CENG 551 Probability and Stochastic Processes for Engineers

Week # 9

Continuous Random Variables: Exponential RV's

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Midterm Exam

- Closed-book,
- One handwritten double-sided A4 formula sheet and calculator permitted
- 2 or 3 questions, each with 2-3 parts
- Planned for 60 minutes
  - Maximum 90 minutes
- Expect questions similar to homework questions

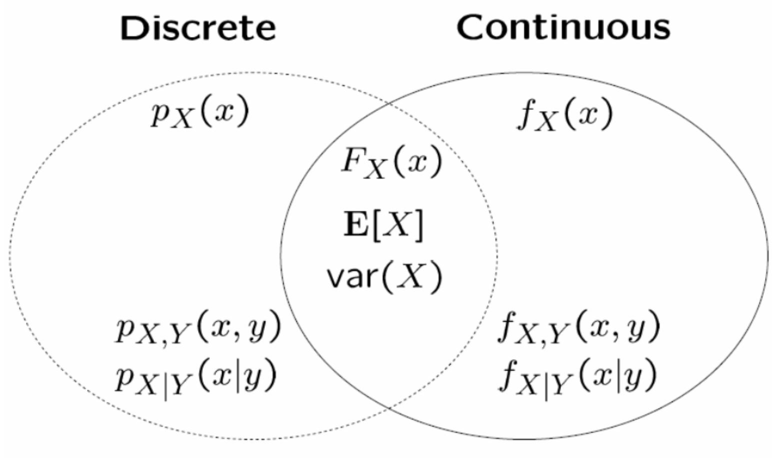
MidTerm Exam

- All topics including today (partially):
  - Set operations,
  - Probability axioms and probability laws
  - Conditional probability, independence
  - Counting rules
  - Discrete random variables (Bernoulli, Binomial, Geometric, Poisson distribution)
  - Expectation, variance
    - Conditional expectation
  - PMF of a (function of) random variable
    - Conditioning and independence.
    - Joint, marginal and conditional PMFs

Plan for the Session

- More on continuous r.v.s
- Derived distributions

Summary of concepts



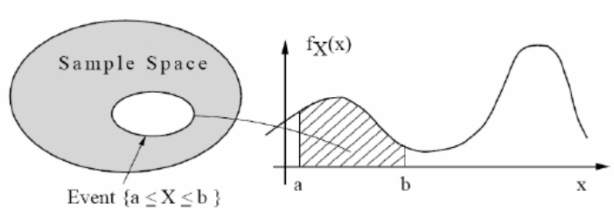
Review

Discrete	Continuous
$p_X(x)$	$f_X(x)$
$p_{X,Y}(x,y)$	$f_{X,Y}(x,y)$
$p_{X\|Y}(x\|y) = \dfrac{p_{X,Y}(x,y)}{p_Y(y)}$	$f_{X\|Y}(x\|y) = \dfrac{f_{X,Y}(x,y)}{f_Y(y)}$
$p_X(x) = \sum_y p_{X,Y}(x,y)$	$f_X(x) = \int_{-\infty}^{\infty} f_{X,Y}(x,y)dy$

$$F_X(x) = \mathbf{P}(X \leq x)$$

$\mathbf{E}[X]$, $\text{var}(X)$

Continuous Random Variables (PDF)



$$\mathbf{P}(a \le X \le b) = \int_a^b f_X(x)dx$$

$$\boxed{\mathbf{P}(x \le X \le x+\delta) \approx f_X(x)\cdot\delta}$$

Probability density function

$$\int_{-\infty}^{\infty} f_X(x)dx = 1$$

Means and Variance

- Analogous to discrete version:

