University of Waterloo

PHYS460A

Thermionic Emission of Electrons

Experiment #4

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Abstract

The Richard-Dushman equation described the emission of electrons from a hot solid. With the addition of Schottky emission, we can compute the work function of a metal based on the relationship between temperature and the electron emission from a filament. These equations are derived from the statistical distributions of electrons in a solid and applying some basic assumptions about the electro-magnetic fields acting on such electrons. Using a circuit setup with a heated filament emitting electrons, along with a coaxial cylindrical anode measuring the emitted current, we explore the relationship between temperature of the wire and the anode current for the purpose of verifying the Richardson-Dushman equation. Using measured values we calculate the work function for the tungsten filament which is calculated to be 5.24 ± .54 eV as compared to an accepted value of 4.5 eV.

1 Introduction

Thermionic emission is the outward flow of charged particles from a solid due to thermal excitation. First observed in the mid-1850's, the phenomena was investigated by a handful of scientists over then next half-century, including Thomas Edison who was interested in why parts of the filament in his light bulb darkened quicker than others. Thermionic emission is a common source of electrons in various applications; such as cathode ray tube, X-ray imaging and radio [].

Thermionic emission is the emission of electrons (or ions) from a hot metal or semiconductor, and can be considered an 'evaporation' process. In analogy to an evaporating liquid, say water, thermionic emission is similar to the random escape of water molecules at temperatures below boiling.

Solids, such as metals or semiconductors, have an energy barrier for electrons leaving the surface. This energy barrier is referred to as the work function, and is dependent on the metal or semiconductor in question. Electrons with sufficient energy and momenta can exit the surface. An applied voltage across this solid serves to *reduce* the effective work function of the solid – promoting greater emission of electrons from the surface. This phenomena is known as Schottky emission and allows for a higher flux of electrons out of the surface.

Thermionic emission is described by the Richardson-Dushman equation which was, which was first described by Owen Richardson in 1901. The equation for the current of electrons emitted from a hot object is of the form,

$$J = A_0 T^2 e^{-w/kT} \tag{1}$$

In this paper we outline the theory behind the Richardson-Dushman equation and Schottky emission, outline a procedure for measuring the work function of a metal based on the relationship between temperature and emitted current, and finally compute the work function for a filament wire and explore the relationship between temperature and current.

2 Theoretical Background

Metals and semiconductors are composed of periodic structures of ions in 3D space – a lattice. The outer shell electrons (usually one electron, but sometimes two or three) are so loosely bound to the original atom that are considered to be shared amongst all ions in the metal – creating a sea of electrons.

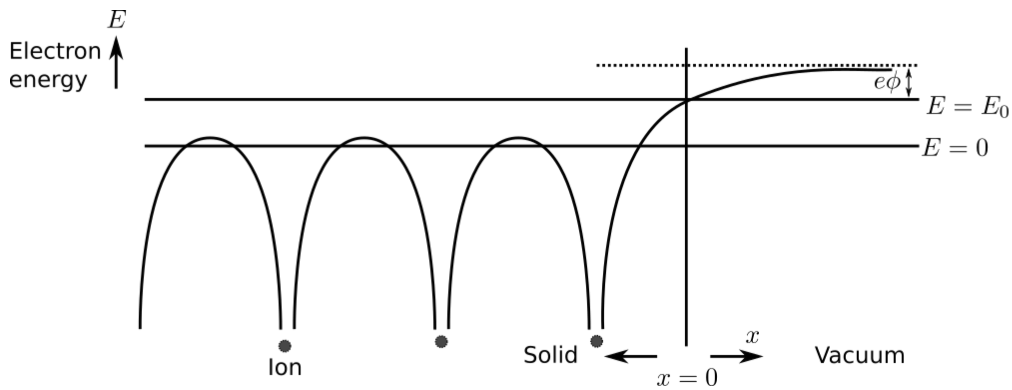

Figure 1: Potential energy diagram for a metal. Electrons with energy $> E_0 + e\phi$ are able to escape from the surface.

