§ 2.5 Cardinality of Sets(基数)

(1) Definition 1 (page 170)

The sets *A* and *B* have the same cardinality if and only if there is a one-to-one correspondence from *A* to ***B***.

When A and B have the same cardinality, we write |A| = |B|.

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(2) Definition 3 (page 171)

A set that is either finite or has the same cardinality as the set of positive integers is countable (可数的或可列的).

A set that is not countable is called uncountable.

When an infinite set S is countable, we denote the cardinality of S by $\aleph_0$ (aleph, the first letter of the Hebrew alphabet).

Don't feel bad if you find this section confusing. When Cantor started talking about sizes of infinity in the

nineteenth century, many mathematicians thought he made no sense.

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(3) Example 1 (page 171)

Show that the set of odd positive integer is a countable set.

Proof:

1	2	3	4	5	6	7	8	9	10	11	12	...
↕	↕	↕	↕	↕	↕	↕	↕	↕	↕	↕	↕	
1	3	5	7	9	11	13	15	17	19	21	23	...

Define the function

$$f(n)=2n-1$$

from Z^+ to the set of odd positive integers.

Now we need to show that f is a one-to-one correspondence

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(4) Further explanation of countable sets.

An infinite set is countable if and only if it is possible to list the elements of the set in a sequence (indexed by the positive integers).

The reason for this is:

a_1, a_2, ….., a_n, ……

where:

$a_1=f(1)$, $a_2=f(2)$, ……, $a_n=f(n)$

(page 171)

(5) Example 3 (page 172)

Show that the set of all integers is countable.

Solution:

All integers can be listed as:

0, 1, -1, 2, -2, 3, -3, …, n, -n, ….

One to one correspondence (from the set of positive integers to the set of all integers):

$$f(n) = \begin{cases} n/2, & \text{if } n \text{ is even} \\ -(n-1)/2, & \text{if } n \text{ is odd} \end{cases}$$

(6) Example 4 (page 172)

Show that the set of positive rational numbers is countable.

Proof

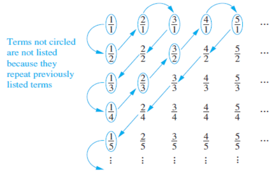

FIGURE 3 The Positive Rational Numbers Are Countable.

(6) Show that the set of real numbers is an uncountable set.

Solution*:* To show that the set of real numbers is uncountable, we suppose that the set of real numbers is countable and arrive at a contradiction. Then, the subset of all real numbers that fall between 0 and 1 would also be countable (because any subset of a countable set is also countable; see Exercise 16).

Under this assumption, the real numbers between 0 and 1 can be listed in some order*.* Let the decimal representation of these real numbers be

$s_1=0.a_{11}a_{12}a_{13}a_{14}\ldots\ldots$,

$s_2=0.a_{21}a_{22}a_{23}a_{24}\ldots\ldots$,

$s_3=0.a_{31}a_{32}a_{33}a_{34}\ldots\ldots$,

………………………………………

$s_n=0.a_{n1}a_{n2}a_{n3}a_{n4}\ldots\ldots$,

……………………………………..

Form a new real number with decimal expansion

s= $0.a_1a_2a_3a_4 \ldots$, where the decimal digits are determined by the following rule:

$a_i = 5$ if $a_{ii} = 4$

$a_i=4$ if $a_{ii} \neq 4$

Because the number is not in the list, the assumption that all the real numbers between 0 and 1 could be listed must be false.

(7) Theorem 1(page 174)

If A and B are countable sets, then A ∪ B is also countable.

(8) Definition 2 (page 170)

If there is a one-to-one function from A to B, the cardinality of A is less than or the same as the cardinality of B and we write $|A| \le |B|$.

Moreover, when $|A| \le |B|$ and A and B have different cardinality, we say that the cardinality of A is less than the cardinality of B and we write $|A| < |B|$.

(9) SCHRÖDER-BERNSTEIN THEOREM

If A and B are sets with $|A| \le |B|$ and $|B| \le |A|$, then $|A| = |B|$.

In other words, if there are one-to-one functions f from A to B and g from B to A, then there is a one-to-one correspondence between A and B.

Example 6 (page 175)

Show that the $|(0, 1)| = |(0, 1]|$.

Proof:

(i) $f: (0,1) \to (0,1]$ (one to one function)
$f(x)=x$

(ii) $g: (0,1] \to (0,1)$ (one to one function)
$g(x)=x/2$

Similarly, $|(0, 1)| = |(0, 1/2)|$.

The theorem seems to be quite straightforward, we might expect that it has an easy proof. However, even though it can be proved without using advanced mathematics, no known proof is easy to explain. Consequently, we omit a proof here. This result is called the Schröder-Bernstein theorem after Ernst Schröder who published a flawed proof of it in 1898 and Felix Bernstein, a student of Georg Cantor, who presented a proof in 1897. However, a proof of this theorem was found in notes of Richard Dedekind dated 1887.

Proof in [AiZiHo09] and [Ve06].

Youtube The Cantor-Schroeder-Bernstein Theorem

Definition 4 (page 175)

A function is computable if there is a computer program in some programming language that finds the values of this function. If a function is not computable we say it is uncomputable.

Set of programs is countable.

There are uncomputable functions

The set of functions f
$f: Z \to Z$
is uncountable (Exercise 38)

The set of functions f
$f: Z \to \{0,1\}$
$f: \{0,1\} \to Z$
countable or not countable?

Hilbert's tenth problem:

for any given Diophantine equation (a polynomial equation with integer coefficients, decide whether the equation has a solution with integer values

$3x^2-2xy-y^2z-7=0$

has an integer solution x=1, y=2,z=-2

$X^2+y^2+1=0$

has no integer solutions

Hilbert's tenth problem has been solved, answer negative: such a general algorithm does not exist.

Combined work of Martin Davis, Yuri Matiyasevich, Hilary Putnam and Julia Robinson, with Matiyasevich completing the theorem in 1970 in his PhD thesis. The theorem is now known as Matiyasevich's theorem or the MRDP theorem.

Discussion in class

1,15,19

Review

Logic

concepts, truth table, laws

Set theory

concepts, laws

Review

Function

inverse, composition, one to one, on to, one to one correspondence

Cardinality

one to one correspondence =

one to one <=

countable: finite set, Z, Q

uncountable: (0,1), (0.1], R