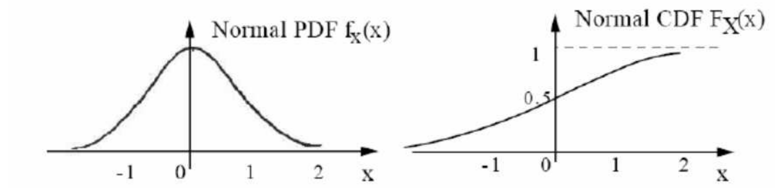
$$\mathbf{E}[X] = \int_{-\infty}^{\infty} x \cdot f_X(x)dx$$

$$\mathbf{E}[g(X)] = \int_{-\infty}^{\infty} g(x)\cdot f_X(x)dx$$

$$\begin{aligned}\text{var}(X) &= \sigma_X^2 \\ &= \int_{-\infty}^{\infty}(x - \mathbf{E}[X])^2 \cdot f_X(x)dx\end{aligned}$$

Standard Gaussian (Normal) PDF

- Standard Normal: $N(0, 1)$
- PDF: $$f_X(x) = \frac{1}{\sqrt{2\pi}} e^{-x^2/2}$$
- Expectation: $E[X] = 0$
- Variance: Var(X)=1



General Gaussian (Normal) PDF

General Normal: $N(\mu, \sigma^2)$

$$f_X(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-(x-\mu)^2/2\sigma^2}$$

It turns out that:

$$\mathbf{E}[X] = \mu \qquad \text{var}(X) = \sigma^2$$

Let $Y = aX + b$ then:

$$\mathbf{E}[Y] = a\mu + b \qquad \text{var}(Y) = a^2\sigma^2$$

Fact: $Y \sim N(a\mu + b, a^2\sigma^2)$

Example

- In a casino there is a Gaussian slot machine. In this game, the machine produces independent identically distributed (IID) numbers $X_1, X_2, \ldots$ that have normal distribution $N(0, \sigma^2)$. For every i, when the number X_i is positive, the player receives from the casino a sum of money equal to X_i. When X_i is negative, the player pays the casino a sum of money equal to $|X_i|$. What is the standard deviation of the net total gain of a player after *n* plays of the Gaussian slot machine?

11

Example

- In a casino there is a Gaussian slot machine. In this game, the machine produces independent identically distributed (IID) numbers $X_1, X_2, \ldots$ that have normal distribution $N(0, \sigma^2)$. For every i, when the number X_i is positive, the player receives from the casino a sum of money equal to X_i. When X_i is negative, the player pays the casino a sum of money equal to $|X_i|$. What is the probability that the absolute value of the net total gain after n plays is greater than $2\sqrt{n}\sigma$?

12

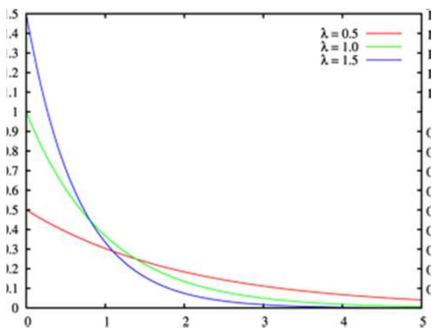
Exponential Distribution

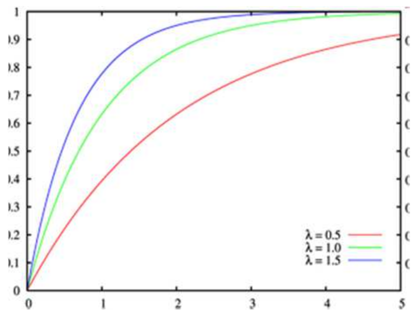
- May be viewed as a continuous version of the *geometric distribution*,
  - Geometric distribution: # of Bernoulli trials necessary for a *discrete* process to change state.
  - Exponential distribution : Amount of time necessary for a *continuous* process to change state.
- Examples: Approximately exponentially distributed variables:
  - the time until a radioactive particle decays,
  - the time between beeps of a Geiger counter;
  - the time it takes before your next telephone call
  - the time until payment of a company debt holders in reduced form credit risk modeling
  - the distance between mutations on a DNA strand;