At the boundary between the solid and vacuum (or other material) is an energy barrier. Electrons with sufficient energy to overcome this barrier can be emitted from the surface. The energy barrier is is dependent on the solid, usually around a few electron-volts. This energy barrier is the **work function**, w, of the metal – or the amount of energy needed to remove one electron. This work function is described as,

$$w_0 = e\phi \tag{2}$$

where e is the elementary charge and ϕ is the electrostatic potential in the vacuum nearby the surface.

The free electrons of a solid have with a distribution of velocities (and thus, momenta) due to random thermal motion. The average energy is given by,

$$\bar{E} = \frac{1}{2}m\bar{v}^2 = \frac{3}{2}kT \tag{3}$$

where m is the mass of the electron, $\bar{v}$ is the average velocity, k is the Boltzmann constant and T is the temperature of the solid.

However, in reality the electrons not only need sufficient energy, but must also be moving in the correct direction to leave the surface. Thus, we describe the requirement for emission in terms of momentum rather than energy. To be ejected, the electron must have enough momentum in the same direction to the surface normal. An electron moving in the y-direction will not be emitted from a surface at $x = 0$ with surface normal in the x-direction (see Figure 1). The electron in this example must be moving with enough momentum in the positive x direction to be emitted.

So, as in Figure 1, the electron needs energy $E = E_0 + e\phi$ to be emitted (this is the minimum requirement, and after emission the electron would have no remaining energy), and the momentum in the direction of the surface normal is,

$$E_0 + e\phi = \frac{p_{x_c}^2}{2m} \tag{4}$$

Thus, the momentum in the positive x-direction required is,

$$p_{x_c} = \sqrt{2m(E_0 + e\phi)} \tag{5}$$

The number of electrons within an infinitesimal slice of the 3D momentum is,

$$\frac{2}{h^3} dp_x dp_y dp_z \tag{6}$$

Now, using the approximation of Fermi's equation we can calculate the temperature-dependent number occupied states,

$$f(E) = e^{-(E-E_0)/kT} \tag{7}$$
$$f(\vec{p}) = e^{-(\vec{p}^2/2m-E_0)/kT} \tag{8}$$

So, our number of occupied states is,

$$N_e = \frac{2}{h^3} e^{-(\vec{p}^2/2m-E_0)/kT} dp_x dp_y dp_z \tag{9}$$

These electrons leaving the solid form an electron flux and volume current density, which we can write as,

$$J = ev_x N_e = \frac{e}{m} p_x N_e \tag{10}$$

where $v_x = p_x/m$ is the velocity in the x-direction and N_e is the number of electrons with sufficient momenta to be emitted per unit volume that we computed above.

Combining the expressions previously derived, we have for current,

$$dJ = \frac{2e}{mh^3} p_x e^{-((p_x^2+p_y^2+p_z^2)/2m-E_0)/kT} dp_x dp_y dp_z \tag{11}$$
$$J = \frac{2e}{mh^3} \int_{p_{x_c}}^{\infty} dp_x \int_{-\infty}^{\infty} dp_y \int_{-\infty}^{\infty} dp_z \left[p_x e^{-((p_x^2+p_y^2+p_z^2)/2m-E_0)/kT} \right] \tag{12}$$

Solving the Gaussian integrals for dp_y, dp_z and the exponential form integral for dp_x with the value for p_{x_c} derived in Equation 5,

$$J = \frac{4\pi emk^2}{h^3} T^2 e^{-e\phi/kT} \tag{13}$$

This equation is known as the Richard-Dushman. We can simplify the constant scalar to a single, universal constant value of,

$$A_0 = \frac{4\pi emk^2}{h^3} \tag{14}$$

which in SI units is,

$$= 1.20173 \times 10^6 \text{ A m}^{-2} \text{ K}^{-2} \tag{15}$$

So, we can write the equation in the simple form of,

$$J = A_0 T^2 e^{-w_0/kT} \tag{16}$$

However, what has been discussed so far accounts only for a solid with no applied electric field. With an electric field across the solid, the forces on the electrons must be considered. The effect will be a reduced work function, $w = w_0 + \Delta w = e(\phi + \Delta\phi)$, so more electrons are able to be emitted.

