HOMEWORK # 2 (Due to October 16, 2009, Tuesday)

1. An urn contains 2 black and 5 brown balls. A ball is selected at random. If the ball drawn is brown, it is replaced and 2 additional brown balls are also put into the urn. If the ball drawn is black, it is not replaced in the urn and no additional balls are added. A ball is then drawn from the urn the second time.
   a. What is the probability that the ball selected at the second stage is brown?
   b. We are given that the ball selected at the second stage was brown. What is the probability that the ball selected at the first stage was also brown?

a) Let B1: Event first ball drawn was Brown
B2: Event second ball drawn was Brown
$P(B2) = P(B2|B1)P(B1) + P(B2|B1^c)\, P(B1^c)$
$= 7/9.5/7 + 5/6.2/7 = 5/9 + 5/21 = \mathbf{50/63}$

b) $P(B1|B2) = \dfrac{P(B2|B1)P(B1)}{P(B2)} = \dfrac{7/9.5/7}{50/63} = \mathbf{7/10}$

2. Suppose that 30 percent of the bottles produced in a certain plant are defective. If a bottle is defective, the probability is 0.9 that an inspector will notice it and remove it from the filling line. If a bottle is not defective, the probability is 0.2 that the inspector will think that it is defective and remove it from the filling line.
   a. If a bottle is removed from the filling line, what is the probability that it is defective?
   b. If a customer buys a bottle that has not been removed from the filling line, what is the probability that it is defective.

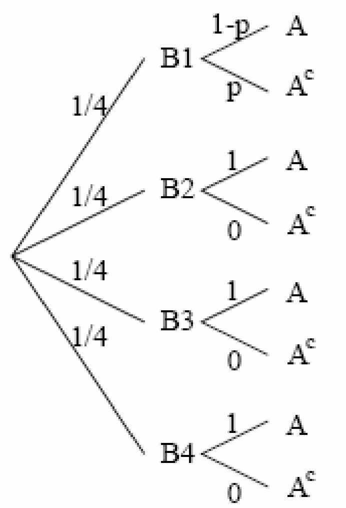
Let D: a bottle is defective
R: Inspector will remove it. We are told that $P(D) = 0.30$, $P(R|D) = 0.9$ and $P(R|D^c) = 0.2$

a) Probability that a bottle is defective given it is removed from the line is $P(D|R)$.
$P(D|R) = \dfrac{P(R|D)P(D)}{P(R)} = \dfrac{P(R|D)P(D)}{P(R|D)P(D)+ P(R|D^c)P(D^c)} = \dfrac{0.9(0.3)}{0.9(0.3)+0.2(0.7)} = \dfrac{0.27}{0.41} = \mathbf{0.6585}$

b) Probability that a bottle is defective given that it has not been removed from the filling line is $P(D|R^c)$.
We know $P(D) = 0.30 = P(D|R)P(R)+ \underline{P(D|R^c)P(R^c)}$
Therefore $P(D|R^c) = \dfrac{P(D)- P(D|R)P(R)}{P(R^c)} = \dfrac{0.3-.6585(.41)}{1-0.41} = \dfrac{0.03}{0.59} = \mathbf{0.0508}$

3. Suppose that you were foolish enough to save your thesis on only one floppy disk, and that this disk got corrupted. To make matters worse, you actually have 3 other old corrupted disks lying around, and it is equally likely that any of the 4 disks holds the corrupted remains of your thesis. Before you take all 4 disks to an expensive disk doctor, your friend across the hall offers to have a look. You know from past experience that the overall probability that your friend will find your paper on any disk is *p*. Given that he searches on disk 1 but cannot find your work, what is the probability that your thesis is on disk *i* for *i* =1, 2, 3, 4?

Let A be the event that your friend searches disk 1 and finds nothing, and let B_i be the event that your thesis is on disk *i*. The sample space is described below.



