M_j(\chi), \end{aligned}$$

$$P_A := U^\star V^\star_{x,y} - W^\star - Q^\star_{A,X} t_n(\chi) - Q^\star_{A,Y} t_{s_{\max}}(\zeta),$$

$$\begin{aligned} F &:= B^\star + \theta_0 s^{(0)}(x,y) + \theta_1 s^{(1)}(x,y) + \theta_2, \\ G &:= B^\star + \theta_0 s^{(2)}(x,y) + \theta_1 y + \theta_2, \\ P_C &:= \begin{pmatrix} \left(R^\star_{x,y} - 1 \right) K_{-1}(x) L_{-1}(y) \\ + \kappa_0 \left(\chi - 1 \right) \left(R^\star_{x,y} G - R'^\star_{x,y} F \right) \\ + \kappa_0^2 K_0(\chi) \left(R^\star_{x,y} G - R''^\star_{x,y} F \right) \end{pmatrix} - Q^\star_{C,X} t_n(\chi) - Q^\star_{C,Y} t_{s_{\max}}(\zeta), \end{aligned}$$

$$
\begin{aligned}
O^\star_{prv} &:= \delta^{-1}\begin{pmatrix} A + U^\star\alpha + V^\star\alpha^2 + W^\star\alpha^3 + B^\star\alpha^4 \\ -\eta O^\star_{mid} - O_{pub} \end{pmatrix},\\
\Pi^\star_\chi &:= (x-\chi)^{-1}\begin{pmatrix} P_A + \kappa_1\left(V^\star - V^\star_{x,y}\right) + \kappa_1^2 P_C + \kappa_1^3\left(R^\star - R^\star_{y,z}\right) + \kappa_1^4\left(A - A_{pub}\right) \\ -(y-\zeta)\Pi^\star_\zeta \end{pmatrix},\\
M^\star_\chi &= \left(x - \omega_{m_I}^{-1}\chi\right)^{-1}\left(R^\star - R'^\star_{y,z} - (y-\zeta) M^\star_\zeta\right),\\
N^\star_\chi &= \left(x - \omega_{m_I}^{-1}\chi\right)^{-1}\left(R^\star - R''^\star_{y,z} - \left(y - \omega_{s_{\max}}^{-1}\zeta\right) N^\star_\zeta\right),
\end{aligned}
$$

and returns

$$
\mathbf{t}^\star = \begin{pmatrix}
[U^\star]_1, [V^\star]_1, [W^\star]_1, [O^\star_{mid}]_1, [O^\star_{prv}]_1, [B^\star]_1, [Q^\star_{A,X}]_1, [Q^\star_{A,Y}]_1, \theta_0, \theta_1, \theta_2, \\
[R^\star]_1, \kappa_0, \\
[Q^\star_{C,X}]_1, [Q^\star_{C,Y}]_1, \chi, \zeta, \\
V^\star_{x,y}, R^\star_{y,z}, R'^\star_{y,z}, R''^\star_{y,z}, \kappa_1, \\
[\Pi^\star_\chi]_1, [\Pi^\star_\zeta]_1, [M^\star_\chi]_1, [M^\star_\zeta]_1, [N^\star_\chi]_1, [N^\star_\zeta]_1
\end{pmatrix}.
$$

The simulated transcript $\mathbf{t}^\star$ involves 17 independent random coins from $\mathbb{F}^*$, and the real transcript $\mathbf{t}$ generated by $\langle \mathcal{P}, \mathcal{V} \rangle$ involves 7 random coins $(r_{U_X}, r_{U_Y}, r_{V_X}, r_{V_Y}, r_{R_Y}, r_{R_Z}, r_{O_{mid}}) \in (\mathbb{F}^*)^7$ and four random polynomials $r_{W_X}, r_{B_X} \in \mathbb{F}[X]$ and $r_{W_Y}, r_{B_Y} \in \mathbb{F}[Y]$ with nonzero coefficients. By adjusting the degrees of the random polynomials, the protocol $\langle \mathcal{P}, \mathcal{V} \rangle$ can generate $\mathbf{t}$ that is statistically indistinguishable from $\mathbf{t}^\star$. For example, we set the degrees of r_{W_X} and r_{W_Y} to 2 and the degrees of r_{B_Y} and r_{B_Z} to 1.

6 Elimination of the verifier preprocess with machine computation

The protocol $\mathtt{IP}_{\mathcal{P},\mathcal{V}}$ for R in Section 4 still requires the verifier $\mathcal{V}$ to preprocess wiring of a circuit, represented by $s^{(0)}, s^{(1)}, s^{(2)} \in \mathbb{F}[X,Y]$. In this section, motivated by verifiable machine computations in [26–28], we eliminate the dependency on the verifier preprocess.

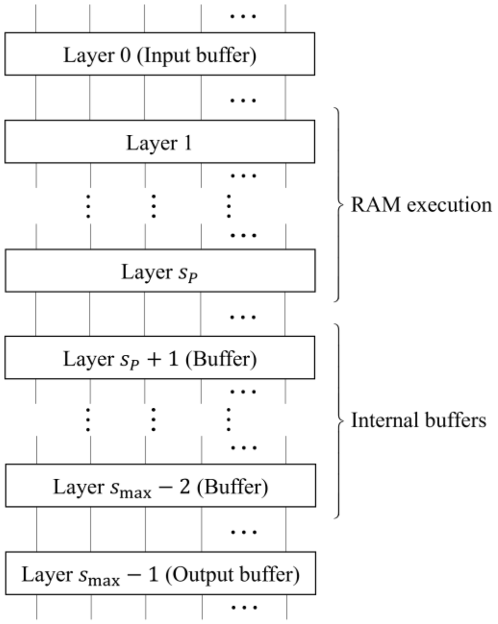


Figure 6. RAM circuit illustration: This circuit comprises $s_{\max}$ layers. The initial and final layers are dedicated to the input and output buffers, respectively. Among the $s_{\max} - 2$ intermediate layers, s_P layers handle RAM execution, with each activation subcircuit is dynamically determined by the input. The remaining layers are activated by internal buffers.

Machine model. We define a subcircuit library $\mathcal{L}$ that is specific to a random-access machine (RAM). Given the number of instructions, s_D, the subcircuit library is defined as $\mathcal{L} := \bigcup_{k=0}^{s_D+2} \mathtt{C}_k$. Each subcircuit $\mathtt{C}_k$ has l_K input wires and l_K outputs wires, where K denotes a length of variables long enough to represent the RAM states.

Out of the $s_D + 3$ subcircuits, subcircuits $\mathtt{C}_0, \mathtt{C}_{s_D+1}, \mathtt{C}_{s_D+2}$ are buffers, which are specialized to pass data to or retrieve data from other circuit components. Specifically, $\mathtt{C}_0$ is an input buffer to transfer the initial machine state $\mathbf{x}$ to other subcircuits, $\mathtt{C}_{s_D+2}$ is an output buffer to return the resulting machine state $\mathbf{y}$, and $\mathtt{C}_{s_D+1}$ is an internal buffer to transfer an intermediate machine state from one subcircuit to another subcircuit.

The rest subcircuits, $\mathtt{C}_i$ for $i = 1, \cdots, s_D$, take as input a machine state and execute the i-th instruction of $\mathcal{M}$. In addition to simply executing each instruction, the subcircuit $\mathtt{C}_i$ for each $i = 1, \cdots, s_D$ checks *1)* whether an input instruction to a subcircuit matches with the subcircuit index, *2)* whether the next program counter is correctly computed, and *3)* whether the next instruction is correctly retrieved from P according to the next program counter.

As a result, a program P with an initial state $\mathbf{x}$, which returns the resulting state $\mathbf{y}$ in s_P machine steps, can be validated by a circuit $\mathcal{C}$ of $s_P + 2$ layers, each comprising $s_D + 3$ subcircuit branches (the additional two layers are for the input and output buffers). However, instead of s_P, we use a constant $s_{\max}$ such that $s_P + 2 \le s_{\max}$ to keep the setup algorithm for $\mathtt{IP}_{P,\mathcal{V}}$ independent of P. Also, we have structured the setup to ensure that the I/O buffers are consistently placed in the first and last layers, respectively (see $\boldsymbol{\sigma}$ in Section 3.3, where $l_{in} = l_{out} = l_K$ in this model). Except for the I/O buffers, in $\mathbf{Q}$, there are s_P layers for the program execution, and the remaining $s_{\max} - s_P - 2$ layers are filled with the internal buffers $\mathtt{C}_{s_D+1}$ (see Figure 6).

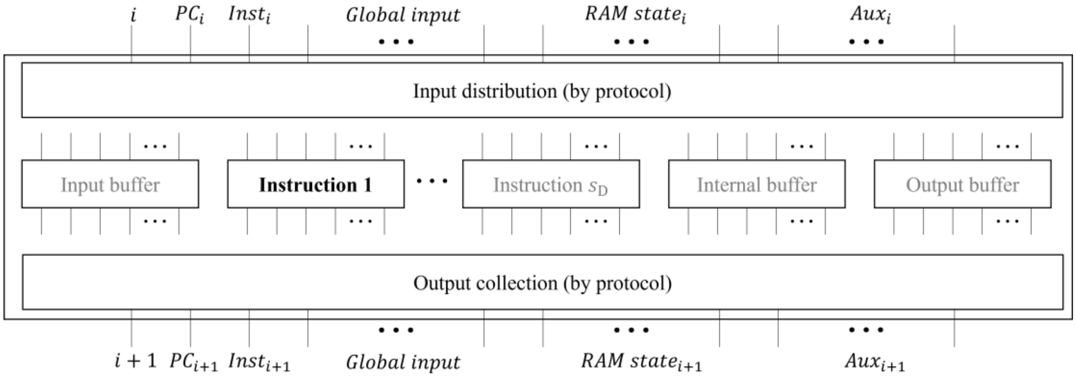


Figure 7. Layer composition example: Each layer comprises s_D+3 subcircuits, with only one activated based on the layer's input. The layer input is distributed to the active subcircuit, and the subcircuit output is collected and returned as the layer output. The input distribution and output collection blocks are driven the back-end protocol rather than implemented subcircuits.

