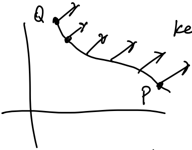
Geodesics, congruence, and Raychaudhuri equation

Parallel transport of a tensor



keep vector const

result depends on the path

Curve $x^\mu(\lambda)$, tensor $T_{\dots}^{\dots}$ $\qquad \dfrac{dT}{d\lambda} = \dfrac{dx^\mu}{d\lambda}\dfrac{\partial T}{\partial x^\mu} = 0$

$$\frac{\mathcal{D}}{d\lambda} = \frac{dx^\mu}{d\lambda}\nabla_\mu$$

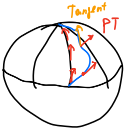
Parallel transport: $\dfrac{\mathcal{D}}{d\lambda} T = \underbrace{\dfrac{dx^\sigma}{d\lambda}}_{\text{tangent vector to the curve}} \nabla_\sigma T = 0$ (definition)

$$\frac{dV^\mu}{d\lambda} + \Gamma^\mu_{\sigma\rho}\frac{dx^\sigma}{d\lambda}V^\rho = 0$$

$$\frac{\mathcal{D}}{d\lambda}\left(g_{\mu\nu}V^\mu W^\nu\right) = 0 \Rightarrow \text{norm is preserved}$$

Geodesics

Tangent vector is PT along the geodesics



$$\frac{\mathcal{D}}{d\lambda}\frac{dx^\mu}{d\lambda} = 0$$

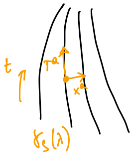
$$\frac{d^2x^\mu}{d\lambda^2} + \Gamma^\mu_{\rho\sigma}\frac{dx^\rho}{d\lambda}\frac{dx^\sigma}{d\lambda} = 0 \text{ - geodesics eq.}$$

In flat space – $\dfrac{d^2x^\mu}{d\lambda^2} = 0$ is an eq. of a straight line

Timelike path, geodesics extremize the proper time (max)

$$\tau = \int\left(-g_{\mu\nu}\frac{dx^\mu}{d\lambda}\frac{dx^\nu}{d\lambda}\right)d\lambda \to x^\mu(\lambda)$$

Geodesics deviation



$$T^\mu = \frac{\partial x^\mu}{\partial \lambda} \qquad X^\mu = \frac{\partial x^\mu}{\partial s} \quad \text{(deviation vector)}$$

(*) X^μ can be always chosen orthogonal to T^μ

$$X^\mu T_\mu = 0$$

$V^\mu = T^\nu \nabla_\nu X^\mu$ - relative velocity on a nearby geodesics $(s+ds)$

$a^\mu = T^\gamma \nabla_\gamma (V^\mu) = T^\gamma \nabla_\gamma (T^\nu \nabla_\nu X^\mu)$ - relative acceleration

$T^\mu \nabla_\mu X^\nu = X^\mu \nabla_\mu T^\nu$ (elements of coordinate basis)

$$a^\mu = T^\gamma \nabla_\gamma X^\nu \nabla_\nu T^\mu + T^\gamma \nabla_\gamma \nabla_\nu T^\mu X^\nu =$$

$$= X^\gamma \nabla_\gamma T^\nu \nabla_\nu T^\mu + T^\gamma \nabla_\gamma \nabla_\nu T^\mu X^\nu - R_{\gamma\beta\nu}{}^\mu X^\beta T^\gamma T^\nu =$$

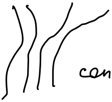
$$= X^\gamma \nabla_\gamma \underbrace{\left(T^\nu \nabla_\nu T^\mu\right)}_{=0,\ \text{geodesic eq.}} - R_{\gamma\beta\nu}{}^\mu X^\beta T^\gamma T^\nu = - R_{\gamma\beta\nu}{}^\mu X^\beta T^\gamma T^\nu =$$

$$= R_{\gamma\nu\beta}{}^\mu X^\beta T^\gamma T^\nu$$

$$a^\mu = \boxed{\frac{D}{d\lambda}\left(\frac{D}{d\lambda} X^\mu\right) = X^\beta R_{\gamma\nu\beta}{}^\mu \frac{\partial x^\gamma}{\partial \lambda} \frac{\partial x^\nu}{\partial \lambda}}$$

In flat space: separation of straight lines grow linearly $(R=0)$

Congruense - a bundle of timelike geodesics $x^\mu(\lambda, s_1, s_2, s_3)$ where each point belongs to a single geodesics



congruence

no congruence

Timelike geodesics

$T^\mu T_\mu = -1$, $\quad T^\mu \nabla_\mu T^\nu = 0$

$\frac{dx^\mu}{ds}$ – 3d subspace orthogonal to T^μ

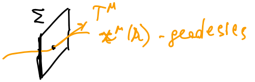
$P^{\mu\nu} = g^{\mu\nu} + T^\mu T^\nu \qquad P^{\mu\nu} T_\nu = 0, \quad P^{\mu\nu} P_\nu{}^\lambda = P^{\mu\lambda}$

