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Short communication

A systematic procedure for deriving RC networks of fractional-order elements emulators using MATLAB

Georgia Tsirimokou

Electronics Laboratory, Physics Department, University of Patras, Rio Patras GR 26504, Greece

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ABSTRACT

The behavior of a fractional-order capacitors can be approximated using appropriately configured RC networks, which implement an integer-order impedance/admittance function. In this paper, the Foster I and II as well as the Cauer I and II structures are systematically derived using the MATLAB software. In addition, the emulation of fractional-order inductors using RC emulators of fractional-order capacitors and Generalized Impedance Converter is performed. The behavior of the derived RC networks is evaluated and compared through a design example, using the OrCAD PSpice software, in order to choose the most appropriate among them.

1. Introduction

Fractional-order calculus has gained significant research interest due to its interdisciplinary nature. It finds application in biochemistry, medicine, electrical engineering etc. For example, the modeling of viscoelasticity, of biological cells and tissues has been performed through the utilization of the fractional-order calculus. Also, fractional-order filters and oscillators offer attractive benefits with regards to their integer-order counterparts [1].

Fractional-order capacitors, also known as Constant Phase Elements (CPEs), are the most important components for realizing fractional-order circuits. Although a significant effort has been performed by researches for implementing such kind of elements [2–5], these are not still commercially available devices. As a result, a way for implementing fractional-order circuits is the substitution of CPEs by appropriately configured RC networks which approximate their behavior [6–11]. Following this approach, a number of fractional-order circuits have been published in the literature [12–22], where various kinds of RC network topologies have been utilized. Another important element for performing fractional-order signal processing is the fractional-order inductor (FI). Its emulation could be performed through the combination of a CPE emulator and a Generalized Impedance Converter (GIC).

The main contributions made in this manuscript are the following: a) The design equations for deriving RC networks that emulate CPEs are presented in a systematic way, b) A performance comparison among the derived RC networks is done in order to evaluate their benefits, c) A systematic way for deriving emulators of fractional-order inductors (FIs) is also presented, and d) The MATLAB code, provided in the Appendix, which could be a powerful tool for an electronics engineer in order to derive the values of the passive components for emulating both CPEs and FIs.

The paper is organized as follows: the technique for approximating a CPE using conventional integer-order impedance/admittance function is presented in Section 2. The Foster and Cauer networks [23,24] for emulating CPEs are given in Section 3, where the procedure for calculating their component values is also discussed. The corresponding procedure for emulating a FI is given in Section 4. The performance of the presented RC networks is evaluated and compared in Section 5, through the OrCAD PSpice software.

2. Approximation of fractional-order capacitor

The impedance of a CPE is described by the following *s*-domain equation:

$$Z(s) = \frac{1}{C_a s^a} \tag{1}$$

The variable a ($0 < a < 1$) in (1) is the order of the CPE, while C_a is (normalized) capacitance expressed in Farad/sec^{1-a}(F/sec^{1-a}). Thus, the value of the capacitance (C) of a CPE (in Farad), at a specific frequency (ω_o), will be calculated using the formula in (2)

E-mail address: tsirimg@upatras.gr

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$$C = \frac{C_a}{\omega_o^{1-a}} \tag{2}$$

Therefore, the capacitance of a CPE is a variable dependent on both frequency and order. The phase (θ) of the impedance is equal to $-a\pi/2$.

There are several methods for obtaining rational approximations of the variable s^a, such as the Carlson's method, Matsuda's method, Oustaloup's method and the continued fraction expansion method [25–28]. According to [9], the last one is an attractive choice optimized in terms of phase and gain error for approximating the variable s^a around a specific frequency ω_o. The corresponding expression is:

$$s^a \cong \omega_o^\alpha \cdot \frac{a_n s^n + a_{n-1}\omega_o s^{n-1} + \cdots + a_1\omega_o^{n-1}s + a_o\omega_o^n}{b_n s^n + b_{n-1}\omega_o s^{n-1} + \cdots + b_1\omega_o^{n-1}s + b_o\omega_o^n} \tag{3}$$

where n is the order of the approximation and ω_o the center frequency where the approximation is performed.

Using (1)-(3), the derived expression for the impedance/admittance of CPE will be:

$$Z(s) = \frac{1}{C\omega_o} \cdot \frac{b_n s^n + b_{n-1}\omega_o s^{n-1} + \cdots + b_1\omega_o^{n-1}s + b_o\omega_o^n}{a_n s^n + a_{n-1}\omega_o s^{n-1} + \cdots + a_1\omega_o^{n-1}s + a_o\omega_o^n} \tag{4a}$$

$$Y(s) = C\omega_o \cdot \frac{a_n s^n + a_{n-1}\omega_o s^{n-1} + \cdots + a_1\omega_o^{n-1}s + a_o\omega_o^n}{b_n s^n + b_{n-1}\omega_o s^{n-1} + \cdots + b_1\omega_o^{n-1}s + b_o\omega_o^n} \tag{4b}$$

3. Foster and Cauer networks for fractional-order capacitor approximation

The expressions in (4) can be approximated by the Foster networks, demonstrated in Figs. 1a-b.