By the constraints in $\mathcal{L}$, only one subcircuit in each layer can be activated (see Figure 7). However, in the view of verifier, the activation of intermediate layers remains nondeterministic. A straightforward effort to resolve this is to make up a large deterministic circuit by injecting multiplexer (MUX) components in each layer that selects one of the outputs from the s_D+3 branches [31], which results in the prover time complexity $O(s_D s_{\max} n)$. Instead, it has been shown that a back-end protocol can play the role of MUX [27, 28]. We will show that our protocol $\mathtt{IP}_{\mathcal{P},\mathcal{V}}$ also replaces the MUX, which results in the prover complexity $O((s_D+s_{\max})n)$.

QAP compiler. We rewrite the subcircuit library $\mathcal{L}$ in the QAP representation. Recall $z_j^{(k)} \in \{u_j^{(k)}, v_j^{(k)}, w_j^{(k)}\}$ for $j \in \{0,\cdots,l^{(k)},\cdots,m^{(k)}\}$ represents the j-th wire polynomials of a subcircuit $\mathtt{c}_k \subset \mathcal{L}$. Since all subcircuits $\mathtt{c}_k$ have l_K input wires and l_K output wires, we can fix the number of connecting wires in the subcircuit $\mathtt{c}_k$, $l^{(k)}=2l_K$ for all k. As $\mathcal{L}$ is a union of $\mathtt{c}_k$ for all k. Thus, the number of wires in $\mathcal{L}$ is parameterized by $l=2l_K$, $l_D=2l_K(s_D+3)$, and $m_D=\sum_{k=0}^{s_D+2} m^{(k)}$. In addition to the definitions in Section 3.1, we use an index set of *connecting position*, $\mathcal{I} := \{0,\cdots,2l_K-1\}$, and disjoint subsets of it, $\mathcal{I}_{in} := \{0,\cdots,l_K-1\}$ and $\mathcal{I}_{out} := \{l_K,\cdots,2l_K-1\}$.

We write the elements of $\mathcal{L}$ as

$$\mathcal{L} = \{o_j\}_{j=0}^{m_D-1},$$

where $o_j \in \{u_j, v_j, w_j\} \subset \mathbb{F}[X]$. Each element o_j is picked from $\mathtt{c}_k$ according to the position indices as follows:

$$\begin{aligned}
(o_j)_{j\in\{0,\cdots,l-1\}} &= \left((z_j^{(0)})_{j\in\mathcal{I}_{in}}, (z_j^{(s_D+2)})_{j\in\mathcal{I}_{out}}\right),\\
(o_j)_{j\in\{l,\cdots,l_D-1\}} &= \left((z_j^{(s_D+2)})_{j\in\mathcal{I}_{in}}, (z_j^{(0)})_{j\in\mathcal{I}_{out}}, (z_j^{(1)})_{j\in\mathcal{I}}, \cdots, (z_j^{(s_D+1)})_{j\in\mathcal{I}}\right),\\
(o_j)_{j\in\{l_D,\cdots,m_D-1\}} &= \left(z_{l^{(0)}}^{(0)},\cdots,z_{m^{(0)}}^{(0)}, z_{l^{(1)}}^{(1)},\cdots,z_{m^{(1)}}^{(1)},\cdots,z_{l^{(s_D+2)}}^{(s_D+2)},\cdots,z_{m^{(s_D+2)}}^{(s_D+2)}\right).
\end{aligned}$$

For the connecting wires, represented by o_j for $j\in\{l,\cdots,l_D-1\}$, we say two wires o_{j_1} and o_{j_2} are in the same *position*, if $j_1 \equiv j_2 \pmod{l_K}$. We put a restriction on the wiring of a circuit such that only two connecting wires at the same position can be connected to each other, as shown in Figure 6.

Recall $\mathcal{C}=(\mathbf{Q},\rho)$ with $\mathbf{Q}=(\mathrm{d}^{(0)},\mathrm{d}^{(1)},\cdots,\mathrm{d}^{(s_{\max}-1)})$, where $\mathrm{d}^{(i)} \in \{\mathrm{c}_0,\cdots,\mathrm{c}_{s_D+2}\}$ indicates the *active subcircuit* of the i-th layer. By the virtue of the enhanced constraints in the subcircuits, unlike the constraint system in

Section 3, the activation is determined by the input to each layer. We again set the wire assignments in all inactive subcircuits to zero.

Synthesizer. When defining wiring of $\mathcal{C}$, the data transfer occurs only between two wires on the same position as shown in Figure 6. We let "$\mathtt{d}^{(g)} \to \mathtt{d}^{(h)}$" for $g \neq h$ denote the data transfer from the g -th layer to the h -th layer. Let $k^{(i)} \in \{1, \cdots, s_D + 2\}$ be the index of active subcircuit in the i -th layer so that $d^{(i)}_{2k^{(i)}l_K + i}$ for $i \in \mathcal{I}$ indicate the assignments to the connecting wires of an active subcircuit. Then, $\mathtt{d}^{(g)} \to \mathtt{d}^{(h)}$ is formally defined as copy constraints between $d^{(g)}_{2k^{(g)}l_K + i_{out}} = d^{(h)}_{2k^{(h)}l_K + i_{in}}$ for all index pairs $(i_{out}, i_{in}) \in \mathcal{I}_{out} \times \mathcal{I}_{in}$ such that $i_{out} \equiv i_{in} \pmod{l_K}$. We also allow multiple compositions of "$\to$", e.g., the wiring of $\mathcal{C}$ in Figure 6 is defined as

$$\begin{aligned}&\mathtt{d}^{(0)} \to \mathtt{d}^{(1)} \to \cdots \to \mathtt{d}^{(s_{\max}-1)} \Leftrightarrow \\ &(\mathtt{d}^{(0)} \to \mathtt{d}^{(1)}) \wedge (\mathtt{d}^{(1)} \to \mathtt{d}^{(2)}) \wedge \cdots \wedge (\mathtt{d}^{(s_{\max}-2)} \to \mathtt{d}^{(s_{\max}-1)}).\end{aligned}$$

Copy constraints. To make the copy constraints $\mathtt{d}^{(g)} \to \mathtt{d}^{(h)}$ deterministic, the following Corollary 2 modifies Lemma 3.

Corollary 2. Assume the zero wire assignments for inactive subcircuits. Given the generators ω_Y and ω_Z of the vanishing sets $\mathcal{Y}$ and $\mathcal{Z}$, define $\omega := \omega_Z^{s_D+2}$ so that $\omega^{2l_K} = 1$. The copy constraints $\mathtt{d}^{(g)} \to \mathtt{d}^{(h)}$ holds if and only if the following equation holds:

$$P_{g \to h}(\boldsymbol{\theta}) = 1, \tag{60}$$

where $P_{g \to h}$ is a Laurent polynomial of indeterminates $\boldsymbol{\theta} = (\theta_0, \theta_1, \theta_2)$ in $\mathbb{F}$ defined as

$$P_{g \to h}(\boldsymbol{\theta}) := \frac{\prod_{k=1}^{s_D+2} \left(\prod_{j \in \mathcal{I}_{out}} \left(d^{(g)}_{2kl_K + j} + \omega_Y^h \theta_0 + \omega^{j-l_K} \theta_1 + \theta_2 \right) \prod_{j \in \mathcal{I}_{in}} \left(d^{(h)}_{2kl_K + j} + \omega_Y^g \theta_0 + \omega^{j+l_K} \theta_1 + \theta_2 \right) \right)}{\prod_{k=1}^{s_D+2} \left(\prod_{j \in \mathcal{I}_{out}} \left(d^{(g)}_{2kl_K + j} + \omega_Y^g \theta_0 + \omega^{j} \theta_1 + \theta_2 \right) \prod_{j \in \mathcal{I}_{in}} \left(d^{(h)}_{2kl_K + j} + \omega_Y^h \theta_0 + \omega^{j} \theta_1 + \theta_2 \right) \right)}. \tag{61}$$

Proof. Due to the zero wire assignments for inactive subcircuits, equation (60) can be reduced to

$$\begin{aligned}&\prod_{j \in \mathcal{I}_{out}} \left(d^{(g)}_{2k^{(g)}l_K + j} + \omega_Y^h \theta_0 + \omega^{j-l_K} \theta_1 + \theta_2 \right) \prod_{j \in \mathcal{I}_{in}} \left(d^{(h)}_{2k^{(h)}l_K + j} + \omega_Y^g \theta_0 + \omega^{j+l_K} \theta_1 + \theta_2 \right) = \\ &\prod_{j \in \mathcal{I}_{out}} \left(d^{(g)}_{2k^{(g)}l_K + j} + \omega_Y^g \theta_0 + \omega^{j} \theta_1 + \theta_2 \right) \prod_{j \in \mathcal{I}_{in}} \left(d^{(h)}_{2k^{(h)}l_K + j} + \omega_Y^h \theta_0 + \omega^{j} \theta_1 + \theta_2 \right)\end{aligned} \tag{62}$$

By the definition, it is straightforward to see that if $\mathtt{d}^{(g)} \to \mathtt{d}^{(h)}$, equation (62) holds. To see the converse, suppose (62) holds. The sets of roots of θ_0 on both sides are given by, respectively,

$$\begin{aligned}&\left\{ -\frac{b^{(g)}_{2k^{(g)}l_K + j} + \omega_Z^{j-l_K} \theta_1 + \theta_2}{\omega_Y^h} \right\}_{j \in \mathcal{I}_{out}} \cup \left\{ -\frac{b^{(h)}_{2k^{(h)}l_K + j} + \omega_Z^{j+l_K} \theta_1 + \theta_2}{\omega_Y^g} \right\}_{j \in \mathcal{I}_{in}}, \\ &\left\{ -\frac{b^{(h)}_{2k^{(h)}l_K + j} + \omega_Z^{j} \theta_1 + \theta_2}{\omega_Y^h} \right\}_{j \in \mathcal{I}_{in}} \cup \left\{ -\frac{b^{(g)}_{2k^{(g)}l_K + j} + \omega_Z^{j} \theta_1 + \theta_2}{\omega_Y^g} \right\}_{j \in \mathcal{I}_{out}}.\end{aligned}$$

