作业7、习题1解：这题虽基础，但我还是推了一些时间:)

(1). $\frac{\partial f}{\partial A} = \frac{\partial}{\partial A}\left(\frac{1}{2}\cdot[Ax+(b-y)]^T\cdot[Ax+(b-y)]\right)$

$= \frac{1}{2}\cdot\frac{\partial}{\partial A}\cdot\left(x^TA^TAx + (b-y)^TAx + x^TA^T(b-y) + (b-y)^T(b-y)\right)$

$= \frac{1}{2}\frac{\partial}{\partial A}\left(x^TA^TAx + (b-y)^TAx + x^TA^T(b-y)\right)$

取 ~~函数~~ $g(A) = (b-y)^TA\cdot x$ (为标量) $dg = d(Tr(g)) = Tr(dg)$

$= Tr(dg^T) = d(Tr\, g^T) = dg^T$

* 其实不用这么麻烦 $g(A)$ 为标量 $g = g^T$ $\frac{\partial g}{\partial A} = \frac{\partial g^T}{\partial A}$

原式 $= \frac{1}{2}\cdot\frac{\partial}{\partial A}\left(x^TA^TAx + 2(b-y)^TAx\right)$

$d(x^TA^TAx) = Tr(d(x^TA^TAx)) = Tr(x^TA^TA\cdot dx + x^TA^T dA\cdot x + x^TdA^T\cdot Ax + dx^T\cdot A^TAx)$

$= Tr(2x^TA^TA\,dx + 2x^TA^TdA\cdot x)$

$= 0 + 2Tr.(x^TA^TdA\cdot x) = \cancel{2}Tr(2xx^TA^TdA)$

由 $df = Tr[(\frac{\partial f}{\partial A})^T dA]$ 可得 $\frac{\partial(x^TA^TAx)}{\partial A} = 2\cdot A\cdot xx^T$

$d((b-y)^TAx) = Tr(d((b-y)^TAx)) = Tr(d(b^TAx) - d(y^TAx))$

$= Tr(b^TdA\cdot x + b^TA\,dx + db^T\cdot Ax - y^TdA\cdot x - y^T\cdot A\,dx - dy^T\cdot Ax)$

$= Tr(b^TdA\cdot x - y^TdA\cdot x)$

$= Tr(x\cdot(b-y)^T dA)$ $\frac{\partial((b-y)^TAx)}{\partial A} = (b-y)\cdot x^T$

综上，原式 $= Axx^T + (b-y)x^T = \frac{\partial f}{\partial A}$

(2). $d(x^TA^TAx) = Tr(2x^TA^TA\,dx) + 0$ 可得 $\frac{\partial(\cancel{2A}\, x^TA^TAx)}{\partial x} = 2A^TAx$

$d((b-y)^TAx) = Tr(b^TA\,dx - y^TA\,dx)$

$= Tr((b-y)^TA\,dx)$ 可得 $\frac{\partial((b-y)^TAx)}{\partial x} = A^T(b-y)$

故原式 $= \frac{1}{2}\cdot 2A^TAx + A^T(b-y) = \frac{\partial f}{\partial x}$

习题2、取 $f=x^TAx$ 为标量 $df=d\,Tr(f)=Tr(df)$

若 $f=f(x)$ $df=Tr(df)=Tr(d(x^TAx))$

$=Tr(x^T\cdot dA\cdot x+x^T\cdot A\,dx+dx^T\cdot Ax)$

$=Tr(x^TA\,dx)+Tr(dx^T\cdot Ax)$

$=Tr(x^TA\,dx)+Tr(x^TA^T dx)=\cancel{2\,Tr(2x^TA\,dx)}$ $Tr(x^T(A+A^T)dx)$

可得 $\frac{\partial f}{\partial x}=\cancel{(2x^TA)^T=2}\,[x^T(A+A^T)]^T$

$=(A+A^T)x$

若 $f=f(A)$ $df=Tr(x^TdA\cdot x)=Tr(x\cdot x^TdA)$

$\frac{\partial f}{\partial A}=(xx^T)^T=x\cdot x^T$

习题3、取 $f(w)=Tr(w^{-1})$ 为标量 $df=\cancel{Tr(df)}\,d(Tr(f))=Tr(df)$

$=Tr(d\,Tr(w^{-1}))$

$=Tr(Tr(d(w^{-1})))$ (麻烦了)

