12.求下列不定积分

(1) $\displaystyle\int \frac{2x-1}{x^2+3x+2}dx.$

解：原式 $=\displaystyle\int \frac{2x+3-4}{x^2+3x+2}dx=\int\frac{2x+3}{x^2+3x+2}dx-\int\frac{4}{x^2+3x+2}dx$

$\displaystyle=\int\frac{d(x^2+3x+2)}{x^2+3x+2}-4\int\frac{1}{(x+1)(x+2)}dx=\int\frac{d(x^2+3x+2)}{x^2+3x+2}-4\left(\int\left(\frac{1}{x+1}-\frac{1}{x+2}\right)dx\right.$

$\displaystyle=\ln(x^2+3x+2)-4\ln\left|\frac{x+1}{x+2}\right|+C.$

(2) $\displaystyle\int \frac{x^{11}}{x^8+3x^4+2}dx.$

解：原式 $=\displaystyle\int\frac{4x^3\cdot x^8}{(x^4+1)(x^4+2)}\times\frac{1}{4}dx=\frac{1}{4}\int\frac{x^8}{(x^4+1)(x^4+2)}d(x^4)$

令 $x^4=t$.

则上式 $=\displaystyle\frac{1}{4}\int\frac{t^2}{(t+1)(t+2)}dt=\frac{1}{4}\int\left(\frac{t^2}{t+1}-\frac{t^2}{t+2}\right)dt=\frac{1}{4}\left[\int\frac{t^2}{t+1}dt-\int\frac{t^2}{t+2}dt\right]$

$\displaystyle=\frac{1}{4}\int\left[(t-1)+\frac{1}{t+1}-(t-2)-\frac{4}{t+2}\right]dt=\frac{1}{4}\int\left(1+\frac{1}{t+1}-\frac{4}{t+2}\right)dt$

$\displaystyle=\frac{1}{4}\left(t+\ln\frac{t+1}{(t+2)^4}\right)+C=\frac{1}{4}\left(x^4+\ln\frac{x^4+1}{(x^4+2)^4}\right)+C$

(3) $\displaystyle\int \frac{dx}{(x^2+1)(x^2+x+1)}.$

解：$\displaystyle\frac{d}{(x^2+1)(x^2+x+1)}=\frac{Ax+B}{(x^2+1)}+\frac{Cx+D}{x^2+x+1}$

展开化简得：$\begin{cases}A+C=0\\A+B+D=0\\A+B+C=0\\B+D=1\end{cases}\Rightarrow\begin{cases}A=-1\\B=0\\C=1\\D=1\end{cases}$

$\displaystyle\therefore\ \frac{-x}{x^2+1}+\frac{x+1}{x^2+x+1}$

原式 $=\displaystyle-\int\frac{x}{x^2+1}dx+\int\frac{(x+1)dx}{x^2+x+1}$

$\displaystyle=-\frac{1}{2}\int\frac{d(x^2+1)}{x^2+1}+\frac{1}{2}\int\frac{d(x^2+x+1)}{x^2+x+1}+\frac{1}{2}\int\frac{dx}{x^2+x}$

$\displaystyle=-\frac{1}{2}\int\frac{d(x^2+1)}{x^2+1}+\frac{1}{2}\int\frac{d(x^2+x+1)}{x^2+x+1}+\frac{1}{2}\cdot\int\frac{dx}{(x+\frac{1}{2})^2+\frac{3}{4}}$

$\displaystyle=-\frac{1}{2}\int\frac{d(x^2+1)}{x^2+1}+\frac{1}{2}\int\frac{d(x^2+x+1)}{x^2+x+1}+\frac{2}{3}\int\frac{dx}{(\frac{2}{\sqrt{3}}x)+\frac{1}{\sqrt{3}})^2+1}$

$\displaystyle=-\frac{1}{2}\int\frac{d(x^2+1)}{x^2+1}+\frac{1}{2}\int\frac{d(x^2+x+1)}{x^2+x+1}+\frac{\sqrt{3}}{2}\times\frac{2}{3}\int\frac{d(\frac{2}{\sqrt{3}}x+\frac{1}{3}}{(\frac{2}{\sqrt{3}}x+\frac{1}{\sqrt{3}})^2}$

$\displaystyle=-\frac{1}{2}\ln|x^2+1|+\frac{1}{2}\ln|x^2+x+1|$

$\displaystyle+\frac{\sqrt{3}}{3}\arctan\left(\frac{2x+1}{\sqrt{3}}\right)+C$

(4) $\displaystyle\int \frac{x^2+1}{(x+1)^2(x-1)}dx.$

解：$\displaystyle\frac{x^2+1}{(x+1)^2(x-1)}=\frac{Ax+B}{(x+1)^2}+\frac{C}{x-1}$

两边同乘以 $(x+1)^2(x-1)$ 得：

$x^2+1=(Ax+B)(x-1)+C(x+1)^2$

展开，得：$\begin{cases}A+C=1\\-A+B+2C=0\\-B+C=1\end{cases}\Rightarrow\begin{cases}A=\frac{1}{2}\\B=-\frac{1}{2}\\C=\frac{1}{2}\end{cases}$