Since $\{j+l_K, \forall j \in \mathcal{I}_{in}\} = \mathcal{I}_{out}$, equating the two sets of roots implies $\mathtt{d}^{(g)} \to \mathtt{d}^{(h)}$. □

By extending Corollary 2, we can define a sufficient and necessary condition for $\mathtt{d}^{(0)} \to \mathtt{d}^{(1)} \to \cdots \to \mathtt{d}^{(s_{\max}-1)}$ as

$$\prod_{h=0}^{s_{\max}-2} P_{h\to h+1}(\boldsymbol{\theta}) = 1. \tag{63}$$

Permutation polynomials. Our protocol $\mathtt{IP}_{\mathcal{P},\mathcal{V}}$ is useful to argue (63) without further modification. All we need to do is replacing the permutation polynomials $s^{(0)}, s^{(1)}, s^{(2)} \in \mathbb{F}_{m_I, s_{\max}}[X,Y]$ for the construction of a recursion polynomial $r \in \mathbb{F}[X,Y,\boldsymbol{\theta}]$ in (17) with the following new definitions:

$$\begin{aligned}
s^{(0)}\left(\omega_Z^i, \omega_Y^h\right) &:= \begin{cases} \omega_Y^{h-1}, & \text{if } \left(h, i \bmod 2l_K\right) \in \{1,\cdots,s_{\max}-1\} \times \mathcal{I}_{in}, \\ \omega_Y^{h+1}, & \text{if } \left(h, i \bmod 2l_K\right) \in \{0,\cdots,s_{\max}-2\} \times \mathcal{I}_{out}, \\ \omega_Y^h, & \textit{otherwise.} \end{cases} \\
s^{(1)}\left(\omega_Z^i, \omega_Y^h\right) &:= \begin{cases} \omega^{i+l_K}, & \text{if } \left(h, i \bmod 2l_K\right) \in \{1,\cdots,s_{\max}-1\} \times \mathcal{I}_{in}, \\ \omega^{i-l_K}, & \text{if } \left(h, i \bmod 2l_K\right) \in \{0,\cdots,s_{\max}-2\} \times \mathcal{I}_{out}, \\ \omega^i, & \textit{otherwise.} \end{cases} \\
s^{(2)}\left(\omega_Z^i, \omega_Y^h\right) &:= \omega^i \Leftrightarrow s^{(2)}(Y,Z) = Z^{s_D+2},
\end{aligned} \tag{64}$$

where $\omega := \omega_Z^{s_D+2}$. Then, given $B(Y,Z)$ as an encoding of $d_j^{(i)}$ (see (13)), the copy constraints in (63) holds if and only if

$$\prod_{h=0}^{s_{\max}-2} P_{h\to h+1}(\boldsymbol{\theta}) = \frac{\prod_{y\in\mathcal{Y}, z\in\mathcal{Z}} B(x,y) + s^{(0)}(x,y)\theta_0 + s^{(1)}(x,y)\theta_1 + \theta_2}{\prod_{y\in\mathcal{Y}, z\in\mathcal{Z}} B(x,y) + y\theta_0 + s^{(2)}(x,y)\theta_1 + \theta_2} = 1.$$

It is clear to see that the equality of this equation can be argued by the copy constraint argument of $\mathtt{IP}_{\mathcal{P},\mathcal{V}}$ along with the newly constructed recursion polynomial r.

Efficiency of machine computation. The permutation polynomials $s^{(0)}(X,Y)$ and $s^{(1)}(X,Y)$ in this verifiable machine computation model are independent of programs and the input instance but only parameterized by the maximum number of machine steps, $s_{\max}$. Thus, in the protocol $\mathtt{IP}_{\mathcal{P},\mathcal{V}}$, appending commitments to the permutation polynomials into the reference string $\boldsymbol{\sigma}$ resolves the reliance of the verifier preprocessing. This approach only adds to the verifier time complexity $O(\log s_D)$ for the evaluation of $s^{(2)}(X,Y)$. The rest factors such as the prover time complexity and the proof size directly inherit those of $\mathtt{IP}_{\mathcal{P},\mathcal{V}}$.

7 Conclusion

In this paper, we have proposed a SNARK with universal setups. We showed that our SNARK satisfies completeness and knowledge soundness, and it can be further enhanced with zero-knowledge. In our SNARK, the prover the complexity is $O((s_D + s_{\max})n \log s_D s_{\max} n)$, with proof size of 19 group elements and 4 field elements, and the verifier time complexity is $O(l)$. Here, s_D and $s_{\max}$ can be considered complementary to each other [26], and in extreme cases where $s_D = O(1)$ with maximized $s_{\max}$, the efficiency of our SNARK asymptotically comparable to other state-of-the-art SNARKs with updatable and universal setups in [6, 7]. Additionally, compared to the other SNARKs, we have reduced the dimensionality of verifier preprocessing data from $O(ns_{\max})$ to $O(s_D s_{\max})$, albeit at the cost of sacrificing updatability of the setup.

Furthermore, we have demonstrated the applicability of our SNARK in verifiable machine computation. Unlike recent machine computation protocols such as MUX-Marlin [27] and SublonK [28], our approach achieves machine computation by merely changing the verifier preprocessed inputs to deterministic ones, without requiring additional auxiliary protocols. This results in proof sizes four to ten times smaller. However, while the CRS sizes of the other two protocols are $O((s_D + s_{\max} \log s_D)n)$, offering an advantage over our CRS size of $O((s_D + n)s_{\max})$.

Our SNARK can effectively support efficient verifiable computation in distributed computing networks, particularly in environments like blockchains where nodes are generally untrusted. In such networks, the preprocessing generated by one verifier node must be verified by others, consuming network resources over time as more preprocesses accumulate. Future research could investigate how the reduced data dimensionality in verifier preprocessing could alleviate this burden, thus improving network efficiency.

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Appendix A – Proof of Lemma 2

It is straightforward to see that the existence of $h(X)$ such that $p(X) = h(X)t_{\mathcal{X}}(X)$ implies $p(x) = 0$ for every $x \in \mathcal{X}$. We show the converse, if $p(x) = 0$ for every $x \in \mathcal{X}$ then there exists such a $h(X)$, by contradiction. Pick any $x_k \in \mathcal{X}$, and suppose that $p(x_k) \neq 0$ but there exists such a $h(X)$. Let $h^{(k)}(X) = h(X)\prod_{x\in\mathcal{X}, x\neq x_k}(X - x)$ so that $p(X) = h^{(k)}(X)(X - x_k)$. Since p and $h^{(k)}$ are polynomials, we can express them as $\sum_{i=0}^{d} p_i X^i$ and $\sum_{i=0}^{d-1} h_i^{(k)} X^i$, respectively. Then, we have $\sum_{i=0}^{d} p_i X^i = h_{d-1}^{(k)} X^d + \sum_{i=1}^{d-1}(h_{i-1}^{(k)} - x_k h_i^{(k)}) X^i - x_k h_0^{(k)}$. This implies a recurrence $p_i = h_{i-1}^{(k)} - x_k h_i^{(k)}$ for $i = 1, \cdots, d-1$ with initial conditions $h_{d-1}^{(k)} = p_d$ and $-x_k h_0^{(k)} = p_0$. Solving the recurrence gives us $\sum_{i=0}^{d} p_i x_k^i = 0$, i.e., $p(x_k) = 0$, which contradicts the supposition $p(x_k) \neq 0$. □

Appendix B – Proof of Corollary 1

It is straightforward to see that if there are such $h_{\mathcal{X}}$ and $h_{\mathcal{Y}}$, $p(X,Y)$ vanishes on $\mathcal{X}\times\mathcal{Y}$. To see the converse, we suppose $p(X,Y)$ vanishes on $\mathcal{X}\times\mathcal{Y}$ and then show there is a unique representation $h_0, h_1, h_2 \in \mathbb{F}[X,Y]$ such that $p(X,Y) = h_0(X,Y)t_{\mathcal{X}}(X) + h_1(X,Y)t_{\mathcal{Y}}(X) + h_2(X,Y)t_{\mathcal{X}}(X)t_{\mathcal{Y}}(X)$.

We first construct h_0. By Lemma 2, $p(X,y)$ for every $y \in \mathcal{Y}$ has a quotient polynomial $h_0^{(y)}$ such that $p(X,y) = h_0^{(y)}(X)t_{\mathcal{X}}(X)$. Let $h_0(X,Y)$ be an interpolating polynomial of data points $(y, h_0^{(y)}(X))$ for all $y \in \mathcal{Y}$ so that $h_0(X,y) = h_0^{(y)}(X)$. We next construct h_1. Let $q(X,Y) := p(X,Y) - h_0(X,Y)t_{\mathcal{X}}(X)$. By Lemma 2, $q(x,Y)$ for every $x \in \mathcal{X}$ has a quotient polynomial $h_1^{(x)}$ such that $q(x,Y) = h_1^{(x)}(Y)t_{\mathcal{Y}}(Y)$. We can obtain an interpolating polynomial $h_1(X,Y)$ of data points $(x, h_1^{(x)}(Y))$ for all $x \in \mathcal{X}$.

