HOMEWORK # 1 (Due to October 2, 2012, Tuesday)

Solutions

1. Let A and B be two events. Use the axioms of probability to prove the following:
   a. $P(A \cap B) \geq P(A) + P(B) - 1$
   b. Show that the probability that one and only one of the events A or B occurs is
   $$P(A) + P(B) - 2P(A \cap B).$$

Note: You may want to argue in terms of Venn diagrams, but you should also provide a complete proof, that is a step-by-step derivation, where each step appeals to an axiom or a logical rule.

(a) We have already proved in lecture and in the course notes that

$$P(A \cup B) = P(A) + P(B) - P(A \cap B).$$

Rearranging, we get

$$P(A \cap B) = P(A) + P(B) - P(A \cup B).$$

Since $(A \cup B)$ is always a subset of Ω, the universal event, therefore, $P(A \cup B) \leq P(\Omega)$ and

$$P(A \cap B) \geq P(A) + P(B) - P(\Omega).$$

Finally, by the normalization axiom, $P(\Omega) = 1$ and

$$P(A \cap B) \geq P(A) + P(B) - 1.$$

(b) We begin by writing

$$\begin{aligned} P(A \text{ or } B, \text{ but not both}) &= P((A^c \cap B) + (A \cap B^c)) \\ &= P(A^c \cap B) + P(A \cap B^c), \end{aligned}$$

where the last equality is from the additivity axiom. Next, we know that $B = (A^c \cap B) \cup (A \cap B)$ and $(A^c \cap B) \cap (A \cap B) = \emptyset$ so that we may apply the additivity axiom to get

$$P(B) = P(A^c \cap B) + P(A \cap B).$$

With rearrangement, this becomes

$$P(A^c \cap B) = P(B) - P(A \cap B).$$

By symmetry, we also have

$$P(B^c \cap A) = P(A) - P(A \cap B).$$

So plugging in for $P(A^c \cap B)$ and $P(B^c \cap A)$, we get

$$\begin{aligned} P(A \text{ or } B, \text{ but not both}) &= P(B) - P(A \cap B) + P(A) - P(A \cap B) \\ &= P(A) + P(B) - 2P(A \cap B). \end{aligned}$$

2. Find $P(A \cup (B^c \cup C^c)^c)$ in each of the following cases:
   a. A, B, C are mutually exclusive events and $P(A)=3/7$.
   b. $P(A)=1/2$, $P(B \cap C)=1/3$, $P(A \cap C)=0$.
   c. $(A^c \cap (B^c \cup C^c)) = 0.65$.

(a) We are given $P(A) = 3/7$, $P(B \cap C) = 0$ and $P(A \cap B \cap C) = 0$. Using De Morgan's laws, we know $(B^c \cup C^c)^c = B \cap C$. Therefore

$$P(A \cup (B^c \cup C^c)^c) = P(A \cup (B \cap C)) = P(A) + P(B \cap C) - P(A \cap (B \cap C)) = \boxed{3/7}.$$

(b) We are given $P(A) = 1/2$, $P(B \cap C) = 1/3$ and $P(A \cap C) = 0$. Therefore, again applying De Morgan's laws,

$$P(A \cup (B^c \cup C^c)^c) = P(A \cup (B \cap C)) = P(A) + P(B \cap C) - P(A \cap (B \cap C)) = \boxed{5/6}$$

where we deduce $A \cap B \cap C = \emptyset$ (and thus $P(A \cap B \cap C) = 0$) because $A \cap C = \emptyset$ and $A \cap B \cap C \subseteq A \cap C$.

(c) We are given $P(A^c \cap (B^c \cup C^c)) = 0.65$ and De Morgan's laws imply $(A^c \cap (B^c \cup C^c))^c = A \cup (B^c \cup C^c)^c$, which is the event of interest. Therefore

$$P(A \cup (B^c \cup C^c)^c) = 1 - P(A^c \cap (B^c \cup C^c)) = \boxed{0.35}$$

3. Anne and Bob each have a deck of playing cards. Each flips over a randomly selected card. Assume that all pairs of cards are equally likely to be drawn. Determine the following probabilities:
   a. the probability that at least one card is an ace,
   b. the probability that the two cards are of the same suit,
   c. the probability that neither card is an ace,
   d. the probability that neither card is a diamond or club.

