CENG 551 Probability and Stochastic Processes for Engineers

Lecture 1

Introduction: Probability Models and Axioms

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Deniz Ozdemir

- Studied
  - Industrial Engineering - Bilkent U.
  - Production & Operations Management – INSEAD (France)
- Worked
  - Technion (Israel), INSEAD (France)
  - McGill University (Canada)
  - UANL (Mexico)
  - Toros University (Mersin)
- Research on
  - Supply chain management
  - Coordination of inventory and production management
  - Reverse logistics

Plan for the Session

- Outline the Subject and Policies
- What to expect from this course?

- Probability Models and Axioms
  - Motivation
  - Sample space of an experiment
    - Examples
  - Axioms of probability
    - More examples

CENG 551 Probability and Stochastic Processes for Engineers

Course Description

Objectives

- Introduction to the theory of probability.
- Develop skills and knowledge about *probability theory* and establish the basic concepts in order to be able to work on advanced probabilistic optimization and stochastic process models.
- Upon completing this course, the student will be able to:
  - Interpret data presented in graphs and frequency tables
  - Present raw data in graphs and frequency tables
  - Compute and interpret certain descriptive statistics
  - Compute probability values and use these in constructing and analyzing probability distributions
  - Introduce basic stochastic processes and their main application areas.
  - Communicate effectively using the terminology of the course.

Course Material

- **Text:** S.M. ROSS. *A First Course in Probability.* Prentice-Hall, Englewood Cliffs, USA, 2002/2005/2008. Ed. 6/7/8 (ROSS)
  - S.M.ROSS. *Introduction to Probability*. *Models* Harcourt Academic Press, San Diego, USA, 1993. (Chp.1-7)
  - S.M.ROSS. *Introduction to Probability*. *And Statistics for Engineers and Scientists* Harcourt Academic Press, San Diego, USA, 2000. (Chp.3-7)

Course Material

- **References:**
  - H.J. LARSON. *Introduction to Probability.* Addison-Wesley, Reading, Massachusetts, USA, 1995.
  - L.L. HELMS. *Introduction to Probability Theory: With Contemporary Applications*. Freeman, San Francisco, USA, 1997.
  - R.A. ROBERTS. *An Introduction to Applied Probability*. Addison-Wesley, Reading, Massachusetts, USA, 1992.
  - E. JAYNES. *Probability Theory, the Logic of Science*. Cambridge University Press, UK, 2003.

Subject info and policies

- Class sessions
  - Tuesdays. 18:30-21:00
  - Lectures & problem solving session
    - Slides posted to "lectures" after the session
    - Lectures.yasar.edu.tr enrollment key: : CENG551EK
    - Webpage: http://dozdemir.yasar.edu.tr/ceng511/

Grading Allocation

Homework assignments	30%
Midterm	30%
Final Exam	40%

Due to University regulations minimum 70% attendance is required for taking exams.

Course Requirements: Homework assignments

- Once +/- every 2 weeks: due to beginning of class
  - 6 homework assignment in total
  - Post & collect via "*lectures*"
- Individual work
  - You can discuss homework assignments with each other, but I expect you to write them up individually.
- No credit will be given for *answers only*
- You may use your calculator.
- 10 points per day late

Course Requirements:Exams

- One midterm and one final exam
  - All the course material up to that point.
  - There will be at least one question that requires a written explanation of a concept.
    - Evaluation will be based on mathematical content,
    - NOT ON spelling, grammar, or sentence construction.
- Make–up tests
  - Only to those students presenting a written excuse, acceptable by the university.

What to expect from the course:

1. Introduction to Probability Theory:
   - The Basic Principle of Counting
   - Permutations
   - Combinations.
   - Axioms of Probability
     - Sample Space and Events
   - Conditional Probability and Independence
     - Conditional Probability
     - Bayes Formula
     - Independent Events

What to expect from the course:

2. Random Variables
   - Expected Value
   - Variance
   - Discrete Random Variables
     - Bernoulli, Binomial, Geometric RV's
     - Poisson RV's
   - Continuous Random Variables
     - Uniform Distribution
     - Normal Distribution
     - Exponential Distribution

What to expect from the course:

3. More advanced distributions
   - Jointly Distributed Random Variables
     - Joint Distribution Function
     - Independent RV's
     - Conditional Distributions
     - Order Statistics
   - Properties of Expectation
   - Moment Generating Functions

What to expect from the course:

4. Some Basic Probabilistic Processes
   - The Bernoulli Random Process
   - The Poisson Process
   - Renewal Processes
   - Markov Processes
5. **Limit Theorems** *(if time permits)*

What to expect from the course: II

2. **Random Variables**
   - Expected Value
   - Variance
   - Discrete Random Variables
     - Bernoulli, Binomial, Geometric RV's
     - Poisson RV's
   - Continuous Random Variables
     - Uniform Distribution
     - Normal Distribution
     - Exponential Distribution

BASIC DEFINITIONS – AXIOMS - MODELS

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Purpose of Course

- Focus has been on solving deterministic computational problems
- This course is about how to deal with solving real world problems that involve uncertainty
- Motivating examples on why its worth studying

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Baby Girl Problem

In a certain country every child is a boy or girl with exactly 50% probability, regardless of the sex of any other children in that (or any other) family. Every family wants to raise a baby girl, so the country has adopted a policy that every family continues to give birth to more children until they have their first baby girl, and then that family must stop. In this fertile population, no family is unable to have another child if they try.

CLAIM: After many generations of this policy the expected # of boys in the population becomes larger than the expected # of girls

TRUE or FALSE???

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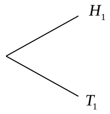
Double or Quarter Game

In the double-or-quarter game, you initially put in $1.00. After every toss of a fair coin, your money doubles if the coin comes up heads, but you lose three quarters of your fortune if the coin comes up tails. You continue to bet your remaining fortune on the next toss.

- CLAIM: This is a good game to play in long run, on average, your fortune increases with each toss.



TRUE or FALSE???

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Motivation

- Why do we study probability theory?
  - An effective model of uncertainty
  - Decision Making under uncertainty
- Examples:
  - Waiting time at a Bank's teller.
  - Value of a stock at a given day.
  - Outcome of a medical procedure.
  - A customer buying behavior.
- One Decision Making Process: Collect Data, Model the Phenomenon, Extrapolate and make decisions.

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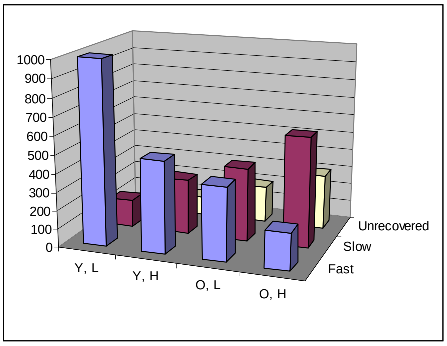
From Frequency to Probability

The time of recovery (Fast, Slow, Unsuccessful) from an ACL knee surgery was seen to be a function of the patient's age (Young, Old) and weight (Heavy, Light). The medical department collected the following data:

	Fast	Slow	Unsuc.	*Total*
Y, L	1000	150	50	*1200*
Y, H	500	300	100	*900*
O, L	400	400	200	*1000*
O, H	200	600	300	*1100*



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From Frequency to Probability

	Fast	Slow	Unsuc.	*Total*
Y, L	1000	150	50	*1200*
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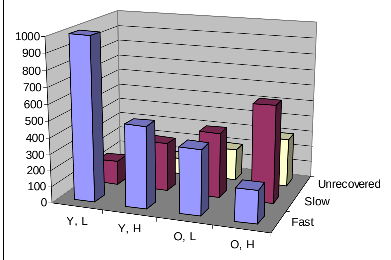
O, L	400	400	200	*1000*
O, H	200	600	300	*1100*



- What is the "likelihood" that a 60 years old man (Old!) will have a successful surgery with a speedy recovery?
- If a patient undergoes an operation, what is the "likelihood" that the result is unsuccessful?
- Need a measure of "likelihood".
- Ingredients: Sample space, Events, Probability.
- **Think of Probability as Frequency....**

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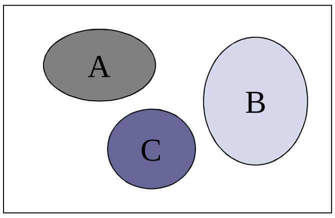
Algebra of Events



Events are collections of points or areas in a space.