Note that B1,B2, B3, and B4 partition the sample space, so applying Bayes' rule, we have

$$\begin{aligned}
P(B_i \mid A) &= \frac{P(B_i)P(A \mid B_i)}{P(B_1)P(A \mid B_1)+P(B_2)P(A \mid B_2)+P(B_3)P(A \mid B_3)+P(B_4)P(A \mid B_4)} \\
&= \frac{\frac{1}{4}P(A \mid B_i)}{\frac{1}{4}((1-p)+1+1+1)} \\
&= \frac{P(A \mid B_i)}{4-p} \\
&= \begin{cases} (1-p)/(4-p) & \text{for } i=1, \\ 1/(4-p) & \text{for } i=2,3,4. \end{cases}
\end{aligned}$$

4. Consider some sample space Ω. Suppose $A,B \subseteq \Omega$. Prove or disprove the following:
   a. A, B independent implies A,B^c independent.
   b. A, B independent implies A^c,B independent.
   c. A, B independent implies A^c, B^c independent

We know that:

$$P(A) = P(A \cap B) + P(A \cap B^c)$$

and therefore

$$\begin{aligned}
P(A \cap B^c) &= P(A) - P(A \cap B) \\
&= P(A) - P(A)\,P(B) \\
&= P(A)(1 - P(B) \\
&= P(A)P(B^c)
\end{aligned}$$

This proves (a) and (b).

For c, we use DeMorgan's Law:

$P(A \cap B) = P((A^c \cup B^c)^c)$

$= 1 - P(A^c \cup B^c)$

$= 1 - P(A^c) - P(B^c) + P(A^c \cap B^c)$

but we also have: $P(A \cap B) = P(A)\,P(B)$

$= (1 - P(A^c))\,(1 - P(B^c))$

$= 1 - P(B^c) - P(A^c) + P(A^c)\,P(B^c)$

This concludes the proof.

5. You are lost in the campus of Yasar, where the population is entirely composed of brilliant students and absent-minded professors. The students comprise two-thirds of the population, and any one student gives a correct answer to a request for directions with probability .3/4 (Assume answers to repeated questions are independent, even if the question and the person asked are the same.) If you ask a professor for directions, the answer is always false.

a. You ask a passer-by whether the exit from campus is East or West. The answer is East. What is the probability this is correct?
b. You ask the same person again, and receive the same reply. Show that the probability that this second reply is correct is ½ .
c. You ask the same person again, and receive the same reply. What is the probability that this third reply is correct?
d. You ask for the fourth time, and receive the answer East again. Show that the probability it is correct is 27/70 .
e. Show that, had the fourth answer been West instead, the probability that East is nevertheless correct is 9/10.

Your friend, Ima Nerd, happens to be in the same position as you are, only she has reason to believe a-priori that, with probability α, East is the correct answer.

f. Show that whatever answer is first received, Ima continues to believe that East is correct with probability α.
g. Show that if the first two replies are the same (that is, either *WW* or *EE*), Ima continues to believe that East is correct with probability α.
h. Show that after three like answers, Ima will calculate as follows (in the obvious notation):

$$P(EastCorrect \mid EEE) = \frac{9\alpha}{11-2\alpha} \qquad P(EastCorrect \mid WWW) = \frac{11\alpha}{9+2\alpha}$$

Without prior bias on whether the exit of campus lies East or West, the exact answers of the passerby are not as important as whether a string of answers is similar or not. Let Rr denote the event that we receive r similar answers and T denote the event that these repeated answers are truthful. Let S denote the event that the questioned passerby is a student. Note that, because a professor always gives a false answer, $T \cap S^c = \varnothing$ and thus $P(T \cap Sc) = 0$.
Therefore,

$$\mathbf{P}(T|R_r) = \frac{\mathbf{P}(T \cap R_r)}{\mathbf{P}(R_r)} = \frac{\mathbf{P}(T \cap R_r \cap S)}{\mathbf{P}(R_r)} = \frac{\mathbf{P}(T \cap R_r|S)\mathbf{P}(S)}{\mathbf{P}(R_r)}$$

where the stated independence of a passerby's successive answers implies $\mathbf{P}(T \cap R_r|S) = \left(\frac{3}{4}\right)^r$
Applying the Total Probability Theorem and again making use of independence, we also deduce

$$\mathbf{P}(R_r) = \mathbf{P}(R_r|S)\mathbf{P}(S) + \underbrace{\mathbf{P}(R_r|S^c)}_{1}\mathbf{P}(S^c) = \left(\left(\frac{3}{4}\right)^r + \left(\frac{1}{4}\right)^r\right)\frac{2}{3} + \frac{1}{3}$$

a) Applying the above formulas for r = 1, we have $P(R_1) = 1$ and thus

$$\mathbf{P}(T|R_1) = \frac{\frac{3}{4}\cdot\frac{2}{3}}{1} = \boxed{\frac{1}{2}}$$

b) For r = 2, the formulas yield $P(R_2) = 3/4$ and thus

$$\mathbf{P}(T|R_2) = \frac{\left(\frac{3}{4}\right)^2\frac{2}{3}}{\frac{3}{4}} = \boxed{\frac{1}{2}}.$$

