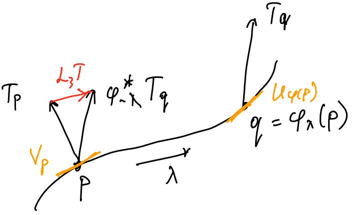
# Conformal symmetry and conformal Killing vector field

## Lie derivative




Map $\varphi : M \to M$

$\varphi^* : V_p \to U_{\varphi(p)}$

$p \to q$ - diffeomorphisms induced by the vector field $\xi$, $\lambda$ - group parameter

$$\mathcal{L}_\xi T^{\mu_1 \dots}{}_{\nu_1 \dots}(p) = \lim_{\lambda \to 0} \frac{\varphi^*_{-\lambda} T^{\mu_1 \dots}{}_{\nu_1 \dots}(q) - T^{\mu_1 \dots}{}_{\nu_1 \dots}(p)}{\lambda}$$

$\xi$ - tangent vector of the curve $\varphi(\lambda)$

3 types of derivatives:

- $\partial_\nu T^{\mu \dots}$ (doesn't preserve tensor properties)
- $\nabla_\nu T^{\mu \dots}$ (preserves tensor properties but changes the rank)
- $\mathcal{L}_\xi T^{\mu \dots}$ (tensor + preserving the rank)

Properties of Lie derivatives

- $\mathcal{L}_\xi f = \partial_\mu f \xi^\mu$
- $\mathcal{L}_\xi \eta^\mu = \partial_\nu \eta^\mu \xi^\nu - \partial_\nu \xi^\mu \eta^\nu$
- $\mathcal{L}_\xi \omega_\mu = \partial_\nu \omega_\mu \xi^\nu + \omega_\nu \partial_\mu \xi^\nu$

$$\mathcal{L}_\xi T^{\mu_1 \dots \mu_k}{}_{\nu_1 \dots \nu_k} = \partial_\lambda T^{\mu_1 \dots \mu_k}{}_{\nu_1 \dots \nu_k} \xi^\lambda - \partial_\lambda \xi^{\mu_1} T^{\lambda \dots \mu_k}{}_{\nu_1 \dots} - \dots$$
$$- \partial_\lambda \xi^{\mu_k} T^{\mu_1 \dots \lambda}{}_{\nu_1 \dots \nu_k} + \partial_{\nu_1} \xi^\lambda T^{\mu_1 \dots \mu_k}{}_{\lambda \dots \nu_k} + \dots + \partial_{\nu_k} \xi^\lambda T^{\mu_1 \dots \mu_k}{}_{\nu_1 \dots \lambda}$$

## Killing vector fields

If $\mathcal{L}_\xi T^{\mu_1 .. \mu_k} = 0$, and $\varphi_\lambda$ is one-parameter group of $T$ determined by the vector field $\xi$, then $\varphi_t$ is a symmetry group of $T$ ,

$$\mathcal{L}_\xi g_{\mu\nu} = \nabla_\nu \xi_\mu + \nabla_\mu \xi_\nu + \nabla_\lambda g_{\mu\nu} \xi^\lambda = \nabla_\nu \xi_\mu + \nabla_\mu \xi_\nu$$

$\xi$ is called a Killing vector of metric

$\varphi_\lambda : \bar{x}^\mu = x^\mu + \xi^\mu d\lambda$ - isometry

If $g_{\mu\nu}$ doesn't depend on $x^\alpha$ then $\xi^\lambda_{(\alpha)} = \delta^\lambda_{(\alpha)}$

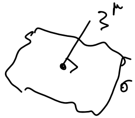
(if $g_{\mu\nu}(x_1) = const(x_1)$ then $\xi_\mu = (\xi_1, 0, 0, 0)$)

## Energy-momentum tensor

$P^\mu = T^\mu{}_\nu \xi^\nu$ (if $\xi^\nu$ is a killing vector)

$\nabla_\mu P^\mu = 0$

$$\int_D \nabla_\mu P^\mu dV_4 = \int_{\partial D} P^\mu d\sigma_\mu = 0 ,$$




$E_0 = \int_\sigma P^0 dV_3$ - conserved

If there is $\xi^\mu = \delta^\mu_0$ - timelike killing vector

$P^0 = T^0{}_\nu \xi^\nu = T^0{}_\nu \delta^\nu_0 = T^0_0$ - energy density

Energy conservation makes sense only if there is a timelike Killing vector field.

