Distributional Impacts of a Local Living Wage Increase with Ability Sorting*

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Abstract

I develop a theoretical model of ability sorting of low-wage workers across multiple markets when one of the markets substantially increases its wage floor, in this context, by installing a living wage. The initial distribution and response of workers and firms to an increase in the wage floor depends on the difference in employment probabilities across markets. An increase in the living wage in the local market can yield positive or negative growth in the probability of employment in the covered and uncovered markets. Because employment levels change in both markets, using the uncovered market as a control actually underestimates the positive employment effects of the living wage in the covered market. Regardless of average effects in both markets, welfare implications are similar to the classical analysis: workers who most want the living wage jobs are hurt by the increase in the wage floor as they are "priced out" of the living wage market and are forced to move to the lower paying sector.

Keywords: Living wage, search, multiple markets, ability sorting

JEL J6, J3

1 Introduction

Starting 1994 in Baltimore, the living wage movement in the United States has spread rapidly, with more than 140 living wage ordinances enacted in cities. These ordinances usually require firms doing business with the city government (or in some cases, the city government itself) to pay its workers a wage that is 50 % to 150 % higher than the federal minimum wage. By raising the wage floor so drastically, advocates argue, all workers with full-time jobs should be able to meet basic living needs to pull his or her family out of poverty.

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As living wage ordinances aim to achieve the same impact as sharp minimum wage increases, one could expect economic outcomes to be similar. Economists have traditionally been skeptical of the prospects of the effectiveness of high wage floors. Adopting the classical minimum wage theory (See Mincer (1976), among many others), the 'covered' living wage sector of affected businesses should decrease employment and push a substantial number of workers into the 'uncovered' minimum wage (or no/lower wage floor) sector, increasing the unemployment level.[1] While workers in the covered sector who earn the new higher wage will benefit, all other workers are hurt as a result.

While the lack of high-quality large-scale data sources has made empirical investigation of living wage ordinances difficult, the studies that have been done yield surprisingly mixed results. Some studies have found that employment in the living wage sector actually increases. Other studies have found increased displacement due to these laws. Wage gains have been observed in some studies and unobserved in others. (See Brenner (2005) and Neumark and Adams (2003, 2005), among others) In fact, this disagreement on the impact of what should be a straightforward application of price floors mirrors the controversy surrounding the impact of a minimum wage increase.[2]

Living wage coverage within the city is not universal. Since the city can only realistically mandate living wage to firms it is directly doing business with, the 'covered' and 'uncovered' sectors are operating within the same city. This geographic proximity of covered and uncovered workers and firms suggests a quasi-experimental approach to measuring the impact, with the covered workers as treatment and uncovered workers as control. The geographic proximity should help to control for many observable and unobservable factors, with only the treatment creating the economically important impact. Indeed, many labor wage studies use adjacent cities or industries with one serving as a control and the other serving as the treatment group. (See Abadie (2005), Angrist and Krueger (1999), and Meyer (1995) for a review of the quasi-experimental literature.)

While this set-up is valid if the markets being compared are isolated from each other, the very geographic or industrial proximity that makes these markets good candidates for quasi-experimental analysis points to the fact that firms and workers should be able to migrate to the adjacent sector with relative ease if better opportunities exist. Proximity, which makes the covered and uncovered sectors useful for comparison of the impact of a structural difference in a policy, also implies that the two markets are virtually interchangeable, and workers who are employable and firms that can operate in one market should be able to change locations or industries with relative ease. When

[1] While the focus in this paper is on living wages, the framework discussed is also relevant to any large disparity in wage floor levels in multiple markets, such as minimum wage levels in the European Union. As of 2010, Luxembourg has the highest minimum wage at 1,683 EUR and Bulgaria has the lowest floor at 123 EUR, an almost 14-fold difference. (Normalized to the same purchasing power across the two nations reduces the difference to 6-fold.)

[2] While considerable controversy still exists (See Neumark and Wascher (2000)), the evidence for strong negative employment effect predicted by traditional economic theory is weak.

firms and workers are mobile, the interpretation of the empirical findings are confounded and must be re-examined. Relative employment change (or lack of change) after one sector experiences a policy change may be driven, at least in part, by interaction with the uncovered sector. In the case of living wage, gains in employment levels comparing differences across two sectors may not be indicative of an actual increase (or decrease) in the treatment sector, but a redistribution of workers and firms across the sectors.

In this paper, I look at the impact of a local living wage ordinance on the covered sector as well as the adjacent uncovered sector. I extend a two-sided search model with endogenous labor supply and labor demand from Ahn et al. (2009) to allow for 1) two labor market sectors and 2) ability sorting. While the classical model forces employment to fall when the wage floor rises above market clearing levels, in the search model, the number of matches (employment level) increases with the number of searching workers and the number of searching firms. An increase in either may result in an overall increase in employment levels.

With multiple markets or sectors, workers and firms are allowed to locate optimally to maximize their expected returns yielding more complex results. A living wage ordinance may drive away some workers into and may attract other workers from the uncovered sector. Firms may open or close vacancies in either market to maximize expected profits. Some workers will be induced to move, and some will stay because workers are differentiated by ability. Workers and firms search for each other, and conditional on matching, generate revenue that is dependent on worker ability. When there is no living wage (or if the living wage does not bind due to worker ability), the worker and the firm split the revenue according to a simple Rubenstein bargaining game. When the living wage ordinance is enacted, it serves as a wage floor in the sense that it "prices out" low-ability workers, as firms that match with these workers will reject the match. Allowing these unwanted workers to move to the uncovered sector changes the match probability and the expected match revenue in both markets. Worker expected income is driven by two elements: ability of the worker and probability of employment. As the worker knows his own ability level and the wage structure in each sector, he know what his wage will be in either sector, conditional on matching. The probability of employment will depend on the initial conditions in each sector, as well as worker and firm responses to the wage floor increase.

In particular, I find that if the covered sector initially has the higher probability of employment compared to the uncovered sector, a living wage ordinance that further raises the wage floor will have ambiguous employment effects in the covered sector but increase employment in the uncovered sector. That is, conditions exist where both sectors may experience positive growth in employment levels and probabilities of employment in the presence of a local living wage ordinance. In contrast, if the covered sector has the lower probability of employment, an increase in the wage floor due

to a living wage ordinance will lead to lower employment levels and probability of employment in the covered sector and increase employment levels and ambiguous change in probability of employment in the uncovered sector. In both cases, the one segment of the working population that is unequivocally hurt is the low-ability workers originally in the covered sector.

The next section presents the two-sector search model. Section 3 provides comparative statics and numerical simulation results, showing how a wage floor hike in sector drives employment and welfare results in both sectors. Section 4 concludes.

2 The model

In this section I present a two-sided search model which is designed to highlight the effect of a city instituting a living wage ordinance. There are two sectors, with Sector A initially having a higher wage floor compared to Sector B. Firms in Sector A contract with the city and are therefore subject to the living wage, whereas firms in Sector B are not. For notational simplicity in the model, I normalize wage floor in Sector B to zero. However, a model in which the two sectors have different wage floors above zero does not qualitatively change the model. There would now be a segment of workers with ability level $\delta_i < \min\{\underline{W}_A, \underline{W}_B\}$, who would be unemployable in either sector. There are $\overline{N}$ total potential workers. Workers decide which sector to locate in to search for a low wage job.

