HJB Equation and Merton's Portfolio Problem

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Overview

1. Problem Statement
2. HJB Equation as Optimal Discounted Value Function PDE
3. Reducing the PDE to an ODE
4. Optimal Allocation and Consumption
5. Insights and Real-World Adaptation

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- Consumption Utility assumed to have constant Relative Risk-Aversion

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- $\gamma =$ (constant) Relative Risk-Aversion $\frac{-x \cdot U''(x)}{U'(x)}$

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- We will solve this problem for $\gamma \neq 1$ ($\gamma = 1$ is easier, hence omitted)

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- Note: $c_t \geq 0$, but π_t is unconstrained

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$$V^*(t, W_t) = \max_{\pi_t, c_t} E[\int_t^T \frac{e^{-\rho s} \cdot c_s^{1-\gamma}}{1-\gamma} \cdot ds + \frac{e^{-\rho T} \cdot \epsilon^\gamma \cdot W_T^{1-\gamma}}{1-\gamma}]$$

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- $V^*(t, W_t)$ satisfies a simple recursive formulation for $0 \leq t < t_1 < T$.

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HJB Equation for Optimal Discounted Value Function

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Rewriting in stochastic differential form, we have the HJB formulation

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Use Ito's Lemma on dV^*, remove the dz_t term since it's a martingale, and divide throughout by dt to produce the HJB Equation in PDE form:

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Let us write the above equation more succinctly as:

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The second-order conditions for Φ are satisfied **under the assumptions** $c_t^* > 0, W_t > 0, \frac{\partial^2 V^*}{\partial W_t^2} < 0$ for all $0 \leq t < T$ (we will later show that these are all satisfied in the solution we derive), and for concave $U(\cdot)$, i.e., $\gamma > 0$

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- The HJB Formulation was key and this solution approach provides a template for similar continuous-time stochastic control problems.

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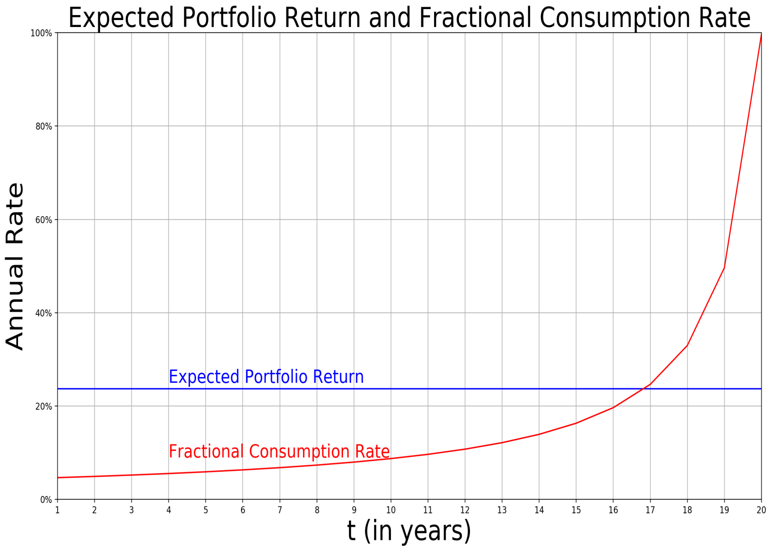
Portfolio Return versus Consumption Rate



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