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Rota's Fubini lectures: The first problem

Daniele Mundici

Department of Mathematics and Computer Science "Ulisse Dini", University of Florence, Viale Morgagni 67/A, I-50134 Florence, Italy

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ABSTRACT

In his 1998 Fubini Lectures, Rota discusses twelve problems in probability that "no one likes to bring up". The first problem calls for a revision of the notion of a sample space, guided by the belief that mention of sample points in a probabilistic argument is bad form and that a "pointless" foundation of probability should be provided by algebras of random variables.

In 1958 Chang introduced MV-algebras to prove the completeness theorem of Łukasiewicz logic $Ł_\infty$. The aim of this paper is to show that MV-algebras provide a solution of Rota's first problem.

The adjunction between MV-algebras and unital commutative C*-algebras equips every MV-algebra A with a natural ring structure, as advocated by Nelson for algebras of random variables. The closed compact set $\mathsf{S}(A) \subseteq [0,1]^A$ of finitely additive probability measures on A (the states of A) coincides with the set of $[0,1]$-valued functions on A whose finite restrictions are consistent in de Finetti's sense. MV-algebras and $Ł_\infty$ thus provide the framework for a generalization (known as ŁIPSAT) of Boole's probabilistic inference problem, and its modern reformulation known as probabilistic satisfiability, PSAT. We construct an affine homeomorphism γ_A of $\mathsf{S}(A)$ onto the weakly compact space of regular Borel probability measures on the maximal spectral space $\boldsymbol{\mu}(A)$. The latter is the most general compact Hausdorff space. As a consequence, for every Kolmogorov probability space $(\Omega, \mathcal{F}_\Omega, P)$, with $\mathcal{F}_\Omega$ the sigma-algebra of Borel sets of a compact Hausdorff space Ω, and P a regular probability measure on $\mathcal{F}_\Omega$,

E-mail address: daniele.mundici@unifi.it.

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there is an MV-algebra A and a state σ of A such that $(\Omega, \mathcal{F}_\Omega, P) \cong (\boldsymbol{\mu}(A), \mathcal{F}_{\boldsymbol{\mu}(A)}, \gamma_A(\sigma))$.

1. Introduction

In the introductory pages of his 1998 Fubini lectures Rota writes:

> I will lay my cards on the table: a revision of the notion of a sample space is my ultimate concern. I hasten to add that I am not about to put forth concrete proposals for carrying out such a revision. We will, however, be guided by a belief that has been a guiding principle of the mathematics of this century. Analysis will play second fiddle to algebra.
>
> [...]
>
> Pointless probability deals with an abstract Boolean σ-algebra Π and with a real-valued function defined on Π which imitates the definition of probability. The problem is to define an algebra of random variables. The setup is not as artificial as it may appear. Among probabilists, mention of sample points in an argument has always been bad form. A fully probabilistic argument must be pointless.
>
> Rota, [59, pp. 57 and 60]

Over the last years, many papers have been devoted to the exploration of probability in the framework of nonclassical logics and their algebras. See, e.g., [1,9,10,18,26–29,48,63]. This thriving literature witnesses the significance of Rota's first problem: logicians and computer scientists do like to bring it up.

Most of the cited papers deal with Gödel-Dummett, Heyting, product, modal and Łukasiewicz logic and their algebras, building on Hájek's seminal work [33] in logic and probability. The common denominator of these papers and monographs goes back to Boole's classical work on logic and probability [3]. Their general aim is to extend the properties of boolean algebras *qua* algebras of yes-no random variables.

Boole's work is also the main archetype of the present paper. Specifically, the following basic properties of boolean algebras and their states (also known as finitely additive probability measures) will serve throughout as a template in our investigation on Rota's first problem: Stone duality [45,61] identifies every boolean algebra A with an algebra of continuous $\{0, 1\}$-valued random variables, or yes-no events, defined over the maximal spectral space $\boldsymbol{\mu}(A)$. Elliott's classification [20,30] for commutative unital approximately finite-dimensional (AF) algebras yields a categorical equivalence between these C*-algebras and boolean algebras. This result discloses the ring-theoretic structure of boolean algebras.[1] To provide a logic-operational definition of probability for boolean events, the Dutch Book theorem [12,14], characterizes the set $\mathsf{S}(A)$ of states of A as the set of maps $\sigma\colon A \to [0, 1]$ whose finite restrictions are consistent (i.e., nondutchbookable)

[1] According to Nelson [56, Chapter 2], addition and multiplication are the bare minimum needed for any algebraic approach to probability theory.

in the sense of de Finetti. See Theorem 2.3(i)⇔(ii) and references therein. Corollary 2.5 shows that de Finetti's consistency[2] is a generalization of boolean consistency. Last, but not least, Carathéodory's extension theorem, [62, pp. 277-280], yields an affine homeomorphism of $\mathsf{S}(A)$ (equipped with the restriction topology of $[0,1]^A$) onto the convex space $\mathsf{P}(A)$ (equipped with the weak topology) of regular Borel probability measures on $\boldsymbol{\mu}(A)$. In this way, countable additivity is made implicit in the finite additivity of finitely consistent maps on A. See Theorem 2.3(ii)⇔(iii).

One major problem restricts the usefulness of Theorem 2.3 as a solution of Rota's first problem: since any boolean algebra A is an algebra of random $\{0,1\}$-valued variables, the measures of $\mathsf{P}(A)$ are only defined on the Borel sets of the totally disconnected compact Hausdorff space $\boldsymbol{\mu}(A)$. This leaves out most random variables of interest to probabilists. For a comprehensive solution of Rota's first problem, Theorem 2.3 must be extended from yes-no events to continuous random variables over arbitrary compact Hausdorff spaces. This is the purpose of the present paper. We will prove that MV-algebras and their states, [7,52] yield the desired extension, thus providing a solution of Rota's first problem. MV-algebras stand to Łukasiewicz infinite-valued logic as boolean algebras stand to boolean logic. Already in his original paper [5], Chang proved that idempotent MV-algebras coincide with boolean algebras.

Theorem 3.1 is the basic preliminary step to extend Theorem 2.3(ii)⇔(iii) to *continuous random variables over any compact Hausdorff space.* Let $\to$ be a continuous $[0,1]$-valued binary operation defined on $[0,1]^2$. Suppose $\to$ has the following properties: (i) $x \to y = 1$ iff $x \leq y$; (ii) $x \to (y \to z) = y \to (x \to z)$. Let $\neg x$ be shorthand for $x \to 0$. Theorem 3.1 states that the algebra $([0,1], 0, \neg, \to)$ is term-equivalent to an MV-algebra. The theorem sheds new light on the meaning of the Łukasiewicz axioms, notably the intriguing axiom $(x \to y) \to y = (y \to x) \to x$. As a matter of fact, for any $[0,1]$-valued function $\to$ on $[0,1]^2$ having properties (i) and (ii), *failure of this axiom implies the discontinuity of* $\to$.

Next, the implicit *ring-theoretic nature* of MV-algebras follows from the adjunction [8, §4] between the category of MV-algebras and the dual of the category of compact Hausdorff spaces—i.e., unital commutative C*-algebras (by Gelfand duality, [19,40]). This is proved in Theorem 4.7. As a consequence, for every unital commutative C*-algebra $\mathcal{A}$, letting $\mathcal{X}_{\mathcal{A}}$ be the compact space of its maximal (closed, two-sided) ideals, the map $\mathcal{A} \mapsto C(\mathcal{X}_{\mathcal{A}}, [0,1])$ induces a categorical equivalence between unital commutative C*-algebras and the full subcategory of MV-algebras isomorphic to $C(X, [0,1])$ for X a compact Hausdorff space.

A *state* of an MV-algebra A is a normalized finitely additive $[0,1]$-valued function on A. By Theorem 5.7(i)⇔(ii), the set of states of any MV-algebra A coincides with the set of $[0,1]$-valued maps on A whose finite restrictions are consistent (i.e., non-dutchbookable) in de Finetti's operational sense. This generalizes Theorem 2.3(i)⇔(ii)

[2] "coerenza" in Italian. This is the substantive used by de Finetti in his original paper [12] where this fundamental notion was introduced and characterized by the Dutch Book theorem.

to MV-algebras. Theorems 5.9-5.11 generalize Theorem 2.3(ii)⇔(iii), yielding canonical affine homeomorphisms between the following compact convex sets:

(a) The set $\mathsf{S}(A)$ of states of A equipped with the restriction topology of the Tychonoff cube $[0,1]^A$.
(b) The set $\mathsf{P}(A)$ (with the weak topology) of regular Borel probability measures on the maximal spectral space $\boldsymbol{\mu}(A)$. By Proposition 4.6, $\boldsymbol{\mu}(A)$ ranges over all compact Hausdorff spaces as A ranges over MV-algebras.
(c) The space of states (with the weak* topology, [19,40]) of the C*-algebra $\mathcal{A} = C(\boldsymbol{\mu}(A), \mathbb{C})$ given by the above mentioned adjunction.

Letting γ_A be the affine homeomorphism of $\mathsf{S}(A)$ onto $\mathsf{P}(A)$, it follows that for every compact Hausdorff space Ω and Kolmogorov probability space $(\Omega, \mathcal{F}_\Omega, P)$, with $\mathcal{F}_\Omega$ the sigma-algebra of Borel sets of Ω and P a regular probability measure on $\mathcal{F}_\Omega$, there is an MV-algebra A and a state σ of A such that $(\Omega, \mathcal{F}_\Omega, P) \cong (\boldsymbol{\mu}(A), \mathcal{F}_{\boldsymbol{\mu}(A)}, \gamma_A(\sigma))$.

This shows that MV-algebras and their states are sufficiently powerful to *recover Kolmogorov probability on arbitrary compact Hausdorff spaces,* while providing an answer to Rota's call for algebras of continuous random variables that do not pay homage to "Her Imperial Majesty the Theory of Commutative Rings", [59, p. 58]. While Nelson advocates addition and multiplication for algebras of random variables, the affine homeomorphisms (a)-(c) show that there is no conflict between these two positions.

Yet another resolved conflict pertains to the finite vs. countable additivity *vexata quaestio* in probability theory. While de Finetti championed finite additivity, Kolmogorov introduced the countable additivity axiom as follows:

> We limit ourselves, arbitrarily, to only those models which satisfy Axiom VI [Countable Additivity]. This limitation has been found expedient in researches of the most diverse sort.
>
> See [43, p. 15 in the English translation].

Via the affine homeomorphisms (a)-(b), states may be identified with regular Borel probability measures. The former are finitely additive, the latter are countably additive. Therefore, the countable additivity *axiom* may be safely thought of as a *consequence* of de Finetti's *definition* of consistency. In the present author's opinion, this attitude is preferable to "arbitrarily" accepting countable additivity as a convenient tool to prove strong theorems in probability theory. For, convenience is not among the admissible acceptance criteria for axioms.

The affine homeomorphisms (a)-(b) also show that the finite vs. countable additivity debate makes no more sense than, say, championing positive Radon measures vs. regular Borel measures in integration theory. One will adopt the more convenient framework depending on the actual problem under consideration.

