Practice Course 08

18/10/2022

Section 2.3
Functions

DEFINITION 5

A function f is said to be *one-to-one*, or an *injunction*, if and only if $f(a) = f(b)$ implies that $a = b$ for all a and b in the domain of f. A function is said to be *injective* if it is one-to-one.

$$\forall a \forall b (f(a) = f(b) \rightarrow a = b) \qquad \forall a \forall b (a \neq b \rightarrow f(a) \neq f(b))$$

DEFINITION 7

A function f from A to B is called *onto*, or a *surjection*, if and only if for every element $b \in B$ there is an element $a \in A$ with $f(a) = b$. A function f is called *surjective* if it is onto.

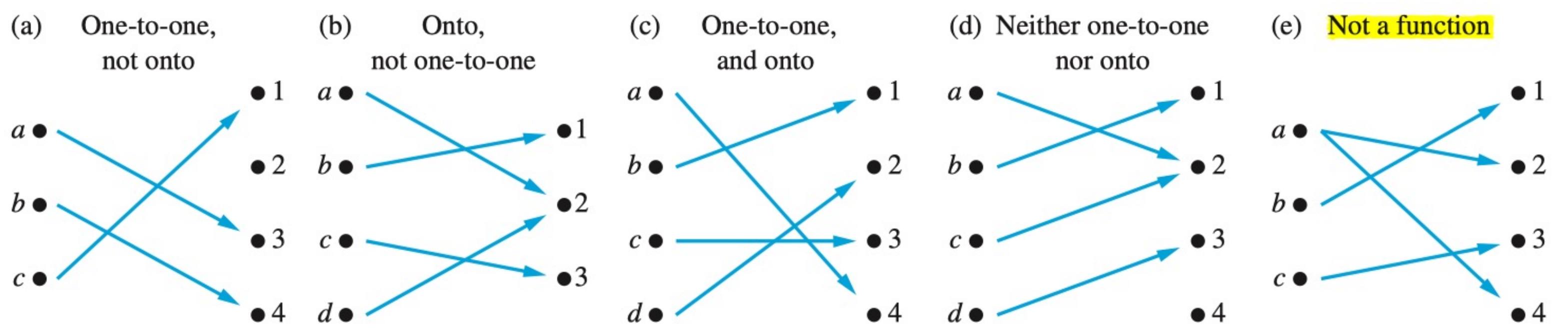
$$\forall y \exists x (f(x) = y)$$

DEFINITION 8

The function f is a *one-to-one correspondence*, or a *bijection*, if it is both one-to-one and onto. We also say that such a function is *bijective*.

The above definitions are particularly useful when judging one-to-one or onto mappings in Chapter 2.5

DEFINITION 1 Let A and B be nonempty sets. A *function* f from A to B is an assignment of exactly one element of B to each element of A. We write $f(a) = b$ if b is the unique element of B assigned by the function f to the element a of A. If f is a function from A to B, we write $f : A \to B$.



Note that a function f is one-to-one if and only if $f(a) \neq f(b)$ whenever $a \neq b$. This way of expressing that f is one-to-one is obtained by taking the contrapositive of the implication in the definition.

15. Determine whether the function $f\colon \mathbf{Z} \times \mathbf{Z} \to \mathbf{Z}$ is onto if

a) $f(m, n) = m + n$.
b) $f(m, n) = m^2 + n^2$.
c) $f(m, n) = m$.
d) $f(m, n) = |n|$.
e) $f(m, n) = m - n$.

15. Determine whether the function $f: \mathbf{Z} \times \mathbf{Z} \to \mathbf{Z}$ is onto if

a) $f(m, n) = m + n.$
b) $f(m, n) = m^2 + n^2.$
c) $f(m, n) = m.$
d) $f(m, n) = |n|.$
e) $f(m, n) = m - n.$

a) Onto b) Not onto c) Onto d) Not onto e) Onto

a) For any n, we can find $m=0 \Rightarrow f(0,n) = 0+n = n$

b) $f(m,n) = m^2 + n^2 \geq 0$, but the codomain is $\mathbb{Z}$ (contain $x < 0$)

c) For any int m we have $f(m,n) = m$

d) Similar to b.

e) For any int m, we can set $n=0 \Rightarrow f(m,0) = m$

23. Determine whether each of these functions is a bijection from **R** to **R**.

a) $f(x) = 2x + 1$

b) $f(x) = x^2 + 1$

c) $f(x) = x^3$

d) $f(x) = (x^2 + 1)/(x^2 + 2)$

a) $f(x) = 2x+1$ $\quad y = 2x+1 \Rightarrow x = g(y) = \frac{y-1}{2}$

one-to-one : $f(a) = f(b) \longrightarrow a = b$.

