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NUMERICAL INVERSION OF THE LAPLACE TRANSFORM BY USE OF JACOBI POLYNOMIALS*

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Abstract. Functional values of a function f are determined from the values $F(s)$ of its Laplace transform at discrete points of s. Evaluation of $F(s)$ at points given by $s = (\beta + 1 + k)\delta$, $k = 0, 1, \cdots$, determine coefficients in an infinite series expansion of $f(t)$ in terms of Jacobi polynomials. The values of β and δ determine the position along the real s-axis at which $F(s)$ is evaluated. An approximation to $f(t)$ is given by using a finite number of terms of the infinite series expansion of $f(t)$. Numerical examples are given and results are compared with some known numerical methods for approximating $f(t)$.

Introduction. The problem of numerically inverting the Laplace transform is known to mathematicians, physicists, and engineers and has been discussed extensively in the mathematical literature [1]–[8]. A single method for numerically inverting the Laplace transform that works equally well for all types of problems encountered is lacking. In many practical problems where the Laplace transform can be evaluated at discrete points along the real axis of the independent variables, the method described here is useful. This method is fast (economical) on the digital computers now available, and it has the advantage that for only a few computations the unknown inverse can be approximated over a large range of values in the t domain.

The Laplace transform of $f(t)$ is defined by the integral

$$F(s) = \int_0^\infty \exp(-st) f(t)\, dt, \quad \operatorname{Re} s \geqq c > 0. \tag{1}$$

For purposes of discussion here it will be assumed that the integral in (1) exists for $\operatorname{Re} s > 0$. A suitable translation of the imaginary axis can be made if this is not the case, and the theory developed here is still applicable.

The inverse Laplace transform is

$$f(t) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \exp(st)\, F(s)\, ds, \tag{2}$$

provided that the integral in (2) converges absolutely for $\operatorname{Re} s > c$, c sufficiently large.

Change of variable. Consider the Laplace transform of $f(t)$ defined by (1) and assume that $F(s)$ is known or can be computed at discrete points along

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the real s-axis. The variable of integration may be changed by the substitution

$$x = 2 \exp(-\delta t) - 1, \tag{3}$$

where δ is a real positive number. It follows that

$$\exp(-st) = (1 + x/2)^{s/\delta}.$$

If this equation is solved for t, then

$$t = -(1/\delta) \log [(1 + x)/2]$$

and a new function g is defined over $(-1, 1)$ by

$$g(x) = f\{-(1/\delta) \log [(1 + x)/2]\} = f(t). \tag{4}$$

In order to extend the domain of definition for g, define $g(1)$ and $g(-1)$ by

$$g(1) = \lim_{x \to 1^-} g(x),$$

and

$$g(-1) = \lim_{x \to -1^+} g(x).$$

Essentially these definitions require that $f(0) = \lim_{t \to 0^+} f(t)$ and $f(\infty) = \lim_{t \to \infty} f(t)$ be finite. If f is continuous, then g is also a continuous function. Substitution of (3) into (1) and some algebraic manipulation give

$$F(s) = (1/2\delta) \int_{-1}^{1} (1 + x/2)^{(s/\delta - 1)} g(x)\, dx. \tag{5}$$

Assume that g can be expanded over $[-1, 1]$ in an infinite series of orthogonal polynomials. The Jacobi polynomials form such a set over $[-1, 1]$. The normalized Jacobi polynomial of degree n is defined by [9]

$$P_n^{(0,\beta)}(x) = \frac{(-1)^n}{2^n n!} (1 - x)^{-\beta} \frac{d^n}{dx^n} [(1 - x)^n (1 + x)^{n+\beta}], \tag{6}$$

where the parameter α which usually appears in this definition is zero and $\beta > -1$. For $n = 0$, $P_n^{(0,\beta)}(x) = 1$. If g can be expanded over $[-1, 1]$ in terms of the Jacobi polynomials, then

$$g(x) = \sum_{n=0}^{\infty} C_n P_n^{(0,\beta)}(x). \tag{7}$$

If the coefficients C_n are known, then $g(x)$ is known, which implies that $f(t)$ can be calculated for any $t = t_0$ by means of (4).

