作业9. 习题1、解(1). 若 $x_1 \geq c$. 则 $\forall k \in \{1,2 \cdots n\}$ $f(x_k; c,\theta) \neq 0$

此时 $f(x_1;c,\theta) \cdot f(x_2;c,\theta) \cdots f(x_n;c,\theta) = (\frac{1}{\theta})^n \cdot e^{\frac{nc-(x_1+x_2+\cdots+x_n)}{\theta}}$

若 $x_1 < c$. $\exists k \in \{1,2,\cdots n\}$ $f(x_k;c,\theta)=0$

此时 $f(x_1;c,\theta) \cdots f(x_n;c,\theta)=0$

故 $L(c,\theta) = \begin{cases} \frac{1}{\theta^n} \cdot \exp\{\frac{nc}{\theta} - \frac{1}{\theta} \cdot \sum_{i=1}^n x_i\} & , c \leq x_1 \\ 0 & , c > x_1 \end{cases}$

利用极大似然法. 可以看到 $L(c,\theta)$ 关于 c 单调递增 而 $c \leq x_1$

故 $\hat{c} = x_1$ 由 $\nabla_\theta L(c,\theta)=0$ 可得

$$-\frac{n}{\theta^n \cdot \theta} \cdot \exp\{\frac{nc}{\theta} - \frac{1}{\theta}\sum_{i=1}^n x_i\} + \frac{1}{\theta^n} \cdot \exp\{\frac{nc}{\theta} - \frac{1}{\theta}\sum_{i=1}^n x_i\} \cdot (-\frac{nc}{\theta^2} + \frac{1}{\theta^2}\sum_{i=1}^n x_i) = 0$$

可得 $\hat{\theta} = \frac{\sum_{i=1}^n x_i}{n} - \hat{c} = \bar{x} - x_1$

(2). $E_{c,\theta}(X) = \int_c^{+\infty} \frac{t}{\theta} e^{-\frac{t-c}{\theta}} dt = \theta e^{\frac{c}{\theta}} \cdot \int_{\frac{c}{\theta}}^{+\infty} k e^{-k} dk$ 取 $k = \frac{t}{\theta}$

$$= \theta e^{\frac{c}{\theta}} [(0 - (-\frac{c}{\theta} e^{-\frac{c}{\theta}})) + (0 - (-e^{-\frac{c}{\theta}}))] = c + \theta$$

$E_{c,\theta}(X^2) = \int_c^{+\infty} \frac{t^2}{\theta} e^{-\frac{t-c}{\theta}} dt = \theta^2 e^{\frac{c}{\theta}} \cdot \int_{\frac{c}{\theta}}^{+\infty} k^2 e^{-k} dk$ 取 $k = \frac{t}{\theta}$

$$= \theta^2 e^{\frac{c}{\theta}} \cdot \left(-k^2 e^{-k}\Big|_{\frac{c}{\theta}}^{+\infty} + 2\int_{\frac{c}{\theta}}^{+\infty} k e^{-k} dk\right)$$

$$= \theta^2 e^{\frac{c}{\theta}} \left(\frac{c^2}{\theta^2} e^{-\frac{c}{\theta}} + \frac{2(c+\theta)}{\theta} e^{-\frac{c}{\theta}}\right) = c^2 + 2c\theta + 2\theta^2$$

$E_{c,\theta}(X) = \frac{1}{n}\sum_{i=1}^n x_i = \bar{x}$ $E_{c,\theta}(X^2) = \frac{1}{n}\sum_{i=1}^n x_i^2$

$\theta^2 = E_{c,\theta}(X^2) - [E_{c,\theta}(X)]^2 = \frac{1}{n}\sum_{i=1}^n x_i^2 - (\frac{1}{n}\sum_{i=1}^n x_i)^2 = \frac{1}{n}\sum_{i=1}^n (x_i - \bar{x})^2$

这个利用公式 $D(X) = E(X^2) - [E(X)]^2$

$\hat{\theta} = \sqrt{\frac{1}{n}\sum_{i=1}^n (x_i - \bar{x})^2}$

$\hat{c} = \bar{x} - \hat{\theta} = \bar{x} - \sqrt{\frac{1}{n}\sum_{i=1}^n (x_i - \bar{x})^2}$

