不定积分的概念与性质
-直接积分法

一、 不定积分的概念

二、 直接积分法--基本积分表

三、不定积分的性质

内容小结

1. 不定积分的概念

- 不定积分的定义
- 不定积分的性质
- 基本积分表

2. 直接积分法:

利用**恒等变形**, **积分性质** 及 **基本积分公式**进行积分 .

$$\text{常用恒等变形方法}\begin{cases}\text{分项积分}\\ \text{加项减项}\\ \text{利用三角公式 , 代数公式 ,}\cdots\end{cases}$$

不定积分

换元积分法

一、第一类换元法

二、第二类换元法

一、第一类换元法

定理1. 设 $f(u)$ 有原函数 $F(u)$, $u=\varphi(x)$ 可导, 则有换元公式

$$\int f[\varphi(x)]\underline{\varphi'(x)\mathrm{d}x} = \int f(u)\mathrm{d}u\Big|_{u=\varphi(x)}$$

即 $$\int f[\varphi(x)]\varphi'(x)\mathrm{d}x = \int f(\varphi(x))\mathrm{d}\varphi(x)$$

(也称**配元法，凑微分法**)

思考与练习 1. 下列各题求积方法有何不同?

(1) $\int \frac{\mathrm{d}x}{4+x} = \int \frac{\mathrm{d}(4+x)}{4+x}$ (2) $\int \frac{\mathrm{d}x}{4+x^2} = \frac{1}{2}\int \frac{\mathrm{d}(\frac{x}{2})}{1+(\frac{x}{2})^2}$

(3) $\int \frac{x}{4+x^2}\mathrm{d}x = \frac{1}{2}\int \frac{\mathrm{d}(4+x^2)}{4+x^2}$

(4) $\int \frac{x^2}{4+x^2}\mathrm{d}x = \int [1-\frac{4}{4+x^2}]\mathrm{d}x$

(5) $\int \frac{\mathrm{d}x}{4-x^2} = \frac{1}{4}\int [\frac{1}{2-x}+\frac{1}{2+x}]\mathrm{d}x$

(6) $\int \frac{\mathrm{d}x}{\sqrt{4x-x^2}} = \int \frac{\mathrm{d}(x-2)}{\sqrt{4-(x-2)^2}}$

例1. 求 $\int \sec x \mathrm{d}x$.

解法1

$$\int \sec x \mathrm{d}x = \int \frac{\cos x}{\cos^2 x} \mathrm{d}x = \int \frac{\mathrm{d}\sin x}{1-\sin^2 x}$$

$$= \frac{1}{2} \int \left[\frac{1}{1+\sin x} + \frac{1}{1-\sin x} \right]\Bigg] \mathrm{d}\sin x$$

$$= \frac{1}{2} \left[\ln\left|1+\sin x\right| - \ln\left|1-\sin x\right| \right] + C$$

$$= \frac{1}{2} \ln\left|\frac{1+\sin x}{1-\sin x}\right| + C$$

解法 2 $\int \sec x \mathrm{d}x = \int \frac{\sec x(\sec x + \tan x)}{\sec x + \tan x} \mathrm{d}x$

$$= \int \frac{\sec^2 x + \sec x \tan x}{\sec x + \tan x} \mathrm{d}x$$

$$= \int \frac{\mathrm{d}(\sec x + \tan x)}{\sec x + \tan x}$$

$$= \ln\left| \sec x + \tan x \right| + C$$

同样可证

$$\int \csc x \mathrm{d}x = \ln\left| \csc x - \cot x \right| + C$$

或 $\int \csc x \mathrm{d}x = \ln\left| \tan \frac{x}{2} \right| + C$

例2. 求 $\int \sin^2 x \cos^2 3x \, \mathrm{d}x$.

解: $\because \sin^2 x \cos^2 3x = [\frac{1}{2}(\sin 4x - \sin 2x)]^2$

$$= \frac{1}{4}\sin^2 4x - \frac{1}{4}\cdot 2\sin 4x \sin 2x + \frac{1}{4}\sin^2 2x$$

$$= \frac{1}{8}(1-\cos 8x) - \sin^2 2x \cos 2x + \frac{1}{8}(1-\cos 4x)$$

$\therefore$ 原式 $= \frac{1}{4}\int \mathrm{d}x - \frac{1}{64}\int \cos 8x \, \mathrm{d}(8x)$

$$- \frac{1}{2}\int \sin^2 2x \, \mathrm{d}(\sin 2x) - \frac{1}{32}\int \cos 4x \, \mathrm{d}(4x)$$

$$= \frac{1}{4}x - \frac{1}{64}\sin 8x - \frac{1}{6}\sin^3 2x - \frac{1}{32}\sin 4x + C$$

sinαcosβ = [sin(α+β)+sin(α-β)]/2

例3. 求 $\displaystyle\int\frac{3}{x^2-4x+5}\,\mathrm{d}x,\ \int\frac{2x-4}{x^2-4x+5}\,\mathrm{d}x,\ \int\frac{6x+1}{x^2-4x+5}\,\mathrm{d}x$