According to [23,24] the expression of the impedance $Z(s)$ of the Foster I network in Fig. 1a, constructed from passive resistors R_i ($i = 0, \ldots, n$) and capacitors C_j ($j = 1, \ldots, n$), is

$$Z(s) = R_o + \sum_{i=1}^{n} \frac{\frac{1}{C_i}}{s + \frac{1}{R_i C_i}} \tag{5}$$

while the expression for the admittance $Y(s)$ of the Foster II network in Fig. 1b is

$$Y(s) = \frac{1}{R_o} + \sum_{i=1}^{n} \frac{\frac{1}{R_i}s}{s + \frac{1}{R_i C_i}} \tag{6}$$

Therefore, the values of passive elements of the network in Fig. 1a can be calculated by equating the coefficients of (5) with those derived from the partial fraction expansion of the $Z(s)$ in (4a), which is given by the general form in (7)

$$Z(s) = k + \sum_{i=1}^{n} \frac{r_i}{s - p_i} \tag{7}$$

with k and r_i being constant terms, and p_i the poles of the impedance. The derived design equations are expressed by (8)

$$\begin{aligned} R_o &= k \\ C_i &= 1/r_i \quad (i = 1 \cdots n) \\ R_i &= \tfrac{1}{C_i |p_i|} \end{aligned} \tag{8}$$

The values of passive elements in the Foster II realization will be calculated by utilizing the partial fraction expansion of the $Y(s)/s$. Using (6), the derived expression is provided in (9)

$$\frac{Y(s)}{s} = \frac{k}{s} + \sum_{i=1}^{n} \frac{r_i}{s - p_i} \tag{9}$$

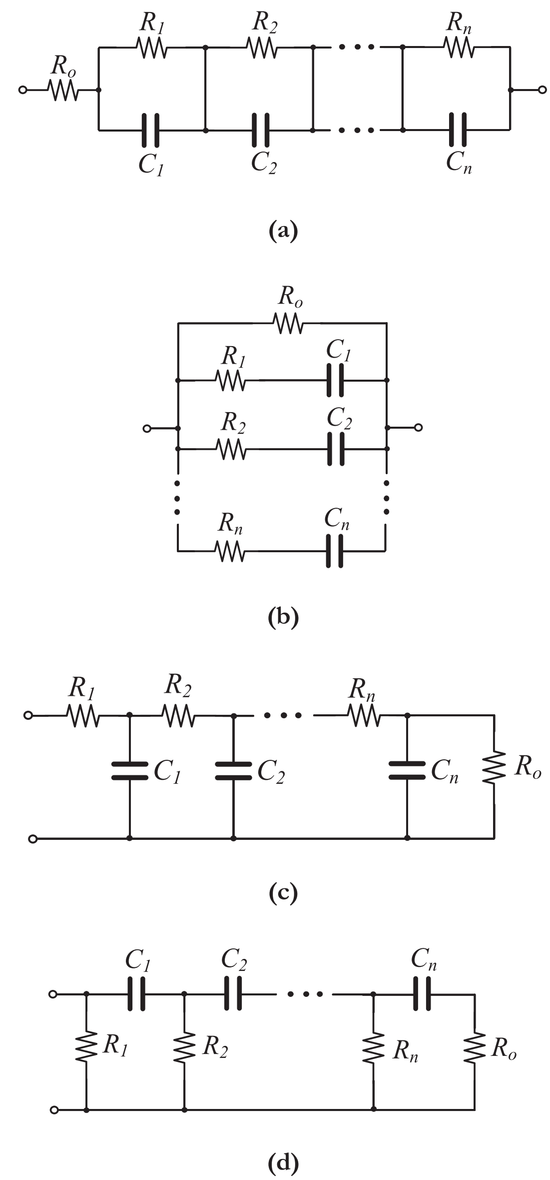

Fig. 1. RC networks for CPE emulation **(a)** Foster I, **(b)** Foster II, **(c)** Cauer I, and **(d)** Cauer II.

Using (4b) and (9), the calculation of the values of passive elements case of the RC network in Fig. 1b will be performed by the set of expressions in (10)

$$\begin{aligned} R_o &= 1/k \\ R_i &= 1/r_i \quad (i = 1 \cdots n) \\ C_i &= \tfrac{1}{R_i |p_i|} \end{aligned} \tag{10}$$

For the Cauer I network in Fig. 1c the expressions of impedance/admittance are:

$$Z(s) = R_1 + \frac{1}{C_1 s +} \frac{1}{R_2 +} \cdots \frac{1}{R_n +} \frac{1}{C_n s +} \frac{1}{R_o} \tag{11}$$

Assuming that both numerator and denominator in (4a) are arranged in the descending powers of s and starting form the highest to lowest power of s, the derived continued fraction expansion of $Z(s)$ will have the form in (12)

$$Z(s) = q_{r1} + \frac{1}{q_{c1}s +} \frac{1}{q_{r2} +} \cdots \frac{1}{q_{rn} +} \frac{1}{q_{cn}s +} \frac{1}{q_{ro}} \tag{12}$$

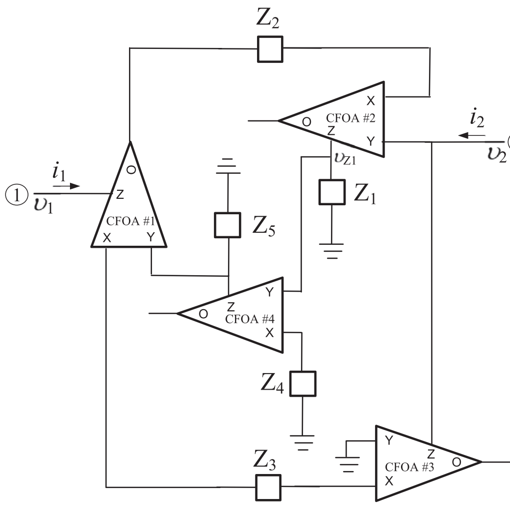

Fig. 2. Generalized Impedance Converter topology using CFOAs.

Table 1
Coefficient expressions of the impedance and admittance polynomials for $n = 5$.