## Minkowski

$$\partial_\mu \xi_\nu + \partial_\nu \xi_\mu = 0 \qquad \partial_\nu \partial_\rho \xi_\mu + \partial_\mu \partial_\rho \xi_\nu = 0$$

$$\partial_\nu \partial_\rho \xi_\mu = 0 \qquad \partial_\rho \partial_\mu \xi_\nu + \partial_\nu \partial_\mu \xi_\rho = 0$$

$$\partial_\mu \partial_\nu \xi_\rho + \partial_\rho \partial_\nu \xi_\mu = 0$$

$$\xi_\mu = C_\mu + \varepsilon_{\mu\nu} x^\nu, \quad \varepsilon_{\mu\nu} = -\varepsilon_{\nu\mu}$$

10 killing vectors: 3d rotations (3) + Lorentz (3) + translations (4)

## 2d sphere

$$ds^2 = d\theta^2 + \sin^2\theta \, d\varphi^2 = (dx^1)^2 + \sin^2 x^1 (dx^2)^2$$

$$\partial_1 \xi_1 = 0 \qquad \partial_2 \xi_1 + \partial_1 \xi_2 = 0 \qquad \partial_2 \xi_2 = 0$$

$$\partial_1 \xi^1 = 0 \qquad \partial_2 \xi^1 + \sin^2\theta \, \partial_1 \xi^2 = 0 \qquad \xi^1 \cos\theta + \partial_2 \xi^2 \sin\theta = 0$$

$$\partial_\theta \xi^\theta = 0 \qquad \partial_\varphi \xi^\theta + \sin^2\theta \, \partial_\theta \xi^\varphi = 0 \qquad \xi^\theta \cos\theta + \partial_\varphi \xi^\varphi \sin\theta = 0$$

$$\xi^\theta = A \sin(\varphi + \alpha), \quad \xi^\varphi = A \cos(\varphi + \alpha) \cot\theta + b$$

$$\xi^\theta = A\cos\alpha \sin\varphi + A\sin\alpha \cos\varphi$$

$$\xi^\varphi = A\cos\alpha \cos\varphi \cot\theta + A\sin\alpha \sin\varphi \cot\theta + b$$

$$\xi_{(1)} = (\sin\varphi, \cos\varphi \cot\theta)$$

$$\xi_{(2)} = (\cos\varphi, \sin\varphi \cot\theta)$$

$$\xi_{(3)} = (0, 1)$$

$$\xi = A\cos\alpha \, \xi_{(1)} + A\sin\alpha \, \xi_{(2)} + b \, \xi_{(3)}$$

## Schwarzschild spacetime

$$R^1 \times R^1 \times S^2$$

$$\xi_{(1)} = (0, 0, \sin\varphi, \cos\varphi \cot\theta) \qquad \xi_{(3)} = (0, 0, 0, 1) \left(\frac{\partial}{\partial\varphi}\right)$$

$$\xi_{(2)} = (0, 0, \cos\varphi, \sin\varphi \cot\theta) \qquad \xi_{(4)} = (1, 0, 0, 0) \left(\frac{\partial}{\partial t}\right)$$