解:

$$\int\frac{3}{x^2-4x+5}\,\mathrm{d}x=\int\frac{3}{(x-2)^2+1}\,\mathrm{d}(x-2)$$

$$=3\arctan(x-2)+C$$

$$\int\frac{2x-4}{x^2-4x+5}\,\mathrm{d}x=\int\frac{1}{x^2-4x+5}\,\mathrm{d}(x^2-4x+5)$$

$$=\ln\left|x^2-4x+5\right|+C$$

$$\int\frac{6x+1}{x^2-4x+5}\,\mathrm{d}x=\int\frac{3(2x-4)+13}{x^2-4x+5}\,\mathrm{d}x$$

$$=3\ln\left|x^2-4x+5\right|+13\arctan(x-2)+C$$

例4. 求 $\int \frac{\mathrm{d}x}{x(x^{10}+1)}$.

提示:

法1 $\int \frac{\mathrm{d}x}{x(x^{10}+1)} = \int \frac{(x^{10}+1)-x^{10}}{x(x^{10}+1)}\mathrm{d}x$

法2 $\int \frac{\mathrm{d}x}{x(x^{10}+1)} = \frac{1}{10}\int \frac{\mathrm{d}\ x^{10}}{x^{10}(x^{10}+1)}$

法3 $\int \frac{\mathrm{d}x}{x(x^{10}+1)} = \int \frac{\mathrm{d}\ x}{x^{11}(1+x^{-10})} = \frac{-1}{10}\int \frac{\mathrm{d}\ x^{-10}}{1+x^{-10}}$

小结 常用简化技巧:

(1) 分项积分: 利用积化和差; 分式分项;

$$1=\sin^2 x+\cos^2 x \text{等}$$

(2) 降低幂次: 利用倍角公式, 如

$$\cos^2 x=\tfrac{1}{2}(1+\cos 2x); \qquad \sin^2 x=\tfrac{1}{2}(1-\cos 2x);$$

$$\begin{cases} \int f(x^n)x^{n-1}\,\mathrm{d}x=\frac{1}{n}\int f(x^n)\,\mathrm{d}\,x^n \\ \int f(x^n)\frac{1}{x}\,\mathrm{d}x=\frac{1}{n}\int f(x^n)\frac{1}{x^n}\,\mathrm{d}\,x^n \end{cases}$$

(3) 统一函数: 利用三角公式; 配元方法

(4) 巧妙换元或配元

二、第二类换元法

第一类换元法解决的问题

$$\underset{\text{难求}}{\int f[\varphi(x)]\varphi'(x)\mathrm{d}x} = \underset{\text{易求}}{\int f(u)\mathrm{d}u}\Big|_{u=\varphi(x)}$$

若所求积分 $\int f(u)\mathrm{d}u$ 难求,

$\int f[\varphi(x)]\varphi'(x)\mathrm{d}x$ 易求,

则得第二类换元积分法 .

定理2．设 $x=\psi(t)$ 是单调可导函数，且 $\psi'(t)\neq 0$，$f[\psi(t)]\psi'(t)$ 具有原函数，则有换元公式

$$\int f(x)\,\mathrm{d}x=\int f[\psi(t)]\psi'(t)\,\mathrm{d}t\,\Big|_{t=\psi^{-1}(x)}$$

其中 $t=\psi^{-1}(x)$ 是 $\underline{x=\psi(t)}$ 的反函数.

证: 设 $f[\psi(t)]\psi'(t)$ 的原函数为 $\Phi(t)$，令

$$F(x)=\Phi[\psi^{-1}(x)]$$

则

$$F'(x)=\frac{\mathrm{d}\Phi}{\mathrm{d}t}\cdot\frac{\mathrm{d}t}{\mathrm{d}x}=f[\psi(t)]\cancel{\psi'(t)}\cdot\frac{1}{\cancel{\psi'(t)}}=f(x)$$

$$\therefore\quad \int f(x)\,\mathrm{d}x=F(x)+C=\Phi[\psi^{-1}(x)]+C$$

$$=\int f[\psi(t)]\psi'(t)\,\mathrm{d}t\,\Big|_{t=\psi^{-1}(x)}$$

例5. 求 $\int \frac{xdx}{\sqrt{x-3}}$.