If we consider a test electron a position x to the right of the solid's surface, we can think of the field lines as coming from a positive test charge at $-x$ through the method of images (as the metal is conducting and field lines must be normal to the surface). The force on the charges would then be,

$$F(x) = \frac{-e^2}{4\pi\epsilon_0 (2x)^2} \tag{17}$$

and the corresponding potential is,

$$P(x) = \frac{-e^2}{16\pi\epsilon_0 x} \tag{18}$$

Note that this is if there is no voltage across the surface. If an accelerating field $\mathcal{E}$ is then applied, there will be an addition of,

$$P(x) = \frac{-e^2}{16\pi\epsilon_0 x} - e\mathcal{E}x \tag{19}$$

Taking the assumption that $P(x \to \infty) = 0$ and finding the maximum of $P(x)$ via the derivative,

$$0 = \frac{e^2}{16\pi\epsilon_0 x_0^2} - e\mathcal{E} \to x_0 = \left(\frac{e}{16\pi\epsilon_0 \mathcal{E}}\right)^{\frac{1}{2}} \tag{20}$$

Calculating $P(x)$ at x_0 is $\Delta w = e\Delta\phi$ and given as,

$$P(x_0) = \Delta w = -\left(\frac{e^3\mathcal{E}}{4\pi\epsilon_0}\right)^{\frac{1}{2}} \tag{21}$$

Thus, when an accelerating field is applied to the solid, it reduces the energy barrier height at the surface of the solid. This allows more electrons to be emitted from the surface, and we can define an effective work function,

$$w = w_0 + \Delta w \tag{22}$$

$$= e\phi - \left(\frac{e^3 E}{4\pi\epsilon_0}\right)^{\frac{1}{2}} \tag{23}$$

This is known as Schottky emission. So, with this change in work function, the Richardson-Dushman equation becomes,

$$\boxed{J = A_0 T^2 e^{-w_0/kT} e^{\sqrt{\frac{e^3 E}{4\pi\epsilon_0}}}} \tag{24}$$

Taking the Richardson-Dushman equation without an external electric field in Equation 16 will allow us to compare our static current density (which we denote from now on as J_o) into the modified Richardson-Dushman equation giving,

$$J = J_0 e^{\sqrt{\frac{e^3 E}{4\pi\epsilon_0}}\Big/KT} \tag{25}$$

Converting this current density into current and then using the fact that $E = \frac{V}{d}$ in an environment with a constant applied electric field we can then further modify the equation to read as,

$$I = I_0 e^{\sqrt{\frac{e^3 V}{4\pi\epsilon_0 d}}\Big/KT} \tag{26}$$

Taking the natural log of both sides gives us a form which would allow for the values of I_o to be found from a plot of $\ln(I)$ vs. $\sqrt{V}$ if the condition of a constant electric field is satisfied. This is shown in,

$$\ln(I) = \ln(I_0) + \left(\sqrt{\frac{e^3}{4\pi\epsilon_0 d}}\Bigg/KT\right)\sqrt{V} \tag{27}$$

Now noting this interesting form and applying a similar methodology to Equation 16 we'll note that taking the current density as current will give us,

$$I_o = A_0 T^2 e^{-w_0/kT} \tag{28}$$

$$\frac{I_o}{T^2} = A_0 e^{-w_0/kT} \tag{29}$$

Now taking the natual log of both sides will give us,

$$\ln\left(I_o/T^2\right) = \ln(A_0) + -w_0/kT \tag{30}$$

3 Experimental Background & Procedure

3.1 Apparatus

It is possible to measure the values in the equations above. Using simple a cylindrical shell anode, which measures the electrons emitted from the filament as a current, we can measure the current volume density J as the current through the anode, I_a (see Fig. 3).

