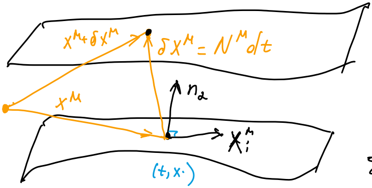
ADM decomposition of the gravitational field and the Hamiltonian formalism of the action

$\Sigma_{t+\Delta t}$



Σ_t, equation $X^\mu = X^\mu(t, x^i)$

Introduce a set of spacelike hypersurfaces Σ_t. At any point we define a normal vector n_α and a tangent vector

$$X_i^\mu = \frac{\partial X^\mu}{\partial x^i} = \partial_i X^\mu$$

(n^μ, X_i^μ) – local 4-frame

* $g_{\mu\nu} X_i^\mu n^\nu = 0$
* induced metric on Σ_t: $h_{ij} = g_{\mu\nu} X_i^\mu X_j^\nu$

$$h_{\mu\nu} = g_{\mu\nu} + n_\mu n_\nu$$

* $g_{\mu\nu} n^\mu n^\nu = -1$

$N^\mu = \dot{X}^\mu$ (definition, shift vector)

$N^\mu = N n^\mu + N^i X_i^\mu$ – decomposition in a local frame

N – lapse function

$N_i = h_{ij} N^j$

$$\boxed{ds^2 = -(N^2 - N_i N^i) dt^2 + 2 N_i dx^i dt + h_{ij} dx^i dx^j}$$

Derivation:

$$ds^2 = g_{\mu\nu} dx^\mu dx^\nu = g_{\mu\nu} \dot{x}^\mu \dot{x}^\nu dt^2 + 2 g_{\mu\nu} \dot{x}^\mu x^\nu_{;i} dt dx^i +$$
$$+ g_{\mu\nu} x^\mu_{;i} x^\nu_{;j} dx^i dx^j = g_{tt} dt^2 + 2 g_{it} dt dx^i + g_{ij} dx^i dx^j$$
$$g_{ij} = x^\mu_{;i} x^\nu_{;j} (h_{\mu\nu} - \cancel{n_\mu n_\nu}) = h_{ij}$$
$$g_{ti} = g_{\mu\nu} (\cancel{N n^\mu} + N^j x^\mu_{;j}) x^\nu_{;i} = g_{\mu\nu} N^j x^\mu_{;j} x^\nu_{;i} = N^j h_{ij} = N_i$$
$$g_{tt} = \dot{x}^\mu \dot{x}^\nu g_{\mu\nu} = N^\mu N^\nu (h_{\mu\nu} - \cancel{n_\mu n_\nu}) = h_{\mu\nu} (\cancel{N n^\mu} + N^i x^\mu_{;i}) \cdot$$
$$\cdot (\cancel{N n^\nu} + N^j x^\nu_{;j}) - N^2 = h_{\mu\nu} N^i x^\mu_{;i} N^j x^\nu_{;j} - N^2 =$$
$$= N_i N^i - N^2$$

$$S = -\frac{1}{16\pi G} \int \sqrt{-g}\, d^4x \, (R + 2\Lambda)$$

$K_{ij} = -\nabla_i h_j$ – extrinsic curvature

$$\boxed{S = -\frac{1}{16\pi G} \int d^4x \sqrt{h}\, N \left[K^2 - K^{ij} K_{ij} + {}^3R(h) + 2\Lambda \right]}$$

Derivation

$$\det g = (N^i N_i - N^2) h - (h h^{ij}) N_i N_j = -N^2 h$$
$$\sqrt{-g} = N \sqrt{h}$$
$$\underline{R = {}^3R - K^{ij} K_{ij} + K^2}$$
$$R = 2 (G_{ab} n^a n^b - R_{ab} n^a n^b)$$
$$R_{ab} n^a n^b = R_{acb}{}^c n^a n^b = -n^a (\nabla_a \nabla_c - \nabla_c \nabla_a) n^c =$$
$$= (\nabla_a n^a)(\nabla_c n^c) - (\nabla_c n^a)(\nabla_a n^c) + \nabla(\ldots) =$$
$$= K^2 - K_{ac} K^{ac} + \nabla(\ldots)$$
$$G_{ab} n^a n^b = \frac{1}{2} ({}^3R - K_{ab} K^{ab} + K^2) \quad \text{(Gauss–Codacci relations)}$$

Hamiltonian formulation

$$H(q,\pi) = \pi\dot{q} - L \, , \quad \pi = \frac{\partial L}{\partial \dot{q}}$$

