Lecture 3 - Transient Response and Transforms

The filters so far considered (Butterworth, Chebyshev and elliptic) were designed with only the amplitude response $|G(j\omega)|$ in mind; the impulse response $g(t)$, and step response, may be poor. If we are concerned with preserving the signal *shape*, then we may have to use a different type of filter (remember, nevertheless, that low-pass filtering will inevitably round the corners of the waveform off and delay it).

Consider a signal split into its Fourier components. To preserve *shape*, all components must get through with the same gain factor and the same delay. (Distortion due to frequency-dependent gain is *amplitude* distortion, that due to frequency-dependent delay is *phase* distortion). If we low-pass filter, there is obviously a gain loss at higher frequencies, hence a "rounding-off" of the shape, but in fact a *gradual* falling-off of $|G(j\omega)|$ does less damage to the shape than a fast fall-off. For all frequencies passed to have the same delay, we need a phase lag ϕ proportional to ω in the pass-band.

Let $\phi = \omega\tau_d$ with τ_d = constant; then an input $A\cos\omega t$ leads to an output $B\cos(\omega t - \phi) = B\cos\omega(t - \tau_d)$ ie. a delay of τ_d for all frequencies, with the amplitude changed from A to B. In fact, for an nth-order all-pole filter, $\phi \to \frac{n\pi}{2}$ as $\omega \to \infty$, and so we cannot have $\phi = \omega\tau_d$ for all ω. But clearly linear phase in the passband is better than non-linear phase (as Fig. 3.1)

3.1 Bessel-Thomson filters

These filters have a maximally-flat time delay; in order to arrive at such a characteristic, we make the derivatives of ϕ, with respect to ω, zero at $\omega = 0$, from the

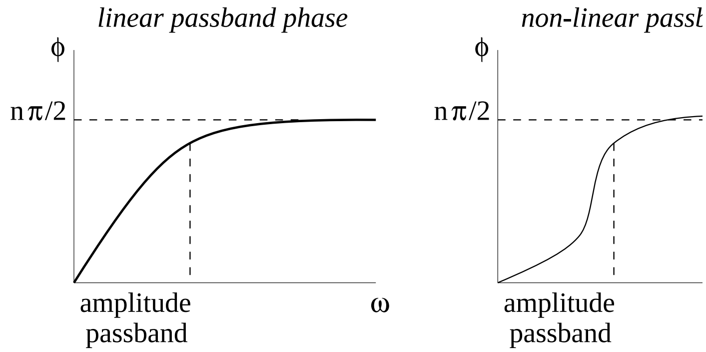

Figure 3.1: *Linear phase in the passband.*

2^{nd} to the $2n^{\text{th}}$, for an nth-order filter. The *first* derivative, $\frac{d\phi}{d\omega}$, is the time delay. Thus:

$$\phi = \tau_d\{\omega + \text{ terms in } \omega^{2n+1} \text{ and higher }\}$$

$$\frac{d\phi}{d\omega} = \tau_d\{1 + \text{ terms in } \omega^{2n} \text{ and higher}\}$$

Consider $n = 3$. A general 3-pole (no zero) transfer function, with low-frequency time delay τ, might be written:

$$G(s) = \frac{1}{1 + s\tau + a_2(s\tau)^2 + a_3(s\tau)^3}$$

$$\text{Therefore} G(j\omega) = \frac{1}{1 + j\omega t - a_2(\omega\tau)^2 - ja_3(\omega\tau)^3}$$

We need to find a_2 and a_3 so that $\frac{d^2\phi}{d\omega^2} \ldots \frac{d^6\phi}{d\omega^6}$ are zero.

$$\phi = -\arg\{G(j\omega)\} = \ tan^{-1}\frac{\omega\tau - a_3(\omega\tau)^3}{1 - a_2(\omega\tau)^2}$$

$$\text{ie.} \quad \phi = \tan^{-1}\frac{x - a_3x^3}{1 - a_2x^2} \qquad \text{if we let} x = \omega\tau$$

$$\frac{d\phi}{d\omega} = \frac{d\phi}{dx} \cdot \frac{dx}{d\omega} = \frac{\tau}{1 + \{\frac{x - a_3x^3}{1 - a_2x^2}\}^2} \cdot \frac{(1 - a_2x^2)(1 - 3a_3x^2) - (x - a_3x^3)(-2a_2x)}{(1 - a_2x^2)^2}$$

$$\text{ie.} \frac{d\phi}{d\omega} = \tau \frac{\{1 - (a_2 + 3a_3)x^2 + 3a_2a_3x^4\} + \{2a_2x^2 - 2a_2a_3x^4\}}{(1 - a_2x^2)^2 + (x - a_3x^3)^2}$$

$$\text{Therefore} \frac{d\phi}{d\omega} = \tau \frac{1 - (3a_3 - a_2)x^2 + a_2a_3x^4}{1 - (2a_2 - 1)x^2 + (a_2^2 - 2a_3)x^4 + a_3^2x^6}$$