We could have a two-dimensional sample space containing 52^2 points, where each axis represents a particular card. However, this sample space would be finer grain than necessary to determine the desired probabilities.

For parts a) and c), we have a sample space of 169 points representing the 169 possible outcomes.

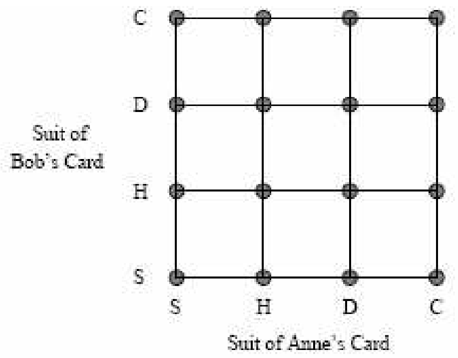
Define event B to be when Bob draws an ace, event A to be when Anne draws an ace. Then we know that

$P(A) = P(B) = \frac{1}{13}$
$P(A \cap B) = \frac{1}{169}$

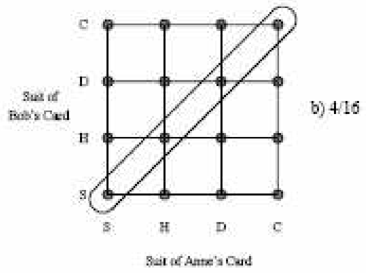
P(at least one card is an ace) $= P(A \cup B) = P(A) + P(B) - P(A \cap B) = \boxed{\frac{25}{169}}$

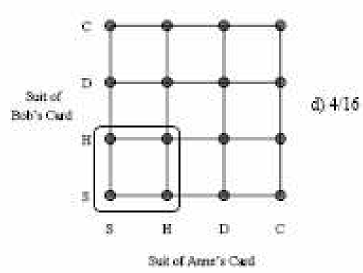
P(neither card is an ace) = 1- P(at least one card is an ace) $= \boxed{\frac{144}{169}}$

For parts b) and d), since we are only interested in the suits of the cards, we represent the sample space as the following 16 points. The horizontal axis represents the suit of Anne's card, and the vertical axis represents the suit of Bob's card. Each of the points is equally likely; therefore, the probability of any particular point occurring is $\frac{1}{16}$.



The probabilities requested can be determined by counting the number of points satisfying each condition and dividing the total by 16, as shown in the figures below.



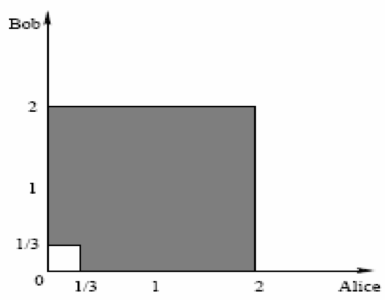


4. Alice and Bob each choose at random a number between zero and two. We assume a uniform probability law under which the probability of an event is proportional to its area. Consider the following events:
   A : The magnitude of the difference of the two numbers is greater than 1/3.
   B : At least one of the numbers is greater than 1/3.
   C : The two numbers are equal.
   D : Alice's number is greater than 1/3.

Find the probabilities *P(B), P(C), P(A∩D).*

P(B)

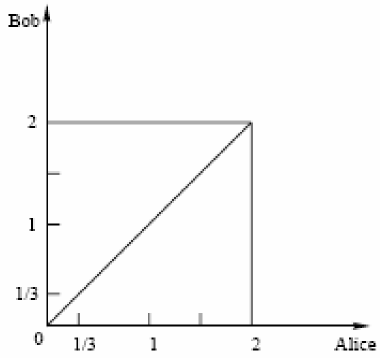
The shaded area in the following figure is the union of Alice's pick being greater than 1/3 and Bob's pick being greater than 1/3.



$$
\begin{aligned}
P(B) &= 1 - P(\textit{both numbers are smaller than } 1/3) \\
&= 1 - \frac{\textit{Area of small square}}{\textit{Total sample area}} \\
&= 1 - \frac{1/3 * 1/3}{4} \\
&= 1 - 1/36 \\
&= \boxed{35/36}
\end{aligned}
$$

P(C)

In the following figure, the line x = y represents the set of points where two numbers are equal.

