A Logical Puzzle

Smullyan posed many puzzles about an island that has two kinds of inhabitants, knights, who always tell the truth, and their opposites, knaves, who always lie. You encounter two people *A* and *B*. What are *A* and *B* if *A* says “*B* is a knight” and *B* says “The two of us are opposite types”

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《孙子算经》 (1500年前)：
今有雉兔同笼，上有三十五头，
下有九十四足，问雉兔各几何？

(94－35×2）÷2=12(兔子数)
总头数（35）－兔子数（12）=鸡数（23）

introduce x and y

Formal Logic

- define a formal language for representing knowledge and for making logical inferences
- construct valid argument

Propositional Logic

2 Propositions (命题)

(1) Definition of proposition

A proposition is a declarative statement that is true or false, but not both. (即:表示判断的语句称为命题)

(2) Example 1 (see page 2)

All the following declarative sentences are propositions:

1. Washington, D.C. is the capital of the United States of America.
2. Toronto is the capital of Canada.
3. 1+1=2.
4. 2+2=3.

Propositions 1 and 3 are true, whereas 2 and 4 are false

(3) Example 2 (not propositions)

Consider the following sentences.

1. What time is it now?
2. Read it carefully!
3. x+1 = 2
4. x+y = z

Sentences 1 and 2 are not declarative sentences.
Sentences 3 and 4 are neither true or false.

(4) The truth value of proposition is true, denoted by T. The false value of proposition is false, denoted by F.

(5) Negation of a Proposition (“非”)

Definition 1

Let p be a proposition. The statement

“It is not the case that p”

is another proposition, called the negation of p.

The negation of p is denoted by ¬p.

The proposition “¬p” is read “not p”.

Truth Table

p	¬p
T	F
F	T

Example 3: Find the negation of “Today is Friday.”

Solution: “It is not the case that today is Friday,”
or “Today is not Friday,”
or “It is not Friday today.”

(6) Conjunction of two propositions (“并且”又称”合取”)

Definition 2

Let p and q be propositions. The proposition “p and q”, denoted as “$p \wedge q$” is the proposition that is true when both of them are true and is false otherwise. The proposition “$p \wedge q$” is called the conjunction of p and q.

Truth table

p	q	$p \wedge q$
T	T	T
T	F	F
F	T	F
F	F	F

Example 5 (page 4)

Find the conjunction of the propositions p and q where p is the proposition “Today is Friday” and q is the proposition “It is raining today.”

Solution:

$p \wedge q$ is the proposition

“Today is Friday and it is raining today.”

When is $p \wedge q$ true?

(7) Disjunction of two propositions (“或者”又称”析取)

Definition 3 (page 4)

Let p and q be propositions. The proposition “p or q”, denoted as $p \vee q$, is the proposition that is false when p and q are both false and true otherwise.

The proposition $p \vee q$ is called the disjunction of p and q.

Truth table

p	q	$p \vee q$
T	T	T
T	F	T
F	T	T
F	F	F

Example 6

Find the disjunction of the propositions p or q where p is the proposition “Today is Friday” and q is the proposition “It is raining today.”

Solution:

$p \vee q$ is the proposition

“Today is Friday or it is raining today.”

When is $p \vee q$ true?

(8) Exclusive or of two propositions (“异或”)

Definition 4

Let p and q be propositions. The exclusive *or* of p and q, denoted by $p \oplus q$, is the proposition that is true when exactly one of p and q is true and is false otherwise

Truth table

p	q	$p \oplus q$
T	T	*F*
T	F	T
F	T	T
F	F	*F*

(9) Implication (“蕴含”或称为：Conditional Statement)

(a) Definition 5

Let p and q be propositions. The *implication* p→q is the proposition that is false when p is true and q is false, and is true otherwise.

In this implication p is called the *hypothesis (or antecedent or premise)* and q is called the *conclusion (or consequence)*

(b) Truth Table

p	q	p→q
T	T	T
T	F	F
F	T	T
F	F	T

Example: P: no Covid cases from now Q: class in classroom from

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(b) Truth Table

p	q	p→q
T	T	T
T	F	F
F	T	T
F	F	T

Example: P: no Covid cases from now Q: class in classroom

Today is Friday it is raining

Remark: a variety of terminology to express p→q (page 6).

