# Restricted 3-Body Problem

## ME303 Fall 2021 Project Assignment II

## Part I

by

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## 1. Introduction

We consider two space objects of masses m1 = nu and m2 = 1 -nu circulating in a plane and a third body of a much smaller (negligible) mass moving in their gravitational field. We can think of these two space objects as the earth and the moon, where the moon moves around the earth with distance 1 (this is a normalized distance). A third object, which is relatively much smaller than the first two, such that it would not make any change in the orbits of the first two, is also moving in the space. We can think of this as a spacecraft.

When we restrict the problem to a planar motion (hence the name restricted), the position of the spacecraft (the lightest body) is determined by the vector (y1(t); y2(t)) which is the position vector with respect to the system of coordinates in which the center of mass of Earth-Moon system is at the origin. We normalize all physical units in such a way that the mass of Moon is m1 = nu = 0.012277471 and the mass of Earth is m2 = 1 - m1 = 1 - nu = nu star. In this system of coordinates the equations of motions can be written as

$$\begin{aligned}
y_1''(t) &= y_1(t) + 2y_2'(t) - \mu^* \frac{y_1(t)+\mu}{D_1} - \mu \frac{y_1(t)-\mu^*}{D_2} \\
y_2''(t) &= y_2(t) - 2y_1'(t) - \mu^* \frac{y_2(t)}{D_1} - \mu \frac{y_2(t)}{D_2} \\
D_1 &= ((y_1+\mu)^2 + y_2^2)^{3/2} \\
D_2 &= ((y_1-\mu^*)^2 + y_2^2)^{3/2} \\
\mu^* &= 1-\mu
\end{aligned}$$

where the initial conditions

$$\begin{aligned}
y_1(0) &= 0.994 \\
y_1'(0) &= 0 \\
y_2(0) &= 0 \\
y_2'(0) &= -2.00158510637908252
\end{aligned}$$

for

$$0 \le t \le T = 17.0652165601579625$$

Here T represents the time required for the spacecraft to complete one full round of orbit.

## 2. Euler Method

The solution of the problem with Euler method having step size T/24000 is given below in figure 1.