I assume that workers are risk neutral and are solely interested in maximizing their expected income. Workers are differentiated by their ability, δ_i, $\delta \in [0, 1]$. Workers in Sector A, numbering N_A, and Sector B, numbering N_B, search for firms, and firms search in each sector for workers. Searching workers in Sector A and Sector B sum up to $\overline{N}$. Entry by firms is endogenous with the number of vacancies posted in Sector A and Sector B denoted as J_A and J_B, respectively. Firms are identical. As in Pissarides (1992), the number of matches in Sector k is Cobb-Douglas on the interior and given by

$$x_k = \min\{\gamma J_k^{\alpha} N_k^{1-\alpha}, J_k, N_k\} \tag{1}$$

where $\alpha \in (0, 1)$ and γ is a normalizing constant. All workers within a sector has the same probability of finding a match, implying that $P_k = x_k/N_k$.[3] Similarly, all firms within a sector have the same probability of matching, given by $q_k = x_k/J_k$.

When a firm and worker i match, a gross revenue value equal to the ability of worker i, δ_i is generated. The firm and worker split the match according to a Rubenstein bargaining game where

[3] All workers, regardless of ability level, have the same probability of matching. Qualitative results remain unchanged if higher ability workers are more likely to match. See the discussion section.

the discount factor may vary for the firm and the worker.[4] The worker's share of the revenue is denoted as β, $\beta \in (0,1)$. In Sector B, which does not have a wage floor, the wage of worker i with ability δ_i conditional on a match is $\beta\delta_i$, and the firm's revenue is $(1-\beta)\delta_i$.

In Sector A, a successful match must pay at least the initial wage floor, $\underline{W}$. A unique subgame perfect equilibrium exists that yields the following expression for wages in Sector A:

$$W_i = \max\{\beta\delta_i, \underline{W}\} \tag{2}$$

Workers search in the sector that will maximize their expected wages.

The expected zero profit conditions for firms operating in Sector A and Sector B are then:

$$q_A E(\max\{\min\{(1-\beta)\delta_i, \delta_i - \underline{W}\}, 0\}|A) - C = 0 \tag{3}$$

$$q_B(1-\beta)E(\delta_i|B) - C = 0 \tag{4}$$

where $E(\delta|k)$ is the expected value of the revenue of the match conditional on being located in Sector k. Firms in Sector A will reject matches where $\delta_i < W_i$.

I bound C, the cost of posting a vacancy between $0 < \underline{C} \leq C \leq \overline{C} < 1$. This ensures that the labor markets of the two sectors exist. Too low a cost of vacancy in one sector will mean that firms can enter virtually costlessly, leading to the costlier sector emptying as both firms and workers flock to the low-cost sector.[5] .

There exist two scenarios to consider.

1. When $P_A > P_B$: the living wage sector has the higher probability of employment.

2. When $P_B > P_A$: the living wage sector has the lower probability of employment.

As I show below, the difference in the probability of employment implies different initial distributions of workers and firms in Sector A and Sector B. When $P_A > P_B$, the covered sector has the tighter labor market. When $P_B > P_A$, the opposite holds. Given these differences in the probabilities and the existence of a wage floor in one sector, workers will sort themselves into the sector that gives them the highest expected income. Assuming uniform distribution of ability, $\delta \sim U(0,1)$ across the population of workers, the following initial distribution of firms and workers emerge.

2.1 When $P_A > P_B$

When the probability of employment is higher in the covered sector, all workers move to Sector A with the exception of workers with ability below $\underline{W}$. This creates a separating equilibrium, as firms

[4] See Ahn, Arcidiacno, and Wessels (2009) for the rigorous derivation.

[5] I assume that $C_A = C_B = C$ for simplicity, however, qualitative results remain the same if $C_A \neq C_B$. For instance, the covered sector may have higher profits due to a contract with the city, expressed as lower vacancy posting cost.

in Sector A that match with low ability workers will find it optimal to reject the match. Workers with ability level between $\underline{W}$ and $\frac{\underline{W}}{\beta}$ will receive $\underline{W}$ from firms in Sector A. All other workers receive $\beta\delta_i$ conditional on matching. See Figure 1.

2.2 When $P_B > P_A$

When the probability of employment is higher in the uncovered sector, all workers move to Sector B with the exception of workers with ability level between $\underline{W}$ and δ^*, where δ^* represents the worker who is just indifferent between locating in Sector A (receiving the higher wage in return for lower probability of matching) or in Sector B. That is, there is a worker indifferent between Sector A and Sector B, defined by her ability level δ^*, where:

$$P_A\underline{W} = P_B\beta\delta^* \tag{5}$$

Workers with ability level near δ^* trade off between the higher wage to be gain from Sector A (due to the wage floor) with the higher probability of employment from Sector B.[6] See Figure 2.

Given the set up described above, a sub-game perfect equilibrium exists in both $P_A > P_B$ and $P_B > P_A$ scenarios.

Proposition 1 *Given equations (1) - (5), parameter vector* $\{\underline{W},\ \overline{N},\ C_A,\ C_B,\ \beta\}$*, and* $\delta \sim U(0,1)$ *there exists a unique sub-game perfect equilibrium in* $\{N_A,\ N_B,\ J_A,\ J_B\ \}$

3 Comparative Statics

As the initial distribution of workers differed depending on which sector initially had the better labor market, the response by workers and firms to a living wage ordinance will also depend on which sector has the higher probability of employment initially. Because workers are maximizing expected wages, the probability of matching with a firm is critical in his location decision. When the wage floor is raised, the primary impact in the local, covered sector is to "price out" workers who have ability levels below $\underline{W}$, as firms will reject the match. These workers must then move to the adjacent, uncovered sector.

When these workers move to the adjacent sector, they change the ability distribution in both sector, which changes the expected revenue draw for firms operating in these sectors. Firms enter or leave as expected revenue increases or decreases. This spurs further movement of other workers.

3.1 When $P_A > P_B$

Proposition 2 *When there is an increase in the wage floor in Sector A:*

[6] When $P_A = P_B$, I assume that $P_B > P_A$ initial distribution holds.

1. *Conditions exist where the employment level (x_A) and probability of employment (P_A) can increase in Sector A.*

2. *Employment level (x_B) and probability of employment (P_B) increase in Sector B.*

3. *Low-ability workers originally in Sector A are hurt.*

Refer back to Figure 1 for an illustration of the impact of a wage floor hike in Sector A when $P_A > P_B$. The impact of a wage floor increase in Sector A on P_A and x_A will depend critically on β and $\underline{W}$. A firm considering opening a vacancy will look at the probability of matching with a worker and the amount of profit generated for the firm, conditional on matching.[7] A large β will decrease the profit generated for the firm which results in a lower P_A (level effect). In addition, I show in the appendix that if β is greater than a critical value, the worker retains too much of the match revenue, pushing firms out of Sector A. This leads to a decline in P_A and x_A as $\underline{W}$ increases (slope effect). On the other hand, if β is smaller than the critical value, the loss in match probability is more than compensated by the gain in expected ability draw, actually attracting more firms.