We can choose a_2 and a_3 so that numerator and denominator have the same x^2 and x^4 coefficients. Then a series expansion of $\frac{d\phi}{d\omega}$ (e.g. by dividing out) has no terms between 1 and ω^6. For this we need:

$$3a_3 - a_2 = 2a_2 - 1 \quad \text{and} \quad a_2a_3 = a_2^2 - 2a_3$$

which gives $a_2 = \frac{2}{5}; a_3 = \frac{1}{15}$

$$\text{Hence} G(s) = \frac{1}{1 + s\tau + \frac{2}{5}(s\tau)^2 + \frac{1}{15}(s\tau)^3}$$

$$\text{and} \frac{d\phi}{d\omega} = \tau\{1 - \frac{1}{225}(\omega\tau)^6 + \frac{1}{1125}(\omega\tau)^8 - \ldots\}$$

In general, for an nth order Bessel-Thomson filter:

$$G(s) = \frac{1}{1 + s\tau + a_2(s\tau)^2 + a_3(s\tau)^3 + \ldots + a_n(s\tau)^n}$$

$$\text{and} \qquad \frac{d\phi}{d\omega} = \tau\{1 - a_n^2(\omega\tau)^{2n} + \ldots\}$$

The coefficients are:

n	a_2	a_3	a_4	a_5	a_6
1	–				
2	$1/3$	–			
3	$2/5$	$1/15$	–		
4	$3/7$	$2/21$	$1/105$	–	
5	$4/9$	$1/9$	$1/63$	$1/945$	–
6	$5/11$	$4/33$	$10/495$	$1/495$	$1/10,395$

These coefficients can also be calculated from Bessel polynomials (hence the name of the filter).

3.1.1 Poles of Bessel-Thomson filter

The poles tend to have larger damping factors than for other types; for example, consider the case $n = 3$ for Butterworth, Chebyshev (0.1dB ripple) and Bessel-Thomson filters. If the poles are scaled so that each filter has the same 3dB cut-off frequency, we obtain the plot of Fig. 3.2:

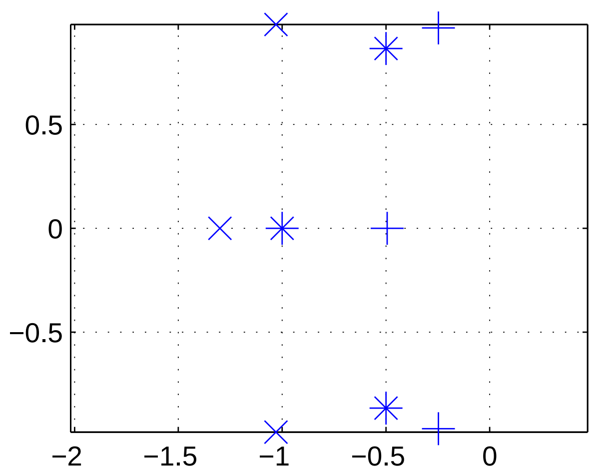

Figure 3.2: *Bessel-Thompson poles (x), Butterworth (*) and Chebyshev Type I (+). Dampings are Butterworth,* $\zeta = 0.5$*, Chebyshev* $(\varepsilon = 0.153)$, $\zeta < 0.5$ *and Bessel-Thomson,* $\zeta = 0.724$.

3.1.2 Impulse response

The impulse response of Bessel-Thomson filters tends towards a Gaussian as the filter order is increased (see below); $g(t) \simeq \frac{1}{2\tau}(1 - \cos \pi \frac{t}{\tau})$. Hence the corresponding step response has practically no overshoot, as in Fig. 3.3.

