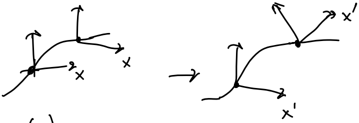
Action, Lagrangian and least action principle

4 dim spacetime, $ds^2 = g_{\mu\nu}(x)\, dx^\mu dx^\nu$, $(+---)$

$g_{\mu\nu} = g_{\nu\mu}$

Poincare invariance: all choices of the local coordinate frame are equivalent

$x^\mu \to x'^\mu(x^\mu)$



Tensors

scalar $\varphi(x)$; $\varphi'(x') = \varphi(x)$

vector $A'^\nu(x') = \dfrac{\partial x'^\nu}{\partial x^\mu} A^\mu(x)$ (contravariant)

$A'_\nu(x') = \dfrac{\partial x^\mu}{\partial x'^\nu} A_\mu(x)$ (covariant)

$A_\nu A^\nu$ - scalar

$B'^\mu{}_{\nu\lambda}(x') = \dfrac{\partial x'^\mu}{\partial x^\sigma} \dfrac{\partial x^\tau}{\partial x'^\nu} \dfrac{\partial x^\rho}{\partial x'^\lambda} B^\sigma{}_{\tau\rho}$

$g'_{\mu\nu}(x') = \dfrac{\partial x^\lambda}{\partial x'^\mu} \dfrac{\partial x^\rho}{\partial x'^\nu} g_{\lambda\rho}(x)$ - metric (ds^2 - scalar)

$\delta^\nu_\mu = (1)$ $\quad g^{\mu\nu} g_{\nu\lambda} = \delta^\mu_\lambda$ $\quad A^\nu = g^{\nu\mu} A_\mu$

$g = \det(g_{\mu\nu})$

$g'(x') = \left(\det\left(\dfrac{\partial x^\mu}{\partial x'^\nu}\right)\right)^2 g(x)$

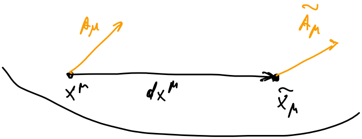
$\sqrt{-g}\, d^4x$ - scalar

$\epsilon^{\mu\nu\lambda\rho}$, $\epsilon^{0123} = 1$ $\quad \dfrac{\epsilon^{\mu\nu\lambda\rho}}{\sqrt{g}}$ - tensor

Covariant derivative

$\nabla_\mu \varphi = \partial_\mu \varphi$ (definition)

$\partial_\mu A_\nu$ - not a tensor



Parallel transport

$$\tilde{A}^\mu(\tilde{x}) = A^\mu(x) - \Gamma^\mu_{\nu\lambda}(x)\, A^\nu(x)\, dx^\lambda$$

$\tilde{A}^\mu, A^\mu, dx^\mu$ – transform as vectors

$$\tilde{A}'^\mu(\tilde{x}') = \frac{\partial x'^\mu(\tilde{x})}{\partial x^\nu}\tilde{A}^\nu(\tilde{x}) = \left(\frac{\partial x'^\mu(x)}{\partial x^\nu} + \frac{\partial^2 x'^\mu(x)}{\partial x^\nu \partial x^\lambda}\right)\tilde{A}^\nu(\tilde{x})$$

$$= \frac{\partial x'^\mu(x)}{\partial x^\nu}\tilde{A}^\nu(\tilde{x}) + \frac{\partial^2 x'^\mu(x)}{\partial x^\nu \partial x^\lambda}\, dx^\lambda A^\nu(x)$$

$$A'^\mu(x') - \Gamma'^\mu_{\nu\lambda} A'^\nu(x')\, dx'^\lambda = \frac{\partial x'^\mu}{\partial x^\nu} A^\nu(x) -$$

$$- \Gamma'^\mu_{\nu\lambda}(x') \frac{\partial x'^\nu}{\partial x^\rho} A^\rho(x) \frac{\partial x'^\lambda}{\partial x^\sigma}\, dx^\sigma = \tilde{A}'^\mu(\tilde{x}')$$

$$\Gamma'^\mu_{\nu\lambda} = \frac{\partial x^\rho}{\partial x'^\nu}\frac{\partial x^\sigma}{\partial x'^\lambda}\frac{\partial x'^\mu}{\partial x^\zeta}\Gamma^\zeta_{\rho\sigma} + \underbrace{\frac{\partial x'^\mu}{\partial x^\rho}\frac{\partial^2 x^\rho}{\partial x'^\nu \partial x'^\lambda}}_{\text{not a tensor part}}$$