For instance, let us consider Boole's problem on probabilistic entailment, [3, Chapter XVI, 4, p. 246], quoted at the beginning of Section 6. In the language of today's computability theory, this problem has well known formulations: PSAT for boolean events, [35, p. 323], [57], and ŁIPSAT for continuous random variables, [4,11,24,25]. Finitely consistent maps on MV-algebras provide a natural approach to both problems, dispensing with what Rota dubbed the "psychological prop" of sample points, [59, p. 79]. This is shown in Theorem 6.1. The deductive algorithms of Łukasiewicz logic $Ł_\infty$ can be directly applied to the ŁIPSAT problem, just as those of boolean logic can be applied to SAT. Furthermore, ŁIPSAT has the same NP-complete complexity as SAT. Combining this result with Theorems 3.1 and 4.7 it follows that Łukasiewicz logic and MV-algebras provide an indispensable tool for a *computable* logic-algebraic approach to *continuous* random variables on compact Hausdorff spaces.

As a second example, Theorem 7.2 shows that multiplication in the *definition* of stochastic independence is the only operation that preserves the consistency of joint books on two sets of algebraically independent random variables. Finitely consistent maps on MV-algebras are the key tool to elucidate the symbiotic relationship between stochastic independence and (probability-free) logic/algebraic independence, a fairly neglected issue in standard textbooks on probability, notwithstanding its foundational interest and mathematical depth.[3]

The excess of ring-theoretic, topological and measure-theoretic structure in the C*-algebra $C(X, \mathbb{C})$ would seem to obscure, rather than clarify, what makes Theorems 6.1 and 7.2 work.

In conclusion, MV-algebras and their states provide a solution of Rota's first problem. Kolmogorov probability spaces $(\Omega, \mathcal{F}_\Omega, P)$ are a natural spinoff of this class of algebras.

2. The boolean archetype: de Finetti, Stone, Carathéodory

Definition 2.1. By a set of *yes-no events* we mean a subset E of some boolean algebra $A = (A, 0, 1, \neg, \wedge, \vee)$. The dependence of E on A will always be clear from the context. By the totality of *possible outcomes* (of these events) we mean the set of homomorphisms of A into the boolean algebra $\{0, 1\} = (\{0, 1\}, 0, 1, \neg, \min, \max)$, where $\neg x = 1 - x$.

The constants and operations of the ambient boolean algebra A give a precise meaning to "the event h and k", occurring when both h and k occur, and to the event "not h", occurring when h doesn't. A mathematical meaning can then be given to more complex expressions like "exactly one of the events 1,X,2 will occur". Further, by saying "for every homomorphism η of A into $\{0, 1\}$", we give a precise meaning to colloquial expressions like "for any possible outcome of the events", or "regardless of the outcome of the events", or even "in any possible world", currently used in probabilistic arguments.

The most important property of $[0, 1]$-valued maps on finite sets of events is given by the following consistency notion, due to de Finetti (see [12, §7, pp. 307-308]:

[3] See Hilbert's 1905 remark in [37, p. 168], on axioms and definitions in probability theory, mentioned at the end of Section 7.

Definition 2.2. Let A be a boolean algebra and $E = \{h_1, \dots h_m\} \subseteq A$. A map $\beta\colon E \to [0,1]$ is *consistent (in A)* if for every $s\colon E \to \mathbb{R}$ there is a homomorphism η of A into $\{0,1\}$ such that $\sum_{i=1}^m s(h_i) \cdot (\beta(h_i) - \eta(h_i)) \geq 0$. We say that β is *inconsistent* if it is not consistent.

As explained by de Finetti in [12, last paragraph, page 308], the rationale behind this definition is as follows: $\beta(h_1), \dots, \beta(h_m)$ is a "book" listing the "betting odds" posted by Bookie on "events" $h_1, \dots, h_m$. Bettor places "stakes" $s(h_1), \dots, s(h_m) \in \mathbb{R}$ on these events, with the understanding that $s(h_i)$ is the prize money she hopes to win in case h_i occurs, by paying Bookie $s(h_i) \cdot \beta(h_i)$ in advance. For some $i = 1, \dots, m$, Bettor may place a stake $s(h_i) < 0$, which results in the swapping of the Bookie/Bettor roles: Bookie advances Bettor the amount $|s(h_i)| \cdot \beta(h_i)$, with the understanding that Bettor will pay him back $|s(h_i)|$ if h_i occurs. Upon giving financial transactions the bookie-to-bettor orientation, the quantity $\sum_{i=1}^m s(h_i) \cdot (\beta(h_i) - \eta(h_i))$ is the balance of Bettor's stakes $s(h_i)$ given Bookie's book β in the "possible world" η.

Naive as it may sound to real-life bookmakers and bettors, this setup is fit to expose (and mathematically certify) the inconsistency of Bookie's book β in terms of Bettor's ability to devise stakes $s(h_1), \dots, s(h_m) \in \mathbb{R}$ guaranteeing her a minimum profit of one zillion, thus bankrupting Bookie "regardless of the outcome of the events $h_1, \dots, h_m$".

Rota's quest for a pointless probability theory based on algebras of random variables has the following first approximation for yes-no events:

Theorem 2.3. *Let A be a boolean algebra and $\sigma\colon A \to [0,1]$ a* function. *The following conditions are equivalent:*

(i) (Finite consistency) *Every finite restriction of σ is consistent.*

(ii) (Finite additivity) *σ has the* finite additivity property*: $\sigma(x \vee y) = \sigma(x) + \sigma(y)$ whenever $x \wedge y = 0$. Further, $\sigma(1) = 1$. In other words, σ is a* finitely additive probability measure on *(for short, a* state[4] *of) A, [39], [42].*

(iii) (Topological countable additivity) *There is precisely one regular Borel probability measure μ_σ on the Borel sets of the Stone space $\boldsymbol{\mu}(A)$ of A such that, upon identifying A with the boolean algebra of characteristic (also known as the indicator) functions of the clopen sets of $\boldsymbol{\mu}(A)$, we have*

$$\sigma(f) = \int\limits_{\boldsymbol{\mu}(A)} f(\mathfrak{m})\, \mathrm{d}\mu_\sigma(\mathfrak{m}) \quad \textit{for all } f \in A. \tag{1}$$

Thus the map $\sigma \mapsto \mu_\sigma$ is an affine isomorphism of the convex set of functions $\sigma\colon A \to [0,1]$ satisfying the equivalent conditions (i)-(ii), onto the convex set $\mathsf{P}(A)$ of regular Borel probability measures on $\boldsymbol{\mu}(A)$.

[4] This terminology will be justified at the outset of Section 5.

Proof. (ii) $\Rightarrow$ (i) Let E be a finite subset of A, and $\beta\colon E \to [0,1]$ a map. By de Finetti's Dutch Book theorem, [12, pp. 309-313], [13, Chapter 1, pp. 7-9], β is consistent iff it can be extended to a state of A.

(i) $\Rightarrow$ (ii) The product space $[0,1]^A$ is compact and so is the set of states of A. For every finite restriction β of σ, from the assumed consistency of β it follows that the set S_β of states of A extending β is nonempty, again by de Finetti's Dutch Book theorem. Further, S_β is closed, by definition of the product topology of $[0,1]^A$. A moment's reflection shows that for every finite family B of finite restrictions of σ, the set S_B of states that simultaneously extend each $\beta \in B$ is nonempty and closed. Compactness yields a state σ^* simultaneously extending *all* finite restrictions of σ. Necessarily $\sigma = \sigma^*$.

A proof of (ii) $\Leftrightarrow$ (iii) is obtained combining Stone duality ([61, I.8.2], [45, Theorem 7.8]) with Carathéodory's extension theorem ([62, pp. 277-280]). □

Nelson's quest for addition and multiplication in any algebra of random variables, [56, Chapter 2] has the following formulation for yes-no events:

Proposition 2.4. *Boolean algebras are categorically equivalent to unital commutative approximately finite dimensional (AF) C*-algebras.*

Proof. This is the (essentially trivial) fragment of Elliott classification for commutative AF algebras, [20,30]. □

The Italian adjective "coerente" (resp., the French "cohérent") adopted by de Finetti in his original paper [12] (resp., in his paper [13]), is translated "consistent" in the present paper, because of the following result, first noted by [16], which is the special case of Theorem 2.2(i)$\Leftrightarrow$(ii) for $\{0,1\}$-valued betting odds:

Corollary 2.5. *Let A be a boolean algebra, $E = \{h_1, \dots h_m\} \subseteq A$, and $\beta\colon E \to [0,1]$ a function. In case $\beta(h_i) = 1$ for all $i = 1, \dots, m$, β is consistent iff there is a homomorphism $\eta\colon A \to \{0,1\}$ such that $\eta(h_1) = \dots = \eta(h_m) = 1$. Therefore, upon coding each $h_i \in E$ by a boolean formula ϕ_i, the finite consistency of β means the existence of a truth-valuation satisfying all ϕ_i, i.e., the logical consistency,[5] of the set of formulas $\{\phi_1, \dots, \phi_m\}$.*

Finally, the following result shows that the inconsistency of a map $\beta\colon E \to [0,1]$ defined on a finite set $E \subseteq A$ of events is largely insensitive of the ambient boolean algebra A. This, together with de Finetti's notorious disinclination to logic, may explain why (ambient) boolean algebras are never mentioned in de Finetti's original papers [12,13] on inconsistency.

[5] "coerenza logica" in Italian.

Proposition 2.6. *A map $\beta\colon E \to [0,1]$ defined on a finite subset E of a boolean algebra A is consistent in A iff it is consistent in every boolean algebra $B \supseteq A$ iff it is consistent in the boolean algebra generated by E in B.*

Proof. For any boolean algebra $B \supseteq A$, any state of A is extendible to a state of B. This is essentially a variant of the Hahn-Banach theorem. See, e.g., [39, Theorem 1.22] for a direct proof. Then apply Theorem 2.2(i)$\Leftrightarrow$(ii). $\square$

3. Axioms for continuous $[0, 1]$-valued implications

One major problem restricts the usefulness of Theorem 2.3: the regular Borel probability measures μ_σ given by Carathéodory's Theorem 2.3(iii) are defined on *totally disconnected* compact Hausdorff spaces, the maximal spectral spaces of boolean algebras. This leaves out most topological spaces of interest to probabilists.

Starting from very basic principles, in the spirit of Rota's Fubini lectures, in Section 5 we will obtain regular Borel probability measures on *arbitrary compact Hausdorff spaces*. The latter are the maximal spectral spaces of an equational class $\mathcal{B}_{[0,1]}$ of algebras which largely extends the equational class $\mathcal{B}_{\{0,1\}}$ of boolean algebras and will be shown to provide sufficiently general and interesting algebras of random variables.