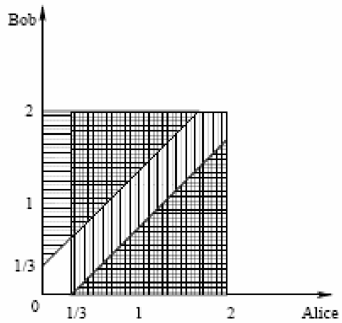


The line has an area of 0. Thus,

$$\begin{aligned} P(C) &= \frac{Area\ of\ line}{Total\ sample\ area} \\ &= \frac{0}{4} \\ &= \boxed{0} \end{aligned}$$

P$(A \cap D)$

Overlapping the diagrams we would get for P(A) and P(D),



$$\begin{aligned} P(A \cap D) &= \frac{double\ shaded\ area}{Total\ sample\ area} \\ &= \frac{5/3 * 5/3 * 1/2 + 4/3 * 4/3 * 1/2}{4} \\ &= \frac{25/18 + 16/18}{4} \\ &= \boxed{41/72} \end{aligned}$$

5. Bob has a peculiar pair of four-sided dice. When he rolls the dice, the probability of any particular outcome is proportional to the product of the outcome of each die. All outcomes that result in a particular product are equally likely.
   a. What is the probability of the product being even?
   b. What is the probability of Bob rolling a 2 and a 3?

We begin by enumerating the sample space Ω and identifying the *relative* probabilities of all outcomes, as shown in the table below, where $p \in [0, 1]$ will need to be determined.

Die 1	Die 2	Product	P(Product)
1	1	1	p
1	2	2	2p
1	3	3	3p
1	4	4	4p
2	1	2	2p
2	2	4	4p
2	3	6	6p
2	4	8	8p
3	1	3	3p
3	2	6	6p
3	3	9	9p
3	4	12	12p
4	1	4	4p
4	2	8	8p
4	3	12	12p
4	4	16	16p
		Total	100p

$$P(\Omega) = 1 = 100p \quad \Rightarrow \quad p = \frac{1}{100} = 0.01$$

(a) Let set A indicate the event that the product is even. Then,

$$P(A) = 2p + 4p + 2p + 4p + 6p + 8p + 6p + 12p + 4p + 8p + 12p + 16p = 84p = \boxed{0.84}$$

(b) Let set B indicate the event of rolling a 2 and 3. Then,

$$P(B) = P(2,3) + P(3,2) = 6p + 6p = 12p = \boxed{0.12}$$

NEXT QUESTIONS ARE FOR IENG & CENG M.S. (BONUS FOR IMIS)

6. Let A, B, C, A_1, ..., A_n be some events. Show the following identities. A mathematical derivation is required, but you can use Venn diagrams to guide your thinking.
   a. $P(A \cup B \cup C) = P(A) + P(B) + P(C) - P(A \cap B) - P(B \cap C) - P(A \cap C) + P(A \cap B \cap C)$
   b. $P\left(\bigcup_{k=1}^{n} A_k\right) = P(A_1) + P(A_1^c \cap A_2) + P(A_1^c \cap A_2^c \cap A_3) + \ldots + P(A_1^c \cap \ldots \cap A_{n-1}^c \cap A_n)$

(a) Define event $E = A \cup B$. Then $E \cap C = (A \cup B) \cap C = (A \cap C) \cup (B \cap C)$.
Therefore $P(E \cap C) = P(A \cap C) + P(B \cap C) - P(A \cap B \cap C)$

$$\begin{aligned} P(A \cup B \cup C) &= P(E \cup C) \\ &= P(E) + P(C) - P(E \cap C) \\ &= P(A \cup B) + P(C) - P(A \cap C) - P(B \cap C) + P(A \cap B \cap C) \\ &= P(A) + P(B) + P(C) - P(A \cap C) - P(B \cap C) - P(A \cap B) + P(A \cap B \cap C) \end{aligned}$$

(b) We will apply an inductive argument.