Using the circuit diagram from Figure 2, the filament wire is heated and generates thermionic emission of electrons which is measured at the anode. A voltmeter and ammeter measure both the current and voltage for both the filament and the anode.

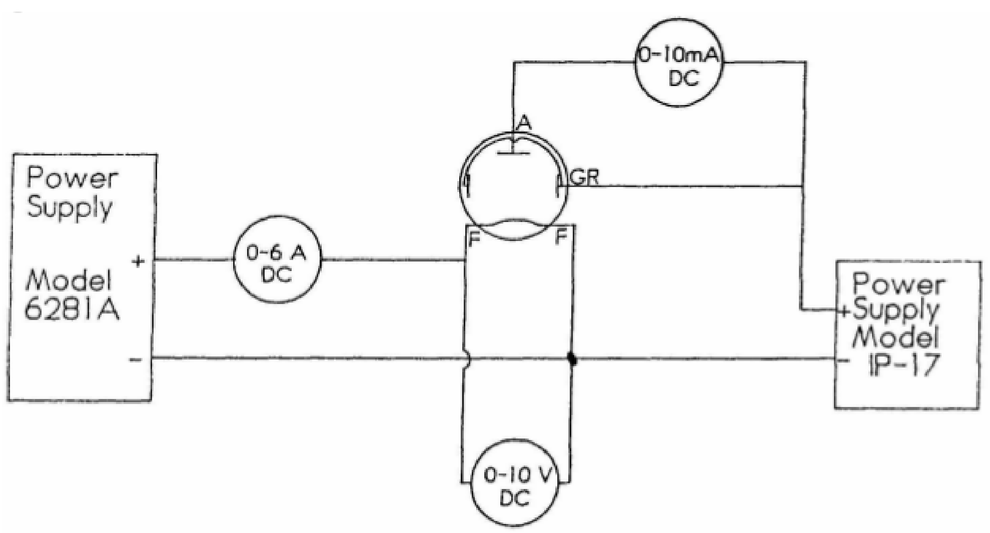

Figure 2: Circuit diagram of experimental setup for the detection of thermionic emission.

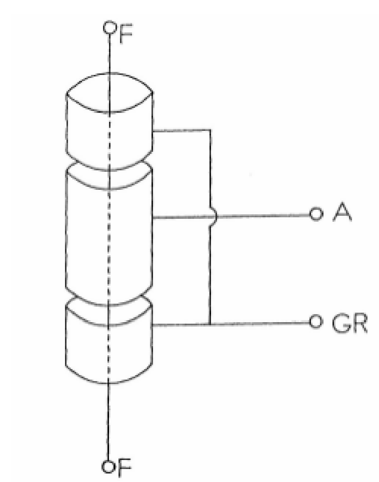

Figure 3: Electrode structure, with filament wire emitting electrons along the axis and coaxial anode and guard rings.

3.2 Experimental Procedure

3.2.1 Data Collection

Our data collection method is based around ramping the current values, and also obtaining a voltage ramp (of V_a for each of the anode current values (I_a)). Data points will be collected for Anode Voltages (V_a) of 300, 250, 200, 150 & 100 V at Anode Currents of (I_a) of 8.00, 4.00, 0.80 & 0.40 mA. The steps for doing so are as follows:

Step 1 Turn off filament power supply.

Step 2 Set the Anode Voltage (V_a) to our first value (300 V), this value is verified through the voltmeter connected to the power supply.

Step 3 Turn on filament power supply.

Step 4 Set the Anode Current (I_a) to our first value (8.00 mA) and verify this value through the anode's ammeter.

Step 5 Record the filament Voltage (I_f) and Current (I_f) by rewiring the ammeter and voltmeter to measure across the filament. These values should stay constant throughout the anode's voltage ramp.

Step 6 Rewire to measure V_a & I_a.