Insertion of the previous series into the integral in (5) yields

$$F(s) = (1/2\delta) \int_{-1}^{1} (1 + x/2)^{(s/\delta-1)} \left[\sum_{n=0}^{\infty} C_n P_n^{(0,\beta)}(x) \right] dx.$$

By substituting $s = (\beta + 1 + k)\delta$ into the previous equation and simplifying terms one has

$$(8) \qquad \delta F[(\beta + 1 + k)\delta] = 2^{(\beta+k+1)} \int_{-1}^{1} (1 + x)^{\beta+k} \left[\sum_{n=0}^{\infty} C_n P_n^{(0,\beta)}(x) \right] dx.$$

The factor $(1 + x)^k$ which appears in (8) may be expressed as a finite linear combination of Jacobi polynomials. That is, $(1 + x)^k$ is given by

$$(9) \qquad (1 + x)^k = a_0 P_0^{(0,\beta)}(x) + a_1 P_1^{(0,\beta)}(x) + \cdots + a_k P_k^{(0,\beta)}(x).$$

For $0 \leqq m \leqq k$, a typical coefficient a_m is a function of k and β. In order to evaluate a_m, multiply both sides of (9) by $(1 + x)^\beta P_m^{(0,\beta)}(x)$ and integrate over $[-1, 1]$. Because of the orthogonality property of the Jacobi polynomials, there is only one nonzero term on the right, and therefore,

$$(10) \qquad \int_{-1}^{1} (1 + x)^k (1 + x)^\beta P_m^{(0,\beta)}(x)\, dx = a_m \frac{2^{\beta+1}}{2m + \beta + 1}.$$

The factor $(2^{\beta+1})/(2m + \beta + 1)$ on the right is the normalization term for the Jacobi polynomials. Let it be denoted by h_m.

The Jacobi polynomial $P_m^{(0,\beta)}(x)$ can be expressed in the form

$$P_m^{(0,\beta)}(x) = b_0 + b_1(1 + x) + \cdots + b_m(1 + x)^m,$$

where the b's can be determined. However, this is not necessary. Substitution of $P_m^{(0,\beta)}(x)$ in this form into the previous integral gives

$$(11) \qquad \begin{aligned} a_m h_m &= \int_{-1}^{1} (1 + x)^{k+\beta}[b_0 + b_1(1 + x) + \cdots + b_m(1 + x)^m]\, dx \\ &= b_0 \frac{2^{k+\beta+1}}{k + \beta + 1} + b_1 \frac{2^{k+\beta+2}}{k + \beta + 2} + \cdots + b_m \frac{2^{k+\beta+m+1}}{k + \beta + m + 1}. \end{aligned}$$

If the unknown a_m is considered as a function of the parameter k, then one may write

$$(12) \qquad a_m h_m = \frac{Q_m(k)}{[k + (\beta + 1)]\,[k + (\beta + 2)] \cdots [k + (\beta + m + 1)]}.$$

$Q_m(k)$ is a polynomial in the symbol "k" of degree m. The explicit expression for $Q_m(k)$ may be determined by the use of (9) and (10). In (10) let $k = m - 1$ and because of the orthogonality of the Jacobi polynomials,

$$\int_{-1}^{1} (1 + x)^{m-1}(1 + x)^\beta P_m^{(0,\beta)}(x)\, dx = 0.$$

Therefore, one of the roots of $Q_m(k)$ must be given by $k = m - 1$. A similar procedure shows that for $k = m - 2, m - 3, \cdots, 1, 0$, the remaining roots of $Q_m(k)$ are determined. Therefore, $Q_m(k)$ is known up to a constant term, and it may be written in factored form as

$$Q_m(k) = A[k - (m - 1)][k - (m - 2)] \cdots k,$$

and A is a constant to be determined.

