CENG 551 Probability and Stochastic Processes for Engineers

Week # 11

Derived Functions

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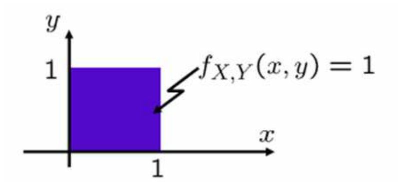
Plan for the Session

- More on derived distributions

Derived Distribution

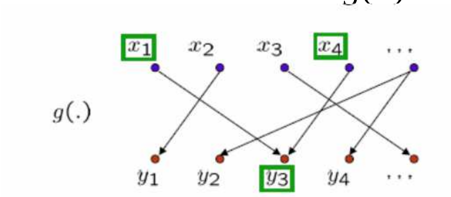
- What is *derived distribution*?
- It is a PMF or PDF of a function of random variables with known probability law.
- Example 0: X and Y



- Let $Z(X, Y) = Y/X$
  - Note: $Z(X, Y)$ is a r.v.

How to find them: Discrete case

- Consider: -a single discrete r.v.: X
  -and a function: $g(X) = Y$



- Obtain probability mass for each possible value of $Y=y$:
  - $p_Y(y) = P(g(X) = y)$
    $= \sum_{x:\, g(x)=y} p_X(x)$

How to find them: Continuous case

- Consider: -a single continuous r.v.: X
  -and a function: $g(X) = Y$
- Two step procedure:

1. Get CDF of Y : $F_Y(y) = P(Y \leq y)$
2. Differentiate to get: $f_Y(y) = \frac{dF_Y}{dy}(y)$

Derived Distribution

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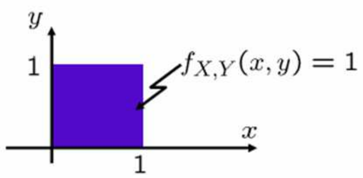
- Example 0: X and Y



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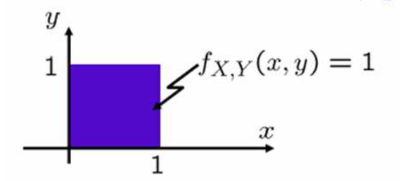
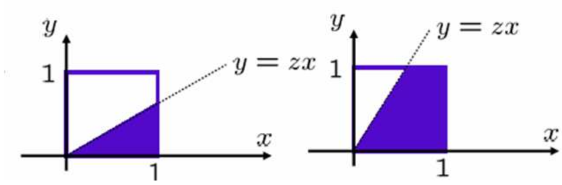
Example 0

- Example: X and Y
- $Z(X, Y) = Y/X$
- Two step procedure:



1. Get CDF of Z : $F_Z(z) = P(Z \le z)$



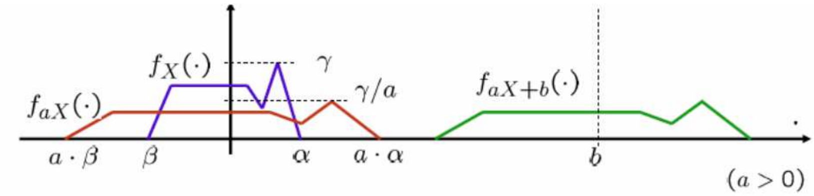
$$F_Z(z) = z/2 \qquad 0 \le z \le 1$$
$$F_Z(z) = 1 - 1/2z \qquad z \ge 1$$

2. Differentiate to get PDF:

Exponential Distribution: Properties

- If $X_1, X_2, \ldots, X_n$ are exponentially distributed, $X_i \sim \text{Exp}(\lambda_i)$ then $Y = \mathbf{min}\ (X_1, X_2, \ldots, X_n\ ) \sim \text{Exp}(\Sigma\lambda_i)$
- BUT $Y = \mathbf{max}\ (X_1, X_2, \ldots, X_n\ ) \cancel{\sim} \text{Exp}(\Sigma\lambda_i)$

Exercise 1: $Y = aX+b$



$$\boxed{f_Y(y) = \frac{1}{|a|} f_X\left(\frac{y-b}{a}\right)}$$

- Use this to check that if is X normal, then $Y = aX+b$ is also normal.