则原式 $=\displaystyle\int\left(\frac{x-1}{2(x+1)^2}+\frac{1}{2(x-1)}\right)dx$

$\displaystyle=\frac{1}{2}\int\frac{x-1}{(x+1)^2}dx+\frac{1}{2}\int\frac{dx}{x-1}$

$\displaystyle=\frac{1}{2}\int\frac{(x+1-2)}{x^2+2x+1}dx+\frac{1}{2}\int\frac{d(x-1)}{x-1}$

$\displaystyle=\frac{1}{4}\int\frac{d(x^2+2x+1)}{x^2+2x+1}-\int\frac{d(x+1)}{(x+1)^2}+\frac{1}{2}\int\frac{d(x-1)}{x-1}=\frac{1}{2}\ln|x+1|+\frac{1}{2}\ln|x-1|+\frac{1}{x+1}+C$

(5) $\displaystyle\int \frac{1}{1+\sin x}dx.$

解: 原式 $=\displaystyle\int\frac{1-\sin x}{1-\sin^2 x}dx=\int\frac{1-\sin x}{\cos^2 x}dx = \tan x - \sec x + C$

(6) $\displaystyle\int \frac{2-\sin x}{2+\cos x}dx.$

解: 原式 $=\displaystyle\int\frac{2}{2+\cos x}dx-\int\frac{\sin x}{2+\cos x}dx$ 令 $t=\tan\frac{x}{2}$

则原式 $=\displaystyle\int\frac{2}{2+\frac{1-t^2}{1+t^2}}\frac{2}{1+t^2}dt+\ln|2+\cos x|+C$

$=\displaystyle\int\frac{4}{3+t^2}dt+\ln|2+\cos x|+C$

$=4\displaystyle\int\frac{1}{t^2+3}dt+\ln|2+\cos x|+C$

$=\displaystyle\frac{4}{3}\int\frac{1}{(\frac{t}{\sqrt{3}})^2+1}dt+\ln|2+\cos x|+C$

$=\displaystyle\frac{4\sqrt{3}}{3}\int\frac{1}{(\frac{t}{\sqrt{3}})^2+1}\cdot\frac{1}{\sqrt{3}}dt+\ln|2+\cos x|$

$=\displaystyle\frac{4\sqrt{3}}{3}\int\frac{1}{(\frac{t}{\sqrt{3}})^2+1}d(\frac{t}{\sqrt{3}})+\ln|2+\cos x|+$

$=\displaystyle\frac{4\sqrt{3}}{3}\arctan\frac{t}{\sqrt{3}}+\ln|2+\cos x|+C$

$=\displaystyle\frac{4\sqrt{3}}{3}\arctan(\frac{\tan\frac{x}{2}}{\sqrt{3}})+\ln|2+\cos x|+C$

(7) $\displaystyle\int\sqrt{\frac{1-x}{1+x}}dx.$

解: 令 $t=\sqrt{\frac{1-x}{1+x}}dx$ 则 $x=\frac{1-t^2}{1+t^2}$ $dx=-\frac{4t}{(t^2+1)^2}$

$\therefore$ 原式 $=-4\displaystyle\int\frac{t^2dt}{(t^2+1)^2}$ 令 $t=\tan\theta$ $\theta\in(-\frac{\pi}{2},\frac{\pi}{2})$

原式 $=-4\displaystyle\int\frac{\tan^2\theta}{\sec^2\theta}d\theta=-4\int\sin^2\theta d\theta=-\int(1-\cos2\theta)d2\theta$

$=\displaystyle\int(\cos2\theta-1)d2\theta=\sin2\theta-2\theta+C$

$=\displaystyle\frac{2t}{t^2+1}-2\arctan t+C=\sqrt{1-x^2}-2\arctan\sqrt{\frac{1-x}{1+x}}+C$

(8) $\displaystyle\int\frac{dx}{\sqrt[3]{(x+1)^2(x-1)^4}}.$

解: $\sqrt[3]{(x+1)^2(x-1)^4}=(x^2-1)\sqrt[3]{\frac{x-1}{x+1}}$

令 $t=\sqrt[3]{\frac{x-1}{x+1}}\Rightarrow x=\frac{1+t^3}{1-t^3}$

原式 $=\displaystyle\int\frac{1}{\frac{(t^3+1)^2-(t^3-1)^2}{(t^3-1)^2}\times t}\times\frac{6t^2}{(t^3-1)^2}dt$

$=\displaystyle\int\frac{6t}{4t^3}dt$

$=\displaystyle\frac{3}{2}\int\frac{1}{t^2}dt$

$=-\displaystyle\frac{3}{2t}+C$

$=-\displaystyle\frac{3}{2}\sqrt[3]{\frac{x-1}{x+1}}+C$

13. 求下列不定积分.

