Homework 20221012
Due Date: 20221019, 5 P.M.

1. Give asymptotic upper and lower bounds for $T(n)$ in each of the following recurrences. Assume that $T(n)$ is constant for $n \le 2$. Make your bounds as tight as possible, and justify your answers.

   a. $T(n) = 2T(n/2) + n^4$.

   b. $T(n) = T(7n/10) + n$.

   c. $T(n) = 16T(n/4) + n^2$.

   d. $T(n) = 7T(n/3) + n^2$.

   e. $T(n) = 7T(n/2) + n^2$.

   f. $T(n) = 2T(n/4) + \sqrt{n}$.

   g. $T(n) = T(n-2) + n^2$.

2. Suppose that we are given a weighted, directed graph $G = (V, E)$ in which edges that leave the source vertex s may have negative weights, all other edge weights are nonnegative, and there are no negative-weight cycles. Argue that Dijkstra's algorithm correctly finds shortest paths from s in this graph.

3. Suppose you are given a directed graph $G = (V, E)$ in which each edge has a cost of either 0 or 1. Also suppose that G has a node r such that there is a path from r to every other node in G. You are also given an integer k. Give a polynomial-time algorithm that either constructs an arborescence rooted at r of cost *exactly* k, or reports (correctly) that no such arborescence exists.

4. Let us say that a graph $G = (V, E)$ is a *near-tree* if it is connected and has at most $n + 8$ edges, where $n = |V|$. Give an algorithm with running time $O(n)$ that takes a near-tree G with costs on its edges, and returns a minimum spanning tree of G. You may assume that all the edge costs are distinct.

5. Consider the Minimum Spanning Tree Problem on an undirected graph $G = (V, E)$, with a cost $c_e \geq 0$ on each edge, where the costs may not all be different. If the costs are not all distinct, there can in general be many distinct minimum-cost solutions. Suppose we are given a spanning tree $T \subseteq E$ with the guarantee that for every $e \in T$, e belongs to *some* minimum-cost spanning tree in G. Can we conclude that T itself must be a minimum-cost spanning tree in G? Give a proof or a counterexample with explanation.