Pricing financial derivatives with the Black-Scholes model

Tom Brusseaux — brusseaux@insa-toulouse.fr
Christopher Huyghues-Beaufond — huyghues@insa-toulouse.fr

National Institute of Applied Sciences (INSA) of Toulouse
Tutor: A. Joulin

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Contents

Abstract

The development of **stochastic methods and algorithms** during the 20th century have allowed researchers to apply them in many domains such as Finance, Insurance, Economics. In parallel, the growth and internationalization of financial markets have highlighted several needs, notably that of options and other financial assets prices. Indeed, in the early 90's financial analysts did not have any explicit manner to price options. They used principally historical data and empirical methods which were not accurate and hard to set up. Hence, in this article we aim to apply stochastic theory on **European call and put options**, to better predict their prices. Thus, we use **Brownian motion**, the **Black Scholes model and partial differential equation**. With these mathematical tools we can price European options at time t assuming that we have enough information relative to the financial markets. We compare the explicit formula with direct simulations and observe the precision but also the limits of the Black Scholes model. We also compare Black-Scholes model to the **Cox-Ross-Rubinstein (CRR)** model which is also used to price options but in discrete time. Our research shows that there is an explicit formula to price European options and that it is more accurate and simpler than the methods used previously. We conclude that Black-Scholes model can be used in concrete cases, but it is still not perfect because it is based on many unrealistic assumptions and only applies to European options. That is why future studies should focus on alternative models which are more realistic.

1 Introduction

The Black-Scholes-Merton model (also called Black-Scholes model) was defined by Robert C. Merton in 1973, based on the previous work of Fischer Black and Myron Scholes, as it is stated in [1]. This work constitutes a major step forward in the domain of financial mathematics and earned the researchers the 1997 Nobel prize in Economic Sciences.

The Black-Scholes model applies to a very specific type of financial asset: European call and put options. These options constitute a contract allowing an investor to buy (call option) or sell (put option) a stock at a predefined price at a fixed date of maturity. This model allows users to compute the price of an option by using the characteristics of the underlying asset. Hence, it uses the maturity T (expiration date of the contract), the strike price K (price fixed at date T), the volatility σ, the interest rate μ and the price of the underlying asset X to predict the expected payoff and then deduce the appropriate price.

The Black-Scholes model uses a specific Gaussian process called Brownian motion defined in 2 to model the evolution of an asset's price. However, the considerable irregularity of this process represents an issue while we do the integration to obtain an explicit formula. To avoid this problem, it's necessary to use the Itō formula. This formula, introduced by Kiyoshi Itō in the 1940's [3], represents one of the major results in stochastic calculus theory by establishing a link between the solutions of stochastic differential equations and partial differential equations of the second order.

The fact that financial markets are constantly moving constitutes another issue which led the model to evolve towards a more complex version. Hence, to give it a more dynamic aspect, a Black-Scholes partial differential equation was introduced. Based on no arbitrage opportunities assumption, we can compute a conditional expectation of the price as a solution of Black-Scholes PDE by taking into account all the previous information (ie. the price of the asset between time t_0 and actual time t). We show in 4 that absence of arbitrage opportunities can be translated by a change of probability, from the historic probability of the asset to the risk neutral probability. It consists in a change of measure using the Girsanov theorem, proved by Cameron-Martin in 1940 and then by Igor Girsanov in 1960 [3].

Finally, our aim is to use all these tools to of stochastic theory to predict the price of European options. To illustrate our work and test the accuracy of the model, we make simulations in which we induce variations in the parameters to observe the evolution of the price. In particular, we compare the approximation of the price by Monte Carlo methods and the one given by the explicit formula, solution of the partial differential equation of Black-Scholes. We also delve into an extension of the Black-Scholes model known as the Cox-Ross-Rubinstein model (CRR). This discrete-time model empowers us to price various derivatives, including European and American options, in contrast to Black-Scholes which is only limited to European option.

2 Brownian motion

We first introduce a crucial tool in finance known as Brownian motion. It serves as a representation of the inherent randomness observed in financial markets. We start by describing a discrete random walk and gradually progress to the concept of Brownian motion.

2.1 Discrete random walk

Let's assume a stochastic process $(X_n)_{n\in\mathbb{N}}$ with its values in $\mathbb{Z}$ such that $X_{n+1} = X_n + U_{n+1}$ and $(U_{n+1})_{n\in\mathbb{N}} \overset{\text{iid}}{\sim} \mathcal{R}(0.5)$ independent from X_0. Thus (X_n) has specific increments $(X_{n+1} - X_n) = U_{n+1}$ with the following properties:

- they are independent because (U_n) are independent,
- they are stationary because (U_n) are identically distributed.

Hence we have $(X_n) = X_0 + \sum_{i=1}^{n} U_i$.
If X_0 is deterministic (let's take $X_0 = 0$ for simplicity), we have:

- $\mathbb{E}[X_n] = \sum_{i=1}^{n} \mathbb{E}[U_i] = 0$,
- $\mathrm{Var}[X_n] = \mathrm{Var}[\sum_{i=1}^{n} U_i] = \sum_{i=1}^{n} \mathrm{Var}[U_i] = n\,\mathrm{Var}[U_0]$,
- $\mathrm{Cov}[X_n, X_l] = \mathrm{Cov}[\sum_{i=1}^{n} U_i, \sum_{j=1}^{l} U_j] = \sum_{i=1}^{n}\sum_{j=1}^{l} \mathrm{Cov}[U_i, U_j] = \sum_{i=1}^{n\wedge l} \mathrm{Cov}[U_i, U_i]$[1] $= n \wedge l$.

This random walk (represented on Figure 1), discrete in time and in space, is the base of the binomial model, also called Cox-Ross-Rubinstein (CRR) model [4] which we define in 5.2.1.

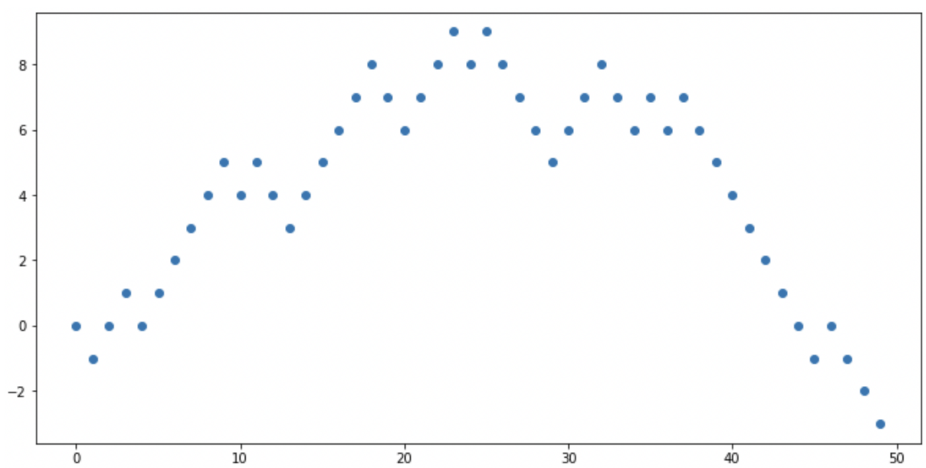


Figure 1: Random walk of 50 steps

[1]We note that $\mathrm{Cov}[U_i, U_j] = 0$ when $i \neq j$.

2.2 Continuous random walk and Wiener process

Let's now take $X_{n+1} = X_n + U_{n+1}$ with $(U_{n+1})_{n\in\mathbb{N}} \overset{\text{iid}}{\sim} \mathcal{N}(0,1)$. The properties remains the same as before, hence we have:

$$\begin{cases} (X_n) = X_0 + \sum_{i=1}^{n} U_i \sim \mathcal{N}(X_0, n), \\ \text{Cov}[X_n, X_l] = n \wedge l \end{cases}$$

When we normalize it, $(X_t^n)_{t\geq 0}$ converge to a limit process B_t when $n \to \infty$. This process $(B_t)_{t\geq 0}$ is called Wiener process (see Figure 2) or Brownian motion (or BM). To make this transition from discrete to continuous time, we need to use the Donsker theorem which establishes the convergence in law of a random walk to a Gaussian stochastic process.

