Practice Course

12/10/2022

summary of quiz

2. (20 points) Determine whether $\neg(q \land (p \to q)) \to \neg p$ is a tautology and explain the reason

Method 01: $\neg(q \land (p \to q)) \to \neg p$

$= \neg(q \land (\neg p \lor q)) \to \neg p$

$= \neg(q) \to \neg p$

Implication $p \to q \equiv \neg p \lor q$

Absorption Law

Method 02:

$(\neg q \lor \neg(p \to q)) \to \neg p$

$(\neg q \lor (p \land \neg q)) \to \neg p$

$= \neg q \to \neg p$

De Morgan Law

Absorption Law

2. (20 points) Determine whether $\neg(q \wedge (p \rightarrow q)) \rightarrow \neg p$ is a tautology and explain the reason

p	q	$\neg q$	$p \rightarrow q$	$q \wedge (p \rightarrow q)$	$\neg(q \wedge (p \rightarrow q))$	$\neg p$	$\rightarrow$
F	F	T	T	F	T	T	T
F	T	F	T	T	F	T	T
T	F	T	F	F	T	F	F
T	T	F	T	T	F	F	T

Note: give a definite answer about True or not.

3. (20 points) Let S(x) be the predicate "x is a student," F(y) the predicate "y is a faculty member," and A (x, y) the predicate "x has asked y a question," where the domain consists of **all people** associated with your school. Use quantifiers to express each of these statements.

a) Every student has asked Professor Lee a question.

b) Some student has **not** asked any faculty member a question.

c) Some student has asked **every** faculty member a question.

d) Some student has never **been** asked a question by a faculty member.

a) $\forall x(S(x) \rightarrow A(x, \text{Professor Lee})$

b) $\exists x\ (S(x) \land \forall y\ (F(y) \rightarrow \neg A(x, y)))$

c) $\exists x\ (S(x) \land \forall y\ (F(y) \rightarrow A(x, y)))$

d) $\exists x\ (S(x) \land \forall y\ (F(y) \rightarrow \neg A(y, x)))$

Note:

1. mind the definition of domain.
2. mind the common use cases of exsistantial and universal quantifier.

A similar example from the PPT of Prof. Xu

3. Translating Mathematical Statements into Statements Involving Nested Quantifiers

(1) Example 6

Translate the statement

"*The sum of two positive integers is always positive*"

into a logical expression.

Answer:

Way1: domain for x and y----$\boxed{\text{all integers}}$

$$\forall x \forall y\ (\ (x>0) \land (y>0) \rightarrow (x+y>0)\ )$$

Way 2: domain for x and y----$\boxed{\text{all positive integers}}$

$$\forall x \forall y\ (\ x+y>0\ )$$

The order of quantifiers (Textbook page 60)

TABLE 1 Quantifications of Two Variables.

Statement	*When True?*	*When False?*
$\forall x \forall y P(x, y)$ $\forall y \forall x P(x, y)$	$P(x, y)$ is true for every pair x, y.	There is a pair x, y for which $P(x, y)$ is false.
$\forall x \exists y P(x, y)$	For every x there is a y for which $P(x, y)$ is true.	There is an x such that $P(x, y)$ is false for every y.
$\exists x \forall y P(x, y)$	There is an x for which $P(x, y)$ is true for every y.	For every x there is a y for which $P(x, y)$ is false.
$\exists x \exists y P(x, y)$ $\exists y \exists x P(x, y)$	There is a pair x, y for which $P(x, y)$ is true.	$P(x, y)$ is false for every pair x, y.

An example (Textbook page 60)

Let $Q(x, y)$ denote "$x + y = 0$." What are the truth values of the quantifications $\exists y \forall x Q(x, y)$ and $\forall x \exists y Q(x, y)$, where the domain for all variables consists of all real numbers?

Solution: The quantification

$$\exists y \forall x Q(x, y)$$

denotes the proposition

"There is a real number y such that for every real number x, $Q(x, y)$."

No matter what value of y is chosen, there is only one value of x for which $x + y = 0$. Because there is no real number y such that $x + y = 0$ for all real numbers x, the statement $\exists y \forall x Q(x, y)$ is false.

The quantification

$$\forall x \exists y Q(x, y)$$

denotes the proposition

"For every real number x there is a real number y such that $Q(x, y)$."

Given a real number x, there is a real number y such that $x + y = 0$; namely, $y = -x$. Hence, the statement $\forall x \exists y Q(x, y)$ is true.