Coefficient	*Expression*
$a_5 = b_o$	$-a^5 - 15a^4 - 85a^3 - 225a^2 - 274a - 120$
$a_4 = b_1$	$5a^5 + 45a^4 + 5a^3 - 1005a^2 - 3250a - 3000$
$a_3 = b_2$	$-10a^5 - 30a^4 + 410a^3 + 1230a^2 - 4000a - 12000$
$a_2 = b_3$	$10a^5 - 30a^4 - 410a^3 + 1230a^2 + 4000a - 12000$
$a_1 = b_4$	$-5a^5 + 45a^4 - 5a^3 - 1005a^2 + 3250a - 3000$
$a_o = b_5$	$a^5 - 15a^4 + 85a^3 - 225a^2 + 274a - 120$

where q_{ri} ($i = 0,\ldots,n$) and q_{cj} ($j = 1, \ldots, n$) are the coefficients of the continued fraction expansion.
Therefore, according to (11) and (12), the design equations are these in (13)

$$\begin{matrix} R_i = q_{ri} \\ C_j = q_{cj} \end{matrix} \quad (i = 0 \ldots n \text{ and } j = 1 \ldots n) \tag{13}$$

The impedance of the Cauer II network in Fig. 1d is:

$$Z(s) = \frac{1}{\frac{1}{R_1} +} \frac{1}{\frac{1}{C_1 s} +} \frac{1}{\frac{1}{R_2} +} \cdots \frac{1}{\frac{1}{R_n} +} \frac{1}{\frac{1}{C_n s} +} \frac{1}{\frac{1}{R_o}} \tag{14}$$

In the case that both numerator and denominator in (4b) are arranged in the ascending powers of s and starting from the lowest to highest power of s, in order to avoid negative terms, the derived continued fraction expansion of $Z(s)$ will have the form in (15)

$$Z(s) = \frac{1}{q_{r1} +} \frac{1}{\frac{q_{c1}}{s} +} \frac{1}{q_{r2} +} \cdots \frac{1}{q_{rn} +} \frac{1}{\frac{q_{cn}}{s} +} \frac{1}{q_{ro}} \tag{15}$$

Comparing the expressions in (14) and (15), the derived design equations will be given by (16)

$$\begin{matrix} R_i = 1/q_{ri} \\ C_j = 1/q_{cj} \end{matrix} \quad (i = 0 \ldots n \text{ and } j = 1 \ldots n) \tag{16}$$

4. Approximation of fractional-order inductor

The realization of a FI emulator could be performed by the floating GIC structure in Fig. 2, which is realized using Current Feedback Operational Amplifiers (CFOAs) as active elements [29]. The implemented equivalent impedance (Z_{eq}) between nodes 1–2 is

Table 2
Passive element values for approximating a CPE with $C = 1$nF at 1 kHz ($a = 0.2$, 0.5, and 0.8) using the Foster I topology.

Element	$a = 0.2$	$a = 0.5$	$a = 0.8$
R_o	63.44 kΩ	14.47 kΩ	2.12 kΩ
R_1	40.89 kΩ	31.43 kΩ	10.00 kΩ
R_2	31.35 kΩ	40.89 kΩ	20.73 kΩ
R_3	34.74 kΩ	67.48 kΩ	50.98 kΩ
R_4	54.62 kΩ	167.68 kΩ	213.87 kΩ
R_5	174.23 kΩ	1.43 MΩ	11.66 MΩ
C_1	244.97 pF	436.55 pF	1.80 nF
C_2	1.74 nF	1.61 nF	3.80 nF
C_3	5.13 nF	3.14 nF	4.96 nF
C_4	11.15 nF	4.55 nF	4.60 nF
C_5	24.80 nF	5.39 nF	1.86 nF

Table 3
Passive element values for approximating a CPE with $C = 1$nF at 1 kHz ($a = 0.2$, 0.5, and 0.8) using the Foster II topology.

Element	$a = 0.2$	$a = 0.5$	$a = 0.8$
R_o	399.28 kΩ	1.75 MΩ	11.96 MΩ
R_1	145.38 kΩ	17.73 kΩ	2.17 kΩ
R_2	463.72 kΩ	151.06 kΩ	118.44 kΩ
R_3	729.15 kΩ	375.39 kΩ	496.82 kΩ
R_4	807.89 kΩ	619.49 kΩ	1.22 MΩ
R_5	619.43 kΩ	805.87 kΩ	2.53 MΩ
C_1	40.33 pF	185.58 pF	536.29 pF
C_2	89.64 pF	219.74 pF	217.23 pF
C_3	194.77 pF	318.33 pF	201.41 pF
C_4	575.66 pF	622.04 pF	263.35 pF
C_5	4.08 nF	2.29 nF	556.27 pF

Table 4
Passive element values for approximating a CPE with $C = 1$nF at 1 kHz ($a = 0.2$, 0.5, and 0.8) using the Cauer I topology.

Element	$a = 0.2$	$a = 0.5$	$a = 0.8$
R_o	100.28 kΩ	903.28 kΩ	8.24 MΩ
R_1	63.44 kΩ	14.47 kΩ	2.12 kΩ
R_2	59.20 kΩ	74.20 kΩ	89.20 kΩ
R_3	55.82 kΩ	142.46 kΩ	360.86 kΩ
R_4	56.69 kΩ	232.41 kΩ	951.59 kΩ
R_5	63.85 kΩ	383.89 kΩ	2.31 MΩ
C_1	200.70 pF	275.00 pF	563.52 pF
C_2	1.27 nF	670.31 pF	372.71 pF
C_3	3.80 nF	1.15 nF	355.35 pF
C_4	9.24 nF	1.85 nF	375.11 pF
C_5	23.92 nF	3.35 nF	469.45 pF

$$Z_{eq} = \frac{Z_2 Z_3 Z_4}{Z_1 Z_5} \tag{17}$$

Setting $Z_1 = R_{a1}$, $Z_2 = R_{a2}$, $Z_3 = R_{a3}$, $Z_4 = R_{a4}$, and $Z_5 = 1/C_\beta s^\beta$ in (17), then

$$Z_{eq} = \frac{R_{a2} R_{a3} R_{a4}}{R_{a1}} C_\beta s^\beta \tag{18}$$

Taking into account that the impedance of an FI of order β ($0 < \beta < 1$) is given by (19) as

$$Z(s) = L_\beta s^\beta \tag{19}$$

where L_β is the (normalized) inductance expressed in Henry/sec$^{1-\beta}$(H/sec$^{1-\beta}$), then the scheme in Fig. 2 will emulate a fractional-inductor with pseudo-inductance given by the expression in (20)

$$L_{\beta,eq} = \frac{R_{a2} R_{a3} R_{a4}}{R_{a1}} C_\beta \tag{20}$$

Table 5
Passive element values for approximating a CPE with C = 1nF at 1 kHz (a = 0.2, 0.5, and 0.8) using the Cauer II topology.