Donsker theorem :
For a sequence $(U_n)_{n\in\mathbb{N}^*}$ of iid variables square integrable, centered and of variance σ^2, we can interpolate piece-wise affine the random walk $\sum_{k=1}^{n} U_k$ by considering $(X_n(t), t \geq 0)_{n\in\mathbb{N}^*}$ such that:

$$X_n(t) = \frac{1}{\sigma\sqrt{n}} \left(\sum_{k=1}^{[nt]} U_k + (nt - [nt]) U_{[nt]+1} \right), \text{ for } t \in [0,1].$$

We consider $\mathcal{C}([0,1])$ the space of continuous functions with real values in $[0,1]$ provided with the Borel sigma-algebra $\mathcal{B}$ and with the infinite norm $||.||_\infty$. Then, X_n converge in law to a Brownian motion $B = (B_t, t \geq 0)$ when $n \to +\infty$.
The BM has the same properties as the processes above and some others (which we will admit). These are:

- It is a centered Gaussian process, i.e. $B_t \sim \mathcal{N}(0, t)$,
- $\text{Cov}[B_s, B_t] = s \wedge t$,
- *(New)* It is almost surely continuous on $\mathbb{R}^+$,
- *(New)* It's increments $(B_{t_{i+1}} - B_{t_i})$ are independent and stationary (i.e. $B_{t_i} - B_{t_j} \sim B_{t_i - t_j}$),
- *(New)* The BM is invariant under scale change (fractal trajectory),
- *(New)* It is nowhere differentiable.

We notice that, if we want $X_0 \neq 0$, we simply have to take $\tilde{B}_t = x + B_t$ with $X_0 = x$ which is equivalent to a translation of x.

Two different definitions of the BM are stated in [3] .

Definition 1:
Let's assume $(B_t)_{t\geq 0}$ a stochastic process continuous in time; it is a Brownian motion if:

- $B_0 = 0$ almost surely,
- It is a centered Gaussian process with covariance function $Cov[B_s, B_t] = s \wedge t$

Definition 2 :
Let's assume $(B_t)_{t\geq 0}$ a stochastic process continuous in time; it is a Brownian motion if:

- It's increments $(B_{t_{i+1}} - B_{t_i})$ are independent and stationary (i.e. $B_{t_i} - B_{t_j} \sim B_{t_i - t_j}$),
- $\forall t > 0$, $B_t \sim \mathcal{N}(0, t)$.

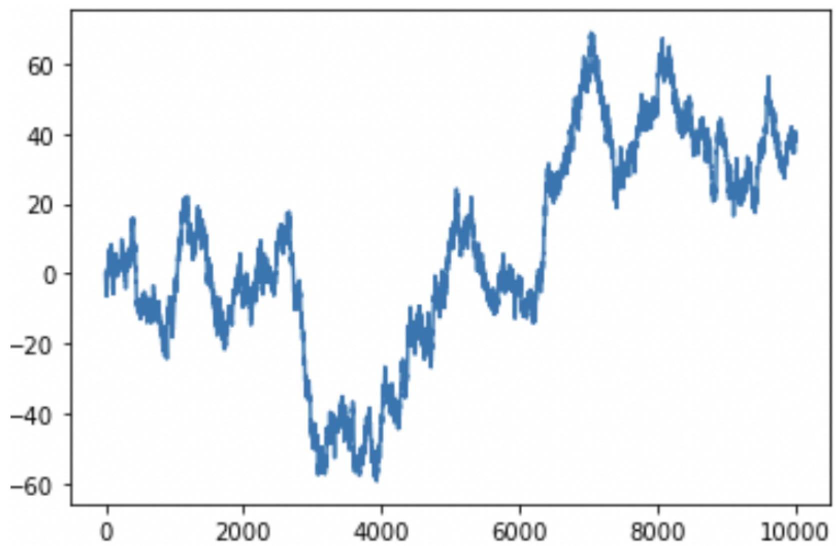


Figure 2: Simulation of a Brownian motion

We use this Brownian motion in option pricing because its behavior is very similar to the market. It fluctuates with a volatility that can be calibrated based on the asset under study, and its value at time t depends on the preceding value.

3 Geometric Brownian Motion

We now present a prominent model, Geometric Brownian Motion, which characterizes stock price behavior through Brownian motion. This model integrates the expected return and volatility of stocks. To lay the groundwork, we discuss the differentiability of Brownian motion and introduce the Itō formula; first steps to understand the Geometric Brownian Motion.

3.1 Theory of Geometric Brownian Motion and Itō formula

As shown previously, the BM is very irregular and then nowhere differentiable. Hence, one can observe that:

$$\begin{cases} \mathbb{E}[B_t^2] = Var(B_t) = t \\ \mathbb{E}[(B_{t+s} - B_t)^2] = \mathbb{E}[B_s^2] = s \text{ (because the BM is stationary).} \end{cases}$$

Thus, if we take $s = \delta t$ a small increment, we obtain

$$\mathbb{E}[(B_{t+\delta t} - B_t)^2] = \mathbb{E}[B_{\delta t}^2] = \delta t$$

Intuitively, we get $(B_{t+\delta t} - B_t) \sim \sqrt{\delta t}$. Hence, we have $\lim_{\delta t \to 0} \frac{B_{t+\delta t} - B_t}{\delta t} \sim \frac{\sqrt{\delta t}}{\delta t} = \frac{1}{\sqrt{\delta t}} = +\infty$ and we conclude that the BM is not differentiable.

This irregularity is the main problem when we want to integrate the expression to build the Geometric Brownian motion, because the process is not with finite variations (ie. not bounded on any interval $[0, t]$).
To deal with that, we need to use the well known Itō formula, introduced in the 1940's by Kiyoshi Itō.

Itō formula [3] **:**
In the one-dimensional case for a martingale M and a function $f : \mathbb{R} \to \mathbb{R} \in C^2$:

$$f(M_t) = f(M_0) + \int_0^t f'(M_s)dM_s + \frac{1}{2}\int_0^t f''(M_s)d[M, M]_s, \ t \geq 0$$

In the case of the Brownian motion, we have $[B, B]_t = \lim_{n\to+\infty} \sum_{i=1}^n (B_{\frac{it}{n}} - B_{\frac{(i-1)t}{n}})^2$. Due to the properties of the BM we know that $(B_{\frac{it}{n}} - B_{\frac{(i-1)t}{n}})_{i\in\{1,\ldots,n\}} \sim \mathcal{N}(0, \frac{t}{n})$ meaning that they are identically distributed and they also are independent.
Hence, due to the strong Law of Great Numbers (LGN) for iid variables we have:

$$\begin{aligned} &\frac{1}{n}\sum_{i=1}^n (B_{\frac{it}{n}} - B_{\frac{(i-1)t}{n}})^2 \underset{n\to+\infty}{\sim} \mathbb{E}[(B_{\frac{t}{n}})^2] = Var(B_{\frac{t}{n}}) = \frac{t}{n} \\ \implies &\sum_{i=1}^n (B_{\frac{it}{n}} - B_{\frac{(i-1)t}{n}})^2 \underset{n\to+\infty}{\overset{a.s.}{\to}} t, \ \forall t \geq 0 \\ \implies &[B, B]_t = t \end{aligned}$$

In the Black-Scholes model introduced by Fischer Black and Myron Scholes in 1973 [3] is, we assumed that stock prices follow a Geometric Brownian Motion (GBM), i.e :

$$dX_t = \mu X_t dt + \sigma X_t dB_t$$

where :

- X_t is the underlying asset's price at time t,
- μ is expected return of the asset (drift),
- σ is the volatility,
- dB_t is a Brownian motion at time t.

This equation describes the dynamics of the underlying asset's price over time; every asset with no dividend should verify this model. However, this model is based on assumptions :

- Expected return (μ) and volatility (σ) are constants.
- There are no dividends and transaction costs.

In addition, we can prove that GBM model has an unique solution X_t because "μX_t" and "σX_t" verify some lipschitz conditions.