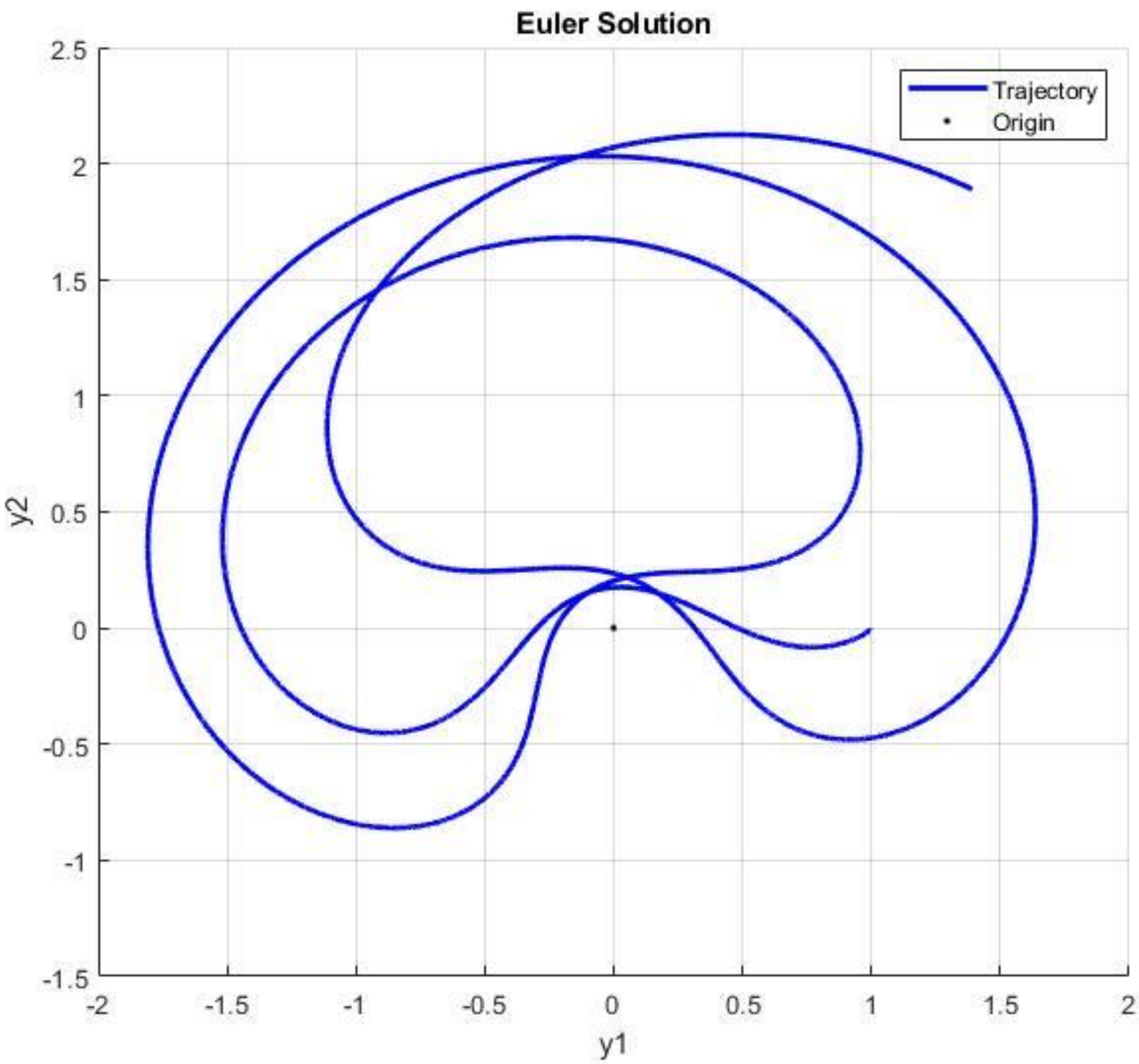


**Figure 1.** Solution with Euler method with h=T/24000

### 3. RK4 Method

The solution of the problem with RK4 method having step size T/6000 is given below in figure 2.

