CENG 551 Probability and Stochastic Processes for Engineers

Week # 13

Problem Solving Session

Dr. Deniz Ozdemir

Deniz.ozdemir@yasar.edu.tr

Web: dozdemir.yasar.edu.tr

Example: Cooking

- Dino, the cook, has good days and bad days with equal frequency. On a good day, the time (in hours) it takes Dino to cook a souffle is described by the PDF

$$f_G(g) = \begin{cases} 2 & if\ \frac{1}{2} \le g \le 1 \\ 0 & otherwise \end{cases}$$

but on a bad day, the time it takes is described by the PDF.

$$f_B(b) = \begin{cases} 1 & if\ \frac{1}{2} \le b \le \frac{3}{2} \\ 0 & otherwise \end{cases}$$

Find the conditional probability that today was a bad day, given that it took Dino less than three quarters of an hour to cook a souffle.

Example: Cooking

- <u>Solution</u>

G: having good day

B: having bad day

T: time to cook

P(B|T≤3/4) = ?

P(G) = P(B) = ½

$$f_G(g) = \begin{cases} 2 & if\ \frac{1}{2} \le g \le 1 \\ 0 & otherwise \end{cases} \qquad f_B(b) = \begin{cases} 1 & if\ \frac{1}{2} \le b \le \frac{3}{2} \\ 0 & otherwise \end{cases}$$

$$f_{X|Y}(x|y) = \frac{f_{X,Y}(x,y)}{f_Y(y)} = \frac{f_X(x)f_{Y|X}(y|x)}{f_Y(y)}$$

$$P(B|T \le 3/4) = \frac{P(T \le 3/4 \mid B)P(B)}{P(T \le 3/4)} = \frac{P(T \le 3/4 \mid B)P(B)}{P(T \le 3/4 \mid B)P(B) + P(T \le 3/4 \mid G)P(G)}$$

$$P(T \le 3/4 \mid B) = \left(\frac{3}{4} - \frac{1}{2}\right) = \frac{1}{8}$$

$$P(T \le 3/4 \mid G) = \left(\frac{3}{4} - \frac{1}{2}\right)\frac{1}{2} = \frac{1}{16}$$

P(B|T≤3/4) =1/3

Exercise:

- Bob is an electrical engineer, and he is equally likely to work between zero and one hundred hours each week (i.e., the time he works is uniformly distributed between zero and one hundred). He gets paid one dollar an hour.
- If Bob works more than fifty hours during a week, there is a probability of 1/2 that he will actually be paid overtime, which means he will receive two dollars an hour for each hour he works longer than fifty hours. Otherwise, he will just get his normal pay for all of his hours that week. Independently of receiving overtime pay, if Bob works more than seventy five hours in a week, there is a probability of 1/2 that he will receive a one hundred dollar bonus, in addition to whatever else he earns.
- Determine the expected value and variance of Bob's weekly salary.

Exercise: At a certain time, the number of people that enter an elevator is a random variable with Poisson PMF, with parameter λ. The weight of each person to get on the elevator is independent of each other person's weight, and is uniformly distributed between 100 and 200 lbs. Let X_i be the fraction of 100 by which the i[th] person exceeds 100 lbs, e.g. if the 7[th] person weighs 175 lbs., then $X_7 = .75$. Let Y be a random variable given by the sum of the X_i.

- What is the probability that at least one person to get on the elevator weighs more than 150 lbs.?