Element	a = 0.2	a = 0.5	a = 0.8
R_o	252.61 kΩ	28.04 kΩ	3.07 kΩ
R_1	399.28 kΩ	1.75 MΩ	11.95 MΩ
R_2	427.84 kΩ	341.39 kΩ	284.01 kΩ
R_3	453.82 kΩ	177.81 kΩ	70.19 kΩ
R_4	446.81 kΩ	108.99 kΩ	26.62 kΩ
R_5	396.68 kΩ	65.98 kΩ	10.95 kΩ
C_1	4.98 nF	3.64 nF	1.77 nF
C_2	789.34 pF	1.49 nF	2.68 nF
C_3	263.17 pF	867.98 pF	2.81 nF
C_4	108.24 pF	539.17 pF	2.67 nF
C_5	41.80 pF	298.71 pF	2.13 nF

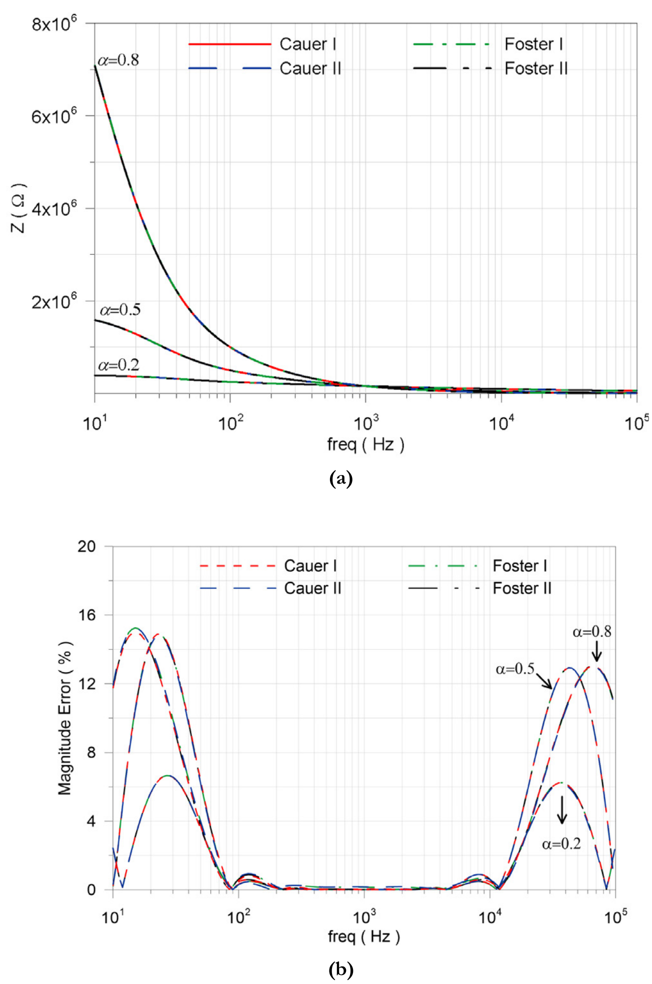

Fig. 3. **(a)** Impedance magnitude responses of the Foster I-II, and Cauer I-II realizations for a = 0.2, 0.5, and a = 0.8, **(b)** magnitude error plots.

It should be mentioned at this point that the value of the inductance (L) of a FI (in Henry), at a specific frequency (ω_o), will be calculated using the formula in (21)

$$L = \frac{L_\beta}{\omega_o^{1-\beta}} \tag{21}$$

5. Simulation and comparison results

A CPE with capacitance C = 1 nF at 1 kHz (i.e. $|Z|$ = 159.15 kΩ) and order a = 0.2, 0.5, and 0.8 (i.e. $\theta = -18^o$, -45^o, and -72^o) will be approximated using the networks in Fig. 1. The fifth-order continued fraction expansion approximation will be employed for this purpose (i.e. n = 5) and the values of coefficients a_i (i = 0. . .5) and b_i (i = 0...5) in (3) are summarized in Table 1 [30].

The calculation of the passive elements values can be performed using MATLAB software. In order to calculate the values of passive elements of the Foster I network in Fig. 1a from the design equations in (8), the following command of MATLAB will be used: [*r p k*] = residue([*num*], [*den*]), where [*num*] and [*den*] are the coefficient matrixes of the numerator and denominator in (4a).

The calculation of the values in the case of the Foster II network in Fig. 1b from (10), will be performed through the partial fraction expansion of the $Y(s)/s$, and therefore the corresponding MATLAB command is: [*r p k*] = residue([*num*], [*den* 0]), where *num* and *den* are the coefficient matrix of the numerator and denominator in (4b).

The passive element values in the case of the Cauer I network in Fig. 1c, will be calculated using the FOMCON toolbox for MATLAB [31]. The employed command is: [*q*] = polycfe([*num*], [*den*]), where *num* and *den* are the coefficient matrix the numerator and denominator in (4a). Having available the values of q_i, the passive element values are calculated from (13).