The collection of all points in the entire space is called U , the *universal set* or the *universal event.*

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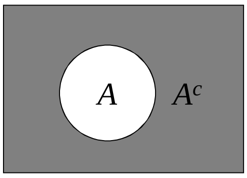
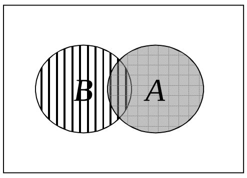
Algebra of Events Continued



Event A^c, the *complement* of event A, is the collection of all points in the universal set which are not included in event A. The *null set* ϕ contains no points and is the complement of the universal set.



The intersection of two events A and B is the collection of all points which are contained both in A and B notated $A \cap B$.

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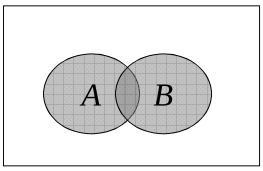
Algebra of Events continued…



The *union* of two events A and B is the collection of all points which are either in A or in B or in both. For the union of events A and B we shall use the notation $A \cup B$

Two events A and B are equal if every point in U which is in A is also in B and every point of U not in A^c is also in B^c; rather A includes B and B includes A.

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7 Axioms of Algebra of Events

$A \cup B = B \cup A$ *Commutative Law*

$A \cup (B \cup C) = (A \cup B) \cup C$ *Associative Law*

$A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$ *Distributive Law*

$(A^c)^c = A$

$(A \cap B)^c = A^c \cup B^c$ *DeMorgan's Law*

$A \cap A^c = \phi$

$A \cap U = A$

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Some Derivable Relations

$A \cup A = A$

$A \cup A \cap B = A$

$A \cup (A^c \cap B) = A \cup B$

$A \cup A^c = U$

$A \cup U = U$

$A \cap \Phi = \Phi$

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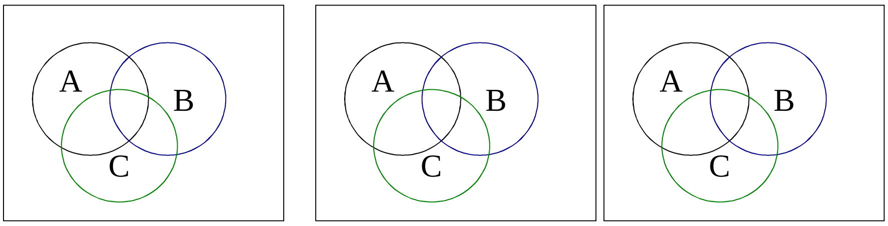
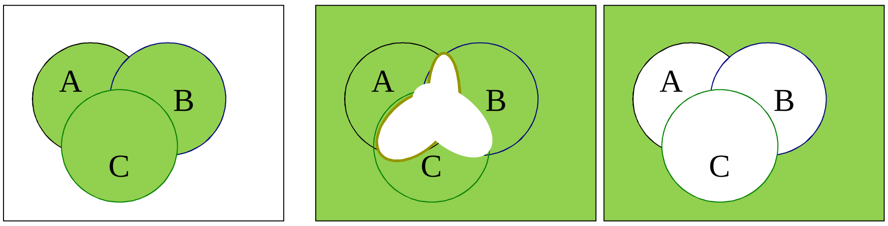
- Express following events in terms of the events A, B and C as well as the operations of complementation, union and intersection:
  - At least one of the events A, B, C occurs;
  - At most one of the events A, B, C occurs;
  - None of the events A, B, C occurs;