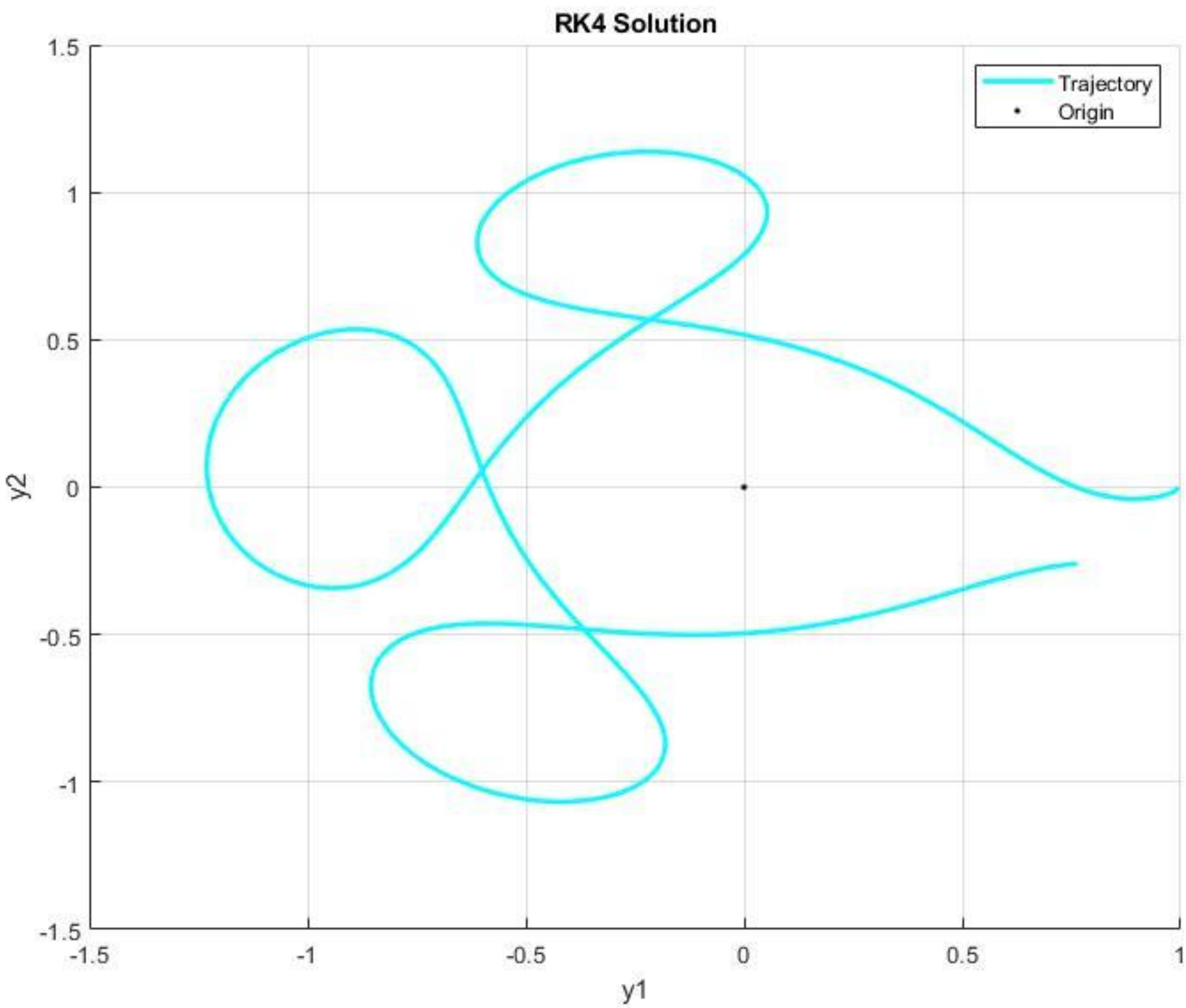


**Figure 2.** Solution with RK4 method with h=T/6000

## 4. ode45 Method

The solution of the problem with ode45 built-in function in MATLAB is given below in figure 3.

