Massachusetts Institute of Technology
Department of Electrical Engineering & Computer Science
6.041/6.431: Probabilistic Systems Analysis
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Recitation 5: Answers
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1. (a) Let X be the number of tosses until the game is over. Noting that X is geometric with probability of success

$$\mathbf{P}(\{HT, TH\}) = p(1-q) + q(1-p),$$

we obtain

$$p_X(k) = (1 - p(1-q) - q(1-p))^{k-1}\left(p(1-q) + q(1-p)\right), \qquad k = 1, 2, \ldots$$

Therefore

$$\mathbf{E}[X] = \frac{1}{p(1-q) + q(1-p)}$$

and

$$\mathrm{var}(X) = \frac{pq + (1-p)(1-q)}{\left(p(1-q) + q(1-p)\right)^2}.$$

(b) The probability that the last toss of the first coin is a head is

$$\mathbf{P}(HT \mid \{HT, TH\}) = \frac{p(1-q)}{p(1-q) + (1-p)q}.$$

2. Let random variable X be the number of trials you need to open the door, and let K_i be the event that the ith key selected opens the door.

(a) In case (1), we have

$$\begin{aligned}
p_X(1) &= \mathbf{P}(K_1) = \frac{1}{5}, \\
p_X(2) &= \mathbf{P}(K_1^c)\mathbf{P}(K_2 \mid K_1^c) = \frac{4}{5} \cdot \frac{1}{4} = \frac{1}{5}, \\
p_X(3) &= \mathbf{P}(K_1^c)\mathbf{P}(K_2^c \mid K_1^c)\mathbf{P}(K_3 \mid K_1^c \cap K_2^c) = \frac{4}{5} \cdot \frac{3}{4} \cdot \frac{1}{3} = \frac{1}{5}.
\end{aligned}$$

Proceeding similarly, we see that the PMF of X is

$$p_X(x) = \frac{1}{5}, \qquad x = 1, 2, 3, 4, 5.$$

We can also view the problem as ordering the keys in advance and then trying them in succession, in which case the probability of any of the five keys being correct is 1/5.

In case (2), X is a geometric random variable with $p = 1/5$, and its PMF is

$$p_X(k) = \frac{1}{5} \cdot \left(\frac{4}{5}\right)^{k-1}, \qquad k \geq 1.$$

(b) In case (1), we have

$$\begin{aligned} p_X(1) &= \mathbf{P}(K_1) = \frac{2}{10}, \\ p_X(2) &= \mathbf{P}(K_1^c)\mathbf{P}(K_2 \mid K_1^c) = \frac{8}{10} \cdot \frac{2}{9}, \\ p_X(3) &= \mathbf{P}(K_1^c)\mathbf{P}(K_2^c \mid K_1^c)\mathbf{P}(K_3 \mid K_1^c \cap K_2^c) = \frac{8}{10} \cdot \frac{7}{9} \cdot \frac{2}{8} = \frac{7}{10} \cdot \frac{2}{9}. \end{aligned}$$

Proceeding similarly, we see that the PMF of X is

$$p_X(x) = \frac{2 \cdot (10 - x)}{90}, \qquad x = 1, 2, \ldots, 10.$$

Consider now an alternative line of reasoning to derive the PMF of X. If we view the problem as ordering the keys in advance and then trying them in succession, the probability that the number of trials required is x is the probability that the first $x - 1$ keys do not contain either of the two correct keys and the xth key is one of the correct keys. We can count the number of ways for this to happen and divide by the total number of ways to order the keys to determine $p_X(x)$. The total number of ways to order the keys is 10! For the xth key to be the first correct key, the other key must be among the last $10 - x$ keys, so there are $10 - x$ spots in which it can be located. There are 8! ways in which the other 8 keys can be in the other 8 locations. We must then multiply by two since either of the two correct keys could be in the xth position. We therefore have $2 \cdot (10 - x) \cdot 8!$ ways for the xth key to be the first correct one and

$$p_X(x) = \frac{2 \cdot (10 - x)8!}{10!} = \frac{2 \cdot (10 - x)}{90}, \qquad x = 1, 2, \ldots, 10,$$

as before.

In case (2), X is again a geometric random variable with $p = 1/5$.

3. The number C of candy bars you need to eat is a geometric random variable with parameter p. Thus the mean is $\mathbf{E}[C] = 1/p$, and the variance is $\mathrm{var}(C) = (1 - p)/p^2$.