$=Tr(d(w^{-1}))$

$df=d\,Tr(w^{-1})=Tr(d(w^{-1}))$

$d(w^{-1})$ 和 dw 怎么联系？

$w\cdot w^{-1}=I$ $d(w\cdot w^{-1})=0_{m\times m}$

(I 与 w 无关) $d(w\cdot w^{-1})=dw\cdot w^{-1}+w\,dw^{-1}=0$

可得 $dw^{-1}=-w^{-1}dw\cdot w^{-1}$

$df=Tr(-w^{-1}dw\cdot w^{-1})=Tr(-w^{-1}\cdot w^{-1}\cdot dw)$

可得 $\frac{\partial f}{\partial w}=(-w^{-1}\cdot w^{-1})^T=-(w^{-2})^T$

习题4、(1) 证明：为了简便，$q=f$（两个符号含义一样了） 取 $A=\ln q=\ln f$

根据链式法则 $\frac{\partial J}{\partial z}=\cancel{\frac{\partial f}{\partial z}\frac{\partial A^T}{\partial f^T}\frac{\partial J}{\partial A}}\ \frac{\partial A^T}{\partial z}\frac{\partial J}{\partial A}$

其中 $\frac{\partial J}{\partial A}=\frac{\partial(-p^TA)}{\partial A}$ $d(-p^TA)=-Tr(d(p^TA))=-Tr(dp^T\cdot A+p^TdA)$

$=-Tr(p^TdA)$ $\frac{\partial J}{\partial A}=(-p^T)^T=-p$

$$f=\begin{bmatrix}\frac{e^{z_1}}{\lambda}\\ \vdots\\ \frac{e^{z_n}}{\lambda}\end{bmatrix}$$

（取 $e^{z_1}+e^{z_2}+\cdots+e^{z_n}=\lambda$，$\lambda$ 不是常数！）

$A=\ln f=[\cancel{*}\,z_1-\ln\lambda,\ z_2-\ln\lambda\cdots,z_n-\ln\lambda]^T$

取 $g_k = z_k - \ln \lambda$ $\quad A^T = [g_1 \cdots g_n]$ $\quad \frac{\partial A^T}{\partial z_1} = [\frac{\partial g_1}{\partial z_1} \cdots \frac{\partial g_n}{\partial z_1}]$

$$\frac{\partial A^T}{\partial z} = \begin{bmatrix} \frac{\partial A^T}{\partial z_1} \\ \vdots \\ \frac{\partial A^T}{\partial z_n} \end{bmatrix} = \begin{bmatrix} \frac{\partial g_1}{\partial z_1} & \cdots & \frac{\partial g_n}{\partial z_1} \\ \vdots & & \vdots \\ \frac{\partial g_1}{\partial z_n} & \cdots & \frac{\partial g_n}{\partial z_n} \end{bmatrix} \qquad \frac{\partial g_k}{\partial z_m} = \begin{cases} 1 - \frac{e^{z_m}}{\lambda}, & m = k \\ -\frac{e^{z_m}}{\lambda}, & m \neq k \end{cases}$$

故 $$\frac{\partial A^T}{\partial z} = I - \begin{bmatrix} \frac{e^{z_1}}{\lambda} & \frac{e^{z_1}}{\lambda} & \cdots & \frac{e^{z_1}}{\lambda} \\ \vdots & \vdots & & \vdots \\ \frac{e^{z_n}}{\lambda} & \frac{e^{z_n}}{\lambda} & \cdots & \frac{e^{z_n}}{\lambda} \end{bmatrix} = I - B$$

$$B = [f \; f \cdots f] = [q \cdots q]$$

$$原式 = (I-B)(-P) = BP - P = \underbrace{[q \cdots q]}_{n个q} \begin{bmatrix} p_1 \\ \vdots \\ p_n \end{bmatrix} - P = (p_1 + \cdots + p_n) q - P$$
$$= (1^T P) q - P = q - P$$

原式得证

(2) 解: J 为标量

$$dJ = d\,Tr(J) = Tr(dJ) = Tr\left(\left(\frac{\partial J}{\partial z}\right)^T \cdot dz\right) = Tr\left((q-P)^T (W dx + dW \cdot x)\right)$$
$$= 0 + Tr\left((q-P)^T dW \cdot x\right) = Tr\left(x \cdot (q-P)^T dW\right) = Tr\left(\left(\frac{\partial J}{\partial W}\right)^T \cdot dW\right)$$