A living wage ordinance in Sector A necessarily decreases the number of workers in Sector A and increases the expected ability draw of a match in Sector A. These two factors pull expected profit for firms operating in Sector A in opposite directions. A firm stays in the market despite a smaller pool of workers if conditional on a match, the revenue the firm gets is large enough. At relatively low levels of the wage floor, an increase in $\underline{W}$ can result in an increase in P_A because the number of workers that receive the wage floor, those with ability $\underline{W} \geq \delta_i \geq \frac{\underline{W}}{\beta}$, is small. For the firm, an increase in the small chance that it will match with these low-ability worker is more than offset by the increase in the *expected* productivity of the match. When $\underline{W}$ is relatively large, however, the proportion of workers that receive the wage floor conditional on matching increases. This translates to a decline in the expected revenue that the firm can take, and eventually, entry is deterred. This is demonstrated in a numerical simulation in Figure 3. The case where P_A and x_A decrease may seem closely related to the classical analysis of the minimum wage, but this is occurring simultaneously as N_A *decreases*.

A wage floor hike in Sector A leads to relatively high ability workers entering Sector B, with no worker exit. This means that N_B and expected ability draw in Sector B increase, which lead to a larger increase in J_B compared to N_B. This leads to an increase in P_B. Unlike the classical covered-uncovered sectors model of the minimum wage, workers originally in the "uncovered sector," Sector B, actually benefit from the excess labor supply created when workers from Sector A are pushed into their sector. The new workers have higher ability compared to the original workers in Sector

[7] Note that q_A can be transformed into a function with the inverse of P_A, such that a high value of P_A implies a low value of q_A.

B. This leads to more firms opening vacancies in Sector B due to the higher expected revenue. The resulting increase in P_B due to the increase in J_B outweighs decline in P_B due to the increase in N_B.

Using $x_A - x_B$ to examine relative employment gains and losses in Sector A is illustrated in a numerical simulation in Figure 4. Since x_B increases in $\underline{W}$, $x_A - x_B$ is not a true measure of the change in Sector A employment level with Sector B employment level serving as a control. Using x_B actually underestimates the increase in x_A.

The difference in employment levels ($x_A - x_B$) is non-monotonic in $\underline{W}$, indicating that observing increasing relative employment in Sector A under a particular wage floor does not necessarily indicate that a similar increase at another level of the wage floor will yield the same result. In particular, as the bargaining power of the worker decreases (that is, when the wage floor would affect larger portions of the working population), the decline in relative employment level is steeper and starts at a lower level of the wage floor.

Denoting the new living wage as $\widehat{\underline{W}} > \underline{W}$, and the new probability of employment in Sectors A and B as $\widehat{P_A}$ and $\widehat{P_B}$, respectively, when Sector A raises its wage floor, workers originally in Sector B will benefit, since relatively high-ability workers (from Sector B's perspective) from Sector A move to Sector B, driving up the expected ability draw, leading to more job openings. Because $\widehat{P_B} > P_B$, workers originally in Sector B (and stay) are made better off. In Sector A, high-ability workers with $\delta_i \geq \frac{\widehat{W}}{\beta}$ who remain in Sector A may be better off if $\widehat{P_A} > P_A$. Workers with $\delta_i < \frac{W}{\beta}$ may benefit if $\widehat{P_A}\widehat{\underline{W}} \geq P_A\underline{W}$, and workers with $\frac{W}{\beta} \leq \delta_i \leq \frac{\widehat{W}}{\beta}$ may benefit if $\widehat{P_A}\widehat{\underline{W}} \geq P_A\beta\delta_i$. Low-ability workers with $\underline{W} \leq \delta_i \leq \widehat{\underline{W}}$, who are forced to move from Sector A to Sector B are made worse off.

3.2 When $P_B > P_A$

Proposition 3 *When there is an increase in the wage floor in Sector A:*

1. *Employment level (x_A) and probability of employment (P_A) decrease in Sector A.*
2. *Employment level (x_B) increases and conditions exists where the probability of employment (P_B) decreases in Sector B.*
3. *Low-ability workers originally in Sector A are hurt.*

Refer again to Figure 2 for an illustration of the impact of a wage floor hike in Sector A when $P_A < P_B$. When the living wage ordinance is enacted, workers of higher ability located to Sector A, as δ^* increases, and lower ability workers are pushed out to Sector B. The net result is that N_A decreases and the expected ability draw increases. The increase in expected ability is not enough to overcome the increase in the wage floor and the decrease in N_A, leading to firm exits in Sector

A. This translates to a decline in x_A, the number of matches, and P_A, the probability of a worker finding a job in Sector A.

When the new wage is instituted in Sector A, Sector B accepts an influx of relatively low ability workers from and loses relatively high ability workers to Sector A. While the expected revenue from the match declines, the increase in match probability from the increase in N_B is enough to entice entry by firms, leading to an increase in J_B. As both J_B and N_B increases, the number of matches, x_B, increases. However, because the rate of firm entry will depend on β, P_B can increase or decrease, as is illustrated in a numerical simulation in Figure 5.

Low-ability workers from Sector A are forced into Sector B as in the previous case, and high-ability workers from Sector B move to Sector A to take advantage of the higher wage floor. As $\widehat{P_B} < P_B$, low-ability workers with $\delta_i < \underline{W}$ are made worse off. Let $\widehat{\delta^*}$ denote the new cut-off ability level at which workers move to Sector B. High-ability workers with $\delta_i > \widehat{\delta^*}$ in Sector B are made worse off. Low-ability workers originally in Sector A with $\underline{W} \leq \delta_i \leq \widehat{\underline{W}}$, who move to Sector B are made worse off because they are forced to accept a lower wage. Workers paid the wage floor who remain in Sector A and collect $\widehat{\underline{W}}$ may benefit if $\widehat{P_A}\widehat{\underline{W}} \geq P_A\underline{W}$. Workers with $\delta_i \leq \widehat{\delta^*}$ move from Sector B to Sector A when the living wage ordinance is in effect, and this segment of the population may benefit if $P_B\beta\delta_i < \widehat{P_A}\widehat{W}$.

4 Discussion

In this paper, I examined the employment level and probability impacts of a wage floor hike in a multiple markets setting, specifically in the context of a living wage ordinance. The analysis shows that the difference in probability of employment across the sectors or markets is critical in determining the initial distribution of workers and wages as well as the impact of a wage floor hike. As demonstrated in the previous section, a local living wage ordinance will have complex employment consequences in both sectors, with changes in probabilities of employment, expected wages, and the movement of workers and firms. Whether workers in the covered and uncovered sectors benefit or are hurt by the living wage law is determined critically by the relative difference in employment probabilities in the two sectors.

The results are more nuanced than the classical minimum wage analysis, which usually results in lower employment levels and higher wage in the "covered" sector, and higher employment and lower wages in the "uncovered" sector. When the sector with higher wage floor (Sector A) has the lower unemployment rate, it forces its low-ability workers into the adjacent sector (Sector B). This has the effect of increasing the expected ability draw of both sectors, which can raise the employment level and the employment rate in both sectors. When Sector B has the lower unemployment rate,

the effect of a wage floor hike in Sector A has mostly negative consequences for workers in Sector A and ambiguous impact on workers in Sector B.