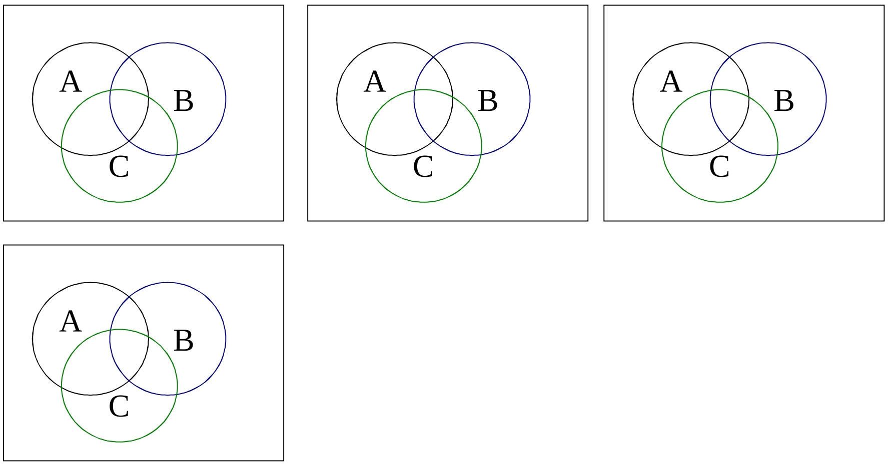
- Express following events in terms of the events A, B and C as well as the operations of complementation, union and intersection:
  - At least one of the events A, B, C occurs; $A \cup B \cup C$
  - At most one of the events A, B, C occurs;

$(A \cap B^c \cap C^c) \cup (B \cap A^c \cap C^c) \cup (C \cap B^c \cap A^c) \cup (A^c \cap B^c \cap C^c)$

  - None of the events A, B, C occurs; $(A \cup B \cup C)^c = A^c \cap B^c \cap C^c$



- All three events A, B, C occur;
- Exactly one of the events A, B, C occurs;
- Events A and B occur, but not C;
- Either event A occurs or, if not, then B also does not occur.



- All three events A, B, C occur; $A \cap B \cap C)$
- Exactly one of the events A, B, C occurs;

$(A \cap B^c \cap C^c) \cup (B \cap A^c \cap C^c) \cup (C \cap B^c \cap A^c)$

- Events A and B occur, but not C;
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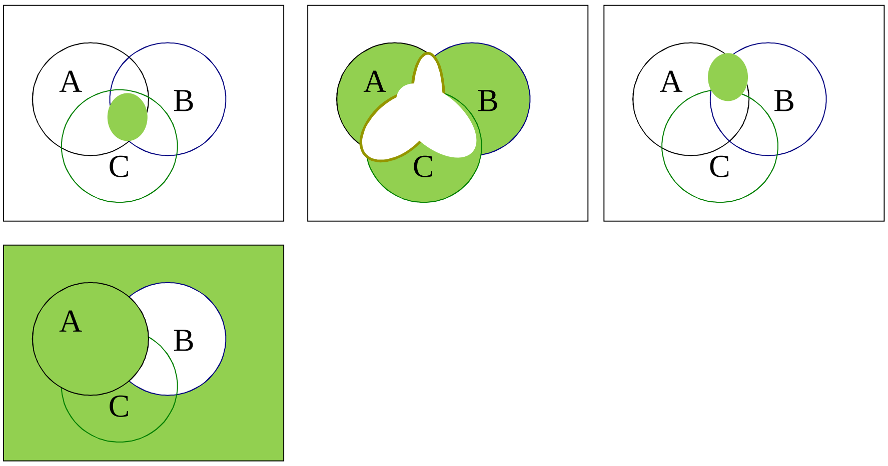
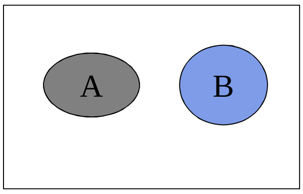
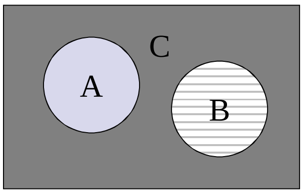


Mutually Exclusive and Collectively Exhaustive



A set of events are mutually exclusive if the set of events do not intersect



A set of events are collectively exhaustive if the sum up to U

e.g. $A \cup B \cup C = U$

From Frequency to Probability

	Fast	Slow	Unsuc.	*Total*
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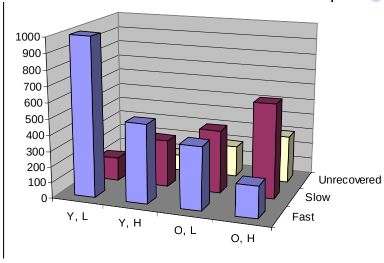


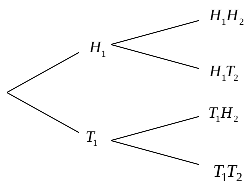
- What is the “likelihood” that a 60 years old man (Old!) will have a successful surgery with a speedy recovery?
- If a patient undergoes an operation, what is the “likelihood” that the result is unsuccessful?
- Need a measure of “likelihood”.
- Ingredients: Sample space, Events, Probability.
- **Think of Probability as Frequency....**

Sample Spaces

Sample Space: The finest-grain **mutually exclusive**, **collectively exhaustive listing** of **all possible outcomes** of a model of an experiment.

