Fermions

Gauging of the Lorentz group

$\omega^{\alpha\beta}$ – Lorentz transformations (small)

$$A \to \Lambda A = \left(1 + \frac{1}{2}\omega^{\alpha\beta}\Sigma_{\alpha\beta}\right)A \quad , \quad \Sigma \text{ – generators}$$

$$[\Sigma_{\alpha\beta}\, \Sigma_{\gamma\delta}] = \eta_{\beta\gamma}\Sigma_{\alpha\delta} + \eta_{\alpha\delta}\Sigma_{\beta\gamma} - \eta_{\alpha\gamma}\Sigma_{\beta\delta} - \eta_{\beta\delta}\Sigma_{\alpha\gamma}$$

$$A^\alpha \to A^\alpha + \omega^\alpha{}_\beta A^\beta \quad \text{– vector}$$

$$[\Sigma_{\alpha\beta}]^\gamma{}_\delta = \delta^\gamma_\alpha \eta_{\beta\delta} - \delta^\gamma_\beta \eta_{\alpha\beta} \quad \text{– generators for vector rep.}$$

$$\Sigma_{\alpha\beta} = \frac{1}{4}[\gamma_\alpha \gamma_\beta] \quad \text{– spinor representation}$$

$$A(x) \to \Lambda(x)A(x) = \left(1 + \frac{1}{2}\omega^{\alpha\beta}(x)\Sigma_{\alpha\beta}\right)A(x)$$

gauge field Γ_μ : $\mathcal{D}_\mu A = \partial_\mu A + \Gamma_\mu A$ (spin connection)

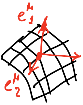
vector: $\Gamma_\mu(x) = \frac{1}{2}\Gamma_\mu^{\alpha\beta}\Sigma_{\alpha\beta}$, $(\Gamma_\mu A)^\alpha = \Gamma_\mu^{\alpha\beta}A_\beta$

spinor: $\Gamma_\mu = \frac{1}{4}\Gamma_\mu^{\alpha\beta}\gamma_\alpha\gamma_\beta$

$$\mathcal{D}_\mu A(x) \to \Lambda(x)(\mathcal{D}_\mu A)(x) \quad \text{– transforms uniformly}$$

$$\Gamma_\mu \to \Lambda\Gamma_\mu\Lambda^{-1} + \Lambda\partial_\mu\Lambda^{-1} = \Gamma_\mu - \frac{1}{2}\left(\partial_\mu \omega + [\Gamma_\mu, \omega]\right)$$

$$\omega(x) = \omega^{\alpha\beta}(x)\Sigma_{\alpha\beta}$$



e^μ_α $(\alpha = 0,1,2,3)$ - tetrad, vierbein

μ – spacetime index, $\alpha, \beta \ldots$ – locally Lorentz index

$$e_{\alpha\mu} e^\mu_\beta = \eta_{\alpha\beta}$$

$$e^\alpha_\mu e^\nu_\alpha = \delta^\nu_\mu$$

$$e^\alpha_\mu e_{\alpha\nu} = g_{\mu\nu}$$

$A^\mu \leftrightarrow A^\alpha e^\mu_\alpha$ – map between spacetime objects and tensors in locally Lorentz frame

Mapping covariant derivatives

$$\mathcal{D}_\nu A^\alpha = e^\alpha_\mu \nabla_\nu A^\mu \quad \text{(definition of } \mathcal{D}_\nu\text{)}$$

$$\partial_\nu A^\alpha + \Gamma_\nu^{\alpha\beta} A_\beta = e^\alpha_\mu \left(\partial_\nu \left(e^\mu_\beta A^\beta\right) + \Gamma^\mu_{\nu\lambda} e^{\beta\lambda} A_\beta\right) =$$

$$= e^\alpha_\mu \left(e^\mu_\beta \partial_\nu A^\beta + A^\beta \partial_\nu e^\mu_\beta + \Gamma^\mu_{\nu\lambda} e^{\beta\lambda} A_\beta\right) =$$

$$\Gamma_\nu^{\alpha\beta} = e^\alpha_\mu \nabla_\nu e^{\beta\mu} \quad \text{– spin connection } (\nabla_\nu \text{ acts only on space})$$

$$\nabla_\nu g^{\mu\nu} \Rightarrow e^\alpha_\mu \nabla_\nu e^{\beta\mu} + e^{\beta\mu} \nabla_\nu e^\alpha_\mu = 0 \quad \left(\Gamma_\nu^{\alpha\beta} = -\Gamma_\nu^{\beta\alpha}\right)$$

$$\hat{D}_\nu e^{\alpha\mu} \equiv \partial_\nu e^{\alpha\beta} + \Gamma^\mu_{\nu\lambda} e^{\alpha\lambda} + \Gamma_\nu^{\alpha\beta} e^\mu_\beta = 0$$