3.2 Explicit Solution

The GBM is an important model in finance because the famous Black-Scholes formula, used to price European options, is derived from it.
By using Ito's formula on $f(x) = \ln(x) \in C^2(]0; +\infty[)$, we find that :

$$f(X_t) = f(X_0) + \int_0^t f'(X_s)\, dX_s + \frac{1}{2}\int_0^t f''(X_s)\, d[X, X]_s$$

where :

$$d[X, X]_s = \sigma^2 X_s^2 ds$$

Thus, we obtain :

$$\begin{aligned}
&\ln(X_t) = \ln(X_0) + \int_0^t \frac{1}{X_s}\, dX_s + \frac{1}{2}\int_0^t (-\frac{1}{X_s^2})\, \sigma^2\, X_s^2\, ds,\\
\Longrightarrow\ &\ln(X_t) = \ln(X_0) + \int_0^t \frac{1}{X_s}\, dX_s - \frac{\sigma^2}{2}\, t,\\
\Longrightarrow\ &\ln(X_t) = \ln(X_0) + \int_0^t \frac{1}{X_s}\, \mu\, X_s\, ds + \sigma\, X_s\, dB_s - \frac{\sigma^2}{2}\, t,\\
\Longrightarrow\ &\ln(X_t) = \ln(X_0) + \mu\, t + \sigma\, B_t - \frac{\sigma^2}{2}\, t.
\end{aligned}$$

Finally, we get :

$$\boxed{X_t = X_0 e^{\mu t + \sigma B_t - \frac{\sigma^2}{2} t}}$$

As we got the solution of the GBM, we observe on the following figure the evolution of many stock prices following this process with same characteristics (μ and σ) :

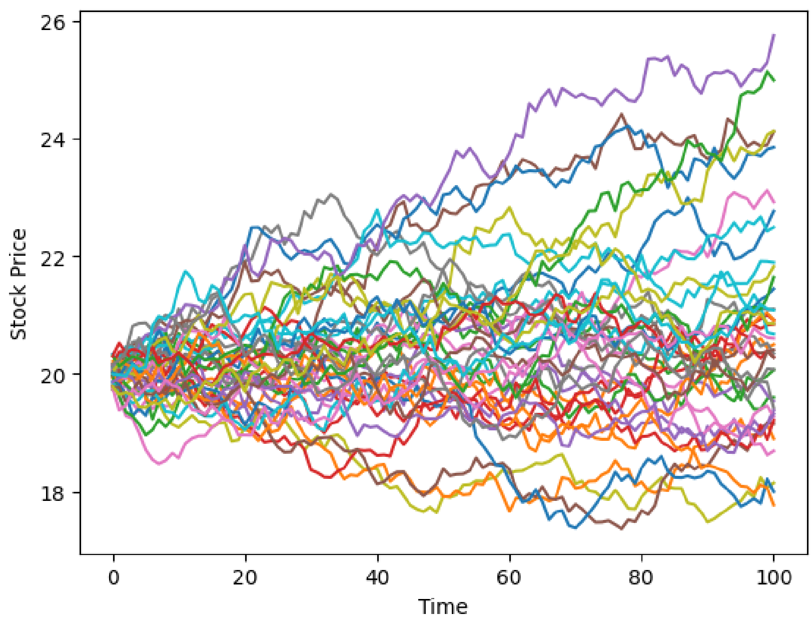


Figure 3: Evolution of multiple stock prices following the GBM ($\mu = 0.05$ and $\sigma = 0.08$)

As shown in Figure 3, stock prices follow different paths, which is crucial for pricing derivatives. GBM models various market scenarios, allowing us to take an "average" to price a derivative.

3.3 Black-Scholes Formula

Modern finance (late 19th and early 20th centuries) have seen the emergence of derivative contracts. There are many different derivatives, but this project focuses on derivatives with a fixed maturation time T and a payoff dependent on the price of the underlying asset at maturity.

Thus, the GBM model enables us to have an explicit formula for the payoff of European option[2].

We can look for example at the call option's payoff in figure 4:

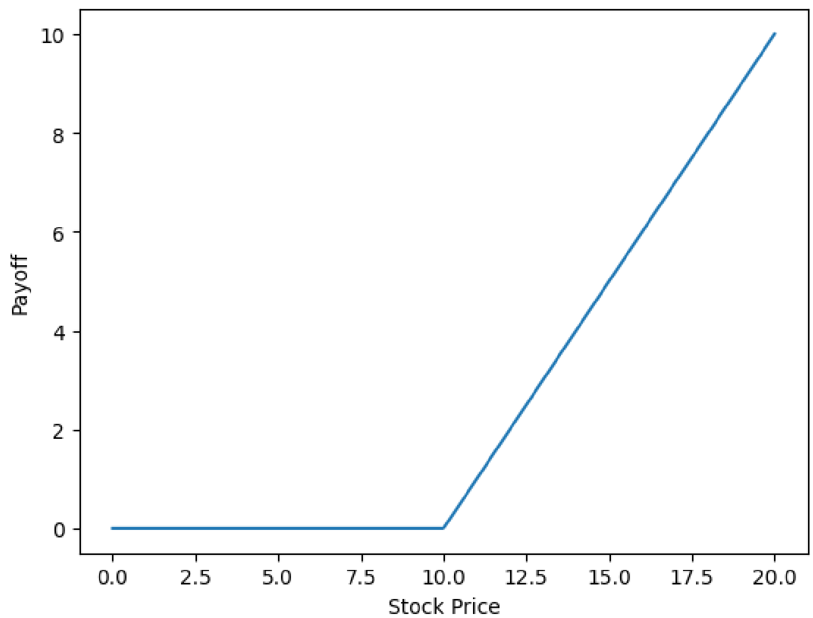


Figure 4: Payoff of an European Call Option at Maturity with Strike Price $K = 10$

Hence the call and put options payoff formulas are :

$$\begin{cases} \text{Call option Payoff} = \max(X_t - K, 0) = (X_t - K)^+, \\ \text{Put option Payoff} = \max(K - X_t, 0) = (K - X_t)^+. \end{cases}$$

The remaining issue is the market price of the option. One way to achieve this is to compute the expectation of the payoff to take into account all the different paths that

[2]A European option is a financial contract based on an underlying asset (commodities, currencies, etc.) that provides the possibility to buy or sell the asset at a specific price (strike) on a specified date (maturity T). Buying the underlying asset constitutes a call option, while selling it constitutes a put option.

the stock price can take, ie. to take $\mathbb{E}[(X_t-K)^+]$ (for a call option). To compute this expectation:

$$\begin{aligned}\mathbb{E}\left[(X_t-K)^+\right] &= \mathbb{E}\left[(X_t-K)\mathbf{1}_{\{X_t\geq K\}}\right] \\ &= \mathbb{E}[X_t\mathbf{1}_{\{X_t\geq K\}}] - K\mathbb{P}\left(X_t\geq K\right)\end{aligned}$$

In addition, we note that :

$$\begin{aligned}X_t\geq K &\iff X_0e^{\mu t+\sigma B_t-\frac{\sigma^2}{2}t}\geq K \\ &\iff B_t\geq \frac{\ln\left(\frac{K}{x_0}\right)-\left(\mu-\frac{\sigma^2}{2}\right)T}{\sigma}\end{aligned}$$

As we know $B_t \overset{law}{=} \sqrt{T}Z$ where Z $\sim\mathcal{N}(0,1)$ then we have that:

$$\begin{aligned}\mathbb{E}[(X_t-K)^+] &= x_0e^{(\mu-\frac{\sigma^2}{2})T}\mathbb{E}[e^{\sigma B_t}\mathbf{1}_{\{B_t\geq\frac{1}{\sigma}(\ln(\frac{K}{x_0})-(\mu-\frac{\sigma^2}{2})T)\}}] - K\mathbb{P}(B_t\geq\frac{1}{\sigma}(\ln(\frac{K}{x_0})-(\mu-\frac{\sigma^2}{2})T)). \\ &= x_0e^{(\mu-\frac{\sigma^2}{2})T}\int_{\mathbb{R}}e^{\sigma\sqrt{t}z}\mathbf{1}_{\{z\geq\frac{1}{\sigma\sqrt{T}}(\ln(\frac{K}{x_0})-(\mu-\frac{\sigma^2}{2})T)\}}e^{-\frac{z^2}{2}}\frac{dt}{\sqrt{2\pi}} - K\int_{\frac{1}{\sigma\sqrt{T}}(\ln(\frac{K}{x_0})-(\mu-\frac{\sigma^2}{2})T)}^{+\infty}e^{\frac{-z^2}{2}}\frac{dz}{\sqrt{2\pi}}. \\ &= x_0e^{(\mu T}\int_{\frac{1}{\sigma\sqrt{T}}(\ln(\frac{K}{x_0})-(\mu-\frac{\sigma^2}{2})T)}^{+\infty}e^{(-\frac{1}{2}(z-\sigma\sqrt{T})^2)}\frac{dz}{\sqrt{2\pi}} - K(1-\mathcal{F}_{\mathcal{N}_{(0,1)}}(\frac{1}{\sigma\sqrt{T}}(\ln(\frac{K}{x_0})-\mu)+\frac{\sigma\sqrt{T}}{2})).\end{aligned}$$