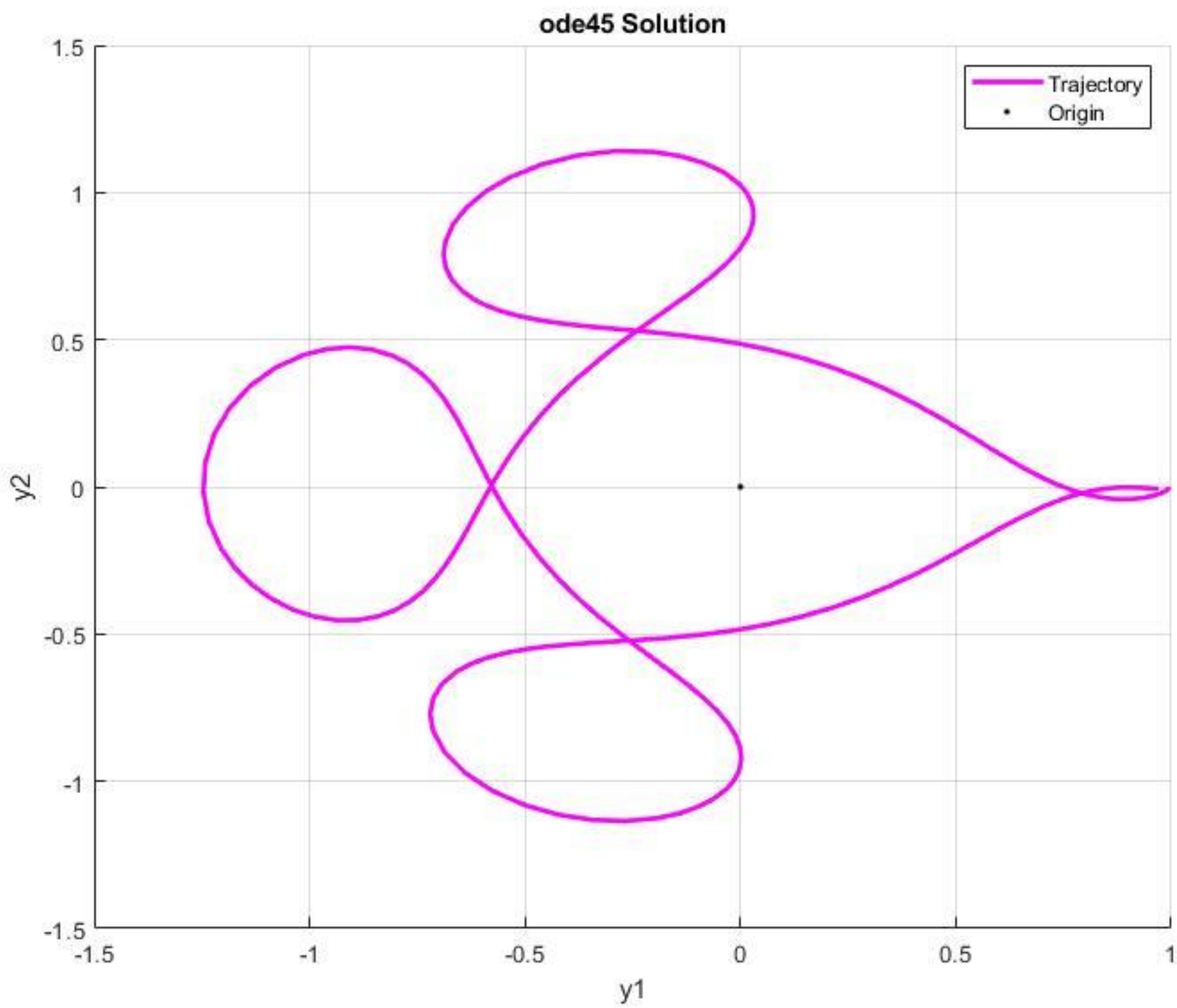


**Figure 3.** Solution with ode45 method

## 5. Conclusions about the Trajectory

The third object having a small mass is following the trajectory determined by the given differential equations and comes back to its initial position. Until it comes to the initial position, its trajectory is a very specific trajectory and does not seem apparent. However, this is indeed very similar to the trajectory of the Apollo 11's trajectory when the first Moon landing was accomplished. The actual trajectory that happened in 1969 is showed in figure 4 below.

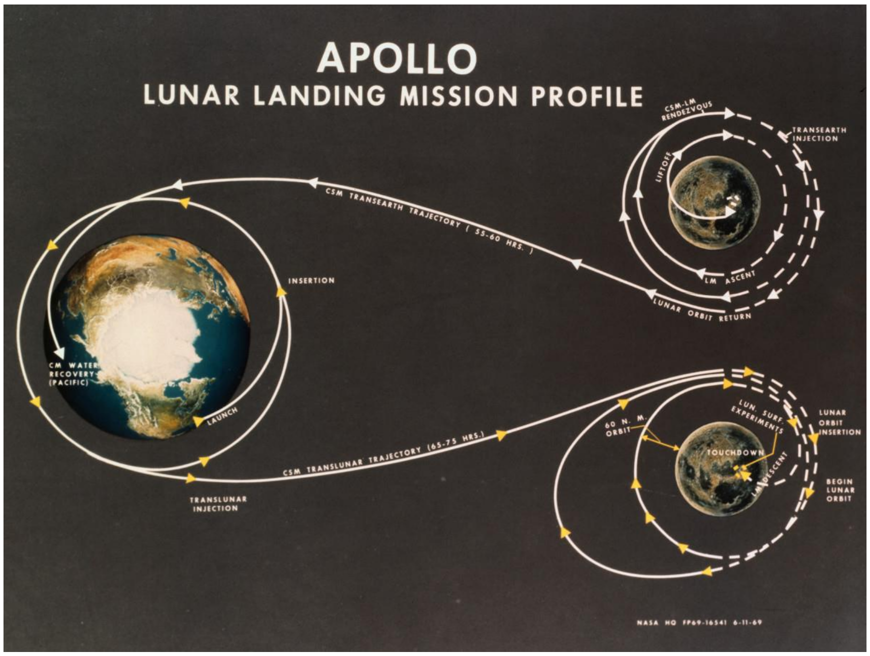


**Figure 4.** Trajectory of the Apollo 11 Spacecraft

However, after the object reaches its initial position and continues to its motion, the trajectory becomes more familiar.

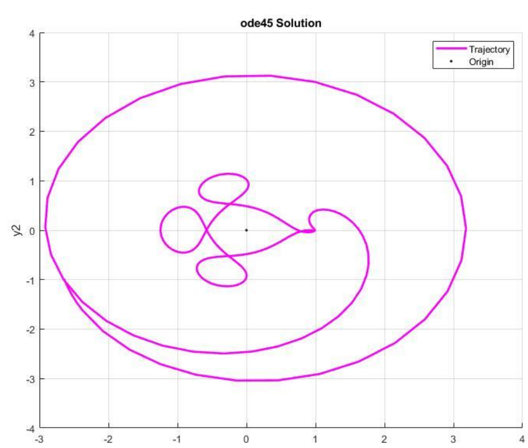


**Figure 5.** Solution with ode45 for T=30

In figure 5 above, the third object starts to orbit the masses 1 and 2 together as if they were a combined mass in a single body. This becomes more apparent as T increases. This can be seen in figure 6 below.