(„~~total~~" covariant derivative)

$$\mathcal{D}_\alpha = e^\mu_\alpha \mathcal{D}_\mu$$

$$S_F = \int d^4x \sqrt{-g} \left(i \bar{\psi} \gamma^\alpha \mathcal{D}_\alpha \psi - m \bar{\psi} \psi\right)$$

$$\mathcal{D}_\alpha \psi = e^\mu_\alpha \left(\partial_\mu + \frac{1}{4} \Gamma_\mu^{\beta\gamma} \gamma_\beta \gamma_\gamma\right) \psi$$

$$i \gamma^\alpha \mathcal{D}_\alpha \psi - m \psi = 0$$

$$i \gamma^\alpha \mathcal{D}_\alpha = i \gamma^\alpha \partial_\alpha + \frac{i}{4} \gamma^\alpha [\gamma_\beta \gamma_\gamma] e_{\gamma\nu} \nabla_\mu e^{\beta\nu} e^{\alpha\mu} =$$

$$= i \left(\gamma^\alpha e^\mu_\alpha \partial_\mu + \frac{1}{2} \gamma^\alpha \left(\nabla_\mu e^\mu_\alpha\right)\right) + \frac{1}{4} \gamma^\alpha \gamma^5 \epsilon_{\alpha\beta\gamma\delta} e^{\beta\mu} e^{\gamma\nu} \nabla_\mu e^\delta_\nu$$

used:

$$\frac{1}{2} \gamma^\alpha [\gamma^\beta \gamma^\gamma] = \left(\eta^{\alpha\beta} \gamma^\gamma - \eta^{\alpha\gamma} \gamma^\beta\right) + i \epsilon^{\alpha\beta\gamma\delta} \gamma_\delta \gamma^5$$

$$\gamma^5 = i \gamma^0 \gamma^1 \gamma^2 \gamma^3$$

$$ds^2 = a^2(\eta) \eta_{\mu\nu} dx^\mu dx^\nu$$

$$e^\alpha_\mu = a(\eta) \delta^\alpha_\mu, \quad \Gamma^\mu_{0\nu} = \frac{a'}{a} \delta^\mu_\nu, \quad \Gamma^0_{ij} = \frac{a'}{a} \delta_{ij}$$

$$\nabla_i e^0_\alpha = \frac{a'}{a^2} \delta_{i\alpha}, \quad \nabla_i e^j_\alpha = \frac{a'}{a^2} \delta^j_i \delta^0_\alpha$$

$$i\gamma^\alpha \mathcal{D}_\alpha = i\left(\frac{1}{a}\gamma^\mu \partial_\mu + \frac{3}{2}\frac{a'}{a^2}\gamma^0\right) \quad (\epsilon\text{-term} \to 0)$$

$$i\left(\frac{1}{a}\gamma^\mu \partial_\mu + \frac{3a'}{2a^2}\gamma^0\right)\psi - m\psi = 0$$

$$\psi = a^{-3/2}\chi$$

$$i\gamma^\mu \partial_\mu \chi - am\chi = 0$$

$$\hat{g}_{\mu\nu} = e^{2\varphi} g_{\mu\nu}$$

$$i\gamma^\alpha \hat{\mathcal{D}}_\alpha \hat{\psi} = e^{-5\varphi/2}\, i\gamma^\alpha \mathcal{D}_\alpha \psi \,, \quad \hat{\psi} = e^{-3\varphi/2}\psi$$

$$S_F(\hat{g}_{\mu\nu}, \hat{\psi}) = S_F(g_{\mu\nu}, \psi) \quad (\text{for } m=0)$$