Substitution of $Q_m(k)$ as given here into (12) gives

$$(13) \qquad a_m h_m = \frac{A[k - (m - 1)]\,[k - (m - 2)] \cdots (k - 1)k}{(k + \beta + 1)\,(k + \beta + 2) \cdots (k + \beta + m + 1)}.$$

However, from (12) it follows that

$$\begin{aligned} A &= b_0 2^{k+\beta+1} + b_1 2^{k+\beta+2} + \cdots + b_m 2^{k+\beta+m+1} \\ &= 2^{k+\beta+1}[b_0 + 2b_1 + \cdots + 2^m b_m]. \end{aligned}$$

Since $P_m^{(0,\beta)}(1) = 1$ for $m = 0, 1, \cdots$, one has

$$P_m^{(0,\beta)}(1) = 1 = b_0 + 2b_1 + \cdots + 2^m b_m .$$

Hence, it follows that $A = 2^{k+\beta+1}$, and from (13) and some algebraic simplification

$$(14) \qquad a_m = 2^k(2m + \beta + 1)\frac{k(k - 1) \cdots [k - (m - 1)]}{(k + \beta + 1)(k + \beta + 2) \cdots (k + \beta + m + 1)}.$$

For $k = 0$ the right side of (14) is replaced by 1.

Substitution of (14) and (9) into (8) gives

$$(15) \qquad \begin{aligned} F[(\beta + 1 + k)\delta] &= \frac{2^{-(\beta+k+1)}}{\delta} \\ &\quad \cdot \int_{-1}^{1} (1 + x)^\beta \sum_{m=0}^{k} a_m P_m^{(0,\beta)}(x) \left[\sum_{n=0}^{\infty} C_n P_n^{(0,\beta)}(x)\right] dx \end{aligned}$$

for $k = 0, 1, \cdots$, where a_m is defined in (14). Integrating termwise in (15) gives only k nonzero terms because of the orthogonality property of the Jacobi polynomials. After the integration has been performed, algebraic simplification gives

$$(16) \qquad \begin{aligned} &\delta F[(\beta + 1 + k)\delta] \\ &\quad = \sum_{m=0}^{k} \frac{k(k - 1) \cdots [k - (m - 1)]}{(k + \beta + 1)\,(k + \beta + 2) \cdots (k + \beta + 1 + m)} C_m . \end{aligned}$$

Again this result is true for $k = 0, 1, \cdots$, and for $k = 0$ the right side of this expression is replaced by $c_0/(\beta + 1)$.

By successively allowing $k = 0, 1, \cdots$, one has the system of equations:

$$
\begin{aligned}
\delta F[(\beta+1)\delta] &= \frac{C_0}{(\beta+1)},\\
\delta F[(\beta+2)\delta] &= \frac{C_0}{(\beta+2)} + \frac{C_1}{(\beta+2)(\beta+3)},\\
\delta F[(\beta+3)\delta] &= \frac{C_0}{(\beta+3)} + \frac{2C_1}{(\beta+3)(\beta+4)}\\
&\qquad + \frac{2C_2}{(\beta+3)(\beta+4)(\beta+5)},\\
\delta F[(\beta+4)\delta] &= \frac{C_0}{(\beta+4)} + \frac{3C_1}{(\beta+4)(\beta+5)}\\
&\qquad + \frac{3\cdot 2C_2}{(\beta+4)(\beta+5)(\beta+6)} + \frac{3!C_3}{(\beta+4)\cdots(\beta+7)}.
\end{aligned} \tag{17}
$$

The coefficient C_0 is determined by allowing $k = 0$ and knowledge of $F(s)$ at $s = (\beta + 1)\delta$. For $k = 1$ the coefficient C_1 is determined from the value (calculated) of C_0 and $F(s)$ at $s = (\beta + 2)\delta$. In a similar manner the remaining coefficients $C_2, C_3, \cdots$ can be determined.