Step 7 Ramp the V_a down from 300 V to 100 V in 50 V steps. V_a is always set without the filament power supply being on.

Step 8 Measure V_a and I_a at each step with the filament supply on.

Step 9 Set the next I_a value. Complete Step 1 to Step 8 with our new I_a

4 Analysis

4.1 Removing Schottky Effects to find I_o

We first wish to look at the relationship between the Anode current (I_a) and the Anode's Voltage (V_a). This is in part to verify the Schottky Effects, and in an effort to obtain estimates for the Anode Current without electrical field from these fits using Equation 27 by comparing $\ln(I_a)$ with $\sqrt{V_a}$. We know that all other terms should be constant since our vacuum tube states that "The guard rings ensure continuous and homogeneous anode-cathode field and eliminates 'fringe' effects." [2]. The temperature is the only other quantity which may change, but it will not per trial as the filament current is monitored during the runs and as we will later see is an empirically known quantity through use of Equation 31 in relation to the filament current.

We also note at this time that the presented V_a is a calculated correction on our V_a, which is calculated by taking the known anode voltage and subtracting off the midpoint of the filament voltage as we can see is necessary from our Circuit Diagram (Figure 2) [3]. We completed the analysis with and without correction and this made no difference to final results as this effect applies to each voltage and any changes are dominated by an anode voltage which are 2 orders of magnitude larger