Our first task is to make $[0, 1]$ into an algebra having the same role in $\mathcal{B}_{[0,1]}$ as the two-element boolean algebra $\{0, 1\}$ has in $\mathcal{B}_{\{0,1\}}$. A preliminary stumbling block stands in the way of constructing such algebra: While either set of boolean operations $\{\neg, \wedge\}$, or $\{\neg, \vee\}$, or even $\{\to\}$, etc., is sufficient to define every operation $o\colon \{0, 1\}^n \to \{0, 1\}$, no finite set of operations defines all operations $e\colon\ [0, 1]^n \to [0, 1]$. As a consequence, we are forced to make a drastic selection of the basic operations on $[0, 1]$. Since implication and inference are symbiotic in logic, recalling from Corollary 2.5 the logical underpinnings of de Finetti's consistency notion, we single out *implication* as the primary operation for the algebras in $\mathcal{B}_{[0,1]}$.

Boolean implication $\to\colon \{0, 1\}^2 \to \{0, 1\}$ can be characterized by the following order property (i): $x \to y = 1$ iff $x \le y$. In this way, $\to$ complies with our intuition of the natural order of the "truth values" $\{0, 1\} = \{\mathsf{False}, \mathsf{True}\}$. As a consequence, the map $\to$ has the following exchange property (ii): $x \to (y \to z) = y \to (x \to z)$, for all $x, y, z \in \{0, 1\}$. This equation states that the order of appearance of the premises x, y is immaterial in the derivation of z. While the continuity of boolean implication is automatic, the continuity of a $[0, 1]$-valued implication defined on $[0, 1]^2$ is to be assumed, thus ensuring that small errors or perturbations in the evaluation of basic random variables have small immediate repercussions on the value of derived random variables.

Remarkably enough, properties (i)-(ii) together with continuity yield the desired extension $\mathcal{B}_{[0,1]}$ of the class of boolean algebras:

Theorem 3.1. *Let $\to$ be a continuous $[0, 1]$-valued map defined on the unit square $[0, 1]^2$. Suppose that for all $x, y, z \in [0, 1]$, $\to$ satisfies the following two conditions:*

(i) $x \to y = 1$ *iff* $x \leq y$ *(order property);*
(ii) $x \to (y \to z) = y \to (x \to z)$ *(premise exchange property).*

Letting $\neg x$ *stand for* $x \to 0$ *we then have:*

(a) *The algebra* $W = ([0,1], 1, \neg, \to)$ *is a* Wajsberg algebra*, i.e.,* W *satisfies the following equations ([7, 4.2.1]):*

$$1 \to x = x$$
$$(x \to y) \to ((y \to z) \to (x \to z)) = 1$$
$$(\neg x \to \neg y) \to (y \to x) = 1$$
$$((x \to y) \to y) = ((y \to x) \to x).$$

(b) *The algebra* $A = ([0,1], 0, \neg, \oplus)$*, with* $x \oplus y = \neg x \to y$*, is an* MV-algebra*, i.e.,* A *satisfies the following equations*[6]*:*

$$x \oplus (y \oplus z) = (x \oplus y) \oplus z$$
$$x \oplus y = y \oplus x$$
$$x \oplus 0 = x$$
$$x \oplus \neg 0 = \neg 0$$
$$\neg\neg x = x$$
$$\neg(\neg x \oplus y) \oplus y = \neg(\neg y \oplus x) \oplus x.$$

Proof. The proof proceeds through the following steps, whose verification is a tedious but straightforward exercise (see [55] for the proof of a stronger result):

$$x \to y = \neg y \to \neg x$$
$$y \leq z \Rightarrow x \to y \leq x \to z$$
$$y \leq z \Rightarrow x \odot y \leq x \odot z$$
$$(y \to x) \to x = \max(x, y)$$
$$(x \to y) \odot z \leq x \to (y \odot z)$$
$$x \to y \leq ((z \to x) \to (z \to y)),$$

with the operation $\odot$ defined by $x \odot y = \neg(\neg x \oplus \neg y)$. □

[6] Compare Chang's original list of equations in [5, p. 468] with the equivalent list of six equations in [7, p. 7]. Actually, Kolařík [44] proved that commutativity of $\oplus$ follows from the remaining five equations.

Remark 3.2. The first three equations in Theorem 3.1(a) are the algebraic counterparts of commonplace properties of implication in most logics existing in the literature (triviality, exchange, contraposition). Theorem 3.1 highlights the significance of the fourth (Łukasiewicz) equation, as a *sine qua non* for the continuity of the implication.

By [7, 4.2.2-4.2.5], Wajsberg algebras are term-equivalent to MV-algebras, just as boolean algebras based on implication and negation are term-equivalent to boolean algebras based on $\neg, \vee, \wedge$. Following boolean algebraic tradition, where the associative-commutative operations $\wedge, \vee$ are more expedient than $\to$, we will henceforth work with MV-algebras. Boolean algebras coincide with MV-algebras satisfying the idempotence axiom $x \oplus x = x$.

4. The natural ring structure of MV-algebras

The *standard* MV-algebra $[0,1] = ([0,1], 0, \neg, \oplus)$ has the operations $\neg x = 1 - x$, $x \oplus y = \min(1, x+y)$, the derived constant $1 = \neg 0$ and the derived operation $x \odot y = \max(0, x+y-1) = \neg(\neg x \oplus \neg y)$. Chang [6] proved that if an equation holds in the standard MV-algebra then it holds in all MV-algebras. (See [7, §2] for a self-contained proof.) Among others, this completeness theorem shows that $[0,1]$ has the same role for MV-algebras as $\{0,1\}$ does for boolean algebras.

For all MV-algebras A, B and homomorphism $\eta\colon A \to B$, the kernel of η, $\mathsf{ker}(\eta)$, is the set of elements $x \in A$ such that $\eta(x) = 0$. *Ideals* are kernels of homomorphisms. We denote by $\boldsymbol{\mu}(A)$ the set of maximal ideals of A. An MV-algebra is *semisimple* if the intersection of all its maximal ideals is $\{0\}$. In particular, every boolean algebra is a semisimple MV-algebra, where the operations $\oplus$ and $\odot$ collapse to $\vee$ and $\wedge$ respectively.

The maximal spectral space $\boldsymbol{\mu}(A)$. For any MV-algebra A the set $\boldsymbol{\mu}(A)$ comes equipped with the *spectral (*also known as the *hull-kernel) topology*. A subbasis of open sets is given by letting a range over all elements of A, and defining O_a as the set of maximal ideals $\mathfrak{m}$ of A such that $a \notin \mathfrak{m}$. Actually, the family of sets O_a forms a basis, because $O_{a\wedge b} = O_a \cap O_b$, where $\wedge$ is the natural infimum operation of A, [7, Proposition 1.1.5]. We denote by $\boldsymbol{\mu}(A)$ the resulting topological space, and call it the *maximal spectral space of* A. A basis of closed sets for the spectral topology of $\boldsymbol{\mu}(A)$ is given sets of the form

$$F_a = \{\mathfrak{m} \in \boldsymbol{\mu}(A) \mid a \in \mathfrak{m}\}. \tag{2}$$

When A is a boolean algebra the maximal spectral topology of A is known as the Stone topology of A.

Proposition 4.1. *The maximal spectral space of any MV-algebra A is a nonempty compact Hausdorff space.*

Proof. $\boldsymbol{\mu}(A) \neq \emptyset$ by [7, 1.2.15]. Since any two maximal ideals $\mathfrak{n} \neq \mathfrak{m} \in \boldsymbol{\mu}(A)$ are incomparable, there are elements $a \in \mathfrak{m} \setminus \mathfrak{n}$ and $b \in \mathfrak{n} \setminus \mathfrak{m}$. Let $a' = a \odot \neg b$ and $b' = b \odot \neg a$. Since by [7, 1.2.14], every maximal ideal is prime, a direct verification shows that $O_{a'}$ and $O_{b'}$ are disjoint open neighborhoods of $\mathfrak{m}$ and $\mathfrak{n}$ respectively, whence $\boldsymbol{\mu}(A)$ is a Hausdorff space.

To complete the proof we must show that any family $\mathcal{F}$ of closed sets of $\boldsymbol{\mu}(A)$ has a nonempty intersection if it has the *finite intersection property*, meaning that every finite subset of $\mathcal{F}$ has nonempty intersection. It is no loss of generality to assume that $\mathcal{F}$ is maximal for the finite intersection property. Let

$$\mathfrak{i} = \{a \in A \mid F_a \in \mathcal{F}\},$$

where $F_a \subseteq \boldsymbol{\mu}(A)$ is the basic closed set defined in (2). In view of the maximality property of $\mathcal{F}$, the following implications are easily verified:

$$b \leq a \in \mathfrak{i} \;\; \Rightarrow \;\; F_b \supseteq F_a \in \mathcal{F} \;\; \Rightarrow \;\; F_b \in \mathcal{F} \;\; \Rightarrow \;\; b \in \mathfrak{i}.$$

For short, $\mathfrak{i}$ is closed under minorants. Trivially, $1 = \neg 0 \notin \mathfrak{i} \ni 0$. Since $\mathcal{F}$ has the finite intersection property and $F_{a \oplus b} = F_a \cap F_b$, then

$$a, b \in \mathfrak{i} \Rightarrow a \oplus b \in \mathfrak{i},$$

whence $\mathfrak{i}$ is closed under the $\oplus$ operation. It follows that $\mathfrak{i}$ is an ideal of A. Let $\mathfrak{m}$ be a maximal ideal of A containing $\mathfrak{i}$, as given by [7, 1.2.11-1.2.12]. Since $\boldsymbol{\mu}(A)$ is a Hausdorff space, the family

$$\mathcal{G} = \{F \subseteq \boldsymbol{\mu}(A) \mid F \text{ is closed and } \mathfrak{m} \in F\}$$

satisfies $\{\mathfrak{m}\} = \bigcap \mathcal{G} \subseteq \bigcap \mathcal{F}$, (actually, $\bigcap \mathcal{G} = \bigcap \mathcal{F}$ by the assumed maximality of $\mathcal{F}$). Thus $\bigcap \mathcal{F} \neq \emptyset$ and $\boldsymbol{\mu}(A)$ is compact. $\square$

Proposition 4.2. *Let A be an MV-algebra. For any $\mathfrak{m} \in \boldsymbol{\mu}(A)$ there is a unique pair $(\omega_\mathfrak{m}, \Omega_\mathfrak{m})$ with $\Omega_\mathfrak{m}$ an MV-subalgebra of the standard MV-algebra $[0,1]$, and $\omega_\mathfrak{m}$ an isomorphism of the quotient MV-algebra $A/\mathfrak{m}$ onto $\Omega_\mathfrak{m}$.*

Proof. By [7, 1.2.10], from the maximality of $\mathfrak{m}$ it follows that the quotient MV-algebra $A/\mathfrak{m}$ is *simple*, i.e., the singleton $\{0\}$ is the only ideal of $A/\mathfrak{m}$. By [7, 3.5.1], $A/\mathfrak{m}$ is isomorphic to a subalgebra $\Omega_\mathfrak{m}$ of $[0,1]$. By [7, 7.2.6], $\Omega_\mathfrak{m}$ is uniquely determined, and so is the isomorphism $\omega_\mathfrak{m}$ of $A/\mathfrak{m}$ onto $\Omega_\mathfrak{m}$. $\square$

By Proposition 4.2, for every $\mathfrak{m} \in \boldsymbol{\mu}(A)$ we may (and will) *identify* the quotient MV-algebra $A/\mathfrak{m}$ with its uniquely isomorphic copy $\Omega_\mathfrak{m} \subseteq [0,1]$. This enables us to define the map ${}^*\colon A \to [0,1]^{\boldsymbol{\mu}(A)}$ by the stipulation

$$a^*(\mathfrak{m}) = a/\mathfrak{m} \in [0,1], \text{ for every } a \in A \text{ and } \mathfrak{m} \in \boldsymbol{\mu}(A). \tag{3}$$

Following [7, p. 72] we let $\mathsf{Rad}(A)$ denote the *radical* of A, i.e., the intersection of all maximal ideals of A.