Where $\mathcal{F}_{\mathcal{N}_{(0,1)}}$ is the distribution function of a centered and reduced Gaussian random variable. If we take $u=z-\sigma\sqrt{T}$ such that $du=dz$, then we have:

$$\begin{aligned}\mathbb{E}[(X_t-K)^+] &= x_0e^{\mu T}\int_{\frac{1}{\sigma\sqrt{T}}(\ln(\frac{K}{x_0})-\mu)-\frac{\sigma\sqrt{T}}{2}}^{+\infty}e^{-\frac{\mu^2}{2}}\frac{du}{\sqrt{2\pi}} - K(1-\mathcal{F}_{\mathcal{N}(0,1)}(\frac{1}{\sigma\sqrt{T}}(\ln(\frac{K}{x_0})-\mu)+\frac{\sigma\sqrt{T}}{2})). \\ &= x_0e^{\mu T}(1-\mathcal{F}_{\mathcal{N}(0,1)}(\frac{1}{\sigma\sqrt{T}}(\ln(\frac{K}{x_0})-\mu)-\frac{\sigma\sqrt{T}}{2}) - K(1-\mathcal{F}_{\mathcal{N}(0,1)}(\frac{1}{\sigma\sqrt{T}}(\ln(\frac{K}{x_0})-\mu)+\frac{\sigma\sqrt{T}}{2})).\end{aligned}$$

Knowing that $1-\mathcal{F}_{\mathcal{N}(0,1)}(d)=\mathcal{F}_{\mathcal{N}(0,1)}(-d)$, we have :

$$\mathbb{E}[(X_t-K)^+] = x_0e^{\mu T}\mathcal{F}_{\mathcal{N}(0,1)}(\frac{1}{\sigma\sqrt{T}}(\ln(\frac{x_0}{K})+\mu)+\frac{\sigma\sqrt{T}}{2}) - K\mathcal{F}_{\mathcal{N}_{(0,1)}}(\frac{1}{\sigma\sqrt{T}}(\ln(\frac{x_0}{K})+\mu)-\frac{\sigma\sqrt{T}}{2}))$$

Finally, we obtain that the formula of the call option's payoff is :

$$C = x_0 e^{\mu T} \mathcal{F}_{\mathcal{N}_{(0,1)}}(d_1) - K \mathcal{F}_{\mathcal{N}_{(0,1)}}(d_2)$$

where :

- C is the call option's price
- $x0$ is the current underlying asset's price (deterministic)
- K is the strike price
- μ is the expected return
- σ is the volatility of the underlying asset
- $d_1 = \frac{1}{\sigma\sqrt{T}}(\ln(\frac{x_0}{K}) + \mu) + \frac{\sigma\sqrt{T}}{2}$
- $d_2 = \frac{1}{\sigma\sqrt{T}}(\ln(\frac{x_0}{K}) + \mu) - \frac{\sigma\sqrt{T}}{2}$

However, we need to take the present value, thus we need to multiply by the discount rate which is $\exp(-\mu T)$. Ultimately, we obtain the well-known Black-Scholes formula:

$$\boxed{C = x_0 \mathcal{F}_{\mathcal{N}_{(0,1)}}(d_1) - K e^{-\mu T} \mathcal{F}_{\mathcal{N}_{(0,1)}}(d_2)}$$

Nevertheless, in Section 4.2 on Arbitrage Opportunity and Martingale, we will demonstrate that the Black-Scholes formula commonly used in practice incorporates $\mu = r$, where r represents the risk-free rate. This reflects the absence of arbitrage opportunities.

3.3.1 Simulation

Finally, we can implement this formula, and we obtain Figure 5.

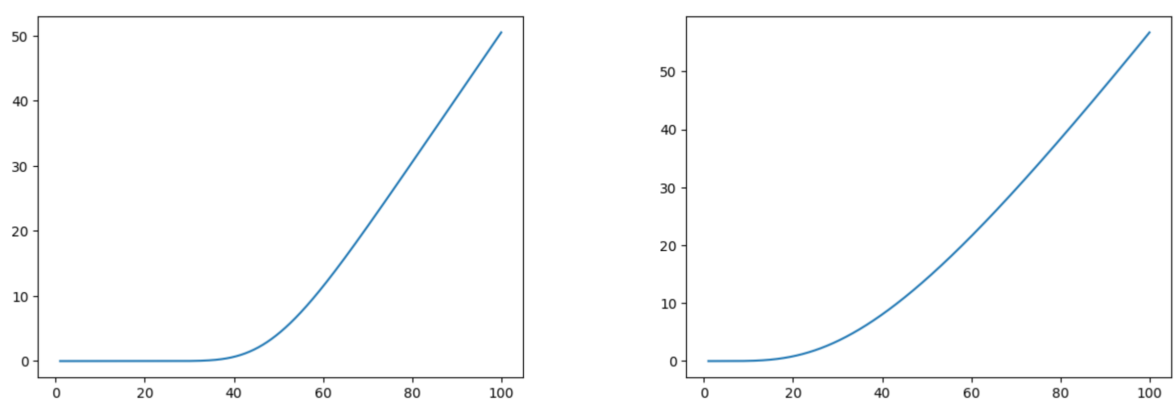


Figure 5: Call option's payoff with $K = 50$, $\sigma = 0.2$, $\mu = 0.01$, $T = 1$ and 10

With this example we can see the impact of time value of money on the price of an European call option. In fact, time value of money is a financial concept that states that a sum of money available today is worth more than the same sum in the future, due to its potential earning capacity. This is because money can earn interest or other returns over time. Hence, a call option with a longer maturity is valued higher than one with a shorter maturity.

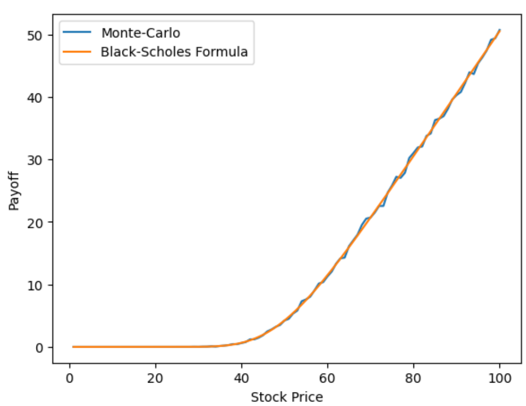
Another common way to price an European call option is by using Monte Carlo methods as defined in [2]. Indeed with the law of large number and using the fact that x_0 is deterministic and that $X_t = X_s e^{\sigma(B_t - B_s)+(\mu-\frac{\sigma^2}{2})(t-s)}\ \forall\ 0 \leq s \leq t$, we have:

$$\mathbb{E}[(X_t - K)^+] \overset{n \text{ large}}{\approx} \frac{1}{n}\sum_{k=1}^{n}(X_t^k - K)^+$$

Then:

$$\begin{cases} \mathbb{E}[(X_t - K)^+] \overset{n \text{ large}}{\approx} \frac{1}{n}\sum_{i=1}^{n}(x_i - K)^+ \\ \text{where } X_t^k \text{ follows a GBM at time } t \end{cases}$$

with $(x_1, x_2, ..., x_{n-1}, x_n)$ a n-sample of X_t.
After implementing it, we obtain Figure 6 which is similar to the Black-Scholes formula result.



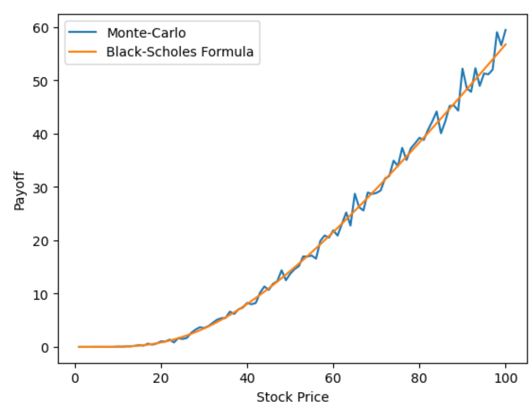


Figure 6: Monte-Carlo and BS call option's payoff for $T = 1$ and $T = 10$ (with $K = 50$, $\sigma = 0.2$, $\mu = 0.01$)

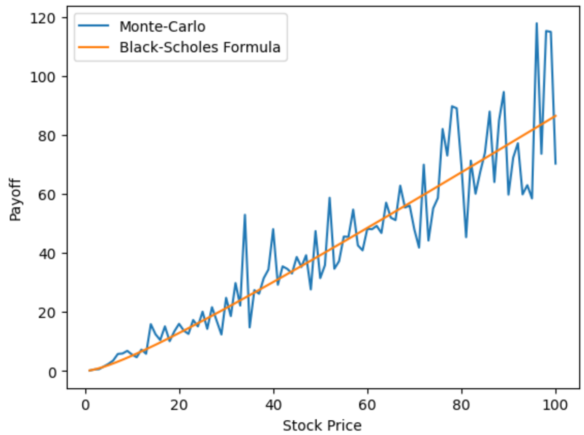


Figure 7: Monte-Carlo and BS call option's payoff for $T = 100$ (with $K = 50$, $\sigma = 0.2$, $\mu = 0.01$)

As T increases, we notice a decrease in the accuracy of the Monte Carlo method (Figure 7). compared to Black-Scholes formula. In fact, we have that $\mathrm{Var}(Z)$ is proportional to T. We can see that for $T = 100$ the accuracy of the Monte-Carlo method becomes really bad.