(1) $\int \frac{\cos 2x}{1+\sin x\cos x}dx.$

解：原式 $= \int \frac{\cos 2x}{1+\frac{1}{2}\sin 2x}dx = \int \frac{2\cos 2x}{2+\sin 2x}dx$

$= \int \frac{d(2+\sin 2x)}{2+\sin 2x} = \ln|2+\sin 2x| + C$

(2) $\int \frac{1}{x^2+2x+5}dx.$

解：原式 $= \int \frac{dx}{(x+1)^2+4} = \frac{1}{4}\int \frac{dx}{(\frac{x+1}{2})^2+1} = \frac{1}{2}\arctan\frac{x+1}{2} + C$

(3) $\int \frac{dx}{\sin^2 x + 2\cos^2 x}.$

解：原式 $= \int \frac{\sec^2 x}{2+\tan^2 x}dx = \int \frac{d(\tan x)}{2+\tan^2 x} = \int \frac{d(\frac{\tan x}{\sqrt{2}})}{1+(\frac{\tan x}{\sqrt{2}})^2}\cdot\frac{1}{\sqrt{2}}$

$= \frac{1}{\sqrt{2}}\arctan(\frac{\tan x}{\sqrt{2}}) + C$

(4) $\int \frac{\sin x}{1+\sin x}dx.$

解：原式 $= \int \frac{\sin x(1-\sin x)}{(1+\sin x)(1-\sin x)}dx = \int \frac{\sin x(1-\sin x)}{\cos^2 x}dx = \int \frac{\sin x}{\cos^2 x}dx - \int \tan^2 x\, dx$

$= \sec x - (\tan x - x) + C$

$= x - \tan x + \sec x + C$

(5) $\int \frac{1}{x^2\sqrt{a^2+x^2}}dx.$

解: 令 $x = a\tan t$. $t \in (-\frac{\pi}{2}, \frac{\pi}{2})$ $dx = a\sec^2 t\,dt$.

原式 $= \frac{1}{a}\int \frac{1}{a^2\tan^2 t\sec t} \cdot a\sec^2 t\,dt = \frac{1}{a^2}\int \frac{\sec t}{\tan^2 t}dt$

$= \frac{1}{a^2}\int \frac{\cos t\,dt}{\sin^2 t} = \frac{1}{a^2}\int (\sin t)^{-2}\,d\sin t$

$= -\frac{1}{a^2}\frac{1}{\sin t} + C = -\frac{1}{a^2}\frac{\sqrt{x^2+a^2}}{x} + C$

(6) $\int \frac{dx}{x\sqrt{1-x^4}}.$

解. 原式 $= \int \frac{x^3}{x^4\sqrt{1-x^4}}dx = \frac{1}{4}\int \frac{dx^4}{x^4(\sqrt{1-x^4})}$ 令 $x^2 = \sin t, t \in (-\frac{\pi}{2}, \frac{\pi}{2})$

则原式 $= \frac{1}{4}\int \frac{d\sin^2 t}{\sin^2 t\cos t} = \frac{1}{4}\int \frac{\sin 2t\,dt}{\sin^2 t\cos t} = \frac{1}{2}\int \frac{dt}{\sin t}$

$= \frac{1}{2}\int \frac{-d\cos t}{\sin^2 t} = -\frac{1}{2}\int \frac{1}{1-\cos^2 t}d\cos t = -\frac{1}{4}\int \frac{d\cos t}{1-\cos t} - \frac{1}{4}\int \frac{d\cos t}{1+\cos t}$

$= \frac{1}{4}\ln|1-\cos t| - \frac{1}{4}\ln|1+\cos t| + C = \frac{1}{4}\ln\left|\frac{1-\sqrt{1-x^4}}{1+\sqrt{1+x^4}}\right| + C = \frac{1}{2}\ln|1-\sqrt{1-x^4}| - \ln|x|$

(7) $\int x^2 \arccos x\,dx.$

解: 原式 $= \frac{1}{3}x^3\arccos x - \int \frac{1}{3}x^3\,d(\arccos x)$

$= \frac{1}{3}x^3\arccos x + \frac{1}{3}\int x^3 \frac{1}{\sqrt{1-x^2}}dx$

$= \frac{1}{3}x^3\arccos x + \frac{1}{6}\int x^2 \frac{1}{\sqrt{1-x^2}}d(x^2)$

$= \frac{1}{3}x^3\arccos x + \frac{1}{6}(\frac{2}{3}(1-x^2)^{\frac{3}{2}} - 2\sqrt{1-x^2}) + C$

