1.4 Predicates (谓词) and Quantifiers (量词)

1. Introduction (page 36)

Question: The meaning of some statements cannot be expressed in propositional logic. How to extend?

Example:

"Every computer connected to the university network is functioning properly"

No rules of propositional logic allows us to conclude the truth of the statement

"MATH3 is functioning properly."

Where MATH3 is one of the computers connected to the university network

Example:

" Everybody loves my baby,
but my baby only loves me " -> my baby is me

2. Predicates

(1) Consider the statement

"*x is greater than 3*".

x---------subject of the statement

"*is greater than 3*"------predicate

(2) P(x) ------ "*x is greater than 3*"

P-----------"is greater than 3"(predicate)

P(x) is also said to be
the value of propositional function P at x.

(3) P(x)--------"x is greater than 3"

P(4)---------true

P(2)--------false

(4) Example 3 (page 38)

Q(x,y)--------"x=y+3"

Q(1,2)--------false

Q(3,0)--------true

(5) In general,

A statement of the form $P(x_1,x_2,\ldots,x_n)$ is the value of the ***propositional function*** P at the n-tuple $(x_1,x_2,\ldots,x_n)$, and P is called a ***predicate.***

2. Universal Quantifier (全称量词)

(1) Universe of Discourse (or domain of discourse, or domain, 个体域)

-----a set containing all the values of a variable

(2) Definition 1 (page 40)

The universal quantification of P(x) is the proposition

"P(x) is true for all values of x in the domain."

------------$\forall x\, P(x)$

----------- $\forall x\, P(x)$ is read as

"for all x P(x)" or

"for every x P(x)"

(3) Example 8 (page 41)

P(x)-----"x+1>x"

domain------all real numbers

How about $\forall x\, P(x)$?

Answer: $\forall x\, P(x)$ ------true

(4) Example 9 (page 41)

Q(x)--------"x<2"

domain------all real numbers

How about $\forall x\, Q(x)$?

Answer: $\forall x\, Q(x)$ ------false

(5) Further explanation

domain-----finite set $\{x_1,x_2,\ldots,x_n\}$

$\forall x\, P(x)$ is the same as

$P(x_1) \wedge P(x_2) \wedge \ldots\ldots \wedge P(x_n)$

(6) Example 11 (page 42)

P(x)------" $x^2<10$ "

domain

----"the positive integers not exceeding 4"

{1,2,3,4}

How about $\forall x\, P(x)$?

Answer:

$\forall x\, P(x)$ is the same as

$P(1) \wedge P(2) \wedge P(3) \wedge P(4)$

--------false

(7) Example 13 (page 42)

$\forall x\, (x^2 \geq x)$

domain ---- all integers

Answer:

$x^2 \geq x$ iff $x(x-1) \geq 0$ iff $x \geq 1$ or $x \leq 0$

$\forall x\, (x^2 \geq x)$--------true

How about

domain ---- all real numbers

$\forall x\, (x^2 \geq x)$ -------- false

(8) How to show $\forall x\, P(x)$ is false?

Try to find a counterexample (反例).

Example :

P(x) ------- "$x^2>0$"

domain ---- all integers

How about $\forall x\, P(x)$?

Answer:

P(0) -------- false

$\forall x\, P(x)$ -------- false

3. Existential Quantifier (存在量词)

(1) Definition 2 (existential quantifier, page 42)

The existential Quantifier of P(x) is the proposition

"There exists ***an element x*** in the domain such that ***P(x) is true***"

--------denoted as $\exists x\, P(x)$

"There is an x such that P(x)"

"There is at least one x such that P(x)"

or "For some x P(x)"

(2) Example 14 (page 43)

P(x)---------"x>3"

domain-----all real numbers

Consider ∃x P(x)

∃x P(x) is true

Why?

(x can be 3.5, 4, …., which makes is P(x) is true)

(3) Example 15 (page 43)

Q(x)-------"x=x+1"

domain---------all real numbers

Consider ∃x Q(x)

For every real number x, Q(x) is false. Therefore,

∃x Q(x) is false

(4)

If the domain is a finite set, i.e., $\{x_1, x_2, \ldots\ldots, x_n\}$, then

∃x P(x) is the same as

$P(x_1) \lor P(x_2) \lor \ldots\ldots \lor P(x_n)$

(5) Example 16 (page 43)

P(x)--------"$x^2>10$"

domain--------"all the positive integers not exceeding 4"

Consider ∃x P(x)

domain={1, 2, 3, 4}

∃x P(x) is the same as

$P(1) \lor P(2) \lor P(3) \lor P(4)$

∃x P(x) is true because P(4) is true.

(6) Summary

Statement	When True?	When False
∀x P(x)	P(x) is true for all x	There is an x for which P(x) is false
∃x P(x)	There is an x for which P(x) is true	P(x) is false for every x

4. Quantifiers with Restricted Domain

Example 17 (page 44)

What do the statements below mean, where the domain in each case consists of the real numbers?

Answer: For the meanings, see page 44.

$\forall x<0\ (x^2>0)$ --------- $\forall x\ (x<0 \rightarrow x^2>0)$

$\forall y\neq 0\ (y^3\neq 0)$ --------- $\forall y\ (y\neq 0 \rightarrow y^3\neq 0)$

$\exists z>0\ (z^2=2)$ ------- $\exists z\ (z>0 \land z^2=2)$

5. Binding Variables (变量约束)

(1) Bound Variable and Free Variable (约束变量和自由变量)

Example 18

(a) $\exists x\,(x+y=1)$

x------bound variable

y------free variable

(b) $\underline{\exists x\,(P(x) \lor Q(x))} \lor \underline{\forall x\, R(x)}$

the scope of bound variable

6. Logical Equivalence of predicates

Definition 3 (Page 45)

Statements involving predicates and quantifiers are logically equivalent if and only if they have the same true value no matter which predicates are substituted into these statements and which domain of discourse is used for the variables in these propositional functions.