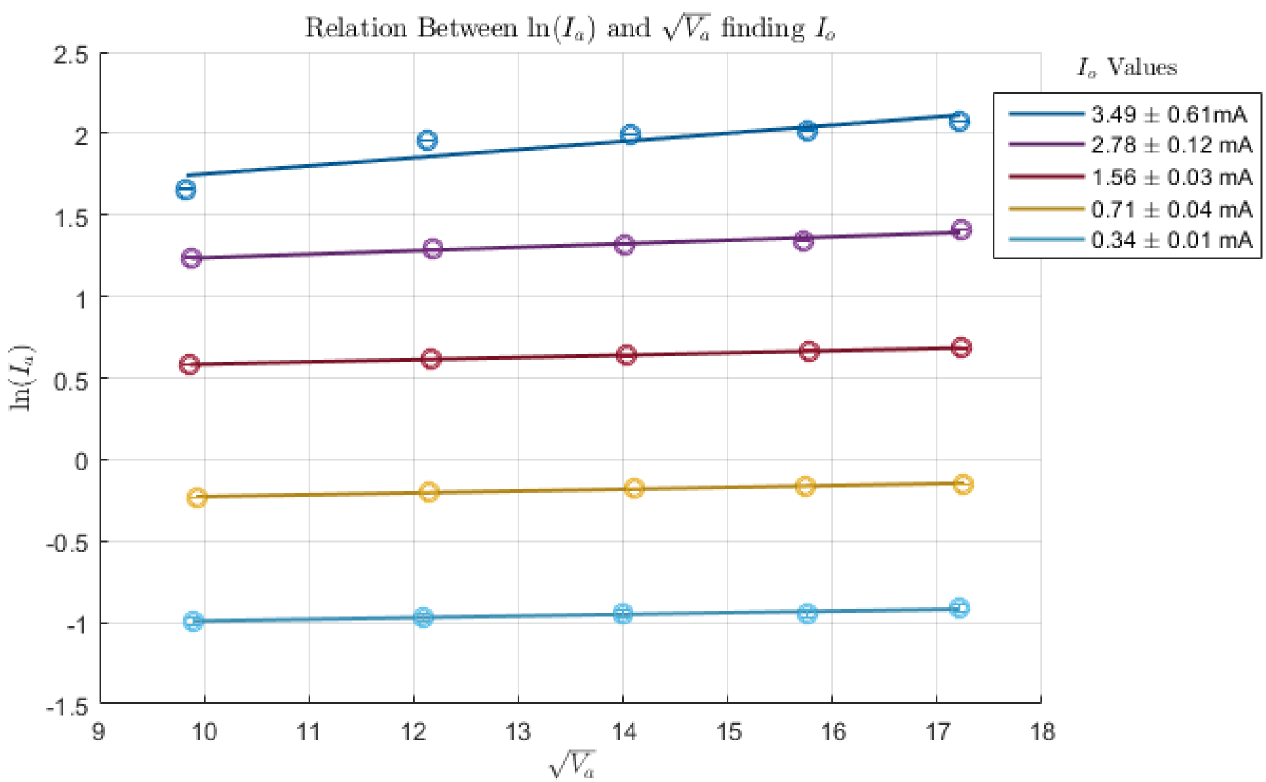

Figure 4: The plots of anode current and voltage to find values for the anode current without effects of an electric field

We present our I_o values in Table 1 which are calculated through a linear regression through using the least squares method. Intersection error estimates are provided by analyzing the standard error of coefficient estimates and scaling them to the final I_o values.

From this analysis it can be seen see that the Schottky Effects appear to be modelled well by the modified Richard-Dushman Equation as the plots of based around linearizing our Schottky modified Richard-Dushman Equation (Equation 27)create a linear fit which has a high accuracy of prediction (goodness of fit (R^2) is greater than .95 and percentage error is less than 5%) for all of the fits other than the highest anode current.

I_a (mA)	I_o (mA)		R^2
8.00	3.49	$\pm$ .61	0.818
4.00	2.78	$\pm$ .12	0.955
2.00	1.56	$\pm$ .03	0.996
0.80	0.71	$\pm$ .04	0.967
0.40	0.34	$\pm$ .01	0.951

Table 1: The values of I_o which are derived from the linear fits. Also included are error estimates and goodness of fit statistics.

This run with the highest anode current appears has much less predictive power, with a low R^2 value of .818 and a high percentage error of 17.46 %. We note that if the lowest voltage value is exclueded, then the R^2 value increases to .964 and the percentage error decreases to 3.16%. This indicates this point may be an outlier. However, two other sets of data were taken which both show the same relation as is seen in Figure 4. This leads to us to the conclusion that this non-linearity appears to be physical. However the origin of this non-uniformity is unclear. It is possible that a reaction creates a film on the filament which changes the work function of the filament for the first couple of data points, but this seems unlikely due to the vacuum containing an inert gas to minimize this effect and large warm-up times were allowed to allow for surface films to dissipate.

4.2 Finding w_o through the Richard-Dushman Equation

Our second graph is a relation based on liearizing the Richard-Dushman equation as presented in Equation 30. This relation is based upon temperature, but we note that this is not a directly measured quantity in our experiment. But we will present an empirically known relation of filament current (I_f) with the filament diameter to find the temperature in Equation 31 [3] We also note the filament diameter is a given quantity with some differences in value depending on documentation source. [3] [2]. Since it is unknown how inaccurate the temperature is, we have not attempted to estimate error. We will also note that it is assumed in the derivation that the temperature is constant across the voltage ramp, which is untrue, but the current was monitored and the deviations appeared to be on the order of $< 1\%$ hence this error has not been included in further discussions.