Proposition 4.3. *For every MV-algebra A, the quotient map yields a homeomorphism of the maximal spectral space of A onto the maximal spectral space of the semisimple MV-algebra $A/\mathsf{Rad}(A)$.*

Proof. By definition of the topology of $\boldsymbol{\mu}(A)$ in view of [7, Proposition 1.2.10]. □

Let $X \neq \emptyset$ be a set and B an MV-algebra of $[0,1]$-valued functions on X, with the pointwise operations of the standard MV-algebra $[0,1]$. We say that B *separates points* (or, B is *separating*), if for any two distinct points $x, y \in X$ there is $f \in B$ such that $f(x) \neq f(y)$.

Proposition 4.4. *For any semisimple MV-algebra A the map * in (3) is an isomorphism of A onto a separating MV-subalgebra A^* of the MV-algebra $C(\boldsymbol{\mu}(A), [0,1])$ of $[0,1]$-valued continuous functions on the maximal spectral space $\boldsymbol{\mu}(A)$, with the pointwise operations of the standard MV-algebra $[0,1]$.*

Proof. By Proposition 4.2, the map * is a homomorphism of A into the MV-algebra $[0,1]^{\boldsymbol{\mu}(A)}$ of all $[0,1]$-valued functions on $\boldsymbol{\mu}(A)$ with the pointwise operations of the standard MV-algebra $[0,1]$. Further, for distinct maximal ideals $\mathfrak{m}, \mathfrak{n} \in \boldsymbol{\mu}(A)$, picking $b \in \mathfrak{m} \setminus \mathfrak{n}$ we have $b^*(\mathfrak{m}) = 0 \neq b^*(\mathfrak{n})$, which shows that A^* separates points. The assumed semisimplicity of A means that for all nonzero $a \in A$ there is a maximal ideal $\mathfrak{m}$ with $a/\mathfrak{m} = a^*(\mathfrak{m}) \neq 0$. Then by [7, Lemma 1.2.3(iii)], the map * is one-one, and hence it is an isomorphism of A onto A^*. To prove the continuity of every function $a^* \colon \boldsymbol{\mu}(A) \to [0,1]$ in A^*, let

$$]r,s[\ \subseteq\ [0,1], \quad r < s \in \mathbb{Q}$$

be a rational open interval, with the intent of showing that the inverse image $a^{*-1}(\,]r,s[\,)$ is open in $\boldsymbol{\mu}(A)$.

Let $U = [0,1] \setminus \{0\}$. By [7, 3.1.9] there is an MV-term τ in one variable X such that, letting $t \colon [0,1] \to [0,1]$ be the (McNaughton) function coded by τ in the one-generator free MV-algebra, we have $t^{-1}(U) = \,]r,s[$. (See [7, §3.2] for the explicit computation of τ from the input pair (r,s).) Let $\tau(a) \in A$ be obtained by inductively applying the MV-algebraic operations coded by the operation symbols in τ, where the variable X is interpreted by the element a. (In [5, 1.16-1.17] the element $\tau(a)$ is denoted $\tau^A(a)$.) Since * is an isomorphism we have

$$(\tau(a))^* = t(a^*) \in A^*.$$

As a consequence,

$$\begin{aligned}
a^{*-1}(]r,s[) &= a^{*-1}(t^{-1}(U)) \\
&= (t(a^*))^{-1}(U) \\
&= \{\mathfrak{m} \in \boldsymbol{\mu}(A) \mid t(a^*(\mathfrak{m})) > 0\} \\
&= \{\mathfrak{m} \in \boldsymbol{\mu}(A) \mid (\tau(a))^*(\mathfrak{m}) > 0\} \\
&= \{\mathfrak{m} \in \boldsymbol{\mu}(A) \mid \tau(a)/\mathfrak{m} > 0\}, \text{ by (3)} \\
&= \{\mathfrak{m} \in \boldsymbol{\mu}(A) \mid \tau(a) \notin \mathfrak{m}\} \\
&= O_{\tau(a)}\,,
\end{aligned}$$

an open set in $\boldsymbol{\mu}(A)$, as desired. This proves the continuity of a^*. □

Recall from the proof of Theorem 3.1 the definition $x \odot y = \neg(\neg x \oplus \neg y)$ of the derived operation $\odot$ of any MV-algebra A.

Proposition 4.5. *Let $X \neq \emptyset$ be a compact Hausdorff space and B a separating subalgebra of the MV-algebra $C(X,[0,1])$ of all continuous $[0,1]$-valued functions on X with the pointwise operations of the standard MV-algebra $[0,1]$. Then B is semisimple. Further, the map*

$$\mathcal{J}\colon x \in X \mapsto \mathcal{J}(x) = \{f \in B \mid f(x) = 0\}$$

is a homeomorphism of X onto the maximal spectral space $\boldsymbol{\mu}(B)$. The inverse map $\mathcal{V} = \mathcal{J}^{-1}$ sends each $\mathfrak{m} \in \boldsymbol{\mu}(B)$ to the only element $\mathcal{V}(\mathfrak{m}) \in X$ of the set $\bigcap\{f^{-1}(0) \mid f \in \mathfrak{m}\}$.

Proof. By [7, Corollary 3.6.8], B is semisimple. By [7, 3.4.3], the map $\mathcal{J}$ sends X one-one onto $\boldsymbol{\mu}(B)$, and $\mathcal{J}^{-1}(\mathfrak{m})$ is the only element of the set $\bigcap\{f^{-1}(0) \mid f \in \mathfrak{m}\}$.

There remains to be proved that $\mathcal{J}$ is a homeomorphism.

By Proposition 4.4, the map * is an isomorphism of B onto a separating MV-algebra B^* of continuous functions over $\boldsymbol{\mu}(B)$. For each $x \in X$ let $\mathsf{ev}_x\colon B \to [0,1]$ be the evaluation homomorphism of the functions of B at x, $\mathsf{ev}_x(f) = f(x)$, for all $f \in B$. Recalling (3), for each $\mathfrak{m} \in \boldsymbol{\mu}(B)$ and $f \in B$ we have

$$f^*(\mathfrak{m}) = f/\mathfrak{m} = \mathsf{ev}_{\mathcal{V}(\mathfrak{m})}(f) = f(\mathcal{V}(\mathfrak{m})),$$

because there is only one homomorphism of the quotient of $B/\mathfrak{m}$ into $[0,1]$, (Proposition 4.2). Thus

$$f^* = f(\mathcal{V}) = f(\mathcal{J}^{-1}).$$

Letting, as above, $U = [0,1] \setminus \{0\}$, we have

$$f^{*-1}(U) = (f(\mathcal{V}))^{-1}(U) = \mathcal{V}^{-1}(f^{-1}(U)).$$

As f ranges over all functions in B, the family of inverse images $f^{-1}(U)$ yields an open basis for the topology of X. (See [7, Theorem 3.4.3(ii) and the remark on page 67] for a proof.) Each set $\mathcal{V}^{-1}(f^{-1}(U))$ coincides with the set $f^{*-1}(U)$, which is open in $\boldsymbol{\mu}(B)$ because f^* is continuous (Proposition 4.4). Thus by Proposition 4.1, $\mathcal{V}\colon\ \boldsymbol{\mu}(B) \to X$ is a continuous one-one map of the compact space $\boldsymbol{\mu}(B)$ onto the (compact) Hausdorff space X. Elementary topology ([21, Theorem 3.1.13]) now shows that $\mathcal{V}$ is a homeomorphism of $\boldsymbol{\mu}(B)$ onto X. Thus $\mathcal{J}$ is a homeomorphism of X onto $\boldsymbol{\mu}(B)$. □

Proposition 4.6. *As A ranges over all MV-algebras, the maximal spectral space $\boldsymbol{\mu}(A)$ ranges over all compact Hausdorff spaces.*

Proof. In case A is not semisimple, by Proposition 4.3 $\boldsymbol{\mu}(A)$ is homeomorphic to the maximal spectral space $\boldsymbol{\mu}(A/\mathsf{Rad}(A))$ of the semisimple MV-algebra $A/\mathsf{Rad}(A)$. Now Propositions 4.1 and 4.5 yield the desired conclusion. □

By definition, the categorical equivalence Γ of [51, Theorem 3.9] sends every unital (always abelian) ℓ-group (G, u) to the MV-algebra $([0, 1], 0, \neg, \oplus)$ where

$$1 = u, \quad \neg x = u - x, \quad \text{and} \quad x \oplus y = (x + y) \wedge u \quad \text{for all } x, y \in [0, u].$$

Further, Γ restricts to the unit interval $[0, u]$ of (G, u) every unital ℓ-homomorphism $(G, u) \to (H, v)$.

The natural ring-theoretic structure underlying every MV-algebra is put in evidence by the following result:

Theorem 4.7. *We refer to [19,40] for C^*-algebras, and to [32] for norm-complete ℓ-groups.*

(i) *There is an adjunction between the category of MV-algebras and the dual of the category of compact Hausdorff spaces—i.e., the category of unital commutative C^*-algebras.*

(ii) (Generalizing Proposition 2.4) *For every unital commutative C^*-algebra $\mathcal{A}$ let $\mathcal{X}_{\mathcal{A}}$ be the compact space of its maximal (closed, two-sided) ideals. Then the map $\mathcal{A} \mapsto C(\mathcal{X}_{\mathcal{A}}, [0, 1])$ induces a categorical equivalence between unital commutative C^*-algebras and the full subcategory of MV-algebras isomorphic to $C(X, [0, 1])$ for X a compact Hausdorff space.*

(iii) *An MV-algebra A is isomorphic to $C(X, [0, 1])$ for X a compact Hausdorff space iff $A = \Gamma(G, u)$ for some archimedean divisible, norm-complete unital ℓ-group (G, u).*

Proof. (i) Combine [8, §4] with Gelfand duality, [19, 1.4.1], [40, § IV.4].

(ii) We first restrict the adjunction in (i) to a categorical equivalence between the full subcategories defined by the fixed objects, i.e., those MV-algebras for which the components of the unit and counit are isomorphisms. By [8, Proposition 4.2], we then obtain precisely the MV-algebras isomorphic to $C(X, [0, 1])$ for X an arbitrary compact Hausdorff space.