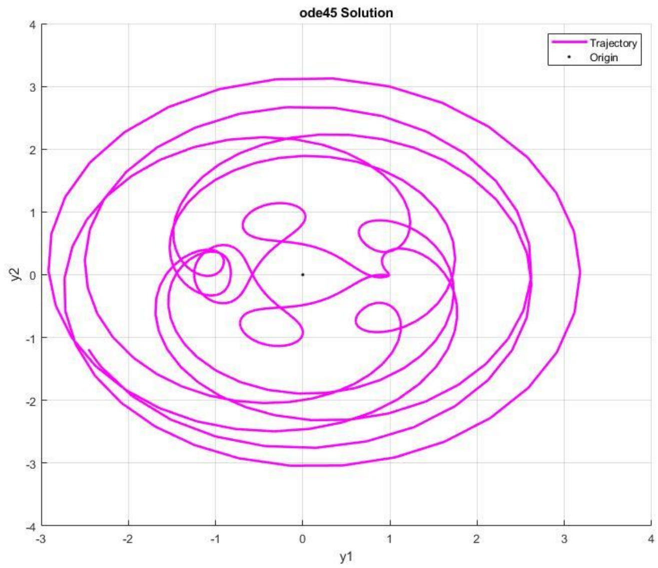


**Figure 6.** Solution with ode45 for T=80

Furthermore, an unwanted example could be the Apollo 11 spacecraft. Although it completed its journey back on Earth, it could be orbiting the Earth and the Moon until the effects of the other objects such as Sun and Mars become more significant, if it had missed the Earth on the way back. One important thing to mention is that the object is getting further away with each completed orbit. This can be seen in the figure 7 below

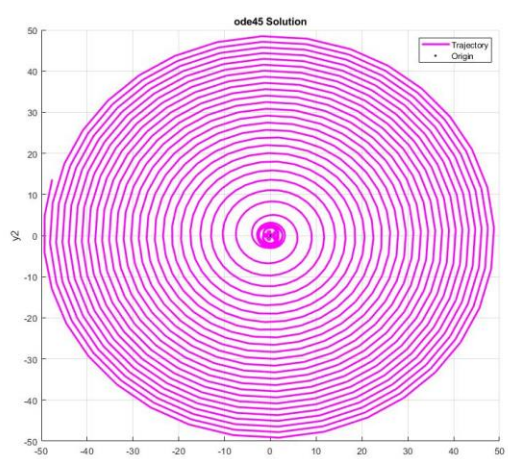


**Figure 7.** Solution with ode45 for T=250

However, one important thing to mention is that the object is getting further away with each completed orbit. A natural example of this phenomenon can be the meteorites coming close to the Earth and passing away. Although they do not get further away, another example can be the circumbinary planets. A circumbinary planet is a planet that orbits two stars instead of one. An illustration is given in the figure 8 below.

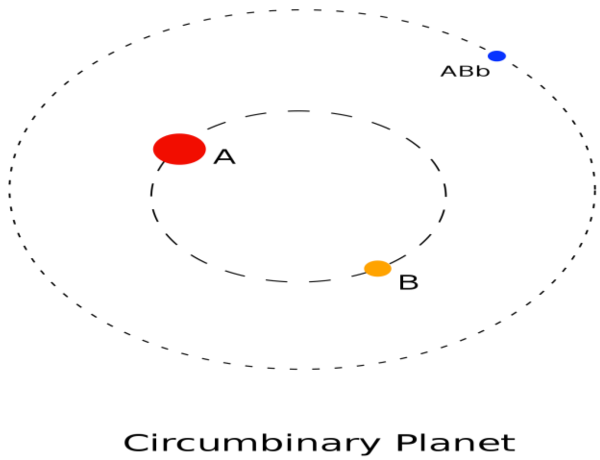


**Figure 8.** An illustration of a binary star system

### 6. The Most Suitable Method

By comparing the solutions and the parameters of different methods, we can choose the most suitable one. To start with, the trajectories with given step sizes are given below in the figure 9.

