CENG 551 Probability and Stochastic Processes for Engineers

Week # 4

Random Variables: Binomial RV's

Dr. Deniz Ozdemir
Deniz.ozdemir@yasar.edu.tr
Web: dozdemir.yasar.edu.tr

Annoncements

- HW # 2 due to FRIDAY
- No class on October 30 (next week)
  - Yes Class on October 23

Plan for the Session

- Expectation
  - Example
- Conditional Expectations

- Bernoulli & Binomial Random Variables

Random Variables

- A random variable (r.v.) associates a unique numerical value with each outcome in the sample space.
- Notation:
  - Random Variable X
  - Experimental Value x
- Mathematically: A function from the sample space to the real numbers:
  - Discrete or Continuous
- Can have several random variables defined on the same sample space.

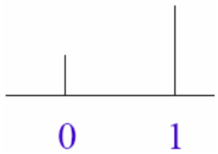
Discrete Random Variables

- **Discrete random variables:** number of possible values is finite or countably infinite: $x_1, x_2, x_3, x_4, x_5, x_6, \ldots$
- *Probability mass function (p.m.f.)*
  - $f(x) = P(X = x)$ (Sum over all possible values =1 always)
- *Cumulative distribution function (c.d.f)*
  - $F(x) = P(X \leq x) = \sum_{k \leq x} f(k)$

Selected Discrete Distributions:

- Single Coin flip P(heads/success) = p
  - *Probability of success*

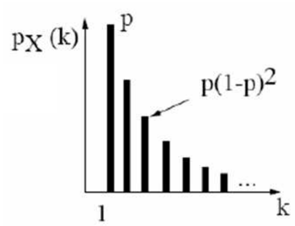
$$f(x) = P(X = x) = \begin{cases} p & \text{if } x = 1 \text{ (success)} \\ 1 - p & \text{if } x = 0 \text{ (failure)} \end{cases}$$



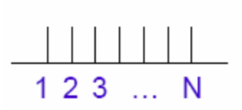
  - *"Bernoulli Distribution"*
- Multiple coin flips
  - x successes out of n trials
  - $f(x) = P(X = x) = \binom{n}{x} p^x (1-p)^{n-x}$ for $x = 1, \ldots, n$
  - *"Binomial distribution"*

Selected Discrete Distributions:

- Number of failures before the first success
  - $f(x)=P(X=x) = (1-p)^{x-1}p, \quad x=1,2,..$
  - *"Geometric Distribution"*



- N equally likely events
  - $f(x)=P(X=x) = 1/N, \quad x=1,2,..$
  - *"Uniform distribution"*



Selected Discrete Distributions:

- *Hypergeometric distribution*
  - Drawing balls from the box without replacing the balls
- *Poisson Distribution*
  - number of occurrences of a rare event
- *Multinomial Distribution*
  - more than two outcomes
- *Negative Binomial Distribution*
  - number of trials to get r successes

Expectation

- The expected value or mean of a discrete r.v. X, denoted by $\mathrm{E}(X)$, is defined as:

$$\boxed{\mathrm{E}[X] = \sum_x x \cdot p_X(x)}$$

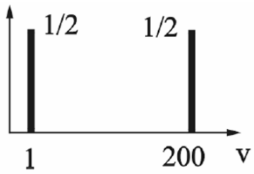
- Interpretations:
  - Center of gravity of pmf.
  - Average in large number of repetitions of the experiment. (to be substantiated later in this course)
- This is essentially a weighted average of the possible values the r.v. can assume, weights = $f(x)$

Properties of Expectations -2

- $Y = g(x)$ then
  - $\mathrm{E}[Y] = \sum_x g(x) \cdot p_X(x)$
- If α, β are constants, then:
  - $\mathrm{E}[\alpha] = \alpha$
  - $\mathrm{E}[\alpha X] = \alpha\, \mathrm{E}[X]$
  - $\mathrm{E}[\alpha X + \beta] = \alpha\, \mathrm{E}[X] + \beta$
  - $\mathrm{E}[X - E(X)] = 0$

Average Speed vs. Average Time

- Traverse a 200 mile distance at constant but random speed V:



- $d = 200$, $T = t(V) = 200/V$
  - $E[V] = 1.\frac{1}{2} + 200.\frac{1}{2} = 100.5$
  - $E[T] = E[t(V)] = \sum_v t(v) p_V(v)$
    $= \frac{200}{1}.\frac{1}{2} + \frac{200}{2}.\frac{1}{2} = 100.5$

Average Speed vs. Average Time vs. Expected Distance

- $E[d] = 200 = E[TV] \neq E[T].E[V]$
- $E[T] \neq 200/E[V]$
- $\mathrm{Var}(V) = \sum_v (v - E[V])^2 p_V(v)$
  $= (1-100.5)^2.\frac{1}{2} + (200-100.5)^2.\frac{1}{2}$
  $\approx 10\,000$
- Standard Deviation $\sigma_V = \sqrt{\mathrm{var}(V)} \approx 100$

Expectations:

- 4 buses carrying 148 job-seeking students arrive at a job fair. The buses carry, respectively, 40, 33, 25, and 50 students. One of the students is randomly selected. Let X denote the number of students that were on the bus carrying this randomly selected student. Also, one of the 4 bus drivers is also randomly selected. Let Y denote the number of students on his bus.