# De sitter

Static Patch coordinates

$$X_0 = \alpha \, sh \frac{t}{\alpha}$$

$$X_1 = \alpha \, ch \frac{t}{\alpha} \cos \chi$$

$$X_2 = \alpha \, ch \frac{t}{\alpha} \sin \chi \cos \theta$$

$$X_3 = \alpha \, ch \frac{t}{\alpha} \sin \chi \sin \theta \cos \varphi$$

$$X_4 = \alpha \, ch \frac{t}{\alpha} \sin \chi \sin \theta \sin \varphi$$

$$-X_0^2 + X_i X^i = \alpha^2$$

$$\xi = \frac{\partial}{\partial t} \, , \qquad \xi = \sin \varphi \frac{\partial}{\partial \theta} + \cot\theta \cos \varphi \frac{\partial}{\partial \varphi}$$

$$L_{AB} = X_A \frac{\partial}{\partial X^B} - X_B \frac{\partial}{\partial X^A}$$

$L_{0i}$ - Lorentz boosts, $i = 1, 2, 3, 4$

$L_{ij}$ - space rot.

$$L_{01} = X_0 \frac{\partial}{\partial X_1} + X_1 \frac{\partial}{\partial X_0} =$$

$$= \frac{\alpha \, ch \frac{t}{\alpha} \cos \chi}{\alpha \, ch \frac{t}{\alpha}} \frac{\partial}{\partial t} +$$

$$+ \alpha \, sh \frac{t}{\alpha} \left( - \frac{\tan \chi}{\alpha \, ch \frac{t}{\alpha}} \frac{\partial}{\partial t} + \frac{1}{\alpha \, ch \frac{t}{\alpha} \cos \chi} \frac{\partial}{\partial \chi} \right) =$$

$$= \mathrm{ch}\,\frac{t}{2} \cos\chi \,\frac{\partial}{\partial\chi} + \frac{\cos\chi}{\mathrm{ch}\,\frac{t}{2}}\,\frac{\partial}{\partial t} \qquad (SO(1,4))$$

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Conformal group | Generators $(SO(d,2))$

$x^\mu \to x^\mu + a^\mu$ | $\partial^\mu$

$x^\mu \to \Lambda^\mu_\nu x^\nu$ | $x_\mu \partial_\nu - x_\nu \partial_\mu$

$x^\mu \to \lambda x^\mu$ | $x^\mu \partial_\mu$

$x^\mu \to \dfrac{x^\mu + b^\mu x^2}{1 + 2 b_\nu x^\nu + b_\mu b^\mu x_\nu x^\nu}$ | $K_\mu = 2 x_\mu x^\nu \partial_\nu - x^2 \partial_\mu$

$$g^{\mu\nu} \to \Omega^2(x)\, g^{\mu\nu}$$

1) $x^\mu \to \dfrac{x^\mu}{x^2}$, 2) $x^\mu \to x^\mu + b^\mu$, 3) $x^\mu \to \dfrac{x^\mu}{x^2}$

$$\Omega(x) = \frac{1}{1 + bx + b^2 x^2}$$

Stationary + axial symmetry

$\xi_{(\varphi)} = (0, 0, 0, 1)$ , $\xi_{(t)} = (1, 0, 0, 0)$

FRW, $k = 0$ – 6 vectors

$\xi_{(k)} = c_k + \varepsilon_{kl} x^l$ $k, l = 1, 2, 3$ $\varepsilon_{kl} = -\varepsilon_{lk}$

## Conformal killing vector fields

$\mathcal{L}_\xi g_{\mu\nu} = \lambda(x) g_{\mu\nu}$ → $\xi$ - conformal killing vector

$\varphi_\lambda^{\alpha} g_{\mu\nu} = c^2(\lambda) g_{\mu\nu}$ $\varphi_\lambda$ generated by $\xi$ – conformal group of the metric

$\mathcal{L}_\xi g_{\mu\nu} = \lim_{\lambda \to 0} \dfrac{c^2 g_{\mu\nu} - g_{\mu\nu}}{\lambda} = g_{\mu\nu} \left( \dfrac{c^2 - 1}{\lambda} \right)$