To solve this problem, we need to reduce the variance to have a more accurate Monte-Carlo simulation. We can remark that :

$$(X_t - K)^+ = (K - X_t)^+ + X_t - K$$

Hence we will use $X_t - K$ as a control variable to reduce the variance.
Let's compute $\mathbb{E}[X_t - K]$:

$$\begin{aligned}
\mathbb{E}[X_T - K] &= \mathbb{E}[X_T] - K \\
&= X_0 e^{\mu T - \frac{\sigma^2}{2}T}\mathbb{E}[e^{\sigma B_T}] - K \\
&= X_0 e^{\mu T - \frac{\sigma^2}{2}T}\int_{\mathbb{R}} e^{\sigma\sqrt{T}z}e^{-\frac{z^2}{2}}\frac{dt}{\sqrt{2\pi}} - K \\
&= X_0 e^{\mu T - \frac{\sigma^2}{2}T}e^{\frac{\sigma^2 T}{2}} - K \\
\mathbb{E}[X_T - K] &= X_0 e^{\mu T} - K
\end{aligned}$$

Then, we use our control variable to compute the expectation of $(K - X_t)^+$ by Monte-Carlo and we obtain the Figure 8 for T = 100 :

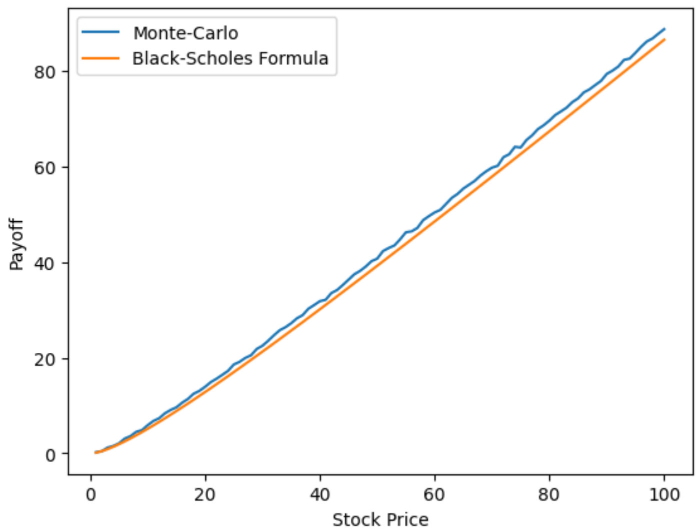


Figure 8: Monte-Carlo with variance reduction and BS call option's payoff for $T = 100$ ($K = 50$, $\sigma = 0.2$, $\mu = 0.01$)

This method allowed us to reduce the variance because each time X_T takes a considerable value (variance of X_T is proportional to T), it is canceled by $(K - X_T)^+$.

4 Black-Scholes Partial Differential Equation (PDE)

In this part, we introduce a more general framework to model the price of different types of derivatives whose values are based on the price of an underlying asset at maturity.

Let's first make some financial assumptions that are necessary to obtain the Black-Scholes PDE. We need to suppose that in the market, there is at least one risky asset (usually called a stock) and a risk-free asset (usually called a bond). We also assume that:

- stock prices follow the GBM, with constant volatility and expected return,
- stocks do not pay dividend,
- we are be able to lend/borrow or buy/sell any amount at any time,
- there are no transaction fees,
- the market is efficient, which means that everyone has access to the same information and prices are fair.

4.1 Girsanov's Theorem

Girsanov's theorem is a fundamental result in stochastic calculus that has profound implications in the field of finance. This theorem provides a crucial link between different probability measures, allowing the passage of stochastic processes to a new measure called the "risk-neutral measure" which is discussed in part 4.2 Arbitrage Opportunity and Martingale.

Girsanov's Theorem:[3]
Let $B = (B_t)_{t\geq 0}$ a Brownian motion (real) and $H = (H_t)_{t\geq 0}$ a stochastic process (continue).
Let $\tilde{B}_t = B_t - \int_0^t H_s d_s$ and the probability defined by his Radon-Nykodym density with respect to $\mathbb{P}$:

$$\mathbb{E}(\frac{d_{\mathbb{Q}}}{d_{\mathbb{P}}}|\mathcal{F}_t) = \exp(\int_0^t H_s dB_s - \frac{1}{2}\int_0^t H_s^2 d_s)$$

Then,

$$(M_t)_{t\geq 0} = exp(\int_0^t H_s dB_s - \frac{1}{2}\int_0^t H_s^2 d_s) \text{ is a martingale with respect to } \mathbb{P}$$

and $(\tilde{B}_t)_{t\geq 0}$ is a $\mathbb{Q}$-Brownian motion.

4.2 Arbitrage Opportunity and Martingale

An arbitrage opportunity means that we can make risk-less profit by trading financial assets. However, the Black-Scholes model assumes that the market is efficient, what implies the absence of arbitrage opportunity.

Let $(X_t)_{t\in[0,T]}$ verify the GBM :

$$dX_t = \mu X_t dt + \sigma X_t dB_t$$

One important property of martingale is that it represents a fair game, i.e. at any point in time, the expected future value of the martingale is equal to its current value, given all available information.

$$\mathbb{E}[X_{t+\Delta t}|\mathcal{F}_t] = X_t, \quad t \geq 0, \Delta \geq 0$$

Thus martingales are closely linked with the no arbitrage opportunity assumption because this implies that that the market is fair.

However, because of "$\mu X_t dt$", "dX_t" is not a martingale. Hence, we use Girsanov's theorem to change the probability measure and make it a martingale.
Let $(\tilde{X}_t)_{t\geq 0}$ be a risk-free asset such that :

$$\tilde{X}_t = e^{-rt} X_t, t \geq 0$$

Then, by using Itô's formula:

$$\begin{aligned} d\tilde{X}_t &= de^{-rt} X_t \\ &= -re^{-rt} X_t dt + e^{-rt} dX_t + 0 \\ &= -re^{-rt} X_t dt + e^{-rt} \mu X_t dt + e^{-rt} \sigma X_t dB_t \end{aligned}$$

Then, $\tilde{X}_t$ verifies the following stochastic differential equation :

$$d\tilde{X}_t = (\mu - r)\tilde{X}_t dt + \sigma \tilde{X}_t dB_t$$

Now, let $(\tilde{B}_t)_{t\geq 0}$ such that :

$$\tilde{B}_t = B_t + \frac{\mu - r}{\sigma} t$$

We see that:

$$\begin{aligned} \sigma \tilde{X}_t d\tilde{B}_t &= (\mu - r)\tilde{X}_t dt + \sigma \tilde{X}_t dB_t \\ \sigma \tilde{X}_t d\tilde{B}_t &= d\tilde{X}_t \end{aligned}$$

Then, if $(\tilde{B}_t)_{t\geq 0}$ is a Brownian motion (it is not the case), then $(\tilde{X}_t)_{t\geq 0}$ is a martingale as a stochastic integral with respect to a Brownian motion.

By using Girsanov's theorem, we can demonstrate that under a certain probability $\mathbb{Q}$, $\tilde{B}_t$ is a $\mathbb{Q}$-Brownian motion.
Let H_s, $\forall s \geq 0$ such that :

$$H_s = -(\frac{\mu - r}{\sigma})$$

Then, $(\tilde{B}_t)_{t\geq 0}$ is a $\mathbb{Q}$-Brownian motion (Girsanov's theorem) where the Radon-Nykodym density of $\mathbb{Q}$ with respect to $\mathbb{P}$ is :

$$\frac{d\mathbb{Q}}{d\mathbb{P}}|_{\mathcal{F}_t} = e^{-(\frac{\mu-r}{\sigma})B_t - \frac{1}{2}(\frac{\mu-r}{\sigma}^2)t}$$

Finally, as $(\tilde{B}_t)_{t\geq 0}$ is a $\mathbb{Q}$-Brownian motion, we have that $(\tilde{X}_t)_{t\geq 0}$ is a $\mathbb{Q}$-martingale.

The $\mathbb{Q}$ probability is the unique probability that makes $\tilde{X}$ a martingale and is called the risk-neutral probability. Consequently, the risk-neutral probability comes from the financial assumption that there are no arbitrage opportunities.

We have now that $(X_t)_{t\geq 0}$ is a $\mathbb{Q}$-martingale (because $\tilde{X}_t = e^{-rt}X_t$).Thus, the drift of X_t under risk-neutral probability has to be r because $\tilde{B}_t = B_t + \frac{\mu-r}{\sigma}t$ and B_t is a Brownian motion.