In the model, all workers have the same ability to match with a vacancy, regardless of ability. It is likely to be the case that higher ability workers have an easier time matching with firms. If this were to be incorporated into the model, the qualitative conclusions would not change, except to exacerbate the welfare loss of low ability workers. Assume for simplicity that the probability of worker i with ability level δ_i matching in Sector k is $P_{k,i} = \gamma(\delta_i) J_k^{\alpha} N_k^{-\alpha}$, where $\gamma(\delta_i)$ is an appropriately scaled normalizing term which is increasing in δ_i. Plugging in these new probabilities into the equilibrium condition (equation (5)) would not change the solution, as the γ_i term on both sides would cancel out. That is, there still exists one unique δ^*. All workers with $\delta_i < \delta^*$ ($\delta_i > \delta^*$) would now have $P_{k,i} < P_k$ ($P_{k,i} > P_k$). Therefore, the degree of change in expected income for both high and low ability workers would increase, through differences in the probabilities of matching.

In addition, because of the mobility of the workforce across the sectors, examining the gap between employment levels may actually *underestimate* the positive employment impact of a living wage ordinance. In both cases, the employment level in Sector B (x_B) increases as the wage floor in Sector A increases. If $x_A - x_B$ is observed to be increasing (or decreasing) in $\underline{W}$, it must be the case that x_A is increasing by more (or decreasing by less) than the difference.

While positive employment effects in Sector A may be observed, workers initially in the covered sector with low ability are unequivocally hurt when living wage law is enacted. They are pushed out of the sector where they were earning more than the market share of their ability since, $\underline{W} > \beta\delta_i$. Ultimately, the intuition that a wage floor hike hurts those who most need the additional income holds in a multiple market setting with endogenous labor and firm mobility.

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5 Appendix A: Figures

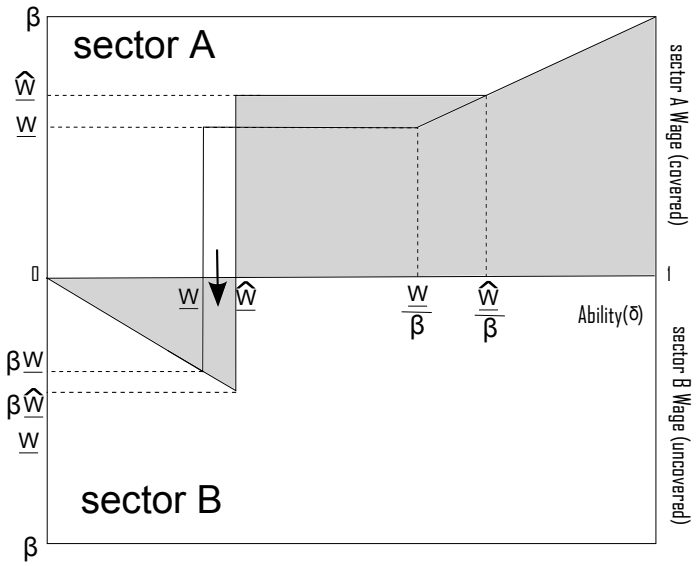

Figure 1: Distribution of workers and wages when $P_A > P_B$

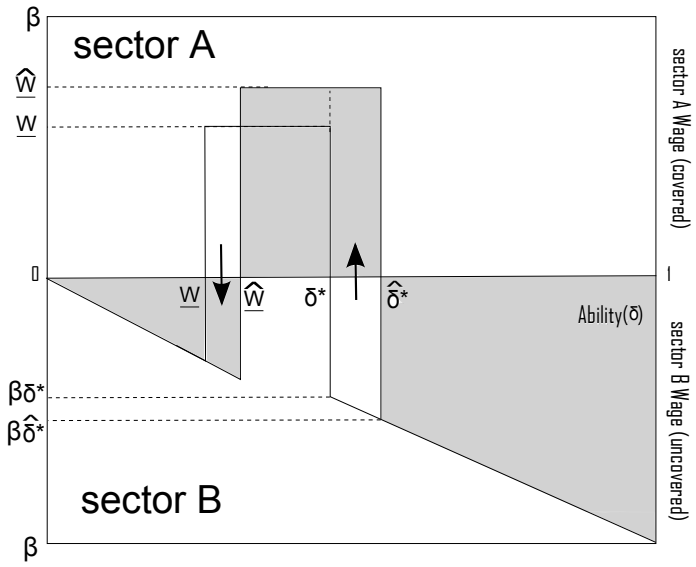

Figure 2: Distribution of workers and wages when $P_B > P_A$

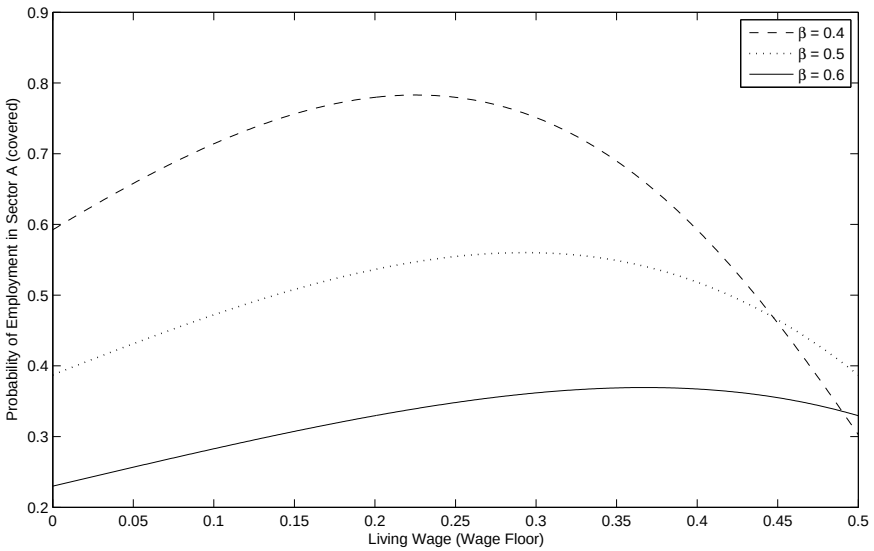

Figure 3: When $P_A > P_B$: Relationship between P_A and $\underline{W}$

Parameter values used in simulation: $\alpha = 0.7$, $\gamma = 0.45$, $C_A = C_B = 0.12$, $\overline{N} = 100$.