(iii) We first observe that a unital ℓ-group (G, u) is divisible archimedean norm-complete iff it is unitally ℓ-isomorphic to an ℓ-group of the form $C(X, \mathbb{R})$ with the constant function 1 as the unit. This is a special case of the characterization [32, Theorem 5.5] for divisible G. The desired result follows now from the preservation properties of the equivalence Γ, [51, §3]. $\square$

5. States and regular Borel probability measures

Following [31, §4], by a *state* of a unital ℓ-group (G, u) we understand a normalized positive homomorphism from (G, u) into $(\mathbb{R}, 1)$, i.e., an additive map $\tau\colon G \to \mathbb{R}$ such that $\tau(u) = 1$ and $\tau(x) \geq 0$ whenever $G \ni x \geq 0$. We let $\mathsf{S}(G, u)$ denote the set of states of (G, u).

The use of the term "state" for ordered groups, as well as for C*-algebras, can be tracked back to quantum mechanics, (see, e.g., [31, p. 72]).

MV-algebraic probability theory hinges upon the following generalization of the "finitely additive measures" defined in Theorem 2.2(ii):

Definition 5.1. ([46,52,58]) A *state* of an MV-algebra A is a map $\sigma\colon A \to [0, 1]$ with $\sigma(1) = 1$, having the *additivity property*: $\sigma(x \oplus y) = \sigma(x) + \sigma(y)$, whenever $x \odot y = 0$. We let $\mathsf{S}(A)$ denote the set of states of A.

For MV-algebras, whence in particular for boolean algebras, this terminology is justified by the following result, which will find use in the proof of Theorem 5.9:

Proposition 5.2. *Let (G, u) be a unital ℓ-group and $A = \Gamma(G, u)$. Then restriction $\sigma \mapsto \sigma \restriction A$ is a one-one affine map of $\mathsf{S}(G, u)$ onto $\mathsf{S}(A)$.*

Proof. [52, Theorem 2.4]. $\square$

For any MV-algebra A we denote by $\mathsf{hom}(A, [0, 1])$ the set of homomorphisms of A into the standard MV-algebra $[0, 1]$.

Proposition 5.3. *Let A be an MV-algebra.*

(i) *Equipped with the restriction of the product topology of the Tychonoff cube $[0, 1]^A$, the set $\mathsf{S}(A)$ is a compact Hausdorff space.*

(ii) *Let A' be shorthand for the semisimple MV-algebra $A/\mathsf{Rad}(A)$. Let ρ denote the quotient map $x \in A \mapsto \rho(x) = x/\mathsf{Rad}(A) \in A'$. Let $\circ$ stand for composition. Then the map $\sigma' \mapsto \sigma' \circ \rho$ is an affine homeomorphism of $\mathsf{S}(A')$ onto $\mathsf{S}(A)$.*

Proof. (i) Immediate by definition of a state, since $[0,1]^A$ is compact.

(ii) Let

$$\partial\mathsf{S}(A) \subseteq [0,1]^A, \quad \text{(resp., } \partial\mathsf{S}(A') \subseteq [0,1]^{A'}\text{)}$$

denote the set of extremal states of A (resp., of A'). Let

$$\mathsf{conv}\,\partial\mathsf{S}(A) \subseteq \mathbb{R}^A, \quad \text{(resp., } \mathsf{conv}\,\partial\mathsf{S}(A') \subseteq \mathbb{R}^{A'}\text{)}$$

be their respective sets of (finite) convex combinations. We also write

$$\mathsf{cl}\,\mathsf{conv}\,\partial\mathsf{S}(A), \quad \text{(resp., } \mathsf{cl}\,\mathsf{conv}\,\partial\mathsf{S}(A')\text{)}$$

for the closure of $\mathsf{conv}\,\partial\mathsf{S}(A)$ in the topological vector space $\mathbb{R}^A$, (resp., for the closure of $\mathsf{conv}\,\partial\mathsf{S}(A')$ in the topological vector space $\mathbb{R}^{A'}$).

By [31, 12.18](a)⇔(b), the extremal states of any unital ℓ-group (H,v) coincide with the unital ℓ-homomorphisms of (H,v) into $(\mathbb{R},1)$. Passing to MV-algebras via the categorical equivalence Γ, we get

$$\partial\mathsf{S}(\Gamma(H,v)) = \mathsf{hom}(\Gamma(H,v),[0,1]),$$

whence in particular

$$\partial\mathsf{S}(A) = \mathsf{hom}(A,[0,1]) \quad \text{and} \quad \partial\mathsf{S}(A') = \mathsf{hom}(A',[0,1]). \tag{4}$$

From [7, 1.2.10] we have a lattice isomorphism Λ of the set of ideals of A' onto the set of ideals of A containing $\mathsf{Rad}(A)$. For any MV-algebras B, C and homomorphisms $\eta_1, \eta_2\colon B \to C$, $\mathsf{ker}(\eta_1) = \mathsf{ker}(\eta_2)$ iff $\eta_1 = \eta_2$. Therefore, for each $\eta \in \mathsf{hom}(A)$ there is a unique $\eta' \in \mathsf{hom}(A')$ with $\mathsf{ker}(\eta) = \Lambda(\mathsf{ker}(\eta'))$. With a self-explanatory notation we can write

$$\eta = \eta' \circ \rho \quad \text{and} \quad \mathsf{hom}(A) = \mathsf{hom}(A') \circ \rho. \tag{5}$$

The map $\eta' \in \mathsf{hom}(A') \mapsto \eta = \eta' \circ \rho$ is a homeomorphism of $\mathsf{hom}(A')$ onto $\mathsf{hom}(A)$, respectively equipped with the restriction topologies of $[0,1]^{A'}$ and $[0,1]^A$. This parallels the homeomorphism

$$\boldsymbol{\mu}(A') \cong \boldsymbol{\mu}(A) \tag{6}$$

of [52, 2.5]. Now (4) and (5) yield the identities

$$
\begin{aligned}
[0,1]^A \supseteq \mathsf{conv}\,\partial\mathsf{S}(A) &= \mathsf{conv}\,\mathsf{hom}(A)\\
&= \mathsf{conv}\,(\mathsf{hom}(A') \circ \rho)\\
&= (\mathsf{conv}\,\mathsf{hom}(A')) \circ \rho\\
&= (\mathsf{conv}\,\partial\mathsf{S}(A')) \circ \rho.
\end{aligned}
$$

From the Krein-Milman theorem we obtain:

$$
\mathsf{S}(A) = \mathsf{cl}\,\mathsf{conv}\,\partial\mathsf{S}(A) = \mathsf{cl}\,((\mathsf{conv}\,\partial\mathsf{S}(A')) \circ \rho) = (\mathsf{cl}\,\mathsf{conv}\,\partial\mathsf{S}(A')) \circ \rho = \mathsf{S}(A') \circ \rho,
$$

showing that the map $\sigma' \mapsto \sigma' \circ \rho$ is the desired affine homeomorphism of $\mathsf{S}(A')$ onto $\mathsf{S}(A)$. $\square$

The following generalization of Definition 2.1 rests upon the results of Section 4, allowing us to identify each element of an MV-algebra A with a $[0,1]$-valued continuous random variable on the compact Hausdorff space $\boldsymbol{\mu}(A)$. When A is not semisimple, one works with its semisimple quotient $A/\mathsf{Rad}(A)$ to neglect infinitesimals. Every boolean algebra is a semisimple (indeed, a hyperarchimedean) MV-algebra, [7, p. 116].

Definition 5.4. Every set E of *random variables* shall be understood as a subset of some MV-algebra $A = (A, 0, \neg, \oplus)$. The dependence of E on A will always be clear from the context. By the totality of *possible outcomes* (of the random variables in E) we mean the set of homomorphisms of A into the standard MV-algebra $[0,1]$.

By Propositions 4.3-4.4 we may safely identify $A/\mathsf{Rad}(A)$ with the separating MV-algebra A^* of $[0,1]$-valued continuous functions defined on $\boldsymbol{\mu}(A/\mathsf{Rad}(A))$ as well as on its homeomorphic copy $\boldsymbol{\mu}(A)$ in (6). Once this is done, for all $f, g \in A$ the operations of A give a precise meaning, e.g., to the $[0,1]$-valued random variables $\neg f = 1 - f$, $f \oplus g = \min(1, f + g)$, and $f \odot g = \max(0, f + g - 1)$. Without reference to some ambient MV-algebra A this would not be possible.

We may then say that f and g are "incompatible" if $f \odot g = 0$. Also, f and g "form a partition" if, in addition, $f \oplus g = 1$. When A is a boolean algebra we recover the usual definition of incompatibility and exhaustiveness of two yes-no events. Further, the expression "for every homomorphism of A into $[0,1]$" gives a precise meaning to such colloquial expressions as "regardless of the outcome", or "in any possible world".

Definition 5.5. Let A be an MV-algebra and $E = \{h_1, \dots h_m\} \subseteq A$. A map $\beta\colon E \to [0,1]$ is *consistent (in A)* if for every $s\colon E \to \mathbb{R}$ there is $\eta \in \mathsf{hom}(A, [0,1])$ such that $\sum_{i=1}^m s(h_i)(\beta(h_i) - \eta(h_i)) \geq 0$. We say that β is *inconsistent* if it is not consistent.

This definition generalizes Definition 2.2 to $[0,1]$-valued random variables. Its rationale is the same as that of Definition 2.2: A list $\beta(h_1), \dots, \beta(h_m)$ of betting odds posted by a bookmaker on $[0,1]$-valued random variables $h_1, \dots, h_m$ is inconsistent iff a shrewd bettor can place stakes $s(h_i), \ldots, s(h_m) \in \mathbb{R}$ causing the bookmaker to go bust whatever the outcome of the h_i.

A modicum of acquaintance with Łukasiewicz logic, [7, §4.4], yields the following generalization of Corollary 2.5, confirming that the adoption of the adjective "consistent" in Definition 5.5 is legitimate:

Corollary 5.6. *When $\beta(h_i) = 1$ for all $h_i \in E = \{h_1, \ldots h_m\} \subseteq A$, β is consistent iff there is a homomorphism $\eta\colon A \to [0,1]$ such that $\eta(h_i) = 1$ for all $i = 1, \ldots, m$. Once each h_i is coded by a formula ϕ_i of Łukasiewicz logic, the consistency of β according to Definition 5.5 boils down to the* logical consistency *of the set of formulas $\{\phi_1, \ldots, \phi_m\}$, i.e., the existence of a truth-valuation assigning value 1 to each formula ϕ_i.*

The following result extends Theorem 2.2 (i)⇔(ii) to MV-algebras:

Theorem 5.7. *Let A be an MV-algebra and $\sigma\colon A \to [0,1]$ a* function*. The following conditions are equivalent:*

(i) *σ is* finitely consistent*, i.e., every finite restriction of σ is consistent.*
(ii) *σ is a state of A.*

Proof. [47, Theorem 2.3]. □

The following extension of Proposition 2.6 to MV-algebras shows that de Finetti's definition of a consistent map is independent of the ambient MV-algebra.