Given h_0 and h_1, we find h_2. Let $r(X,Y) := p(X,Y) - h_0(X,Y)t_{\mathcal{X}}(X) - h_1(X,Y)t_{\mathcal{Y}}(Y)$. We can see that $r(X,y) = 0$ for every $y \in \mathcal{Y}$, since $p(X,y) = h_0^{(y)}(X)t_{\mathcal{X}}(X)$ and $h_0(X,y) = h_0^{(y)}(X)$. Applying Lemma 2 to $r(X,y) = 0$ implies there exists h_3 such that $r(X,Y) = h_3(X,Y)t_{\mathcal{Y}}(Y)$. Similarly, as it holds true that $r(x,Y) = 0$ for every $x \in \mathcal{X}$, there also exists h_4 such that $h_3(X,Y) = h_4(X,Y)t_{\mathcal{X}}(X)$. Combining them we finally obtain $r(X,Y) = h_2(X,Y)t_{\mathcal{X}}(X)t_{\mathcal{Y}}(Y)$, where $h_2(X,Y) = h_3(X,Y)h_4(X,Y)$.

Letting $h_{\mathcal{X}}(X,Y) = h_0(X,Y)$ and $h_{\mathcal{Y}}(X,Y) = h_1(X,Y) + h_2(X,Y)t_{\mathcal{X}}(X)$ concludes the proof. □

Appendix C– Proof of Proposition 1

We show that if $(\mathbf{a}, (\mathbf{b}(y), \mathbf{c}(y))) \in R$, the transcript $\mathbf{t}$ produced by $\langle \mathcal{P}, \mathcal{V} \rangle$ satisfies (59). It is quite straightforward to see that if the transcript satisfies the verification equations for the arithmetic argument in (33), the copy constraint argument in (51), and the binding polynomial in (56), respectively, equation (59) holds. So, we show that each premise is true.

Completeness of the arithmetic constraint argument. To see that if $\mathbf{d}(Y)$ satisfies the arithmetic constraint (12), the transcript $\mathbf{t}$ produced by $\langle \mathcal{P}, \mathcal{V} \rangle$ satisfies (33), we need to see that all cross terms incurred by the random coin tosses $(r_{U_X}, r_{U_Y}, r_{V_X}, r_{V_Y}) \in (\mathbb{F}^*)^4$, $r_{W_X} \in \mathbb{F}[X]$, and $r_{W_Y} \in \mathbb{F}[Y]$ by $\mathcal{P}$ are cancelled out in the equation (33).

By the bilinearity of the pairing, we can rewrite (33) by

$$e\left(\begin{bmatrix} V_{x,y}U(x,y)-W(x,y) \\ -t_n(\chi)Q_{A,X}-t_{s_{\max}}(\zeta)Q_{A,Y} \\ +\kappa_1\left(V-V_{x,y}\right) \end{bmatrix}_1,[1]_2\right)=e\left(\left[\Pi_{A,\chi}\left(x-\chi\right)+\Pi_{A,\zeta}\left(y-\zeta\right)\right]_1,[1]_2\right). \tag{65}$$

From (23) and (26), we have

$$V_{x,y}U(x,y)-W(x,y)=\begin{pmatrix} u(x,y)v(\chi,\zeta)-w(x,y) \\ +u(x,y)\left(r_{V_X}t_n(\chi)+r_{V_Y}t_{s_{\max}}(\zeta)\right)+v(\chi,\zeta)\left(r_{U_X}t_n(x)+r_{U_Y}t_{s_{\max}}(y)\right) \\ +\left(r_{V_X}t_n(\chi)+r_{V_Y}t_{s_{\max}}(\zeta)\right)\left(r_{U_X}t_n(x)+r_{U_Y}t_{s_{\max}}(y)\right) \\ -r_{W_X}(x)t_n(x)-r_{W_Y}(y)t_{s_{\max}}(y) \end{pmatrix}. \tag{66}$$

From (25), we have

$$\begin{aligned} t_n(\chi)Q_{A,X}+t_{s_{\max}}(\zeta)Q_{A,Y}=t_n(\chi)&\begin{pmatrix} q_0(x,y)+r_{U_X}v(x,y)+r_{V_X}u(x,y)-r_{W_X}(x) \\ +r_{U_X}r_{V_X}t_n(x)+r_{U_Y}r_{V_X}t_{s_{\max}}(y) \end{pmatrix} \\ +t_{s_{\max}}(\zeta)&\begin{pmatrix} q_1(x,y)+r_{U_Y}v(x,y)+r_{V_Y}u(x,y)-r_{W_Y}(y) \\ +r_{U_X}r_{V_Y}t_n(x)+r_{U_Y}r_{V_Y}t_{s_{\max}}(y) \end{pmatrix}. \end{aligned} \tag{67}$$

By subtracting (67) from (66), we obtain

$$\begin{pmatrix} V_{x,y}U(x,y)-W(x,y) \\ -t_n(\chi)Q_{A,X}-t_{s_{\max}}(\zeta)Q_{A,Y} \end{pmatrix}=\begin{pmatrix} u(x,y)v(\chi,\zeta)-w(x,y)-t_n(\chi)q_0(x,y)-t_{s_{\max}}(\zeta)q_1(x,y) \\ -r_{W_X}(x)\left(t_n(x)-t_n(\chi)\right)-r_{W_Y}(y)\left(t_{s_{\max}}(y)-t_{s_{\max}}(\zeta)\right) \\ +v(\chi,\zeta)\left(r_{U_X}t_n(x)+r_{U_Y}t_{s_{\max}}(y)\right)-v(x,y)\left(r_{U_X}t_n(\chi)+r_{U_Y}t_{s_{\max}}(\zeta)\right) \end{pmatrix}. \tag{68}$$

By substituting (68) and (26), the left-hand side of (65) can be rewritten as

$$e\left(\begin{bmatrix} u(x,y)v(\chi,\zeta)-t_n(\chi)q_0(x,y)-t_{s_{\max}}(\zeta)q_1(x,y) \\ -r_{W_X}(x)\left(t_n(x)-t_n(\chi)\right)-r_{W_Y}(y)\left(t_{s_{\max}}(y)-t_{s_{\max}}(\zeta)\right) \\ +v(\chi,\zeta)\left(r_{U_X}t_n(x)+r_{U_Y}t_{s_{\max}}(y)\right)-v(x,y)\left(r_{U_X}t_n(\chi)+r_{U_Y}t_{s_{\max}}(\zeta)\right) \\ +\kappa_1\left(V(x,y)-V(\chi,\zeta)\right) \end{bmatrix}_1,[1]_2\right). \tag{69}$$

By the definitions in (27)-(30), all terms in (69) are canceled out by the terms in the right-hand side of (65). In other words, equation (65) holds.

Completeness of the copy constraint argument. To see if $b(X,Y)$ satisfies the copy constraint (12) and $r(X,Y)$ satisfying (17) is well-defined, the transcript $\mathbf{t}$ produced by $\langle\mathcal{P},\mathcal{V}\rangle$ satisfies (51), we need to see that all cross terms incurred by the random coin tosses $r_{B_X}\in\mathbb{F}[X]$ and $r_{B_Y}\in\mathbb{F}[Y]$ by $\mathcal{P}$ are cancelled out in the equation (51).

By the bilinearity of the pairing, we can rewrite (51) as

$$e\left([\mathrm{LHS}]_1,[1]_2\right)=e\left([\mathrm{RHS}]_1,[1]_2\right), \tag{70}$$

where

$$\mathrm{RHS}:=\begin{pmatrix}\Pi_{C,\chi}\left(x-\chi\right)+\Pi_{C,\zeta}\left(y-\zeta\right)\\ +\kappa_2\left(M_\chi\left(x-\omega_{m_I}^{-1}\chi\right)+M_\zeta\left(y-\zeta\right)\right)\\ +\kappa_2^2\left(M_\chi\left(x-\omega_{m_I}^{-1}\chi\right)+N_\zeta\left(y-\omega_{s_{\max}}^{-1}\zeta\right)\right)\end{pmatrix}, \tag{71}$$

and

$$\mathrm{LHS}:=\kappa_1^2\begin{pmatrix}\left(R_{x,y}-1\right)K_{-1}(x)L_{-1}(y)\\ +\kappa_0\left(\chi-1\right)\left(R_{x,y}G-R'_{x,y}F\right)\\ +\kappa_0^2K_0(\chi)\left(R_{x,y}G-R''_{x,y}F\right)\\ -t_{m_I}(\chi)Q_{C,X}-t_{s_{\max}}(\zeta)Q_{C,Y}\end{pmatrix}+\begin{pmatrix}\kappa_1^3\left(R-R_{x,y}\right)\\ +\kappa_2\left(R-R'_{x,y}\right)\\ +\kappa_2^2\left(R-R''_{x,y}\right)\end{pmatrix}. \tag{72}$$

To further unfold the first term in (72), we will need the following unfolded expressions based on the definitions in (34)-(40) and (48):

$$\left(R_{x,y}-1\right)K_{-1}(x)L_{-1}(y)=\begin{pmatrix}\left(r(\chi,\zeta)-1\right)K_{-1}(x)L_{-1}(y)\\ +\left(r_{R_X}t_{m_I}(\chi)+r_{R_Y}t_{s_{\max}}(\zeta)\right)K_{-1}(x)L_{-1}(y)\end{pmatrix}, \tag{73}$$

$$\begin{aligned}R_{x,y}G-R'_{x,y}F=&\left(r(\chi,\zeta)+r_{R_X}t_{m_I}(\chi)+r_{R_Y}t_{s_{\max}}(\zeta)\right)\left(g(x,y)+r_{B_X}(x)t_{m_I}(x)+r_{B_Y}(y)t_{s_{\max}}(y)\right)\\ &-\left(r(\omega_{m_I}^{-1}\chi,\zeta)+r_{R_X}t_{m_I}(\chi)+r_{R_Y}t_{s_{\max}}(\zeta)\right)\left(f(x,y)+r_{B_X}(x)t_{m_I}(x)+r_{B_Y}(y)t_{s_{\max}}(y)\right)\\ &=r(\chi,\zeta)g(x,y)-r(\omega_{m_I}^{-1}\chi,\zeta)f(x,y)\\ &+\left(r_{B_X}(x)t_{m_I}(x)+r_{B_Y}(y)t_{s_{\max}}(y)\right)r_{D_1}(\chi,\zeta)\\ &+\left(r_{R_X}t_{m_I}(\chi)+r_{R_Y}t_{s_{\max}}(\zeta)\right)g_D(x,y),\end{aligned} \tag{74}$$