Exponential Distribution: Properties

- A random variable X is exponentially distributed with parameter λ

PDF: $f_X(x) = \begin{cases} \lambda e^{-\lambda x} & x \geq 0 \\ 0, & \text{otherwise} \end{cases}$ CDF $F_X(x) = \begin{cases} e^{-\lambda x} & x \geq 0 \\ 0 & \text{otherwise} \end{cases}$





Exponential Distribution: Properties

- X ~Exp(λ)
- PDF: $f_X(x) = \begin{cases} \lambda e^{-\lambda x} & x \geq 0 \\ 0, & \text{otherwise} \end{cases}$
- Expectation: $E[X]$ =
- Variance: Var(X)=

Example: Bus Stop

- Suppose the waiting time until the next bus at a particular bus stop is exponentially distributed, with parameter λ = 1/15 . Suppose that a bus pulls out just as you arrive at the stop. Find the probability that: You *wait more than 15 minutes* for a bus.

(a) The probability that you wait more than 15 minutes is:

$$\int_{15}^{\infty} \frac{1}{15} e^{-\frac{x}{15}}\, dx = -e^{-\frac{x}{15}} \Big|_{15}^{\infty} = e^{-1}.$$

Example: Bus Stop

- Suppose the waiting time until the next bus at a particular bus stop is exponentially distributed, with parameter $\lambda = 1/15$. Suppose that a bus pulls out just as you arrive at the stop. Find the probability that You *wait between 15 and 30 minutes* for a bus.

The probability that you wait between 15 and thirty minutes is:

$$\int_{15}^{30} \frac{1}{15} e^{-\frac{x}{15}}\, dx = -e^{-\frac{x}{15}}\Big|_{15}^{30} = e^{-1} - e^{-2}.$$

Joint PDF $f_{X,Y}(x, y)$

$$p_{X,Y}(x,y) = \mathbf{P}(X = x \text{ and } Y = y)$$

$$f_{X,Y}(x,y)$$

$$\mathbf{P}(A) = \iint_A f_{X,Y}(x,y)\,dx\,dy$$

- Interpretation

$$\mathbf{P}(x \le X \le x+\delta, y \le Y \le y+\delta) \approx f_{X,Y}(x,y)\cdot\delta^2$$

- Expectation

$$\mathbf{E}[g(X,Y)] = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} g(x,y) f_{X,Y}(x,y)\,dx\,dy$$

Joint PDF $f_{X,Y}(x, y)$ II

$$\mathbf{P}(A) = \iint_A f_{X,Y}(x,y)\,dx\,dy$$

- From the joint to the marginal:

$$f_X(x) \cdot \delta \approx \mathbf{P}(x \le X \le x + \delta) =$$

$$\boxed{\int_{-\infty}^{\infty} \int_{x}^{x+\delta} f_{X,Y}(t,y)\,dt\,dy \approx \int_{-\infty}^{\infty} f_{X,Y}(x,y)\,dy \cdot \delta}$$

- X and Y are called independent iff:

$$\boxed{f_{X,Y}(x,y) = f_X(x) \cdot f_Y(y)}$$

Exponential Distribution: Properties

- If $X_1, X_2, \ldots, X_n$ are exponentially distributed, $X_i \sim \text{Exp}(\lambda_i)$ then $Y =$ **min** $(X_1, X_2, \ldots, X_n) \sim \text{Exp}(\Sigma\lambda_i)$

- BUT $Y =$ **max** $(X_1, X_2, \ldots, X_n)$ $\cancel{\sim \text{Exp}(\Sigma\lambda_i)}$

Conditioning

- Recall, again: $\mathbf{P}(x \leq X \leq x+\delta) \approx f_X(x) \cdot \delta$
- Thus, the definition: $f_{X|Y}(x|y) = \dfrac{f_{X,Y}(x,y)}{f_Y(y)}$
- Conditioning is a “section”of the joint PDF, normalized.
- Independence gives: $f_{X|Y}(x|y) = f_X(x)$

11/20/2012 21

Exponential Distribution: Properties

- $P(X \geq E[X]) = ?$

  $\mathbf{P}(X \geq \mathbf{E}[X]) = \frac{1}{e}$

- $P(X \geq k + t \mid X>t) = ?$

  $\mathbf{P}(X > t+k | X > t) \;=\; e^{-\lambda(k)}$

Memoryless Property!