Corollary 5.8. *Let A be an MV-algebra. A map $\beta\colon E \to [0,1]$ defined on a finite set $E \subseteq A$ is consistent in the MV-algebra A iff it is consistent in every MV-algebra $B \supseteq A$ iff it is consistent in the MV-algebra generated by E in B.*

Proof. For any two MV-algebras $A \subseteq B$, any state of A is extendible to a state of B. This follows from the analogous result for unital ℓ-groups, [31, Corollary 4.3], in view of the preservation properties of the Γ functor, [51, §3]. Then apply Theorem 5.7(i)⇔(ii). □

The following result extends Theorem 2.2 (ii)⇔(iii) to semisimple MV-algebras:

Theorem 5.9. *Let A be a semisimple MV-algebra identified, via Proposition 4.4, with the separating MV-algebra A^* of continuous $[0,1]$-valued functions on $\boldsymbol{\mu}(A)$. Then for every map $\sigma\colon A \to [0,1]$ the equivalent conditions (i) and (ii) of Theorem 5.7 are also equivalent to the following condition:*

(iii) *There is a (necessarily unique) regular Borel probability measure μ_σ on the compact Hausdorff space $\boldsymbol{\mu}(A)$ such that*

$$\sigma(f) = \int_{\boldsymbol{\mu}(A)} f(\mathfrak{m})\,\mathrm{d}\mu_\sigma(\mathfrak{m}) \quad \textit{for every} \quad f \in A. \tag{7}$$

Proof. (iii)⇒(ii) Trivially, $\sigma(1) = 1$. As for the additivity property, suppose $f, g \in A = A^* \subseteq G_A$ and $0 = f \odot g = \max(0, f+g-1)$. From $f+g \leq 1$ we get $f \oplus g = \min(1, f+g) = f + g$. By hypothesis, $\sigma(f \oplus g) = \sigma(f + g) = \sigma(f) + \sigma(g)$. Thus σ is a state of A.

(ii)⇒(iii) (Compare with [46, Corollary 29] and [58, Proposition 1.1].) For every $f = f^* \in A^*$ and maximal ideal $\mathfrak{m}$ of A, we may write

$$f(\mathfrak{m}) = f/\mathfrak{m} \in [0, 1]$$

without danger of confusion. We next introduce the following increasingly large extensions of A:

- $(G_A, 1)$**:** the unital ℓ-group of functions on $\boldsymbol{\mu}(A)$ generated by A in $\mathbb{R}^{\boldsymbol{\mu}(A)}$, with (the distinguished strong order) unit $1 = \neg 0 =$ the constant function 1 on $\boldsymbol{\mu}(A)$. Equivalently, $\Gamma(G_A, 1) = A$, ([51, §3]). Each $f \in G_A$ is continuous.
- $(\mathbb{Q}G_A, 1)$**:** the divisible hull of $(G_A, 1)$, (see [2, 1.6.8-1.6.9]). Up to isomorphism, this is the unital rational vector lattice generated by $(G_A, 1)$ in $\mathbb{R}^{\boldsymbol{\mu}(A)}$. (See [64, p. 372] for vector lattices with a distinguished unit.)
- $C(\boldsymbol{\mu}(A), \mathbb{R})$**:** the Banach algebra of real-valued continuous functions on $\boldsymbol{\mu}(A)$ with the sup norm $||\cdot||_\infty$.
- $C(\boldsymbol{\mu}(A), \mathbb{C})$**:** the unital commutative C*-algebra of complex-valued continuous functions on $\boldsymbol{\mu}(A)$.

We have a unique unital ℓ-embedding $(G_A, 1) \hookrightarrow (\mathbb{Q}G_A, 1)$. Also, $\mathbb{Q}G_A$ is norm-dense in $C(\boldsymbol{\mu}(A), \mathbb{R})$, by the lattice version of the Stone-Weierstrass theorem, [36, Theorem 7.29], [64, pp. 8-10]. Furthermore, $C(\boldsymbol{\mu}(A), \mathbb{R})$ is uniquely embedded into $C(\boldsymbol{\mu}(A), \mathbb{C})$. The restriction map is an affine isomorphism of the set $\mathsf{S}(C(\boldsymbol{\mu}(A), \mathbb{C}))$ of states of $C(\boldsymbol{\mu}(A), \mathbb{C})$ onto the set $\mathsf{S}(C(\boldsymbol{\mu}(A), \mathbb{R}))$ of normalized positive linear functionals on the Banach algebra $C(\boldsymbol{\mu}(A), \mathbb{R})$.

Claim 1. *The restriction map is an affine isomorphism of* $\mathsf{S}(C(\boldsymbol{\mu}(A), \mathbb{R}))$ *onto the set* $\mathsf{S}(\mathbb{Q}G_A, 1)$ *of states of the unital* ℓ*-group* $(\mathbb{Q}G_A, 1)$*. (See Definition 5.1.)*

As a matter of fact, the positivity and additivity of every state τ of $(\mathbb{Q}G_A, 1)$ yields

$$f \leq g \Rightarrow \tau(f) \leq \tau(g) \quad \text{for all} \quad f, g \in \mathbb{Q}\, G_A \subseteq C(\boldsymbol{\mu}(A), \mathbb{R}).$$

For any rational number λ the constant function $\lambda\, 1$ on $\boldsymbol{\mu}(A)$ belongs to $\mathbb{Q}\, G_A$. Therefore, whenever an element $g \in \mathbb{Q}\, G_A$ satisfies the inequality $||g||_\infty \leq \lambda$ we have

$$|\tau(g)| = |\tau((g \vee 0) - (-g \vee 0))|$$

$$\begin{aligned}
&= |\tau(g \vee 0) - \tau(-g \vee 0)| \\
&\leq \tau(g \vee 0) + \tau(-g \vee 0) \\
&\leq \tau(\lambda 1) \text{ by the additivity of } \tau, \quad \text{since } (g \vee 0) \wedge (-g \vee 0) = 0 \\
&\quad \text{ and } \sup_{\mathfrak{m} \in \boldsymbol{\mu}(A)} |g(\mathfrak{m})| = ||g||_\infty \leq \lambda \\
&= \lambda, \text{ because } \tau(1) = 1 \text{ and } \tau \text{ is additive.}
\end{aligned}$$

Thus for all $f \in \mathbb{Q}\, G_A$ and $\epsilon > 0$ there is $\delta > 0$ (notably, $\delta = \epsilon$) such that

$$\text{for all } g \in \mathbb{Q}\, G_A, \;\; ||f - g||_\infty < \delta \Rightarrow |\tau(f) - \tau(g)| = |\tau(f - g)| < \epsilon.$$

Since τ is continuous and $\mathbb{Q}G_A$ is norm-dense in $C(\boldsymbol{\mu}(A), \mathbb{R})$, the stipulation

$$\bar{\tau}(\lim f_i) = \lim \tau(f_i), \quad (f_i \in \mathbb{Q}\, G_A)$$

uniquely extends τ to a *map* $\bar{\tau}\colon C(\boldsymbol{\mu}(A), \mathbb{R})) \to \mathbb{R}$. Trivially, $\bar{\tau}$ is additive and normalized, $\bar{\tau}(1) = 1$. To complete the verification that $\bar{\tau}$ belongs to $\mathsf{S}(C(\boldsymbol{\mu}(A), \mathbb{R}))$, let $0 \leq g \in C(\boldsymbol{\mu}(A), \mathbb{R})$, with the intent of showing $\bar{\tau}(g) \geq 0$. Again by the Stone-Weierstrass theorem, g is the (norm) limit of a sequence $g_1, g_2, \ldots$ of elements of $\mathbb{Q}G_A$. It follows that $\lim(g_i \vee 0) = g$, with each function $g_i \vee 0$ still a member of $\mathbb{Q}G_A$. Since τ is positive, $\tau(g_i \vee 0) \geq 0$, whence $0 \leq \lim \tau(g_i \vee 0) = \bar{\tau}(\lim(g_i \vee 0)) = \bar{\tau}(g)$. The positivity of $\bar{\tau}$ is thus proved. The proof of Claim 1 is concluded by noting that every member of $\mathsf{S}(C(\boldsymbol{\mu}(A), \mathbb{R}))$ is the unique extension of its restriction to $\mathbb{Q}G_A$. The inverse of the extension map $\tau \mapsto \bar{\tau}$ is the desired affine isomorphism of $\mathsf{S}(C(\boldsymbol{\mu}(A), \mathbb{R}))$ onto $\mathsf{S}(\mathbb{Q}G_A, 1)$.

Claim 2. *The restriction map is an affine isomorphism of* $\mathsf{S}(\mathbb{Q}G_A, 1)$ *onto* $\mathsf{S}(G_A, 1)$.

This is so because every state $\sigma \in \mathsf{S}(G_A, 1)$ uniquely extends to a state $\sigma_{\mathbb{Q}} \in \mathsf{S}(\mathbb{Q}G_A, 1)$ by stipulating that for all $f \in G_A$ and $\lambda \in \mathbb{Q}$, $\sigma_{\mathbb{Q}}(\lambda f) = \lambda\sigma(f)$. All states of $(\mathbb{Q}G_A, 1)$ are obtained in this way.

Claim 3. *The restriction map is an affine isomorphism of* $\mathsf{S}(G_A, 1)$ *onto* $\mathsf{S}(A)$.

See Proposition 5.2.

Let now $\mathsf{P}(A)$ denote the convex set of regular Borel probability measures on $\boldsymbol{\mu}(A)$. The Riesz-Markov-Kakutani representation theorem [60, 2.14], [64, p. 119 and references therein] yields an affine isomorphism $\tau \in \mathsf{S}(C(\boldsymbol{\mu}(A), \mathbb{R})) \mapsto \nu_\tau \in \mathsf{P}(A)$ such that

$$\tau(f) = \int_{\boldsymbol{\mu}(A)} f(\mathfrak{m})\, \mathrm{d}\nu_\tau(\mathfrak{m}) \quad \text{for all} \quad f \in C(\boldsymbol{\mu}(A), \mathbb{R}).$$

Using the affine isomorphisms of Claims 1-3 we may write

$$\mathsf{P}(A) \cong \mathsf{S}(A) \cong \mathsf{S}(G_A, 1) \cong \mathsf{S}(\mathbb{Q}G_A, 1) \cong \mathsf{S}(C(\boldsymbol{\mu}(A), \mathbb{R})) \cong \mathsf{S}(C(\boldsymbol{\mu}(A), \mathbb{C})). \tag{8}$$

We then have an affine isomorphism sending every state σ of A to the regular Borel probability measure $\mu_\sigma \in \mathsf{P}(A)$ such that

$$\sigma(f) = \int_{\boldsymbol{\mu}(A)} f(\mathfrak{m})\, \mathrm{d}\mu_\sigma(\mathfrak{m}) \quad \text{for all} \quad f \in C(\boldsymbol{\mu}(A), \mathbb{R}).$$

The *inclusions* $A \subseteq (G_A, 1) \subseteq (\mathbb{Q}G_A, 1) \subseteq C(\boldsymbol{\mu}(A), \mathbb{R}) \subseteq C(\boldsymbol{\mu}(A), \mathbb{C})$ together with (8) ensure that μ_σ is the unique regular Borel probability measure on $\boldsymbol{\mu}(A)$ satisfying (7). □

