CME 241: Reinforcement Learning for Stochastic Control Problems in Finance

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1 Understanding Risk-Aversion through Utility Theory

Definition: *Risk-Aversion*
Personality-based degree of risk fear.

Definition: *Risk-Premium*
Compensation for taking the risk.

We seek a "valuation formula" for the amount we'd pay that:

- Increases one-to-one with the Mean of the outcome
- Decreases as the Variance of the outcome (i.e.. Risk) increases
- Decreases as our Personal Risk-Aversion increases

Remark: Our satisfaction to better outcomes grows non-linearly. We express this satisfaction non-linearity as a mathematical function: *Utility*

Notations:

- Utility $U(X)$ is a concave function, i.e. $\mathbb{E}[U(X)] < U(\mathbb{E}[X])$
- Certainty-Equivalent Value: $x_{CE} = U^{-1}(\mathbb{E}[U(X)])$
- Absolute Risk-Premium $\pi_A = \mathbb{E}[X] - x_{CE}$
- Relative Risk-Premium $\pi_R = \frac{\pi_A}{\mathbb{E}(X)} = 1 - \frac{x_{CE}}{\mathbb{E}(X)}$
- Absolute Risk-Aversion $A(X) = -\frac{U''(X)}{U'(X)}$ so that $\pi_A \approx \frac{1}{2}A(\mathbb{E}[X])\sigma_X^2$
- Relative Risk-Aversion $R(X) = -\frac{U''(X)X}{U'(X)}$ so that $\pi_R \approx \frac{1}{2}R(\mathbb{E}[X])\sigma^2_{\frac{X}{\mathbb{E}[X]}}$

x_{CE} denotes certain amount we'd pay to consume an uncertain outcome

1.1 CARA - Constant Absolute Risk Aversion

Theoretical example

- Utility function: $U(x) = \frac{1-e^{-ax}}{a}$ for $a \neq 0$
- Absolute risk-aversion: $A(x) = a$
- for $x \sim \mathcal{N}(\mu, \sigma^2)$:

$$\mathbb{E}[U(X)] = \begin{cases} \frac{1-e^{-a\mu+\frac{a^2\sigma^2}{2}}}{a} & \text{for } a \neq 0 \\ \mu & \text{for } a = 0 \end{cases}$$

$$x_{CE} = \mu - \frac{a\sigma^2}{2} \tag{1}$$

$$\pi_A = \frac{a\sigma^2}{2} \tag{2}$$

Portfolio Application Example

- **Context:** We are given \$1 to invest and hold for a horizon of 1 year. Investment choices are π of risky asset and $(1-\pi)$ of riskless asset.
  Let's note W our wealth, and $U(W)$ our utility of wealth.
- With a CARA utility function, we have

$$U(W) = \frac{1-e^{-aW}}{a} \text{for } a \neq 0$$

- We suppose that:

  Risky Asset Annual Return $\sim \mathcal{N}(\mu, \sigma^2)$
  Riskless Asset Annual Return $= r$
- We therefore have

$$W \sim \mathcal{N}(1 + r + \pi(\mu - r), \pi^2\sigma^2)$$

- Which gives us using (1)

$$x_{CE} = 1 + r + \pi(\mu - r) - \frac{a\pi^2\sigma^2}{2}$$

- This reaches a maximum for

$$\pi^* = \frac{\mu - r}{a\sigma^2}$$

1.2 CRRA - Constant Relative Risk Aversion

Theoretical example

- Utility function: $U(x) = \frac{x^{1-\gamma}-1}{1-\gamma}$ for $\gamma \neq 1$
- Absolute risk-aversion: $R(x) = \gamma$
- for $log(x) \sim \mathcal{N}(\mu, \sigma^2)$:

$$\mathbb{E}[U(X)] = \begin{cases} \frac{e^{\mu(1-\gamma)+\frac{\sigma^2}{2}(1-\gamma)^2}-1}{1-\gamma} & \text{for } \gamma \neq 1 \\ \mu & \text{for } \gamma = 1 \end{cases}$$

$$x_{CE} = e^{\mu+\frac{\sigma^2}{2}(1-\gamma)} \quad (3)$$

$$\pi_R = 1 - e^{-\frac{\sigma^2\gamma}{2}} \quad (4)$$

Portfolio Application Example

- **Context:** We consider Merton's 1969 Portfolio problem:

$$dR_t = rR_t dt$$

$$dS_t = \mu S_t dt + \sigma S_t dz_t$$

  We are given \$1 to invest with continuous rebalancing for 1 year. We denote W_t our wealth, and π the constant fraction of W_t to allocate to our risky asset.
  We have:

$$dW_t = (r + \pi(\mu - r))W_t dt + \pi\sigma W_t dz_t$$

- With a CRRA utility function, we have:

$$U(W) = \frac{W^{1-\gamma}-1}{1-\gamma} \text{for } \gamma \neq 1$$

- With Itô's Lemma:

$$log(W) \sim \mathcal{N}(r + \pi(\mu - r) - \frac{\pi^2\sigma^2}{2}, \pi^2\sigma^2)$$

- Which gives us using (3)

$$x_{CE} = r + \pi(\mu - r) - \frac{\pi^2\sigma^2}{2} + \frac{\pi^2\sigma^2}{2}(1-\gamma)$$

$$x_{CE} = r + \pi(\mu - r) + \gamma\frac{\pi^2\sigma^2}{2}$$

- This reaches a maximum for

$$\pi^* = \frac{\mu - r}{\gamma\sigma^2}$$