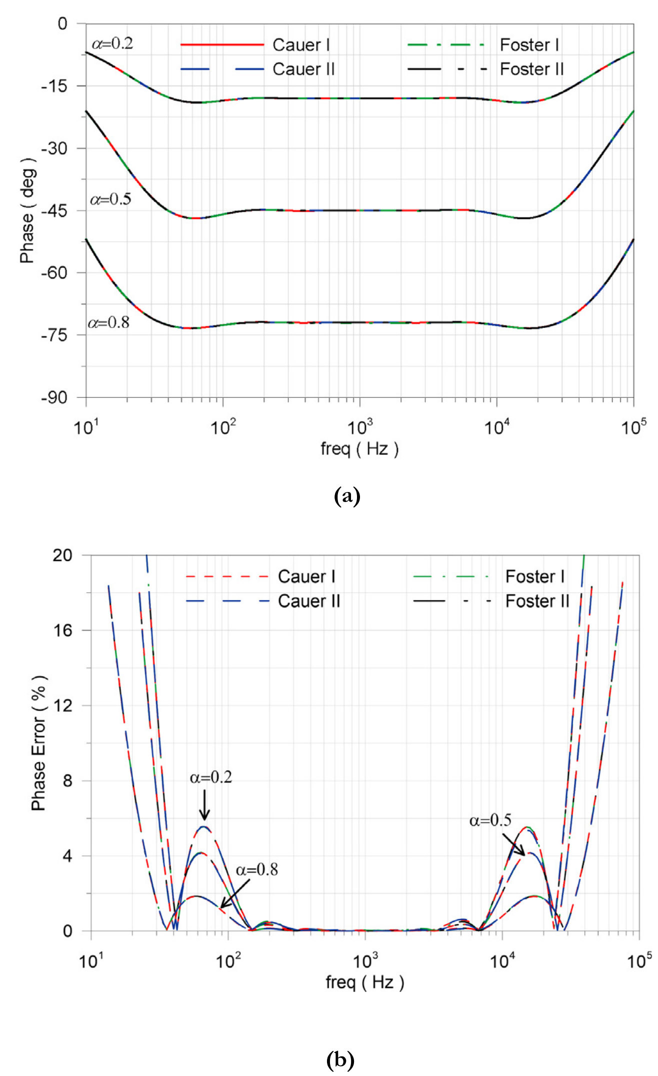

Fig. 4. **(a)** Impedance phase responses of the Foster I-II, and Cauer I-II realizations for a = 0.2, 0.5, and a = 0.8, **(b)** error phase plots.

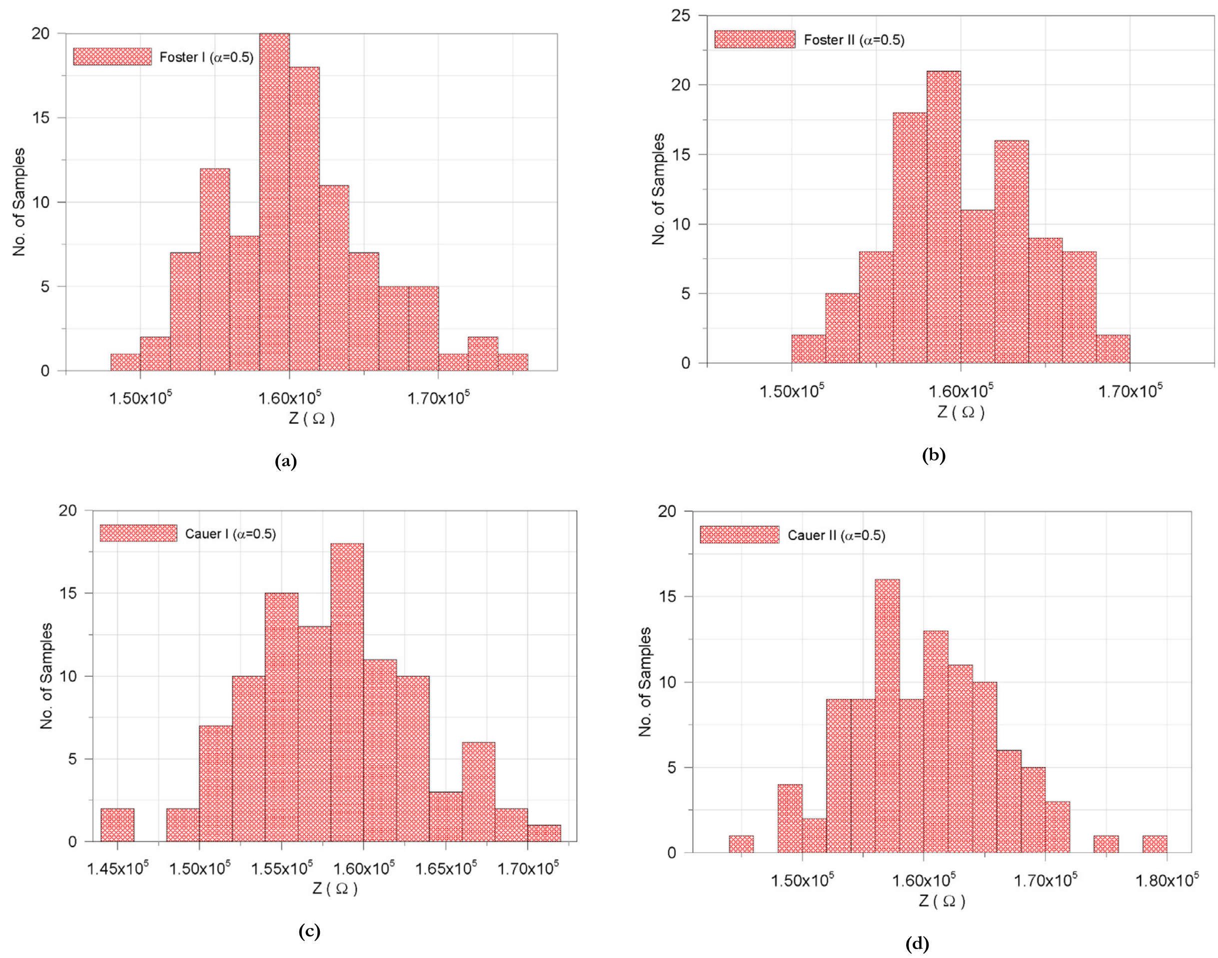

Fig. 5. Monte-Carlo results of the impedance magnitude (a = 0.5) **(a)** Foster I, **(b)** Foster II, **(c)** Cauer I, and **(d)** Cauer II realizations.

In the case of the Cauer II network given in Fig. 1d, the correspondingMATLAB command is: [q] = polycfe([*num_comp*], [*den_comp* 0]) where *num_comp* and *den_comp* are the coefficient matrixes of the numerator and denominator in (4b), when they are arranged in the ascending powers of s. As a second step, the design equations in (16) will be utilized for calculating the element values.