(c) For r = 3, the formulas yield $P(R_3) = 15/24$ and thus

$$\mathbf{P}(T|R_3) = \frac{\left(\frac{3}{4}\right)^3\frac{2}{3}}{\frac{15}{24}} = \boxed{\frac{9}{20}}$$

(d) For r = 4, the formulas yield $P(R_4) = 35/64$ and thus

$$\mathbf{P}(T|R_4) = \frac{\left(\frac{3}{4}\right)^4\frac{2}{3}}{\frac{35}{64}} = \boxed{\frac{27}{70}}$$

(e) As soon as we receive a dissimilar answer from the same passerby, we know that this passerby is a student; a professor will always give the same (false) answer. Let D denote the event of receiving the first dissimilar answer. Given D on the fourth answer, either the student has provided three truthful answers followed by one untruthful answer, occurring with probability $\left(\frac{3}{4}\right)^3 \frac{1}{4}$, or the student has provided three untruthful answers followed by one truthful answer, occurring with probability $\left(\frac{1}{4}\right)^3 \frac{3}{4}$. Note that event T corresponds to the former; thus,

$$\mathbf{P}(T|R_3 \cap D) = \frac{\left(\frac{3}{4}\right)^3 \frac{1}{4}}{\left(\frac{3}{4}\right)^3 \frac{1}{4} + \left(\frac{1}{4}\right)^3 \frac{3}{4}} = \boxed{\frac{9}{10}}$$

In parts (a) - (d), notice the decreasing trend in the probability of the passer-by being truthful as the number of similar answers grows. Intuitively, our confidence that the passerby is a professor grows as the sequence of similar answers gets longer, because we know a professor will always give the same (false) answer while a student has a chance to answer either way. However, as part (e) demonstrates, the first indication that the passerby is a student will boost our confidence that the previous string of similar answers are truthful, because any single answer by the student has a 3-to-1 chance of being a truthful one.

For the remainder of this problem, let E and W represent the events that a passerby provides East and West, respectively, as an answer and let T_E represent the event that East is the correct answer. We are told Ima's a-priori bias is $P(T_E) = \varepsilon$.

(f) Using Bayes's Rule and all the arguments used in parts (a) - (e), we have

$$\mathbf{P}(T_E|E) = \frac{\mathbf{P}(E|T_E)\mathbf{P}(T_E)}{\mathbf{P}(E)} = \frac{\left(\frac{2}{3}\right)\left(\frac{3}{4}\right)\epsilon}{\left(\frac{2}{3}\right)\left(\frac{3}{4}\right)\epsilon + \left(\left(\frac{2}{3}\right)\left(\frac{1}{4}\right) + \frac{1}{3}\right)(1-\epsilon)} = \boxed{\epsilon}$$

$$\mathbf{P}(T_E|W) = \frac{\mathbf{P}(W|T_E)\mathbf{P}(T_E)}{\mathbf{P}(W)} = \frac{\left(\left(\frac{2}{3}\right)\left(\frac{1}{4}\right) + \frac{1}{3}\right)\epsilon}{\left(\left(\frac{2}{3}\right)\left(\frac{1}{4}\right) + \frac{1}{3}\right)\epsilon + \left(\frac{2}{3}\right)\left(\frac{3}{4}\right)(1-\epsilon)} = \boxed{\epsilon}$$

In particular, we have used that $P(E) = P(E|T_E)P(T_E)+P(E|T^c_E)P(T^c_E)$ (and similarly for P(W)).