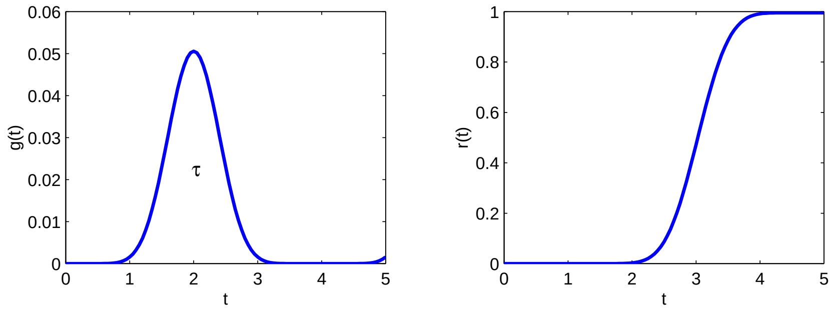

Figure 3.3: *Bessel $n = 5$ filter. Impulse response (left) and step response (right).*

3.1.3 Delay characteristics

The graph of phase lag vs. $\omega\tau$ for Bessel-Thomson filters (up to $n = 7$) is in Fig. 3.4. For the larger values of n, $\phi = \omega\tau$ (ie. constant delay) holds more exactly up to a point where the signal gets through in any case. Thus, for large n, the delay is flat over a large range of frequencies.

Attenuation frequencies, for example 20dB, can be determined by examining the high-frequency behaviour of the filter:

$$\text{For an } n \text{ th order filter,} \qquad \lim_{\omega\to\infty} |G(j\omega)| = \frac{1}{a_n(\omega\tau)^n}$$

For $n = 1$,

$$\frac{1}{\omega\tau} = \frac{1}{10} \to \omega\tau = 10$$

For $n = 2$,

$$\frac{1}{\frac{1}{3}(\omega\tau)^2} = \frac{1}{10} \to \omega\tau = 5.48$$

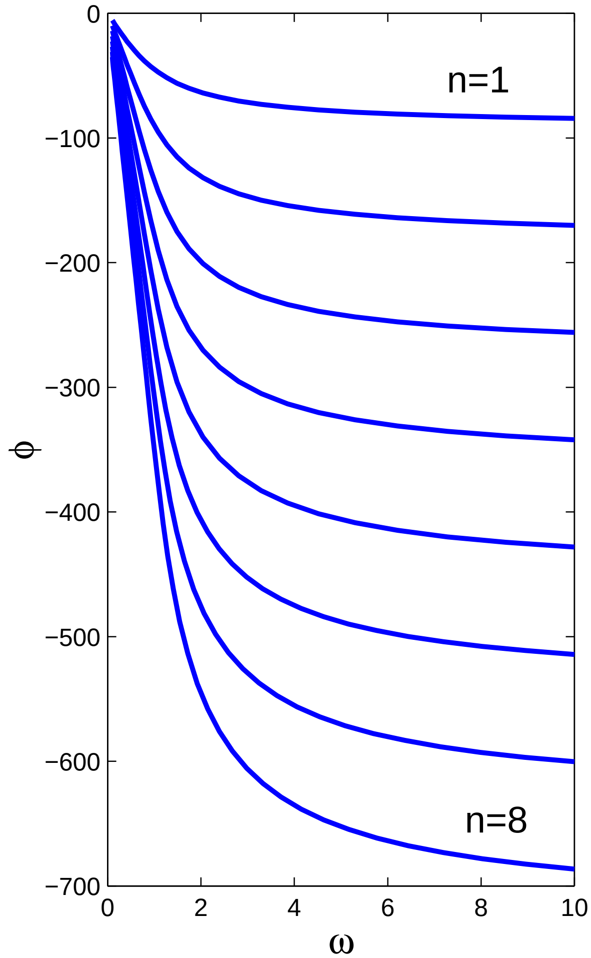

Figure 3.4: *Bessel filter delays.*

3.1.4 Comparison of delays for Bessel-Thomson & Butterworth filters

Fig. 3.5 gives a comparison of the time delays for 6th-order Bessel-Thomson and Butterworth low-pass filters. Only with the Bessel filter will there be minimal waveform distortion in the pass-band.