可得 $\frac{\partial J}{\partial W} = [x \cdot (q-P)^T]^T = (q-P) \cdot x^T$ 原式得证

习题5.(1)解: $dL = Tr(dL) = Tr\left(-\frac{1}{2}\cdot d\left(\sum_t (x_t-\mu)^T\Sigma^{-1}(x_t-\mu)\right)\right)$

$$= -\frac{1}{2}\cdot\sum_t Tr\left(d\left((x_t-\mu)^T\Sigma^{-1}(x_t-\mu)\right)\right)$$

$$= -\frac{1}{2}\sum_t Tr\left(d\left(x_t^T\Sigma^{-1}x_t - \mu^T\Sigma^{-1}x_t - x_t^T\Sigma^{-1}\mu + \mu^T\Sigma^{-1}\mu\right)\right)$$

$$= -\frac{1}{2}\sum_t Tr\left(-d\mu^T\cdot\Sigma^{-1}x_t - x_t^T\Sigma^{-1}d\mu + d\mu^T\cdot\Sigma^{-1}\mu + \mu^T\Sigma^{-1}d\mu\right)$$

$$= -\frac{1}{2}\sum_t Tr\left(\left(-x_t^T(\Sigma^{-1})^T - x_t^T\Sigma^{-1} + \mu^T(\Sigma^{-1})^T + \mu^T\Sigma^{-1}\right)d\mu\right)$$

则 $\frac{\partial L}{\partial \mu} = -\frac{1}{2}\sum_t\cdot\left(-\Sigma^{-1}\cdot x_t - (\Sigma^{-1})^T x_t + \Sigma^{-1}\cdot\mu + (\Sigma^{-1})^T\mu\right)$

$$= \frac{1}{2}\sum_t\left((\Sigma^{-1}) + (\Sigma^{-1})^T\right)\cdot(x_t-\mu) = \sum_t\cdot(\Sigma^{-1})\cdot(x_t-\mu)$$

$\boxed{\text{这题本来直接链式法则就可! 我麻烦了}}$

(2)解: ~~参考(1)中的方法,~~ $\cancel{dL = -\frac{1}{2}\sum_t Tr\left(-d\mu^T\Sigma^{-1}x - x^T\Sigma^{-1}d\mu + d\mu^T\Sigma^{-1}\mu + \mu^T\Sigma^{-1}d\mu\right)}$

看错了,

$$\cancel{= -\frac{1}{2}\sum_t Tr\left(-x^T(\Sigma^{-1})^T d\mu - x^T\Sigma^{-1}d\mu + \mu^T(\Sigma^{-1})^T d\mu + \mu^T\Sigma^{-1}d\mu\right)}$$

$$\cancel{\frac{\partial L}{\partial \mu} = -\frac{1}{2}\sum_t}$$

$$dL = -\frac{N}{2}Tr(d\ln|\Sigma|) - \frac{1}{2}\sum_t Tr\left(d\left((x_t-\mu)^T\Sigma^{-1}(x_t-\mu)\right)\right)$$

$$= Tr\left(-\frac{N}{2}\cdot\frac{1}{|\Sigma|}\cdot\Sigma^* d\Sigma\right) - \frac{1}{2}\cdot\sum_t Tr\left((x_t-\mu)^T d\Sigma^{-1}(x_t-\mu)\right)$$

$\Sigma\cdot\Sigma^{-1} = I$

$d\Sigma\cdot\Sigma^{-1} + \Sigma d\Sigma^{-1} = 0$

$$= Tr\left(-\frac{N}{2}\frac{1}{|\Sigma|}\Sigma^* d\Sigma - \frac{1}{2}\sum_t (x_t-\mu)(x_t-\mu)^T d\Sigma^{-1}\right)$$