$$\begin{aligned}R_{y,z}G-R''_{y,z}F=&\left(r(\chi,\zeta)+r_{R_X}t_{m_I}(\chi)+r_{R_Y}t_{s_{\max}}(\zeta)\right)\left(g(x,y)+r_{B_X}(x)t_{m_I}(x)+r_{B_Y}(y)t_{s_{\max}}(y)\right)\\ &-\left(r(\omega_{m_I}^{-1}\chi,\omega_{s_{\max}}^{-1}\zeta)+r_{R_X}t_{m_I}(\chi)+r_{R_Y}t_{s_{\max}}(\zeta)\right)\left(f(x,y)+r_{B_X}(x)t_{m_I}(x)+r_{B_Y}(y)t_{s_{\max}}(y)\right)\\ &=r(\chi,\zeta)g(x,y)-r(\omega_{m_I}^{-1}\chi,\omega_{s_{\max}}^{-1}\zeta)f(x,y)\\ &+\left(r_{B_X}(x)t_{m_I}(x)+r_{B_Y}(y)t_{s_{\max}}(y)\right)r_{D_2}(\chi,\zeta)\\ &+\left(r_{R_X}t_{m_I}(\chi)+r_{R_Y}t_{s_{\max}}(\zeta)\right)g_D(x,y),\end{aligned} \tag{75}$$

$$
\begin{aligned}
t_{m_I}(\chi)Q_{C,X} + t_{s_{\max}}(\zeta)Q_{C,Y} &= \left(q_2(x,y)+\kappa_0 q_4(x,y)+\kappa_0^2 q_6(x,y)\right)t_{m_I}(\chi) \\
&+\left(q_3(x,y)+\kappa_0 q_5(x,y)+\kappa_0^2 q_7(x,y)\right)t_{s_{\max}}(\zeta) \\
&+\left(r_{R_X} t_{m_I}(\chi)+r_{R_Y} t_{s_{\max}}(\zeta)\right)K_{-1}(x)L_{-1}(y) \\
&+\kappa_0\begin{pmatrix}(x-1)\left(r_{B_X}(x)t_{m_I}(\chi)+r_{B_Y}(y)t_{s_{\max}}(\zeta)\right)r_{D_1}(x,y) \\ +(x-1)\left(r_{R_X} t_{m_I}(\chi)+r_{R_Y} t_{s_{\max}}(\zeta)\right)g_D(x,y)\end{pmatrix} \\
&+\kappa_0^2\begin{pmatrix}K_0(x)\left(r_{B_X}(x)t_{m_I}(\chi)+r_{B_Y}(y)t_{s_{\max}}(\zeta)\right)r_{D_2}(x,y) \\ +K_0(x)\left(r_{R_X} t_{m_I}(\chi)+r_{R_Y} t_{s_{\max}}(\zeta)\right)g_D(x,y)\end{pmatrix}.
\end{aligned}
\tag{76}
$$

By linearly combining (73)-(76) and substituting (42), (45), and (46), we can rewrite the first term in (72) as

$$
\begin{pmatrix}\left(R_{x,y}-1\right)K_{-1}(x)L_{-1}(y) \\ +\kappa_0(\chi-1)\left(R_{x,y}G-R'_{x,y}F\right) \\ +\kappa_0^2 K_0(\chi)\left(R_{x,y}G-R''_{x,y}F\right) \\ -t_{m_I}(\chi)Q_{C,\chi}-t_{s_{\max}}(\zeta)Q_{C,Y}\end{pmatrix}
=
\begin{pmatrix}
\left(r(\chi,\zeta)-1\right)K_{-1}(x)L_{-1}(y) \\
\kappa_0(\chi-1)\left(r(\chi,\zeta)g(x,y)-r(\omega_{m_I}^{-1}\chi,\zeta)f(x,y)\right) \\
\kappa_0^2 K_0(\chi)\left(r(\chi,\zeta)g(x,y)-r(\omega_{m_I}^{-1}\chi,\omega_{s_{\max}}^{-1}\zeta)f(x,y)\right) \\
-t_{m_I}(\chi)\left(q_2(x,y)+\kappa_0 q_4(x,y)+\kappa_0^2 q_6(x,y)\right) \\
-t_{s_{\max}}(\zeta)\left(q_3(x,y)+\kappa_0 q_5(x,y)+\kappa_0^2 q_7(x,y)\right) \\
+\kappa_0(\chi-1)\left(r_{B_X}(x)t_{m_I}(x)+r_{B_Y}(y)t_{s_{\max}}(y)\right)r_{D_1}(\chi,\zeta) \\
-\kappa_0(x-1)\left(r_{B_X}(x)t_{m_I}(\chi)+r_{B_Y}(y)t_{s_{\max}}(\zeta)\right)r_{D_1}(x,y) \\
+\kappa_0(\chi-x)\left(r_{R_X} t_{m_I}(\chi)+r_{R_Y} t_{s_{\max}}(\zeta)\right)g_D(x,y) \\
+\kappa_0^2 K_0(\chi)\left(r_{B_X}(x)t_{m_I}(x)+r_{B_Y}(y)t_{s_{\max}}(y)\right)r_{D_2}(\chi,\zeta) \\
-\kappa_0^2 K_0(x)\left(r_{B_X}(x)t_{m_I}(\chi)+r_{B_Y}(y)t_{s_{\max}}(\zeta)\right)r_{D_2}(x,y) \\
+\kappa_0^2\left(K_0(\chi)-K_0(x)\right)\left(r_{R_X} t_{m_I}(\chi)+r_{R_Y} t_{s_{\max}}(\zeta)\right)g_D(x,y)
\end{pmatrix}.
\tag{77}
$$

All terms in (77) can be cancelled out by a linear combination of $\pi_{C,\chi}(x)$, $\pi_{C,\zeta}(x,y)$, defined in (42), and $\pi_{i,\chi}(x)$ and $\pi_{i,\zeta}(x,y)$ for $i=4,5$, defined in (45).

Similarly, the remaining terms in (72) can be cancelled out by a linear combination of $\pi_{i,\chi}(x)$ and $\pi_{i,\zeta}(x,y)$ for $i=1,2,3$ defined in (44).

Finally, the definitions of $\Pi_{C,\chi}$, $\Pi_{C,\zeta}$, M_χ, M_ζ, N_χ, N_ζ in (47) imply equation (70) holds.

The above derivation of completeness holds only when all polynomials are well-defined. However, there is nonzero probability that the polynomial $r(X,Y)$ satisfying (17) is not well-defined, depending on the samples $\boldsymbol{\theta}$. According to Lemma 1, this probability is not greater than $3s_{\max}m_I/|\mathbb{F}^*|$, where the numerator represents the maximum number of combinations of $(\boldsymbol{\theta},x,y)\in(\mathbb{F}^*)^5$ that make zero denominator of the recurrence formula for r.

We omit the completeness of this argument, as it straightforwardly follows Lemma 2. □

Appendix D– Proof of Proposition 2

We begin the proof by transforming the transcript components in (22), $[O_{pub}]_1$, and $[A]_1$ into the affine prover strategy of generic adversaries below.

Definition 6 (Affine prover strategy). *Let $\mathcal{A}$ be a polynomial-time adversary. In generic group model, any group elements $[G]_1$ returned by $\mathcal{A}$ can be expressed as a linear combinations of the entries in $(\boldsymbol{\sigma}_{A,C}, \boldsymbol{\sigma}_B)$. Formally, let*

$$\mathbf{T} := \left(\mathrm{A}, \Gamma, \Delta, \mathrm{H}, X, Y\right) \tag{78}$$

be a vector of indeterminates. Given the trapdoor $\boldsymbol{\tau}$ in (21), any $[G]_1 \in \mathbb{G}_1$ generated by $\mathcal{A}$ can be expressed as $G(\boldsymbol{\tau})$, where $G(\mathbf{T})$ is a degree-bounded Laurent polynomial defined by

$$\begin{aligned}
G(\mathbf{T}) &= G^{(xy)}(X,Y) + G^{(\delta)}\delta + G^{(\eta)}\eta \\
&+\gamma^{-1}\begin{pmatrix} \displaystyle\sum_{j=0}^{l_{in}-1} G_j^{(M)}\left(L_0(Y)\left(\mathrm{A}u_j(X)+\mathrm{A}^2 v_j(X)+\mathrm{A}^3 w_j(X)\right)+M_j(X)\right) \\ \displaystyle +\sum_{j=l_{in}}^{l-1} G_j^{(M)}\left(L_{-1}(Y)\left(\mathrm{A}u_j(X)+\mathrm{A}^2 v_j(X)+\mathrm{A}^3 w_j(X)\right)+M_j(X)\right) \end{pmatrix} \\
&+\eta^{-1}\sum_{j=0}^{m_I-1} G_j^{(K)}(Y)\left(\left(\mathrm{A}u_{j+l}(X)+\mathrm{A}^2 v_{j+l}(X)+\mathrm{A}^3 w_{j+l}(X)\right)+\mathrm{A}^4 K_j(X)\right) \\
&+\delta^{-1} G^{(o)}_{l_D:m_D}(\mathrm{A},X,Y) + \delta^{-1} G^{(zk)}(\mathrm{A},X,Y),
\end{aligned}$$

where

$$\begin{aligned}
G^{(xy)}(X,Y) &:= \sum_{h=0}^{\max(2n-2,2m_I-3)} \sum_{i=0}^{2s_{\max}-2} G_{h,i}^{(xy)} X^h Y^i, \\
G_{I:J}^{(o)}(\mathrm{A},X,Y) &:= \sum_{j=I}^{J-1} G_{j-I}^{(c)}(Y)\left(\mathrm{A}u_j(X)+\mathrm{A}^2 v_j(X)+\mathrm{A}^3 w_j(X)\right), \\
G^{(zk)}(\mathrm{A},X,Y) &:= \begin{pmatrix} \displaystyle\sum_{k=1}^{3} \mathrm{A}^k\left(G_k^{(zk_n)}(X)t_n(X)+G_k^{(zk_{s_{\max}})}(Y)t_{s_{\max}}(Y)\right) \\ +\mathrm{A}^4\left(G_4^{(zk_{m_I})}(X)t_{m_I}(X)+G_4^{(zk_{s_{\max}})}(Y)t_{s_{\max}}(Y)\right) \end{pmatrix}.
\end{aligned} \tag{79}$$