$$T = 60.2\sqrt{B(1 + 83 \times 10^{-6}B)}, B = I_f/d^{3/2} \tag{31}$$

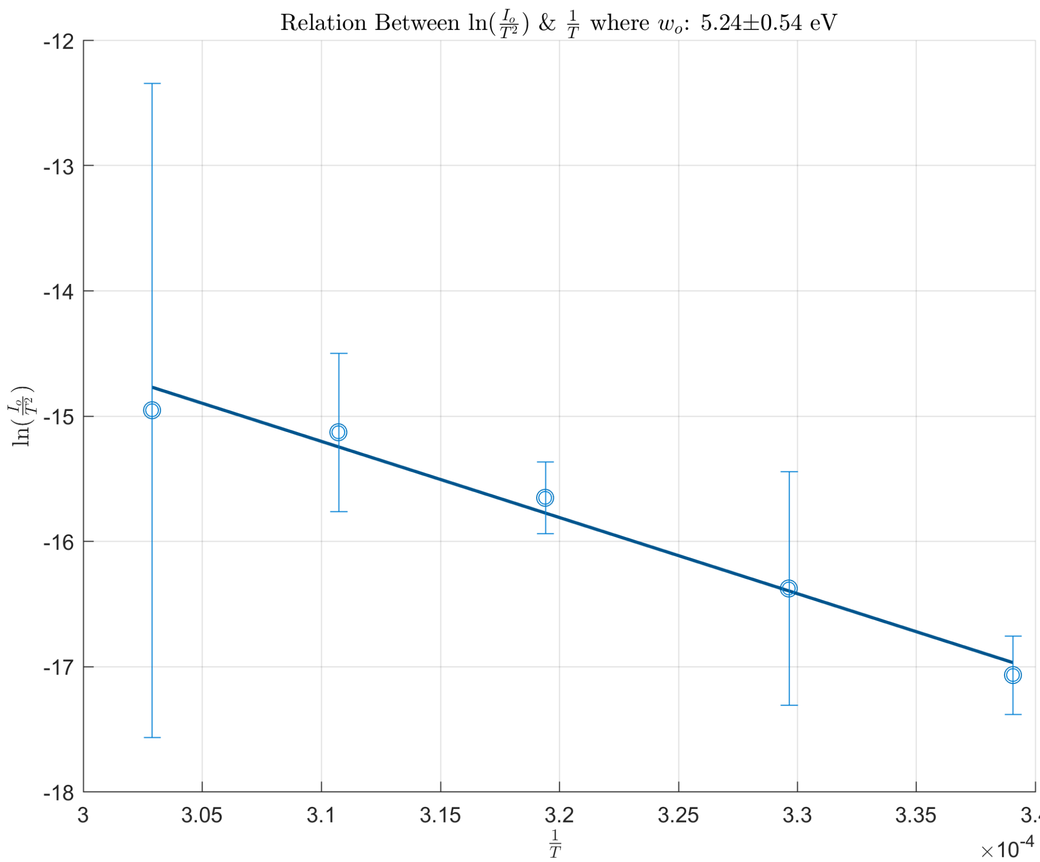

Figure 5: The plots of initial current scaled with temperature give us a relation for the work function

We see that we are able to achieve a relation with high levels of linearity as well in Figure 5, with a high level of linearity, due to the fact that the giving us a goodness of fit of 0.977. This indicates that the Richard-Dushman equation holds well for this tube. However the calculated value of w_o which is $5.24 \pm .54$ eV does not agree with the accepted work function of tungsten which is 4.5 eV in literature. Our fitting error of 10%, does not explain the deviation from the accepted value. We believe that this deviation is caused by the unknown nature of temperature in this fit, there is no way to verify the quality of the empirically known temperature equation or the given wire diameter. Either of these issues may have caused systematic errors which are unaccounted for in our calculation of the w_o's and its error.

5 Conclusion

The purpose of this experiment was to take a look at thermonic emission, which explains a fundamental statistical/quantum property of metals that occurs in our daily lives through X-Rays and incandescent light bulbs. The emission of electrons with applied current is a relatively simple property to test, and finding their emission energies can be completed by using the phenomena of Schottky emission. With these details we can find the work function of a metal.

In this lab we first investigate Schottky emission and validate that it is occurring for this field setup. We are able to find that our observations follow the theory of Schottky emission for all but one data point in our trials. It is unclear as to why this data point is deviating from theory.

The next investigation is into Richard-Dushman equation, with the goal of finding the work function. We are able to find that the phenomena is well modelled by the theory and we are able to find a work function of $5.24 \pm .54$ eV which deviates 14% from the accepted value of 4.5eV. This deviation which is not covered by our error bounds is likely due to a lack of accounting for the errors which occur in temperature estimates.

References

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