STA510, fall 2016. Compulsory assignment 2.

A report with solutions is to be handed in to me within Thursday, November 3. The report may be delivered attached to an email to me, or preferably on its's:learning. Remember to write your NAME on the report/file.

Each student is to work independently with the problems and deliver her/his own report.

The report should include an R script file with R code for the solution of the problems, and a short explanation of the statistical background for the solutions.

The statistical explanation could very well be hand written and scanned.

NOTE: All R code must be free of bugs and it must be ready to run it for checks.

Problem 1

A reliability system consists of 15 independent components. The probability that a component works is 0.9 for each. The system works if at least 11 of the components work.

a) Find the probability that the system works.

In another context the 15 probabilities are:

0.9, 0.9, 0.7, 0.8, 0.7, 0.9, 0.7, 0.7, 0.8, 0.8, 0.7, 0.9, 0.8, 0.9, 0.8

b) Find a simulation estimate of the probability that system works in this situation. Also find the standard error corresponding to the estimate.

Problem 2

We have that

$$\int_0^1 4\sqrt{1-x^2}dx = \pi. \tag{1}$$

a) Use (1) to show that $\mathrm{E}\Big(4\sqrt{1-U^2}\Big) = \pi$ when U is a uniform random variable on $[0,1]$. Also, show that $\mathrm{Var}\Big(4\sqrt{1-U^2}\Big) = 32/3 - \pi^2$.

b) Use (1) to make a Monte Carlo estimate of π; write the `R` code.

Problem 3

Let $N(t)$ be a Poisson process with intensity $\lambda = 2$.

a) Produce `R` code to simulate realizations of $N(t)$ in the interval $[0, 20]$.

b) What is the expected value and standard deviation of $N(5)$ and of $N(20)$?

c) Plot 30 independent realizations of the process in the same figure.
Comment on the pattern of the realizations in relation to the results in b).

Problem 4

Two different ways to perform a specific task are considered. The task consists of two jobs each lasting an exponentially distributed amount of time with expectation one week. Let T_1 and T_2 be the the times job 1 and 2 lasts. The time to finish the task is then $Y = T_1 + T_2$.

The two ways to perform the task results in strong correlation between the job times T_1 and T_2. In one case negative correlation and in the other positive.

The following procedure to generate correlated exponentially distributed variables is to be used:

1. *Generate two $N(0,1)$ random variables, X_1 and X_2, with correlation ρ.*
2. *Let $U_1 = \phi^{-1}(X_1)$ and $U_2 = \phi^{-1}(X_2)$, where ϕ^{-1} is the inverse function of $\phi(x) = P(X \leq x)$ for $X \sim N(0,1)$. (See the* `R` *function* `pnorm` *regarding $\phi^{-1}(u)$.)*
3. *Let $T_1 = -\ln(U_1)$ and $T_2 = -\ln(U_2)$.*

a) Explain why T_1 and T_2 are exponentially distributed with mean one.

b) Simulate values of (T_1, T_2) and plot a scatter plot, T_2 v.s. T_1, when $\rho = -0.9$ and when $\rho = 0.9$. Estimate the correlation between T_1 and T_2 and check the scatter plots of U_2 v.s. U_1 in both cases. Use the Choleski factorization technique to simulate X_1 and X_2. (Hint: it might be helpful to know that matrix multiplication in `R` is done by the operator $\% * \%$; see pages 5 and 6 in the text book.)

c) Simulate $m = 10^4$ values of $Y = T_1 + T_2$ for $\rho = -0.9$ and for $\rho = 0.9$. In both cases estimate the distribution of Y by drawing a histogram.

d) Estimate $P(Y \geq 4)$ and the corresponding standard error in both cases.
Does correlation affect $\mathrm{E}(Y)$?