Double Pendulum

ME303 Fall 2021 Project Assignment II

Part II

by

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1. Introduction

In this problem, we will investigate the double pendulum motion. There are two bobs with masses m_1 and m_2. The mass m_1 is connected to the wall with a rigid massless string of length L_1 and mass m_2 is connected to the mass m_1 with a rigid massless string of length L_2. The system is shown in figure 1 and the forces acting on the bodies are shown in figure 2 and 3.

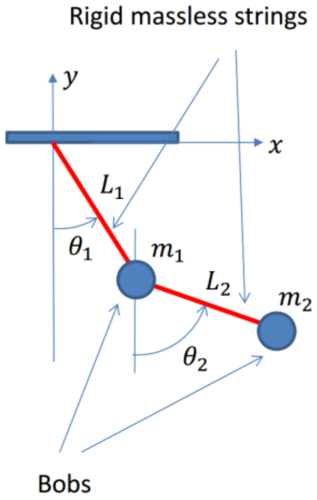

Figure 1. The system

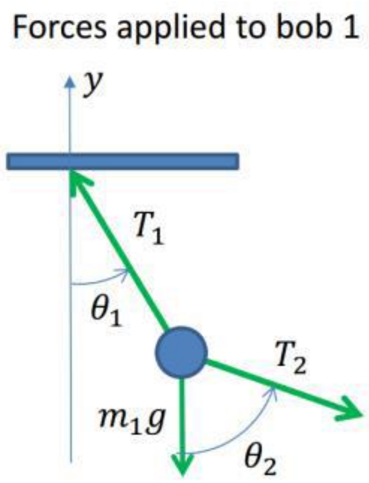

Figure 2. The forces acting on the bob1

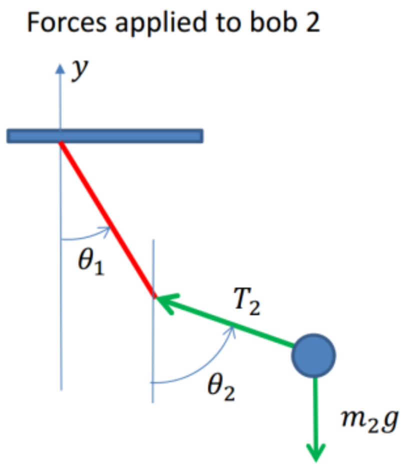

Figure 3. The forces acting on the bob2

After writing and simplifying the equations of motion the following differential equations are found. It is more convenient to write in the matrix form and we will make us of this.

$$\mathbf{A}\begin{pmatrix}\ddot{\theta}_1\\ \ddot{\theta}_2\end{pmatrix} = \mathbf{B}$$

$$\mathbf{A} = \begin{pmatrix}A_{11} & A_{12}\\ A_{21} & A_{22}\end{pmatrix} = \begin{pmatrix}L_1 & \mu_2 L_2\cos(\theta_1-\theta_2)\\ L_1\cos(\theta_1-\theta_2) & L_2\end{pmatrix}$$

$$\mathbf{B} = \begin{pmatrix}B_1\\ B_2\end{pmatrix} = \begin{pmatrix}-g\sin\theta_1 - \mu_2 L_2\sin(\theta_1-\theta_2)\,\dot{\theta}_2^2\\ -g\sin\theta_2 + L_1\sin(\theta_1-\theta_2)\,\dot{\theta}_1^2\end{pmatrix}$$

$$\boxed{\ddot{\theta}_1 = \frac{A_{22}B_1 - A_{12}B_2}{A_{11}A_{22} - A_{12}A_{21}}, \qquad \ddot{\theta}_2 = \frac{-A_{21}B_1 + A_{11}B_2}{A_{11}A_{22} - A_{12}A_{21}}}$$

In order to solve this set of differential equations numerically, we will write them as a system of first order differential equations. If we let,

$$z_1 = \theta_1, \qquad z_2 = \dot{\theta}_1, \qquad z_3 = \theta_2, \qquad z_4 = \dot{\theta}_2$$

Then, the equations are:

$$
\boxed{
\begin{aligned}
\dot{z}_1 &= z_1 \\
\dot{z}_2 &= \frac{A_{22}B_1 - A_{12}B_2}{A_{11}A_{22} - A_{12}A_{21}} \\
\dot{z}_3 &= z_4 \\
\dot{z}_4 &= \frac{-A_{21}B_1 + A_{11}B_2}{A_{11}A_{22} - A_{12}A_{21}}
\end{aligned}
}
$$

$$
\begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix} = \begin{pmatrix} L_1 & \mu_2 L_2 \cos(z_1 - z_3) \\ L_1 \cos(z_1 - z_3) & L_2 \end{pmatrix}
$$

$$
\begin{pmatrix} B_1 \\ B_2 \end{pmatrix} = \begin{pmatrix} -g \sin z_1 - \mu_2 L_2 \sin(z_1 - z_3)\, z_4^2 \\ -g \sin z_3 + L_1 \sin(z_1 - z_3)\, z_2^2 \end{pmatrix}
$$

with the following mass and length values along with the given initial conditions to solve this problem

$$m_1 = m_2 = 1 \text{ kg}, \qquad L_1 = L_2 = 10 \text{ cm}$$

Initial conditions:

$$\theta_1(0) = \theta_2(0) = 90^0$$
$$\dot{\theta}_1(0) = \dot{\theta}_2(0) = 0$$

2. Results

After we solve the differential equations, we get the results as follows for T=20 seconds shown in the figure 4. The colors are the same for the rk4 and the ode45 for easier comparison.