(c) When is p→q false?

How about the case that p is false?

(d) q→p is called the converse of p→q (逆命题).

¬q→¬p is called contrapositive of p→q (逆否命题).

¬p→¬q is called inverse of p→q (否命题).

(e) Example 9 (see page 8)

What are the converse, contrapositive, inverse of the implication

“The home team wins whenever it is raining.”?

Solution:

The implication can be rewritten as:

“If it is raining, then the home team wins”

Then …………………

(10) Biconditional (“当且仅当”又称”等价”)

Definition 6

Let p and q be propositions. The biconditional p↔q is the proposition that is true when p and q have the same truth values, and is false otherwise.

Truth table

p	q	p↔q
T	T	T
T	F	F
F	T	F
F	F	T

p↔q -------------------”p if and only if q”

p↔q has the same truth table of (p→q) Λ (q→p)

Example 10 (see page 9)

Let p be the statement “*You can take the flight*” and let q be the statement “*You buy a ticket.*”

Then p↔q is the statement

“*You can take the flight if and only if you buy a ticket.*”

(11)Truth Tables of compound Propositions

Example 11 Construct the truth table of the compound proposition (p∨¬q) →(p∧q)

Solution: The Truth table is:

p	q	¬q	p∨¬q	p∧q	(p∨¬q) →(p∧q)
T	T	F	T	T	T
T	F	T	T	F	F
F	T	F	F	F	T
F	F	T	T	F	F

3. Precedence of Logical Operations (Page 11)

Operator	Precendence
$\neg$	1
$\wedge$	2
$\vee$	3
$\rightarrow$	4
$\leftrightarrow$	5

Example: (1) $p \wedge q \vee r$ means $(p \wedge q) \vee r$ rather than $p \wedge (q \vee r)$

(2) $p \vee q \rightarrow r$ is the same as $(p \vee q) \rightarrow r$.

Remark: no problem if you want to use some redundant parentheses (but not too many).

Applications of Logics

4. Translating English Sentences (Page 17)

Example 1 How can this English sentence be translated into a logical expression.

"You can access the internet from the campus only if (page 17) you are a computer science major or you are not a freshman."

Solution:

a------"You can access the internet from the campus."

b------"You are a computer science major."

c------"You are a freshman."

Then this sentence can be expressed as

$a \rightarrow (b \vee \neg c)$

"You can access the internet from the campus only if (page 17) you are a computer science major or you are not a freshman."

a only if $(b \vee \neg c)$ = $a \rightarrow (b \vee \neg c)$

"You can access the internet from the campus if (page 17) you are a computer science major or you are not a freshman."

a if $(b \vee \neg c)$ = $(b \vee \neg c) \rightarrow a$

if p then q = $p \rightarrow q$ p is sufficient for q
q if p = $p \rightarrow q$
p only if q = $p \rightarrow q$ q is necessary for p

p if and only if q sufficient and necessary
p if q = $q \rightarrow p$
p only if q = $p \rightarrow q$

x=0 and y=0 if and only if x+y=0
x=0 and y=0 if x+y=0 False
x=0 and y=0 only if x+y=0 True

Example 2 How can this English sentence be translated into logical expression? (Page 17)

"You cannot ride the roller coaster (过山车) if you are under 4 feet tall unless you are older than 16 years old."

Solution:

q--------*"You can ride the roller coaster."*

r---------*"You are under 4 feet tall."*

s--------*"You are older than 16 years old."*

Then the sentence can be translated to

$(r \wedge \neg s) \rightarrow \neg q$.

a unless b = $\neg b \rightarrow a$

$\neg s \rightarrow (r \rightarrow \neg q)$

p unless q
p except if q $\neg q \rightarrow p$

p: we have on line class, q: COVID under control
we have on line class unless COVID under control
if COVID not under control we have on line class

p: x+y=0 q: x≠0 or y≠0
x+y=0 unless x≠0 or y≠0
if x=0 and y=0 then x+y=0

same as p or q