By Definition 6 and the bilinearity of pairing, we can rewrite the verification equation (59) as $P(\boldsymbol{\tau}) = 0$, where P is defined by

$$P(\mathbf{T}) := \begin{pmatrix} P_B(\mathbf{T}) + \kappa_2 \mathrm{LHS}(\mathbf{T}) + \kappa_2^2\left(R(\mathbf{T}) - R'_{x,y}\right) + \kappa_2^3\left(R(\mathbf{T}) - R''_{x,y}\right) \\ -\kappa_2\left(\Pi_\chi(\mathbf{T})\left(X-\chi\right) + \Pi_\zeta(\mathbf{T})\left(Y-\zeta\right)\right) \\ -\kappa_2^2\left(M_\chi(\mathbf{T})\left(X-\omega_{m_I}^{-1}\chi\right) + M_\zeta(\mathbf{T})\left(Y-\zeta\right)\right) \\ -\kappa_2^3\left(N_\chi(\mathbf{T})\left(X-\omega_{m_I}^{-1}\chi\right) + N_\zeta(\mathbf{T})\left(Y-\omega_{s_{\max}}^{-1}\zeta\right)\right) \end{pmatrix} \tag{80}$$

with

$$P_B(\mathbf{T}) := \begin{pmatrix} A(\mathbf{T}) + U(\mathbf{T})\mathrm{A} + V(\mathbf{T})\mathrm{A}^2 + W(\mathbf{T})\mathrm{A}^3 + B(\mathbf{T})\mathrm{A}^4 \\ -O_{pub}(\mathbf{T})\gamma - O_{mid}(\mathbf{T})\mathrm{H} - O_{prv}(\mathbf{T})\Delta \end{pmatrix},$$

$$\mathrm{LHS}(\mathbf{T}) := \begin{pmatrix} P_A(\mathbf{T}) + \kappa_1\left(V(\mathbf{T}) - V_{x,y}\right) + \kappa_1^2 P_C(\mathbf{T}) \\ +\kappa_1^3\left(R(\mathbf{T}) - R_{x,y}\right) + \kappa_1^4\left(A(\mathbf{T}) - A_{pub}\right) \end{pmatrix},$$

$$P_A(\mathbf{T}) := V_{x,y}U(\mathbf{T}) - W(\mathbf{T}) - t_n(\chi)Q_{A,X}(\mathbf{T}) - t_{s_{\max}}(\zeta)Q_{A,Y}(\mathbf{T}),$$

$$P_C(\mathbf{T}) := \begin{pmatrix} \left(R_{y,z} - 1\right)L_{-1}(y)K_{-1}(z) \\ +\kappa_0\left(\xi - 1\right)\left(R_{x,y}G(\mathbf{T}) - R'_{x,y}F(\mathbf{T})\right) \\ +\kappa_0^2 K_0(\xi)\left(R_{x,y}G(\mathbf{T}) - R''_{x,y}F(\mathbf{T})\right) \\ -t_{m_I}(\chi)Q_{C,X}(\mathbf{T}) - t_{s_{\max}}(\zeta)Q_{C,Y}(\mathbf{T}) \end{pmatrix},$$

$$A_{pub} := \sum_{j=0}^{l-1} a_j M_j(\chi),$$

$$F(\mathbf{T}) := B(\mathbf{T}) + \theta_0 s^{(0)}(X,Y) + \theta_1 s^{(1)}(X,Y) + \theta_2,$$
$$G(\mathbf{T}) := B(\mathbf{T}) + \theta_0 s^{(2)}(X,Y) + \theta_1 Y + \theta_2,$$

and the other polynomial terms in (80) follow Definition 6.

1) Witness-extended emulation

We run an emulator that iterates witness-extended emulations on $\mathrm{IP}_{\mathcal{P}_\mathcal{A}^*,\mathcal{V}}$ according to Definition 5. These emulations run a deterministic $\mathcal{P}_\mathcal{A}^*$ using the fixed affine strategy $(\mathbf{a},\mathbf{b}(Y),\mathbf{c}(Y))$ of $\mathcal{A}$ as input while using for the random challenges generated by $\mathcal{A}$. Specifically, for each emulation, the emulator rewinds the protocol to a certain point in the transcript $\mathbf{t}$. From that point, the deterministic $\mathcal{P}_\mathcal{A}^*$ overwrites the remaining part of the transcript by allowing $\mathcal{V}$ to pick new, distinct challenges. Each emulated transcript is included in a sample space.

The emulator proceeds until it collects at least N sample transcripts that are accepted by $\mathcal{V}$. Since $\mathrm{IP}_{\mathcal{P}_\mathcal{A}^*,\mathcal{V}}$ yields an acceptable transcript with a probability, the expected runtime of the emulator is polynomial in n, $s_{\max}$, m_I, and l, which together determine the size of R, provided that N itself is polynomial. Our goal in the remainder of this proof is to show that there exists $N = cnm_I s_{\max}$ for a constant c that implies the extraction of a valid witness in R.

In the extraction procedure, every transcript in the sample space yields a sample of the equation $P(\boldsymbol{\tau}) = 0$, which each transcript induces a different instantiation of the polynomial $P(\mathbf{T})$ in (80) with distinct coefficients. In certain cases, we interpolate useful polynomials from the emulated samples to derive new polynomial equations.

2) Equation separation by κ_2

Suppose the emulator has obtained N samples of $P(\boldsymbol{\tau}) = 0$. As defined in (80), every P from the emulation contains random elements $\kappa_2 \in \mathbb{F}$. We will treat them as indeterminates after defining an event where each equation $P(\boldsymbol{\tau}) = 0$ could hold only for the picked values of κ_2. By Lemma 1, this probability is upper bounded to $3/|\mathbb{F}^*|$, where the numerator equals the sum of the highest exponents of κ_2 in $P(\mathbf{T})$. Since we have N samples, the probability that at least one equation sample fall into this event is not greater than $3N/|\mathbb{F}^*|$.

Suppose that the N equation samples did not fall into this event. Then, for each sample, we can separate $P(\boldsymbol{\tau})=0$ into four equations as follows,

$$P_B(\boldsymbol{\tau})=0, \tag{81}$$

$$\mathrm{LHS}(\boldsymbol{\tau})=\Pi_\chi(\boldsymbol{\tau})\left(x-\chi\right)+\Pi_\zeta(\boldsymbol{\tau})\left(y-\zeta\right), \tag{82}$$

$$\begin{aligned} R(\boldsymbol{\tau})-R'_{x,y} &= M_\chi(\boldsymbol{\tau})\left(x-\omega_{m_I}^{-1}\chi\right)+M_\zeta(\boldsymbol{\tau})\left(y-\zeta\right), \\ R(\boldsymbol{\tau})-R''_{x,y} &= N_\chi(\boldsymbol{\tau})\left(x-\omega_{m_I}^{-1}\chi\right)+N_\zeta(\boldsymbol{\tau})\left(y-\omega_{s_{\max}}^{-1}\zeta\right). \end{aligned} \tag{83}$$

3) Polynomial reconstruction by $\boldsymbol{\tau}$

Suppose the emulator has collected N samples of (81)-(83). We define an event where each equation sample could hold only for the picked values of $\boldsymbol{\tau}$. By Lemma 1, this probability is upper bounded to $(2n+2s_{\max}+2m_I)/|\mathbb{F}^*|$, where the numerator is an upper bound to the sum of the highest exponents of $\boldsymbol{\tau}$ in (81)-(83). Since we have N samples, the probability that at least one equation sample fall into this event is not greater than $(2n+2s_{\max}+2m_I)N/|\mathbb{F}^*|$.

Suppose that the N equation samples did not fall into this event, under which the evaluated equations (81)-(83) imply polynomial equations, i.e., the evaluated equations are satisfied for all samples $\boldsymbol{\tau}$ in the event. Then, from each polynomial version of the evaluated equations (81)-(83), we can extract sub-polynomials that involve specific indeterminates of our interest.

