第4次作业

习题1.(i) 解 $B=A^TA=\begin{bmatrix}14 & -2 & 0\\ -2 & 6 & -4\\ 0 & -4 & 3\end{bmatrix}$ $|B|_1=14>0$ $|B|_2=80>0$ $|B|_3=16>0$

B为正定矩阵，可进行cholesky分解.

$B=GG^T$ $g_{11}=\sqrt{14}$ $g_{21}=\frac{a_{21}}{g_{11}}=-\frac{2}{\sqrt{14}}$ $g_{31}=\frac{a_{31}}{g_{11}}=0$

$g_{22}=\sqrt{a_{22}-g_{21}^2}=\sqrt{\frac{40}{7}}$ $g_{32}=\frac{a_{32}-g_{31}\cdot g_{21}}{g_{22}}=-\frac{\sqrt{70}}{5}$

$g_{33}=\sqrt{a_{33}-g_{31}^2-g_{32}^2}=\frac{\sqrt{5}}{5}$ 可得 $G=\begin{bmatrix}\sqrt{14} & 0 & 0\\ -\frac{2}{\sqrt{14}} & \frac{2\sqrt{70}}{7} & 0\\ 0 & -\frac{\sqrt{70}}{5} & \frac{\sqrt{5}}{5}\end{bmatrix}$

(ii) 证明：设 $G=U\Sigma V^T$ U,V正交且 $U,V\in R^{n\times n}$ $\Sigma=diag(\delta_1,\cdots\delta_n)$

$\delta_1\geq\delta_2\cdots\geq\delta_n\geq 0$

$A^TA=GG^T=U\Sigma^2U^T$

A^TA的奇异值为 $\delta_1^2\cdots\delta_n^2$

$\|A^TA\|_2=\sqrt{\lambda_{max}((A^TA)^TA^TA))}=\sqrt{\lambda_{max}(U\Sigma^4U^T)}=\sqrt{\lambda_{max}(\Sigma^4)}=\delta_1^2$

$\|G\|_2=\sqrt{\lambda_{max}(G^TG)}=\sqrt{\lambda_{max}(V\Sigma^2V^T)}=\sqrt{\lambda_{max}(\Sigma^2)}=\delta_1$

$\|A\|_2=\sqrt{\lambda_{max}(A^TA)}=\sqrt{\lambda_{max}(U\Sigma^2U^T)}=\sqrt{\lambda_{max}(\Sigma^2)}=\delta_1$

故 $\|A^TA\|_2=\|G\|_2^2=\|A\|_2^2$

习题2.(i) 解：$A^TA=\begin{bmatrix}160 & -136 & -56\\ -136 & 148 & 80\\ -56 & 80 & 52\end{bmatrix}=B$ $|\lambda E-B|=(\lambda-36)(\lambda-324)\lambda=0$

$\lambda_1=324$ $\lambda_2=36$ $\lambda_3=0$

$\lambda_1=324$对应特征向量 $\lambda_2=36$对应 $\lambda_3=0$对应（needs单位化）

$V_1=\cancel{\begin{bmatrix}1\\2\\-2\end{bmatrix}}\ \frac{1}{3}\begin{bmatrix}-2\\2\\1\end{bmatrix}$ $V_2=\frac{1}{3}\begin{bmatrix}2\\1\\2\end{bmatrix}$ $V_3=\frac{1}{3}\begin{bmatrix}1\\2\\-2\end{bmatrix}$

$\delta_1=\sqrt{\lambda_1}=18$ $\delta_2=\sqrt{\lambda_2}=6$ $\delta_3=\sqrt{\lambda_3}=0$

$V=[V_1\ \ V_2\ \ V_3]$

$\Sigma=diag_{4\times3}(\delta_1,\delta_2,\delta_3)$ $u_1=\frac{AV_1}{\delta_1}=\frac{1}{2}\begin{bmatrix}1\\1\\1\\1\end{bmatrix}$ $u_2=\frac{AV_2}{\delta_2}=\frac{1}{2}\begin{bmatrix}-1\\1\\-1\\1\end{bmatrix}$

$u_3,u_4\in Null(A^T)$

$Null(A^T)=span\{\begin{bmatrix}-1\\0\\1\\0\end{bmatrix},\begin{bmatrix}0\\-1\\0\\1\end{bmatrix}\}$

要使2U为Hadamard矩阵.

$(2U)(2U)^T=4\cdot E=(2U)^T(2U)$

即 $U\cdot U^T=E=U^TU$

故U应为正交矩阵.