If N coefficients are calculated, then $g(x)$ may be approximated by $g(x) \approx \sum_{n=0}^{N} C_n P_n^{(0,\beta)}(x)$. Since $x = 2\exp(-\delta t) - 1$, the Jacobi polynomials may be expressed as functions of t directly. From (4) it then follows that

$$f(t) \approx \sum_{n=0}^{N} C_n P_n^{(0,\beta)}[2\exp(-\delta t) - 1]. \tag{18}$$

Application of method. Theoretically, $f(t)$ can be determined for all values of t from knowledge of $F(s)$ at discrete points along the real s-axis. However, numerical errors limit the number of terms in (18) that can be accurately computed. Therefore, the accuracy of the approximation to $f(t)$ may be increased by selecting the position along the real s-axis at which $F(s)$ is evaluated. The points at which $F(s)$ is evaluated ($s = (\beta + 1 + k)\delta$ for $k = 0, 1, 2, \cdots$) are determined by β and δ. Thus, β and δ should be selected so that (in some sense) the "best" approximation possible is obtained.

It is well known that large s corresponds to small t and small s corresponds to large t, [3]. This fact is a guideline to follow and for asymptotic values of t the values of β and δ can be selected accordingly. Of more general interest, however, is the approximation of $f(t)$ for values of t which are not asymptotic. Therefore, for a given error norm, β and δ should be selected in order that the error is minimized.

Error bounds. Since the series in (7) converges uniformly (g continuous), it may be truncated after N terms to give an approximation valid for $x \in [-1, 1]$. Thus, there exists an $n_0 \geqq 0$ such that the terms in (7) are approaching zero. The rate of convergence of (7) may be used (or (18) in the t-space) as a criterion for selecting β and δ. First, however, some definitions are needed.

DEFINITION 1. Let g be continuous over $[-1, 1]$ and define $\epsilon_n(x)$ by

$$|\epsilon_n(x)| = \left| g(x) - \sum_{k=0}^{n} C_k P_k^{(0,\beta)}(x) \right|. \tag{19}$$

DEFINITION 2. The norm of the error in the approximation for $g(x)$ is defined by

$$\| \epsilon_n(x) \| = \int_{-1}^{1} |\epsilon_n(x)|^2 \, dx. \tag{20}$$

The theorem that follows gives an estimate of the error.

THEOREM 1. *Let g be continuous over $[-1, 1]$ and ϵ_n defined by (19). Assume that there exists a real number r, $0 < r < 1$, and there exists a positive integer p such that for $n \geqq p$,*

$$| C_{n+m} P_{n+m}^{(0,\beta)}(x) | \leqq r^m \, | C_n P_n^{(0,\beta)}(x) |,$$

for $m = 0, 1, \cdots$. Under these hypotheses it follows that for $n \geqq p$,

$$| \epsilon_n(x) | \leqq C_{n+1} P_{n+1}^{(0,\beta)}(x) | / (1 - r)^2. \tag{21}$$

Proof. Rewrite (19) in the form

$$| \epsilon_n(x) | = | C_{n+1} P_{n+1}^{(0,\beta)}(x) + C_{n+2} P_{n+2}^{(0,\beta)}(x) + \cdots |.$$

Application of the triangle inequality to this expression gives

$$| \epsilon_n(x) | \leqq | C_{n+1} P_{n+1}^{(0,\beta)}(x) | + | C_{n+2} P_{n+2}^{(0,\beta)}(x) | + \cdots .$$

Under the hypothesis of the theorem, use of the geometric series and some algebraic manipulation give the result in (21).

If $K = \max_{\beta,\delta} \{ | C_{p+1} P_{p+1}^{(0,\beta)}(x) | \}$, then it follows from (21) that $| \epsilon_n(x) | \leqq K/(1 - r)$. Hence, the following theorem gives a bound on the error norm $\| \epsilon_n(x) \|$.

THEOREM 2. *If $\epsilon_n(x)$ is defined by (19) and K is given as above, then for g continuous over $[-1, 1]$,*

$$\| \epsilon_n(x) \| \leqq 2K^2/(1 - r)^2.$$

Proof of Theorem 2 follows from the definition of $\| \epsilon_n(x) \|$ if $\epsilon_n(x)$ given in terms of K is substituted into (2).

A result similar to Theorem 2 holds in the t-space for any interval $(0, T)$.