We conclude that the efficient market assumption implies there is no arbitrage opportunity, meaning that we need to use the risk-neutral probability. So, we should have $\mu = r$ the risk-free rate. For the rest of our study, we will assume that we use this specific probability and take $\mu = r$.

4.3 Self Financing condition and Black-Scholes PDE

We now use the no arbitrage opportunity assumption to derive the Black-Scholes PDE. Under this assumption, it should not be possible to construct a portfolio of assets without initial investment that generates risk-free profit. Hence, we assume that every portfolio verifies the self-financing condition [3]. Thus, it means that any cash inflows or outflows are entirely offset by the changes in the values of the assets in the portfolio.

Let's consider a risky asset X which follows the GBM model such that:

$$dX_t = \mu X_t dt + \sigma X_t dB_t, X_0 \geq 0$$

and a risk-free asset A such that :

$$\begin{aligned} dA_t &= rA_t dt \\ \implies A_t &= A_0 e^{-rt} \end{aligned}$$

Let ξ and η be adapted process with bounded variation which represents the amount of unity invested at time t in the assets X and A. Then, we have a portfolio with the following value V_t:

$$V_t = \xi_t X_t + \eta_t A_t \tag{1}$$

Now, we can translate the self-financing assumption mathematically by :

$$X_t d\xi_t + A_t d\eta_t = 0$$

It means that a change in the value of $X_t d\xi_t$ is offset by the change in the value of $A_t d\eta_t$ and vice-versa.
We can show that the self-financing assumption is equivalent to $dV_t = \xi_t dX_t + \eta_t dA_t$.
In fact, using Itô's formula, we have:

$$dV_t = X_t d\xi_t + \xi_t dX_t + A_t d\eta_t + \eta_t dA_t$$

Thus,

$$X_t d\xi_t + A_t d\eta_t = 0 \iff dV_t = \xi_t dX_t + \eta_t dA_t$$

As always in finance, we want to look at the present value[4], so we define $\tilde{V}$ and $\tilde{X}$ as the discounted process such that $\forall t \geq 0$:

$$\tilde{V_t} = e^{-rt} V_t$$

[3]Self-financing means that the portfolio's value is entirely explained by the values of the assets held in the portfolio and the associated cash flows, without any additional external funding or cash injections.

[4]Discounted value of future cash flows

$$\tilde{X}_t = e^{-rt}X_t$$

Then, the self-financing assumption becomes :

$$d\tilde{V}_t = \xi_t d\tilde{X}_t$$

We use once again Itō formula and we have:

$$\begin{cases} d\tilde{V}_t = e^{-rt}(dV_t - rV_t dt), \\ d\tilde{X}_t = e^{-rt}(dX_t - rX_t dt). \end{cases}$$

$$\begin{aligned}
&\implies d\tilde{V}_t = \xi_t d\tilde{X}_t, \\
&\iff e^{-rt}(dV_t - rV_t dt) = \xi_t e^{-rt}(dX_t - rX_t dt), \\
&\iff dV_t = \xi_t dX_t - \xi_t r X_t dt + rV_t dt, \\
&\iff dV_t = \xi_t dX_t + r dt \eta_t A_t, \\
&\iff dV_t = \xi_t dX_t + \eta_t dA_t, \\
&\iff X_t d\xi_t + A_t d\eta_t = 0 \text{ (self-financing assumption.)}
\end{aligned}$$

Due to these equations, we can show that $\tilde{V}_t$ is a martingale under the risk-neutral probability $\mathbb{Q}$ by the previous argument with Girsanov's theorem. We now derive the Black Scholes PDE.

Let's consider a finite time horizon $T > 0$ such that we have $V_t = g(X_t, t)\, \forall t \in [0, T]$, with $g \in C^2((0, +\infty) \times [0, T])$ and $g(x, T) = f(x)$. f represents the payoff of a derivative whose value is based on the value of an underlying asset x and which has a maturity T.
With the Itō formula we have on one side:

$$\begin{aligned}
g(X_T, T) &= g(X_0, 0) + \int_0^T \partial_1 g(X_s, s) dX_s + \int_0^T \partial_2 g(X_s, s) ds + \frac{1}{2}\int_0^T \partial^2_{1,1} g(X_s, s) d[X, X]_s\,, \\
&= g(X_0, 0) + \int_0^T \partial_1 g(X_s, s)[rX_s ds + \sigma X_s dB_s] + \int_0^T \partial_2 g(X_s, s) ds + \frac{1}{2}\int_0^T \partial^2_{1,1} g(X_s, s)\sigma^2 X_s^2 ds\,, \\
&= g(X_0, 0) + \int_0^T [rX_s \partial_1 g(X_s, s) + \partial_2 g(X_s, s) + \frac{\sigma^2}{2} X_s^2 \partial^2_{1,1} g(X_s, s)] ds + \int_0^T \sigma X_s \partial_1 g(X_s, s) dB_s\,.
\end{aligned}$$

On the other side, we know that $dV_t = \xi_t dX_t + \mu_t dA_t$, so if we integrate it we have:

$$\begin{aligned}
V_T &= V_0 + \int_0^T \xi_s dX_s + \int_0^T \mu_s dA_s\,, \\
&= V_0 + \int_0^T rX_s \xi_s ds + \int_0^T \sigma X_s \xi_s dB_s + \int_0^T \mu_s r A_s ds\,, \\
&= V_0 + \int_0^T [rX_s \xi_s + \mu_s r A_s] ds + \int_0^T \sigma X_s \xi_s dB_s\,.
\end{aligned}$$

By uniqueness of the decomposition of V_T, the two previous equations must be equal. Hence, by identification of the integrals we conclude that :

$$\begin{cases} rX_s\partial_1 g(X_s,s) + \partial_2 g(X_s,s) + \frac{\sigma^2}{2}X_s^2\partial_{1,1}^2 g(X_s,s) = rX_s\xi_s + \mu_s rA_s \\ \sigma X_s \partial_1 g(X_s,s)dB_s = \sigma X_s \xi_s dB_s \end{cases}$$

With the Equation (1) we have that $\mu_t = \frac{V_t - \xi_t X_t}{A_t}$. Hence we can write $rA_s\mu_s = r(g(X_s,s) - \xi_s X_s)$ and the first equation above simplifies itself. Finally, we obtain the Black-Scholes PDE:

$$\boxed{\begin{gathered} rg(x,t) = \partial_t g(x,t) + rx\partial_x g(x,t) + \frac{1}{2}x^2\sigma^2\partial_{x,x}^2 g(x,t)\,, \\ \text{with } \xi_t = \partial_x g(X_t,t) \text{ and } \mu_t = \frac{g(X_t,t) - X_t\partial_x g(X_t,t)}{A_0 e^{rt}}. \end{gathered}}$$

4.4 Solution of Black-Scholes PDE

One of the advantages of the Black-Scholes PDE is that we can derive a solution. $\forall T > 0, \tilde{V}_t = e^{-rt}V_t$ is a martingale so we have:

$$\begin{aligned} &\mathbb{E}[\tilde{V}_T|\mathcal{F}_t] = \tilde{V}_t,\ \forall t \in [0,T]\,, \\ \Rightarrow\, &e^{-rt}\mathbb{E}[g(X_T,T)|\mathcal{F}_t] = e^{-rt}g(X_t,t)\,, \\ \Rightarrow\, &g(X_t,t) = e^{-r(T-t)}\mathbb{E}[g(X_T,T)|\mathcal{F}_t]\,. \end{aligned}$$

Knowing that the solutions of the GBM are Markov's processes and using the property of Markov we have:

$$\begin{aligned} &g(X_t,t) = e^{-r(T-t)}\mathbb{E}[g(X_T,T)|X_t],\ \forall t \in [0,T] \text{ and } \forall T > 0\,, \\ \Rightarrow\, &g(x,t) = e^{-r(T-t)}\mathbb{E}[g(X_T,T)|X_t = x],\ \forall t \in [0,T], \forall T > 0 \text{ and } \forall x > 0\,, \\ \Rightarrow\, &g(x,t) = e^{-r(T-t)}\mathbb{E}[f(X_T)|X_t = x],\ \forall t \in [0,T], \forall T > 0 \text{ and } \forall x > 0\,. \end{aligned}$$

Then, we obtain a solution of the Black-Scholes PDE :

$$\boxed{g(x,t) = e^{-r(T-t)}\mathbb{E}[f(X_T)|X_t = x],\ \forall t \in [0,T], \forall T > 0 \text{ and } \forall x > 0\ .}$$

If f represents the payoff of a derivative whose value is based on the value of an underlying asset x at maturity T, this solution clearly represents the price of that derivative.