EOM: $\dot{q} = \frac{\delta H}{\delta \pi} \, , \quad \dot{\pi} = -\frac{\delta H}{\delta q}$

$$H = \int_{\Sigma_t} \mathcal{H} \, , \quad \mathcal{H} = \pi\dot{q} - \mathcal{L}(q)$$

$$\mathcal{L}_{KG} = \frac{1}{2}\dot{\varphi}^2 - (\partial_i\varphi)^2 - m^2\varphi^2)$$

$$\pi = \dot{\varphi}$$

$$\mathcal{H}_{KG} = \frac{1}{2}\left(\pi^2 + (\partial_i\varphi)^2 + m^2\varphi^2\right)$$

EM field, $\mathcal{L} = -\frac{1}{4}F_{\mu\nu}^2$

$V = -A_a n^a \, , \quad {}^3A_a = h_a{}^b A_b$ (on Σ_t)

$$\mathcal{L}_{EM} = \frac{1}{2}\left(\dot{A}_i + \partial_i V\right)^2 - \frac{1}{2}\left(\vec{\nabla}\times\vec{A}\right)^2$$

$$\pi_i = \dot{A}_i + \partial_i V = -E_i$$

$\pi_V = 0$ (no $\dot{V}$ term)

Resolution – V is not a dynamical field, only A_i

$$H = \pi_i \dot{A}^i - \mathcal{L} = \frac{1}{2}\pi_i^2 + \frac{1}{2}B_i^2 - \pi_i\partial^i V =$$

$$= \frac{1}{2}\pi_i^2 + \frac{1}{2}B_i^2 + V\partial_i\pi^i + \partial_i(\dots) \, , \quad \vec{B} = \vec{\nabla}\times\vec{A}$$

$\frac{\delta H}{\delta V} = 0 \Rightarrow \partial_i E^i = 0$ (constraint)

$$\left.\begin{aligned} \dot{A}_i &= -E_i - \partial_i V \\ \dot{\pi}_i &= -\dot{E}_i = -(\nabla\times(\nabla\times A))_i \end{aligned}\right\}$$ Maxwell equations

GR | $K_{ab} = \frac{1}{2N}\left(\dot{h}_{ab} - \mathcal{D}_a N_b - \mathcal{D}_b N_a\right)$, $\mathcal{D}_a$ - cov. ∂ on Σ_t

$$\pi^{ij} = 16\pi G \frac{\delta S}{\delta \dot{h}_{ij}} = \sqrt{h}\left(K h^{ij} - K^{ij}\right)$$

$$\frac{\delta S}{\delta \dot{N}_i} = \frac{\delta S}{\delta \dot{N}} = 0 \Rightarrow N_i, N \text{ - non-dynamical fields}$$

$(16\pi G \equiv 1)$

$$H = \pi^{ij}\dot{h}_{ij} - S = -\sqrt{h}N\,{}^3R + N\frac{1}{\sqrt{h}}\left(\pi^{ij}\pi_{ij} - \frac{1}{2}\pi^2\right) +$$

$$+ 2\pi^{ij}\mathcal{D}_i N_j = \sqrt{h}\left(\underline{N}\left(-{}^3R + h^{-1}\pi^{ij}\pi_{ij} - \frac{1}{2}h^{-1}\pi^2\right) -\right.$$

$$\left.- 2\underline{N_i}\,\mathcal{D}_j\left(h^{-\frac{1}{2}}\pi^{ij}\right)\right) + \underbrace{\mathcal{D}_a(\ldots)}_{\text{boundary term}}$$

* Constraint equations:

$$-{}^3R + h^{-1}\pi^{ij}\pi_{ij} - \frac{1}{2}h^{-1}\pi^2 = 0 \qquad \text{(non-linear)}$$

$$\mathcal{D}_i\left(h^{-\frac{1}{2}}\pi^{ij}\right) = 0$$

* EOM

$$\dot{h}_{ab} = \frac{\delta H}{\delta \pi^{ab}} = 2h^{-\frac{1}{2}}N\left(\pi_{ab} - \frac{1}{2}\pi h_{ab}\right) + 2\mathcal{D}_{(a}N_{b)}$$

$$\dot{\pi}_{ab} = -\frac{\delta H}{\delta h_{ab}} = -Nh^{\frac{1}{2}}\left({}^3R^{ab} - \frac{1}{2}{}^3R\,h^{ab}\right) + \frac{1}{2}Nh^{-\frac{1}{2}}h^{ab}\left(\pi_{cd}\pi^{cd} -\right.$$

$$\left.- \frac{1}{2}\pi^2\right) - 2Nh^{-\frac{1}{2}}\left(\pi^{ac}\pi_c{}^b - \frac{1}{2}\pi\pi^{ab}\right) + h^{\frac{1}{2}}\left(\mathcal{D}^a\mathcal{D}^b N - h^{ab}\mathcal{D}_c\mathcal{D}^c N\right)$$

$$+ h^{\frac{1}{2}}\mathcal{D}_c\left(h^{-\frac{1}{2}}N^c\pi^{ab}\right) - 2\pi^{c(a}\mathcal{D}_c N^{b)}.$$