THEOREM 3. *If* $e_n(t) = \epsilon_n(x)$ *and* x *and* t *are related by* (3), *then*

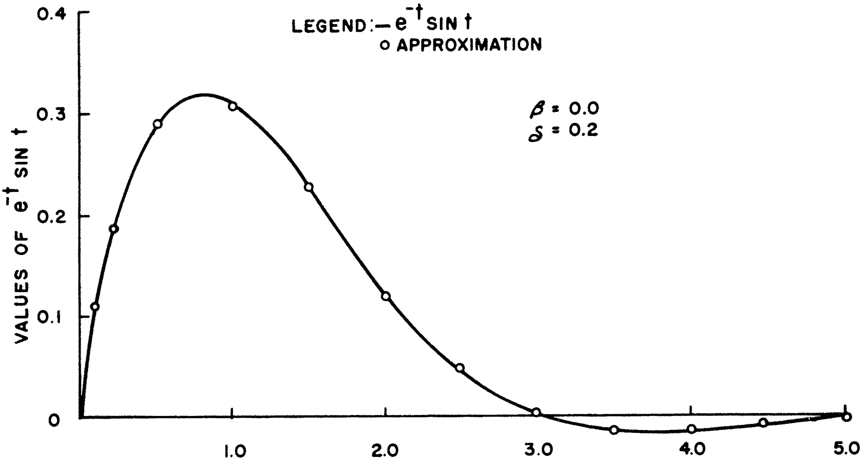

FIG. 1. *Approximations for* $f(t) = e^{-t} \sin t$

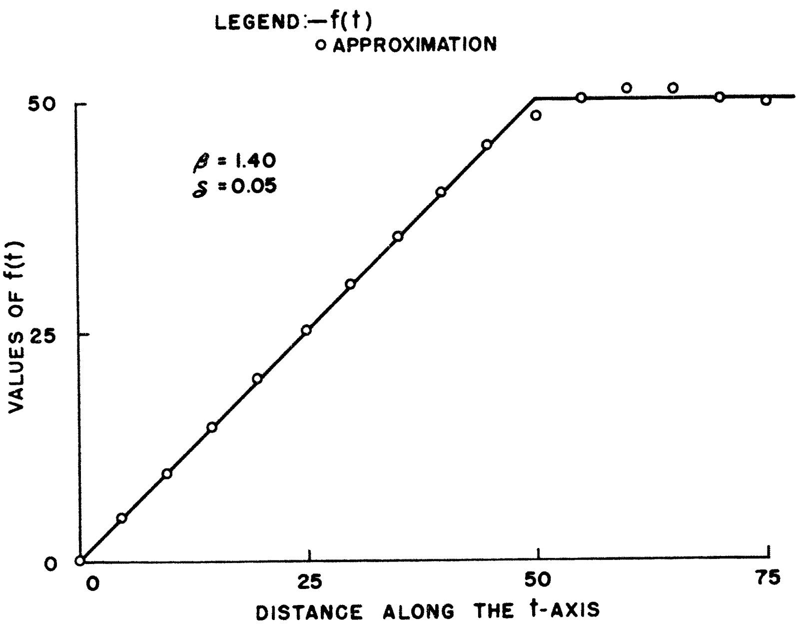

FIG. 2. *Approximations for* $f(t) = \begin{cases} t & \text{for } 0 \leqq t \leqq 50, \\ 50 & \text{for } 50 \leqq t \end{cases}$

$$\int_0^T |e_n(t)|^2\, dt \leqq \frac{1}{\delta} e^{\delta T} K^2/(1-r)^2.$$

Numerical examples. The examples given in the following paragraphs indicate the results of this inversion scheme. They were selected because they have poles at various positions in the complex plane, they have been used in the literature as examples of different inversion schemes, or because the functions (in the t-space) do not always have "gentle" slope.

For the first example consider the Laplace transform defined by $F(s) = 1/[(s+1)^2+1]$. The known inverse is $f(t) = \exp(-t)\sin t$. The results are shown in Fig. 1. For this calculation $\beta = 0.0$ and $\delta = 0.2$; 11 terms were used in the approximating function defined in (18).