$$A^\mu(\tilde{x}) - \tilde{A}^\mu(\tilde{x}) = \nabla_\nu A^\mu dx^\nu$$ – definition of cov. ∂

$$\nabla_\nu A^\mu = \partial_\nu A^\mu + \Gamma^\mu_{\lambda\nu} A^\lambda$$

$$\nabla_\mu B_\nu = \partial_\mu B_\nu - \Gamma^\lambda_{\mu\nu} B_\lambda \qquad (\nabla_\rho A^\mu B_\mu)$$

$$\nabla_\mu (AB) = \nabla_\mu A \cdot B + A \nabla_\mu B$$

Riemann geometry: Parallel transport commutes with raising and lowering indices

$$\nabla_\mu g_{\nu\lambda} = 0$$

$$\partial_\mu g_{\nu\lambda} = \Gamma^\rho_{\nu\mu} g_{\rho\lambda} + \Gamma^\rho_{\lambda\mu} g_{\nu\rho}, \qquad \Gamma_{\mu\nu} = \Gamma_{\nu\mu} \text{ (no torsion)}$$

$$\Gamma^{\mu}_{\nu\lambda} = \frac{1}{2} g^{\mu\rho}\left(\partial_\nu g_{\rho\lambda} + \partial_\lambda g_{\rho\nu} - \partial_\rho g_{\nu\lambda}\right)$$

Going to locally Lorentz frame:

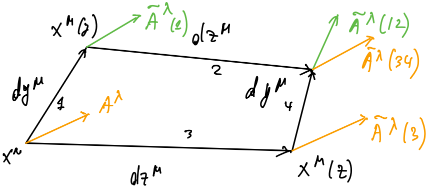
$$x^\mu \to x^{\mu\prime} = x^\mu + \frac{1}{2}\Gamma^{\mu}_{\nu\lambda}(0) x^\nu x^\lambda$$ - always possible

$$g_{\mu\nu}(0) = \eta_{\mu\nu}, \quad \Gamma(0) = 0$$

A tensor independent of the reference frame - curvature

$$\nabla_\mu \nabla_\nu A^\lambda - \nabla_\nu \nabla_\mu A^\lambda = A^\sigma R^\lambda{}_{\sigma\mu\nu}$$

$$R^\mu{}_{\nu\rho\lambda} = \partial_\lambda \Gamma^\mu{}_{\nu\rho} - \partial_\rho \Gamma^\mu_{\nu\lambda} + \Gamma^\mu_{\sigma\lambda}\Gamma^\sigma_{\nu\rho} - \Gamma^\mu_{\sigma\rho}\Gamma^\sigma_{\nu\lambda}$$



$1 \to 2$

$3 \to 4$

$$\tilde{A}^\lambda(12) - \tilde{A}^\lambda(34) = A^\sigma R^\lambda{}_{\sigma\mu\nu}\, dz^\mu dy^\nu$$

Properties:

$$(\underbrace{(\mu\nu)(\lambda\rho)}) \quad \text{- symmetries}$$

$$R_{\rho\mu\nu\lambda} + R_{\rho\lambda\mu\nu} + R_{\rho\nu\lambda\mu} = 0$$

$$\nabla_\rho R^\lambda{}_{\sigma\mu\nu} + \nabla_\nu R^\lambda{}_{\sigma\rho\mu} + \nabla_\mu R^\lambda{}_{\sigma\nu\rho} = 0 \quad \text{(Bianchi)}$$

$$R_{\mu\nu} = R^\lambda{}_{\mu\lambda\nu}, \quad R = g^{\mu\nu}R_{\mu\nu}$$

<u>Gravitational field equations from least action principle</u>

In mechanics:

$$S = \int_{t_1}^{t_2} L(q, \dot{q}, t)\, dt, \quad L = K - V \qquad L = \frac{1}{2} m \dot{x}^2 - V(x)$$ (a particle in a potential)

$$\delta S = 0 \rightarrow m\ddot{x} = V'(x)$$

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}_i} - \frac{\partial L}{\partial q_i} = 0$$