Corollary 5.10. *Let A be a semisimple MV-algebra, identified with the separating MV-algebra $A^* \subseteq C(\boldsymbol{\mu}(A), [0, 1])$ of Proposition 4.4.*

(i) *The map $\sigma \mapsto \mu_\sigma$ of Theorem 5.9(iii) is an affine homeomorphism of the convex set $\mathsf{S}(A)$ onto the convex set $\mathsf{P}(A)$ (equipped with the weak topology) of regular Borel probability measures on $\boldsymbol{\mu}(A)$.*
(ii) *Let $C(\boldsymbol{\mu}(A), \mathbb{C})$ be the unital commutative C^*-algebra of all continuous complex-valued functions on $\boldsymbol{\mu}(A)$. Then every state σ of A uniquely extends to the positive linear functional $\bar{\sigma}$ of $C(\boldsymbol{\mu}(A), \mathbb{C})$ given by*

$$\bar{\sigma}(g) = \int_{\boldsymbol{\mu}(A)} g(\mathfrak{m})\, \mathrm{d}\mu_\sigma(\mathfrak{m}) \quad \text{for every } g \in C(\boldsymbol{\mu}(A), \mathbb{C}).$$

The map $\sigma \mapsto \bar{\sigma}$ is an affine homeomorphism of the state space $\mathsf{S}(A)$ onto the set of states of the C^*-algebra $C(\boldsymbol{\mu}(A), \mathbb{C})$, equipped with the weak topology.*

Proof. (i) By inspection of the proof of Theorem 5.9, recalling the definition of the topologies of $\mathsf{S}(A)$ and $\mathsf{P}(A)$. (ii) Follows from (i) and Gelfand duality, [19,40]. □

Part (i) in the following result, together with Theorem 5.7, is the last step in the extension of Theorem 2.2 to all MV-algebras:

Theorem 5.11. *For any MV-algebra A we have:*

(i) *(Generalizing Corollary 5.10(i).) The map*

$$\eta \in \mathsf{hom}(A, [0, 1]) \subseteq \mathsf{S}(A) \mapsto \text{(Dirac) point mass at } \mathsf{ker}(\eta) \in \mathsf{P}(A)$$

uniquely extends to an affine homeomorphism γ_A of $\mathsf{S}(A)$ onto $\mathsf{P}(A)$.

(ii) *(Generalizing Corollary 5.10(ii).) The map*

$$\eta \in \mathsf{hom}(A, [0,1]) \mapsto \text{evaluation at the point of } \boldsymbol{\mu}(A) \text{ given by } \mathsf{ker}(\eta)$$

uniquely extends to an affine homeomorphism of the state space $\mathsf{S}(A)$ *onto the state space of the* C^**-algebra* $\mathcal{A} = C(\boldsymbol{\mu}(A), \mathbb{C})$.

Proof. (i) Equipped with the restriction topology of the Tychonoff cube $[0,1]^A$, the set $\mathsf{hom}(A, [0,1])$ is a compact Hausdorff subspace of the state space $\mathsf{S}(A)$. The latter is compact by Proposition 5.3(i). By definition of the maximal spectral topology of A, the map

$$\eta \in \mathsf{hom}(A, [0,1]) \mapsto \mathsf{ker}(\eta) \in \boldsymbol{\mu}(A) \tag{9}$$

is a homeomorphism of $\mathsf{hom}(A, [0,1])$ onto $\boldsymbol{\mu}(A)$. Arguing as in the proof of Proposition 5.3, the quotient map $A \to A/\mathsf{Rad}(A)$ induces homeomorphisms

$$\mathsf{hom}(A/\mathsf{Rad}(A), [0,1]) \cong \mathsf{hom}(A, [0,1]) \cong \boldsymbol{\mu}(A) \cong \boldsymbol{\mu}(A/\mathsf{Rad}(A)),$$

in such a way that we may safely assume that A is semisimple.

Let $\gamma_A \colon \sigma \mapsto \mu_\sigma$ be the *affine isomorphism* (8) between the convex sets $\mathsf{S}(A)$ and $\mathsf{P}(A)$ constructed in the proof of Theorem 5.9 (ii)⇒(iii). Then γ_A sends each $\eta \in \mathsf{hom}(A, [0,1])$ to the point mass $\delta_{\mathsf{ker}(\eta)} \in \mathsf{P}(A)$ localized at the maximal ideal $\mathsf{ker}(\eta)$. Every regular Borel probability measure is the weak limit of a net of convex combinations of point mass probability measures. Correspondingly, by the Krein-Milman theorem, every state of A belongs to the closure in $[0,1]^A$ of the convex hull of the set $\partial\mathsf{S}(A)$ of extremal states. By [52, 2.5], $\partial\mathsf{S}(A)$ is homeomorphic to $\boldsymbol{\mu}(A)$. By (9), $\boldsymbol{\mu}(A)$ is homeomorphic to $\mathsf{hom}(A, [0,1])$. This completes the proof of (i).

(ii) Gelfand duality [19,40] shows that every extremal state E of the commutative C*-algebra $\mathcal{A} = C(X, \mathbb{C})$ amounts to evaluation at a uniquely determined point x_E in X. Furthermore, the map $E \mapsto x_E$ is a homeomorphism of the set of extremal states of $\mathcal{A}$ equipped with the weak* topology, onto the compact Hausdorff space X. Now use (i). □

6. Computing on continuous random variables

Let us revert to Rota's first problem.

> The alternative definitions of probability, by means of a sample space and by means of an algebra of random variables are equivalent. Mathematicians opt for the definition in terms of random variables, because they do not wish to miss a chance to appeal to Her Imperial Majesty, the Theory of Commutative Rings.
>
> Rota, [59, p. 58]

In the foregoing section it has been proved that the following structures provide two equivalent basic approaches to problems in probability theory:

— a C*-algebra of continuous complex-valued functions equipped with a state, or with a regular Borel probability measure on its maximal spectral space;
— an MV-algebra A equipped with a state (i.e., a finitely consistent map), or with a regular Borel probability measure on its maximal spectral space.

In this section and in the next one, two time-honored problems in probability theory will be approached using MV-algebras and their finitely consistent maps, rather than the Banach algebra $C(X, \mathbb{R})$ and its normalized positive Radon measures, or the C*-algebra $C(X, \mathbb{C})$ and its states. Indeed, the excess of structure in $C(X, \mathbb{R})$ and $C(X, \mathbb{C})$ would seem to obscure what makes Theorems 6.1 and 7.2 below work.

Boole on probabilistic entailment. Boole [3, Chapter XVI, 4, p. 246] writes:

> the object of the theory of probabilities might be thus defined. Given the probabilities of any events, of whatever kind, to find the probability of some other event connected with them.

The following result gives a general solution of this problem for $[0, 1]$-valued random variables, and a fortiori for the yes-no events considered by Boole.

Theorem 6.1. *Let $\beta\colon E \to [0, 1]$ be a map on a subset $E = \{h_1, \ldots, h_m\}$ of an MV-algebra A. Let $\mathsf{S}_\beta = \{\sigma \in \mathsf{S}(A) \mid \sigma \supseteq \beta\}$ be the set of states of A extending β. For $h \in A$, let $\mathsf{S}_\beta(h) = \{\sigma(h) \mid \sigma \supseteq \beta\}$ be the set of probabilities assigned to h by the states extending β. Then $\mathsf{S}_\beta(h) = \emptyset$ iff β is inconsistent iff no probability can be assigned to h given β. When β is consistent, $\mathsf{S}_\beta(h)$ is a nonempty closed (possibly a singleton) interval contained in $[0, 1]$.*

Proof. By Proposition 5.3(ii) it is no loss of generality to assume A semisimple. If β is inconsistent, then $\mathsf{S}_\beta = \emptyset = \mathsf{S}_\beta(h)$ by Theorem 5.7, and we are done. So assume β is consistent and fix $j = 1, \ldots, m$. The set of states τ of A such that $\tau(h_j) = \beta(h_j)$ is the intersection of the compact convex set $\mathsf{S}(A)$ with the closed convex subset of $[0, 1]^A$ given by

$$\mathcal{H}_j = \{\chi \in [0, 1]^A \mid \chi(h_j) = \beta(h_j)\}.$$

Thus S_β coincides with the compact convex (*a fortiori* connected) set

$$\mathsf{S}(A) \cap \mathcal{H}_1 \cap \cdots \cap \mathcal{H}_m.$$

Let the function $\psi_h\colon \mathsf{S}(A) \to [0, 1]$ be defined by $\psi_h(\sigma) = \sigma(h)$, for all $\sigma \in \mathsf{S}(A)$. Then $\mathsf{S}_\beta(h) = \mathrm{range}(\psi_h \restriction \mathsf{S}_\beta)$, where, as the reader will recall, "$\restriction$" is shorthand for restriction.

Since ψ_h is a continuous $[0,1]$-valued function defined on the compact connected set S_β, its range $\mathsf{S}_\beta(h)$ is a closed connected subset of $[0,1]$. Since β is consistent, Theorem 5.7 (i)⇒(ii) ensures that S_β is nonempty. It follows that the set $\mathsf{S}_\beta(h)$ is a nonempty closed (possibly a singleton) interval in $[0,1]$. $\square$

In [13, pp. 10-13], as well as in [14, 3.10.1, Theorem on page 112, part(b)], de Finetti proved the boolean fragment of Theorem 6.1. In [14] he called his result (restricted to yes-no events) the "Fundamental Theorem of Probability".

Furthermore, de Finetti gave a necessary and sufficient condition for $\mathsf{S}_\beta(h)$ to be a singleton. See [13, pp. 9-10] or [14, 3.10.1, Theorem on page 112, part(a)].

PSAT and ŁIPSAT. The modern algorithmic version of Boole's problem on probabilistic entailment asks to compute the closed interval of possible probabilities of an event h, when events $h_1, \ldots, h_m$ are assigned *rational* probabilities $p_1, \ldots, p_m$, and each event h_i is coded by a boolean formula ϕ_i. This problem is known in the literature as "the optimization form of probabilistic satisfiability", or "probabilistic entailment" [35, p. 323], [57, p. 75]. One has the preliminary task of *deciding* whether the assignment $\phi_i \mapsto p_i, (i = 1, \ldots, m)$ is consistent, as a necessary and sufficient condition for the *existence* of a probability of the new event h. By Corollary 2.5, this latter problem, known as PSAT, is a generalization of the boolean satisfiability problem SAT. PSAT has the same computational complexity of SAT, [35,57]. The handbook chapter [35] is devoted to the algorithmic theory of the PSAT problem and probabilistic entailment. Combining Theorems 5.7, 6.1, and 5.9-5.11, PSAT has a natural extension, known as ŁIPSAT, to $[0,1]$-valued random variables h_i coded by formulas ϕ_i in Łukasiewicz infinite-valued logic. Also ŁIPSAT is NP-complete, [4]. See [11,24,25] for recent developments.

