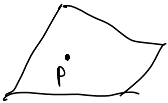
Riemannian normal coordinates



$g_{\mu\nu}(p) = \eta_{\mu\nu}$ (locally Lorentz frame)

$\partial_\rho g_{\mu\nu}\big|_p = 0, \quad \Gamma_{(p)} = 0$

$$g_{\mu\nu}(x) = \eta_{\mu\nu} - \frac{1}{3} R_{\mu\rho\nu\sigma} x^\rho x^\sigma + \dots O(x^3)$$

derivation:

$$g_{\mu\nu}(x) = g_{\mu\nu}(p) + \underbrace{\partial_\rho g_{\mu\nu}(p)}_{=0} x^\rho + \frac{1}{2} \partial_\rho \partial_\sigma g_{\mu\nu}(p) x^\rho x^\sigma + \dots$$

$$R_{\mu\nu\lambda\rho} = \frac{1}{2}\left(\partial_\nu \partial_\lambda g_{\mu\rho} + \partial_\mu \partial_\rho g_{\nu\lambda} - \partial_\nu \partial_\rho g_{\mu\lambda} - \partial_\mu \partial_\lambda g_{\nu\rho}\right)$$

$$x^\nu x^\rho R_{\mu\nu\lambda\rho} = x^\nu x^\rho \left(\partial_\nu \partial_\lambda g_{\mu\rho} - \partial_\mu \partial_\lambda g_{\nu\rho}\right)$$

$$\partial_\rho \partial_\sigma g_{\mu\nu} = -\frac{1}{3}\left(R_{\mu\rho\nu\sigma} + R_{\mu\sigma\nu\rho}\right)$$

Inverse metric

$$g^{\mu\nu} = g^{\mu\nu} + \frac{1}{3} R_{\alpha\beta\gamma\delta} x^\alpha x^\gamma g^{\beta\mu} g^{\delta\nu} + \dots$$

$$-g = 1 - \frac{1}{3} x^\mu x^\nu R_{\mu\nu} - \frac{1}{6} x^\mu x^\nu x^\sigma \nabla_\mu R_{\nu\sigma}$$

$$\partial_\rho \partial_\sigma g_{\mu\nu} = \alpha R_{\mu\rho\nu\sigma}$$

$$x^\nu x^\rho R_{\mu\nu\lambda\rho} = x^\nu x^\rho \alpha \left(R_{\mu\nu\rho\lambda} - R_{\nu\mu\rho\lambda}\right)$$

$$R_{abcd} + R_{acdb} + R_{adbc} = 0$$

$$R_{\nu\mu\rho\lambda} = -\underset{\mu\ \rho}{R_{\nu\rho\lambda\mu}} - R_{\nu\lambda\mu\rho}$$

$$g_{\mu\nu} = g_{\mu\nu} - \frac{1}{3} x^\rho x^\sigma R_{\mu\rho\nu\sigma} - \frac{1}{6} x^\lambda x^\rho x^\sigma \nabla_\lambda R_{\mu\rho\nu\sigma} +$$

$$+ \frac{1}{180} x^\alpha x^\beta x^\gamma x^\delta \left(8 g^{\rho\sigma} R_{\mu\alpha\beta\rho} R_{\nu\gamma\delta\sigma} - 9 \nabla_\alpha \nabla_\beta R_{\mu\gamma\nu\delta} \right) + \ldots$$