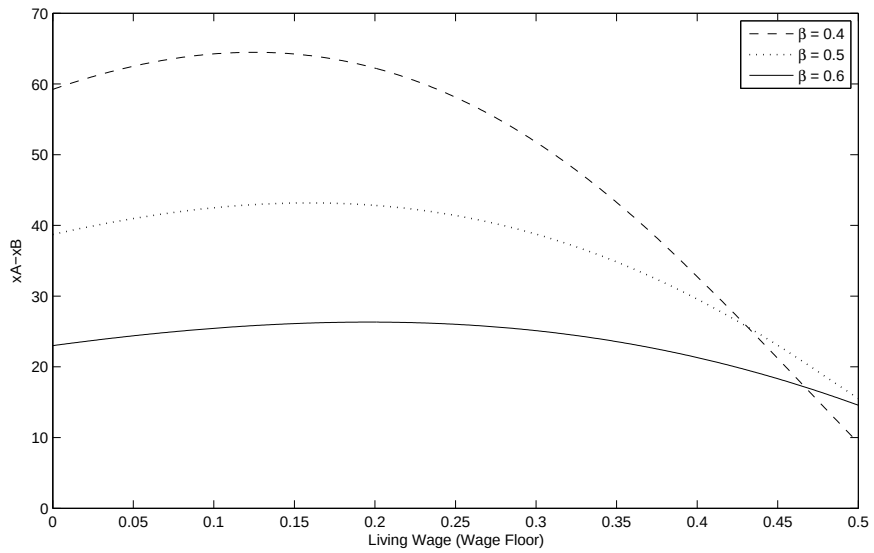

Figure 4: When $P_A > P_B$: Relationship between $x_A - x_B$ and $\underline{W}$

Parameter values used in simulation: $\alpha = 0.7$, $\gamma = 0.45$, $C_A = C_B = 0.12$, $\overline{N} = 100$.

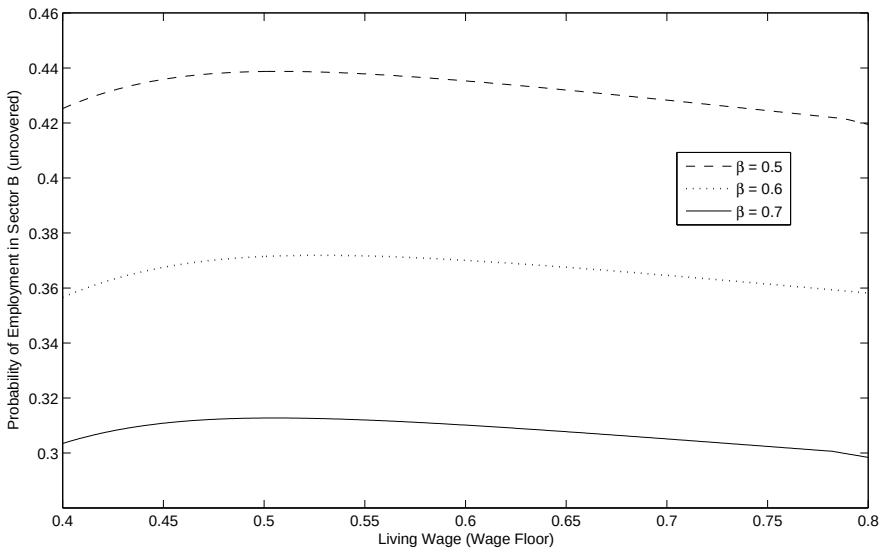

Figure 5: When $P_B > P_A$: Relationship between P_B and $\underline{W}$

Parameter values used in simulation: $\alpha = 0.4$, $\gamma = 0.45$, $C_A = C_B = 0.1$, $\overline{N} = 100$

6 Appendix B: Proofs

6.1 Proof of Existence of Equilibrium

When $P_A > P_B$, set $\underline{W} = \epsilon$ such that ϵ is a very small value greater than zero. When $\underline{W}$ is small it serves as a separating equilibrium, driving low ability workers to Sector B and high ability workers to Sector A. When $\beta < \frac{\underline{W}^2}{1-2\underline{W}+2\underline{W}^2}$, N_A decreases and J_A increases. When $\beta \geq \frac{\underline{W}^2}{1-2\underline{W}+2\underline{W}^2}$, as $\underline{W}$ increases, N_A and J_A both decrease. In both cases, N_B is uniquely identified by $\overline{N} - N_A$ and J_B is shown to be increasing as $\underline{W}$ increases. See proof of proposition 2 for details. Therefore, for each value of $\underline{W}$, there exists a unique triplet $\{N_A, J_A, J_B\}$ when $P_A > P_B$.

As $\underline{W}$ increases, there may exist a critical value beyond which $P_B > P_A$. When $P_B > P_A$, J_A and J_B decrease in $\underline{W}$ and N_A increases in $\underline{W}$. See proof of proposition 3 for details. Therefore, for each value of $\underline{W}$, there exists a unique triplet $\{N_A, J_A, J_B\}$ when $P_B > P_A$.

When the equilibrium flip occurs as $\underline{W}$ increases, we must make certain that there is no flip back from $P_B > P_A$ to the previous regime of $P_A > P_B$ for a stable equilibrium to exist. Taking total derivative of the indifference condition with respect to $\underline{W}$ and rearranging:

$$\frac{d\ P_B}{d\ \underline{W}} = \frac{1}{\beta\delta^*}\frac{d\ P_A}{d\ \underline{W}} - \frac{P_B}{\delta^*}\frac{d\ \delta^*}{d\ \underline{W}}$$

A sufficient condition for the equilibrium not switching (i.e.$P_B > P_A$ condition does not flip) is $\frac{d\ P_B}{d\ \underline{W}} \geq \frac{d\ P_A}{d\ \underline{W}}$.

$$\frac{1}{\beta\delta^*}\frac{d\ P_A}{d\ \underline{W}} - \frac{P_B}{\delta^*}\frac{d\ \delta^*}{d\ \underline{W}} \geq \frac{d\ P_A}{d\ \underline{W}}$$

Rearranging:

$$\frac{\beta P_B}{1-\beta\delta^*}\left(\frac{1}{\overline{N}}\frac{d\ N_A}{d\ \underline{W}}+1\right) \geq \frac{d\ P_A}{d\ \underline{W}}$$

Since we know that $\frac{d\ P_A}{d\ \underline{W}} < 0$ and $\frac{d\ N_A}{d\ \underline{W}} > 0$, the condition always holds. Therefore, equilibrium does not flip back as $\underline{W}$ is increased. QED

7 Proof Details for $P_A > P_B$

Noting some identities:

$$\begin{aligned}
N_B &= \underline{W}\overline{N}, \quad \frac{d\ N_B}{d\ \underline{W}} = \overline{N} > 0 \\
N_A &= (1-\underline{W})\overline{N}, \quad \frac{d\ N_A}{d\ \underline{W}} = -\overline{N} < 0 \\
E(\delta|B) &= \frac{\underline{W}}{2}, \quad \frac{d\ E(\delta|B)}{d\ \underline{W}} = \frac{1}{2} > 0 \\
E(\delta|A) &= \frac{1+\underline{W}}{2}, \quad \frac{d\ E(\delta|A)}{d\ \underline{W}} = \frac{1}{2} > 0
\end{aligned}$$

7.1 Proof that $\frac{d\ q_B}{d\ \underline{W}} < 0$, $\frac{d\ P_B}{d\ \underline{W}} > 0$, $\frac{d\ N_B}{d\ \underline{W}} > 0$, $\frac{d\ J_B}{d\ \underline{W}} > 0$, and $\frac{d\ x_B}{d\ \underline{W}} > 0$

Note that the expected zero profit condition for Sector B is:

$$\begin{aligned}
q_B(1-\beta)E(\delta|B) - C_B &= 0 \\
q_B(1-\beta)\frac{\underline{W}}{2} - C_B &= 0
\end{aligned}$$

Therefore, as $\underline{W}$ increases, q_B must decrease to maintain the zero profit condition. Since q_B and P_B are negatively related, P_B must increase. Since we know $\frac{d\ N_B}{d\ \underline{W}} > 0$, yet q_B is decreasing, J_B must be increasing. Since both N_B and J_B are increasing, x_B must increase as well.