Sequential Sample Space

Event $\left\{ \begin{matrix} H_n \\ T_n \end{matrix} \right\} : \left\{ \begin{matrix} \mathit{Heads} \\ \mathit{Tails} \end{matrix} \right\}$ on the nth toss of the coin.

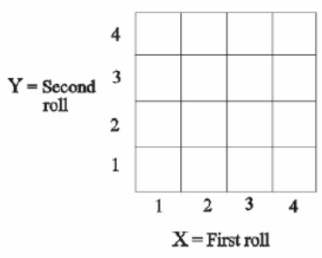


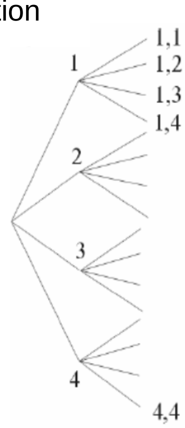
H_1H_2 Is finest grain event type for two tosses

H_1 Is coarser grain event for two tosses

Sample Space Example (1)

- Two rolls of a tetrahedral die
  - Sample space vs. sequential description

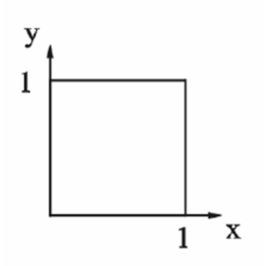




Sample Space Example (2)

- A continuous sample space:

(x, y) such that $0 \le x, y \le 1$



Axioms of Probability

- Sample Space: List of all possible outcomes
- Event: a subset of the sample space
- Every outcome is an event and certain set of outcomes are events
- Probability is assigned to events
- Axioms:
  1. For Any Event A, $P(A) \ge 0$
  2. $P(U) = 1$ (Normalization)
  3. If $A \cap B = \phi$, then $P(A \cup B) = P(A) + P(B)$

Axioms of Probability

- Axioms:
  - For Any Event A, $P(A) \geq 0$
  - $P(U) = 1$ (Normalization)
  - If $A \cap B = \phi$, then $P(A \cup B) = P(A) + P(B)$

From this and the prior Axioms one can determine the probability measure of an event by simply summing up all the measures for each of the finest grain events that the event consists of.

$$P(\{s_1, s_2, \ldots, s_k\}) = P(s_1) + \cdots + P(s_k)$$

- Axiom 3 needs strengthening

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Axioms of Probability

- Axioms:
  - For Any Event A, $P(A) \geq 0$
  - $P(U) = 1$ (Normalization)
  - If $A \cap B = \phi$, then $P(A \cup B) = P(A) + P(B)$
- Exercise:
  - If $A \cap B \neq \phi$, then $P(A \cup B) = ?$

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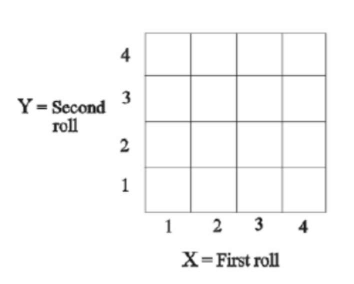
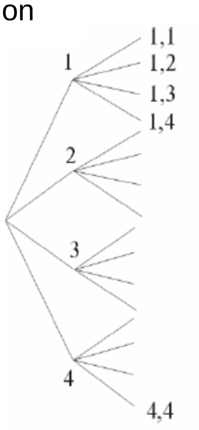
Sample Space Example (1)

- Two rolls of a tetrahedral die
  - Sample space vs. sequential description





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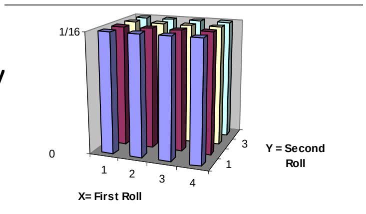
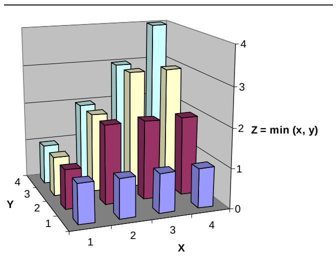
Example (1) revisited