The theory presented here requires that $f(0)$ and $f(\infty)$ be finite. Thus,

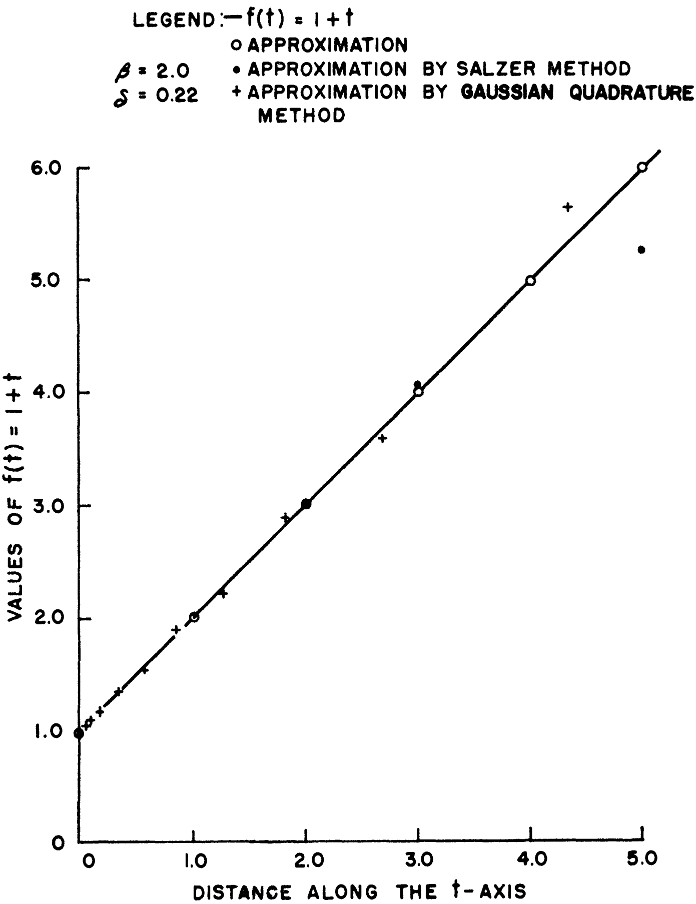

FIG. 3. *Approximations for $f(t) = 1 + t$ if ten terms are used*

for the Laplace transform $F(s) = 1/s^2$ which has an inverse $f(t) = t$, the theory is not applicable. However, the Laplace transform $F_1(s) = [1 - \exp(-st)]/s^2$ has an inverse $f_1(t) = t$ for $0 \leqq t \leqq T$ and $f_1(t) = T$ for $T \leqq t$. Hence for $T \to \infty$ and sT sufficiently large one has $\exp(-st) \ll 1$ and $F_1(s) \approx F(s)$. Fig. 2 shows the results obtained for $T = 50$. For these calculations $\beta = 1.40$, $\delta = 0.05$, and 11 terms were used in (18). For this approximation of $f(t)$ the range of values of t used is quite extensive with $0 \leqq t \leqq 75$.

As it was explained previously, for sT sufficiently large, $\exp(-st) \ll 1$. If this is true, then on the register of a computer $F_1(s) = F(s)$. That is, for sufficiently large s, the technique can be applied to $F(s) = 1/s^2$. Fig. 3 shows the results obtained for the approximation to $f(t) = t$, where $0 \leqq t \leqq 5$ and $\beta = 2.0$ and $\delta = 0.22$. Two other known methods were also used to numerically invert $F(s) = 1/s^2$. One of these methods is due to Salzer [7], [8] and the other method uses a Gaussian type quadrature [1], [2], [4]. In each of the approximation schemes a 10-point quadrature [10 terms in (18)] was used. That is, $F(s)$ was evaluated at 10 points along the real s-axis. Tables used for these comparisons were obtained from [1], [7].