Exercise 1: $Y = aX+b$

- if is X normal, then $Y = aX+b$ is also normal.

General Normal: $N(\mu, \sigma^2)$

$$f_X(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-(x-\mu)^2/2\sigma^2}$$

Example: Difference of Exp r.v.s

- Romeo and Juliet are to meet. Romeo is late by X minutes. Juliet is late by Y. X and Y are independent exponential r.vs.
- What is the PDF of the waiting time for each other?
- X, Y ~*Exp*.

$$f_X(x) = \lambda e^{-\lambda x},\ x \geq 0$$
$$f_Y(y) = \lambda e^{-\lambda y},\ y \geq 0$$

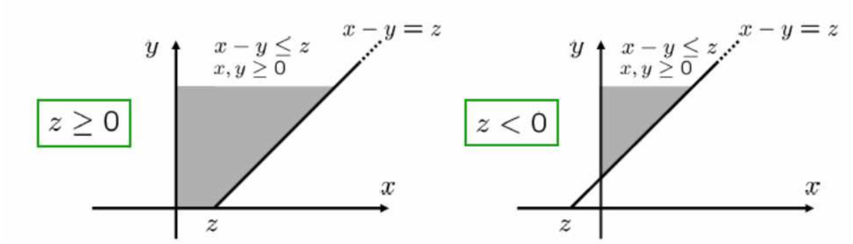
- Let $Z = X\text{-}Y$, find $f_Z(z)$

Example: Difference of Exp r.v.s

- X, Y independents,

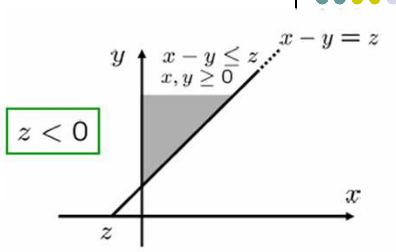
$$f_{X,Y}(y) = \lambda^2 e^{-\lambda(x+y)} \quad x, y \geq 0$$

- $Z = X\text{-}Y$, Compute $F_Z(z) = P(X\text{-}Y \leq z)$
  - Integration region varies for 2 cases:



Example: Difference of Exp r.v.s

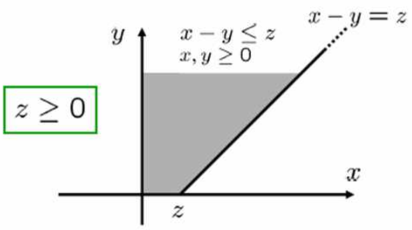
- So for $z < 0$



$$F_Z(z) = \mathbf{P}(X - Y \le z)$$
$$= \int_0^\infty \left(\int_{x-z}^\infty f_{X,Y}(x,y)dy \right) dx$$
$$= \int_0^\infty \lambda e^{-\lambda x} \left(\int_{x-z}^\infty \lambda e^{-\lambda y} dy \right) dx$$
$$= \int_0^\infty \lambda e^{-\lambda x} e^{-\lambda(x-z)} dx \quad = \frac{1}{2} e^{\lambda z}$$

Example: Difference of Exp r.v.s

- For $z \ge 0$



- Fact: Z~-Z (same distribution). So,

$$F_Z(z) = \mathbf{P}(Z \le z) = \mathbf{P}(-Z \ge -z) = \mathbf{P}(Z \ge -z)$$
$$= 1 - F_Z(-z) \quad = 1 - \frac{1}{2} e^{-\lambda z}$$

Example: Difference of Exp r.v.s

- We thus have:

$$F_Z(z) = \begin{cases} 1 - \frac{1}{2}e^{-\lambda z}, & \text{if } z \geq 0 \\ \frac{1}{2}e^{\lambda z}, & \text{if } z < 0 \end{cases}$$

- Differentiate:

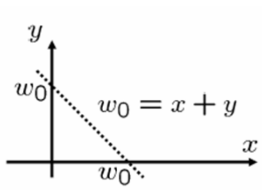
$$f_Z(z) = \begin{cases} \frac{\lambda}{2}e^{-\lambda z}, & \text{if } z \geq 0 \\ \frac{\lambda}{2}e^{\lambda z}, & \text{if } z < 0 \end{cases}$$

- Rewrite, to obtain a two-sided exponential PDF:

$$f_Z(z) = \frac{\lambda}{2}e^{-\lambda |z|}$$

Distribution of $X + Y$

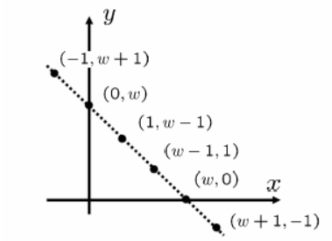
- Let X and Y be two r.v.s, and let $W=X+Y$
- Points where the value $W = w_0$ is some constant lie on the following line:



- Idea
  - Discrete case: add probabilities of all points on this line.
  - Continuous case: integrate the joint density on this line.

$X + Y$: Independent Discrete rv.

- Let X and Y be integer-valued, independent.
- Then $W=X+Y$ is also integer-valued
- Picture:

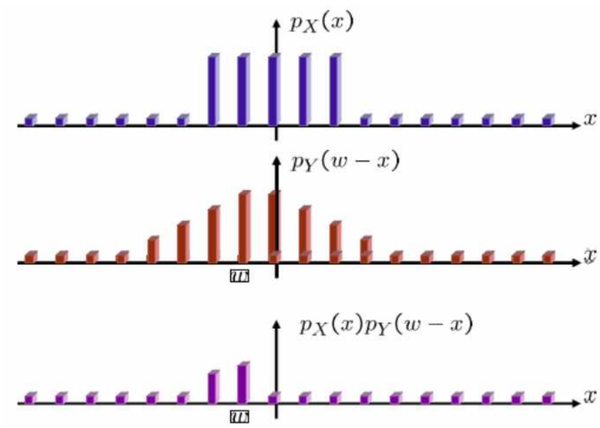


- Thus:

$$
\begin{aligned}
p_W(w) &= \mathbf{P}(X+Y=w) \\
&= \sum_x \mathbf{P}(X=x)\mathbf{P}(Y=w-x) \\
&= \sum_x p_X(x)p_Y(w-x)
\end{aligned}
$$

Obtaining $P_W(w)$ by convolution

$$p_W(w) = \sum_x p_X(x)p_Y(w-x)$$

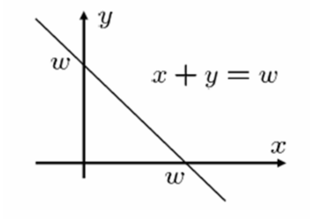


$X + Y$: Independent Continuous rv.

- Let X and Y are independent continuous variables.

$$f_{X,Y}(x,y) = f_X(x) \cdot f_Y(y)$$

- Picture



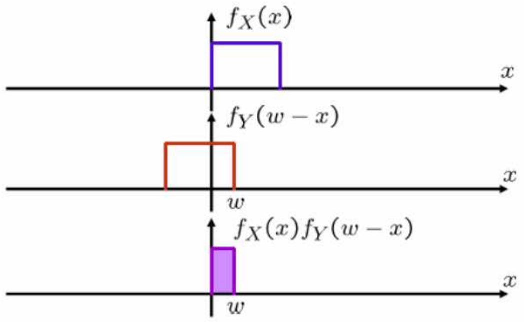
- Then the density of W=X+Y is given by

$$f_W(w) = \int_{-\infty}^{\infty} f_X(x) f_Y(w-x)\,dx$$

$X + Y$ Example: Independent Uniform

- Let X and Y be independent, uniform on [0, 1]:
- Find the density of W=X+Y .
- Convolution idea applies:

$$f_W(w) = \int_{-\infty}^{\infty} f_X(x) f_Y(w-x)\,dx$$



Two Independent Normals

- Let X and Y be independent, normal r.v.s:

