Useful Formulas for Algorithm Analysis

Important Summation Formulas

1. $\sum_{i=l}^{u} 1 = \underbrace{1+1+\ldots+1}_{u-l+1 \text{ times}} = u - l + 1$ (l, u are integer limits, $l \le u$); $\sum_{i=1}^{n} 1 = n$

2. $\sum_{i=1}^{n} i = 1 + 2 + \ldots + n = \frac{n(n+1)}{2} \approx \frac{1}{2}n^2$ [Arithmetic series]

3. $\sum_{i=1}^{n} i^2 = 1^2 + 2^2 + \ldots + n^2 = \frac{n(n+1)(2n+1)}{6} \approx \frac{1}{3}n^3$ $\sum_{i=1}^{n} i^3 = \frac{n^2(n+1)^2}{4} \approx \frac{n^4}{4}$

4. $\sum_{i=1}^{n} i^k = 1^k + 2^k + \cdots + n^k \approx \frac{1}{k+1} n^{k+1}$

When necessary use the precise result; if appropriate use the approximate result.

5. $\sum_{i=0}^{n} a^i = 1 + a + \cdots + a^n = \frac{a^{n+1} - 1}{a - 1}$ $(a \ne 1)$; [Geometric series]

5a. $\sum_{i=0}^{n} 2^i = 2^{n+1} - 1$

6. $\sum_{i=1}^{n} i a^i = \frac{n a^{n+2} - n a^{n+1} - a^{n+1} + a}{(a-1)^2}$ $[= \Theta(n a^n)]$, $a \ne 1$

6a. $\sum_{i=1}^{n} i 2^i = 1 * 2 + 2 * 2^2 + \cdots + n 2^n = (n-1) 2^{n+1} + 2$

6b. $\sum_{i=0}^{n} \frac{i}{2^i} = 2 - \frac{n+2}{2^n}$

6c. $\sum_{i=0}^{n} \frac{1}{2^i} = 1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \ldots = 2 - 2^{-n} = 2 - \frac{1}{2^n}$ [prove using binary number representation]

7. $\sum_{i=1}^{n} \frac{1}{i} = 1 + \frac{1}{2} + \cdots + \frac{1}{n} \approx \ln n + \gamma$, where $\gamma \approx 0.5772\ldots$ (Euler's constant)

8. $\sum_{i=1}^{n} \lg i \approx n \lg n$

Sum Manipulation Rules

1. $\sum_{i=l}^{u} ca_i = c \sum_{i=l}^{u} a_i$

2. $\sum_{i=l}^{u} (a_i \pm b_i) = \sum_{i=l}^{u} a_i \pm \sum_{i=l}^{u} b_i$

3. $\sum_{i=l}^{u} a_i = \sum_{i=l}^{m} a_i + \sum_{i=m+1}^{u} a_i$, where $l \le m < u$

$$\sum_{i=b}^{c} a_i = \sum_{i=a}^{c} a_i - \sum_{i=a}^{b-1} a_i, \quad a < b \le c \qquad !!!$$

Approximation of a Sum by a Definite Integral *

$$\int_{l-1}^{u} f(x)dx \le \sum_{i=l}^{u} f(i) \le \int_{l}^{u+1} f(x)dx \quad \text{for a nondecreasing } f(x)$$

$$\int_{l}^{u+1} f(x)dx \le \sum_{i=l}^{u} f(i) \le \int_{l-1}^{u} f(x)dx \quad \text{for a nonincreasing } f(x)$$

Floor and Ceiling Formulas

The floor of a real number x, denoted $\lfloor x \rfloor$, is defined as the greatest integer not larger than x (e.g., $\lfloor 3.8 \rfloor = 3$, $\lfloor -3.8 \rfloor = -4$, $\lfloor 3 \rfloor = 3$). The ceiling of a real number x, denoted $\lceil x \rceil$, is defined as the smallest integer not smaller than x (e.g., $\lceil 3.8 \rceil = 4$, $\lceil -3.8 \rceil = -3$, $\lceil 3 \rceil = 3$).

1. $x - 1 < \lfloor x \rfloor \le x \le \lceil x \rceil < x + 1$
2. $\lfloor x + n \rfloor = \lfloor x \rfloor + n$ and $\lceil x + n \rceil = \lceil x \rceil + n$ for real x and integer n
3. $\lfloor n/2 \rfloor + \lceil n/2 \rceil = n$ *
4. $\lceil \lg(n+1) \rceil = \lfloor \lg n \rfloor + 1$

Miscellaneous

1. $n! \approx \sqrt{2\pi n}\left(\frac{n}{e}\right)^n$ as $n \to \infty$ (Stirling's formula)

where e is the base of the natural logarithms, e ≈ 2.71828

1. Basic Equations of Algebra

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Equation no.1 $(x+y)^2 = x^2 + 2xy + y^2$

Equation no.2 $(x-y)^2 = x^2 - 2xy + y^2$

Equation no.3 $(x^2 - y^2) = (x+y)(x-y)$

Equation no.4* $(x^3 + y^3) = (x+y)(x^2 - xy + y^2)$

Equation no.5* $(x^3 - y^3) = (x-y)(x^2 + xy + y^2)$

Equation no.6 $(x+y)^3 = x^3 + 3x^2y + 3xy^2 + y^3$

Equation no.7 $(x-y)^3 = x^3 - 3x^2y + 3xy^2 - y^3$

Equation no.8 (Roots of quadratic equation) $x = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$