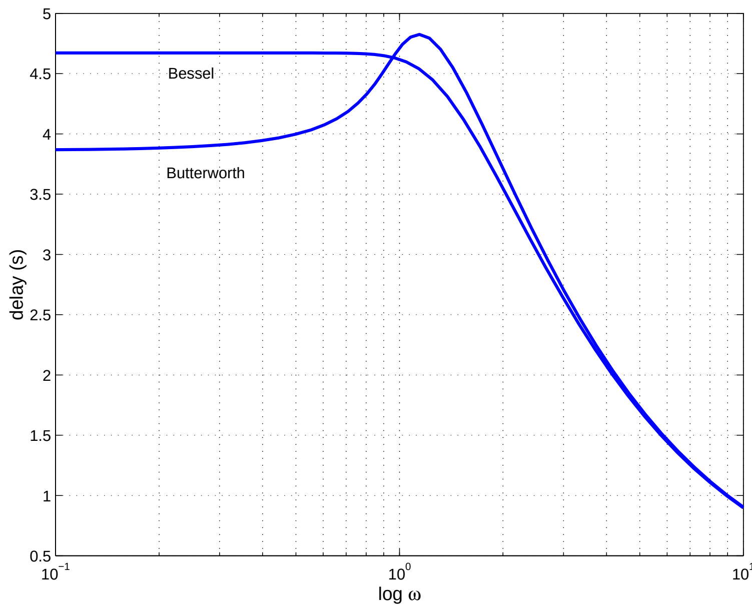

Figure 3.5: *Delays for Butterworth & Bessel filters.*

3.1.5 Design of Bessel-Thomson low-pass filters

The main specification for Bessel-Thomson filters is usually that of a constant delay (or linear phase response) over as large a band of frequencies as possible (this requirement is often specified as the percentage deviation allowed from a given delay at a specific frequency). As shown on the previous page, there will be a minimum value of n needed to meet that specification. Once n is determined, the designer can check the steepness of the amplitude response for the low-pass filter, but this is usually a secondary requirement. The main problem is caused by the added circuit complexity for large values of n, but this is true for all filter types.

3.2 Analogue Frequency Transformations

We have now derived design procedures for low-pass filters using Butterworth, Chebyshev or elliptic approximations to an ideal low-pass characteristic. The high-pass (HP), band-pass (BP) and band-stop (BS) transfer functions can all be obtained through well-known frequency transformations.

3.2.1 High-pass filters

We can transform low-pass (cutting off *from* ω_c) to high-pass (cutting off *up to* ω_c) by replacing s in the low-pass transfer function by ω_c^2/s to give the high-pass transfer function. The amplitude response

$$|G(j\omega)|_{LP} = \frac{1}{(1 + H\{(\frac{\omega}{\omega_c})^2\})^{\frac{1}{2}}}$$

is then mirrored about ω_c in a log. plot since

$$|G(j\omega)|_{HP} = \frac{1}{(1 + H\{(\frac{\omega_c}{\omega})^2\})^{\frac{1}{2}}}$$

(eg. response at $\frac{\omega_{LP}}{\omega_c} = 2$ is now the same as that at $\frac{\omega_{HP}}{\omega_c} = \frac{1}{2}$).

Reminder:
ω_c = 3dB cut-off frequency for Butterworth filter
ω_c = pass-band edge frequency for Chebyshev filter
ω_c = geometric mean $\sqrt{\omega_p \omega_s}$ for elliptic filter (usually)

Transfer function of high-pass filter

In general the HP poles are different from the LP poles since HP pole $= \frac{\omega_c^2}{LPpole}$ (however, for Butterworth filters, the LP and HP poles are the same since the coefficients are the same in reverse order – see example on next page).

Delay characteristics

These do not transform in the same way as the amplitude response under the non-linear $(s \rightarrow \frac{\omega_c^2}{s})$ frequency transformation given on the previous page; for example, the *low-pass* phase lag of a Bessel-Thomson filter is approximately $\omega\tau$ (τ constant). When this filter is transformed to a HPF, we have, instead a phase-*lead* of $\approx \frac{1}{\omega\tau}$, ie. a time advance of $\frac{1}{\omega^2\tau}$, which is non-uniform across the pass-band.