From the polynomial version of (81), according to Definition 6, we extract terms that only involve $\{A^k X^h Y^i\}_{h=0,i=0,k=0}^{\infty,\infty,\infty}$ to construct

$$P_B^{(xy)}(\mathrm{A},X,Y):=\left(\begin{array}{l} A^{(xy)}(X,Y)+U^{(xy)}\left(X,Y\right)\mathrm{A}+V^{(xy)}\left(X,Y\right)\mathrm{A}^2+W^{(xy)}\left(X,Y\right)\mathrm{A}^3+B^{(xy)}\left(X,Y\right)\mathrm{A}^4 \\ -\left(\begin{array}{l}\displaystyle\sum_{j=0}^{l_{in}-1}O_{pub,j}^{(M)}\left(L_0(Y)\left(\mathrm{A}u_j(X)+\mathrm{A}^2v_j(X)+\mathrm{A}^3w_j(X)\right)+M_j(X)\right) \\ \displaystyle+\sum_{j=l_{in}}^{l-1}O_{pub,j}^{(M)}\left(L_{-1}(Y)\left(\mathrm{A}u_j(X)+\mathrm{A}^2v_j(X)+\mathrm{A}^3w_j(X)\right)+M_j(X)\right)\end{array}\right) \\ \displaystyle-\sum_{j=0}^{m_I-1}O_{mid,j}^{(K)}(Y)\left(\left(\mathrm{A}u_{j+l}(X)+\mathrm{A}^2v_{j+l}(X)+\mathrm{A}^3w_{j+l}(X)\right)+\mathrm{A}^4K_j(X)\right) \\ -O_{prv,l_D:m_D}^{(o)}(\mathrm{A},X,Y)+\delta^{-1}O_{prv}^{(zk)}(\mathrm{A},X,Y), \end{array}\right), \tag{84}$$

where $O_{pub}^{(M)}$, $O_{mid}^{(K)}$, and $O_{prv}^{(zk)}$ follow the definitions in (79). Since the polynomial version of (81) holds, we have $P_B^{(xyz)}(\mathrm{A},X,Y,Z)=0$. By Assumption 1 and the linearly independence of $K_j(Z)$ for distinct j, we can conclude that

$$
\begin{aligned}
U^{(xyz)}(X,Y) &= \begin{pmatrix} \sum_{j=0}^{l-1} O_{pub,j}^{(ML)}(Y)u_j(X) + \sum_{j=l}^{m_I+l-1} O_{mid,j-l}^{(K)}(Y)u_j(X) + \sum_{j=m_I+l}^{m_D-1} O_{prv,j-(m_I+l)}^{(c)}(Y)u_j(X) \\ +O_{prv,1}^{(zk_n)}(X)t_n(X) + G_{prv,1}^{(zk_{s_{\max}})}(Y)t_{s_{\max}}(Y) \end{pmatrix}, \\
V^{(xyz)}(X,Y) &= \begin{pmatrix} \sum_{j=0}^{l-1} O_{pub,j}^{(ML)}(Y)v_j(X) + \sum_{j=l}^{m_I+l-1} O_{mid,j-l}^{(K)}(Y)v_j(X) + \sum_{j=m_I+l}^{m_D-1} O_{prv,j-(m_I+l)}^{(c)}(Y)v_j(X) \\ +O_{prv,2}^{(zk_n)}(X)t_n(X) + G_{prv,2}^{(zk_{s_{\max}})}(Y)t_{s_{\max}}(Y) \end{pmatrix}, \\
W^{(xyz)}(X,Y) &= \begin{pmatrix} \sum_{j=0}^{l-1} O_{pub,j}^{(ML)}(Y)w_j(X) + \sum_{j=l}^{m_I+l-1} O_{mid,j-l}^{(K)}(Y)w_j(X) + \sum_{j=m_I+l}^{m_D-1} O_{prv,j-(m_I+l)}^{(c)}(Y)w_j(X) \\ +O_{prv,3}^{(zk_n)}(X)t_n(X) + G_{prv,3}^{(zk_{s_{\max}})}(Y)t_{s_{\max}}(Y) \end{pmatrix},
\end{aligned}
\tag{85}
$$

$$
A^{(xy)}(X,Y) = \sum_{j=0}^{l-1} O_{pub,j}^{(M)} M_j(X), \tag{86}
$$

and

$$
B^{(xy)}(X,Y) = \sum_{j=0}^{m_I-1} O_{mid,j}^{(K)}(Y)K_j(X) + O_{prv,4}^{(zk_{m_I})}(X)t_{m_I}(X) + O_{prv,4}^{(zk_{s_{\max}})}(Y)t_{s_{\max}}(Y), \tag{87}
$$

where $\mathbf{O}_{pub}^{(ML)}(Y) := (O_{pub,0}^{(M)}L_0(Y), \cdots, O_{pub,l_{in}-1}^{(M)}L_0(Y), O_{pub,l_{in}}^{(M)}L_{-1}(Y), \cdots, O_{pub,l-1}^{(M)}L_{-1}(Y))$.

From the polynomial version of (82), we extract terms that only involve $\{X^hY^i\}_{h=0,i=0}^{\infty,\infty}$ to obtain

$$
\begin{pmatrix} P_A^{(xy)}(X,Y) + \kappa_1\left(V^{(xy)}(X,Y) - V_{x,y}\right) + \kappa_1^2 P_C^{(xy)}(X,Y) \\ +\kappa_1^3\left(R^{(xy)}(X,Y) - R_{x,y}\right) + \kappa_1^4\left(A^{(xy)}(X,Y) - A_{pub}\right) \end{pmatrix} = \begin{pmatrix} \Pi_\chi^{(xy)}(X,Y)(X-\chi) \\ +\Pi_\zeta^{(xy)}(X,Y)(Y-\zeta) \end{pmatrix}. \tag{88}
$$

By Corollary 1, equation (88) implies that

$$
\begin{pmatrix} P_A^{(xy)}(\chi,\zeta) + \kappa_1\left(V^{(xy)}(\chi,\zeta) - V_{x,y}\right) + \kappa_1^2 P_C^{(xy)}(\chi,\zeta) \\ +\kappa_1^3\left(R^{(xy)}(\chi,\zeta) - R_{x,y}\right) + \kappa_1^4\left(A^{(xy)}(\chi,\zeta) - A_{pub}\right) \end{pmatrix} = 0. \tag{89}
$$

From the polynomial versions of the equations in (83), we extract terms that only involve $\{X^hY^i\}_{h=0,i=0}^{\infty,\infty}$ to get, respectively,

$$
\begin{aligned}
R^{(xy)}(X,Y) - R'_{x,y} &= M_\chi^{(xy)}(X,Y)\left(X - \omega_{m_I}^{-1}\chi\right) + M_\zeta^{(xy)}(X,Y)(Y-\zeta), \\
R^{(xy)}(X,Y) - R''_{x,y} &= N_\chi^{(xy)}(X,Y)\left(X - \omega_{m_I}^{-1}\chi\right) + N_\zeta^{(xy)}(X,Y)\left(Y - \omega_{s_{\max}}^{-1}\zeta\right).
\end{aligned}
\tag{90}
$$

By Corollary 1, equations in (90) imply that, respectively,

$$
\begin{aligned}
R^{(xy)}(\omega_{m_I}^{-1}\chi, \zeta) &= R'_{x,y}, \\
R^{(xy)}(\omega_{m_I}^{-1}\chi, \omega_{s_{\max}}^{-1}\zeta) &= R''_{x,y}.
\end{aligned}
\tag{91}
$$

4) Equation separation by κ_1

Suppose the emulator has obtained N samples of (89). Every sample contains $\kappa_1 \in \mathbb{F}$. We will treat κ_1 as an indeterminate after defining an event where each sample of (89) could hold only for the picked value κ_1. By Lemma 1, this probability is upper bounded to $4/|\mathbb{F}^*|$, where the numerator equals the highest exponent of κ_1 in the (89). Since we have N samples, the probability that at least one equation sample fall into this event is not greater than $4N/|\mathbb{F}^*|$.

Suppose that the N equation samples did not fall into this event. Then, for each sample, we can separate the equation (89) into three equations as follows,

$$R^{(xy)}(\chi,\zeta)=R_{x,y}, \tag{92}$$

$$V^{(xy)}(\chi,\zeta)=V_{x,y}, \tag{93}$$

$$A^{(xy)}(\chi)=A_{pub}, \tag{94}$$

$$P_A^{(xy)}(\chi,\zeta)=0, \tag{95}$$

$$P_C^{(xy)}(\chi,\zeta)=0. \tag{96}$$

By substituting (86), equation (94) implies that

$$\sum_{j=0}^{l-1} O_{pub,j}^{(M)} M_j(\zeta)=A_{pub} \tag{97}$$

By substituting (93), we can rewrite (95) as

$$U^{(xy)}(\chi,\zeta)V^{(xy)}(\chi,\zeta)-W^{(xy)}(\chi,\zeta)=t_n(\chi)Q_{A,X}^{(xy)}(\chi,\zeta)+t_{s_{\max}}(\zeta)Q_{A,Y}^{(xy)}(\chi,\zeta), \tag{98}$$

By substituting (91)-(92), we can rewrite (96) as

$$\begin{pmatrix} \left(R^{(xy)}(\chi,\zeta)-1\right)K_{-1}(\chi)L_{-1}(\zeta) \\ +\kappa_0(\chi-1)\begin{pmatrix} R^{(xy)}(\chi,\zeta)G^{(xy)}(\chi,\zeta) \\ -R^{(xy)}(\omega_{m_I}^{-1}\chi,\zeta)F^{(xy)}(\chi,\zeta)\end{pmatrix} \\ +\kappa_0^2K_0(\chi)\begin{pmatrix} R^{(xy)}(\chi,\zeta)G^{(xy)}(\chi,\zeta) \\ -R^{(xy)}(\omega_{m_I}^{-1}\chi,\omega_{s_{\max}}^{-1}\zeta)F^{(xy)}(\chi,\zeta)\end{pmatrix}\end{pmatrix}=t_{m_I}(\chi)Q_{C,X}^{(xy)}(\chi,\zeta)+t_{s_{\max}}(\zeta)Q_{C,Y}^{(xy)}(\chi,\zeta), \tag{99}$$

where

$$\begin{aligned} F^{(xy)}(X,Y) &:= B^{(xy)}(X,Y)+\theta_0 s^{(0)}(X,Y)+\theta_1 s^{(1)}(X,Y)+\theta_2, \\ G^{(xy)}(X,Y) &:= B^{(xy)}(X,Y)+\theta_0 s^{(2)}(X,Y)+\theta_1 Y+\theta_2. \end{aligned}$$