7.2 Proof that $\frac{d\ q_A}{d\ \underline{W}} \gtreqless 0$, $\frac{d\ P_A}{d\ \underline{W}} \gtreqless 0$, $\frac{d\ N_A}{d\ \underline{W}} < 0$, $\frac{d\ J_A}{d\ \underline{W}} \gtreqless 0$, and $\frac{d\ x_A}{d\ \underline{W}} \gtreqless 0$

The expected zero profit condition for Sector A is:

$$\begin{aligned}
q_A\left\{ Pr\left(\underline{W} \leq \delta \leq \frac{\underline{W}}{\beta}|A\right)\left(E\left(\delta|\underline{W} \leq \delta \leq \frac{\underline{W}}{\beta}\right) - \underline{W}\right) + \Pr\left(\delta > \frac{\underline{W}}{\beta}|A\right)(1-\beta)E\left(\delta|\delta > \frac{\underline{W}}{\beta}\right)\right\} - C_A &= 0 \\
q_A\left\{\frac{\underline{W}}{\beta}\frac{1-\beta}{1-\underline{W}}\left(\frac{\underline{W}}{\beta}\frac{1+\beta}{2} - \underline{W}\right) + (1-\beta)\frac{1-\frac{\underline{W}}{\beta}}{1-\underline{W}}\frac{1+\frac{\underline{W}}{\beta}}{2}\right\} - C_A &= 0
\end{aligned}$$

This simplifies to:

$$\frac{1-\beta}{1-\underline{W}}\frac{q_A}{2}\left(1-\frac{\underline{W}^2}{\beta}\right) - C_A = 0$$

Taking total derivatives with respect to $\underline{W}$ and simplifying

$$\frac{d\ q_A}{d\ \underline{W}} = q_A \left(\frac{1}{1-\underline{W}} + \frac{2\underline{W}}{1-\frac{\underline{W}^2}{\beta}} \right)$$

That is

$$\frac{d\ q_A}{d\ \underline{W}} \geq 0 \text{ iff } \frac{1}{1-\underline{W}} + \frac{2\underline{W}}{1-\frac{\underline{W}^2}{\beta}} \geq 0$$

Assume $\beta < \frac{\underline{W}^2}{1+2\underline{W}-2\underline{W}^2}$. Then $\frac{d\ q_A}{d\ \underline{W}} < 0$, which implies that $\frac{d\ P_A}{d\ \underline{W}} \geq 0$. Since P_A increases as either J_A or N_A increases, and since N_A always decreases, J_A must increase. There exists cases where $\frac{d\ x_A}{d\ \underline{W}} \geq 0$.

$$\frac{d\ x_A}{d\ \underline{W}} = \gamma \cdot \left(\alpha J_A^{\alpha-1} \frac{dJ_A}{d\underline{W}} N_A^{1-\alpha} + (1-\alpha) J_A^{\alpha} \frac{dN_A}{d\underline{W}} N_A^{-\alpha} \right)$$

Note that as α approaches 1, $\frac{d\ x_A}{d\ \underline{W}} \geq 0$ almost surely.

Assume $\beta \geq \frac{\underline{W}^2}{1+2\underline{W}-2\underline{W}^2}$, $\frac{d\ q_A}{d\ \underline{W}} \geq 0$.

Similar to above, we know $\frac{d\ P_A}{d\ \underline{W}} < 0$ and $\frac{d\ N_A}{d\ \underline{W}} < 0$. Further,

$$\begin{aligned}
\frac{d\ q_A}{d\ \underline{W}} &= (\alpha-1)\frac{q_A}{J_A}\frac{d\ J_A}{d\ \underline{W}} + (1-\alpha)\frac{q_A}{N_A}(-\overline{N}) \\
q_A \left(\frac{1}{1-\underline{W}} + \frac{2\underline{W}}{1-\frac{\underline{W}^2}{\beta}} \right) &= (\alpha-1)\frac{q_A}{J_A}\frac{d\ J_A}{d\ \underline{W}} + (\alpha-1)\frac{q_A}{N_A}\overline{N} \\
\frac{1}{1-\underline{W}} + \frac{2\underline{W}}{1-\frac{\underline{W}^2}{\beta}} + (1-\alpha)\frac{\overline{N}}{N_A} &= (\alpha-1)\frac{1}{J_A}\frac{d\ J_A}{d\ \underline{W}}
\end{aligned}$$

Because the right hand side is positive, $\frac{d\ J_A}{d\ \underline{W}} < 0$. Since J_A and N_A decrease, it must be that x_A decreases as well. QED

8 Proof Details for $P_B > P_A$

$$\begin{aligned}
F_1 &= q_A(\delta^* - \underline{W}) - 2C \\
F_2 &= \frac{q_B}{Z}\left\{\underline{W}^2(1-\beta) + (1-\delta^{*2})(1-\beta)\right\} - 2C \\
F_3 &= P_A\underline{W} - P_B\beta\delta^*
\end{aligned}$$

where $Z = \underline{W} + 1 - \delta^*$. Using the implicit function theorem

List of partial derivatives follows:

$$
\begin{aligned}
\frac{\partial F_1}{\partial J_A} &= (\alpha-1)\frac{q_A}{J_A}\frac{N_A}{\overline{N}} < 0 \\
\frac{\partial F_1}{\partial J_B} &= 0 \\
\frac{\partial F_1}{\partial N_A} &= (2-\alpha)\frac{q_A}{\overline{N}} > 0 \\
\frac{\partial F_1}{\partial \underline{W}} &= 0 \\
\frac{\partial F_2}{\partial J_A} &= 0 \\
\frac{\partial F_2}{\partial J_B} &= (\alpha-1)\frac{q_B}{J_B}(1-\beta)\left(Z+\frac{2\underline{W}}{Z}\right) < 0 \\
\frac{\partial F_2}{\partial N_A} &= \frac{q_B}{N_B}(1-\beta)\left((\alpha-2)Z+\frac{2\alpha\underline{W}}{Z}\right) < 0 \\
\frac{\partial F_2}{\partial \underline{W}} &= -q_B(1-\beta)\frac{N_A}{N_B} < 0 \\
\frac{\partial F_3}{\partial J_A} &= \alpha\frac{P_A}{J_A}\underline{W} > 0 \\
\frac{\partial F_3}{\partial J_B} &= -\alpha\beta\frac{P_B}{J_B}\left(\frac{N_A}{\overline{N}}+\underline{W}\right) < 0 \\
\frac{\partial F_3}{\partial N_A} &= -\alpha\frac{P_A}{N_A}\underline{W}-\frac{P_B}{N_B}(Z+\alpha\beta) < 0 \\
\frac{\partial F_3}{\partial \underline{W}} &= P_A - P_B\beta > 0
\end{aligned}
$$

