2.3 Functions

1. Introduction

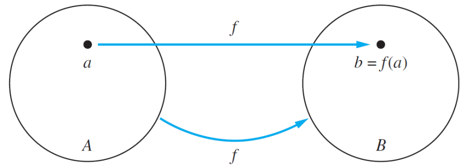
(1) What is function?

Definition 1 (page 139)

Let A and B be sets. A function from A to B is an assignment of exactly one element of B to each element of A.

We write f(a)=b if b is the unique element of B assigned by the function f to the element a of A.

If f is a function from A to B, we write $f: A \to B$.



Example (page 139)

Assignment of Grades in a Discrete Mathematics Class

(2) Domain (定义域), codomain (共域) and range (值域)

Definition 2 (page 139)

If f is a function from A to B, we say that A is the *domain* of f and B is the *codomain* of f.

If $f(a)=b$, we say that b is the image (像) of a and a is a pre-image (原像) of b.

The range of f is the set of all images of elements of A. Also, if f is a function from A to B, we say f *maps* A to B.

(3) Example 4 (page 140)

Let $f: Z \to Z$ assign the square of an integer to this integer, i.e., $f(x)=x^2$.

What is the domain, codomain and range of function f?

Answer:

See book.

(4) Definition 3 (page 141)

Let f_1 and f_2 be functions from A to R. Then f_1+f_2 and $f_1 f_2$ are also functions from A to R defined by

$(f_1+f_2)(x) = f_1(x) + f_2(x)$

$(f_1 f_2)(x) = f_1(x) f_2(x)$

Example 6 (page 141)

Let f_1 and f_2 be functions from R to R such that $f_1(x)=x^2$ and $f_2(x)=x-x^2$.

What are the functions f_1+f_2 and $f_1 f_2$?

(5) Definition 4 (page 136)

Let *f* be a function from the set *A* to the set *B and* let *S* be a subset of *A*. The *image* of S is the subset of B that consists of the images of the elements of S.

The notation:

f(S) = { f(s) | s∈S }

Example 7 (page 141)

A={a,b,c,d,e}

B={1,2,3,4}

f(a)=2, f(b)=1, f(c)=4, f(d)=1, f(e)=1

S={b,c,d}

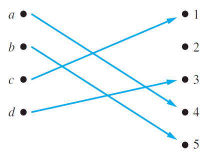
What is *f(S)*?

2. One-to-one and Onto functions

(1) One-to-one function (or injective, 单射)

Definition 5 (page 141)

A function *f* is said to be one-to-one, if and only if *f(x)=f(y)* implies that *x=y* for all *x* and *y* in the domain of *f*. Not many-to-one.



A function is said to be an *injection* if it is one-to-one.

(2) Examples (pages 142)

(a) Example 8

Determine whether the function f from {a,b,c,d} to {1,2,3,4,5} with f(a)=4, f(b)=5, f(c)=1, and f(d)=3 is one to one.

Answer:

Function *f is one-to-one.*

(b) Example 9

Determine whether the function $f(x)=x^2$ from the set of integers to the set of integers is one-to-one.

Answer:

No

However, if the domain is Z^+, then *f* is one-to-one.

(c) Example 10 (page 142)

Determine whether the function f(x)=x+1 from the set of real numbers to itself is one to one.

Solution:

Yes.

(3) Some conditions that guarantee that a function is one to one (page 143)

(a) Definition 6 (strictly increasing or descreasing function)

A function *f* whose domain and codomain are subset of the set of real numbers is called strictly increasing if $f(x)<f(y)$ whenever $x<y$ and x and y are in the domain of *f*.

strictly descreasing?

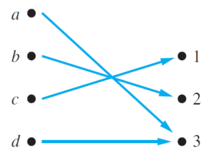
(b)

If a function is either strictly increasing or strictly decreasing, it must be one to one.

(4) Onto (or surjective function, 满射)

Definition 7 (page 143)

A function *f* from *A* to *B* is called onto, or surjective, if and only if for every element $b \in B$ there is an element $a \in A$ with $f(a)=b$.



A function *f* is called a surjection if it is onto.