Design of high-pass filter

Design a Butterworth high-pass filter with a 3dB cut-off frequency of 2kHz and at least 30dB attenuation at 500Hz:

1. Translate the specifications to those for the equivalent or "prototype" LPF:

$$\omega_{LP} = \frac{\omega_c^2}{\omega_{HP}} \qquad\qquad \omega_c = 4000\pi \text{ rads/sec}$$

   Let $[\omega_{LP}]_a$ be the 30dB cut-off frequency for the LPF.

$$[\omega_{LP}]_a = \frac{(4 \times 10^3\pi)^2}{10^3\pi} = 16 \times 10^3\pi \text{ rads/sec}$$

2. Design the prototype LPF to meet the required specification.

$$n \simeq \frac{logA}{\log \frac{[\omega_{LP}]_a}{\omega_c}} \simeq \frac{\log 10\sqrt{10}}{\log 4} = 2.49$$

n=3 meets the specification

3. For a 3rd-order Butterworth LPF, $|G(s)|_{LP} = \frac{1}{1+2(\frac{s}{\omega_c})+2(\frac{s}{\omega_c})^2+(\frac{s}{\omega_c})^3}$

$$|G(s)|_{HP} = |G(\frac{\omega_c^2}{s})|_{LP} = \frac{1}{1 + 2(\frac{\omega_c}{s}) + 2(\frac{\omega_c}{s})^2 + (\frac{\omega_c}{s})^3}$$

$$\text{ie.}\quad |G(s)|_{HP} = \frac{(\frac{s}{\omega_c})^3}{1+2(\frac{s}{\omega_c})+2(\frac{s}{\omega_c})^2+(\frac{s}{\omega_c})^3}$$

the same as $|G(s)|_{LP}$ except for 3 zeroes at the origin.

$$\text{Finally,}\quad |G(j\omega)|_{LP} = \frac{1}{\sqrt{1+(\frac{\omega}{\omega_c})^6}} \quad \text{and thus} \quad |G(j\omega)|_{HP} = \frac{1}{\sqrt{1+(\frac{\omega_c}{\omega})^6}}$$

Check: For $\omega = 1000\pi,\quad |G|_{HP} = \frac{1}{\sqrt{1+(4)^6}} = 0.0156$, ie. 36dB attenuation.

3.3 Band-pass filters

We can transform a low-pass filter to a band-pass filter by replacing s in the low-pass transfer function by $s + \frac{\omega_n^2}{s}$ to give the band-pass transfer function (for band-stop filters, $s \rightarrow \frac{s}{s+\omega_n^2}$). ω_n is the geometric mean of the upper and lower cut-off frequencies, ie. $\omega_n = \sqrt{\omega_l \omega_u}$; the BPF has the same *total* bandwidth as the original LPF.

3.3.1 Frequency-response characteristics of band-pass filters

$$|G(j\omega)|_{LP} = \frac{1}{\sqrt{1+H\{(\frac{\omega}{\omega_c})^2\}}}$$

If $s \rightarrow s + \frac{\omega_n^2}{s}$, then $\omega \rightarrow \omega - \frac{\omega_n^2}{\omega}$.

$$\text{Therefore}\quad |G(j\omega)|_{BP} = \frac{1}{[1+H\{(\frac{\omega-\frac{\omega_n^2}{\omega}}{\omega_c})^2\}]^{\frac{1}{2}}}$$

Note that putting $\omega = r\omega_n$ or $r^{-1}\omega_n$ leads to the same G (for any H-function) and so the frequency response of the BPF is symmetrical on a *logarithmic* plot, as indicated in Fig. 3.6. Under the transform $\omega_{LP} = \omega_{BP} - \omega_n^2/\omega_{BP}$ so