Yes

$f(x_1) = f(x_2) \Rightarrow 2x_1 + 1 = 2x_2 + 1 \Rightarrow x_1 = x_2$

onto : $f(x) = 2x+1 = m \in \boxed{R} \Rightarrow x = \frac{m-1}{2}$.

Yes

i.e. for all elements in R, we can find a x

b) one-to-one : $f(x_1) = f(x_2) \Rightarrow 2x_1^2 + 1 = 2x_2^2 + 1$

No

$\Rightarrow x_1^2 = x_2^2 \quad x_1 \overset{?}{=} x_2$

onto : method 1: $f(x) = x^2 + 1 = m \in R \Rightarrow x^2 = m-1$ exist x ?

No

method 2. $f(x) = x^2 + 1 \in [1, +\infty)$ not all of R

c) one-to-one : $f(x_1) = f(x_2) \Rightarrow x_1^3 = x_2^3 \Rightarrow x_1 = x_2$

Yes

onto : $f(x) = x^3 = m \in R \Rightarrow x = m^{\frac{1}{3}}$

Yes

d) one-to-one: $f(x_1) \quad f(x_2) \Rightarrow \frac{x_1^2+1}{x_1^2+2} = \frac{x_2^2+1}{x_2^2+2}$

No.

$$\Rightarrow (x_1^2+1)(x_2^2+2) = (x_1^2+2)(x_2^2+1)$$

$$2x_1^2 + x_2^2 = x_1^2 + 2x_2^2$$

$$\Rightarrow x_1^2 \neq x_2^2 \qquad x_1 \overset{?}{=} x_2$$

onto : $f(x) = \frac{x^2+1}{x^2+2} = m \Rightarrow x^2+1 = mx^2 + 2m$

No.

$$(1-m)x^2 = 2m-1$$

$$x^2 = \frac{2m-1}{1-m} = \frac{2m-2+1}{1-m}$$

$$= -1 + \frac{1}{1-m}$$

33. Suppose that g is a function from A to B and f is a function from B to C.

a) Show that if both f and g are one-to-one functions, then $f \circ g$ is also one-to-one.

b) Show that if both f and g are onto functions, then $f \circ g$ is also onto.

Q33. $A \xrightarrow{g} B \qquad B \xrightarrow{f} C$

a) $\boxed{\text{By definition of one-to-one : if } x \neq y \text{ then } f(g(x)) \neq f(g(y))}$

proof: if $x \neq y$

g is one-to-one $\Longrightarrow g(x) \neq g(y)$

f is one-to-one $\Rightarrow f(g(x)) \neq f(g(y))$

Here, we view $g(x)$ and $g(y)$ as different elements of B.

b) $\boxed{\forall z \exists x \left(f(g(x)) = z \right)}$

proof: f is onto: $\Rightarrow$ exist $y \in B$ Such that $f(y) = z$

g is onto and $y \in B \Rightarrow$ exist $x \in A$ Such that $g(x) = y$

$\Rightarrow z = f(y) = f(g(x))$

Section 2.5
Cardinality of Sets

DEFINITION 1 The sets A and B have the same *cardinality* if and only if there is a one-to-one correspondence from A to B. When A and B have the same cardinality, we write $|A| = |B|$.

DEFINITION 2 If there is a one-to-one function from A to B, the cardinality of A is less than or the same as the cardinality of B and we write $|A| \leq |B|$. Moreover, when $|A| \leq |B|$ and A and B have different cardinality, we say that the cardinality of A is less than the cardinality of B and we write $|A| < |B|$.

DEFINITION 3 A set that is either finite or has the same cardinality as the set of positive integers is called *countable*. A set that is not countable is called *uncountable*. When an infinite set S is countable, we denote the cardinality of S by $\aleph_0$ (where $\aleph$ is aleph, the first letter of the Hebrew alphabet). We write $|S| = \aleph_0$ and say that S has cardinality "aleph null."

THEOREM 1 If A and B are countable sets, then $A \cup B$ is also countable.

