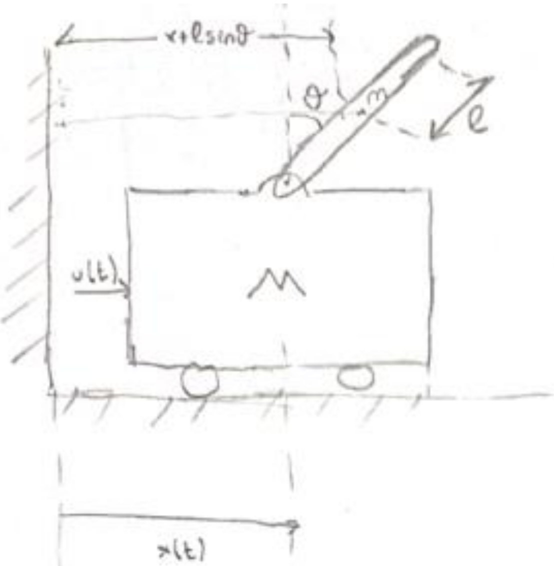
Tolga Çatak
2018405144

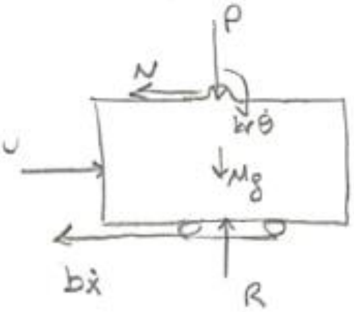
The nonlinear model is built using the equations below.



→ Defined dimensions

where M is the mass of the cart
m is the mass of the pendulum
l is the half length of the pendulum

FBD of the Cart

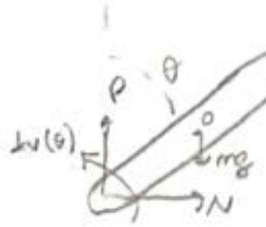


$$\underset{\rightarrow +}{\sum F_x} = u - N - b\dot{x} = Ma_{cart} = M\ddot{x} \qquad (1)$$

$$\Rightarrow M\ddot{x} + b\dot{x} + N = u$$

where b is the coefficient of the friction
P, R, N are the action-reaction forces
u is the external force applied.

FBD of the Pendulum



$$\underset{+\circlearrowright}{\sum M_o} = -Nl\cos\theta + Pl\cos\theta - k_v\dot{\theta} = I\ddot{\theta} \qquad (2)$$

$$\underset{\rightarrow +}{\sum F_x} = N = m a_{pend_x} = m\frac{d^2}{dt^2}(x + l\sin\theta)$$

$$= m\frac{d}{dt}(\dot{x} + l\cos\theta\dot{\theta})$$

$$N = m(\ddot{x} - \dot{\theta}^2 l\sin\theta + \ddot{\theta} l\cos\theta) \qquad (3)$$

where k_v is the coefficient of the moment friction

$$\underset{\uparrow +}{\sum F_y} = P - mg = m a_{pend_y} = m\frac{d^2}{dt^2}(l\cos\theta)$$

$$= m\frac{d}{dt} = m(-\dot{\theta} l\sin\theta)$$

$$P - mg = m(-\dot{\theta}^2 l\cos\theta - \ddot{\theta} l\sin\theta) \qquad (4)$$

Substituting (3) into (1), we get

$$(M+m)\ddot{x} + b\dot{x} + ml\ddot{\theta}\cos\theta - ml\dot{\theta}^2\sin\theta = u \qquad (5) \rightarrow \text{EOM 1}$$

Substituting (3) and (4) into (2), we get

$$(I+ml^2)\ddot{\theta} = mgl\sin\theta - ml\ddot{x}\cos\theta - kv\dot{\theta} \qquad (6) \rightarrow \text{EOM 2}$$

2) In the nonlinear model, EOM 1 and EOM 2 are modeled.

The parameters are assigned as in the lecture notes. The input is given with the pulse generator block with a amplitude of 100N, period of 100s and pulse width of 0.01% of the period.

Then, the results are found as given in the figure 1 and 2 below.