27. Determine the truth value of each of these statements if the domain for all variables consists of all integers.

a) $\forall n \exists m (n^2 < m)$

b) $\exists n \forall m (n < m^2)$

c) $\forall n \exists m (n + m = 0)$

d) $\exists n \forall m (nm = m)$

27. Determine the truth value of each of these statements if the domain for all variables consists of all integers.

a) $\forall n \exists m (n^2 < m)$ **b)** $\exists n \forall m (n < m^2)$

c) $\forall n \exists m (n + m = 0)$ **d)** $\exists n \forall m (nm = m)$

a) $n = \cdots -3\ -2\ -1\ \ 0\ \ 1\ \ 2\ \ 3\ \ 4 \cdots$

True

$n^2 = 9\ 4\ 1\ 0\ 1\ 4\ 9\ 16 \cdots$

no matter how large n might be, we can always find an int $m > n^2$

b) $m = \cdots -3\ -2\ -1\ \ 0\ \ 1\ \ 2\ \ 3 \cdots$

True

$m^2 = 9\ \ 4\ \ 1\ \ 0\ \ 1\ \ 4\ \ 9$

Once $n < \min(m^2) = 0$, then $n < m^2$

eg. There exists: $n = -1$

Note: mind the definition of domain x

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b) $\exists n \forall m (n < m^2)$

c) $\forall n \exists m (n + m = 0)$

d) $\exists n \forall m (nm = m)$

c). $n = -3, -2, -1, 0, 1, 2, 3.$

Set $m = -n \Rightarrow m + n = 0.$

d) Set $n = 1$ or 0

True.

Note: mind the definition of domain x

Section 2.1
Sets

Some useful definitions

Subset

We see that $A \subseteq B$ if and only if the quantification

$$\forall x(x \in A \to x \in B)$$

Proper subset

$$\forall x(x \in A \to x \in B) \wedge \exists x(x \in B \wedge x \notin A)$$

Power set

Given a set S, the *power set of* S is the set of all subsets of the set S. The power set of S is denoted by $\mathcal{P}(S)$.

Cartesian product

$$A \times B = \{(a, b) \mid a \in A \wedge b \in B\}.$$

7. For each of the following sets, determine whether 2 is an element of that set.

a) $\{x \in \mathbf{R} \mid x \text{ is an integer greater than } 1\}$

b) $\{x \in \mathbf{R} \mid x \text{ is the square of an integer}\}$

c) $\{2, \{2\}\}$

d) $\{\{2\}, \{\{2\}\}\}$

e) $\{\{2\}, \{2, \{2\}\}\}$

f) $\{\{\{2\}\}\}$

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c) $\{2, \{2\}\}$

d) $\{\{2\}, \{\{2\}\}\}$

e) $\{\{2\}, \{2, \{2\}\}\}$

f) $\{\{\{2\}\}\}$

a) Yes b) No c) Yes d) No e) No f) No

Note:
2 is an element, while {2} is a set.

11. Determine whether each of these statements is true or false.

a) $x \in \{x\}$ **b)** $\{x\} \subseteq \{x\}$ **c)** $\{x\} \in \{x\}$

d) $\{x\} \in \{\{x\}\}$ **e)** $\emptyset \subseteq \{x\}$ **f)** $\emptyset \in \{x\}$

11. Determine whether each of these statements is true or false.

a) $x \in \{x\}$ **b)** $\{x\} \subseteq \{x\}$ **c)** $\{x\} \in \{x\}$

d) $\{x\} \in \{\{x\}\}$ **e)** $\emptyset \subseteq \{x\}$ **f)** $\emptyset \in \{x\}$

a) True b) True c) False d) True e) True f) False

39. Explain why $A \times B \times C$ and $(A \times B) \times C$ are not the same.

39. Explain why $A \times B \times C$ and $(A \times B) \times C$ are not the same.

AxBxC: (a1, b1, c1), (a2, b2, c2)…(an, bn, cn)
(AxB)xC: ((a1, b1), c1), ((a2, b2), c2)…((an, bn), cn)

The is a one-to-one correspondence between AxBxC and (AxB)xC
Note that the Cartesian products A × B and B × A are not equal, unless A = ∅ or B = ∅ (so that A × B = ∅) or A = B.

Section 2.2
Set Operations

Some useful definitions

Union	$A \cup B = \{x \mid x \in A \vee x \in B\}.$
Intersection	$A \cap B = \{x \mid x \in A \wedge x \in B\}.$
Cardinality of a union	$\lvert A \cup B \rvert = \lvert A \rvert + \lvert B \rvert - \lvert A \cap B \rvert.$
Difference	$A - B = \{x \mid x \in A \wedge x \notin B\}.$
Complement	$\overline{A} = \{x \in U \mid x \notin A\}.$

Some useful set identities

$A \cup (B \cup C) = (A \cup B) \cup C$ $A \cap (B \cap C) = (A \cap B) \cap C$	Associative laws
$A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$ $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$	Distributive laws
$\overline{A \cap B} = \overline{A} \cup \overline{B}$ $\overline{A \cup B} = \overline{A} \cap \overline{B}$	De Morgan's laws
$A \cup (A \cap B) = A$ $A \cap (A \cup B) = A$	Absorption laws

Example

19. Show that if A and B are sets, then

a) $A - B = A \cap \overline{B}$.