By this solution, we now have a way to price derivatives (whose value are based on value at maturity) by implementing a Monte-Carlo method to compute the expectation.

Furthermore, we also observe that if we take the payoff function $f(X_T) = (X_T - K)^+$, we obtain the Black-Scholes formula for an European call option, except that now we take the risk-free rate because of the assumption of the absence of arbitrage opportunity. :

$$\boxed{C = x_0 \mathcal{F}_{\mathcal{N}_{(0,1)}}(d_1) - K e^{-rT} \mathcal{F}_{\mathcal{N}_{(0,1)}}(d_2)}$$

We conclude that Black-Scholes PDE gives us a more general framework to model the evolution of the price of derivatives (with value based on the price at maturity). Conversely to the Black-Scholes formula which is only used to price European option, with Black-Scholes PDE we can price many derivatives such as European options, digital options or barrier options (with an European style).

5 Limitations of the Black-Scholes model

5.1 Implied Volatility

An important assumption of the Black-Scholes model is that volatility is constant, however, in reality it's not true. In fact, volatility of financial asset can vary in the time due to economic or politic events.

Therefore, we need to define a "new" volatility which is called the implied volatility. This volatility is derived from the Black-Scholes formula, using the price of the option, the strike price, the maturity and the initial stock price.

If the model holds and in particular, volatility is constant we should observe in reality that for multiple European call options with same maturity but a different strike price, we have the same implied volatility.

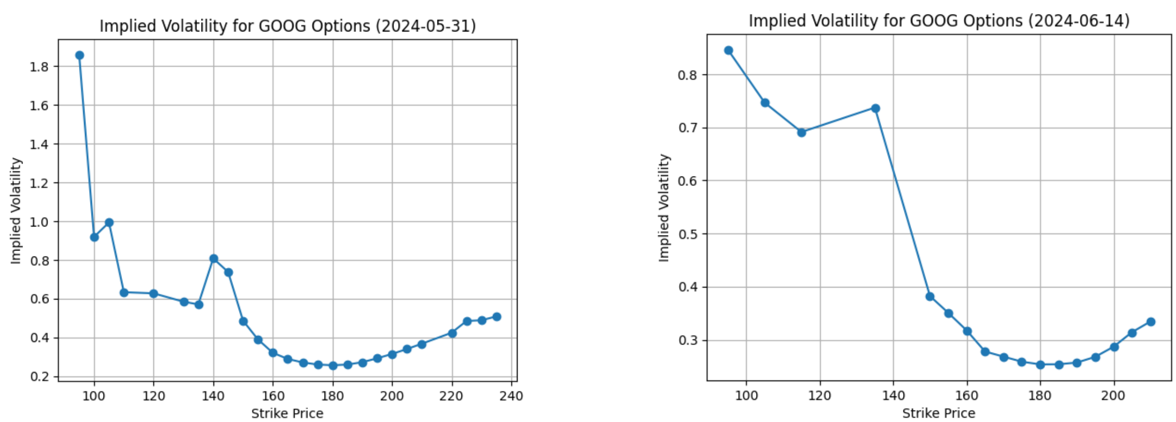


Figure 9: Comparison of implied volatility for European call option with different maturities on Google Stock (2024/05/12)

As shown in Figure 9, implied volatility is not constant, thus, it shows that the Black-Scholes model does not hold i.e. it does not fully capture the real behavior of market prices. That is why it is interesting to study other models such as Heston model which is a stochastic volatility model:

$$
\begin{aligned}
dS_t &= \mu S_t dt + \sqrt{v_t} S_t dB_t^1 \\
d\sqrt{v_t} &= -\theta \sqrt{v_t} dt + \delta dB_t^2
\end{aligned}
$$

where :

- S_t is the stock price at time t,

- $\sqrt{v_t}$ is the volatility of the asset,
- μ is the expected return of the asset,
- θ is the variance of the asset when t tends to infinity,
- B_t^1 and B_t^2 are brownian motion with a correlation ρ,
- δ is the volatility of the volatility of the asset.

This model shares the same structure as the Geometric Brownian Motion (GBM), but with the volatility $\sqrt{v_t}$ governed by another stochastic differential equation (SDE), specifically an Ornstein-Uhlenbeck process.

5.2 Opening on other models

The Black-Scholes model provides us a framework to model the price evolution of derivatives whose values are based on the value of an underlying asset x and which has a maturity T. However, this type of derivatives is very restrictive.
This is why other models, better suited for derivatives lacking a fixed maturity T, exist, such as American options.[5].

5.2.1 Cox-Ross-Rubinstein (CRR) model

A common model which is used to price derivatives with floating maturity is the Cox-Ross-Rubinstein (CRR) model which is a discrete time model. This model was developed by John Cox, Stephen Ross, and Mark Rubinstein in 1979 as an extension of the original Black-Scholes model.

In the CRR model, the price of the underlying asset changes in discrete time intervals based on a binomial distribution. At each interval, the asset price can either increase by a certain factor or decrease by another factor (cf. Figure 10). The model operates under the assumption of a risk-neutral environment, where the expected return on the asset equals to the risk-free rate.
The inputs of the CRR model are the current price of the underlying asset, the volatility of the underlying asset, the risk-free rate and the number of steps.
The principle of the CRR model is to recursively compute the option price at each nodes of the tree from expiration back to present. Thus, we consider multiple paths that represents the stock price through time.

Algorithm:[4]
At each node in time, we assume that stock price either goes up with probability p by

[5] Similar to European options but with the right to sell/buy at any time

u or go down by d with probability $1-p$. (with $0 < p < 1$)
Under the no arbitrage opportunity assumption, p needs to be defined such that :

$$p = \frac{e^{(r\,\Delta_t)} - d}{u - d}$$

with :

- r the risk-free rate,
- σ the volatility of the underlying asset,
- Δ_t the discretization step.

In addition, we also define u and d such that :

$$u = \exp(\sigma\sqrt{\Delta_t})$$

$$d = \frac{1}{u}$$

Then, we can draw the following representation (binomial tree with 3 periods) which represents the stock's price paths:

$$\begin{tikzcd}[column sep=tiny]
 & & & S_0 \arrow[dll, dash, "p"'] \arrow[drr, dash, "1-p"] & & & \\
 & S_{1,1} = S_0 * u \arrow[dl, dash, "p"'] \arrow[dr, dash, "1-p"] & & & & S_{1,2} = S_0 * d \arrow[dl, dash, "p"'] \arrow[dr, dash, "1-p"] & \\
S_{2,1} = S_0 * u^2 & & S_{2,2} = S_0 * u * d & & S_{2,3} = S_0 * d * u & & S_{2,4} = S_0 * d^2
\end{tikzcd}$$

The second step is to recursively compute the option price at each nodes of the tree from expiration back to present. In the case of an European call option with strike price K, we get:

$$\begin{tikzcd}[column sep=tiny]
 & & & C_0 = e^{-r\Delta_t}((1-p) * C_{1,2} + p * C_{1,1}) \arrow[dll, dash, "p"'] \arrow[drr, dash, "1-p"] & & & \\
 & C_{1,1} = e^{-r\Delta_t}((1-p) * C_{2,2} + p * C_{2,1}) \arrow[dl, dash, "p"'] \arrow[dr, dash, "1-p"] & & & & C_{1,2} = e^{-r\Delta_t}((1-p) * C_{2,4} + p * C_{2,3}) \arrow[dl, dash, "p"'] \arrow[dr, dash, "1-p"] & \\
C_{2,1} = (S_{2,1} - K)^+ & & C_{2,2} = (S_{2,2} - K)^+ & & C_{2,3} = (S_{2,3} - K)^+ & & C_{2,4} = (S_{2,4} - K)^+
\end{tikzcd}$$

For example, we implemented the CRR model for an European call option with strike price 50, an initial stock price of 100, $r = 0.01$ and $\sigma = 0.2$. We obtain the following tree for the stock price (cf. Figure 10).