Do you think $E[X]$ and $E[Y]$ are equal?

$$E[X] = \frac{40}{148}\cdot 40+\frac{33}{148}\cdot 33+\frac{25}{148}\cdot 25+\frac{50}{148}\cdot 50 \approx 39.3$$
$$E[Y] = \frac{1}{4}\cdot 40+\frac{1}{4}\cdot 33+\frac{1}{4}\cdot 25+\frac{1}{4}\cdot 50 = 37$$

Variance and Standard Deviation

- The variance of an r.v. X, denoted by $\mathrm{Var}(X)$, or simply σ^2, is defined as:
  - $\mathrm{Var}(X) = \sigma^2 = E[(X - E[X])^2]$

    $\mathrm{Var}(X) = E(X^2 - 2E[X]X + (E[X])^2)$

    $= E(X^2) - 2E[X].E(X) + E((E[X])^2)$

    $= E(X^2) - 2(E[X])^2 + (E[X])^2$

    $= E(X^2) - (E[X])^2$
- The standard deviation (SD) is the square root of the variance.
- Note that the variance is in the square of the original units, while the SD is in the original units.

Conditional Expectations

Conditional Expectation

- Recall definition of expectation:

$$\boxed{\mathbf{E}[X] = \sum_x x \cdot p_X(x)}$$

where $p_X(x) = \mathbf{P}(X = x)$

- Denote conditional probability:

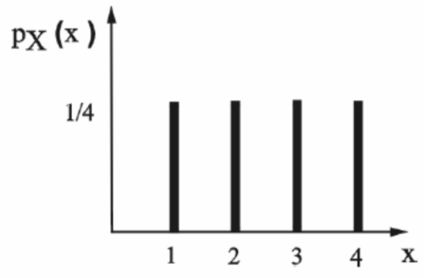
$$P(X=x|A) = p_{X|A}(x)$$

$$\boxed{\mathbf{E}[X|A] = \sum_x x \cdot p_{X|A}(x)}$$

Conditional Expectation

- Example:

$E[X|X\geq 2]$ =?



Example II:

- You are waiting for bus number 1 in the bus stop. Every minute one bus arrives. Probability that the arrived bus is No:1 is p.
  - What is the expected waiting time?
  - What is the probability that you are going to wait x minutes?

X: Waiting time for the bus at the stop

$f(x)=P(X=x) = (1-p)^{x-1}p, \quad x=1,2,..$

$$E[X] = \sum_{k=1}^{\infty} k \cdot p_X(k) = \sum_{k=1}^{\infty} k \cdot (1-p)^{k-1}p$$

Useful things to remember:

- Stirling Approximation: $n! \sim n^{n+1/2} e^{-n} \sqrt{2\pi}$

$$\sum_{k=0}^{n} k = \frac{n(n+1)}{2}$$

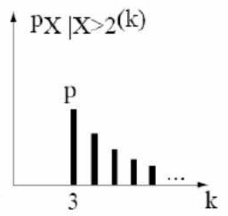
$$\sum_{k=0}^{n} k^2 = \frac{n(n+1)(2n+1)}{6}$$

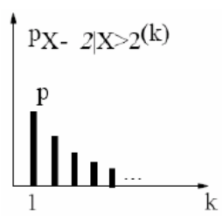
$$\sum_{k=0}^{n} x^k = \frac{1-x^{k+1}}{1-x}$$

$$\sum_{k=0}^{\infty} x^k = \begin{cases} \infty & \text{if } x \geq 1 \\ \dfrac{1}{1-x} & \text{if } x < 1 \end{cases}$$

Example II:

- You are waiting for bus number 1 in the bus stop. Every minute one bus arrives. Probability that the arrived bus is No:1 is p.
  - What is the expected waiting time conditioned on the fact that you have already waited 2 minutes?





Memoryless property: Given that $X > 2$, the r.v. $X-2$ has same (geometric) PMF.

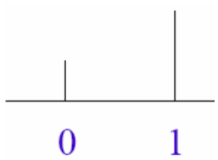
$$\mathbf{E}[X \mid X \geq 2] = 2 + \mathbf{E}[X]$$

Bernoulli and Binomial Random variables

Selected Discrete Distributions:

- Single Coin flip P(heads/success) = p
  - *Probability of success*

$$f(x) = P(X = x) = \begin{cases} p & \text{if } \mathrm{x} = 1 \text{ (success)} \\ 1 - p & \text{if } \mathrm{x} = 0 \text{ (failure)} \end{cases}$$



  - *"Bernoulli Distribution"*
- Multiple coin flips
  - x successes out of n trials
  - $f(x) = P(X = x) = \binom{n}{x} p^x (1-p)^{n-x}$ for x = 1,…, n
  - *"Binomial distribution"*

Properties of Bernoulli Distribution

- PMF: $f(x) = P(X = x) = \begin{cases} p & \text{if } x = 1 \text{ (success)} \\ 1-p & \text{if } x = 0 \text{ (failure)} \end{cases}$
- Expectation:
  - $E[X] = p$
  - $E[X^k] = p$
- Variance:
  - $E(X^2) - (E[X])^2 = p - p^2 = p(1-p)$

Properties of Binomial Distribution

- PMF: $f(x) = P(X = x) = \binom{n}{x} p^x (1-p)^{n-x}$
- Expectation:
  - $E[X] = np$
  - $E[X^k] = pnE[Y+1]^{k-1}$ where $Y \sim \text{Bin}(n-1,p)$
- Variance:
  - $E(X^2) - (E[X])^2 = pn[(n-1)p] - n^2p^2 = np(1-p)$

Exercise: Binomial Distribution

- What is the ratio of $P(X = k)/P(X = k\text{-}1)$?
  - When does $P(X = k) < P(X = k\text{-}1)$?
  - When does $P(X = k) > P(X = k\text{-}1)$?

- Proposition: If x is a binomial r.v. with parameters (n, p) where $0 < p < 1$, then as k goes to from 0 to n, $P(X = k)$ first increases monotonically and then decreases monotonically, reaching its largest value when k is the largest integer less than or equal to $(n+1)p$.

Next Time..

- Geometric Distribution
- Joint PMF of two random variables
- Independence
- Poisson PMF