Base Case:
$P(A_1) = P(A_1)$
Inductive Step:
Assume $P(\cup_{k=1}^{n-1} A_k) = P(A_1) + P(A_1^c \cap A_2) + P(A_1^c \cap A_2^c \cap A_3) + \cdots + P(A_1^c \cap \cdots \cap A_{n-2}^c \cap A_{n-1})$.

$$\begin{aligned} P(\cup_{k=1}^{n} A_k) &= P(\cup_{k=1}^{n-1} A_k \cup A_n) \\ &= P(A_1) + P(A_1^c \cap A_2) + P(A_1^c \cap A_2^c \cap A_3) + \cdots + P(A_1^c \cap \cdots \cap A_{n-2}^c \cap A_{n-1}) \\ &\quad + (P(A_n) - P(\cup_{k=1}^{n-1} A_k \cap A_n)) \\ &= P(A_1) + P(A_1^c \cap A_2) + P(A_1^c \cap A_2^c \cap A_3) + \cdots + P(A_1^c \cap \cdots \cap A_{n-2}^c \cap A_{n-1}) \\ &\quad + P((\cup_{k=1}^{n-1} A_k)^c \cap A_n) \\ &= P(A_1) + P(A_1^c \cap A_2) + P(A_1^c \cap A_2^c \cap A_3) + \cdots + P(A_1^c \cap \cdots \cap A_{n-1}^c \cap A_n). \end{aligned}$$

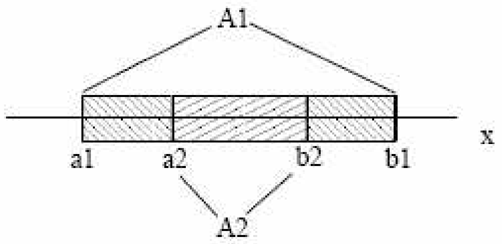
7. Consider an experiment whose sample space is the real line. Let $\{a_n\}$ an increasing sequence of numbers that converges to a and $\{b_n\}$ a decreasing sequence that converges to b. Show that

$$\lim_{n \leftarrow \infty} P([a_n, b_n]) = P([a, b])$$

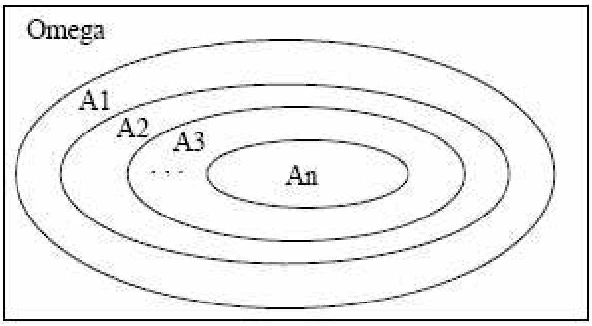
Here, the notation $[a,b]$ stands for the closed interval $\{x \mid a \leq x \leq b\}$.
Note: This result seems intuitively obvious. The issue is to derive it **using the axioms of probability theory.**

Solution I:

First, consider the set $A_1 = \{x \mid a_1 \leq x \leq b_1\}$ and the set $A_2 = \{x \mid a_2 \leq x \leq b_2\}$. Since a_n is an increasing sequence, $a_1 \leq a_2$ and since b_n is a decreasing sequence, $b_1 \geq b_2$. As we see in the following diagram, we have $A_2 \subset A_1$.



Continuing this argument up to A_n, we get the following picture:



Finally, $A_\infty = \{x \mid a \leq x \leq b\}$. So, by the above picture, $A_\infty = \lim_{n\to\infty} A_n = (\bigcup_{i=1}^{\infty} A_i^c)^c$.
Observe that $A_n, n \geq 1$ is a decreasing sequence and that $A_n^c, n \geq 1$ is an increasing sequence.
Now, we define the events $B_n, C_n, n \geq 1$ as follows:

$$C_n = A_n^c$$

$$\begin{aligned} B_1 &= C_1 \\ B_2 &= C_2 \cap C_1^c \\ B_n &= C_n \cap C_{n-1}^c \end{aligned}$$