Following the procedures early described the calculated values of passive elements for Foster and Cauer realizations, for a = 0.2, 0.5, and 0.8, are summarized in the Tables 2–5, respectively. In order to facilitate the reader, the complete MATLAB code is provided in the Appendix A.

The impedance magnitude responses obtained using the OrCAD PSpice simulator, are depicted in Fig. 3a, while the corresponding error plots are given in Fig. 3b. The corresponding ranges of approximation with a 5% accuracy, for a = 0.2, 0.5, and 0.8, are (41 Hz–25.2 kHz), (54 Hz–18.6 kHz), (44 Hz–23.1 kHz), respectively. The phase responses are demonstrated in Fig. 4a; from the provided error plots it is derived that the corresponding ranges for 5% accuracy are (77 Hz–13.1 kHz), (32 Hz–30.1 kHz), and (23 Hz–42.3 kHz).

Assuming that the tolerance of both resistors and capacitors is equal to 10%, the sensitivity performance of the RC networks has been studied using the Monte-Carlo analysis offered by the OrCAD PSpice. In Figs. 5 and 6 the corresponding plots for impedance magnitude and phase at a = 0.5 are given. The simulated values of the standard deviation of the magnitude and phase of the impedance at the center frequency (i.e. 1 kHz) were (23.1 kΩ, 0.93°), (23.7 kΩ, 0.89°), (24 kΩ, 0.91°), and (22.9 kΩ, 0.87°), for Foster I, Foster II, Cauer I, and Cauer II, respectively.

The performance of the RC networks in Fig. 1 will be compared through the utilization of the results in Table 6. According to these, the Foster I and Cauer I realizations offer the minimum total resistance; in addition, the Cauer I offers the minimum resistor value spread among them. On the other hand, the Foster II realization offers the minimum total capacitance value. Also, all the topologies under comparison offer almost the same level of sensitivity with regards to the magnitude and phase of the impedance. Thus, according to the demand, the designer could choose between the Foster II (min. capacitance) and Cauer I (min. resistance) schemes. It should be also mentioned that both of them offer the capability for employing grounded capacitors in the case that a grounded CPE will be emulated, minimizing the effect of parasitics.

As a next step, the realization of a FI with inductance L = 10 mH at 1 kHz (i.e. $|Z|$ = 62.8Ω) and order β = 0.2, 0.5, and 0.8 (i.e. phase 18°, 45°, and 72°). Assuming that R_{a1} = 10 kΩ, R_{a2} = R_{a3} = R_{a4} = 1 kΩ, then using (8) and (9) the corresponding values of the pseudo-capacitance will be 109.28 μF/sec$^{0.8}$, 7.93 μF/sec$^{0.5}$, and 0.57 μF/sec$^{0.2}$, respectively. Employing a Foster II structure for emulating the pseudo-capacitance, the element values of the corresponding RC network are summarized in Table 7. The required CFOAs will be simulated by employing the AD844 discreet IC component model offered by the OrCAD PSpice [32]. The obtained magnitude and phase responses of the impedances are demonstrated in Figs. 7 and 8, respectively, where for β = 0.2, 0.5, and 0.8, the correspond-

