§ 1.3 Propositional Equivalence (命题等价)

1. Introduction

Example 1 (page 21)

$p \vee \neg p$ is always true. It is a tautology. (永真公式)

$p \wedge \neg p$ is always false. It is a contradiction. (永假公式)

Definition 1 (see page 21)

(1) A compound proposition that is always *true,* no matter what the truth values of the propositions that occur in it, is called a tautology.

(2) A compound proposition that is always *false*, no matter what the truth values of the propositions that occur in it, is called a contradiction.

(3) A proposition that is neither a tautology nor a contradiction is called a contingency (中性公式).

2. Logical Equivalence

(1) Example 2 (page 22)

Show that $\neg(p \vee q)$ and $\neg p \wedge \neg q$ are logically equivalent.

Truth Table

p	q	$p \vee q$	$\neg(p \vee q)$	$\neg p$	$\neg q$	$\neg p \wedge \neg q$
T	T	T	F	F	F	F
T	F	T	F	F	T	F
F	T	T	F	T	F	F
F	F	F	T	T	T	T

(2) Definition 2 (logical equivalence)

The propositions p and q are called logically equivalent if $p \leftrightarrow q$ is a tautology.

The notation $p \equiv q$ denotes that p and q are logically equivalence.

$p \equiv T$ denotes that p is a tautology.

(3) More Examples

Example 3 (page 23)

Show that $p \to q$ and $\neg p \vee q$ are logically equivalent.

Solution:

We construct the truth table for these propositions in the table below. Since the truth values of $p \to q$ and $\neg p \vee q$ agree, these propositions are logically equivalence.

p	q	$\neg p$	$\neg p \vee q$	$p \to q$
T	T	F	T	T
T	F	F	F	F
F	T	T	T	T
F	F	T	T	T

Example 4

Show that the propositions $p \vee (q \wedge r)$ and $(p \vee q) \wedge (p \vee r)$ are logically equivalent.

Solution: By constructing truth table (page 27).

p	q	r	$q \wedge r$	$p \vee (q \wedge r)$	$p \vee q$	$p \vee r$	$(p \vee q) \wedge (p \vee r)$
T	T	T	T	T	T	T	T
T	T	F	F	T	T	T	T
T	F	T					
T	F	F					
F	T	T					
F	T	F	F	F	T	F	F
F	F	T					
F	F	F					

(4) Some Important Equivalences

(a) Table 5 (Logical Equivalence)

$p \wedge T \equiv p$ Identity laws
$p \vee F \equiv p$ (同一律)

$p \vee T \equiv T$ Domination laws
$p \wedge F \equiv F$ (零律)

$p \vee p \equiv p$ Idempotent laws
$p \wedge p \equiv p$ (幂等律)

$\neg(\neg p) \equiv p$ Double negation law (双重否定律)

$p \vee q \equiv q \vee p$ Commutative laws
$p \wedge q \equiv q \wedge P$ (交换律)

$(p \vee q) \vee r \equiv p \vee (q \vee r)$ Associative laws
$(p \wedge q) \wedge r \equiv p \wedge (q \wedge r)$ (结合律)

$p \vee (q \wedge r) \equiv (p \vee q) \wedge (p \vee r)$ Distributive laws
$p \wedge (q \vee r) \equiv (p \wedge q) \vee (p \wedge r)$ (分配律)

$\neg(p \wedge q) \equiv \neg p \vee \neg q$ De Morgan's laws
$\neg(p \vee q) \equiv \neg p \wedge \neg q$ (德摩根律)

$p \vee (p \wedge q) \equiv p$ Absorption laws
$p \wedge (p \vee q) \equiv p$ (吸收律)

$p \vee \neg p \equiv T$ (排中律) Negation laws
$p \wedge \neg p \equiv F$ (矛盾律)

(b) Table 6 (Logical Equivalence Involving Implication)

$p \to q \equiv \neg p \vee q$
$p \to q \equiv \neg q \to \neg p$
$p \vee q \equiv \neg p \to q$
$p \wedge q \equiv \neg(p \to \neg q)$
$\neg(p \to q) \equiv p \wedge \neg q$
$(p \to q) \wedge (p \to r) \equiv p \to (q \wedge r)$
$(p \to r) \wedge (q \to r) \equiv (p \vee q) \to r$
$(p \to q) \vee (p \to r) \equiv p \to (q \vee r)$
$(p \to r) \vee (q \to r) \equiv (p \wedge q) \to r$

(c) Table 6 (Logical Equivalence Involving Biconditionals)

$p \leftrightarrow q \equiv (p \to q) \wedge (q \to p)$
$p \leftrightarrow q \equiv \neg q \leftrightarrow \neg p$
$p \leftrightarrow q \equiv (p \wedge q) \vee (\neg p \wedge \neg q)$
$\neg(p \leftrightarrow q) \equiv p \leftrightarrow \neg q$

(d) Questions:
How to verify these equivalences?
Answer: One way is by constructing the truth table.

(5) Extension of De Morgan's Law

$\neg(p \wedge q) \equiv \neg p \vee \neg q$ can be extended to

$\neg(p_1 \wedge p_2 \wedge \ldots \wedge p_n) \equiv \neg p_1 \vee \neg p_2 \vee \ldots \neg p_n$

$\neg(p \vee q) \equiv \neg p \wedge \neg q$ can be extended to

$\neg(p_1 \vee p_2 \vee \ldots \vee p_n) \equiv \neg p_1 \wedge \neg p_2 \wedge \ldots \neg p_n$

(6) Constructing New Logical Equivalence (page 26)
Example 6: Show that $\neg(p \to q)$ and $p \wedge \neg q$ are logically equivalent.
Proof:

$\neg(p \to q)$	
$\equiv \neg(\neg p \vee q)$	by Example 3
$\equiv \neg(\neg p) \wedge \neg q$	by the second De Morgan law
$\equiv p \wedge \neg q$	by the double negation law

Example 7: Show that $\neg(p \vee (\neg p \wedge q))$ and $\neg p \wedge \neg q$ are logically equivalent.
Proof:

$\neg(p \vee (\neg p \wedge q))$	
$\equiv \neg p \wedge \neg(\neg p \wedge q))$	by the second De Morgan law
$\equiv \neg p \wedge (\neg(\neg p) \vee \neg q)$	by the first De Morgan law
$\equiv \neg p \wedge (p \vee \neg q)$	by the double negation law
$\equiv (\neg p \wedge p) \vee (\neg p \wedge \neg q)$	by the second distributed law
$\equiv F \vee (\neg p \wedge \neg q)$	because $\neg p \wedge p \equiv F$
$\equiv (\neg p \wedge \neg q) \vee F$	by the communicative law for disjunction
$\equiv \neg p \wedge \neg q$	by the identify law for F

Example 8: Show that $(p \wedge q) \to (p \vee q)$ is a tautology.
Solution:

$(p \wedge q) \to (p \vee q)$	
$\equiv \neg(p \wedge q) \vee (p \vee q)$	by Example 3
$\equiv (\neg p \vee \neg q) \vee (p \vee q)$	by the first De Morgan law
$\equiv (\neg p \vee p) \vee (\neg q \vee q)$	by the associative and communicative law for disjunction
$\equiv T \vee T$	by example 1 and the communicative law for disjunction
$\equiv T$	by domination law

Study the equivalences and make sure you can prove them.