由于Hadmard矩阵中的元素为±1

易得 $u_3=\frac{1}{2}\begin{bmatrix}-1\\0\\1\\0\end{bmatrix}+\frac{1}{2}\begin{bmatrix}0\\-1\\0\\1\end{bmatrix}=\frac{1}{2}\begin{bmatrix}-1\\-1\\1\\1\end{bmatrix}$

同理，易得 $u_4 = -\frac{1}{2}\begin{bmatrix}-1\\0\\1\\0\end{bmatrix} + \frac{1}{2}\begin{bmatrix}0\\-1\\0\\1\end{bmatrix} = \frac{1}{2}\begin{bmatrix}1\\-1\\-1\\1\end{bmatrix}$

$U = [u_1\ u_2\ u_3\ u_4] = \frac{1}{2}\begin{bmatrix}1 & -1 & -1 & 1\\1 & 1 & -1 & -1\\1 & -1 & 1 & -1\\1 & 1 & 1 & 1\end{bmatrix}$ $V = \frac{1}{3}\begin{bmatrix}-2 & 2 & 1\\2 & 1 & 2\\1 & 2 & -2\end{bmatrix}$

$\Sigma = \begin{bmatrix}18 & 0 & 0\\0 & 6 & 0\\0 & 0 & 0\\0 & 0 & 0\end{bmatrix}$ $A = U\Sigma V^T$

(ii) A有两个非零奇异值 故 $rank(A) = 2$

$R(A) = span(u_1, u_2) = span\{\frac{1}{2}\begin{bmatrix}1\\1\\1\\1\end{bmatrix}, \frac{1}{2}\begin{bmatrix}-1\\1\\-1\\1\end{bmatrix}\}$

$N(A) = span(v_{r+1} \cdots v_3) = span\{\frac{1}{3}\begin{bmatrix}1\\2\\-2\end{bmatrix}\}$

$\|A\|_2 = \delta_1 = 18$ $\|A\|_F = \sqrt{Tr(A^TA)} = \sqrt{\delta_1^2 + \delta_2^2} = 6\sqrt{10}$

(iii) 解：$k=0$ 时 $rank(A_k) = 0$ $A_k = 0_{3\times 4}$ $r_0 = \|A^T\|_2 = 18$

$A^T = (U\Sigma V^T)^T = V\Sigma U^T$

$k=1$ 时 根据 Eckhart-Young 定理

$A_1 = \cancel{\delta_1 u_1 v_1^T}$ $\bar{\delta}_1 \bar{u}_1 \bar{v}_1^T = \delta_1 v_1 u_1^T$

$= 18 \cdot \frac{1}{3}\begin{bmatrix}-2\\2\\1\end{bmatrix}[\frac{1}{2}\ \frac{1}{2}\ \frac{1}{2}\ \frac{1}{2}] = \begin{bmatrix}-6 & -6 & -6 & -6\\6 & 6 & 6 & 6\\3 & 3 & 3 & 3\end{bmatrix}$

$r_1 = \|A^T - A_1\|_2 = \bar{\delta}_2 = 6$

$k \geq 2$ 时 $rank(A^T) = rank(A) = 2$ $A_k = A^T$ $r_k = 0$

习题3、(i) 设 A 的SVD 分解为 $A = U\Sigma V^T$ U, V 正交 $\Sigma_{n\times n} = diag_{n\times n}(\delta_1 \cdots \delta_n)$
$\delta_1 \geq \delta_2 \cdots \geq \delta_n \geq 0$

A可逆 $rank(A) = n$ $\delta_t > 0$ $t = 1 \cdots n$

~~$Av_1 =$~~ $A = \delta_1 u_1 v_1^T + \delta_2 u_2 v_2^T + \cdots + \delta_n u_n v_n^T$

由 $Av_i = \delta_i u_i$ 可得 $\delta_i A^{-1} u_i = A^{-1} A v_i$ 即 $A^{-1} u_i = \frac{1}{\delta_i} v_i$

根据奇异值分解的原理

故 $A^{-1} = [V_n \cdots V_2\ V_1]\,\mathrm{diag}_{n\times n}\left(\frac{1}{\delta_n}, \frac{1}{\delta_{n-1}} \cdots \frac{1}{\delta_2}, \frac{1}{\delta_1}\right)[U_n \cdots U_2\ U_1]^T$

(ii) Q正交 $Q = Q E_n \cdot E_n = Q \cdot E_n \cdot E_n^T$ 为Q的SVD分解

则Q的所有奇异值为1

(iii) 设B的奇异值分解为 $B = U\Sigma V^T$

$A = QU\Sigma V^T Q^T = (QU)\Sigma(QV)^T$

$(QU)(QU)^T = QU \cdot U^T Q^T = Q \cdot Q^T = E_n$

$(QU)^T(QU) = U^T Q^T Q U = U^T U = E_n$

QU为正交矩阵

$(QV)(QV)^T = QV \cdot V^T Q^T = E_n$

$(QV)^T(QV) = V^T Q^T Q V = E_n$

QV为正交矩阵

故A与B的奇异值相同