习题2. 解: (1) $L(\theta) = f(x_1;\theta) \cdots f(x_n;\theta)$

$H(\theta) = \ln L(\theta) = \sum_{k=1}^{n} \ln f(x_k;\theta) = -n\ln\theta + \frac{1-\theta}{\theta}\sum_{i=1}^{n}\ln x_i$

令 $H'(\theta)=0$ 可得 $-\frac{n}{\theta} + (-\frac{1}{\theta^2})(\sum_{i=1}^{n}\ln x_i) = 0$ 得 $\hat{\theta} = -\frac{1}{n}\sum_{i=1}^{n}\ln x_i$

故 θ 的极大似然估计量为 $\hat{\theta} = -\frac{1}{n}\sum_{i=1}^{n}\ln x_i$

(2) 证明: ~~E(~~ 对 $\forall i \in \{1,\cdots n\}$

$E(\ln x_i) = \int_0^1 \ln x_i \frac{1}{\theta} x_i^{\frac{(1-\theta)}{\theta}} dx_i$

$= \frac{1}{\theta}\cdot\int_0^1 \ln x \cdot x^{\frac{1-\theta}{\theta}} dx = \int_0^1 \ln x \, d(x^{\frac{1}{\theta}})$

$= \ln x \cdot x^{\frac{1}{\theta}} \Big|_0^1 - \int_0^1 x^{\frac{1}{\theta}-1} dx = -\theta$

$E(\hat{\theta}) = E(-\frac{1}{n}\sum_{i=1}^{n}\ln x_i) = -\frac{1}{n}\cdot\sum_{i=1}^{n}E(\ln x_i) = -\frac{1}{n}\cdot(-n\theta) = \theta$

故 $\hat{\theta}$ 为 θ 的无偏估计

习题3. 解: 样本的条件概率函数

$q(x|\mu) = \pi(x_1|\mu)\cdots\pi(x_n|\mu)$

$= \frac{1}{\delta^n(\sqrt{2\pi})^n}\exp\left[-\frac{1}{2\delta^2}\sum_{i=1}^{n}(x_i-\mu)^2\right]$

后验分布密度函数 $h(\mu|x) = \frac{q(x|\mu)\cdot\pi(\mu)}{\int_{-\infty}^{+\infty} q(x|\mu)\cdot\pi(\mu)d\mu}$

可得 $h(\mu|x) \propto \exp\left[-\frac{1}{2\delta^2}\sum_{i=1}^{n}(x_i-\mu)^2\right]\exp\left[-\frac{1}{2\delta_\mu^2}(\mu-\mu_0)^2\right]$

~~= exp~~

我们要使 $h(\mu|x) \propto \exp\left[\frac{-(a-b)^2}{2c^2}\right]$ a可以被μ替换

找到b, c, 这样就可以找到了 $\mu|x \sim N(b, c^2)$

(结构类似)

$\exp\left[-\frac{1}{2\delta^2}\sum_{i=1}^{n}(x_i-\mu)^2 - \frac{1}{2\delta_\mu^2}(\mu-\mu_0)^2\right] =$ (见项)

作为个之前根本没学过概率论的小白，结合一点自己的理解，根据看到的方法，尝试一下。

由上一页，$h(\mu|x) \propto \exp\left[-\frac{1}{2\delta^2}\sum_{i=1}^{n}(x_i-\mu)^2\right]\cdot\exp\left[-\frac{1}{2\delta_\mu^2}(\mu-\mu_0)^2\right]$

我们可以更进一步地去相似，只要满足 $\boxed{\text{消掉的式子与}\mu\text{无关}}$

对 $-\frac{1}{2\delta^2}\sum_{i=1}^{n}(x_i-\mu)^2-\frac{1}{2\delta_\mu^2}(\mu-\mu_0)^2$ 而言，去掉任何与 μ 相关的式子，

