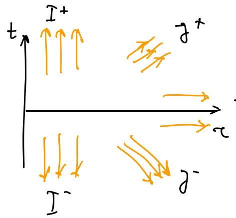
Causal structure of spacetime and Penrose diagrams

Spacetimes: we consider

1) 4d Real, connected, C^∞ Hausdorff manifold on which metric tensor g can be defined. (3+1) causal.
2) Solution to GR
3) ~~det~~ $g_{\mu\nu} \neq 0$
4) no naked singularities, there is a global hyperbolic region where initial conditions can be defined



$I^\pm$ – timelike past or future infinity

I^0 – spacelike infinity

$\mathcal{J}^\pm$ – null future or past ∞

$t \pm \tau = \text{const}, \quad t \mp \tau \to \mp\infty$

Penrose diagram of Minkowski space

$$ds^2 = dt^2 - d\tau^2 - \tau^2 \left(d\theta^2 + \sin^2\theta \, d\varphi^2\right)$$

$$(t, \tau, \theta, \varphi) \to (\psi, \xi, \theta, \varphi)$$

$$\begin{cases} t + \tau = \tan \dfrac{\psi + \xi}{2} \\ t - \tau = \tan \dfrac{\psi - \xi}{2} \end{cases}$$

$$t = \frac{\sin\psi}{2\cos\frac{\psi+\xi}{2}\cos\frac{\psi-\xi}{2}}$$

$$\tau = \frac{\sin\xi}{2\cos\frac{\psi+\xi}{2}\cos\frac{\psi-\xi}{2}}$$

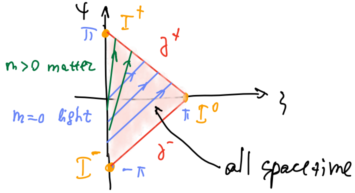
$$ds^2 = \frac{1}{4\cos^2\frac{\psi+\xi}{2}\cos^2\frac{\psi-\xi}{2}} \left(d\psi^2 - d\xi^2 - \sin^2\xi \left(d\theta^2 + \sin^2\theta \, d\varphi^2\right)\right)$$

$\tau = 0 \to \xi = 0$

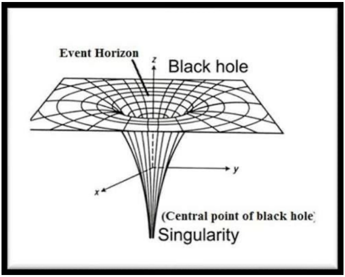
$I^+ \to \psi = \pi, \; \xi = 0$

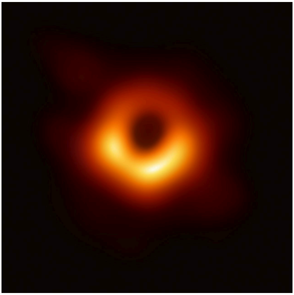
$I^- \to \psi = -\pi, \ \xi = 0$

$I_0 \to \psi = 0, \ \xi = \pi$



Schwarzschild spacetime





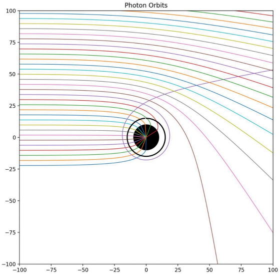
$$ds^2 = \left(1-\frac{2M}{r}\right)dt^2 - \left(1-\frac{2M}{r}\right)^{-1}dr^2 - r^2\left(d\theta^2 + \sin^2\theta \, d\varphi^2\right)$$

$r=0, \ r=2M$ - singular

$$R^2 - T^2 = \left(\frac{r}{2M} - 1\right)e^{\frac{r}{2M}}$$

$$\frac{T}{R} = \operatorname{th}\frac{t}{4M} \quad (r > 2M)$$

$$\frac{R}{T} = \operatorname{th}\frac{t}{4M} \quad (r < 2M)$$



Kruskal coordinates

$$ds^2 = \frac{32M^3}{r} e^{-\frac{r}{2M}} \left(dT^2 - dR^2\right) - r^2 \left(d\theta^2 + \sin^2\theta \, d\varphi^2\right)$$

To compact coordinates:

