Assignment no. 4
Bellman Ford’s DP algorithm: Single-Source Shortest Paths(General Weights)

We now consider the single-source shortest path problem when some or all of the edges of the directed graph G may have negative length (weight). We have seen that Djikstra’s ShortestPaths algorithm does not necessarily give the correct results on such graphs.

We wish to solve the problem when negative edge lengths are permitted. However, it is required that the graph has no cycle of negative length. E.g., in the graph denoted by: c(1,2)=1, c(2,1)=-2 and c(2,3)=1, the length of the shortest path from vertex 1 to vertex 3 is $-\alpha$. The corresponding shortest path is 1,2,1,2,1,2,……,1,2,3. Thus, it is necessary to ensure that shortest paths consist of a finite number of edges.

When there is no cycle of negative length in a directed graph of n vertices, there is a shortest path between any two vertices of at most n-1 edges. Note that a path that has more than n-1 edges have a repetition of a least one vertex, and hence must contain a cycle. If it contains a cycle, this cycle can be deleted, which results in a path with the same source and destination. This path is then cycle-free and has a length that is no more than the original path, as the length of the eliminated cycle was non-negative (at least zero).

Let $dist^{l}[u]$ be the length of a shortest path from the source vertex v to vertex u containing at most l edges. Thus, $dist^{1}[u] = cost[v,u]$, $1<=u<=n$. Since there are no cycles with negative length, we can limit our search for shortest paths to paths with at most n-1 edges. Thus, our goal is to determine $dist^{n-1}[u]$ for all u. There are two cases:

Case 1: The shortest path (SP) from v to u has exactly n-1 edges. Then, this SP is made up of a SP from v to some vertex j followed by the edge (j,u). All vertices i such that (i,u) is an edge are candidates for j. Thus, the candidate vertex i that minimizes the term $dist^{n-2}[i]+cost(i,u)$ is the value for j.

Case 2: The shortest path (SP) from v to u has at most n-2 edges. Then, $dist^{n-1}[u] = dist^{n-2}[u]$.

Thus, the following recurrence follows:
$dist^{n-1}[u] = \min \{ dist^{n-2}[u], \ \min \{ dist^{n-2}[i] + cost\ (i,u)\}\}$

Generalizing,
$dist^{k}[u] = \min \{ dist^{k-1}[u], \ \min \{ dist^{k-1}[i] + cost\ (i,u)\}\}$ ------------------- (1)

Thus, this recurrence can be used to compute $dist^{k}$ from $dist^{k-1}$ for k = 2, 3,.., n-1.

For the directed graph of 4 vertices (n=4) denoted by the following cost matrix, find the shortest path from vertex 1 to vertex 3:
$c\langle 1,2\rangle=2$; $c(1,3)=1$; $c\langle 2,4\rangle= 3$; $c\langle 4,3\rangle = -6$.

Step 1: For k=2, calculate $dist^{k}[u]$ for u=2, 3, 4

Step 2: For k=3, calculate $dist^k$ [u] for u=2, 3, 4
From the above calculations, you can find the shortest paths from 1 to vertices 2, 3 and 4. The lengths of the shortest paths are given by $dist^{n-1}$ [u] for u=2, 3, 4.