2. Properties of Exponents

Property no.1 $a^m . a^n = a^{m+n}$

Property no.2 $(a^m)^n = a^{mn}$ ($a \neq 0$ if m or n is negative or zero)

Property no.3 $(ab)^m = a^m . b^m$ ($ab \neq 0$ if $m \leq 0$)

Property no.4 $\frac{a^m}{a^n} = a^{m-n}$ ($a \neq 0$) $ax^2 + bx + c = 0$

Property no.5 $\left(\frac{a}{b}\right)^m = \frac{a^m}{b^m}$ ($b \neq 0$ if $m \leq 0$)

3. Properties of Logarithms

Property no.1 $\log_a 1 = 0$

Property no.2 $\log_a a = 1$

Property no.3 $\log_a xy = \log_a x + \log_a y$

Property no.4 $\log_a \left(\frac{x}{y}\right) = \log_a x - \log_a y$

Property no.5 $\log_a \left(\frac{1}{x}\right) = -\log_a x$

Property no.6 $\log_a (x^n) = n \log_a x$

Property no.7 $\log_b a = \frac{\log_c a}{\log_c b}$!!!

Property no.8 $(y = x^a) \Leftrightarrow (\log y = a \log x)$

Property no.9 The base-b logarithm of x, $\log_b x$, is the power to which you need to raise b in order to get x.
Symbolically,

$\log_b x = y$	means	$b^y = x$
Logarithmic form		Exponential form

!!!

4. Limits

Let be any function of n. We say that f(n) tends to a limit a as n tends to infinity if f(n) is nearly equal to a when n is large. The following formal definition makes this notion more precise:

> Definition The function f(n) is said to tend to the limit a as n tends to infinity if for any positive real number x, no matter how small, f(n) differs from a by less than x for all sufficiently large values of n.

When tends to a limit a as n tends to infinity, we write: $\lim_{n\to\infty} f(n) = a.$

I. Theorems on Limits

Everywhere n instead of x

Theorem no.1 $\lim_{x\to c}(mx+b) = mc+b$

Theorem no.2 $\lim_{x\to c} bf(x) = b\lim_{x\to c} f(x)$

Theorem no.3 $\lim_{x\to c}[f(x)+g(x)] = \lim_{x\to c} f(x) + \lim_{x\to c} g(x)$

Theorem no.4 $\lim_{x\to c}[f(x)-g(x)] = \lim_{x\to c} f(x) - \lim_{x\to c} g(x)$

Theorem no.5 $\lim_{n\to\infty}[f(x)g(x)] = [\lim_{n\to\infty} f(x)][\lim_{n\to\infty} g(x)]$

Theorem no.6 $\lim_{n\to\infty}\frac{f(x)}{g(x)} = \frac{\lim_{n\to\infty} f(x)}{\lim_{n\to\infty} g(x)}$!!!

II. The l'Hôpital's Rule

L'Hôpital's Rule can be applied to the following "forms" of limit:

1. lim f(n) / g(n), where both f(n) and g(n) approach zero: $\lim_{n\to\infty}\frac{f(n)}{g(n)} = \frac{0}{0} \Rightarrow \lim_{n\to\infty}\frac{f'(n)}{g'(n)}$

2. lim f(n) / g(n), where both f(n) and g(n) approach infinity: $\lim_{n\to\infty}\frac{f(n)}{g(n)} = \frac{\infty}{\infty} \Rightarrow \lim_{n\to\infty}\frac{f'(n)}{g'(n)}$

5. Standard Derivatives

I. Basic Formulas

1. The derivative of a constant is zero: $\frac{d}{dx}c = 0$

2. For any constant c , $\frac{d}{dx}[cf(x)] = cf'(x)$.

3. Sum formula : $\frac{d}{dx}[f(x)+g(x)] = f'(x)+g'(x)$

4. Product Formula : $\frac{d}{dx}[f(x).g(x)] = f(x)g'(x)+g(x)f'(x)$!!!

5. Quotient Formula : $\frac{d}{dx}\left(\frac{f(x)}{g(x)}\right) = \frac{g(x)f'(x)-f(x)g'(x)}{[g(x)]^2}$!!!

6. Power Formula : $\frac{d}{dx}(x^n) = nx^{n-1}$

II. Derivatives of Exponential and Logarithmic Functions

1. $\frac{d}{dx}(e^x) = e^x$

2. $\frac{d}{dx}(a^x) = a^x \ln a$

3. $\frac{d}{dx}(\log_a x) = \frac{1}{x \ln a} \quad (x \neq 0)$!!!

4. $\frac{d}{dx}(\ln x) = \frac{1}{x} \quad (x \neq 0)$

III. Other Derivatives

1. $\frac{d}{dx}x = 1$

2. $\frac{d}{dx}x^n = nx^{n-1} \quad (x > 0, n \geq 0)$

3. $\frac{d}{dx}\frac{1}{x} = -\frac{1}{x^2} \quad (x \neq 0)$

4. $\frac{d}{dx}(\sqrt{x}) = \frac{1}{2\sqrt{x}} \quad (x > 0)$

IV Derivative of a complex function

Let y = f(u) where u itself is a function of x, u = u(x)

!!!

y' = f'(u).u'(x)

Formulas marked with red vertical line must be known by heart.