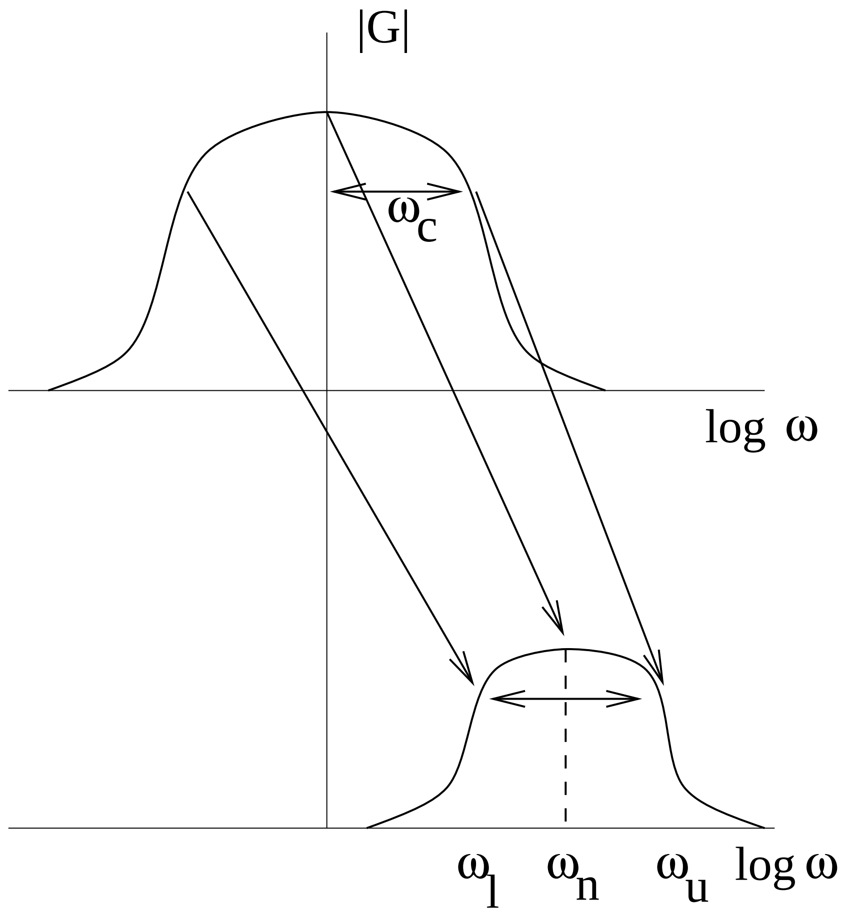

Figure 3.6: *Lowpass to bandpass transform.*

LPF	BPF
0	ω_n
$-\omega_c$	ω_l
ω_c	ω_u
$-\infty$	0

Note: this **does not** mean that $\omega_u - \omega_l = 2\omega_c$, despite what it looks like! In fact, if you work through using the *transform* it is easy to see that $\omega_l - \omega_u = \omega_c$.

Phase characteristics

Band-pass filters derived from low-pass prototypes using $s \rightarrow s + \frac{\omega_n^2}{s}$ have characteristics which are symmetrical (for amplitude) or anti-symmetrical (for phase) about ω_n on a logarithmic scale. Let $G(s)_{LP}$ be the low-pass transfer function; $G(-j\omega)_{LP}$ =complex conjugate of $G(j\omega)_{LP}$, will have the same amplitude but opposite phase.

$$G(s)_{BP} = G(s + \frac{\omega_n^2}{s})_{LP}$$

Consider $s = jr\omega_n$ (factor r *above* centre frequency) and $s = jr^{-1}\omega_n$ (factor r *below* centre frequency).

$$G(jr\omega_n)_{BP} = G\{jr\omega_n + \frac{\omega_n^2}{jr\omega_n}\}_{LP} = G\{j(r - r^{-1})\omega_n\}_{LP}$$

$$G(jr^{-1}\omega_n)_{BP} = G\{jr^{-1}\omega_n + \frac{\omega_n^2}{jr^{-1}\omega_n}\}_{LP} = G\{j(r^{-1} - r)\omega_n\}_{LP} = \{G(jr\omega_n)_{BP}\}^*$$

Therefore these two frequencies get passed with the same amplitude but opposite phase, as can be seen in Fig. 3.7. Note that the component at ω_n is not phase-shifted.

Transfer function of BPFs (approximate treatment)

Let $s = p$ be a pole of $G(s)_{LP}$. Then $s + \frac{\omega_n^2}{s} = p$ is the equation for the poles of $G(s)_{BP}$. i.e.

$$s^2 - ps + \omega_n^2 = 0$$

Poles at

$$s = \frac{1}{2} \pm \sqrt{(\frac{p}{2})^2 - \omega_n^2} = \frac{p}{2} \pm j\omega_n\sqrt{1 - (\frac{p}{2\omega_n})^2}$$

(NB: p is *complex*!)