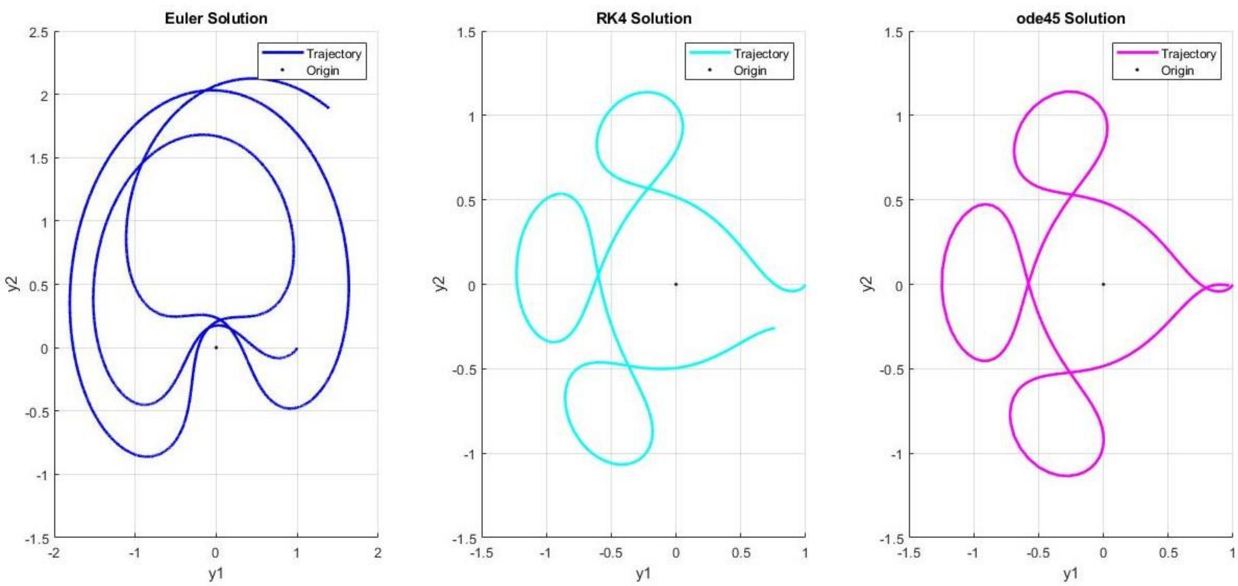


**Figure 9.** All solutions together with given step sizes

As can be seen, even though the Euler method uses a much smaller step size (4 times smaller than the RK4 method step size), it does not provide an acceptable solution.

We can understand the difference if we compare in two different cases where they have the same step sizes. When the step size of both methods is equal to T/6000, the Euler solution diverges almost immediately, whereas the RK4 solution is close to the ode45 solution which can be accepted as the actual solution for now. The results are shown in figure 10 below.