5) Polynomial interpolation based on emulated samples

Suppose the emulator has obtained N emulated transcripts. For the extraction, we interpolate the following polynomials: $\tilde{A}(X)$, $\tilde{V}(X,Y)$, and $\tilde{R}(X,Y)$ respectively based on the samples A_{pub}, $V_{x,y}$, and $R_{x,y}$

respectively in (92)-(94). Our goal is to conclude that

$$\tilde{A}(X) = A^{(xy)}(X), \tag{100}$$

$$\tilde{V}(X,Y) = V^{(xy)}(X,Y), \tag{101}$$

$$\tilde{R}(X,Y) = R^{(xy)}(X,Y), \tag{102}$$

and to this end, the interpolation requires more samples than the number of roots of the right-hand sides. Since the degrees of the right-hand sides are finite, there exists a constant c such that $N = cnm_I s_{\max}$ samples imply (100)-(102). We then define the following polynomials:

$$\tilde{p}_A(X,Y) := \tilde{V}(X,Y)U^{(xy)}(X,Y) - W^{(xy)}(X,Y) - \left(t_n(X)Q_{A,X}^{(xy)}(X,Y) + t_{s_{\max}}(Y)Q_{A,Y}^{(xy)}(X,Y)\right), \tag{103}$$

$$\tilde{p}_C(X,Y) := \begin{pmatrix} \left(\tilde{R}(X,Y)-1\right)K_{-1}(X)L_{-1}(Y) \\ +\kappa_0\left(X-1\right)\begin{pmatrix} \tilde{R}(X,Y)G^{(xy)}(X,Y) \\ -\tilde{R}(\omega_{m_I}^{-1}X,Y)F^{(xy)}(X,Y) \end{pmatrix} \\ +\kappa_0^2 K_0(X)\begin{pmatrix} \tilde{R}(X,Y)G^{(xy)}(X,Y) \\ -\tilde{R}(\omega_{m_I}^{-1}X,\omega_{s_{\max}}^{-1}Y)F^{(xy)}(X,Y) \end{pmatrix} \end{pmatrix} - \begin{pmatrix} t_{m_I}(X)Q_{C,X}^{(xy)}(X,Y) \\ +t_{s_{\max}}(Y)Q_{C,Y}^{(xy)}(X,Y) \end{pmatrix}. \tag{104}$$

Our extraction objective is to show $\tilde{p}_A(X,Y) = 0$ and $\tilde{P}_C(X,Y) = 0$ using the results (100)-(102) and the N samples of (98)-(99). Since the two equations have finite degrees, there exists a constant c such that $N = cnm_I s_{\max}$ samples achieve this objective.

Suppose the emulator has obtained N instances of $\tilde{A}(\chi) = A_{pub}$, $\tilde{V}(\chi,\zeta) = V_{x,y}$, $\tilde{R}(\chi,\zeta) = R_{x,y}$, $\tilde{p}_A(\chi,\zeta) = 0$, and $\tilde{p}_C(\chi,\zeta) = 0$. We should check each equation sample does not identically hold with respect to χ,ζ,ξ but holds only for the picked values of χ,ζ. By Lemma 1, this probability is upper bounded to $N/|\mathbb{F}^*|$. Since we have N instances of 5 equations in total, the probability that at least one equation sample fall into this event is not greater than $5N^2/|\mathbb{F}^*|$.

Suppose that the $5N$ equation instances did not fall into this event. Then, we can conclude that

$$\tilde{P}_A(X,Y) = 0, \tag{105}$$

$$\tilde{p}_C(X,Y) = 0. \tag{106}$$

6) Witness extraction

We extract the witness in R that simultaneously satisfies both arithmetic and copy constraints.

From the results (86) and (100), by the linear independence of $\{M_j(X)\}_{j=0}^{m_I}$, we can conclude that

$$\mathbf{O}_{pub}^{(M)} = \mathbf{a}, \tag{107}$$

where $\mathbf{a}$ is the vector of public input. We also denote the adversary strategy by

$$\mathbf{b}(Y) := (O^{(K)}_{mid,0}(Y), O^{(K)}_{mid,1}(Y), \cdots, O^{(K)}_{mid,m_I-1}(Y)) \tag{108}$$

and

$$\mathbf{c}(Y) := (O^{(c)}_{prv,0}(Y), O^{(c)}_{prv,1}(Y), \cdots, O^{(c)}_{prv,m_D-(l+m_I)-1}(Y)).$$

From (105), by applying Corollary 1 and substituting (101), we can conclude that, for all $h \in \{0,\cdots,n-1\}$ and $i \in \{0,\cdots,s_{\max}-1\}$ and $x_h := \omega_n^h$ and $y_i := \omega_{s_{\max}}^i$, it holds that

$$V^{(xy)}(x_h, y_i) U^{(xy)}(x_h, y_i) = W^{(xy)}(x_h, y_i). \tag{109}$$

From (106), by applying Corollary 1 and substituting (102), we can conclude that, for all $j \in \{0,\cdots,m_I-1\}$ and $i \in \{0,\cdots,s_{\max}-1\}$ and $z_j := \omega_{m_I}^h$ and $y_i := \omega_{s_{\max}}^i$, it holds that

$$\begin{pmatrix} \left(R^{(xy)}(z_j, y_i)-1\right) K_{-1}(z_j) L_{-1}(y_i) \\ +\kappa_0 \left(z_j-1\right)\left(R^{(xy)}(z_j, y_i) G^{(xy)}(z_j, y_i) - R^{(xy)}(z_{j-1}, y_i) F^{(xy)}(z_j, y_i)\right) \\ +\kappa_0^2 K_0(z_j)\left(R^{(xy)}(z_j, y_i) G^{(xy)}(z_j, y_i) - R^{(xy)}(z_{j-1}, y_{i-1}) F^{(xy)}(z_j, y_i)\right) \end{pmatrix} = 0. \tag{110}$$

We again apply Lemma 1 to separate (110) by κ_0. Among a total of N samples, the probability that there is at least one sample for which the following polynomial equation of indeterminates K_0 does not hold, but the equation (110) still holds, is not greater than $2N/|\mathbb{F}^*|$:

$$\begin{pmatrix} \left(R^{(xy)}(z_j, y_i)-1\right) K_{-1}(z_j) L_{-1}(y_i) \\ +\mathrm{K}_0 \left(z_j-1\right)\left(R^{(xy)}(z_j, y_i) G^{(xy)}(z_j, y_i) - R^{(xy)}(z_{j-1}, y_i) F^{(xy)}(z_j, y_i)\right) \\ +\mathrm{K}_0^2 K_0(z_j)\left(R^{(xy)}(z_j, y_i) G^{(xy)}(z_j, y_i) - R^{(xy)}(z_{j-1}, y_{i-1}) F^{(xy)}(z_j, y_i)\right) \end{pmatrix} = 0. \tag{111}$$

Under the event where (111) holds for all N samples, we can obtain four separated equations as follows,

$$\begin{aligned} &\left(R^{(xy)}(z_j, y_i)-1\right) K_{-1}(z_j) L_{-1}(y_i) = 0, \\ &\left(z_j-1\right)\left(R^{(xy)}(z_j, y_i) G^{(xy)}(z_j, y_i) - R^{(xy)}(z_{j-1}, y_i) F^{(xy)}(z_j, y_i)\right), \\ &K_0(z_j)\left(R^{(xy)}(z_j, y_i) G^{(xy)}(z_j, y_i) - R^{(xy)}(z_{j-1}, y_{i-1}) F^{(xy)}(z_j, y_i)\right). \end{aligned} \tag{112}$$

We write $r_{i,j} := R^{(xy)}(z_j, y_i)$, $g_{i,j} := G^{(xy)}(z_j, y_i)$, and $f_{i,j} = F^{(xy)}(z_j, y_i)$. Equations in (112) imply that

$$\begin{cases} r_{s_{\max}-1, m_I-1} = 1, \\ r_{i,j} g_{i,j} = r_{i,j-1} f_{i,j}, \quad \text{for } (i,j) \in \{0,\cdots,s_{\max}-1\} \times \{1,\cdots,m_I-1\}, \\ r_{i,j} g_{i,j} = r_{i-1,j-1} f_{i,j}, \quad \text{for } (i,j) \in \{0,\cdots,s_{\max}-1\} \times \{0\}. \end{cases}$$

By solving this recursion, we obtain,

$$\prod_{j=0}^{s_{\max}-1}\prod_{k=0}^{m_I-1} B^{(xy)}(z_j, y_i) + \theta_0 s^{(0)}(y_i, z_j) + \theta_1 s^{(1)}(y_i, z_j) + \theta_2 = \prod_{j=0}^{s_{\max}-1}\prod_{k=0}^{m_I-1} B^{(xy)}(z_j, y_i) + \theta_0 s^{(2)}(z_j, y_i) + \theta_1 y_i + \theta_2. \tag{113}$$

Suppose the emulator has obtained N samples of (113), each contains random elements $\theta_0, \theta_1, \theta_2 \in \mathbb{F}^*$. We check that each equation sample does not identically hold with respect to $\theta_0, \theta_1, \theta_2$ but holds only for the picked values of $\theta_0, \theta_1, \theta_2$. By Lemma 1, this probability is upper bounded to $3/|\mathbb{F}^*|$, where the numerator equals the sum of the highest exponents of $\theta_0, \theta_1, \theta_2$ in (113). Since we have N samples, the probability that at least one equation sample fall into this event is not greater than $3N/|\mathbb{F}^*|$.

Suppose that the N equation instances of (113) did not fall into this event. Then, by Lemma 3, $B^{(xy)}(X, Y)$ satisfies the copy constraint.

Simultaneously, substituting (85) into (109) implies that $(\mathbf{a}, (\mathbf{b}(Y), \mathbf{c}(Y)))$ satisfies the arithmetic constraint.

Note that by equation (87), we can extract $\mathbf{b}(Y)$ in (108) from the evaluations $B^{(xy)}(z_j, y_i)$ in (113). Thus, we can conclude that $(\mathbf{a}, (\mathbf{b}(Y), \mathbf{c}(Y))) \in R$, where R is defined in (20).

7) Counting the probability of extraction failure

The extraction so far have assumed that all the intermediate polynomials are well constructed so that the relevant equations with those polynomials are valid for any samples of $\boldsymbol{\tau}$, θ_0, θ_1, θ_2, κ_0, χ, ζ, κ_0, κ_1, κ_2.

The extraction fails when the polynomials are not well-constructed, and the probability of this is bounded above by

$$\Pr\left[Extraction \ \ failure\right] \le \frac{N\left(5N + 2(n + s_{\max} + m_I) + 12\right)}{\left|\mathbb{F}^*\right|}.$$