第5次作业

习题1. 解：$A=LU$　$Ax=b \Rightarrow LUx=b \Rightarrow x=U^{-1}L^{-1}b$

$$x=U^{-1}L^{-1}b=\begin{bmatrix}2&4\\0&-3\end{bmatrix}^{-1}\begin{bmatrix}1&0\\2&1\end{bmatrix}^{-1}\begin{bmatrix}5\\3\end{bmatrix}$$

习题2. 解：A能LU分解的充要条件：A的前n阶顺序主子式都不为0 $(A\in R^{n\times n})$

取不能进行LU分解的矩阵 $A=\begin{bmatrix}0&1&1\\1&1&0\\1&0&1\end{bmatrix}$

若我们非要对其LU分解 $A=\begin{bmatrix}1&0&0\\L_{21}&1&0\\L_{31}&L_{32}&1\end{bmatrix}\begin{bmatrix}u_{11}&u_{12}&u_{13}\\0&u_{22}&u_{23}\\0&0&u_{33}\end{bmatrix}$

$$\text{该式}=\begin{bmatrix}u_{11}&u_{12}&u_{13}\\L_{21}\cdot u_{11}&L_{21}\cdot u_{12}+u_{22}&L_{21}\cdot u_{13}+u_{23}\\L_{31}u_{11}&L_{31}\cdot u_{12}+L_{32}\cdot u_{22}&L_{31}\cdot u_{13}+L_{32}\cdot u_{23}+u_{33}\end{bmatrix}$$

$$\begin{cases}u_{11}=0 & u_{12}=1 & u_{13}=1\\ L_{21}\cdot u_{11}=1 & L_{21}\cdot u_{12}+u_{22}=1 & L_{21}\cdot u_{13}+u_{23}=0\\ L_{31}\cdot u_{11}=1 & \cdots\end{cases}$$

对于 $\begin{cases}u_{11}=0\\L_{21}\cdot u_{11}=1\\L_{31}\cdot u_{11}=1\end{cases}$ 该方程不可能成立　A不能进行LU分解，这便是方程解集不可能出现的情况

对A的行重排，再进行LU分解即可解决该问题

$$A=\begin{bmatrix}0&1&1\\1&1&0\\1&0&1\end{bmatrix}\xrightarrow{R_2\leftrightarrow R_1}\begin{bmatrix}1&1&0\\0&1&1\\1&0&1\end{bmatrix}\xrightarrow{R_3+(-1)R_1}\begin{bmatrix}1&1&0\\0&1&1\\0&-1&1\end{bmatrix}\xrightarrow{R_3+(1)R_2}\begin{bmatrix}1&1&0\\0&1&1\\0&0&2\end{bmatrix}$$

$P_1=\begin{bmatrix}0&1&0\\1&0&0\\0&0&1\end{bmatrix}$　$P_2(P_1A)=U$　$A=P_1^{-1}P_2^{-1}U$　$P_2=\begin{bmatrix}1&0&0\\0&1&0\\-1&1&1\end{bmatrix}$

$P_1^{-1}=P$　$P_2^{-1}=L$　$A=PLU$

$Ax=b$　$Pz_1=b$　~~$z_1=LUx$~~

一步一步迭代求解即可

~~$Lz_2=$~~ $z_1=LUx=Lz_2$　$z_2=Ux$

习题3. 解：$A=\begin{bmatrix}1&2&2\\2&1&2\\1&2&1\end{bmatrix}$　$\alpha_1=\begin{bmatrix}1\\2\\1\end{bmatrix}$　$w_1=\dfrac{\alpha_1-\|\alpha_1\|_2e_1}{\|\alpha_1-\|\alpha_1\|_2e_1\|_2}=\dfrac{[1-\sqrt{6},2,1]^T}{\sqrt{12-2\sqrt{6}}}$

$$H_1=I-2w_1w_1^T=\frac{1}{6-\sqrt{6}}\begin{bmatrix}\sqrt{6}-1&2\sqrt{6}-2&\sqrt{6}-1\\2\sqrt{6}-2&2-\sqrt{6}&-2\\\sqrt{6}-1&-2&5-\sqrt{6}\end{bmatrix}$$