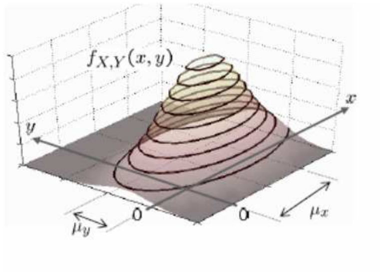
$$X \sim N(\mu_x, \sigma_x^2) \qquad Y \sim N(\mu_y, \sigma_y^2)$$

$$f_{X,Y}(x,y) = f_X(x) \cdot f_Y(y)$$
$$= \frac{1}{2\pi\sigma_x\sigma_y} \exp\left\{-\frac{(x-\mu_x)^2}{2\sigma_x^2} - \frac{(y-\mu_y)^2}{2\sigma_y^2}\right\}$$

- PDF is constant on ellipses:

$$-\frac{(x-\mu_x)^2}{2\sigma_x^2} - \frac{(y-\mu_y)^2}{2\sigma_y^2} = c^2$$

- Circles, when $\sigma_x = \sigma_y$



Sum of Two Independent Normals

- Let X, Y be independent, std. normal r.v.s:

$$X \sim N(0, \sigma_x^2) \qquad Y \sim N(0, \sigma_y^2)$$

- Find the density of W=X+Y .

$$f_W(w) = \int_{-\infty}^{\infty} f_X(x) f_Y(w-x)dx$$
$$= \frac{1}{2\pi\sigma_x\sigma_y} \int_{-\infty}^{\infty} e^{-x^2/2\sigma_x^2} e^{-(w-x)^2/2\sigma_y^2} dx$$
$$= ce^{-\gamma w^2}$$

- Conclusion: W is Normal with $\mu_w = 0$, $\sigma_w^2 = \sigma_x^2 + \sigma_y^2$

Other functions: I

- For X a random variable uniformly distributed between -1 and 1, find the PDF of

$Y = \sqrt{|X|}$

Get CDF of Y:

Differentiate to get PDF

Other functions: II

- For X a random variable uniformly distributed between -1 and 1, find the PDF of

$Y = \ln|X|$

Get CDF of Y:

Differentiate to get PDF

Maximum of uniform

- You are allowed to take a certain test three times, and your final score will be the maximum of the test scores. Your score in test i, where i=1, 2, 3 takes one of the values from i to 10 with equal probability $1/(11-i)$, independently of the scores in other tests. What is the PMF of the final score?

Joint PDF Example

- Let continuous random variables X, Y and Z be independent and identically distributed (IID) according to the uniform distribution [0,1]. Consider two new random variables:
  - $V = XY$
  - $W = Z^2$. Derive the joint PDF $f_{V,W}(v,w)$.

Continuous Bayes' Rule

$$f_{X|Y}(x|y) = \frac{f_{X,Y}(x,y)}{f_Y(y)} = \frac{f_X(x) f_{Y|X}(y|x)}{f_Y(y)}$$

$$f_{X,Y}(x,y) = f_{Y|X}(y\,|\,x)\, f_X(x) = f_{X|Y}(x\,|\,y)\, f_Y(y)$$

- Combined with convolution results from derived probabilities can be used in different areas/

$$f_{Y|X}(y\,|\,x) = f_N(y-x)$$

Continuous Bayes' Rule

- Potential Application
- In forecasting: $Y = X + E$

Signal (observation)

Additive noise

independent

- Then:

$$f_{X,Y}(x,y) = f_{Y|X}(y\,|\,x)\, f_X(x) = f_{Y|X}(y\,|\,x)\, f_N(y-x)$$

$$f_{Y|X}(y\,|\,x) = f_N(y-x)$$

- Remarkable fact:
  - If X and E are normal, then $f_{Y|X}(y|x)$ [as well as $f_{X|Y}(x|y)$] is a normal PDF, for any given y [x]

Next Time

- Transforms
  - Definition of Transforms
  - Why transforms?
- Moment Generation Functions
  - Moment Generating Property
  - Examples
  - Application to Sums of Independent rv.s