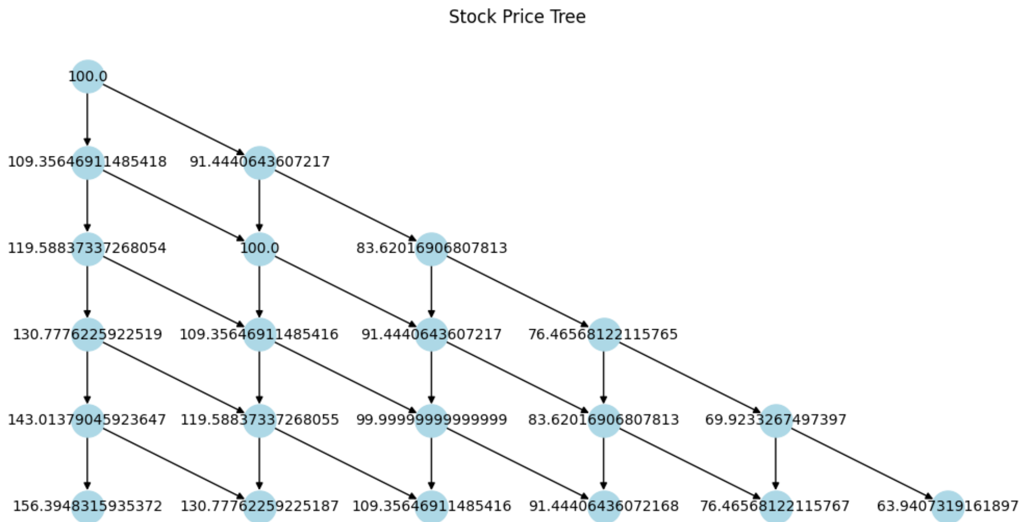


Figure 10: Stock Price Tree

And then we obtain the call option value on this tree (cf. Figure 11) :

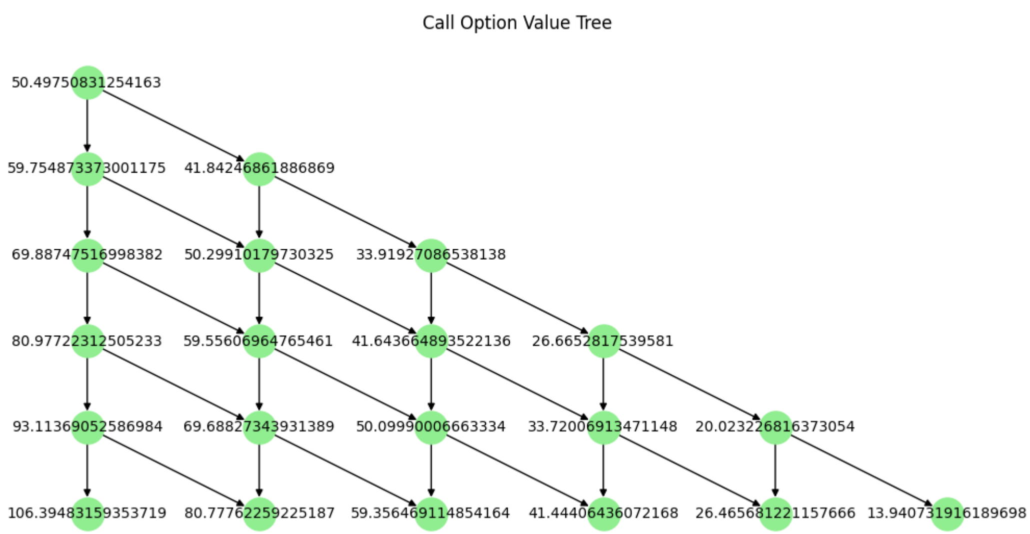


Figure 11: Call Option Value Tree

Finally, we obtain the price of the call option which is €50.49. An important property of the CRR model with European call option is that it can be shown that it converges to the Black-Scholes formula, as it is proven in [5]. We can look at the error of the CRR model compared to Black Scholes formula with respect to n the number of steps on Figure 12.

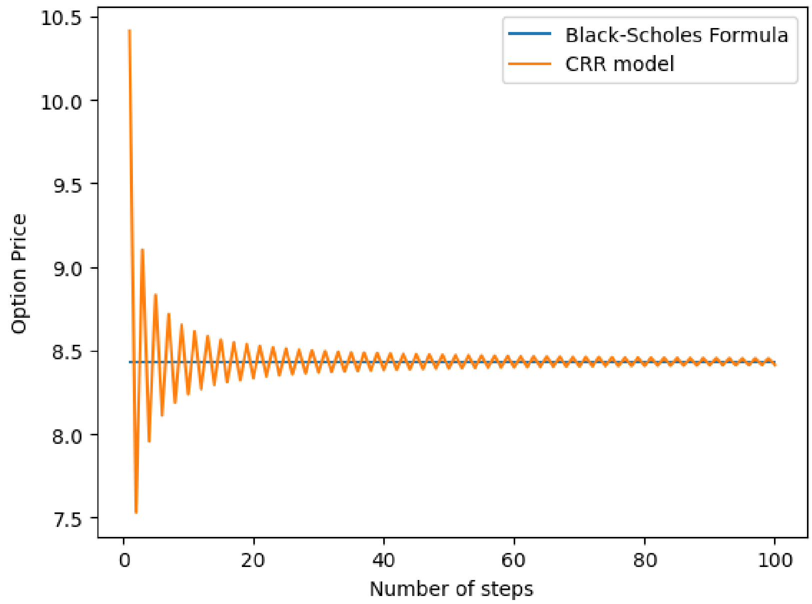


Figure 12: Comparison of Black-Scholes formula and CRR model for an option ($S_0 = 100, K = 100, T = 1, \sigma = 0.2, r = 0.01$) with respect to n

We conclude that CRR model is accurate compared to the Black-Scholes model, on top of the fact that it allows us to price derivatives with floating maturity.

For American options, the process is the same as for European options. The only difference is that when we go back from expiration to present, we need to take the maximum between the payoff of an European option and the payoff of the American option if we exercise it now.

Let $V(i,j)$ represent the price at each node (i,j), where i represents the number in the length at each time t and j represents the number of steps. Then, when we do the recursive calculation, the price at each nodes is :

$$V(i,j) = \max\left(\exp(-r \cdot dt) \cdot [p \cdot V(i,j+1) + (1-p) \cdot V(i+1,j+1)], \text{Payoff}(i,j)\right) \tag{2}$$

with $\text{Payoff}(i,j) = \max(V(i,j) - K, 0)$

Then, we can implement it and make the calculation to obtain the results shown on Figure 13 bellow.

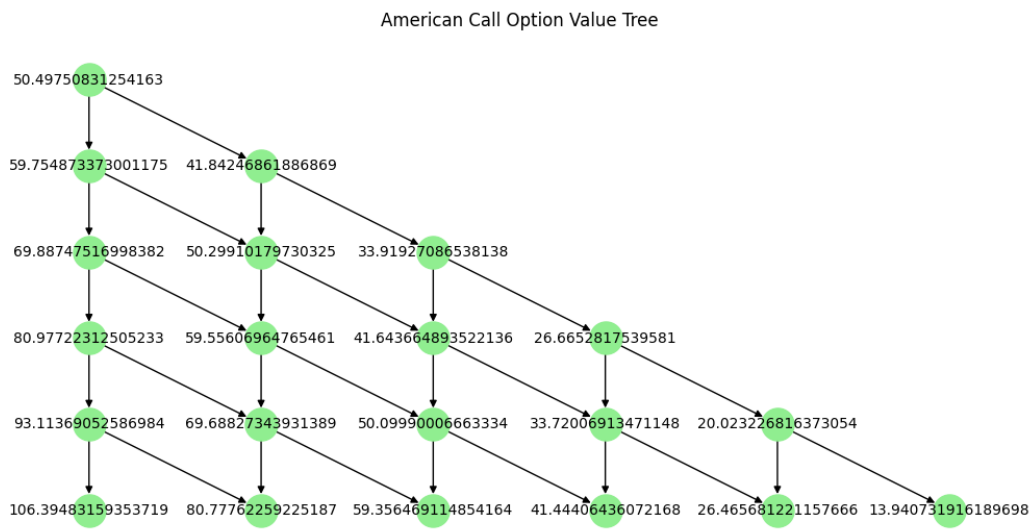


Figure 13: American Call Option Value ; $S_0 = 100, K = 50, \sigma = 0.2, r = 0.01$

Finally, CRR model provides us a framework to price a much larger variety of derivatives. Furthermore, CRR model is easier to understand than the Black-Scholes model, that's why this model is widely used in Finance.

6 Conclusion

In conclusion, we have seen that the Black-Scholes model is a good tool to price European options thanks to the existence of an explicit solution. However, it is very restrictive because it relies on many (non realistic) assumptions and can be applied only to a specific type of derivatives.

We have also seen that the Black-Scholes PDE, derived from BS model, offer a more general approach and allows to deal with different type of financial derivatives (derivatives whose values are based on the value of an underlying asset x and which has a maturity T) . Nevertheless, it does not have an explicit formula and it is hence harder to use.

In addition, we show that the Black-Scholes model does not hold because volatility is not constant in reality, that is why it leads us to explore other models such as stochastic volatility model.

Finally, we learned about the existence of other pricing models such as Cox-Ross-Rubinstein model, which is less restrictive than the Black-Scholes model. In fact, CRR model allows us to price derivatives with a fixed and a floating maturity.

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