Remark 6.2. The literature cited at the outset of the present paper features quite a few examples of contemporary research on *logic*-algebraic probability, showing the relevance and vitality of Rota's problems. All these papers investigate the relations between probability and nonclassical logics—with a view of generalizing to random variables Boole's program of computing/reasoning on yes-no events and their probabilities.

By Theorem 3.1, Ł_∞-formulas (and only Ł_∞-formulas) code *continuous* $[0,1]$-valued random variables on compact Hausdorff spaces. The deductive-algorithmic machinery of Ł_∞ allows computations on these formulas, yielding the above mentioned NP-completeness results.

Unital commutative C*-algebras are grandees of what Rota dubbed the Empire of Her Majesty the Theory of Commutative Rings. As such, they are not immediately presentable by strings of symbols, as MV-algebras are via Ł_∞-formulas. Interestingly enough, in their paper [49] the authors construct an *infinitary* equational class of MV-algebras equipped with an infinitary operation. While this class is shown to be categorically equivalent to unital commutative C*-algebras, the authors' did not aim at an algorithmic theory of unital commutative C*-algebras.

Rational Pavelka logic is the precursor of a wealth of variants of MV-algebras and Łukasiewicz logic, allowing 2-divisibility, or *infinitely* many constant symbols for every rational in $[0,1]$, or even a multiplication connective. See [18,22,23,34,50, and references therein]. Once rational constants are coded as pairs of binary digits, the computational complexity of these logics exceeds that of $Ł_\infty$. The same holds for all extensions of $Ł_\infty$ with a product connective or for logics allowing 2-divisibility. Owing to their being categorically equivalent to abelian ℓ-groups, the Di Nola-Lettieri "perfect" MV-algebras of [17] provide logics for nonarchimedean random variables. The extension of (suitable modifications of) the results of this paper to these logics and algebras is an interesting research program.

7. Stochastic vs. logic-algebraic independence

The random variable "percentage of European glaciers melt by the end of this century" is independent of the yes-no event "Brazil will win the next FIFA World Cup" without bothering to assign them a probability. Independence comes before probability.

> There is, thus, independence in a vague and intuitive sense, and there is "independence" in the narrow but well-defined sense that the rule of multiplication of probabilities is applicable. It was the vague and intuitive notions that provided for a long time the main motivation and driving force behind probability theory. And while an impressive formalism was being created, mathematicians (with very few exceptions) remained aloof because it was not clear to them what the objects were to which the formalism was applicable.
>
> Mark Kac, [41, p. 10]

One cannot but agree with Kac. No probabilistic preliminaries are needed to assert the logical independence of two events. Two random variables or events are "independent" if there is no logical relation whatsoever between them. Logic and algebra do provide a precise definition of this vague and intuitive notion of independence. The reader may refer to [61, §13] or [45, Chapter 4, 11.3] for boolean algebraic independence, and to [53] for semisimple tensor products in MV-algebras, the main tool for the mathematization of MV-algebraic independence.

By [7, 3.6.4], if A is an MV-subalgebra of a semisimple MV-algebra C then A is semisimple. So let us suppose A and B are MV-subalgebras of a semisimple MV-algebra C.

Definition 7.1. We say that A and B are *independent subalgebras* of C if the semisimple tensor product $A \otimes B$ is embeddable into C. Finite subsets $E, F \subseteq C$ of $[0,1]$-valued random variables are said to be *independent* if so are the MV-algebras generated by E and F in C.

For each $a \in E$ and $b \in F$ the $[0,1]$-valued random variable "a and b" is the *pure tensor* $a \otimes b \in A \otimes B$. For any map $\circledast\colon\ [0,1]^2 \to [0,1]$, the $\circledast$-*book* $\alpha \circledast \beta$ determined by consistent maps $\alpha\colon E \to [0,1]$ and $\beta\colon F \to [0,1]$, assigns the value $\alpha(a) \circledast \beta(b) \in [0,1]$ to each pure tensor $a \otimes b$.

In particular, when A, B, C are boolean algebras with A and B subalgebras of C, the tensor product $A \otimes B$ boils down to their free product ([45, 11.1], called "boolean product" in [61, p. 37]), and the pure tensor $a \otimes b$ coincides with the meet $a \wedge b \in C$. In this way one recovers the usual definition of boolean algebraic independence, [61, §13], [45, Chapter 4, 11.3].

The existence of $\alpha \circledast \beta$ as a *map* is not guaranteed in general. However, in the particular case when $\circledast$ coincides with product we have:

Theorem 7.2. *Product is the only continuous map* $\circledast\colon [0,1]^2 \to [0,1]$ *having the following property: For all independent subalgebras* A *and* B *of a semisimple MV-algebra* C *and consistent* $[0,1]$*-valued maps* α *and* β *on finite subsets* $E \subseteq A$ *and* $F \subseteq B$*, the* $\circledast$*-book* $\alpha \circledast \beta$ *(exists and) is consistent.*

Proof. [54, Theorem 2.3]. $\square$

Example 7.3. Let us suppose that in the definition of stochastic independence multiplication is replaced by another operation $\circledast\colon [0,1]^2 \to [0,1]$. Let $\{x_1 \dots, x_n\}$ be a free generating set of the free n-generator boolean algebra F_n. Then the events x_i form an *independent set* in F_n. For every sequence $\epsilon_i \in \{-1, 1\}$, letting $\epsilon_i x_i = \neg x_i$ or x_i according as $\epsilon_i = -1$ or $\epsilon_i = 1$, we have $\epsilon_1 x_1 \wedge \cdots \wedge \epsilon_n x_n \neq 0$. (See [61, §13] and [45, Chapter 4, 11.3].) By de Finetti's Dutch Book Theorem 2.3(i)$\Leftrightarrow$(ii), the map β assigning probability $1/n$ to each event x_i is consistent, because it is (uniquely) extendible to the state of F_n which assigns the value $1/2^n$ to every atom of F_n. It follows that the extension $\beta^{\circledast}$ of β given by $\beta^{\circledast} = \beta \cup \{x_1 \wedge \cdots \wedge x_n \mapsto (\dots(((\frac{1}{n} \circledast \frac{1}{n}) \circledast \frac{1}{n}) \circledast \frac{1}{n}) \circledast \cdots \circledast \frac{1}{n})\}$ is inconsistent.

In a similar way, consistency is lost if addition is replaced by another operation $\boxplus : [0,1]^2 \to [0,1]$ to axiomatize the probability of disjunctions of incompatible events. This, too, is a consequence of Theorem 2.3(i)$\Leftrightarrow$(ii).

The additivity of probability for disjunctions of incompatible events is an *axiom*, while the "rule of multiplication of probabilities" is the *definition* of independent events. As Hilbert noted in 1905:

> Wir fassen das einfach als Definitionen auf, wiewohl im gegenwärtigen Zustande der Entwicklung besonders die Bezeichnungen "Axiom" und Definition noch etwas durcheinandergehen.
>
> (We simply take this as definitions, although in the present state of development, especially the terms axiom and definition are still a bit confused.)
>
> D. Hilbert, [37, p. 168].

Theorem 5.7 shows that the additivity of the probability of the random variable $a \oplus b$ (in particular, the yes-no event $a \vee b$) for incompatible a and b is a corollary of the definition of de Finetti's consistency, Definition 5.5. Theorem 7.2 similarly shows that the multiplicativity of the probability of the random variable $a \otimes b$ (in particular, the yes-no event $a \wedge b$) for logically-algebraically independent a and b, is a corollary of the same definition.

8. Two reconciliations

Nelson/Rota on commutative rings of random variables. While algebras of random variables in Nelson's book [56, Chapter 2] are preliminarily equipped with addition and multiplication, Rota advocated algebras of random variables that do not pay homage to Her Imperial Majesty the theory of commutative rings. Combining Theorem 4.7(iii) and Theorems 5.9-5.11 we see that there is no conflict between these two positions:

From every MV-algebra A, a ring-theoretic structure is uniquely recovered in the C-algebra $C(\boldsymbol{\mu}(A), \mathbb{C})$ via the following canonical embeddings of the semisimple MV-algebra $A' = A/\mathsf{Rad}(A)$:*

$$A' \hookrightarrow C(\boldsymbol{\mu}(A), [0,1]) \hookrightarrow C(\boldsymbol{\mu}(A), \mathbb{R}) \hookrightarrow C(\boldsymbol{\mu}(A), \mathbb{C}).$$

The homeomorphism of Proposition 4.3 identifies the maximal spectral spaces $\boldsymbol{\mu}(A)$ and $\boldsymbol{\mu}(A')$.

De Finetti/Kolmogorov on finite/countable additivity. While de Finetti championed finite additive probabilities, Kolmogorov introduced the countable additivity axiom as an expedient tool to prove theorems. The following result shows that finite and countable additivity are two faces of the same coin:

For any MV-algebra A, Theorem 5.7 identifies the states of A with the finitely consistent maps on A. Via the affine homeomorphism $\gamma_A \colon \mathsf{S}(A) \cong \mathsf{P}(A)$ of Theorem 5.11, the regular Borel probability measures on the maximal spectral space $\boldsymbol{\mu}(A)$ may be identified with the states of A. The former are countably additive, the latter are finitely additive. By Propositions 4.4 and 4.6, as A ranges over all MV-algebras, $\boldsymbol{\mu}(A)$ ranges over all compact Hausdorff spaces. Conversely, for every compact Hausdorff space Ω and Kolmogorov probability space $(\Omega, \mathcal{F}_\Omega, P)$, with $\mathcal{F}_\Omega$ the sigma-algebra of Borel sets of Ω and P a regular probability measure on $\mathcal{F}_\Omega$, there is an MV-algebra A and a state σ of A such that $(\Omega, \mathcal{F}_\Omega, P) \cong (\boldsymbol{\mu}(A), \mathcal{F}_{\boldsymbol{\mu}(A)}, \gamma_A(\sigma))$.

Identifications like the one induced by the affine homeomorphism γ_A are customary in functional analysis: thus, e.g., positive Radon measures are routinely [38, p. 70] identified with their corresponding regular Borel measures. It is pointless to argue whether the countable additivity of the latter is preferable to the finite additivity of the former. As another example, (proof of Theorem 2.3 (ii)⇔(iii)), Carathéodory theorem *extends* every finitely additive probability measure of a boolean algebra A to a regular Borel probability measure on the Stone space $\boldsymbol{\mu}(A)$. For this to make sense, A is preliminarily identified with the boolean algebra of clopens of $\boldsymbol{\mu}(A)$.

Recalling Hilbert's 1905 excerpt at the end of Section 7 and inverting the adage "old theorems never die, they turn into definitions", the results of this paper make both Kolmogorov's countable additivity *axiom*, and the *definition* of independence into *consequences* of de Finetti's notion of consistency.

In de Finetti's own words (italics in the original),

> Dimostrate le proprietá fondamentali del calcolo classico delle probabilitá, ne scende che tutti i risultati di tale calcolo non sono che *conseguenze* della definizione che abbiamo data della *coerenza*.
>
> (Having proved the fundamental properties of the classical probability calculus, it follows that all its results are nothing else but *consequences* of the definition of *consistency* given in this paper.)
>
> [12, § 16, page 328]

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