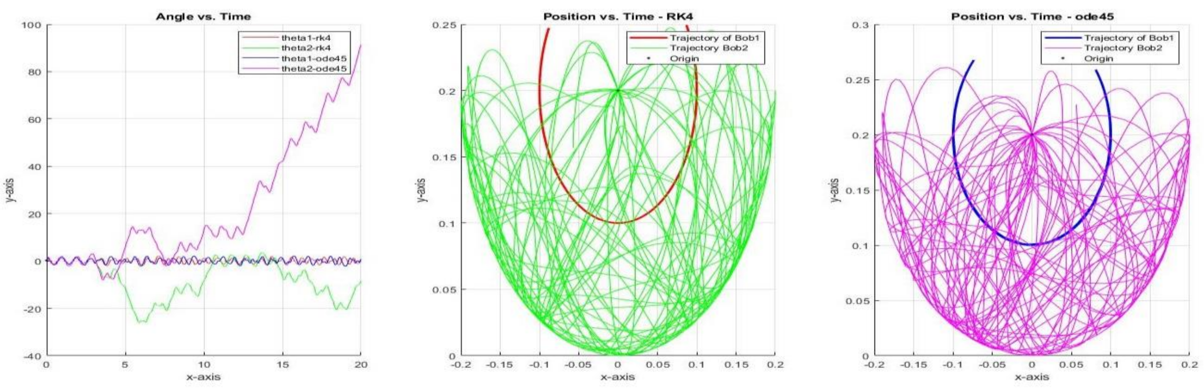

Figure 4. Angle vs. Time — Position vs. Time – RK4 — Position vs. Time – ode45

Although at first glance, it may seem like that the results are too different, this is not the case when we look at them carefully.

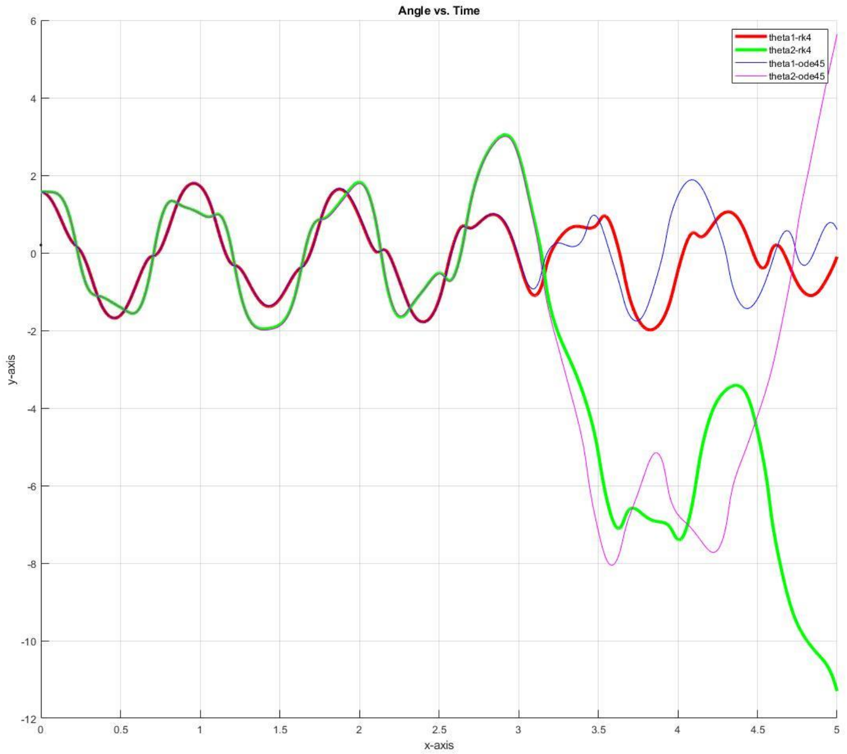

Figure 5. Angle vs. Time graph for t=5s

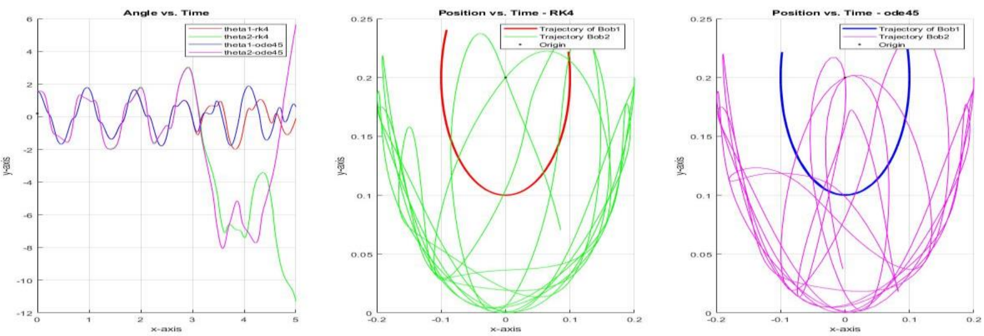

Figure 6. Angle vs. Time Position vs. Time – RK4 Position vs. Time – ode45 for t=5s

As can be seen from figure 5 and figure 6, the results are almost identical until t=3s and they start to diverge after t=3s. We had also seen a similar case in the Restricted 3-Body Problem. This is due to the differences in the methods. However, desπte the very different looking graphs the results are similar even after t=3s. Since the angles of the second rod is cumulative meaning that if it rotates 2 times around the bob1 the result is not 0 again but 4π instead; after a small deviation, the results go on different ways but in reality, they do not differ that much.