11/20/2012 22

Example: Computer Chips

- Computer chips can be expected to fail after operating for a random amount of time. Suppose, in particular, that

  P(chip still works at time t)= $e^{-\alpha t}$, $t \geq 0$. (1)

Consider now that we have a manufacturing process that produces a mix of "good" and "bad" chips. The lifetime of good chips satisfies Eq. (1). The lifetime of bad chips satisfies the same relation except that α is replaced by 1000α. Assume that the fraction of good chips is p and the fraction of bad chips $1-p$.

- Find the probability that a randomly selected chip is still functioning after τ time units of operation.

A_t: Chip still works at time t.
B: The chip is bad.
G: The chip is good.

We can specify that $\mathbf{P}(A_t|G) = e^{-\alpha t}$ and $\mathbf{P}(A_t|B) = e^{-1000\alpha t}$.

the definition of conditional probability and the total probability theorem,

$$\mathbf{P}(A_t) = \mathbf{P}(G)\mathbf{P}(A_t|G) + \mathbf{P}(B)\mathbf{P}(A_t|B) = pe^{-\alpha t} + (1-p)e^{-1000\alpha t}$$

By using the definition of conditional probability, we get:

$$\mathbf{P}(B|A_t) = \frac{\mathbf{P}(B \cap A_t)}{\mathbf{P}(A_t)}.$$

Furthermore, $\mathbf{P}(B \cap A_t) = \mathbf{P}(B)\mathbf{P}(A_t|B) = (1-p)e^{-1000\alpha t}$. Therefore,

$$\mathbf{P}(B|A_t) = \frac{(1-p)e^{-1000\alpha t}}{pe^{-\alpha t} + (1-p)e^{-1000\alpha t}}.$$

Example: Computer Chips II

- Computer chips can be expected to fail after operating for a random amount of time. Suppose, in particular, that

$$P(\text{chip still works at time } t) = e^{-\alpha t},\ t \geq 0. \quad (1)$$

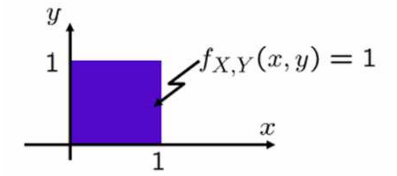
Consider now that we have a manufacturing process that produces a mix of "good" and "bad" chips. The lifetime of good chips satisfies Eq. (1). The lifetime of bad chips satisfies the same relation except that α is replaced by 1000α. Assume that the fraction of good chips is p and the fraction of bad chips $1 - p$.

In order to weed out bad chips, every chip is tested for τ time units before leaving the factory, and only chips that do not fail during the testing period are shipped to customers.

Calculate a formula for the probability that a customer receives a bad chip (as a function of the constants α, p, and τ).

Derived Distribution

- What is *derived distribution*?
- It is a PMF or PDF of a function of random variables with known probability law.
- Example: X and Y



- Let $g(X, Y) = X/Y$
  - Note: $g(X, Y)$ is a r.v.
- Obtaining the PDF for $g(X, Y)$ involves deriving a distribution.

Why do we derive distributions?