- Let every possible outcome have probability 1/16
  - $P(X = 1) = 1/4$



- Define $Z = min(X, Y)$

$$P(Z = 1) = 7/16$$
$$P(Z = 2) = 5/16$$
$$P(Z = 3) = 3/16$$
$$P(Z = 4) = 1/16$$



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Discrete Uniform Law

- Let all sample points be equally likely
- Then,

$$P(A) = \frac{\text{number of elements of } A}{\text{total number of sample points}}$$

- Just count …

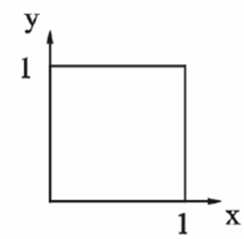
Hands on..

- Consider an experiment in which a fair coin is tossed once and a balanced die is rolled once.
  - Describe the sample space for this experiment.
  - What is the probability that a head will be obtained on the coin and an odd number will be obtained on the die?

Sample Space Example (2)

- A continuous sample space:

(x, y) such that $0 \le x, y \le 1$



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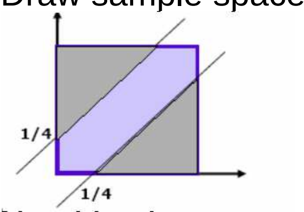
Example (2) revisited

- Each of two people choose a number between zero and one. What is the probability that they are at most 1/4 apart?
- Draw sample space and event of interest:



- Need to choose a probability law:
  - Choose **uniform** law: probability = area
- The probability is: 1-(3/4)(3/4) = 7/16

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Another try…

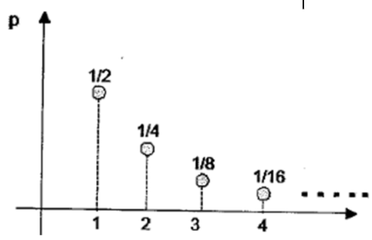
- Out of the students in a class, 60% are geniuses, 70% love chocolate, and 40% fall into both categories. Determine the probability that a randomly selected student is neither a genius nor a chocolate lover.

Another try…

- Romeo and Juliet have a date at a given time, and each will arrive at the meeting place with a delay between 0 and 1 hour, with all pairs of delays being equally likely. The first to arrive will wait for 15 minutes and will leave if the other has not yet arrived. What is the probability that they will meet?

Note on Infinite Sample Spaces

- Sample space: $\{1,2,..\}$
  - We are given $P(n)= 2^{-n}$
  - Find P(outcome is even)
- Solution:



$P(\{2, 4, 6, \ldots\}) = P(2) + P(4) + P(6)+\ldots$

$= 1/\ 2^2 +1/\ 2^4 + 1/2^6\ldots=1/3$

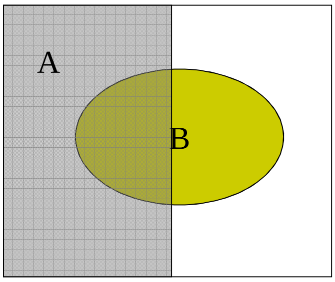
- Axiom needed:
  - If $A_1, A_2,\ldots$are (a possibly infinite collection of) disjoint events, then:

$P(A_1\cup A_2 \cup \ldots) = P(A_1)+ P(A_2)+ \ldots$

Probability and the "Real World"

- Probability is a branch of math:
  - Axioms $\Rightarrow$Theorems
  - One theorem: Frequency of event A is $P(A)$
- But are probabilities frequencies?
  - P(coins toss yields head) = ½
  - P(Don Quixote is written by Cervantes) = 0.95
  - P(a piece of equipment aboard the space shuttle fails) = $1/10^{-8}$
- Probability models as a way of describing uncertainty:
  - Use for consistent reasoning
  - Use for predictions, decisions

Conditional Probability; an intuitive Taste



$$P(A \mid B) = \frac{P(AB)}{P(B)}$$

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