If we look closely to the figure below 7, in which the angles are given in their reference angles (between 0 and 2π), the results are more similar. However, again after t=3s, the results seem to be diverging from each other. This is due to the variable step size in the ode45 function.

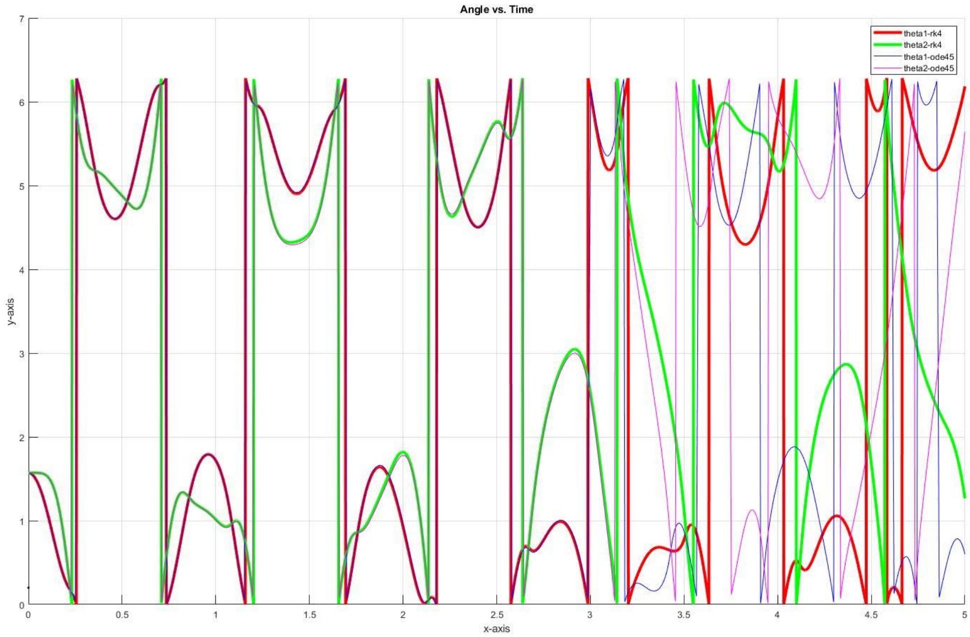

Figure 7. Angle vs. Time graph for t=5s with reference angles

After explaining the reason for the difference in the graphs, we will explain how the results vary depending on the parameters. To be able to observe the changes, we will use t=10s for the figures, since otherwise the figures become too complicated and unreadable.

2.1 Change of the mass of the Bob1 (m_1)

If we increase the mass of the bob1 relative to bob2, the relationship becomes more like a star and a planet. The bob2 rotates in a continuous manner around the bob1 and the position of the bob2 is affected less from bob1. The continuous rotation of the bob1 from t=1.2s to t=3.3s can be seen from both the angle graph on the left and the trajectory on the right in the figure 8.

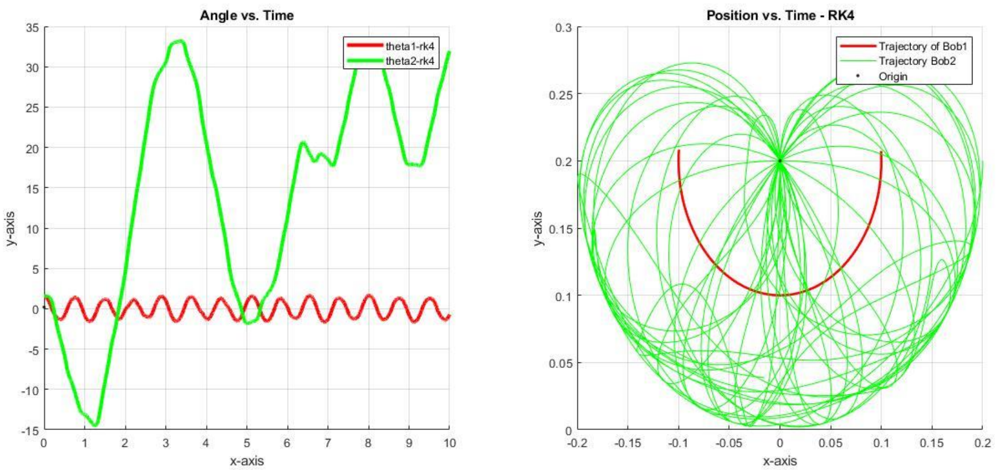

Figure 8. Angle vs. Time and Position vs. Time graph for default values except m_1=10kg

These changes become more apparent as can be seen from figure 9 if we keep increasing the mass of the bob1.