(8) $\int \frac{1-x^7}{x(1+x^7)}dx.$

解. 原式 $= \int \frac{1}{x(1+x^7)}dx - \int \frac{x^7}{x(1+x^7)}dx$

$= \frac{1}{7}\int \frac{x^6}{x^7(1+x^7)}dx - \frac{1}{7}\int \frac{x^6}{1+x^7}dx$

$= \frac{1}{7}\int \frac{d(x^7)}{x^7(1+x^7)} - \frac{1}{7}\int \frac{d(x^7)}{1+x^7}$

$= \frac{1}{7}(\int \frac{d(x^7)}{x^7(1+x^7)} - \int \frac{d(x^7)}{1+x^7})$

$= \frac{1}{7}\int (\frac{1}{x^7} - \frac{1}{1+x^7})\,d(x^7) - \frac{1}{7}\int \frac{d(x^7)}{1+x^7}$

$= \frac{1}{7}\ln\left|\frac{x^7}{1+x^7}\right| - \frac{1}{7}\ln|1+x^7| + C$

$= \ln|x| - \frac{2}{7}\ln|1+x^7| + C$

(9) $\int \frac{dx}{\sin^3 x \cos^5 x}$.

解: 原式 $= \int \frac{dx}{\tan^3 x \cos^8 x} = \int \frac{\sec^6 x \, d(\tan x)}{\tan^3 x}$

$= \int \frac{(1+\tan^2 x)^3 \, d(\tan x)}{\tan^3 x}$

$= -\frac{1}{2}\tan^{-2} x + 3\ln|\tan x| + \frac{3}{2}\tan^2 x + \frac{1}{4}\tan^4 x + C$.

(10) $\int \frac{\cot x dx}{1+\sin x}$.

解: 原式 $= \int \frac{\cot x \csc x}{1+\csc x} dx$

$= -\int \frac{d(1+\csc x)}{1+\csc x} = -\ln|1+\csc x| + C$

(11) $\int \frac{\arctan x}{x^2(1+x^2)} dx$.

解: 原式 $= \int \frac{\arctan x}{x^2} dx - \frac{\arctan x}{1+x^2} dx$

$= -\frac{1}{x}\arctan x + \int \frac{1}{x(1+x^2)} dx - \frac{1}{2}(\arctan x)^2$

$= -\frac{1}{x}\arctan x + \int \frac{x}{x^2(1+x^2)} dx - \frac{1}{2}(\arctan x)^2$

$= -\frac{1}{x}\arctan x + \frac{1}{2}\left(\int \frac{1}{x^2} dx^2 - \int \frac{1}{1+x^2} d(1+x^2) \right) - \frac{1}{2}(\arctan x)^2$

$= -\frac{1}{x}\arctan x + \ln|x| - \ln(1+x^2) - \frac{1}{2}(\arctan x)^2 + C$.

$\frac{1}{2}$

(12) $\int \frac{1}{(1+2x^2)\sqrt{x^2+1}}dx.$ 令: $x=\tan t$

解: 原式 $= \int \frac{\sec^2 t}{(2\tan^2 t+1)\sec t}dt$

$= \int \frac{\sec t}{\tan^2 t+\sec^2 t}dt = \int \frac{\cos t}{\sin^2 t+1}dt$

$= \arctan(\sin t)+C = \arctan\left(\frac{x}{\sqrt{1+x^2}}\right)+C$

(13) $\int \frac{xe^x}{\sqrt{e^x-1}}dx.$

解: 原式 $= 2\int x\,d(\sqrt{e^x-1})$

$= 2x\sqrt{e^x-1} - 2\int\sqrt{e^x-1}\,dx$

$\int\sqrt{e^x-1}\,dx = \int t\frac{2t}{1+t^2}dt = 2\int\left(1-\frac{1}{1+t^2}\right)dt$ (令 $t=\sqrt{e^x-1}$)

$= 2t-2\arctan t+C$

$= 2\sqrt{e^x-1} - 2\arctan\sqrt{e^x-1}+C$

故原式 $= 2x\sqrt{e^x-1} - 4\sqrt{e^x-1} + 4\arctan\sqrt{e^x-1} + C.$

(14) $\int \frac{dx}{e^x+e^{-x}}.$

解: 原式 $= \int \frac{e^x dx}{(e^x)^2+1}.$

$= \int \frac{1}{(e^x)^2+1}d(e^x)$

$= \arctan e^x + C.$