The probability that there are k people in an elevator is given by:

$$P(k) = e^{-\lambda}\frac{\lambda^k}{k!}$$

Let A denote the event that at least one person to get on the elevator weighs more than 150 pounds. The probability of A is 1 minus the probability that no one weighs more than 150 pounds.

$$\begin{aligned} P(A) &= 1 - \sum_{k=0}^{\infty} e^{-\lambda}\frac{\lambda^k}{k!}\left(\frac{1}{2}\right)^k \\ &= 1 - e^{-\lambda}\sum_{k=0}^{\infty}\frac{(\lambda/2)^k}{k!} \\ &= 1 - e^{-\frac{\lambda}{2}} \end{aligned}$$

Exercise

- Random variables X and Y have the joint PDF:

$$f_{X,Y}(x,y) = \begin{cases} 2 \quad if \quad x > 0 \quad \underline{and} \quad y > 0 \quad \underline{and} \quad x + y \leq 1 \\ 0 \quad \text{otherwise} \end{cases}$$

- Let A be the event $Y \leq 0.5$.
- Let B be the event $Y > X$.
- Calculate $P(B|A)$.

- because the density function is uniform over the indicated region, probabilities of events defined on this region are proportional to areas.

EXERCISES FOR JANUARY 8

- Exercise: At a certain time, the number of people that enter an elevator is a random variable with Poisson PMF, with parameter λ. The weight of each person to get on the elevator is independent of each other person's weight, and is uniformly distributed between 100 and 200 lbs. Let X_i be the fraction of 100 by which the i^{th} person exceeds 100 lbs, e.g. if the 7^{th} person weighs 175 lbs., then $X_7 = .75$. Let Y be a random variable given by the sum of the X_i.
- Find the generating function for Y .

- Exercise: At a certain time, the number of people that enter an elevator is a random variable with Poisson PMF, with parameter λ. The weight of each person to get on the elevator is independent of each other person's weight, and is uniformly distributed between 100 and 200 lbs. Let X_i be the fraction of 100 by which the i^{th} person exceeds 100 lbs, e.g. if the 7^{th} person weighs 175 lbs., then $X_7 = .75$. Let Y be a random variable given by the sum of the X_i.
- Find the expected value of Y using generating function .

12/26/2012

- Exercise: At a certain time, the number of people that enter an elevator is a random variable with Poisson PMF, with parameter λ. The weight of each person to get on the elevator is independent of each other person's weight, and is uniformly distributed between 100 and 200 lbs. Let X_i be the fraction of 100 by which the ith person exceeds 100 lbs, e.g. if the 7th person weighs 175 lbs., then X_7 = .75. Let Y be a random variable given by the sum of the X_i.
- Verify the expected value of Y using the conditional expectation.

Exercise

- Everynight, Jill leaves a bar and walks on a street that looks like a straight line and at time t after she starts walking she is at position X_t which follows a Gausian distribution with mean μt and variance $t\sigma^2_X$. Most of the nights her cat compaines Jill, where cat's position at time t, is Z_t, as $Z_t - X_t = Y_t$, such that Y_t is a random variable with Gausian distributión with mean 0 and variance σ^2_Y. The variables X_t and Y_t are independent.
  - Calculate the probability density function (PDF) of cat's position, Z_t, as function of t.

Exercise

- Let $X_1, X_2,...$ be independent normal random variables with mean 2 and variance 4.
  - If δ is a small positive number, we have $P(|X_1| \leq \delta) \approx \alpha\delta$, for some constant α. Find the value of α.

Exercise

- Let X_1, X_2,... be independent normal random variables with mean 2 and variance 4. Let N be a geometric random variable which is independent of the X_i, with parameter $p = 2/3$. (In particular, $E[N]= 3/2$ and $E[N^2] = 3$.)
  - If δ is a small positive number, we have $P(|X_1| \leq \delta) \approx \alpha\delta$, for some constant α. Find the value of α.
  - Find $E[X_1 N]$.
  - Find $E[X_1 + \ldots + X_N \mid N \geq 2]$.
  - Write down the transform associated with $N + X_1 + \ldots + X_N$.

Exercise

Alice and Bob flip bias coins independently. Alice's coin comes up heads with probability 1/4, while Bob's coin comes up head with probability 3/4. Each stop as soon as they get a head; that is, Alice stops when she gets a head while Bob stops when he gets a head. What is the PMF of the total amount of flips until both stop? (That is, what is the PMF of the combined total amount of flips for both Alice and Bob until they stop?)

The End