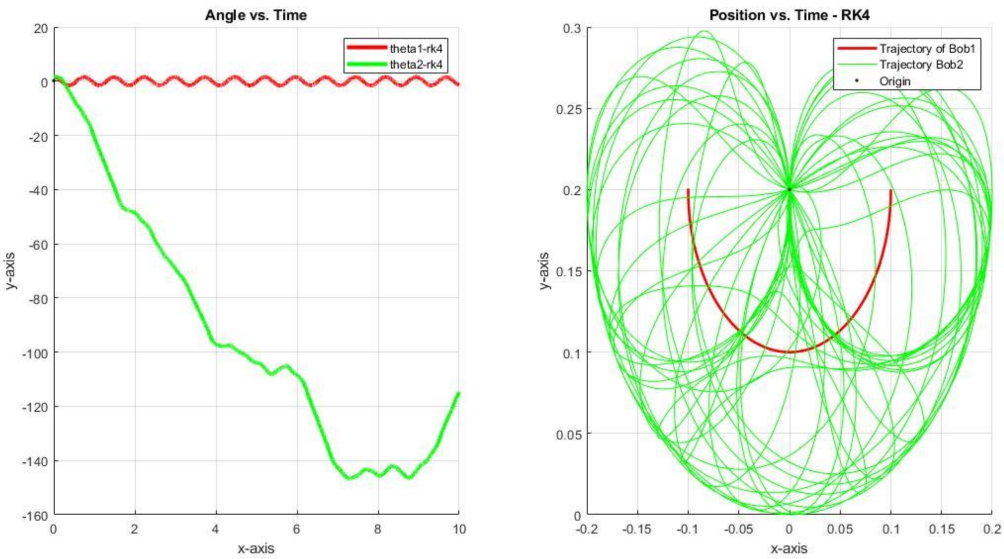

Figure 9. Angle vs. Time and Position vs. Time graph for default values except m_1=100kg

2.2 Change of the mass of the Bob2 (m_2)

If we increase the mass of the bob2 relative to bob1, the motion becomes less random (The word "random" should not be misunderstood. The motion is still deterministic with the motions of equations but not predictable trivially. From now on, the use random will refer to this meaning). The angles of the two bobs are more alike. Although the angles diverge after t=4s, they become periodic again after t=6s. However, bob1 still has some affects on the motion of the bob2 and makes it change location and direction as shown in figure 10.

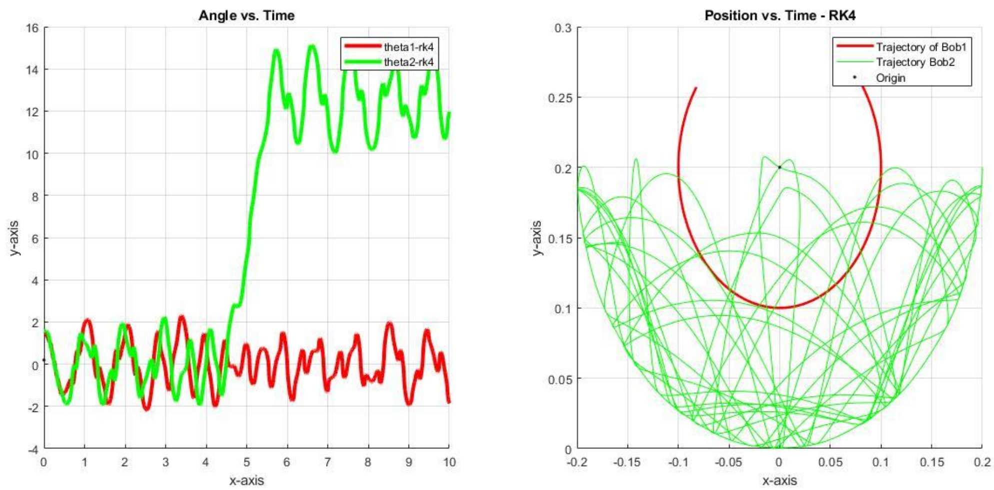

Figure 10. Angle vs. Time and Position vs. Time graph for default values except m_2=10kg

If we further increase the mass of the bob2, the bob1 becomes less important and act as if does not exist, meaning that the motion is more like a single pendulum. The angles are almost the same and the motion of the bob2 is disturbed very slightly. These relations are shown in figure 11 below.