Fig. 4 shows the results obtained for $f(t) = J_0(t)$. The approximations again use 10 terms. $F(s)$ was evaluated at points determined by $\beta = 3.0$ and $\delta = 0.5$. Values of t are for $0 \leqq t \leqq 5$. For a specific value of t a different choice of β and δ gives better results. For this example it was found that for $J_0(2)$, the values $\beta = 4.0$ and $\delta = 0.6$ give the approximation $J_0(2) \approx 0.223896$, while $J_0(2) = 0.223891$ (rounded to six decimal digits).

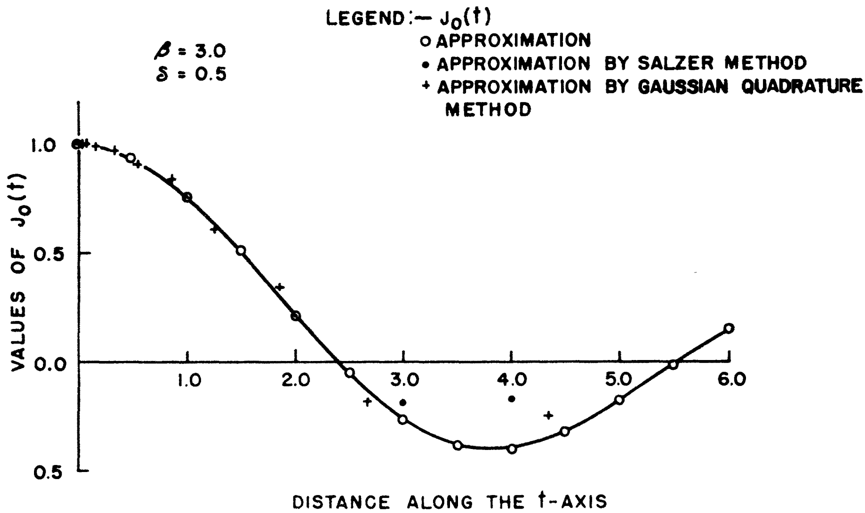

FIG. 4. *Approximations for $J_0(t)$ if ten terms are used*

The next example is for the Laplace transform $F(s) = \exp(-\frac{1}{2}\sqrt{s})$. The inverse is given by

$$f(t) = \frac{\exp(-t/16)}{4(\pi t^3)^{1/2}}.$$

This example is given by Bellman, et al., [2] and illustrates the difficulty involved in numerically inverting a Laplace transform which has an inverse with a "steep" slope. Ten terms in (18) and a 10-point quadrature were used for these calculations. One of the advantages of the method described here is illustrated in this example. This is the fact that $f(t)$ may be approxi-

