## § 1.5 Nested Quantifiers (量词嵌套)

1. Introduction

nested quantifiers

-------occur within the scope of other quantifiers

$\forall x\ \exists y(x+y=0)$

(1) Example 1 (page 57)

domain for x and y

--------all real numbers

$\forall x\ \forall y\ (x+y=y+x)$------commutative law

-------true

$\forall x\ \exists y\ (x+y=0)$------true

$\forall x\ \forall y\ \forall z\ (x+(y+z)=(x+y)+z)$------true

-------associative law

(2) Example 2 (page 58)

$\forall x\ \forall y\ ((x>0) \wedge (y<0) \rightarrow (xy<0))$

domain----all real numbers

-----English meaning

-----value…….true

2. The order of Quantifiers

(1) Example 3 (page 58)

P(x,y)-----"x+y=y+x"

domain for all variables

------all real numbers

How about

$\forall x\ \forall y\ P(x,y)$----------true

$\forall y\ \forall x\ P(x,y)$----------true

We have:

$\forall x\ \forall y\ P(x,y) \equiv \forall y\ \forall x\ P(x,y)$

(2) Example 4 (page 59)

Q(x,y)------"x+y=0"

universe of discourse----all real numbers

How about $\exists y\ \forall x\ Q(x,y)$ and $\forall x\ \exists y\ Q(x,y)$?

Answer:

(a) $\exists y\ \forall x\ Q(x,y)$----There is a real number y such that for every real number x, Q(x,y).

----false

(b) $\forall x\ \exists y\ Q(x,y)$----For every real number x there is a real number y such that Q(x,y).

-----true

(c) $\exists y\ \forall x\ Q(x,y)$-----$\forall x\ \exists y\ Q(x,y)$

( not equivalent)

(3) Summary (page 60, Table 1)

Statement When true? When false?

$\forall x\ \forall y\ P(x,y)$

$\forall y\ \forall x\ P(x,y)$

---

$\forall x\ \exists y\ P(x,y)$

---

$\exists x \forall y\ P(x,y)$

---

$\exists x\ \exists y\ P(x,y)$

$\exists y\ \exists x\ P(x,y)$

Further,

**(a) If $\exists y \forall x\, P(x,y)$ is true,**
**then $\forall x \exists y\, P(x,y)$ is true.**
**(b) If $\forall x \exists y\, P(x,y)$ is true,**
**then it is not necessary for**
**$\exists y \forall x\, P(x,y)$ to be true.**

**Please see Exercise 28 and 30 at the end of this chapter (page 113).**

**(4) Example 5 (page 59)**

**$Q(x,y,z)$-----"x+y=z"**
**domain-----all real numbers**
**How about $\forall x\, \forall y\, \exists z\, Q(x,y,z)$ and**
**$\exists z\, \forall x\, \forall y\, Q(x,y,z)$**

**Answer:**
**$\forall x\, \forall y\, \exists z\, Q(x,y,z)$ is true.**
**$\exists z\, \forall x\, \forall y\, Q(x,y,z)$ is false.**

## 3. Translating Mathematical Statements into Statements Involving Nested Quantifiers

**(1) Example 6**

**Translate the statement**
***"The sum of two positive integers is always positive"***
**into a logical expression.**

**Answer:**

**Way1: domain for x and y----all integers**
$\forall x \forall y\, ( (x>0) \wedge (y>0) \rightarrow (x+y>0) )$

**Way 2: domain for x and y----all positive integers**
$\forall x \forall y\, ( x+y>0 )$

**(2) Example 7**

**Translate the statement**
***"Every real number except zero has a multiplicative inverse"***

**Answer:**

**Domain for x and y-----all real numbers**

$$\forall x\, ( (x\neq 0) \rightarrow \exists y\, (xy=1) )$$

## 4. Translating from Nested Quantifiers into English

**(1) Example 9 (page 61)**

$$\forall x\, ( C(x) \vee \exists y\, (C(y) \wedge F(x,y)) )$$

**$C(x)$----"x has a computer"**
**$F(x,y)$----"x,y are friends"**
**universe of discourse for both x and y**
**-----------all students in the school**

**What does the formula mean?**

**(2) Example 10 (page 61)**

$$\exists x\, \forall y\, \forall z\, ( (F(x,y) \wedge F(x,z) \wedge (y\neq z)) \rightarrow \neg F(y,z) )$$

**$F(a,b)$----a and b are friends**
**domain for x, y and z**
**----- all students in your school**

**What does this formula mean?**

## 5. Translating English Sentences Into Logical Expression

(1) Example 11 (page 62)

"*If a person is female and is parent, then this person is someone's mother*."

domain------all people

Answer:

also can be expressed as

"*For every person, if person x is a female and person x is a parent, then there exists a person y such that person x is the mother of person y*."

F(x)-----x is female; P(x)-----x is a parent

M(x,y)-----x is the mother of y

Then, the formula is:

$\forall x\,((F(x) \land P(x)) \rightarrow \exists y\, M(x,y))$ or

$\forall x\, \exists y\,((F(x) \land P(x)) \rightarrow M(x,y))$

(2) Example 12 (page 62)

"*Everyone has exactly one best friend*"

domain--------all people

Answer:

"*For every person x, person x has exactly one best friend*"

B(x,y) -----y is the best friend of x

$\exists y\,( B(x,y) \land \forall z\,((z \neq y) \rightarrow \neg B(x,z)))$

$\forall x$(………the above formula…………….)

## 6. Negating Nested Quantifiers

(1) Example 14 (page 63)

Express the negation of $\forall x \exists y\,(xy=1)$ so that no negation precedes a quantifier.

Answer:

$\neg \forall x \exists y\,(xy=1)$

$\equiv \exists x\, \neg \exists y\,(xy=1)$

$\equiv \exists x\, \forall y\, \neg (xy=1)$

$\equiv \exists x\, \forall y\,(xy \neq 1)$

## Discussion (in theory class)

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