If $\omega_n >> |p|$, as is often the case (narrow-band filter), poles are at $s \simeq \pm j\omega_n + \frac{p}{2}$

ie. the pole pattern is reproduced at half-scale,
but with the origin transferred to $\pm j\omega_n$

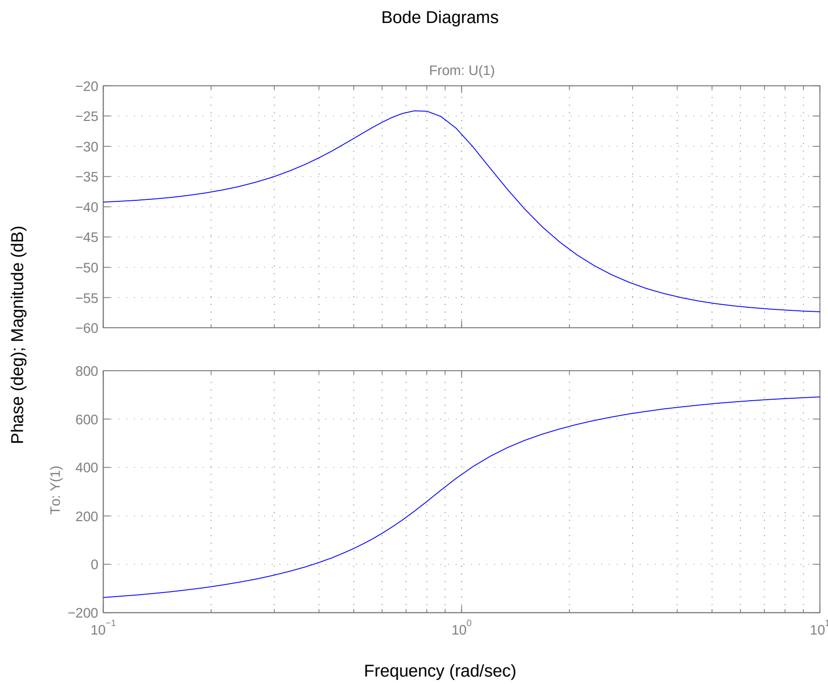

Figure 3.7: *Band-pass filter Bode plot - amplitude (top) and phase (bottom).*

Design of band-pass filters

Design a Butterworth band-pass filter that meets the following specifications:

- lower cut-off frequency f_l = 600 Hz
- upper cut-off frequency f_u = 900 Hz
- Minimum attenuation of 50dB for $f \leq 200$Hz and $f \geq 2700$Hz

Answer

1. Translate the band-pass specifications to those of a low-pass prototype:

$$\omega_c = \omega_u - \omega_l = 1800\pi - 1200\pi = 600\pi \text{ rads/sec}$$

$$\omega_n^2 = \omega_l.\omega_u = (1200\pi).(1800\pi) \quad \text{ie} \;\; \omega_n = 1470\pi \text{ rads/sec}$$

For $\omega = 400\pi$ or 5400π, $\omega_{LP} = \omega_{BP} - \frac{\omega_n^2}{\omega_{BP}} = 5400\pi - \frac{1210.1800\pi^2}{5400\pi}$

ie. $\omega_{LP} = 5000\pi$ rads/sec (or f = 2500Hz)

2. Design the prototype LPF to meet the required specification.

$$n \simeq \frac{logA}{log\frac{[\omega_{LP}]_a}{\omega_c}} = \frac{log100\sqrt{10}}{log\frac{5000\pi}{600\pi}} = 2.71$$

n = 3 meets the specification

Therefore, we have $|G(s)|_{LP} = \frac{1}{1+2(\frac{s}{\omega_c})+2(\frac{s}{\omega_c})^2+(\frac{s}{\omega_c})^3}$

$$|G(s)|_{BP} = |G(s+\frac{\omega_n^2}{s})|_{LP} = \frac{1}{1+2(\frac{s^2+\omega_n^2}{s\omega_c})+2(\frac{s^2+\omega_n^2}{s\omega_c})^2+(\frac{s^2+\omega_n^2}{s\omega_c})^3}$$

i.e. $$|G(s)|_{BP} = \frac{ks^3}{s^6+a_5s^5+a_4s^4+a_3s^3+a_2s^2+a_1s+a_0}$$

$$\begin{array}{lll} where & k = \omega_c^3 & \\ & a_0 = \omega_n^6 & a_3 = \omega_c^3 + 4\omega_c\omega_n^2 \\ & a_1 = 2\omega_c\omega_n^4 & a_4 = 2\omega_c^2 + 3\omega_n^2 \\ & a_2 = 2\omega_c^2\omega_n^2 + 3\omega_n^4 & a_5 = 2\omega_c \end{array}$$

3.4 Conclusion

The frequency transformations, although highly non-linear, are appropriate for transforming the standard low-pass filters (except for Bessel-Thomson filters) to high-pass, band-pass and band-stop filters. This is because the frequency response of the filters being transformed is an approximation to a piecewise constant characteristic in the frequency bands of interest. Thus, taking the example of the elliptic low-pass filter of Fig. 3.8, the non-linearity of the mapping affects the spacing of the ripple peaks and valleys but has no effect on the amplitude of these ripples. Therefore, the filters designed using this frequency transformation preserve the equiripple character of the prototype filter. Figure 3.9 shows Butterworth Bode plots for low-pass, high-pass and band-pass filters. All are $n = 5$ filters.