Thus, each B_n consists of elements that are not in the previous events and are consequently mutually exclusive. Furthermore:

$$\begin{aligned} \bigcup_{i=1}^{n} B_i &= C_1 \bigcup (C_2 \cap C_1^c) \bigcup ... \bigcup (C_n \cap C_{n-1}^c) \\ \bigcup_{i=1}^{n} B_i &= \bigcup_{i=1}^{n} C_i \\ \bigcup_{i=1}^{\infty} B_i &= \bigcup_{i=1}^{\infty} C_i \end{aligned}$$

So, by additivity,

$$\begin{aligned} P(\bigcup_{i=1}^{\infty} C_i) &= P(\bigcup_{i=1}^{\infty} B_i) = \Sigma_{i=1}^{\infty} P(B_i) \\ &= \lim_{n\to\infty} \Sigma_{i=1}^{n} P(B_i) = \lim_{n\to\infty} P(\bigcup_{i=1}^{n} B_i) \\ &= \lim_{n\to\infty} P(\bigcup_{i=1}^{n} C_i) = \lim_{n\to\infty} P(C_n) \\ &= \lim_{n\to\infty} P(A_n^c) \end{aligned}$$

Also:

$$P(\bigcup_{i=1}^{\infty} C_i) = P(\bigcup_{i=1}^{\infty} A_i^c) = P[((\bigcup_{i=1}^{\infty} A_i^c)^c)^c]$$

$$\begin{aligned} P[((\bigcup_{i=1}^{\infty} A_i^c)^c)^c] &= 1 - P[(\bigcup_{i=1}^{\infty} A_i^c)^c] = 1 - P(A_\infty) \\ \lim_{n\to\infty} P(A_n^c) &= \lim_{n\to\infty} (1 - P(A_n)) = 1 - \lim_{n\to\infty} P(A_n) \end{aligned}$$

Therefore:

$$\begin{aligned} 1 - P(A_\infty) &= 1 - \lim_{n\to\infty} P(A_n) \\ P(A_\infty) &= \lim_{n\to\infty} P(A_n) \\ P(\{x \mid a \le x \le b\}) &= \lim_{n\to\infty} P(\{x \mid a_n \le x \le b_n\}) \\ P([a,b]) &= \lim_{n\to\infty} P([a_n, b_n]) \end{aligned}$$

Alternative Solution:

A more general version of this problem is the following. Let $A_1, A_2, \ldots$ be a countably infinite sequence of events such that $A_{n+1} \subset A_n$ for all $n = 1, 2, \ldots$. Let $A = \bigcap_n A_n$. [1] Show that $\lim_n \mathbf{P}(A_n) = \mathbf{P}(A)$.

Solution. First note that for all n

$$\begin{aligned} \mathbf{P}(A_n) &= \mathbf{P}((A_n \cap A^c) \cup (A_n \cap A)) \\ &= \mathbf{P}(A_n \cap A^c) + \mathbf{P}(A_n \cap A) \qquad \text{[additivity]} \\ &= \mathbf{P}(A_n \cap A^c) + \mathbf{P}(A). \end{aligned}$$

Then note that similarly for all n

$$\mathbf{P}(A_1 \cap A^c) = \mathbf{P}\left(\bigcup_{i=1}^{n-1}(A_i \cap A_{i+1}^c) \cup (A_n \cap A^c)\right)$$
$$= \sum_{i=1}^{n-1} \mathbf{P}(A_i \cap A_{i+1}^c) + \mathbf{P}(A_n \cap A^c). \qquad \text{[additivity]}$$

Taking the limit at both sides, we have [2]

$$\mathbf{P}(A_1 \cap A^c) = \sum_{i=1}^{\infty} \mathbf{P}(A_i \cap A_{i+1}^c) + \lim_n \mathbf{P}(A_n \cap A^c)$$
$$= \mathbf{P}\left(\bigcup_i (A_i \cap A_{i+1}^c)\right) + \lim_n \mathbf{P}(A_n \cap A^c) \qquad \text{[additivity]}$$
$$= \mathbf{P}(A_1 \cap A^c) + \lim_n \mathbf{P}(A_n \cap A^c).$$

Thus, we obtain

$$\lim_n \mathbf{P}(A_n \cap A^c) = 0,$$

which gives the result

$$\lim_n \mathbf{P}(A_n) = \lim_n \mathbf{P}(A_n \cap A^c) + \mathbf{P}(A) = \mathbf{P}(A).$$

This completes the proof.