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a) $A - B = A \cap \overline{B}$.

LHS=RHS= $\{x \mid x \in A \wedge x \notin B\}$

7. Prove the domination laws in Table 1 by showing that

a) $A \cup U = U$. **b)** $A \cap \emptyset = \emptyset$.

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a) $A \cup U = U$. **b)** $A \cap \emptyset = \emptyset$.

$$
\begin{aligned}
A \cup U &= \{x \mid x \in A \vee x \in U\} \\
&= \{x \mid x \in A \vee \text{True}\} \\
&= \{x \mid \text{True}\} \\
&= U
\end{aligned}
$$

$$
\begin{aligned}
A \cap \phi &= \{x \mid x \in A \wedge x \in \phi\} \\
&= \{x \mid x \in A \wedge \text{False}\} \\
&= \{x \mid \text{False}\} \\
&= \phi
\end{aligned}
$$

Example (textbook page 131)

Use set builder notation and logical equivalences to establish the first De Morgan law $\overline{A \cap B} = \overline{A} \cup \overline{B}$.

Solution: We can prove this identity with the following steps.

$$
\begin{aligned}
\overline{A \cap B} &= \{x \mid x \notin A \cap B\} && \text{by definition of complement} \\
&= \{x \mid \neg(x \in (A \cap B))\} && \text{by definition of does not belong symbol} \\
&= \{x \mid \neg(x \in A \wedge x \in B)\} && \text{by definition of intersection} \\
&= \{x \mid \neg(x \in A) \vee \neg(x \in B)\} && \text{by the first De Morgan law for logical equivalences} \\
&= \{x \mid x \notin A \vee x \notin B\} && \text{by definition of does not belong symbol} \\
&= \{x \mid x \in \overline{A} \vee x \in \overline{B}\} && \text{by definition of complement} \\
&= \{x \mid x \in \overline{A} \cup \overline{B}\} && \text{by definition of union} \\
&= \overline{A} \cup \overline{B} && \text{by meaning of set builder notation}
\end{aligned}
$$

31. Let A and B be subsets of a universal set U. Show that $A \subseteq B$ if and only if $\overline{B} \subseteq \overline{A}$.

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$A \subseteq B$ definition of $A \subseteq B$

$\equiv \forall x (x \in A \to x \in B)$ Contrapositive Law

$\equiv \forall x (x \notin B \to x \notin A)$

$\equiv \forall x (x \in \overline{B} \to x \in \overline{A})$ definition of $\overline{B} \subseteq \overline{A}$

$\equiv \overline{B} \subseteq \overline{A}$

Textbook page 28

TABLE 7 Logical Equivalences Involving Conditional Statements.

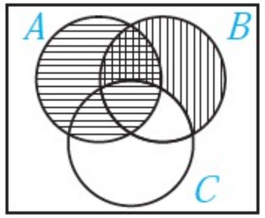
$p \to q \equiv \neg p \vee q$
$p \to q \equiv \neg q \to \neg p$
$p \vee q \equiv \neg p \to q$
$p \wedge q \equiv \neg(p \to \neg q)$
$\neg(p \to q) \equiv p \wedge \neg q$
$(p \to q) \wedge (p \to r) \equiv p \to (q \wedge r)$
$(p \to r) \wedge (q \to r) \equiv (p \vee q) \to r$
$(p \to q) \vee (p \to r) \equiv p \to (q \vee r)$
$(p \to r) \vee (q \to r) \equiv (p \wedge q) \to r$

27. Draw the Venn diagrams for each of these combinations of the sets A, B, and C.

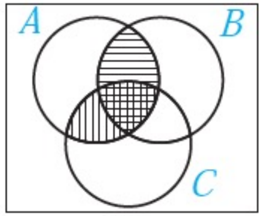
a) $A \cap (B - C)$

b) $(A \cap B) \cup (A \cap C)$

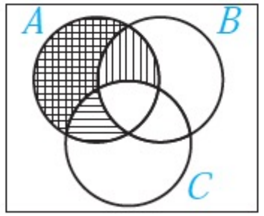
c) $(A \cap \overline{B}) \cup (A \cap \overline{C})$