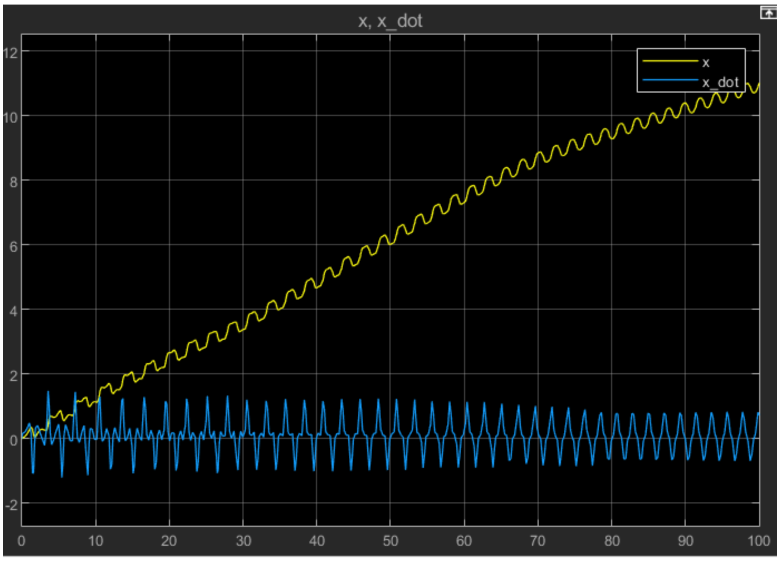


Figure 1. Position (x) and velocity (x_dot) vs. time graph

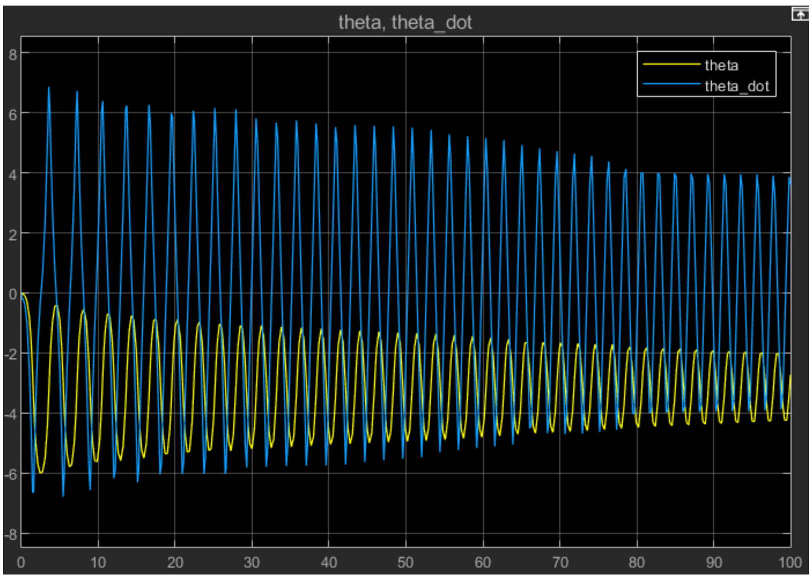


Figure 2. Angle (θ) and angular velocity (θ_dot) vs. time graph

As seen from the figure 1, when the impulse is given to the cart, it starts to move to the right. The oscillations occur due to the pendulum swinging on top of the cart. However, since there is a small friction coefficient, the velocity of the cart decreases with time and the distance increases with a decreasing rate of change.

As seen from the figure 2, with the given impulse, the pendulum first deviates from the rest position and starts to oscillate. However, since there is a small coefficient of moment friction, the pendulum does not reach the upright position (2π) and its angular speed and the peak points of the angles decrease with time. The system has no control mechanism thus it doesn't come back to its initial position (unstable equilibrium position). It is also worth noting that the θ oscillates around π radians pointing downward (stable equilibrium position).

When the friction coefficients are increased 10 times, the decrease of the velocity and the angular velocity can be seen better in figure 3 and figure 4.

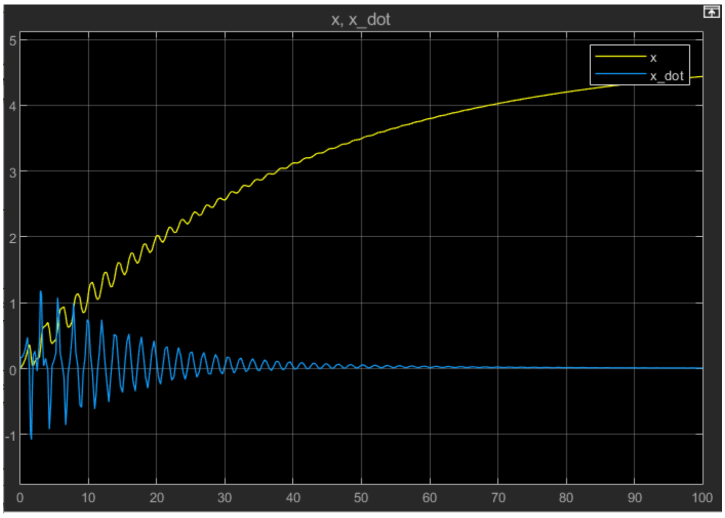


Figure 3. Position (x) and velocity (x_dot) vs. time graph with higher friction coefficients