习题3. 解: $A=\begin{bmatrix}1&2&2\\2&1&2\\1&2&1\end{bmatrix}$ $\alpha_1=\begin{bmatrix}1\\2\\1\end{bmatrix}$ $\alpha_2=\begin{bmatrix}2\\1\\2\end{bmatrix}$ $\alpha_3=\begin{bmatrix}2\\2\\1\end{bmatrix}$

由施密特正交化法 $\beta_1=\alpha_1=\begin{bmatrix}1\\2\\1\end{bmatrix}$

$$\beta_2=\alpha_2-\frac{\langle\beta_1,\alpha_2\rangle}{\langle\beta_1,\beta_1\rangle}\beta_1=\begin{bmatrix}1\\-1\\1\end{bmatrix}$$

$$\beta_3=\alpha_3-\frac{\langle\beta_1,\alpha_3\rangle}{\langle\beta_1,\beta_1\rangle}\beta_1-\frac{\langle\beta_2,\alpha_3\rangle}{\langle\beta_2,\beta_2\rangle}\beta_2=\begin{bmatrix}\frac{1}{2}\\0\\-\frac{1}{2}\end{bmatrix}$$

单位化 $q_1=\frac{1}{\sqrt6}\begin{bmatrix}1\\2\\1\end{bmatrix}$ $q_2=\frac{1}{\sqrt3}\begin{bmatrix}1\\-1\\1\end{bmatrix}$ $q_3=\frac{1}{\sqrt2}\begin{bmatrix}1\\0\\-1\end{bmatrix}$

$$Q=(q_1\ q_2\ q_3)=\begin{bmatrix}\frac{1}{\sqrt6}&\frac{1}{\sqrt3}&\frac{1}{\sqrt2}\\\frac{2}{\sqrt6}&-\frac{1}{\sqrt3}&0\\\frac{1}{\sqrt6}&\frac{1}{\sqrt3}&-\frac{1}{\sqrt2}\end{bmatrix}$$

$\alpha_1=|\beta_1|q_1$

$\alpha_2=\langle\alpha_2,q_1\rangle q_1+|\beta_2|q_2$

$\alpha_3=\langle\alpha_3,q_1\rangle q_1+\langle\alpha_3,q_2\rangle q_2+|\beta_3|q_3$

$$R=\begin{bmatrix}|\beta_1|&\langle\alpha_2,q_1\rangle&\langle\alpha_3,q_1\rangle\\0&|\beta_2|&\langle\alpha_3,q_2\rangle\\0&0&|\beta_3|\end{bmatrix}=\begin{bmatrix}\sqrt6&\sqrt6&\frac{7}{\sqrt6}\\0&\sqrt3&\frac{\sqrt3}{3}\\0&0&\frac{\sqrt2}{2}\end{bmatrix}$$

$$A=\underbrace{\begin{bmatrix}\frac{1}{\sqrt6}&\frac{1}{\sqrt3}&\frac{1}{\sqrt2}\\\frac{2}{\sqrt6}&-\frac{1}{\sqrt3}&0\\\frac{1}{\sqrt6}&\frac{1}{\sqrt3}&-\frac{1}{\sqrt2}\end{bmatrix}}_{Q}\underbrace{\begin{bmatrix}\sqrt6&\sqrt6&\frac{7}{\sqrt6}\\0&\sqrt3&\frac{\sqrt3}{3}\\0&0&\frac{\sqrt2}{2}\end{bmatrix}}_{R}$$

$Ax=b$ $QRx=b$

$Rx=Q^Tb$

$$Q^Tb=\begin{bmatrix}\frac{8}{\sqrt6}\\\frac{2}{\sqrt3}\\-\sqrt2\end{bmatrix}$$

后向代入法求 $Rx=Q^Tb$ 可得 $\begin{bmatrix}\frac{7}{3}\\\frac{4}{3}\\-2\end{bmatrix}=X$