解: 令 $\sqrt{x-3}=t$, 得 $x=3+t^2$ 则

$$\int \frac{xdx}{\sqrt{x-3}}=\int \frac{t^2+3}{t}2tdt$$

$$=\int \left(2t^2+6\right)dt$$

$$=\frac{2t^3}{3}+6t+C$$

$$=\frac{2}{3}(x+6)\sqrt{x-3}+C$$

例6. 求 $\displaystyle\int \frac{dx}{\sqrt{x}+\sqrt[3]{x}}$.

解: 令 $\sqrt[6]{x}=t$, 得 $x=t^6$ 则,

$$\text{原式}=\int \frac{1}{t^3+t^2}6t^5dt$$

$$=6\int \frac{t^3}{t+1}dt=6\int \frac{t^3+1-1}{t+1}dt$$

$$=6\left(\frac{t^3}{3}-\frac{t^2}{2}+t-\ln|t+1|\right)+C$$

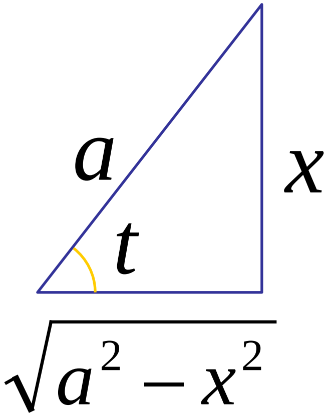
$$=2\sqrt{x}-3\sqrt[3]{x}+6\sqrt[6]{x}-6\ln\left|\sqrt[6]{x}+1\right|+C$$

例7. 求 $\int \sqrt{a^2 - x^2}\,\mathrm{d}x \;\; (a > 0)$.

解: 令 $x = a\sin t,\;\; t \in [-\frac{\pi}{2}, \frac{\pi}{2}]$, 则

$$\sqrt{a^2 - x^2} = \sqrt{a^2 - a^2\sin^2 t} = a\cos t$$

$$\mathrm{d}x = a\cos t\,\mathrm{d}t$$



$$\therefore \text{原式} = \int a\cos t \cdot a\cos t\,\mathrm{d}t = a^2\int \cos^2 t\,\mathrm{d}t$$

$$= a^2\left(\frac{t}{2} + \frac{\sin 2t}{4}\right) + C$$

$$\sin 2t = 2\sin t\cos t = 2\cdot\frac{x}{a}\cdot\frac{\sqrt{a^2 - x^2}}{a}$$

$$= \frac{a^2}{2}\arcsin\frac{x}{a} + \frac{1}{2}x\sqrt{a^2 - x^2} + C$$

例8. 求 $\int \frac{\mathrm{d}x}{\sqrt{x^2+a^2}} \quad (a>0).$

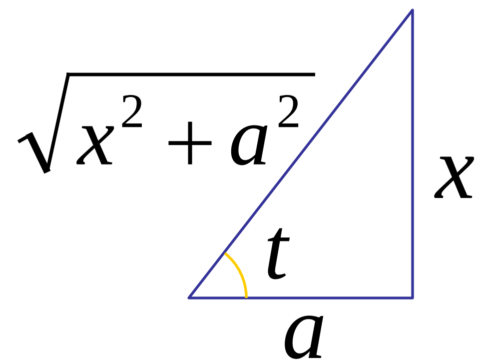
解: 令 $x = a\tan t,\ t\in(-\frac{\pi}{2},\frac{\pi}{2})$，则

$$\sqrt{x^2+a^2} = \sqrt{a^2\tan^2 t + a^2} = a\sec t$$

$$\mathrm{d}x = a\sec^2 t\,\mathrm{d}t$$

$$\therefore \text{原式} = \int \frac{a\sec^2 t}{a\sec t}\mathrm{d}t = \int \sec t\,\mathrm{d}t$$

$$= \ln\left|\sec t + \tan t\right| + C_1$$



$$= \ln\left[\frac{\sqrt{x^2+a^2}}{a} + \frac{x}{a}\right] + C_1$$

$$= \ln\left[x + \sqrt{x^2+a^2}\right] + C \qquad (C = C_1 - \ln a)$$

例9. 求 $\int \frac{\mathrm{d}x}{\sqrt{x^2-a^2}} \quad (a>0).$

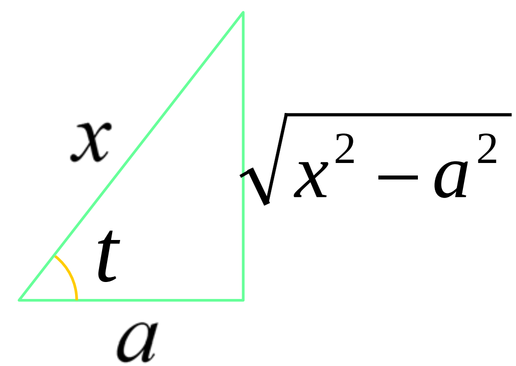
解: 当$x>a$时, 令 $x = a\sec t,\ t\in(0,\frac{\pi}{2})$, 则