We use the notation $S \equiv T$ to indicate the two statements S and T involving predicates and quantifiers are logically equivalent.

Equivalences of propositions can be "lifted"

e.g. $P(x) \land Q(x) \equiv Q(x) \land P(x)$

How about quantifiers?

Example 19 (page 45)

Show that $\forall x\,(P(x) \land Q(x))$ and $\forall x\, P(x) \land \forall x\, Q(x)$ are logically the same?

Answer: We need to show

(1), if $\forall x(P(x) \land Q(x))$ is true, then $\forall xP(x) \land \forall xQ(x)$ is true.

(2), if $\forall xP(x) \land \forall xQ(x)$ is true, then $\forall x(P(x) \land Q(x))$ is true.

Suppose that $\forall x(P(x) \land Q(x))$ is true. This means that if *a* is in the domain, then $P(a) \land Q(a)$ is true. Hence, P(a) is true and Q(a) is true. Because P(a) is true and Q(a) is true for every element in the domain, we can conclude that $\forall xP(x)$ and $\forall xQ(x)$ are both true. This means that $\forall xP(x) \land \forall xQ(x)$ is true.

Next, suppose that $\forall xP(x) \land \forall xQ(x)$ is true. It follows that $\forall xP(x)$ is true and $\forall xQ(x)$ is true. Hence, if a is in the domain, then P(a) is true and Q(a) is true [because P(x) and Q(x) are both true for all elements in the domain]. It follows that for all a, $P(a) \land Q(a)$ is true. Therefore, $\forall x(P(x) \land Q(x))$ is true.

We can now conclude that

$$\forall x\,(P(x) \land Q(x)) \equiv \forall x\, P(x) \land \forall x\, Q(x)$$

5. Negating Quantified Expressions

(1)

------ ***"Every student in the class has taken a course in calculus"***

$\forall x\, P(x)$

Here, P(x)--------"x has taken a course in calculus"

--------The negation is

"It is not the case that Every student in the class has taken a course in calculus"

or *"There is a student in the class who has not taken a course in calculus"*

$\exists x\, \neg P(x)$

-------- $\neg \forall x\, P(x) \equiv \exists x\, \neg P(x)$

(2)

------ "*There is a student in the class who has taken a course in calculus*"

$\exists x\, Q(x)$

Here, Q(x)--------"x has taken a course in calculus"

--------The negation is

"It is not the case that there is a student in the class who has taken a course in calculus"

or "*Every student in the class has not taken a course in calculus*"

$\forall x\, \neg Q(x)$

------- $\neg \exists x\, Q(x) \equiv \forall x\, \neg Q(x)$

(3) Example 21 (page 41)

What is the negations of the statements

$\forall x\, (x^2>x)$ and $\exists x\, (x^2=2)$

Answer:

$\neg\forall x\, (x^2>x) \equiv \exists x\, \neg(x^2>x)$

$\equiv \exists x\, (x^2 \le x)$ ……….(1)

$\neg \exists x\, (x^2=2) \equiv \forall x\, \neg(x^2=2)$

$\equiv \forall x\, (x^2 \ne 2)$ ………..(2)

--------The truth values of these statements depends on the universe of discourse.

For (1), use [0.5, 3] and [2, 5] to check

For (2), use [0,1] and [1,2] to check

(4) Example 22 (page 48)

Show that $\neg\forall x\, (P(x) \to Q(x))$ and $\exists x\, (P(x) \land \neg Q(x))$ are logically equivalent.

Proof:

$\neg\forall x\, (P(x) \to Q(x))$

$\equiv \exists x\, \neg(P(x) \to Q(x))$

$\equiv \exists x\, (P(x) \land \neg Q(x))$

6. Translating from English into logical expression

Example 23 (page 48) Express the statement "***<u>Every student in this calss has studied calculus</u>***" in predicates and quantifiers.

Answer:

(1) C(x)-----"x has studied calculus"

domian:

-----all the students in the class

$\forall x\, C(x)$

(2) Way 2:

domain-------all people

"For every person x, if person x is in this class then x has studied calculus."

S(x)-------person x is in this class

C(x)------person x has studied calculus

$\forall x\, (S(x) \to C(x))$ (correct)

$\forall x\, (S(x) \land C(x))$ (wrong, why?)

Example 24 (page 49)

Express the statements below in predicates and quantifiers.

"Some students in this class has visited Mexico" -------------------(1)

and

"Every student in this class has visited Canada or Mexico" -------------------(2)

Answer: (a) For (1),

Way 1:

M(x)------x has visited Mexico.

domian------all the students in this class

$\exists x\, M(x)$

Way 2:

domain-------all people

S(x)------"x is a student in this class"

$\exists x\, (S(x) \wedge M(x))$--------correct

$\exists x\, (S(x) \rightarrow M(x))$---------wrong

For (b)

Way 1:

C(x)------"x has visited Canada."

M(x)------"x has visited Mexico"

domain

-------"all the students in this class"

$\forall x\, (C(x) \vee M(x))$

Way 2:

domain-------"all people"

S(x)-------"x is a student in this class."

C(x)------"x has visited Canada."

M(x)------"x has visited Mexico"

$\forall x\, (S(x) \rightarrow (C(x) \vee M(x)))$

Discussion

Exercises 1,7,15,43,45