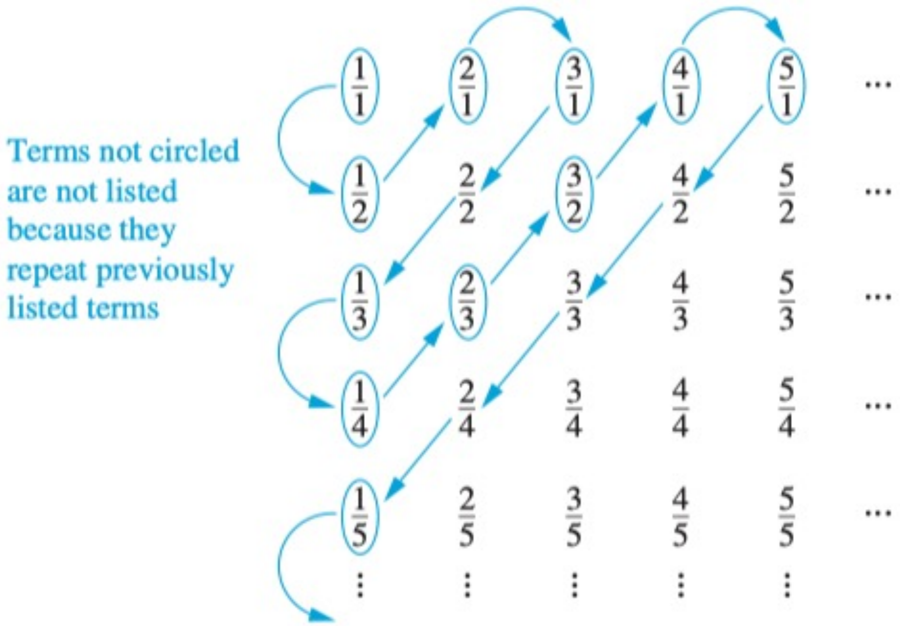


FIGURE 3 **The Positive Rational Numbers Are Countable.**

1, 1/2, 2, 3, 1/3, 1/4, 2/3, 3/2, 4, 5,

1. Determine whether each of these sets is finite, countably infinite, or uncountable. For those that are countably infinite, exhibit a one-to-one correspondence between the set of positive integers and that set.

a) the negative integers
b) the even integers
c) the integers less than 100
d) the real numbers between 0 and $\frac{1}{2}$
e) the positive integers less than 1,000,000,000
f) the integers that are multiples of 7

a) Countably infinite
b) Countably Infinite
c) Countably infinite
d) Uncountable
e) Finite
f) Countably infinite

16. Show that a subset of a countable set is also countable.

15. Show that if A and B are sets, A is uncountable, and $A \subseteq B$, then B is uncountable.

Q16. Suppose $A \subseteq B$. and B were Countable.

with $B = \{ b_1, b_2, b_3 \dots \}$.

Since $A \subseteq B$. we can list the elements of A using the order same with B. So A is countable.

19. Show that if A, B, C, and D are sets with $|A| = |B|$ and $|C| = |D|$, then $|A \times C| = |B \times D|$.

Q19.

$|A| = |B| \iff$ bijection f from A to B $(a \to f(a))$

$|C| = |D| \iff$ bijection g from C to D $(c \to f(c))$

$|A \times C|$ $|B \times D|$ we can define a bijection mapping

(a, c) $(f(a), g(c))$ (a, c) to $(f(a), g(c))$

Show that the $|(0, 1)| = |(0, 1]|$.

Solution: It is not at all obvious how to find a one-to-one correspondence between $(0, 1)$ and $(0, 1]$ to show that $|(0, 1)| = |(0, 1]|$. Fortunately, we can use the Schröder-Bernstein theorem instead. Finding a one-to-one function from $(0, 1)$ to $(0, 1]$ is simple. Because $(0, 1) \subset (0, 1]$, $f(x) = x$ is a one-to-one function from $(0, 1)$ to $(0, 1]$. Finding a one-to-one function from $(0, 1]$ to $(0, 1)$ is also not difficult. The function $g(x) = x/2$ is clearly one-to-one and maps $(0, 1]$ to $(0, 1/2] \subset (0, 1)$. As we have found one-to-one functions from $(0, 1)$ to $(0, 1]$ and from $(0, 1]$ to $(0, 1)$, the Schröder-Bernstein theorem tells us that $|(0, 1)| = |(0, 1]|$.

33. Use the Schröder-Bernstein theorem to show that $(0, 1)$ and $[0, 1]$ have the same cardinality

Q33 $f: (0,1)$ to $[0,1]$ $\quad f(x) = x$

$g: [0,1]$ to $(0,1)$. $\quad g(x) = \frac{1}{2}x + \frac{1}{2}$

Then use Schroder - Bernstein Theorem.

1, Let A and B be two finite sets, prove that A and B have the same number of elements if and only if there is a one to one correspondence.

2. Let S={1,3,5,7,8,9,12,13,…} (the set of odd positive integers plus 8 and 12), find a one to one correspondence from Z+ to S.

3. Let A={a,b}, prove the set of all finite sequences over A is countable.