(g) Likewise, given two consecutive and similar answers from the same passerby, we have

$$\mathbf{P}(T_E|EE) = \frac{\left(\frac{2}{3}\right)\left(\frac{3}{4}\right)^2\epsilon}{\left(\frac{2}{3}\right)\left(\frac{3}{4}\right)^2\epsilon + \left(\left(\frac{2}{3}\right)\left(\frac{1}{4}\right)^2 + \frac{1}{3}\right)(1-\epsilon)} = \boxed{\epsilon}$$

$$\mathbf{P}(T_E|WW) = \frac{\left(\left(\frac{2}{3}\right)\left(\frac{1}{4}\right)^2 + \frac{1}{3}\right)\epsilon}{\left(\left(\frac{2}{3}\right)\left(\frac{1}{4}\right)^2 + \frac{1}{3}\right)\epsilon + \left(\frac{2}{3}\right)\left(\frac{3}{4}\right)^2(1-\epsilon)} = \boxed{\epsilon}$$

(h) Finally, given three consecutive and similar answers from the same passerby,

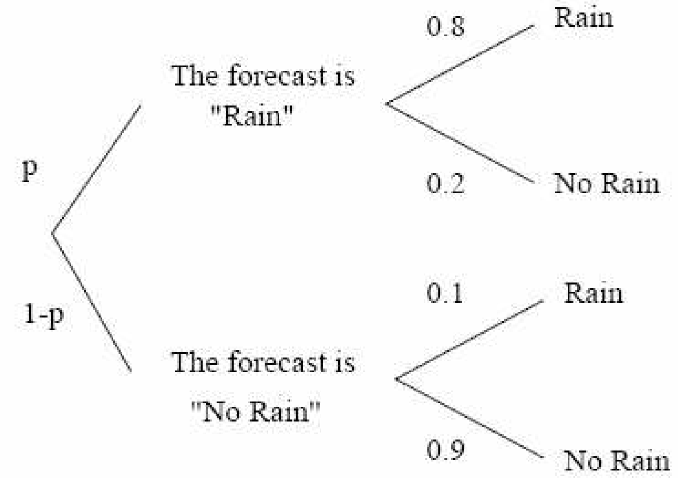
$$\mathbf{P}(T_E|EEE) = \frac{\left(\frac{2}{3}\right)\left(\frac{3}{4}\right)^3\epsilon}{\left(\frac{2}{3}\right)\left(\frac{3}{4}\right)^3\epsilon + \left(\left(\frac{2}{3}\right)\left(\frac{1}{4}\right)^3 + \frac{1}{3}\right)(1-\epsilon)} = \boxed{\frac{9\epsilon}{11-2\epsilon}}$$

$$\mathbf{P}(T_E|WWW) = \frac{\left(\left(\frac{2}{3}\right)\left(\frac{1}{4}\right)^3 + \frac{1}{3}\right)\epsilon}{\left(\left(\frac{2}{3}\right)\left(\frac{1}{4}\right)^3 + \frac{1}{3}\right)\epsilon + \left(\frac{2}{3}\right)\left(\frac{3}{4}\right)^3(1-\epsilon)} = \boxed{\frac{11\epsilon}{9+2\epsilon}}$$

Notice that the E, EE and EEE answers to parts (f) - (h) match the answers to parts (a)-(c) when $\varepsilon = 1/2$, or when Ima's prior bias does not favor either possibility

6. Before leaving to work, Victor checks the weather report before deciding on carrying an umbrella or not. If the forecast is "rain", the probability of actually having rain that day is 80%. On the other hand, if the forecast is "no rain" the probability of actually raining is equal to 10%. During fall and winter the forecast is 70% of the time "rain" and during summer and spring it is 20%.
   a. One day, Victor missed the forecast and it rained. What is the probability that the forecast was "rain" if it was during the winter? What is the probability that the forecast was "rain" if it was during the summer?
   b. The probability of Victor missing the morning forecast is equal to 0.2 on any day in the year. If he misses the forecast, Victor will flip a fair coin to decide on carrying an umbrella or not. On the day he sees the forecast, if it says "rain" he will always carry an umbrella, and if it says "no rain", he will never carry an umbrella. Are the events "Victor is carrying an umbrella", and "The forecast is no rain" independent? Does your answer depend on the season?
   c. Victor is carrying an umbrella and it is not raining. What is the probability that he saw the forecast?

(a) The tree representation during the winter can be drawn as the following:



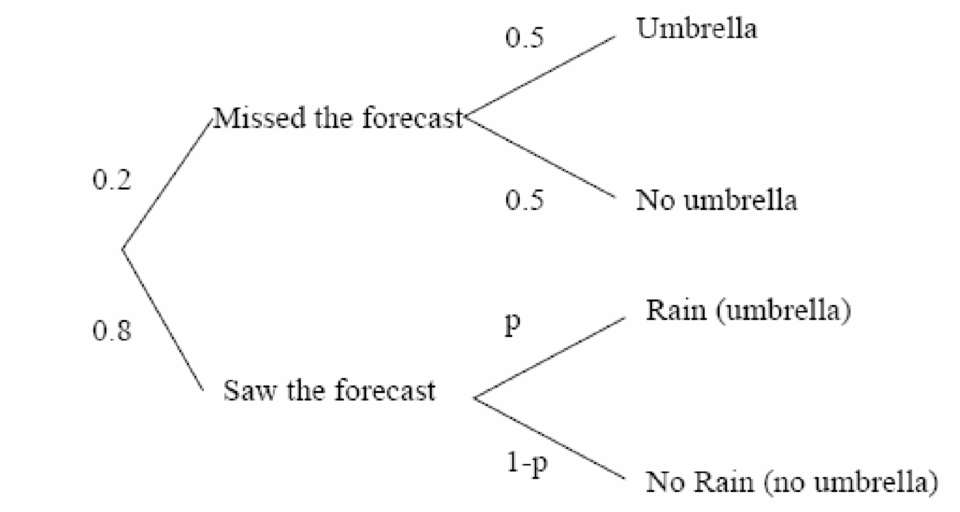
Let A be the event that the forecast was "Rain",
Let B be the event that it rained, Let p be the probability that the forecast says "Rain",
If it is in the winter, p = 0.7,

$$P(A \mid B) = \frac{P(B \mid A)P(A)}{P(B)} = \frac{(0.8)(0.7)}{(0.8)(0.7)+(0.1)(0.3)} = \frac{56}{59}$$

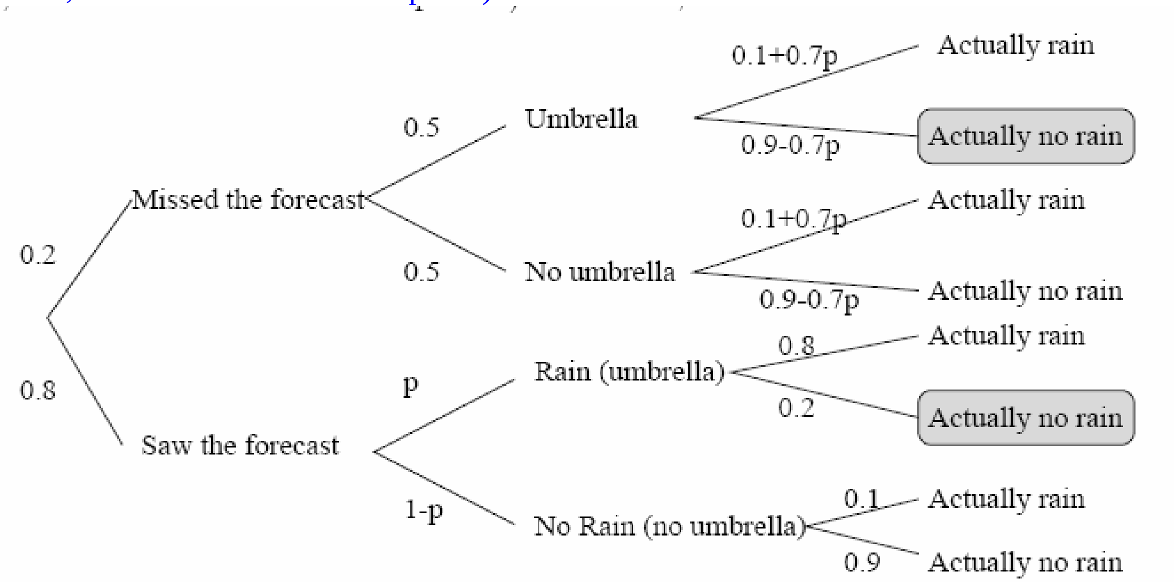
Similarly, if it is in the summer, p = 0.2,

$$P(A \mid B) = \frac{P(B \mid A)P(A)}{P(B)} = \frac{(0.8)(0.2)}{(0.8)(0.2)+(0.1)(0.8)} = \frac{2}{3}$$

(b) Let C be the event that Victor is carrying an umbrella.
Let D be the event that the forecast is no rain.
The tree diagram in this case is:



P (D) = 1− p
P (C) = (0.8)p + (0.2)(0.5) = 0.8p + 0.1
P (C | D) = (0.8)(0) + (0.2)(0.5) = 0.1
Therefore, P (C) = P (C | D) if and only if p = 0. However, p can only be 0.7 or 0.2, which implies the event C and D can never be independent, and this result does not depend on the season.
(c) Let us first find the probability of raining if Victor missed the forecast.
P (actually rains | missed forecast) = (0.8)p + (0.1)(1 − p) = 0.1 + 0.7p.
Then, we can extend the tree in part b) as follows