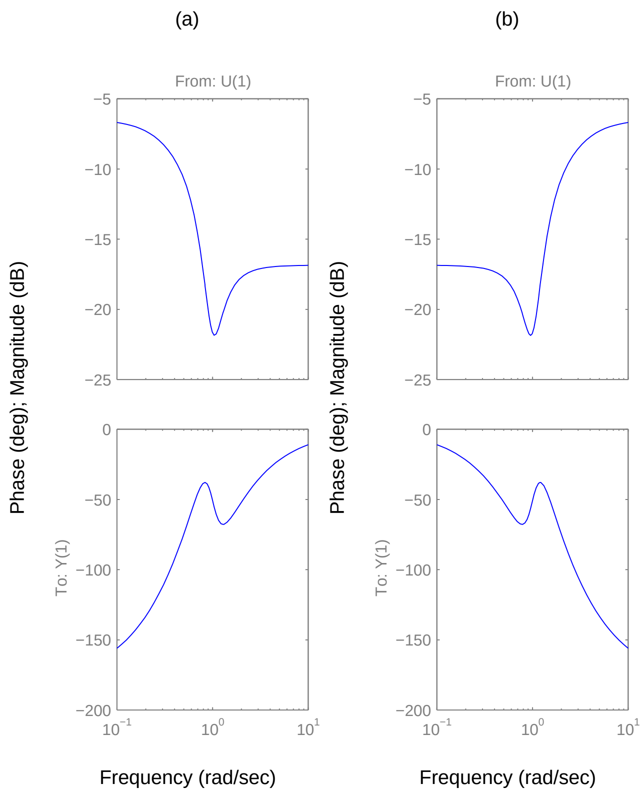

Figure 3.8: *Elliptic filters, $n = 3$. (a) lowpass, (b) highpass.*

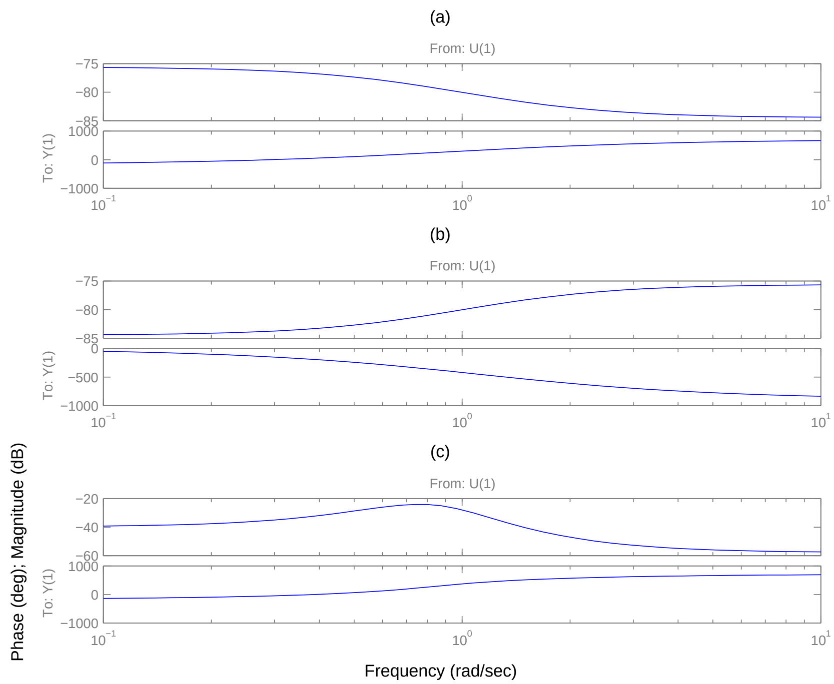

Figure 3.9: *Butterworth filters,* $n = 5$*. (a) lowpass, (b) highpass & (c) bandpass.*