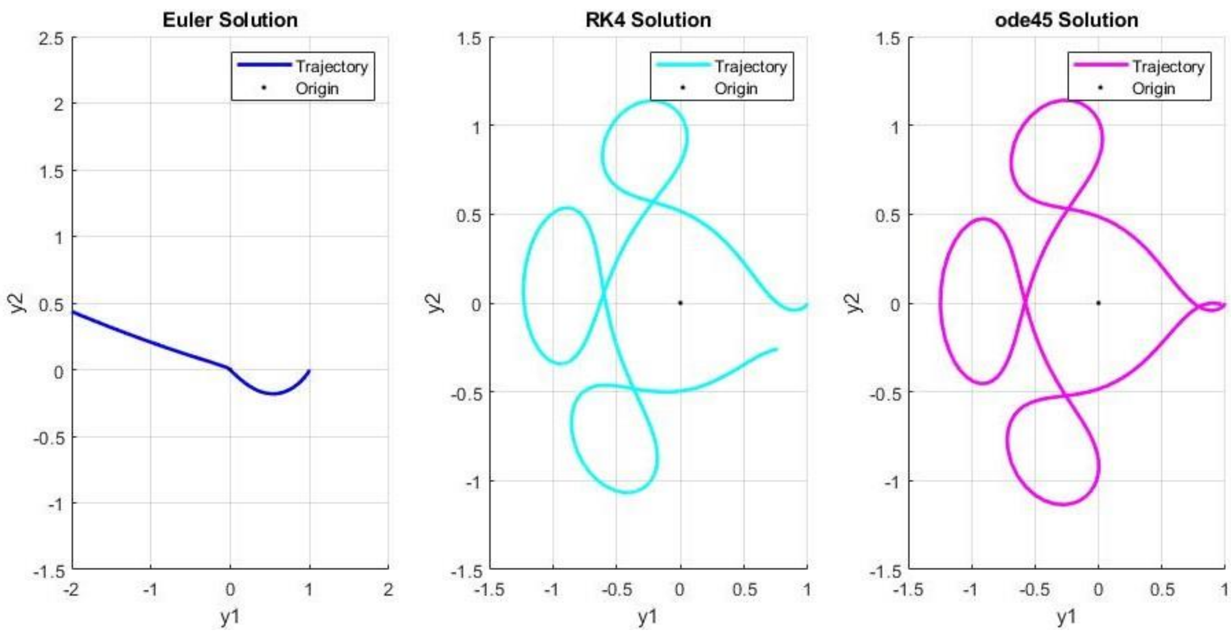


**Figure 10.** All solutions together with step sizes h=T/6000

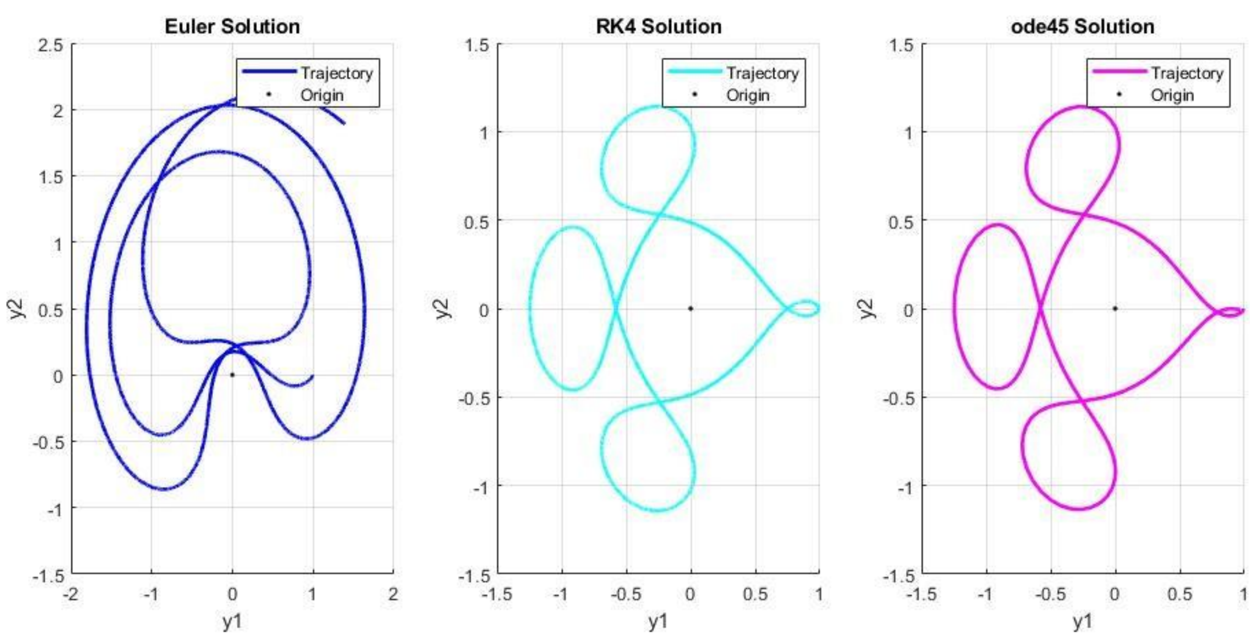


**Figure 11.** All solutions together with step sizes h=T/24000

When the step size of both methods is equal to T/24000, the Euler solution still diverges and does not provide a proper trajectory, however, the RK4 solution is almost the same as the ode45 solution as can be seen in figure 11 above.

Even when the step size of the Euler method is equal to T/384000 (64 times smaller than the RK4 method step size), the solution of the RK4 method is closer to the ode45 solution, which is shown in figure 12 below.