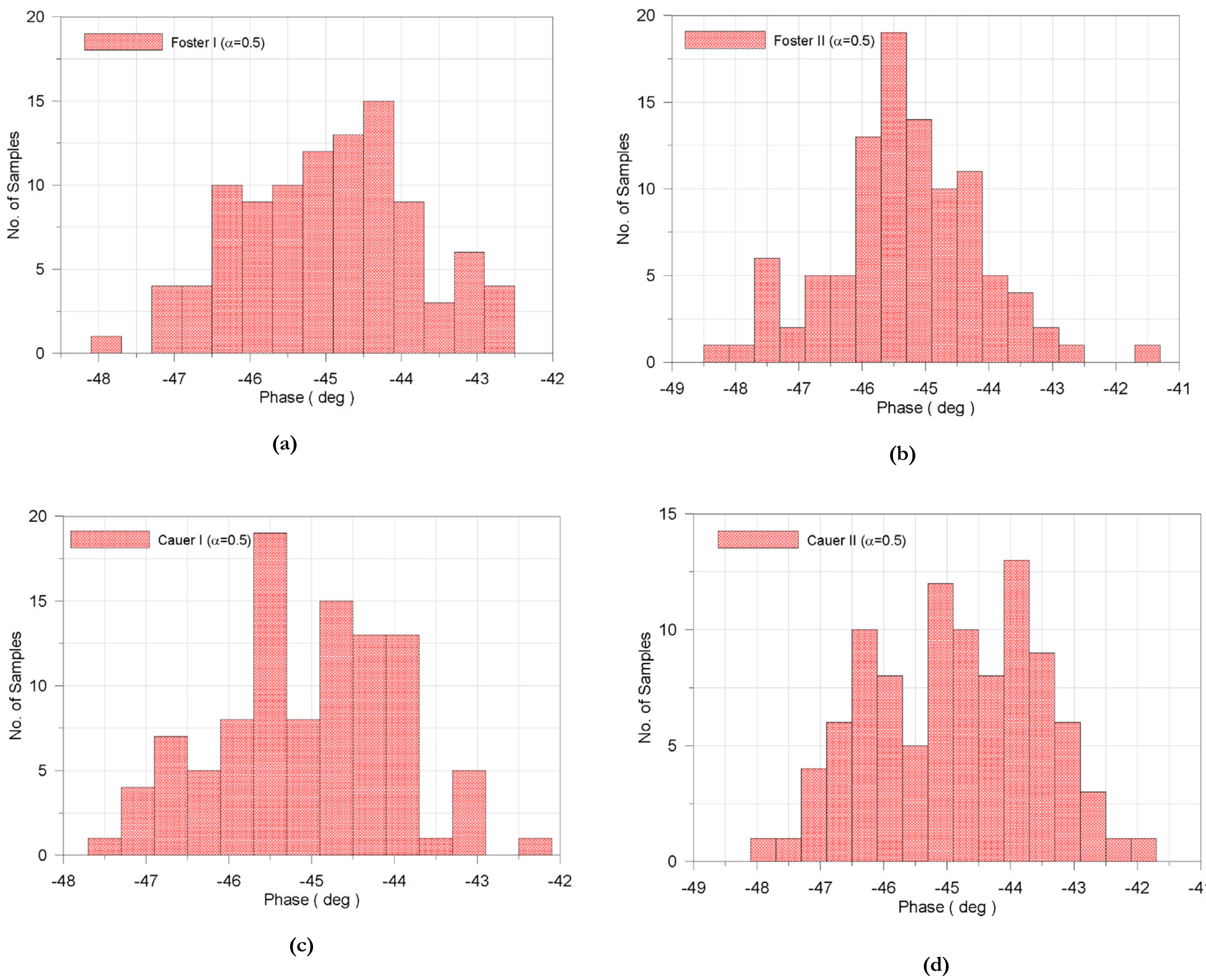

Fig. 6. Monte-Carlo results of the impedance phase (a = 0.5) **(a)** Foster I, **(b)** Foster II, **(c)** Cauer I, and **(d)** Cauer II realizations.

Table 6
Performance characteristics of the RC networks in Fig. 1.

Element	*Foster I*	*Foster II*	*Cauer I*	*Cauer II*
Total resistance (a = 0.2)	399.3 kΩ	3.16 MΩ	399.3 kΩ	2.38 MΩ
Total resistance (a = 0.5)	1.75 MΩ	3.72 MΩ	1.75 MΩ	2.47 MΩ
Total resistance (a = 0.8)	11.96 MΩ	16.33 MΩ	11.96 MΩ	12.34 MΩ
Resistance spread (a = 0.2)	5.6	5.6	1.8	1.8
Resistance spread (a = 0.5)	98.8	98.8	62.4	62.4
Resistance spread (a = 0.8)	5500	5510	3887	3892
Total capacitance (a = 0.2)	43.06 nF	4.98 nF	38.43 nF	47.94 nF
Total capacitance (a = 0.5)	15.13 nF	3.63 nF	7.29 nF	6.83 nF
Total capacitance (a = 0.8)	17.02 nF	1.77 nF	2.13 nF	12.06 nF
Capacitance spread (a = 0.2)	101.6	102	119.6	124.5
Capacitance spread (a = 0.5)	12.4	12.4	12.2	12.2
Capacitance spread (a = 0.8)	2.7	2.7	1.6	1.6
Stand. dev. of impedance (a = 0.2)	2.2 kΩ	1.8 kΩ	2.3 kΩ	6.9 kΩ
Stand. dev. of impedance (a = 0.5)	5.1 kΩ	4.1 kΩ	5.2 kΩ	6.0 kΩ
Stand. dev. of impedance (a = 0.8)	7.9 kΩ	8.6 kΩ	11.8 kΩ	9.5 kΩ
Stand. dev. of phase (a = 0.2)	0.6°	0.5°	0.5°	1.2°
Stand. dev. of phase (a = 0.5)	1.1°	1.2°	1.1°	1.4°
Stand. dev. of phase (a = 0.8)	1.1°	1.1°	1.4°	1.3°

ing ranges of approximation with a ±5% accuracy are the same as in the case of CPE.

6. Conclusion

The emulation of the impedance of a CPE, using Foster and Cauer RC networks, has been exploited in this work. Using a fifth-order continued fraction expansion of the impedance expression the emulation of a CPE with ±5% accuracy in both magnitude and phase within the range $[\omega_o/13, 13\omega_o]$, $[\omega_o/19, 19\omega_o]$, and $[\omega_o/23, 23\omega_o]$, for a = 0.2, 0.5, and 0.8, is possible, where ω_o is the frequency of interest. The provided results of comparison show that the Foster II and Cauer I structures are the most suitable in the sense that the first one offers minimum total capacitance, while

Table 7
Passive element values for approximating a FI with L = 10 mH at 1 kHz (β = 0.2, 0.5, and 0.8) using the Foster II topology and R_{a1} = 10 kΩ, R_{a2} = R_{a3} = R_{a4} = 1 kΩ).

Element	β = 0.2	β = 0.5	β = 0.8
R_o	3.99 kΩ	17.51 kΩ	119.58 kΩ
R_1	1.45 kΩ	0.18 kΩ	0.022 kΩ
R_2	4.64 kΩ	1.51 kΩ	1.18 kΩ
R_3	7.29 kΩ	3.75 kΩ	4.97 kΩ
R_4	8.10 kΩ	6.19 kΩ	12.22kΩ
R_5	6.19 kΩ	8.06 kΩ	25.31 kΩ
C_1	4.03 nF	18.56 nF	53.63 nF
C_2	8.96 nF	21.97 nF	21.73 nF
C_3	19.50 nF	31.83 nF	20.14 nF
C_4	57.60 nF	62.20 nF	26.33 nF
C_5	408.20 nF	229.10 nF	55.63 nF

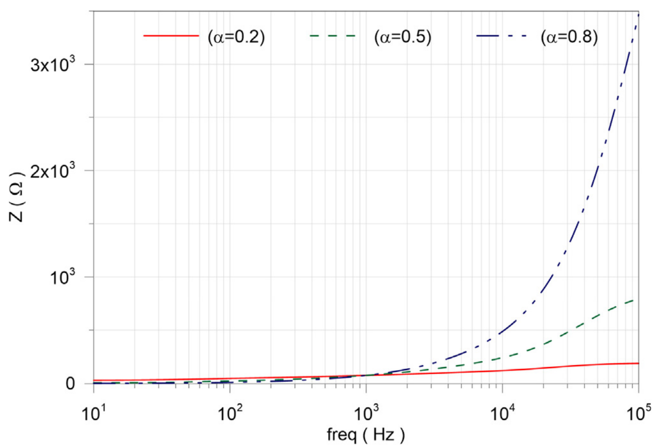

Fig. 7. Impedance magnitude responses of FI with β = 0.2, 0.5, and a = 0.8.