$\underline{d\Sigma^{-1} = -\Sigma^{-1}\cdot d\Sigma\cdot\Sigma^{-1}}$

故 $\frac{\partial L}{\partial \Sigma} = -\frac{N}{2|\Sigma|}(\Sigma^*)^T - \frac{1}{2}\sum_t(-1)(\Sigma^{-1})^T(x_t-\mu)\cdot(x_t-\mu)^T(\Sigma^{-1})^T$

$$= -\frac{N}{2|\Sigma|}\cdot(\Sigma^*)^T + \frac{1}{2}\cdot(\Sigma^{-1})^T\cdot\left(x_1\cdot x_1^T + \cdots + x_N\cdot x_N^T - N\mu\mu^T - N\mu\mu^T + N\mu\mu^T\right)(\Sigma^{-1})^T$$

~~习题6.解~~ $= -\frac{N}{2|\Sigma|}|\Sigma|\cdot(\Sigma^{-1})^T + \frac{1}{2}\cdot(\Sigma^{-1})^T\cdot\left(\sum_{k=1}^N x_k x_k^T - N\mu\mu^T\right)(\Sigma^{-1})^T$

$\frac{\partial L}{\partial \Sigma} = 0$ 可得 $N\Sigma^T = \sum_{k=1}^N x_k x_k^T - N\mu\mu^T$

$$\Sigma = \frac{1}{N}\left(\sum_{k=1}^N(x_k x_k^T) - N\mu\mu^T\right)$$

$$= \frac{1}{N}\left(\sum_{k=1}^N(x_k x_k^T) - \cancel{N}\cdot\frac{1}{N}\sum_{i=1}^N\sum_{j=1}^N(x_i x_j^T)\right)$$

$\boxed{\text{没学过相关理论, 我不知道 X 的数量是否就是样本数N}}$

习题6. 取 $A=X^k$. $|A|=|X|^k\neq 0$ A 可逆 $A^*=|A|\cdot A^{-1}=|X|^k\cdot X^{-k}$

$d|A|=Tr(A^*\cdot dA)$ $dA=dX^k$ 在换序后,有.
(积的分)

$$d|A|=Tr(A^*dX^k)$$

$$=\begin{cases} Tr\Big(\big(\underbrace{X^{k-1}A^*+X^{k-2}A^*X+\cdots+A^*X^{k-1}}_{k\text{项}}\big)dX\Big), & k\geq 1 \\ Tr\Big(\big(\underbrace{X^{k+1}A^*+X^{k+2}A^*X^{-1}+\cdots+A^*X^{k+1}}_{|k|\text{项}}\big)dX^{-1}\Big) & k\leq -1 \end{cases}$$

$X\cdot X^{-1}=I \Rightarrow dX^{-1}=-X^{-1}dX\,X^{-1}$

$$=\begin{cases} Tr(k\cdot|X|^k\cdot X^{-1}dX) & k\geq 1 \\ Tr(-|k|\cdot|X|^k\cdot X\cdot X^{-1}dX\cdot X^{-1}) & k\leq -1 \end{cases}$$

$k=0$ 时 $\frac{\partial |X^k|}{\partial X}=0$

$$=\begin{cases} Tr(k\cdot|X|^kX^{-1}dX) & k\geq 1 \\ Tr(-|k|\cdot|X|^k\cdot X^{-1}dX) & k\leq -1 \end{cases}$$

综上, 可得 $\frac{\partial|X^k|}{\partial X}=\underset{\downarrow\,(-1)^k}{\pm}|k|\cdot|X|^k\cdot(X^{-1})^T=\cancel{|k|\cdot|X|^k X^{-T}}$

$(-1)^k|k|\cdot|X|^k\cdot(X^T)^{-1}$

$=k\cdot|X|^k\cdot X^{-T}$

习题7. $Tr(AXBX^TC)$ 为标量.

设 $f(X)=Tr(AXBX^TC)$ $df=dTr(AXBX^TC)=Tr(d(AXBX^TC))$

$=Tr(A\cdot dX\cdot BX^TC+AXB\,dX^TC)$

$=Tr(dX\cdot BX^TC\cdot A+dX^T\cdot C\cdot AXB)$

$=Tr(BX^TCA\cdot dX)+Tr(B^TX^TA^TC^TdX)$

$=Tr(BX^TCA\cdot dX)+Tr(B^TX^TA^TC^TdX)$

$=Tr((BX^TCA+B^TX^TA^TC^T)dX)$

$\frac{\partial f}{\partial X}=(BX^TCA)^T+(B^TX^TA^TC)^T=A^TC^TXB^T+CAXB$