可得 $-\frac{1}{2\delta^2}\sum_{i=1}^{n}(\mu^2-2\mu x_i)-\frac{1}{2\delta_\mu^2}\cdot(\mu^2-2\mu\mu_0)$

即 $\mu^2\cdot\left(-\frac{n}{2\delta^2}-\frac{1}{2\delta_\mu^2}\right)+\mu\left(\frac{\sum_{i=1}^{n}x_i}{\delta^2}+\frac{\mu_0}{\delta_\mu^2}\right)$

$=-\frac{1}{2}\left[\mu^2\left(\frac{n}{\delta^2}+\frac{1}{\delta_\mu^2}\right)-2\mu\left(\frac{\sum_{i=1}^{n}x_i}{\delta^2}+\frac{\mu_0}{\delta_\mu^2}\right)\right]$

取 $A^2=\frac{n}{\delta^2}+\frac{1}{\delta_\mu^2}$ $B=\frac{\sum_{i=1}^{n}x_i}{\delta^2}+\frac{\mu_0}{\delta_\mu^2}$

$=-\frac{1}{2}\left[A^2\mu^2-2B\mu\right]=-\frac{1}{2}\cdot A^2\left(\mu-\frac{B}{A^2}\right)^2+\frac{B^2}{2A^2}$

由于 A,B 皆与 μ 无关，故 $\frac{B^2}{2A^2}$ 与 μ 无关

故该式 $\propto -\frac{1}{2}A^2\left(\mu-\frac{B}{A}\right)^2=-\frac{1}{2}\cdot\frac{\left(\mu-\frac{B}{A^2}\right)^2}{\cancel{A^2}}$

即 $h(\mu|x)\propto\exp\left[-\frac{1}{2}\frac{\left(\mu-\frac{B}{A^2}\right)^2}{\frac{1}{A^2}}\right]$

即 $\mu|x \sim N\left(\frac{B}{A^2},\frac{1}{A^2}\right)=N\left(\frac{\frac{\sum_{i=1}^{n}x_i}{\delta^2}+\frac{\mu_0}{\delta_\mu^2}}{\frac{n}{\delta^2}+\frac{1}{\delta_\mu^2}},\frac{1}{\frac{n}{\delta^2}+\frac{1}{\delta_\mu^2}}\right)$

这题做得真过瘾~

习题4. 解：~~$\Gamma(\alpha,\beta)$~~ = $X\sim\Gamma(\alpha,\beta)$ 即为 $f(x)=\frac{1}{\beta^\alpha\Gamma(\alpha)}x^{\alpha-1}e^{-\cancel{\frac{x}{\beta}}\beta x}$，$x>0$

这个在LEC20 PPT 还写错了:)

(这个能否在题目中告诉我公式呢??!)

$\pi(\lambda)\propto\lambda^{\alpha-1}e^{-\beta\lambda}$

泊松分布 $f(x)=e^{-\lambda}\frac{\lambda^x}{x!}$ $x\geq 0$

$q(x|\lambda)=p(x_1;\lambda)\cdots p(x_n;\lambda)$

$=\frac{\lambda^{x_1+\cdots+x_n}}{x_1!\cdots x_n!}e^{-n\lambda}$

$\propto\lambda^{x_1+\cdots+x_n}\cdot e^{-n\lambda}$

$h(\lambda|x)=\frac{q(x|\lambda)\cdot\pi(\lambda)}{\int_{-\infty}^{\infty}q(x|\lambda)\cdot\pi(\lambda)}$

$\propto q(x|\lambda)\cdot\pi(\lambda)$

$$h(\lambda|x) \propto g(x|\lambda)\cdot\pi(\lambda) = \lambda^{x_1+\cdots+x_n}\cdot e^{-n\lambda}\cdot \lambda^{\alpha-1}\cdot e^{-\beta\lambda}$$

$$= \lambda^{x_1+\cdots+x_n+\alpha-1} e^{-n\lambda-\beta\lambda}$$

$$\lambda|x \sim \Gamma(x_1+\cdots+x_n+\alpha,\ n+\beta)$$

λ的后验期望估计 $\hat{\lambda} = \dfrac{x_1+\cdots+x_n+\alpha}{n+\beta}$