Energy conditions

Energy-momentum tensor

$G_{\mu\nu} = T_{\mu\nu}$

For any metric one can find matter forming it

Problems: closed timelike curves
causality violations

Restrictions on matter?

Isotropic fluid

$u^\mu = \frac{dx^\mu}{d\lambda}$

$$T^{\mu\nu} = \begin{pmatrix} \rho & & & \\ & p & & \\ & & p & \\ & & & p \end{pmatrix}$$ - in flat spacetime

$u^\mu = (1,0,0,0)$ - velocity vector for stationary fluid

$T^{\mu\nu} = (p+\rho) u^\mu u^\nu - p\eta^{\mu\nu}$ - change the reference frame

In curved space: $T^{\mu\nu} = (p+\rho) u^\mu u^\nu - p g^{\mu\nu}$

locally Lorentz frame

$\nabla^\mu T^{\mu\nu} = 0$

$$T^{\mu\nu} = \frac{2}{\sqrt{-g}} \frac{\delta}{\delta g_{\mu\nu}} S_m(g, \varphi)$$

Example 1 scalar field

$$\mathcal{L} = \frac{1}{2} (\partial_\mu \varphi)^2 + (3R - m^2)\varphi^2$$

$$\Box\varphi + m^2\varphi - 3R\varphi = 0$$

$$T_{\mu\nu} = \nabla_\mu \varphi \nabla_\nu \varphi - \frac{1}{2} g_{\mu\nu} \left((\nabla\varphi)^2 - m^2\varphi^2 \right) + 3 \left(g_{\mu\nu}\Box - \nabla_\mu \nabla_\nu + G_{\mu\nu} \right) \varphi^2$$

Example 2: $F_{\mu\nu}$

$$\mathcal{L} = -\frac{1}{4} F^{\mu\nu} F_{\mu\nu}$$

$$\nabla^\mu F_{\mu\nu} = 0$$

$$T_{\mu\nu} = F_{\mu\gamma} F_\nu{}^\gamma - \frac{1}{4} g_{\mu\nu} F^{\alpha\beta} F_{\alpha\beta}$$

Weak energy condition (WEC)

$T_{\mu\nu} u^\mu u^\nu \geq 0$ $\quad u^\mu$ - timelike, to future

$\rho \geq 0, \quad \rho + P \geq 0$

$G_{\mu\nu} u^\mu u^\nu \geq 0$

geometrical meaning : small ball, keep surface area const

$$\frac{V_{curved}}{V_{Minkowski}} \geq 1$$

Strong energy condition (SEC)

$$T_{\mu\nu} - \frac{1}{2} T_\rho{}^\rho g_{\mu\nu}) V^\mu V^\nu \geq 0$$

$$\left((\rho + P) u^\mu u^\nu + \frac{\rho - P}{2} g^{\mu\nu}\right) V_\mu V_\nu \geq 0$$

$$\begin{cases} \rho + P \geq 0 \\ \rho + 3P \geq 0 \end{cases}$$

geometric form: $R_{\mu\nu} V^\mu V^\nu \geq 0$

Raychaudhuri equation

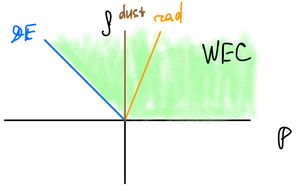
$$\boxed{\frac{d\theta}{d\lambda} = g^{\mu\nu} \frac{D B_{\mu\nu}}{d\lambda} = -\frac{1}{3}\theta^2 - \sigma_{\mu\nu}\sigma^{\mu\nu} + \omega_{\mu\nu}\omega^{\mu\nu} - R_{\mu\nu} V^\mu V^\nu}$$

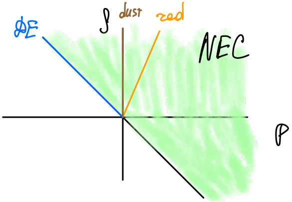
$$V^\mu B_{\mu\nu} = 0 \Rightarrow \sigma_{\mu\nu} = \sigma_{ij} \Rightarrow \sigma_{\mu\nu}\sigma^{\mu\nu} > 0$$

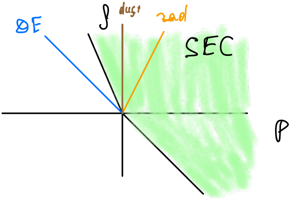
If $\omega_{\mu\nu} = 0$

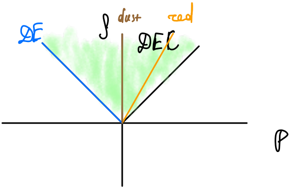
$$\frac{D\theta}{d\lambda} \leq -\frac{1}{3}\theta^2 - R_{\mu\nu} V^\mu V^\nu$$

- timelike and null geodesics converge, given SEC is satisfied









Dominant Energy condition (DEC)

$T_{\mu\nu}\xi^{\mu}V^{\nu} \geq 0$ $\quad V, \xi$ – $\forall$ timelike vectors (to future)

WEC + [$-T_{\mu\nu}\xi^{\mu}$ – future pointed timelike vector]

WEC + $V^{\gamma} T_{\mu\gamma} T^{\mu}{}_{\nu} V^{\nu} \leq 0$

- no superluminality
- well defined initial value problem
- flux of matter is causal

$T_{00} \geq |T_{\mu\nu}|$, T_{00} is dominant

$G_{\mu\nu} V^{\mu}\xi^{\nu} \geq 0$ – no clear geometric interpretation

Null energy condition (NEC)

$T_{\mu\nu}k^{\mu}k^{\nu} \geq 0$ $\quad k_{\mu}k^{\mu} = 0$ (null vector)

$p + \rho \geq 0$ (for observers on null geodesics)

$R_{\mu\nu}k^{\mu}k^{\nu} \geq 0$ – null convergence condition (Penrose theorem)

Cosmological applications

$$ds^2 = dt^2 + a^2_{(t)}(dx_i\, dx^i)$$

$$G_{\mu\nu} + \Lambda g_{\mu\nu} = 8\pi T_{\mu\nu}$$

$$a = e^{Ht}, \quad H = \sqrt{\frac{\Lambda}{3}}$$

* Cosmic no-hair conjecture (Wald, 1983)
The expanding Universe will locally approach dS, given SEC and DEC for $T_{\mu\nu}$ of matter

$$\frac{d\theta}{d\lambda} = \Lambda - 8\pi\left(T_{\mu\nu} - \frac{T}{2} g_{\mu\nu}\right)u^\mu u^\nu - \sigma_{\mu\nu}\sigma^{\mu\nu} - \frac{1}{3}\theta^2$$

$$\frac{d\theta}{d\lambda} \leq \Lambda - \frac{1}{3}\theta^2, \quad \theta^2 \geq 3\Lambda \Big|_{\lambda_0}$$

$$\theta(\lambda) \leq \frac{\sqrt{3\Lambda}}{\tanh\left(H(\lambda - \lambda_0)\right)} \to \sqrt{3\Lambda} \quad \text{(exponentially fast)}$$

$$T_{\mu\nu} u^\mu v^\nu < \theta^2 - 3\Lambda \to 0, \quad T^{00} \to 0$$

$$\text{dec} \to |T_{\mu\nu}| < T_{00} \to 0$$

The conjecture implies homogeneity of Λ-dominated Universe at late times.