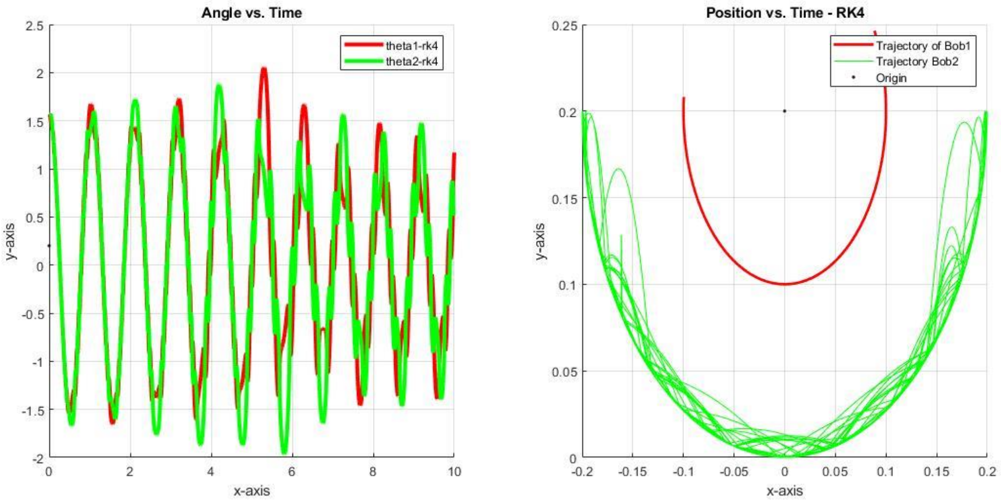

Figure 11. Angle vs. Time and Position vs. Time graph for default values except m_2=100kg

2.3 Change of the Length of the Rod1 (L_1)

As we increase the length of the rod1, the motion of the bob1 becomes more predictable and it starts to move as if it is a single pendulum with longer L_1. This predictable periodic motion can be seen from the figure with the angle 1 line being periodic. However, the motion of the bob2 is still not very predictable as it keeps swinging back and forth and keep rotation around the bob1 as in the figure 12.

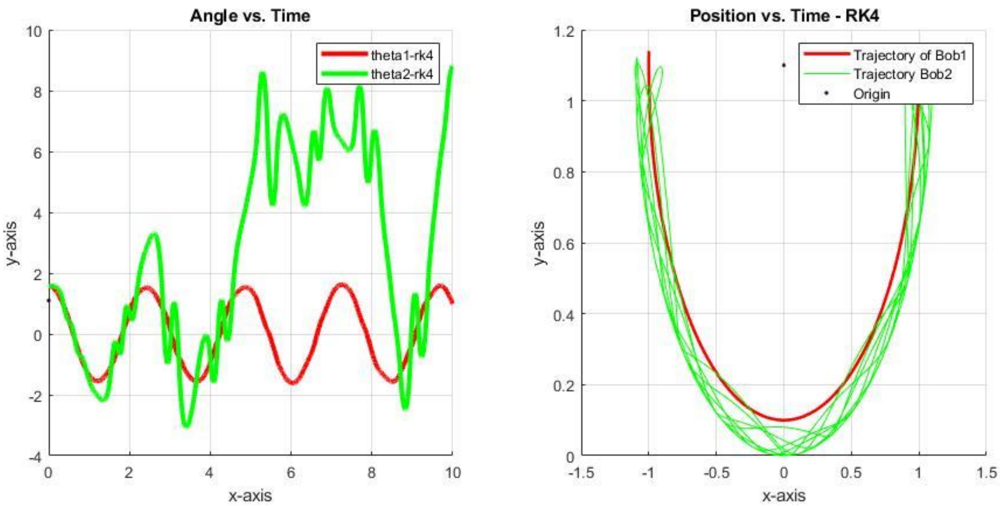

Figure 12. Angle vs. Time and Position vs. Time graph for default values except L_1=1m

2.4 Change of the Length of the Rod2 (L_2)

As we increase the length of the rod2, the effect of the bob1 on the motion of bob2 becomes less significant. As the length of the rod1 is relatively small compared to length of the rod2, the motion of the can be thought as a single pendulum with the initial position being the position of the bob2. This means that the direction and rotation is not very affected. Bob2 keeps swinging with only small disturbances. The periodic motion of the bob2 can be seen from the angle graph in the figure 13 below.

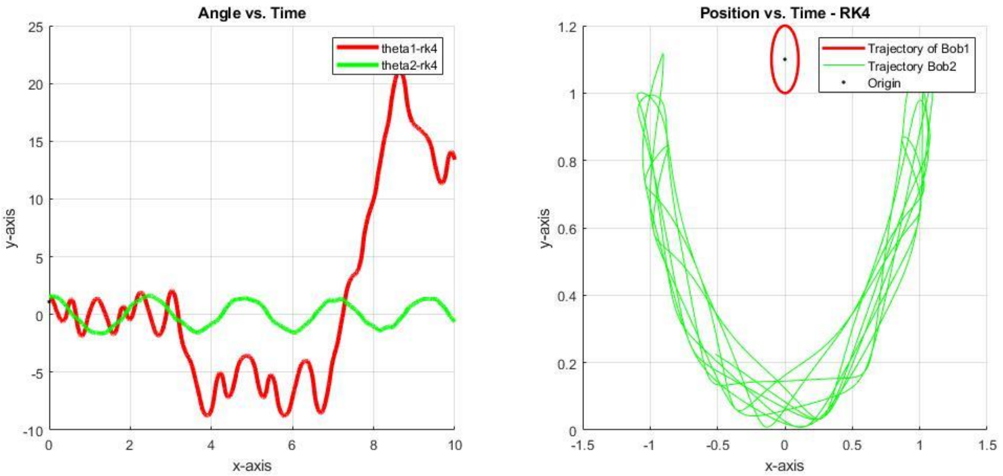

Figure 13. Angle vs. Time and Position vs. Time graph for default values except L_2=1m

2.5 Change of the Gravity Constant

If we were to conduct this experiment on Mars or on Juπter, we would have the same motion with different periods. Although it is harder with the double pendulum the period of a single pendulum is inversely proportional to the square-root of the gravity constant with the following formula. This formula can provide an insight.

$$T = 2\pi\sqrt{\frac{L}{g}}$$

When the gravity constant is equal to 3.721m/s^2 (Mars), the motion is slower and has fewer oscillations as seen in the figure 14.