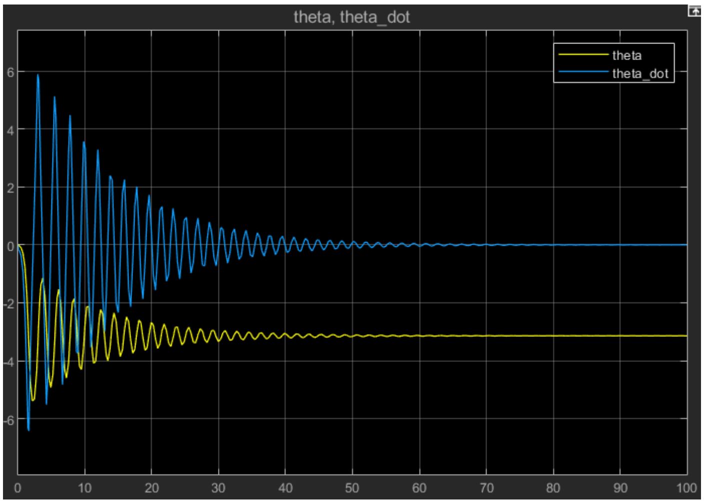


Figure 4. Angle (θ) and angular velocity (θ_dot) vs. time graph with higher friction coefficients

As seen in figure 3 and 4, the cart eventually comes to a stop with a zero velocity and the pendulum comes to a stop around π radians.

b) Since we are linearizing the model around $\theta = 0$, we can do the following assumptions:

$$\left.\begin{aligned} \sin\theta &= \theta \\ \cos\theta &= 1 \\ \dot{\theta}^2 &= 0 \end{aligned}\right\}$$

Assuming the pendulum is at 0 degrees when it is at upright position. Also, it starts from the rest position with a small input so that $\dot{\theta}$ is small enough.

Then, EOM 1 becomes:

$$(M+m)\ddot{x} + b\dot{x} + ml\ddot{\theta} = u \rightarrow \text{EOM } 1$$

And, EOM 2 becomes:

$$(I+ml^2)\ddot{\theta} = mgl\theta - ml\ddot{x} - kv\dot{\theta} \rightarrow \text{EOM } 2$$

These updated and linearized EOM 1 and EOM 2 are modeled in Simulink

For the linearized model, the results are found as given in figure 5 and 6.

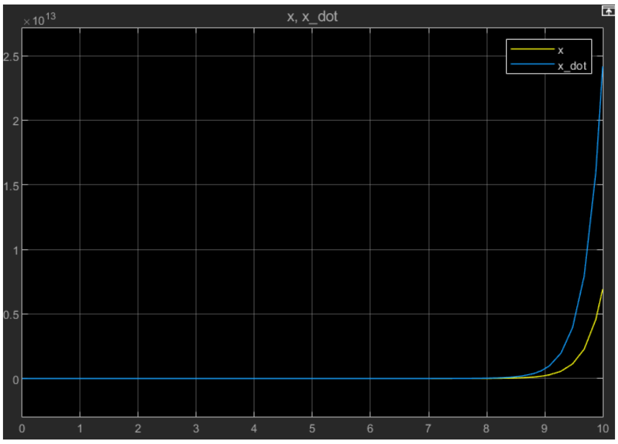


Figure 5. Position (x) and velocity (x_dot) vs. time graph with

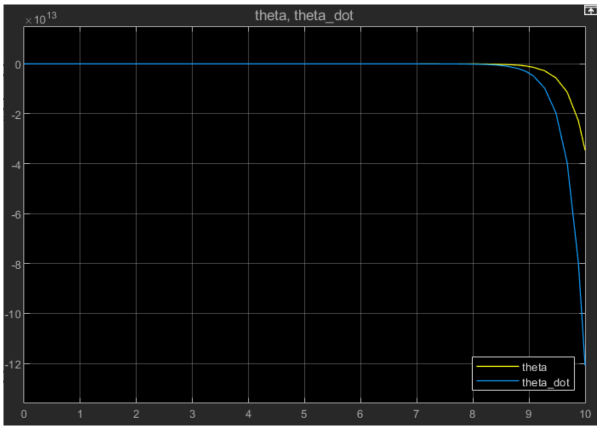


Figure 6. Position (x) and velocity (x_dot) vs. time graph with

As seen above, the linearized model diverges and reaches very high values (practically infinity). This is because we made the linearization around θ=0. However, when the pendulum starts to oscillate and depart from θ=0, the model is not valid anymore and the errors only get bigger.

c) To obtain the transfer function we use the linearized equations, and take the Laplace transform.