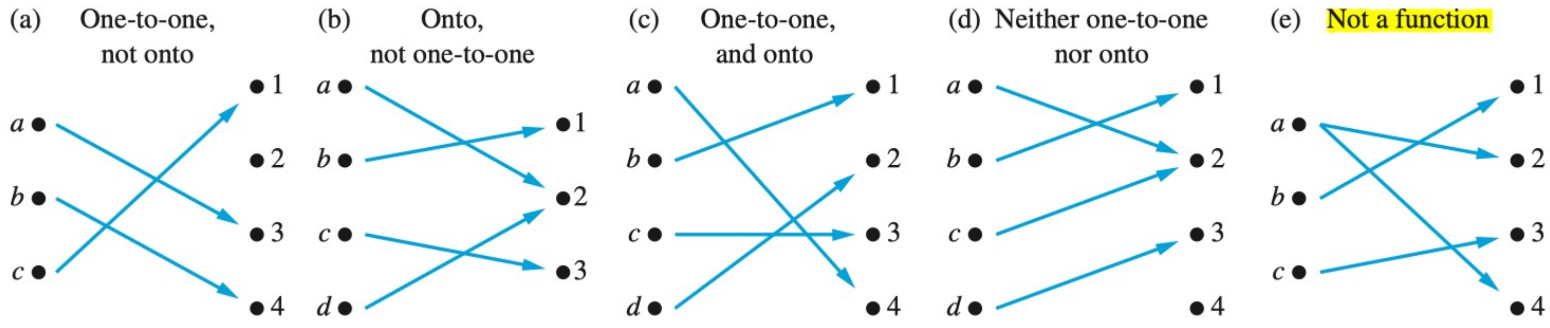
b) The desired set is the entire shaded portion.



c) The desired set is the entire shaded portion.



Section 2.3
Functions



Note that a function f is one-to-one if and only if $f(a) \neq f(b)$ whenever $a \neq b$. This way of expressing that f is one-to-one is obtained by taking the contrapositive of the implication in the definition.

1. Why is f not a function from $\mathbf{R}$ to $\mathbf{R}$ if

a) $f(x) = 1/x$?

b) $f(x) = \sqrt{x}$?

c) $f(x) = \pm\sqrt{(x^2 + 1)}$?

a) $f(0)$ is not defined.
b) $f(x)$ is not defined for $x < 0$.
c) $f(x)$ is not well-defined because there are two distinct values assigned to each x.

13. Which functions in Exercise 12 are onto?

From Z to Z (all integers)

a) $f(n) = n - 1$

b) $f(n) = n^2 + 1$

c) $f(n) = n^3$

d) $f(n) = \lceil n/2 \rceil$

Onto : (a) and (d)

d: F(2n)=ceil(2n/2)=ceil(n)=n   (n is an integer)

15. Determine whether the function $f\colon \mathbf{Z} \times \mathbf{Z} \to \mathbf{Z}$ is onto if

a) $f(m, n) = m + n$.
b) $f(m, n) = m^2 + n^2$.
c) $f(m, n) = m$.
d) $f(m, n) = |n|$.
e) $f(m, n) = m - n$.

15. Determine whether the function $f\colon \mathbf{Z} \times \mathbf{Z} \to \mathbf{Z}$ is onto if

a) $f(m, n) = m + n$.
b) $f(m, n) = m^2 + n^2$.
c) $f(m, n) = m$.
d) $f(m, n) = |n|$.
e) $f(m, n) = m - n$.

a) Onto b) Not onto c) Onto d) Not onto e) Onto

a) For any n, we can find $m=0 \Rightarrow f(0,n) = 0+n = n$

b) $f(m,n) = m^2+n^2 \geq 0$, but the codomain is $\mathbb{Z}$ (contain $x<0$)

c) For any int m we have $f(m,n) = m$

d) Similar to b.

e) For any int m, we can set $n=0 \Rightarrow f(m,0) = m$

23. Determine whether each of these functions is a bijection from **R** to **R**.

a) $f(x) = 2x + 1$

b) $f(x) = x^2 + 1$

c) $f(x) = x^3$

d) $f(x) = (x^2 + 1)/(x^2 + 2)$

a) Yes b) No c) Yes d) No

25. Let $f: \mathbf{R} \to \mathbf{R}$ and let $f(x) > 0$ for all $x \in \mathbf{R}$. Show that $f(x)$ is strictly decreasing if and only if the function $g(x) = 1/f(x)$ is strictly increasing.

31. Let $f(x) = \lfloor x^2/3 \rfloor$. Find $f(S)$ if

a) $S = \{-2, -1, 0, 1, 2, 3\}$.

b) $S = \{0, 1, 2, 3, 4, 5\}$.

c) $S = \{1, 5, 7, 11\}$.

d) $S = \{2, 6, 10, 14\}$.

a) $f(S) = \{0, 1, 3\}$
b) $f(S) = \{0, 1, 3, 5, 8\}$
c) $f(S) = \{0, 8, 16, 40\}$
d) $f(S) = \{1, 12, 33, 65\}$

33. Suppose that g is a function from A to B and f is a function from B to C.

a) Show that if both f and g are one-to-one functions, then $f \circ g$ is also one-to-one.

b) Show that if both f and g are onto functions, then $f \circ g$ is also onto.