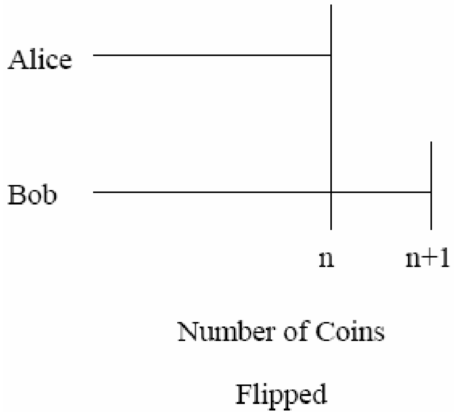


Therefore, given that Victor is carrying an umbrella and it is not raining, we are looking at the two shaded cases.

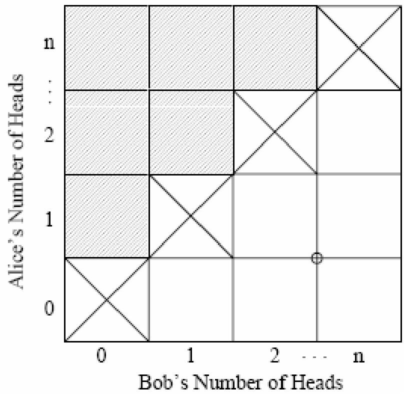
P (saw forecast | umbrella and not raining) = $\dfrac{(0.8)p(0.2)}{(0.8)p(0.2)+(0.2)(0.5)(0.9-0.7p)}$
In fall and winter, p = 0.7, Probability = 112/153
In summer and spring, p = 0.2 Probability = 8/27

7. Alice and Bob love to challenge each other to coin tossing contests. On one particular day, Alice brings 2n + 1 fair coins, and lets Bob toss n + 1 coins, while she tosses the remaining n coins. Show that the probability that after all the coins have been tossed Bob will have gotten more heads that Alice is 1/2.

Through the first n coins, Alice and Bob are equally likely to have flipped the same number of heads as the other (since they are using fair coins and each flip is independent of the other flips). Given this, when Bob flips his last coin but Alice doesn't flip a coin, Bob has a ½ chance of getting a head and thus having more heads than Alice.



Let's look at it another way. First define the following events:
1) A = the number of heads Alice tossed 2) B = the number of heads Bob tossed
Now suppose both Bob and Alice toss n coins. A sample space of interest is shown below where the shaded area represents Alice having more heads than Bob and the unshaded and uncrossed area represents Bob having more heads than Alice.



Since each coin is fair and each toss is independent of all the other tosses, each box in the diagram has an equal amount of probabilistic weight. We wish to show that
$P(B > A) = P(A > B)$
Well, each box has the same weight and there is the same number of boxes on each side of the diagonal. Consequently the above equation holds, meaning Alice and Bob are equally likely to have flipped more heads than the other. Furthermore, Alice and Bob are equally likely to have flipped the same number of heads, again since each box with an X in it has the same probabilistic weight.
Now Bob picks up the last coin. Given that both Alice and Bob are equally likely to have the same number of heads, the event Bob having more heads than Alice boils down to Bob getting a head on the last coin flip. Since this coin is fair and the flip is independent of past flips, this probability is simply 1/2.
An alternative solution is shown as follows,
Let B be the event that Bob tossed more heads,
let X be the event that after each has tossed n of their coins, Bob has more heads than Alice,
let Y be the event that under the same conditions, Alice has more heads than Bob,
and let Z be the event that they have the same number of heads.
Since the coins are fair, we have $P(X) = P(Y)$, and also $P(Z) = 1-P(X)-P(Y)$. Furthermore, we see that
$P(B \mid X) = 1$, $P(B \mid Y) = 0$, $P(B \mid Z) = 1/2$.
Now we have, using the theorem of total probability,

$P(B) = P(X)\, P(B \mid X) + P(Y)\, P(B \mid Y) + P(Z)\, P(B \mid Z)$
$= P(X) + 1/2\, P(Z)$
$= ½\, [(P(X) + P(Y) + P(Z)]$
$=1/2$ as required.