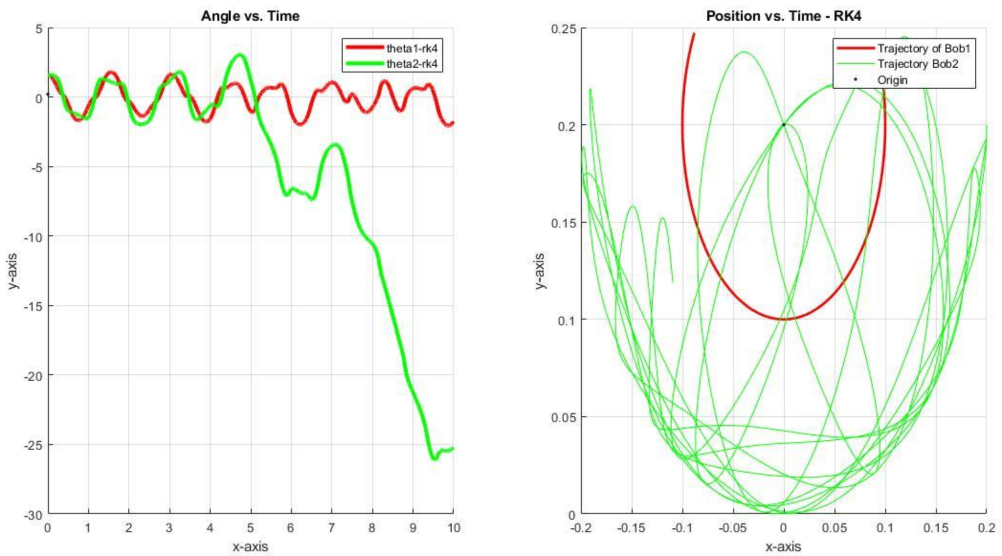

Figure 14. Angle vs. Time and Position vs. Time graph for default values except g=3.721m/s^2

On the contrary, when the gravity constant is equal to 24.79m/s^2 (Juπter) the motion is faster and has more oscillations for the same time T as seen below in the figure 15.

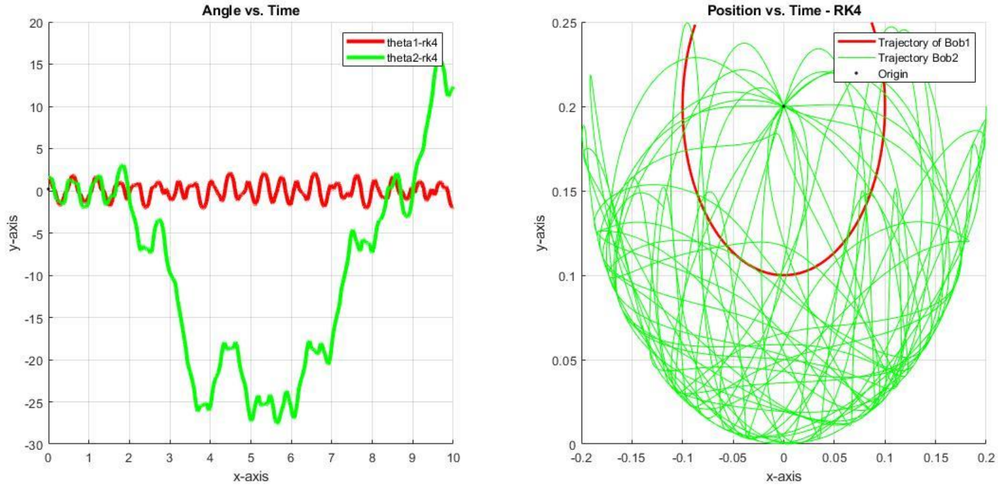

Figure 15. Angle vs. Time and Position vs. Time graph for default values except g=24.79m/s^2

If we compare the angle vs. time graphs on both figures, we can see that their motion is the same except their period.

2.6 Change of the Initial Conditions

The motion is not random (same use as before) and periodic for small angles. In fact, it is very similar to a single pendulum until very close values to π/2 for both angles. An example of initial values of 2π/5 for both angles are given below in figure 16.

