HOMEWORK # 4-5 (Due to December 25, 2012, Tuesday)

1. You are allowed to take a certain test three times, and your final score will be the maximum of the test scores. Your score in test i, where i=1, 2, 3 takes one of the values from i to 10 with equal probability $1/(11-i)$, independently of the scores in other tests. What is the PMF of the final score?

2. Random variables X and **Y** have the joint PMF

$$p_{X,Y}(x,y) = \begin{cases} c(x^2+y^2) & , \quad x \in \{1,2,4\} \quad \text{and} \quad y \in \{1,3\} \\ 0 & , \quad \text{otherwise.} \end{cases}$$

   (a) What is the value of the constant c?
   (b) What is $P(Y<X)$?
   (c) What is $P(Y>X)$?
   (d) What is $P(Y = X)$?
   (e) What is $P(Y =3)$?
   (f) Find the marginal PMFs $p_X(x)$ and $p_Y(y)$.
   (g) Find the expectations E[X] and E[Y].
   (h) Find the variances var(X) and var(Y).

3. You just rented a large house and the realtor gave you five keys, one for the front door and the other four for each of the four side and back doors of the house. Unfortunately, all keys look identical, so to open the front door, you are forced to try them at random. Find the mean and the variance of the number of trials you will need to open the door, under the following alternative assumptions:
   a) after an unsuccessful trial, you mark the corresponding key so that you never try it again, or
   b) at each trial, you are equally likely to choose any key

4. A contestant on a quiz show is presented with two questions, question 1 and 2, which he is to attempt to answer in some order chosen by him. If he decides to try question i, i =1, 2 first, then he will be allowed to go on to the other question j, $j \neq i$, only if his answer to i is correct. If his initial answer is incorrect, he is not allowed to answer the other question. The contestant is to receive v_i dollars if he answers question i correctly, i =1, 2. Thus, for instance, he will receive $v_1 + v_2$ dollars if both questions are correctly answered. If the probability that he knows the answer to question i is p_i, i =1, 2, which question should he attempt first so as to maximize his expected winnings? Assume that the events E_i, i =1, 2, that he knows the answer to question i, are independent events.

5. Professor May B. Right often has her science facts wrong, and answers each of her students' questions incorrectly with probability 1/4, independently of other questions. In each lecture Professor Right is asked either 1 or 2 questions with equal probability.
   a) What is the probability that Professor Right gives wrong answers to all the questions she gets in a given lecture?
   b) Given that Professor Right gave wrong answers to all the questions she was asked in a given lecture, what is the probability that she got two questions?
   c) Let X and Y be the number of questions asked and the number of questions answered correctly in a lecture, respectively. What are the mean and variance of X and the mean and the variance of Y?
   d) Give a neatly labeled sketch of the joint PMF $p_{X,Y}(x,y)$.
   e) Let Z = X+ 2Y. What are the expectation and variance of Z?

f) For the remaining parts of this problem, assume that Professor Right has 20 lectures each semester and each lecture is independent of any other lecture

g) The university where Professor Right works has a peculiar compensation plan. For each lecture, she gets paid a base salary of \$1,000 plus \$40 for each question she answers and an additional \$80 for each of these she answers correctly. In terms of random variable Z, she gets paid \$1000 + \$40Z per lecture. What are the expected value and variance of her semesterly salary?

6. Determined to improve her reputation, Professor Right decides to teach an additional 20-lecture class in her specialty (math), where she answers questions incorrectly with probability 1/10 rather than 1/4. What is the expected number of questions that she will answer wrong in a randomly chosen lecture (math or science).

7. Joe Lucky plays the lottery on any given week with probability p, independently of whether he played on any other week. Each time he plays, he has a probability q of winning, again independently of everything else. During a fixed time period of n weeks, let X be the number of weeks that he played the lottery and Y the number of weeks that he won.

(a) What is the probability that he played the lottery any particular week, given that he did not win anything that week?
(b) Find the conditional PMF $p_{Y|X}(y|x)$.
(c) Find the joint PMF $p_{X,Y}(x, y)$.
(d) Find the marginal PMF $p_Y(y)$. *Hint:* One possibility is to start with the answer to part (c), but the algebra can be messy. But if you think intuitively about the procedure that generates Y, you may be able to guess the answer.
(e) Find the conditional PMF $p_{X|Y}(x|y)$. Do this algebraically using previous answers.
(f) Rederive the answer to part (e) by thinking as follows: For each one of the $n - Y$ weeks that he did not win, the answer to part (a) should tell you something.

8. Let X and Y be independent random variables that take values in the set {1,2,3}.
Let $V = 2X + 2Y$ and $W = X - Y$.

(a) Assume that P({$X = k$}) and P({$Y = k$}) are positive for any $k \in \{1,2,3\}$. Can V and W be independent? Explain.(*No calculations needed*.)

For the remaining parts of this problem, assume that X and Y are equally like to have values 1, 2, or 3

(b) Find and plot $p_V(v)$. Also, determine E[V] and var(V).
(c) Find and show in a diagram $p_{V,W}(v, w)$.
(d) Find $E[V|W > 0]$.
(e) Find the conditional variance of W given the event {V =8}.
(f) Find and plot the conditional PMF $p_{X|V}(x|v)$, for all values.

9. The joint PMF of discrete random variables X and Y is given by

$$p_{X,Y}(x,y) = \begin{cases} Cx^2\sqrt{y}, & \text{if } x = -5, -4, \ldots, 4, 5 \text{ and } y = 0, 1, \ldots, 10; \\ 0 & \text{otherwise.} \end{cases}$$

Here, C is some constant. What is $E[XY^3]$?

Hint: This question admits a short answer and explanation. Don't spend time doing calculations.