$P^{\mu\nu} P_{\mu\nu} = 3$ (induced metric)

$$\frac{\mathcal{D}}{d\lambda}\left(\frac{dx^\mu}{ds}\right) = T^\nu \nabla_\nu \left(\frac{dx^\mu}{ds}\right) = \frac{dx^\nu}{ds} \underbrace{\nabla_\nu T^\mu} = B^\mu{}_\nu \rightarrow \underline{\text{eq. on } B}$$

$T^\mu B_{\mu\nu} = 0$ (geodesics eq.)

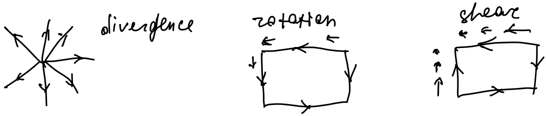
$B_{\mu\nu}$ is a tensor in 3d space Σ orthogonal to T^μ with induced metric $P_{\mu\nu}$



irreps

$B_{\mu\nu} = \sigma_{\mu\nu} + \omega_{\mu\nu} + \frac{1}{3}\theta P_{\mu\nu}$, (symmetric + antisymmetric + trace)

(expansion, shear, rotation)
divergence



$$\frac{\mathcal{D} B_{\mu\nu}}{d\lambda} = T^\gamma \nabla_\gamma B_{\mu\nu} = T^\gamma \nabla_\gamma \nabla_\nu T_\mu = T^\gamma \nabla_\nu \nabla_\gamma T_\mu + T^\lambda [\nabla_\lambda, \nabla_\nu] T_\mu =$$

$$= \nabla_\nu (T^\gamma \cancel{\nabla_\gamma T_\mu}) - \nabla_\nu T^\gamma \nabla_\gamma T_\mu - V^\gamma R^\sigma{}_{\mu\gamma\nu} T_\sigma =$$

$$\boxed{\frac{\mathcal{D} B_{\mu\nu}}{d\lambda} = - B_{\mu\gamma} B^\gamma{}_\nu - R_{\sigma\mu\gamma\nu} T^\sigma T^\gamma}$$ (Raychaudhuri equation)

Trace:

$$g^{\mu\nu} B_{\mu\lambda} B^\lambda{}_\nu = B_{\mu\nu} B^{\nu\mu} = (\sigma_{\mu\nu} + \frac{1}{3}\theta P_{\mu\nu} + \omega_{\mu\nu})(\sigma^{\mu\nu} + \frac{1}{3} P_{\mu\nu}\theta - \omega^{\mu\nu})$$

$$= \sigma_{\mu\nu}\sigma^{\mu\nu} - \omega_{\mu\nu}\omega^{\mu\nu} + \frac{1}{3}\theta^2$$

$$\boxed{\frac{d\theta}{d\lambda} = g^{\mu\nu}\frac{DB_{\mu\nu}}{d\lambda} = -\frac{1}{3}\theta^2 - \sigma_{\mu\nu}\sigma^{\mu\nu} + \omega_{\mu\nu}\omega^{\mu\nu} - R_{\mu\nu}T^\mu T^\nu}$$

$$T^\mu B_{\mu\nu} = 0 \Rightarrow \sigma_{\mu\nu} = \sigma_{ij} \Rightarrow \sigma_{\mu\nu}\sigma^{\mu\nu} > 0$$

If $\omega_{\mu\nu} = 0$

$$\frac{d\theta}{d\lambda} \leq -\frac{1}{3}\theta^2 - R_{\mu\nu}T^\mu T^\nu$$

$$R_{\mu\nu}T^\mu T^\nu = 8\pi G\left(T_{\mu\nu}T^\mu T^\nu - \frac{1}{2}T^\mu_\mu T^\nu T_\nu\right) > 0 \quad (\text{SEC})$$

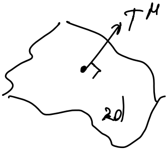
$$P + \rho > 0, \quad \rho + 3P > 0$$

Given this condition, geodesics always focus!

Null geodesics!

$$T^\mu T_\mu = 0, \quad \theta = \nabla_\mu T^\mu \quad (\text{expansion or divergence})$$

$$\nabla_\nu T_\mu = \frac{1}{2}\theta P_{\mu\nu} + \sigma_{\mu\nu} + \omega_{\mu\nu}$$



Same derivation but with $\frac{1}{2}\theta$:

$$\frac{d\theta}{d\lambda} = -\frac{1}{2}\theta^2 - \sigma_{\mu\nu}\sigma^{\mu\nu} + \omega_{\mu\nu}\omega^{\mu\nu} - R_{\mu\nu}T^\mu T^\nu$$