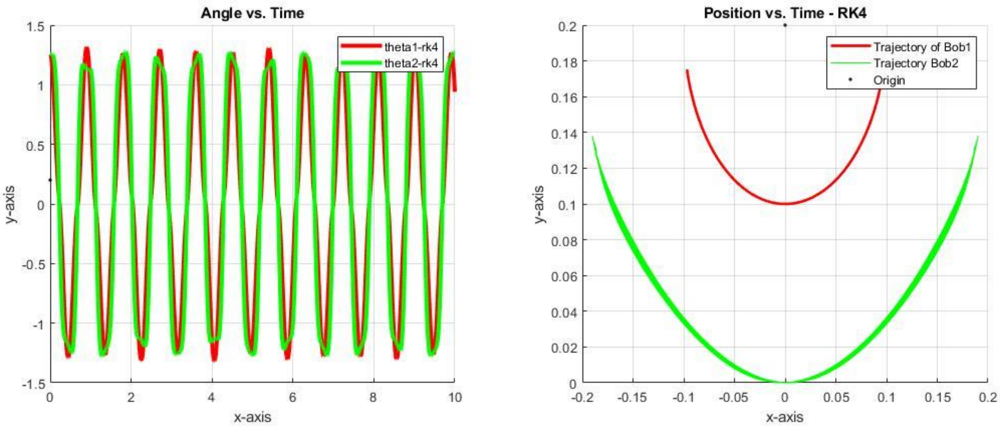

Figure 16. Angle vs. Time and Position vs. Time graph for default values except $\theta_1(0)=2\pi/5$ and $\theta_2(0)=2\pi/5$

However, with increasing angles the motion becomes less predictable. The transition from predictable to random motion begins as the energy of the system is high enough to allow the bob2 to complete a full rotation around the bob1.

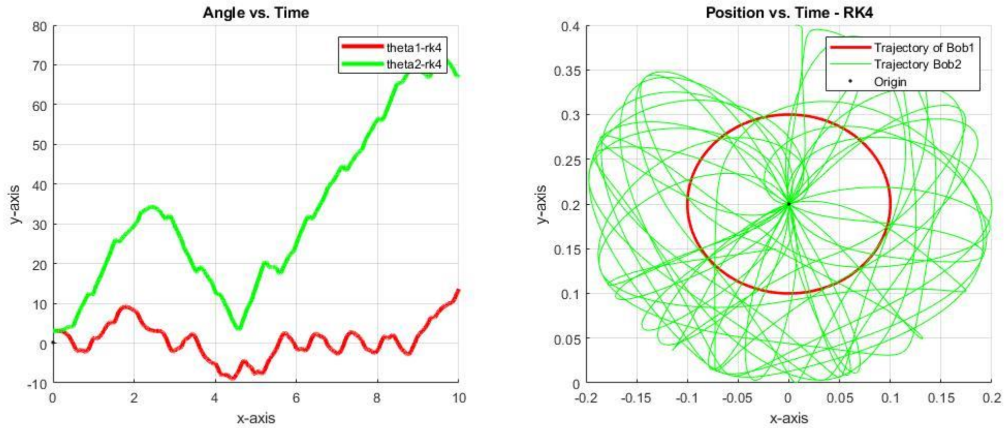

Figure 17. Angle vs. Time and Position vs. Time graph for default values except $\theta_1(0)=0.99\pi$ and $\theta_2(0)= 0.99\pi$

An example is given with the initial angles of 0.99π in figure 17 above. (It does not move if they are vertical, as expected.)

Since the potential energy that can be provided to system is limited, we will also investigate the effect of the initial velocities of the bobs.

When the initial angular velocities are low, the motion is similar to the small angle motions in the figure 16 above. For intermediate values, the motion is random again. Two examples of intermediate angular velocities are given below in figures 18 and 19 respectively.

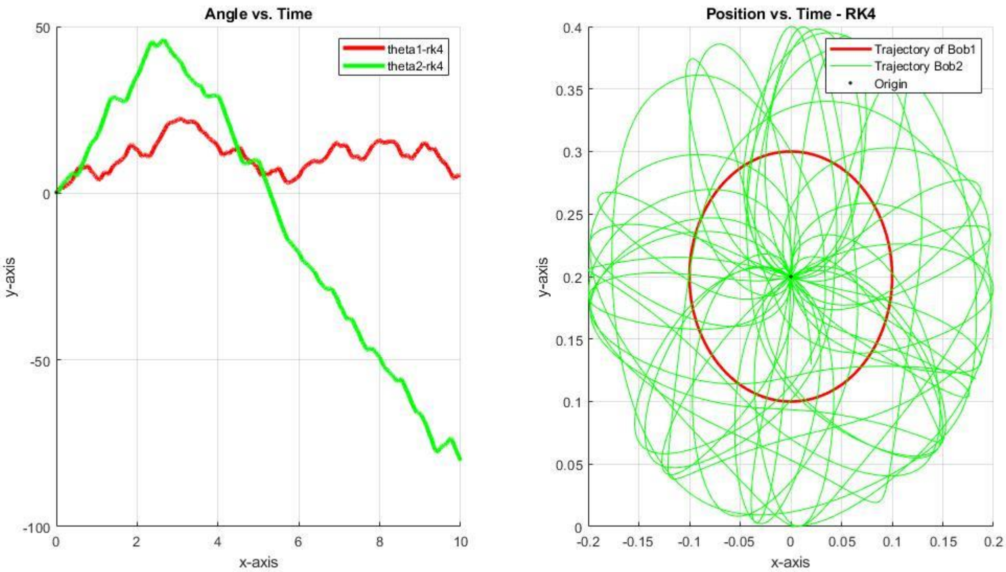

Figure 18. Angle vs. Time and Position vs. Time graph for default values except $\dot{\theta}_1$=25rad/s