(5) Examples (page 143)

(a) Example 12

Let f be the function from *{a,b,c,d}* to *{1,2,3}* defined by $f(a)=3$, $f(b)=2$, $f(c)=1$, and $f(d)=3$.

Is *f* an onto function?

Answer:

Yes.

How about the answer if the codomain is {1,2,3,4}?

(b) Example 13

Is the function $f(x)=x^2$ from the set of integers to the set of integers onto?

Answer:

No.

(c) Example 14

Is the function $f(x)=x+1$ *from the set of integers to the set of integers onto?*

Answer:

Yes.

(6) One-to-one correspondence (or bijection, 一一对应, 双射)

Definition 8 (page 144)

The function f is one-to-one correspondence or a bijection if it is both one-to-one and onto.

(7) Example 16 (page 144)

Let f be the function from {a,b,c,d} to {1,2,3,4} with

f(a)=4, f(b)=2, f(c)=1, and f(d)=3.

Is f a bijective?

Answer:

Yes.

(8) Identity function (恒等函数)

Let A be a set. The identity function on A

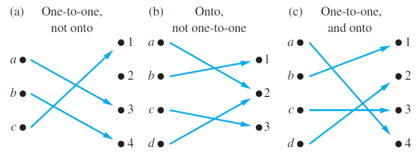
i_A: A→A where $i_A(x)=x$

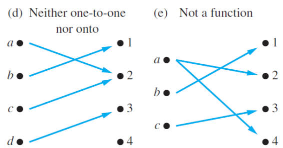
where x∈A.

(9) Examples of Different Types of Correspondences

Please Figure 5 (page 144).

(9) Examples of Different Types of Correspondences





4. Inverse functions (反函数) and compositions of functions (函数的复合)

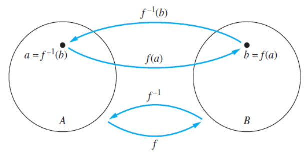
(1) Introduction

Consider a one-to-one correspondence f from the set A to the set B.

(1) Since f is an onto function, every element of B is the image of some element in A.

(2) Because f is also one-to-one, every element of B is the image of a unique element of A.

Therefore, we can define a new function from B to A that reverse the correspondence given by f.



(2) Definition 9 (inverse function, page 145)

(1) Let f be a one-to-one correspondence from the set A to the set B. The inverse function of f is the function that assigns to an element b belonging to B the unique element a such that f(a)=b.

(2) The inverse function of f is denoted by f^{-1}.

(3) Hence $f^{-1}(b)=a$ when f(a)=b.

Please note:

if f is not a one-to-one correspondence, we cannot define an inverse function of f.

(why? page 145.)

(3) Examples (page 140)

(a) Example 16

Let f be a function from {a,b,c} to {1,2,3} such that f(a)=2, f(b)=3, and f(c)=1. Is f invertible, and if it is, what is its inverse?

Solution:

Yes.

(b) Let f be the function from the set of integers to the set of integers such that f(x)=x+1. Is f invertible, and if it is, what is its inverse?

Solution:

Yes.

$f^{-1}(y)=y-1$

(c) Example 18

Let f be the function from R to R with $f(x)=x^2$. Is f invertible?

Answer:

Since f(1)=f(-1)=1, f is not one to one.

Hence, f is not invertible.

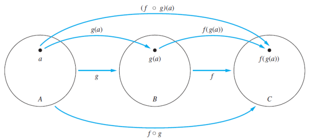
If f is the function from the set of all nonnegative real numbers to the set of all nonnegative real numbers

$f^{-1}(y)$=sqrt(y)

(4) Definition 10 (composition of two functions, 函数复合)

Let g be a function from the set A to the set B and let f be a function from the set B to the set C. The composition of the function f and g, denoted by $f \circ g$, is defined by

$$(f \circ g)(a) = f(g(a))$$



(5) Examples (page 147)

(a) Let g be the function from the set {a,b,c} to itself such that g(a)=b, g(b)=c, and g(c)=a.

Let f be the function from the set {a,b,c} to the set {1,2,3} such that f(a)=3, f(b)=2, and f(c)=1.