$$(M+m)s^2X(s) + bsX(s) + mls^2\Phi(s) = U(s)$$

$$(I+ml^2)s^2\Phi(s) = mgl\Phi(s) - mls^2X(s) - kvs\Phi(s)$$

$$X(s) = \frac{mgl\Phi(s) - (I+ml^2)s^2\Phi(s) - kvs\Phi(s)}{mls^2}$$

$$\left[(M+m)s+b\right]\left[\Phi(s)\left(\frac{mgl-(I+ml^2)s^2-kvs}{mls}\right)\right] + mls^2\Phi(s) = U(s)$$

$$\Phi(s)\left(\left[(M+m)s+b\right]\left[\frac{mgl-(I+ml^2)s^2-kvs}{mls}\right] + mls^2\right) = U(s)$$

$$\frac{\Phi(s)}{U(s)} = \frac{m l s}{\left((M+m)s+b\right)\left(mgl-(I+ml^2)s^2-k_v s\right)+m^2 l^2 s^3}$$

$$\frac{\Phi(s)}{U(s)} = \frac{m l s}{\left(\left[m^2 l^2-(M+m)(I+ml^2)\right]s^3-\left[(M+m)k_v+b(I+ml^2)\right]s^2+\left[(M+m)(mgl)-k_v b\right]s+bmgl\right)}$$

d) Using the numerical values of M, m, b, k_v, L, I and taking g as $9.81 m/s^2$, zeros and poles are found.

Zeros: Values for which the numerator of the transfer function becomes zero $\Rightarrow s = 0$

Poles: Values for which the the denominator of the transfer function becomes zero $\Rightarrow s_1 = -2.5910$, $s_2 = -0.0027$, $s_3 = 2.5815$

After the transfer function is obtained, its impulse response is found as in the figure 7 below. Also, the poles and the zeros are found using MATLAB and plotted in figure 8 (both with equating to zero and with rlocus function).

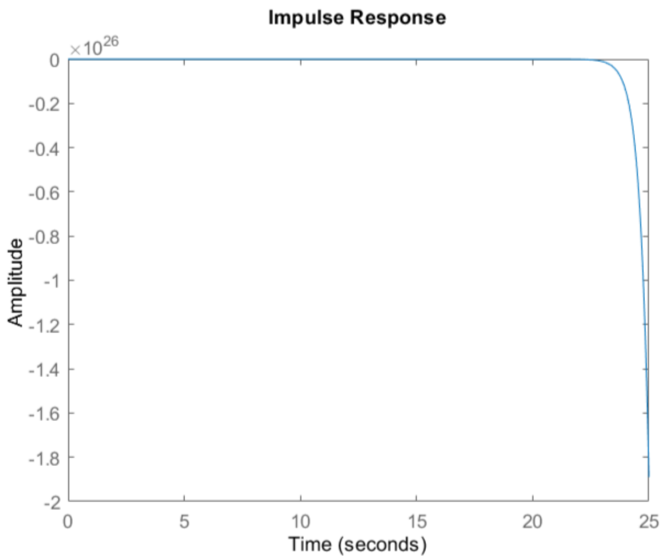


Figure 7. Impulse response of the transfer function

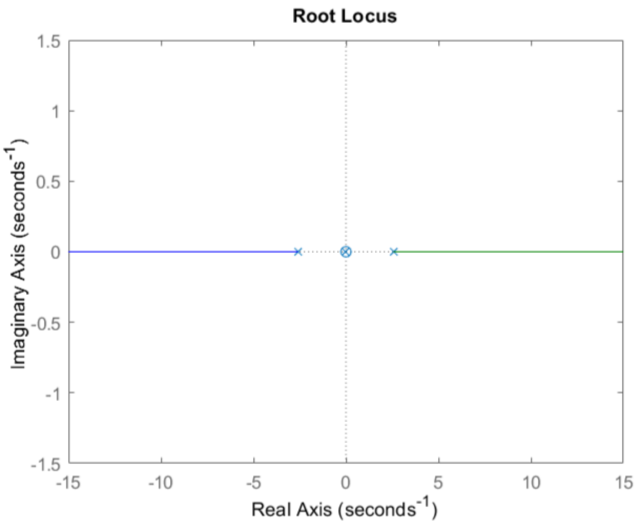


Figure 8. Root Locus of the transfer function

d) Using the numerical values of M, m, b, k_v, L, I and taking g as $9.81 m/s^2$, zeros and poles are found.

Zeros: Values for which the numerator of the transfer function becomes zero $\Rightarrow s = 0$

Poles: Values for which the the denominator of the transfer function becomes zero $\Rightarrow s_1 = -2.5910$, $s_2 = -0.0027$, $s_3 = 2.5815$

When all the poles of a transfer function is smaller than zero, the system is stable. However, when even one pole (suppose a) is larger than zero the system becomes unstable as it goes to infinity in the time domain with exp(a*t) and the effect of other poles becomes unsignificant. This expected behavior is also seen in the scope of the linear Simulink model in figure 5 and 6. The graph of the impulse response confirms this divergent behavior of the system. Furthermore, having a zero at zero means that the output will be zero when the input is zero.