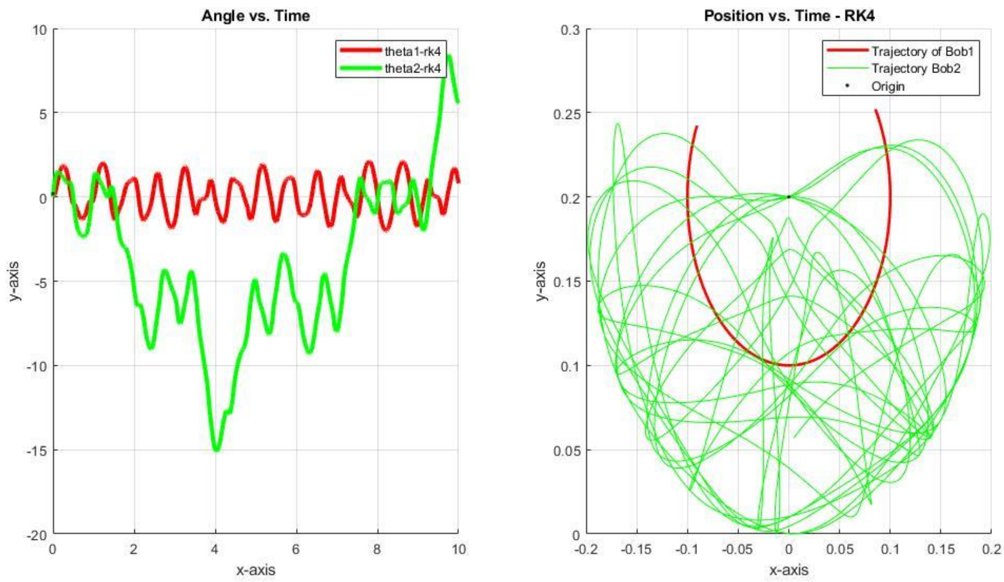

Figure 19. Angle vs. Time and Position vs. Time graph for default values except $\dot{\theta}_2$=25rad/s

However, for higher initial angular velocities, the motion is like a circular motion. The important thing here is that to give enough initial velocities so that the tensions in the rods are never zero even at the top so that the bobs keep rotating around the center. The examples are given below in figure 20 and figure 21.

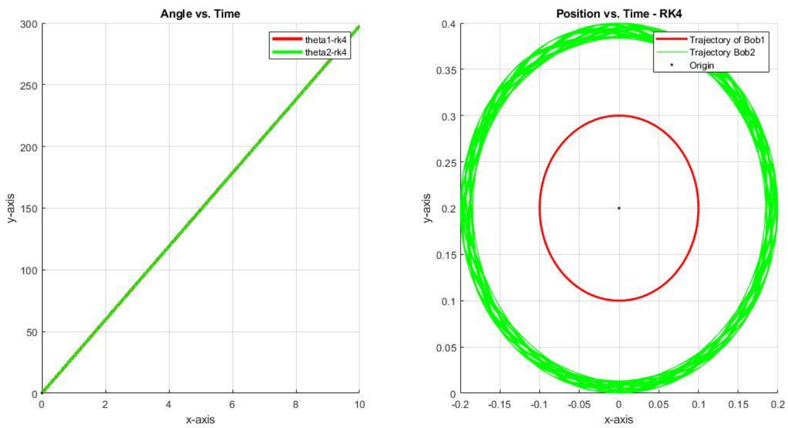

Figure 20. Angle vs. Time and Position vs. Time graph for default values except $\dot{\theta}_1$=50rad/s

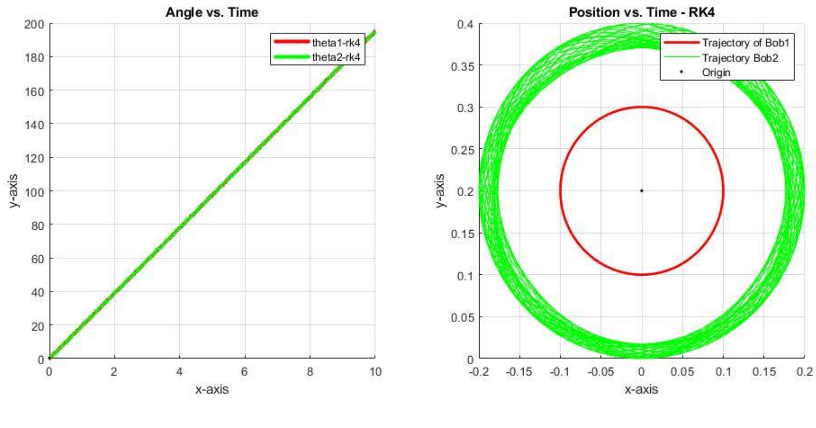

Figure 21. Angle vs. Time and Position vs. Time graph for default values except $\dot{\theta}_2$=50rad/s

References

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