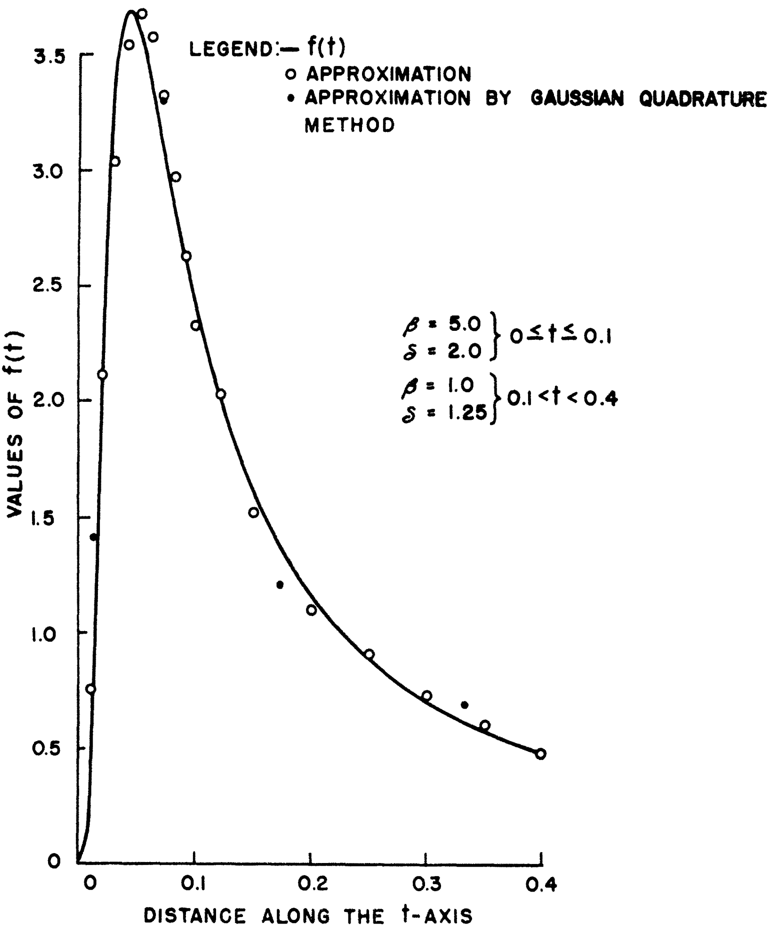

FIG. 5 *Approximations for* $f(t) = \frac{\exp(-t/16)}{4(\pi t^3)^{1/2}}$

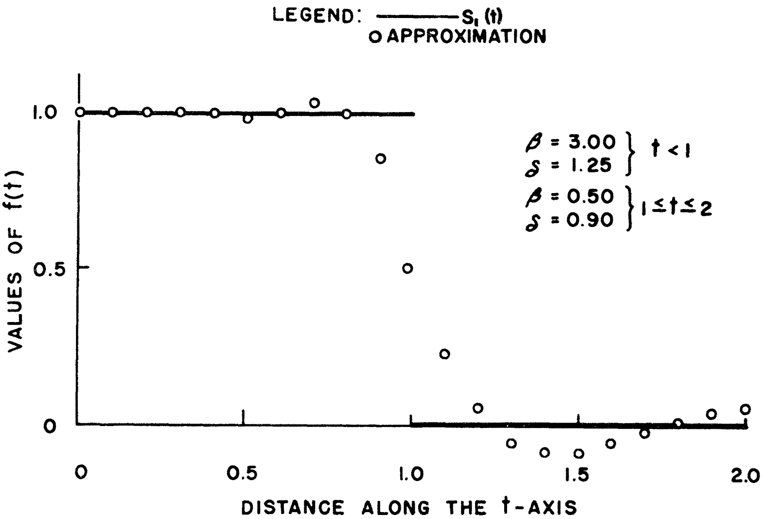

FIG. 6. *Approximations for* $f(t) = S_0(t)$

mated at values of t which lie sufficiently close so that the outline of $f(t)$ is well described, as shown in Fig. 5.

The previous examples have been for continuous functions in the t-space. The infinite series representation for these functions converges uniformly and the termwise integration in (15) is valid. Consider the step function given by $S_1(t)$ for values of t in $(0, 2)$. Although it has not been shown that the termwise integration in (15) may be performed without altering the results, a "rough" outline of $f(t)$ may still be obtained in this particular example. Fig. 6 shows these results.

Numerical errors. Numerical round-off and cancellation errors limit the number of coefficients c_n that can be accurately calculated from the system of equations in (17). By the use of multiple precision arithmetic, the number of coefficients that may be accurately computed is increased. The exact number of coefficients which can be accurately computed depends on a particular problem. Experience has shown that for these examples and for ones similar, about 12 to 14 coefficients may be accurately calculated using single precision arithmetic on a Control Data 1604 computer.

The Jacobi polynomials were calculated using the recurrence relation found in [9, p. 71].

Conclusions. The method for numerically inverting Laplace transforms that has been described here is applicable to many problems of practical interest. Round-off and cancellation errors must be considered when calculating the coefficients that appear in the series approximation for $f(t)$. For a small number of calculations $f(t)$ may be approximated over a wide range of values. A general guide for the user of this method is to select β and δ such that $-0.5 \leqq \beta \leqq 5.0$ and $0.05 \leqq \delta \leqq 2.0$. For t such that $t > 0.1$, a more realistic value of β is $\beta \leqq 2.0$. The required computer time is only a fraction of a second for computation of 15 Jacobi polynomials and 15 coefficients on a Control Data 1604 computer.

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