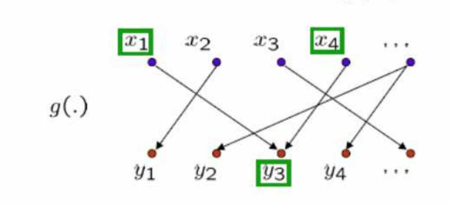
- Sometimes we don't need to. Example:
  - Computing expected values.

$$\mathbf{E}[g(X,Y)] = \iint g(x,y) f_{X,Y}(x,y)\,dx\,dy$$

- But often they're useful. Examples:
  - Maximum of several r.v.s. (delay models)
  - Minimum of several r.v.s (failure models).
  - Sum of several r.v.s. (multiple arrivals)

How to find them: Discrete case

- Consider: -a single discrete r.v.: X
  -and a function: $g(X) = Y$



- Obtain probability mass for each possible value of $Y=y$:
  - $p_Y(y) = P(g(X) = y)$
    $= \Sigma_{x:\, g(x)=y}\, p_X(x)$

How to find them: Continuous case

- Consider: -a single continuous r.v.: X
  -and a function: $g(X) = Y$
- Two step procedure:

1. Get CDF of Y : $F_Y(y) = P(Y \leq y)$
2. Differentiate to get: $f_Y(y) = \frac{dF_Y}{dy}(y)$

- Why go to the CDF?

Example 1:

- X uniform on $[0, 2]$
- Find PDF of $Y = X^3$

- Solution: Two step procedure:

1. Get CDF of Y :

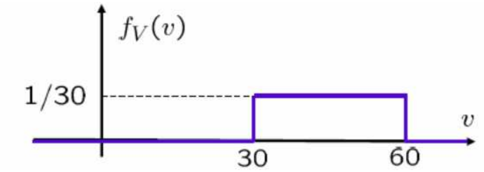
$$F_Y(y) = P(Y \le y) = P(X^3 \le y) = P(X \le y^{1/3}) = (1/2)\, y^{1/3}$$

2. Differentiate to get:

$$f_Y(y) = \frac{dF_Y}{dy}(y) = \frac{1}{6y^{2/3}}$$

Example 2:

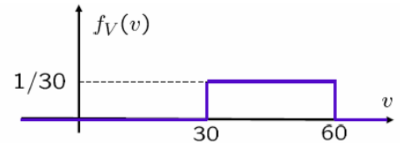
- Joan is driving from Boston to New York (200 km). Her speed is uniformly distributed between 30 and 60 km/h. What is the distribution of the duration of the trip?
- PDF of the velocity V:



- Let time: $T(V) = 200/V$
- Find $f_T(t)$.

Example 2:

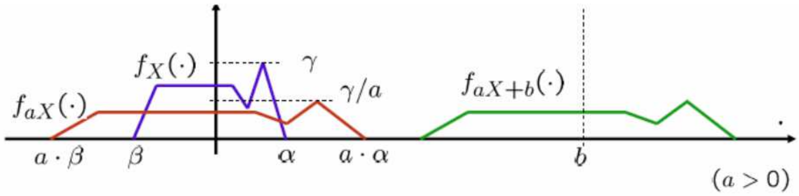
- Joan is driving from Boston to New York (200 km). Her speed is uniformly distributed between 30 and 60 km/h. What is the distribution of the duration of the trip?
- PDF of the velocity V:



- Let time: $T(V) = 200/V$
- Solution: Two step procedure:

1. Get CDF of T :
   $F_T(t) = P(T \leq t)$
2. Differentiate to get:

Exercise 1: $Y = aX+b$



$$\boxed{f_Y(y) = \frac{1}{|a|} f_X\left(\frac{y-b}{a}\right)}$$

- Use this to check that if is X normal, then $Y = aX+b$ is also normal.

Exercise 2: $Y = \min\ (X_1, X_2, \ldots, X_n)$

Show that

If $X_1, X_2, \ldots, X_n$ are exponentially distributed, $X_i \sim \text{Exp}(\lambda_i)$ then $Y = \mathbf{min}\ (X_1, X_2, \ldots, X_n) \sim \text{Exp}(\Sigma_i \lambda_i)$

Next Time

- More on derived distributions