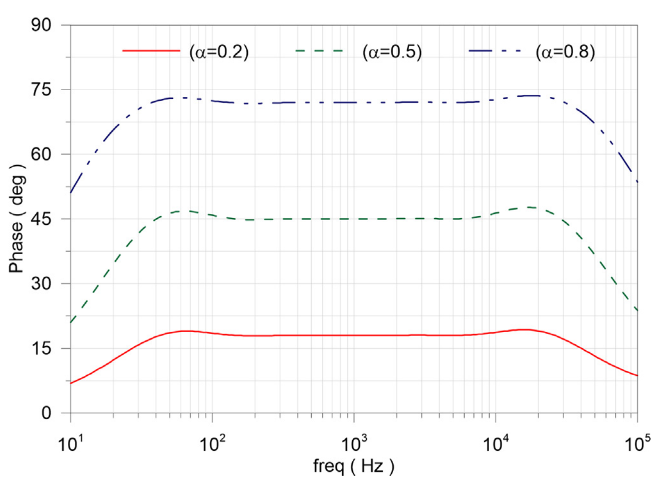

Fig. 8. Impedance phase responses of FI with β = 0.2, 0.5, and a = 0.8.

the second one minimum total resistance. Both of them have reasonable sensitivity characteristics. In addition, emulation of FIs of various orders using an appropriately configured GIC scheme demonstrated the facilitation of the design procedure presented in this paper.

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Appendix A. MATLAB code for calculating the passive element values of RC networks that emulate CPEs.

```
%% Specifications
a=0.5;
cap=1E-09;
fo=1000;
wo=2*pi*fo;
imped=1/(cap*wo);
%% Coefficients of the second-order continued fraction
expansion
a5=(-(a^5)-15*(a^4)-85*(a^3)-225*(a^2)-274*a-120);
a4=wo*(5*(a^5)+45*(a^4)+5*(a^3)-1005*(a^2)-3250*a-3000);
a3=(wo^2)*(-10*(a^5)-30*(a^4)+410*(a^3)+1230*(a^2)-400
0*a-12000);
a2=(wo^3)*(10*(a^5)-30*(a^4)-410*(a^3)+1230*(a^2)+4000
*a-12000);
a1=(wo^4)*(-5*(a^5)+45*(a^4)-5*(a^3)-1005*(a^2)+3250*a-
3000);
ao=(wo^5)*((a^5)-15*(a^4)+85*(a^3)-225*(a^2)+274*a-120);
b5=(imped)*((a^5)-15*(a^4)+85*(a^3)-225*(a^2)+274*a-120);
b4=(imped)*wo*(-5*(a^5)+45*(a^4)-5*(a^3)-1005*(a^2)+32
50*a-3000);
b3=(imped)*(wo^2)*(10*(a^5)-30*(a^4)-410*(a^3)+1230*(a
^2)+4000*a-12000);
b2=(imped)*(wo^3)*(-10*(a^5)-30*(a^4)+410*(a^3)+1230*(
a^2)-4000*a-12000);
b1=(imped)*(wo^4)*(5*(a^5)+45*(a^4)+5*(a^3)-1005*(a^2)-
3250*a-3000);
bo=(imped)*(wo^5)*(-(a^5)-15*(a^4)-85*(a^3)-225*(a^2)-2
74*a-120);
```

```
%% Foster I form
disp(' ########################################
####')
disp(' Foster I form ')disp(' ######################
###################')
% Impedance of CPE
num=[b5 b4 b3 b2 b1 bo];
den=[a5 a4 a3 a2 a1 ao];
% Partial Fractional Expansion for the impedance (Foster I)
[r p k]=residue(num,den);
% Calculation of passive element values for Foster I (rzero, ri,
ci)
rzero_fi=k(1:1)
for n=[1,2,3,4,5];
r_fi=r(n:n)/abs(p(n:n))
c_fi=1/r(n:n)
end
```

```
%% Foster II form
disp(' ########################################
####')
disp(' Foster II form ')
disp(' ########################################
####')
% Impedance of CPE
num=[b5 b4 b3 b2 b1 bo 0];
den=[a5 a4 a3 a2 a1 ao];
% Partial Fractional Expansion for the admittance (Foster II)
[r p ]=residue(den,num);
% Calculation of passive element values for Foster II (rzero, ri,
ci)
rzero_fii=1/r(6:6)
for n=[1,2,3,4,5];
r_fii=1/r(n:n)
c_fii=r(n:n)/abs(p(n:n))
end
```

```
%% Cauer I form
disp(' ########################################
####')
disp(' Cauer I form ')
disp(' ########################################
####')
% Impedance of CPE
num=[b5 b4 b3 b2 b1 bo];
den=[a5 a4 a3 a2 a1 ao];
% CFE for the impedance (descending order)
[q,expr]=polycfe(num,den);
% Calculation of resistors values for Cauer I (rzero and ri)
rzero_ci=q{11}
for n=[1,3,5,7,9]
r_ci=q{n}
end
% Calculation of capacitors values for Cauer I (ci)
for m=[2,4,6,8,10]
c_ci=q{m}(1:1)
end
```

```
%% Cauer II form
disp(' ########################################
####')
disp(' Cauer II form ')
disp(' ########################################
####')
% Impedance of CPE
num=[bo b1 b2 b3 b4 b5];
den=[ao a1 a2 a3 a4 a5 0];
% CFE for the impedance (ascending order)
[q,expr]=polycfe(num,den);
% Calculation of resistors values for Cauer II (rzero and ri)
rzero_cii=1/q{11}(1:1)
for n=[1,3,5,7,9]
r_cii=1/q{n}(1:1)
end
% Calculation of capacitors values for Cauer II (ci)
for m=[2,4,6,8,10]
c_cii=1/q{m}(1:1)
end
```

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