Note that the necessary and sufficient conditions for $\frac{\partial F_2}{\partial N_A} < 0$ is $\beta > \frac{\underline{W}}{1+\underline{W}-\sqrt{\frac{2\alpha\underline{W}}{2-\alpha}}}$.

$$
\begin{bmatrix} \frac{d\,J_A}{d\,\underline{W}} \\ \frac{d\,J_B}{d\,\underline{W}} \\ \frac{d\,N_A}{d\,\underline{W}} \end{bmatrix} = B^{-1} \begin{bmatrix} \frac{\partial F_1}{\partial \underline{W}} \\ \frac{\partial F_2}{\partial \underline{W}} \\ \frac{\partial F_3}{\partial \underline{W}} \end{bmatrix}
$$

where

$$
B^{-1} = \frac{1}{Det(B)} \begin{bmatrix} \frac{\partial F_2}{\partial J_B}\frac{\partial F_3}{\partial N_A} - \frac{\partial F_3}{\partial J_B}\frac{\partial F_2}{\partial N_A} & \frac{\partial F_3}{\partial J_B}\frac{\partial F_1}{\partial N_A} & -\frac{\partial F_2}{\partial J_B}\frac{\partial F_1}{\partial N_A} \\ \frac{\partial F_3}{\partial J_A}\frac{\partial F_2}{\partial N_A} & \frac{\partial F_1}{\partial J_A}\frac{\partial F_3}{\partial N_A} - \frac{\partial F_3}{\partial J_A}\frac{\partial F_1}{\partial N_A} & -\frac{\partial F_1}{\partial J_A}\frac{\partial F_2}{\partial N_A} \\ -\frac{\partial F_3}{\partial J_A}\frac{\partial F_2}{\partial J_B} & -\frac{\partial F_1}{\partial J_A}\frac{\partial F_3}{\partial J_B} & \frac{\partial F_1}{\partial J_A}\frac{\partial F_2}{\partial J_B} \end{bmatrix}
$$

and $Det(B)$ can be written as

$$
Det(B) = \frac{\partial F_1}{\partial J_A}\frac{\partial F_2}{\partial J_B}\frac{\partial F_3}{\partial N_A} - \frac{\partial F_1}{\partial J_A}\frac{\partial F_2}{\partial N_A}\frac{\partial F_3}{\partial J_B} - \frac{\partial F_1}{\partial N_A}\frac{\partial F_2}{\partial J_B}\frac{\partial F_3}{\partial J_A}
$$

8.1 Proof that determinant is negative

$$
\begin{aligned}
Det(B) \quad = \quad & (\alpha-1)\frac{q_A}{J_A}\frac{q_B}{J_B}(1-\beta)\frac{1}{\overline{N}} \times \Bigg\{ N_A(\alpha-1)\left(Z+\frac{2\underline{W}}{Z}\right)\left(-\alpha\frac{P_A}{N_A}\underline{W}-\frac{P_B}{N_B}\beta(Z+\alpha\beta)\right) \\
& +\frac{N_A}{N_B}\left((\alpha-2)Z+\frac{2\alpha\underline{W}}{Z}\right)\alpha\beta P_B\left(\frac{N_A}{\overline{N}}+\underline{W}\right) \\
& -(2-\alpha)\left(Z+\frac{2\underline{W}}{Z}\right)\alpha P_A\underline{W}\Bigg\}
\end{aligned}
$$

Simplifying the term inside the curly brackets

$$
-\alpha P_A\underline{W}\Big(Z+\tfrac{2\underline{W}}{Z}\Big)+\frac{N_A}{N_B}P_B(1-\alpha)\Big(Z+\tfrac{2\underline{W}}{Z}\Big)(Z+\alpha\beta)-\frac{N_A}{N_B}P_B\delta^*\beta\alpha\left(2Z-\alpha\Big(Z+\tfrac{2\underline{W}}{Z}\Big)\right)
$$

Rearranging

$$
P_B\beta\frac{N_A}{N_B}(1-\alpha)\Big(Z+\tfrac{2\underline{W}}{Z}\Big)(Z+\alpha\beta)-\alpha P_A\underline{W}\frac{N_A}{N_B}\left(\frac{N_B}{N_A}\Big(Z+\tfrac{2\underline{W}}{Z}\Big)+2Z-\alpha\Big(Z+\tfrac{2\underline{W}}{Z}\Big)\right)
$$

This term must be greater than 0. Note that $P_B\beta > P_B\beta\delta^* = P_A\underline{W}$ from the indifference condition. Therefore, I need

$$
(1-\alpha)Z+\alpha\beta-\alpha^2\beta \geq \alpha\frac{N_A}{N_B}+\frac{2\alpha Z}{\Big(Z+\frac{2\underline{W}}{Z}\Big)}-\alpha^2
$$

Since it is clear that $-\alpha^2\beta \geq -\alpha^2$, we only need

$$
(1-\alpha)Z+\alpha\beta \geq \alpha\frac{N_A}{N_B}+\frac{2\alpha Z}{\Big(Z+\frac{2\underline{W}}{Z}\Big)}
$$

Using $Z=\frac{N_B}{\overline{N}}$ and isolating α yields

$$
\alpha < \frac{N_A N_B}{\frac{N_B}{\overline{N}}+(2-\beta)\frac{N_A}{\overline{N}}+N_A N_B}
$$

Using the definition of the zero profit condition from City A and plugging in for N_A and N_B

$$
\alpha < \frac{\overline{N}^2 C_A(1-C_A)}{1-(1-\beta)C_A+\overline{N}^2 C_A(1-C_A)} = \frac{1}{\zeta+1} < 1
$$

where $\zeta = \frac{1-(1-\beta)C_A}{\overline{N}^2 C_A(1-C_A)}$. As long as $0 < C_A^L < C < C_A^H < 1$, $\zeta > 0$, and the above expression holds. Then, $Det(B) < 0$. QED

8.2 Proof of derivatives

$$
\begin{aligned}
\frac{d\ J_A}{d\ \underline{W}} &= \frac{1}{Det(B)}\left(\frac{\partial F_3}{\partial J_B}\frac{\partial F_1}{\partial N_A}\frac{\partial F_2}{\partial \underline{W}} - \frac{\partial F_2}{\partial J_B}\frac{\partial F_1}{\partial N_A}\frac{\partial F_3}{\partial \underline{W}}\right) < 0 \\
\frac{d\ J_B}{d\ \underline{W}} &= \frac{1}{Det(B)}\left(\frac{\partial F_2}{\partial \underline{W}}\left\{\frac{\partial F_1}{\partial J_A}\frac{\partial F_3}{\partial N_A} - \frac{\partial F_3}{\partial J_A}\frac{\partial F_1}{\partial N_A}\right\} - \frac{\partial F_3}{\partial \underline{W}}\frac{\partial F_1}{\partial J_A}\frac{\partial F_2}{\partial N_A}\right) > 0 \\
\frac{d\ N_A}{d\ \underline{W}} &= \frac{1}{Det(B)}\left(-\frac{\partial F_1}{\partial J_A}\frac{\partial F_3}{\partial J_B}\frac{\partial F_2}{\partial \underline{W}} + \frac{\partial F_1}{\partial J_A}\frac{\partial F_2}{\partial J_B}\frac{\partial F_3}{\partial \underline{W}}\right) < 0 \\
\frac{d\ N_B}{d\ \underline{W}} &= -\frac{d\ N_A}{d\ \underline{W}} > 0
\end{aligned}
$$