In field theory:

$$dt \rightarrow d^4x, \quad S = \frac{1}{2}\int d^4x \left((\partial_\mu \varphi)^2 - m^2 \varphi^2 \right)$$

$$\delta S = 0 \rightarrow \Box \varphi + m^2 \varphi = 0$$

In curved space

Action – scalar $(\sqrt{-g}\, d^4x, \ R)$

$$\boxed{S_{EH} = -\frac{1}{16\pi G}\int d^4x\, R \sqrt{-g}}$$ → the only possibility in 4d w. EOM with 2 derivatives

degrees of freedom – metric $g_{\mu\nu}$

$$\frac{\delta S}{\delta g_{\mu\nu}} = 0 \rightarrow \text{EOM}$$

$$\det(M + \delta M) = \det M \left(1 + \operatorname{Tr}(M^{-1}\delta M) + o(\delta M) \right)$$

$$\delta g = g\, g^{\mu\nu}\delta g_{\mu\nu}$$

$$\delta S_\Lambda = -\Lambda \int d^4x\, \delta(\sqrt{-g}) = -\frac{1}{2}\Lambda \int d^4x \sqrt{-g}\, g^{\mu\nu}\delta g_{\mu\nu}$$

$$\delta S_{EH} = \delta S_1 + \delta S_2 + \delta S_3$$

$$\delta S_1 = -\frac{1}{16\pi G}\int d^4x\, R\, \delta(\sqrt{-g}) = -\frac{1}{32\pi G}\int d^4x \sqrt{-g}\, R\, g^{\mu\nu}\delta g_{\mu\nu}$$

$$\delta S_2 = -\frac{1}{16\pi G}\int d^4x \sqrt{-g}\, R_{\mu\nu}\delta g^{\mu\nu}$$

$$\delta S_3 = -\frac{1}{16\pi G}\int d^4x \sqrt{-g}\, g^{\mu\nu}\delta R_{\mu\nu}$$

$$g_{\rho\lambda}\,\delta g^{\mu\rho} = -g^{\mu\rho}\,\delta g_{\rho\lambda} \quad | \cdot g^{\lambda\nu}$$

$$\delta g^{\mu\nu} = -g^{\mu\rho}\,\delta g_{\rho\lambda}\, g^{\lambda\nu}$$

$$\delta S_2 = +\frac{1}{16\pi G}\int d^4x\,\sqrt{-g}\,R^{\mu\nu}\,\delta g_{\mu\nu}$$

$$\delta R^{\mu}{}_{\nu\lambda\rho} = \partial_\lambda \delta\Gamma^{\mu}_{\nu\rho} - \partial_\rho \delta\Gamma^{\mu}_{\nu\lambda} + \delta\Gamma^{\mu}_{\sigma\lambda}\Gamma^{\sigma}_{\nu\rho} + \Gamma^{\mu}_{\sigma\lambda}\delta\Gamma^{\sigma}_{\nu\rho} - \delta\Gamma^{\mu}_{\sigma\rho}\Gamma^{\sigma}_{\nu\lambda} - \Gamma^{\mu}_{\sigma\rho}\delta\Gamma^{\sigma}_{\nu\lambda}$$

$$= \nabla_\lambda\left(\delta\Gamma^{\mu}_{\nu\rho}\right) - \nabla_\rho\left(\delta\Gamma^{\mu}_{\nu\lambda}\right) \qquad (\nabla_\rho \text{ - non-perturbed})$$

$$\delta R_{\mu\nu} = \nabla_\lambda\left(\delta\Gamma^{\lambda}_{\mu\nu}\right) - \nabla_\nu\left(\delta\Gamma^{\lambda}_{\mu\lambda}\right)$$

$$\delta S_3 = -\frac{1}{16\pi G}\int d^4x\,\sqrt{-g}\,g^{\mu\nu}\left[\nabla_\lambda\left(\delta\Gamma^{\lambda}_{\mu\nu}\right) - \nabla_\nu\left(\delta\Gamma^{\lambda}_{\mu\lambda}\right)\right] =$$

$$= -\frac{1}{16\pi G}\int d^4x\,\sqrt{-g}\,\nabla_\lambda\left(g^{\mu\nu}\delta\Gamma^{\lambda}_{\mu\nu} - g^{\mu\lambda}\delta\Gamma^{\sigma}_{\mu\sigma}\right) \text{ - } \underline{\text{tot. } \partial}$$

(boundary term)

$$\boxed{R^{\mu\nu} - \frac{1}{2}g^{\mu\nu}R = 8\pi G\Lambda g^{\mu\nu}}$$

$$G^{\mu\nu} = 8\pi G\,T^{\mu\nu}, \quad T^{\mu\nu} = \frac{\delta(\mathcal{L}\sqrt{-g})}{\delta g_{\mu\nu}}$$ - energy-momentum tensor of matter

10 equations on metric components