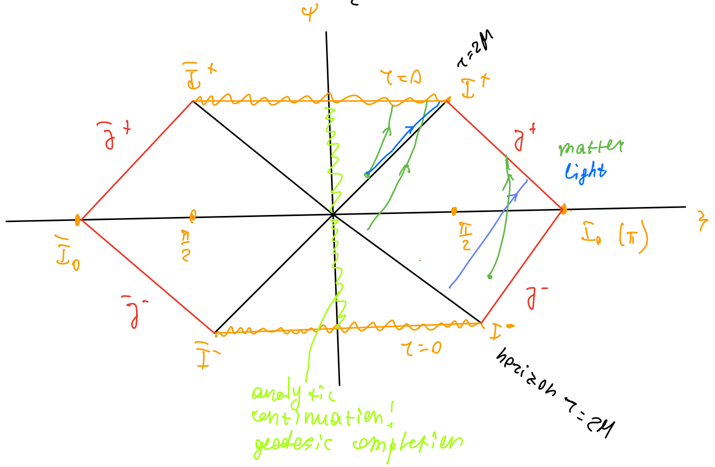
$$(T, R, \theta, \varphi) \to (\psi, \xi, \theta, \varphi)$$

$$T + R = \operatorname{th}\left(\frac{\psi+\xi}{2}\right)$$

$$T - R = \operatorname{th}\left(\frac{\psi-\xi}{2}\right)$$

$$ds^2 = \Omega^2 \left(d\psi^2 - d\xi^2\right) - r^2\left(d\theta^2 + \sin^2\theta \, d\varphi^2\right)$$

$$\Omega^2 = \frac{32M^3}{r} e^{-\frac{r}{2M}} \frac{1}{4\cos^2\frac{\psi+\xi}{2}\cos^2\frac{\psi-\xi}{2}}$$



FRW spacetime

$$ds^2 = dt^2 - a^2(t)\left(\frac{dr^2}{1-kr^2} + r^2 d\Omega^2\right) =$$

$$= dt^2 - a^2(t)\left(d\chi^2 + f^2(\chi)\, d\Omega^2\right)$$

$$f(\chi) = r = \begin{cases} \sin\chi & (k=1) \\ \chi & (k=0) \\ \mathrm{sh}\,\chi & (k=-1) \end{cases}$$

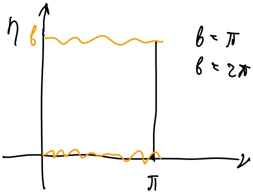
$d\eta = \frac{dt}{a}$ - conformal time

$$ds^2 = c^2(\eta)\left(d\eta^2 - d\chi^2 - f^2(\chi)\, d\Omega^2\right)$$

MD stage: $c(\eta) = -\frac{E}{3}(1-\cos\eta)$, $t = -\frac{E}{3}(\eta - \sin\eta)$, $E<0$

$(k=-1)$ $c(\eta) = -\frac{E}{3}(1-\mathrm{ch}\,\eta)$, $t = -\frac{E}{3}(\eta - \mathrm{sh}\,\eta)$, $E>0$

$c(\eta) = \eta^3$, $t = \frac{\eta^3}{3}$ (flat)



$b = \pi$ - RD

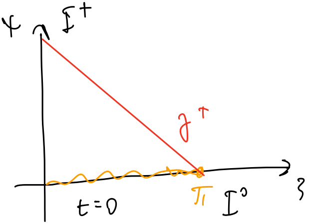
$b = 2\pi$ - MD

$k = +1$

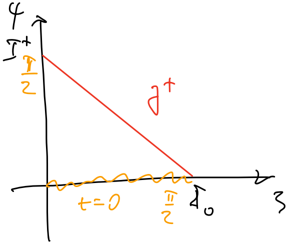
$\underline{k=0}$

$$\eta \pm \chi = \tan\left(\frac{\psi \pm \xi}{2}\right)$$

$$ds^2 = \frac{c^2(\eta)}{4\cos^2\left(\frac{\psi+\xi}{2}\right)\cos^2\left(\frac{\psi-\xi}{2}\right)}\left(d\psi^2 - d\xi^2 - \sin^2\xi\, d\Omega^2\right)$$



k=0



K=−1

Desitter

$$\eta_{AB}\, z^A z^B = -\alpha^2, \quad \alpha = \left(\frac{3}{\Lambda}\right)^{\frac{1}{2}}$$

(5d)

hyperboloid

$$ds^2 = \eta_{AB}\, dz^A\, dz^B$$

$$\eta_{AB} = (1,-1,-1,-1,-1)$$

$\boxed{k=0}$

$$z^0 = a\,\mathrm{sh}\,\frac{t}{\alpha} + \frac{1}{2\alpha} e^{t/\alpha} |x|^2$$

$$z^4 = a\,\mathrm{ch}\,\frac{t}{\alpha} - \frac{1}{2\alpha} e^{t/\alpha} |x|^2$$

$$z^i = x^i e^{t/\alpha}$$

$$z^0 + z^4 > 0$$