8.3 Proof $\frac{d\ J_B}{d\ \underline{W}} > 0$

Compare $\frac{d\ J_B}{d\ \underline{W}}$ with $Det(B)$.

$$
\begin{aligned}
\frac{d\ J_B}{d\ \underline{W}} &= \frac{1}{Det(B)}\left(\frac{\partial F_2}{\partial \underline{W}}\left\{\frac{\partial F_1}{\partial J_A}\frac{\partial F_3}{\partial N_A} - \frac{\partial F_3}{\partial J_A}\frac{\partial F_1}{\partial N_A}\right\} - \frac{\partial F_3}{\partial \underline{W}}\frac{\partial F_1}{\partial J_A}\frac{\partial F_2}{\partial N_A}\right) \\
Det(B) &= \frac{\partial F_1}{\partial J_A}\frac{\partial F_2}{\partial J_B}\frac{\partial F_3}{\partial N_A} - \frac{\partial F_1}{\partial J_A}\frac{\partial F_2}{\partial N_A}\frac{\partial F_3}{\partial J_B} - \frac{\partial F_1}{\partial N_A}\frac{\partial F_2}{\partial J_B}\frac{\partial F_3}{\partial J_A} < 0 \\
&= \frac{\partial F_2}{\partial J_B}\left\{\frac{\partial F_1}{\partial J_A}\frac{\partial F_3}{\partial N_A} - \frac{\partial F_3}{\partial J_A}\frac{\partial F_1}{\partial N_A}\right\} - \frac{\partial F_1}{\partial J_A}\frac{\partial F_2}{\partial N_A}\frac{\partial F_3}{\partial J_B} < 0
\end{aligned}
$$

Note that the second term in the curly brackets in the $\frac{d\ J_B}{d\ \underline{W}}$ expression is negative (including the minus sign). The second term of $Det(B)$ is positive (including the minus sign). Since we know $Det(B) < 0$ and $\frac{\partial F_2}{\partial J_B} < 0$, the term in curly brackets must be positive. Therefore, $\frac{d\ J_B}{d\ \underline{W}} > 0$. QED

8.4 Proof that $\frac{d\ P_A}{d\ \underline{W}} \leq 0$

$$
\begin{aligned}
\frac{d\ P_A}{d\ \underline{W}} &= \alpha\frac{P_A}{J_A}\frac{d\ P_A}{d\ \underline{W}} - \alpha\frac{P_A}{N_A}\frac{d\ N_A}{d\ \underline{W}} \\
&= \frac{\alpha P_A}{Det(B)}\left\{\frac{1}{J_A}\frac{\partial F_1}{\partial N_A}\left(\frac{\partial F_3}{\partial J_B}\frac{\partial F_2}{\partial \underline{W}} - \frac{\partial F_2}{\partial J_B}\frac{\partial F_3}{\partial \underline{W}}\right) - \frac{1}{N_A}\frac{\partial F_1}{\partial J_A}\left(\frac{\partial F_3}{\partial J_B}\frac{\partial F_2}{\partial \underline{W}} - \frac{\partial F_2}{\partial J_B}\frac{\partial F_3}{\partial \underline{W}}\right)\right\} \\
&= \frac{\alpha P_A \eta}{Det(B)}\left\{\frac{1}{J_A}\frac{\partial F_1}{\partial N_A} + \frac{1}{N_A}\frac{\partial F_1}{\partial J_A}\right\} \\
&= \frac{\alpha \eta P_A q_A}{Det(B) J_A \overline{N}}
\end{aligned}
$$

where $\eta = \frac{\partial F_3}{\partial J_B}\frac{\partial F_2}{\partial \underline{W}} - \frac{\partial F_2}{\partial J_B}\frac{\partial F_3}{\partial \underline{W}}$. Since $Det(B) < 0$, $\frac{d\ P_A}{d\ \underline{W}} \leq 0$. QED

8.5 Proof that $\frac{d\ P_B}{d\ \underline{W}}$ can be negative

$$
\begin{aligned}
\frac{d\ P_B}{d\ \underline{W}} &= \alpha P_B \left\{ \frac{1}{J_B} \frac{d\ J_B}{d\ \underline{W}} - \frac{1}{N_B} \frac{d\ N_B}{d\ \underline{W}} \right\} \\
&= \alpha P_B \left\{ \frac{1}{J_B} \frac{d\ J_B}{d\ \underline{W}} + \frac{1}{N_B} \frac{d\ N_A}{d\ \underline{W}} \right\} \\
&= \frac{\alpha P_B}{Det(B)} \Big\{ \frac{1}{J_B} \Big(\frac{\partial F_2}{\partial \underline{W}} \Big[\frac{\partial F_1}{\partial J_A} \frac{\partial F_3}{\partial N_A} - \frac{\partial F_3}{\partial J_A} \frac{\partial F_1}{\partial N_A} \Big] \\
&\quad - \frac{\partial F_3}{\partial \underline{W}} \frac{\partial F_1}{\partial J_A} \frac{\partial F_2}{\partial N_A} \Big) + \frac{1}{N_B} \Big(\frac{\partial F_1}{\partial J_A} \frac{\partial F_2}{\partial J_B} \frac{\partial F_3}{\partial \underline{W}} - \frac{\partial F_1}{\partial J_A} \frac{\partial F_3}{\partial J_B} \frac{\partial F_2}{\partial \underline{W}} \Big) \Big\}
\end{aligned}
$$

Note that the first term (which includes the square brackets) and the third and fourth terms are positive. Only the second term is negative. The second term and third term can be compared to show that the second term is dominated by the third term in absolute value.

$$
\begin{aligned}
\left| \frac{1}{J_B} \frac{\partial F_3}{\partial \underline{W}} \frac{\partial F_1}{\partial J_A} \frac{\partial F_2}{\partial N_A} \right| &\leq \left| \frac{1}{N_B} \frac{\partial F_1}{\partial J_A} \frac{\partial F_3}{\partial \underline{W}} \frac{\partial F_2}{\partial J_B} \right| \\
\left| (\alpha - 2)Z + \frac{2\alpha \underline{W}}{Z} \right| &\leq \left| (\alpha - 1)Z + \frac{2(\alpha - 1)\underline{W}}{Z} \right| \\
2Z &\leq Z + \frac{2\underline{W}}{Z} \\
\underline{W} < \underline{W} + (1 - \sqrt{2\underline{W}}) &\leq \delta^* \leq \frac{\underline{W}}{\beta}
\end{aligned}
$$

As long as the above condition holds, $\frac{d\ P_B}{d\ \underline{W}} \leq 0$ since $Det(B) < 0$. Note that as $\underline{W}$ approaches $\frac{1}{2}$, the condition simplifies to $0 \leq \delta^*$ which holds always. QED