What is the composition of f and g, and what is the composition of g and f?

(b) Let f and g be the function from the set of integers to the set of integers defined by

$$f(x)=2x+3 \text{ and } g(x)=3x+2.$$

What is the composition of f and g?

What is the composition of g and f?

Remark:

Note that even if $f \circ g$ and $g \circ f$ are defined for functions f and g in Example 23, $f \circ g$ and $g \circ f$ are not equal.

In other words, the commutative law does not hold for the composition of functions

(c) The composition of a function and its inverse function

If $f: A \rightarrow B$ and f is a one-to-one correspondence,

then (i) $f^{-1} \circ f = i_A$ and $f \circ f^{-1} = i_B$

(ii) $(f^{-1})^{-1} = f$

Why? Page 147

5. The graph of functions

(1) Definition 11 (page 148)

Le f be the function from the set A to the set B. The graph of the function f is the set of ordered pairs

$\{ (a,b) \mid a \in A \text{ and } f(a)=b \}$.

(2) Examples (page 148)

Example 24

Display the graph of the function f(n)=2n+1 from the set of integers to the set of integers.

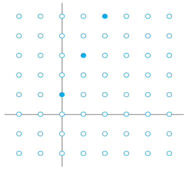


FIGURE 8 The Graph of $f(n) = 2n + 1$ from Z to Z.

Display the graph of the function $f(x)=x^2$ from the set of integers to the set of integers.

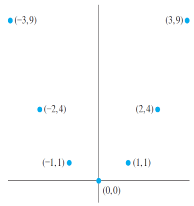
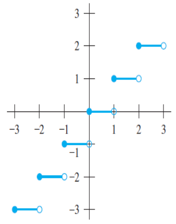


FIGURE 9 The Graph of $f(x) = x^2$ from Z to Z.

6. Some important functions

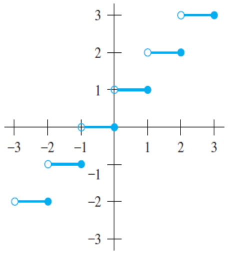
(1) Definition12 (floor function and ceiling function)

The floor function assigns to the real numbers x the largest integer that is less than or equal to x. The value of the floor function at x is denoted by $\lfloor x \rfloor$.



(a) $y = \lfloor x \rfloor$

The ceiling function assigns to the real number x the smallest integer that is greater than or equal to x. The value of the ceiling function at x is denoted by $\lceil x \rceil$.



(b) $y = \lceil x \rceil$

(2) Example 26 (page 149)

These are some values of the floor and ceiling functions.

$\lfloor 0.5 \rfloor=0$ $\lceil 0.5 \rceil=1$ $\lfloor -0.5 \rfloor=-1$ $\lceil -0.5 \rceil=0$

$\lfloor 3.1 \rfloor=3$ $\lceil 3.1 \rceil=4$ $\lfloor 7 \rfloor=7$ $\lceil 7 \rceil=7$

(3) Useful properties of the floor and ceiling function (page 150)

(1a) $\lfloor x \rfloor = n$ if and only if $n \le x < n+1$

(1b) $\lceil x \rceil = n$ if and only if $n-1 < x \le n$

(1c) $\lfloor x \rfloor = n$ if and only if $x-1 < n \le x$

(1d) $\lceil x \rceil = n$ if and only if $x \le n < x+1$

and ....

(1a') $\lfloor x \rfloor = n$ if and only if $x=n+r$ where $0 \le r < 1$

(4) Example 27 (page 150)

Prove that if x is a real number, then

$\lfloor 2x \rfloor = \lfloor x \rfloor + \lfloor x+0.5 \rfloor$

(5) Example 28 (page 145)

Prove or disprove that

$\lceil x+y \rceil = \lceil x \rceil + \lceil y \rceil$

for all real numbers x and y.

Answer:

This statement is false.

Counterexample:

x=0.5 and y=0.5

(7) The factorial function

$f: N \rightarrow Z^+$

$f(n)=n!$

i.e.,

$f(n) = 1 \times 2 \times \ldots \times (n-1) \times n$

(and $f(0)=0!=1$)

Discussion in class

15,23,33