$c^2(\lambda) = 1 + a\lambda + \dots$

## Spacetimes

- Einstein universe : $ds^2 = dt^2 - \frac{1}{\Lambda}\left( \frac{dr^2}{1 - r^2} + r^2 d\Omega^2 \right)$
- Milne : $ds^2 = dt^2 - a^2 t^2 (dx_i^2)$
- Rindler : $ds^2 = a^2 x^2 dt^2 - dx_i^2$ (uniform acceleration in x)
- Open Einstein U. : $ds^2 = dt^2 - a^2 \left( \frac{dr^2}{1 + r^2} + r^2 d\Omega^2 \right)$

conformally connected : 1) flat → FRW ($k = 0, 1$) → Einstein + dS
2) Milne → FRW ($k = -1$) → Open E. → → static dS → Rindler

Conformally related metrics

$$\hat{g}_{\mu\nu} = e^{2\varphi(x)} g_{\mu\nu}$$

$$\Gamma^{\mu}_{\nu\lambda} = \frac{1}{2} g^{\mu\rho} \left( \partial_\nu g_{\rho\lambda} + \partial_\lambda g_{\rho\nu} - \partial_\rho g_{\nu\lambda} \right)$$

$$\hat{\Gamma}^{\mu}_{\nu\lambda} = \Gamma^{\mu}_{\nu\lambda} + \delta^{\mu}_{\lambda} \partial_\nu \varphi + \delta^{\mu}_{\nu} \partial_\lambda \varphi - g_{\nu\lambda} g^{\mu\rho} \partial_\rho \varphi$$

$$\hat{R}_{\mu\nu} = R_{\mu\nu} - 2 \nabla_\mu \nabla_\nu \varphi - g_{\mu\nu} g^{\lambda\rho} \nabla_\lambda \nabla_\rho \varphi + 2 \partial_\mu \varphi \partial_\nu \varphi -$$
$$- 2 g_{\mu\nu} g^{\rho\lambda} \partial_\lambda \varphi \partial_\rho \varphi$$

$$\hat{R} = e^{-2\varphi} \left( R - 6 g^{\mu\nu} \nabla_\mu \nabla_\nu \varphi - 6 g^{\mu\nu} \partial_\mu \varphi \partial_\nu \varphi \right)$$

$$\int \hat{R} \sqrt{-\hat{g}}\, d^4x = \int e^{2\varphi} R \sqrt{-g}\, d^4x + 6 \int (\partial_\mu \varphi)^2 \sqrt{-g}\, d^4x\, e^{2\varphi}$$

$$\int -\frac{1}{2} \partial_\mu \chi \partial_\nu \chi \hat{g}^{\mu\nu} d^4x \sqrt{-\hat{g}} = \int -\frac{1}{2} e^{2\varphi} (\partial_\mu \chi)^2 \sqrt{-g}$$

$$\int \left( -\frac{1}{2} (\partial_\mu \chi)^2 + \frac{1}{6} \hat{R} \chi^2 \right) \sqrt{-\hat{g}} \to \int \left( -\frac{1}{2} (\partial_\mu \psi)^2 + \frac{1}{6} R \psi^2 \right) \sqrt{-g}\, d^4x$$

$$\psi = e^{+\varphi} \chi$$

$$W_{\mu\nu\rho\sigma} = R_{\mu\nu\rho\sigma} - \frac{1}{2} \left( g_{\mu\rho} R_{\nu\sigma} - g_{\mu\sigma} R_{\nu\rho} - g_{\nu\rho} R_{\mu\sigma} + g_{\nu\sigma} R_{\mu\rho} \right) +$$
$$+ \frac{1}{6} R \left( g_{\mu\rho} g_{\nu\sigma} - g_{\mu\sigma} g_{\nu\rho} \right)$$

$W^{\mu}{}_{\nu\rho\sigma}$ - conformal