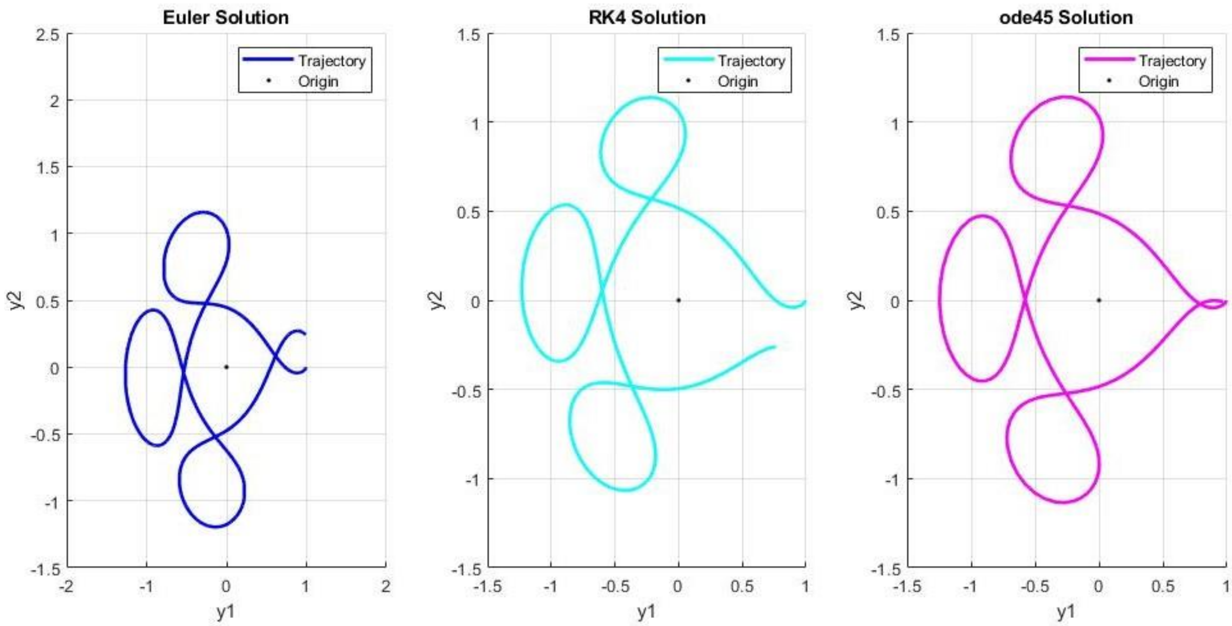


**Figure 12.** All solutions together with step sizes h=T/384000 for Euler and h=T/6000 for RK4

Therefore, we can say the RK4 method is a more suitable method for analyzing trajectories, not only from the point of view of accuracy but also efficiency although RK4 method makes more calculations in one iteration. If we make the calculations for the last case, the number of calculations in the RK4 method is 1.2*10^5 (20 calculations for every 6000 iterations) while the number of calculations in the Euler method is 1.536*10^6 (4 calculations for every 384000 iterations). Considering that the Euler solution is does not proper solution with a number of calculations 12.8 times the RK4 solution, it is safe to say that our claim is correct.

To compare the RK4 method and the ode45 method, we need to understand how the ode45 method works. The RK4 method is a $4^{th}$ order method as the name suggests. However, the ode45 method uses $5^{th}$ order RK method for the solution. The ode45 method uses a variable step size rather than using a fixed step size as in the case of the RK4 method. The '4' in the ode45 refers to the $4^{th}$ order use of variable step selection. This allows ode45 function to use larger step sizes than the RK4 method which increases efficiency.

The errors of the two methods can be seen in the figure 13 below. Although it may seem like the error is larger for the ode45, the error at the true steps grows more slowly than the error for RK4. This is not surprising considering that the ode45 is a higher order solution. The large misleading errors between the integration points are caused due to the interpolation.

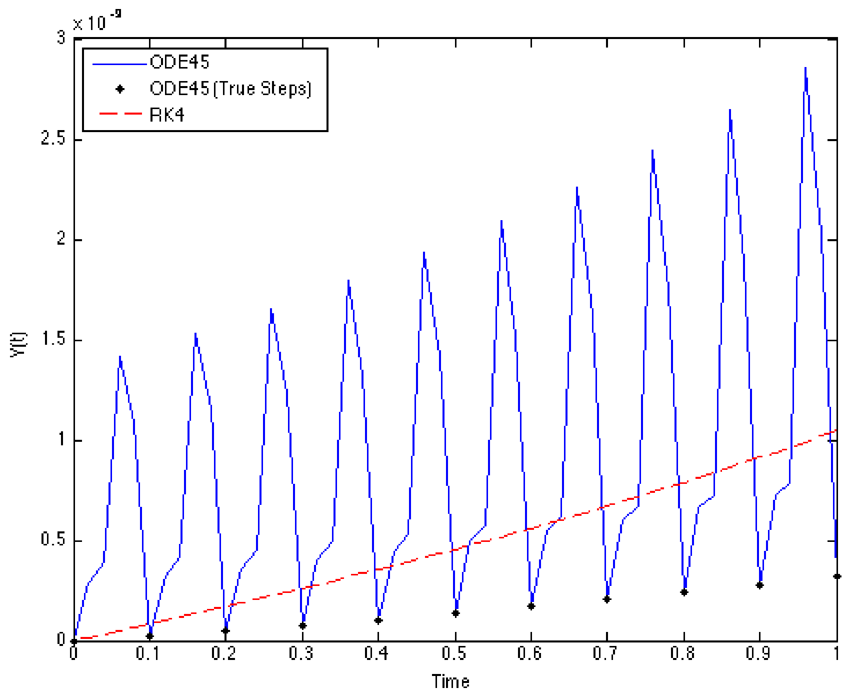


**Figure 13.** Error sizes for the ode45 and the Euler

To conclude, the ode45 method is the most suitable method both from the point of view of accuracy and the efficiency. It also has some other features since it is a built-in function. But personally, I would like to also express my own opinions. I believe that the RK4 method is better from the learning point of view since it allows us to see what is happening behind the scenes and helps us understand the fundamentals.

## References


[1] «Circumbinary planet,» Wikipedia, [Çevrimiçi]. Available: https://en.wikipedia.org/wiki/Circumbinary_planet. [Erişildi: 21 01 2022].

[2] «Apollo 11,» Wikipedia, [Çevrimiçi]. Available: https://en.wikipedia.org/wiki/Apollo_11. [Erişildi: 21 01 2022].

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[5] T. Edelhofer, «difference in ode45 and rk4 code,» MathWorks, [Çevrimiçi]. Available: https://www.mathworks.com/matlabcentral/answers/416953-difference-in-ode45-and-rk4-code. [Erişildi: 21 01 2022].