$$\sqrt{x^2-a^2} = \sqrt{a^2\sec^2 t - a^2} = a\tan t$$

$$\mathrm{d}x = a\sec t\tan t\,\mathrm{d}\,t$$

$$\therefore \text{原式} = \int \frac{a\sec t\tan t}{a\tan t}\mathrm{d}\,t = \int \sec t\,\mathrm{d}\,t$$

$$= \ln\left|\sec t + \tan t\right| + C_1$$



$$= \ln\left|\frac{x}{a} + \frac{\sqrt{x^2-a^2}}{a}\right| + C_1$$

$$= \ln\left|x + \sqrt{x^2-a^2}\right| + C \qquad (C = C_1 - \ln a)$$

当$x<-a$时，令$x=-u$，则$u>a$，于是

$$\int \frac{\mathrm{d}x}{\sqrt{x^2-a^2}} = -\int \frac{\mathrm{d}u}{\sqrt{u^2-a^2}} = -\ln\left|u+\sqrt{u^2-a^2}\right|+C_1$$

$$= -\ln\left|-x+\sqrt{x^2-a^2}\right|+C_1$$

$$= -\ln\left|\frac{a^2}{-x-\sqrt{x^2-a^2}}\right|+C_1$$

$$= \ln\left|x+\sqrt{x^2-a^2}\right|+C \quad (C=C_1-2\ln a)$$

$$x>a\text{时}, \int \frac{\mathrm{d}x}{\sqrt{x^2-a^2}} = \ln\left|x+\sqrt{x^2-a^2}\right|+C$$

例10. 求 $\int \frac{\sqrt{a^2-x^2}}{x^4}\mathrm{d}x$.

解: 令 $x=\frac{1}{t}$,则 $\mathrm{d}x=\frac{-1}{t^2}\mathrm{d}\,t$

$$原式=\int \frac{\sqrt{a^2-\frac{1}{t^2}}}{\frac{1}{t^4}}\cdot\frac{-1}{t^2}\mathrm{d}\,t=-\int (a^2t^2-1)^{\frac{1}{2}}\left|t\right|\mathrm{d}\,t$$

当 $x>0$ 时,

$$原式=-\frac{1}{2a^2}\int (a^2t^2-1)^{\frac{1}{2}}\,\mathrm{d}(a^2t^2-1)$$

$$=-\frac{(a^2t^2-1)^{\frac{3}{2}}}{3a^2}+C\ =-\frac{(a^2-x^2)^{\frac{3}{2}}}{3a^2x^3}+C$$

当 $x<0$ 时, 类似可得同样结果 .

小结:

1. 第二类换元法常见类型:

(1) $\int f(x, \sqrt[n]{ax+b})\,\mathrm{d}x$， 令 $t = \sqrt[n]{ax+b}$

(2) $\int f\left(x, \sqrt[n]{\frac{ax+b}{cx+d}}\right)\mathrm{d}x$， 令 $t = \sqrt[n]{\frac{ax+b}{cx+d}}$

(3) $\int f(x, \sqrt{a^2-x^2})\,\mathrm{d}x$，令 $x = a\sin t$ 或 $x = a\cos t$

(4) $\int f(x, \sqrt{a^2+x^2})\,\mathrm{d}x$，令 $x = a\tan t$

(5) $\int f(x, \sqrt{x^2-a^2})\,\mathrm{d}x$，令 $x = a\sec t$

(6) $\int f(a^x)\,\mathrm{d}x$，令 $t = a^x$

(7) 分母中因子次数较高时, 可试用**倒代换**

2. 常用基本积分公式的补充

$$(16)\quad \int \tan x\,\mathrm{d}x = -\ln|\cos x| + C$$

$$(17)\quad \int \cot x\,\mathrm{d}x = \ln|\sin x| + C$$

$$(18)\quad \int \sec x\,\mathrm{d}x = \ln|\sec x + \tan x| + C$$

$$(19)\quad \int \csc x\,\mathrm{d}x = \ln|\csc x - \cot x| + C$$

(20) $$\int \frac{1}{a^2+x^2}\,\mathrm{d}\,x = \frac{1}{a}\arctan\frac{x}{a}+C$$

(21) $$\int \frac{1}{x^2-a^2}\,\mathrm{d}\,x = \frac{1}{2a}\ln\left|\frac{x-a}{x+a}\right|+C$$

(22) $$\int \frac{1}{\sqrt{a^2-x^2}}\,\mathrm{d}\,x = \arcsin\frac{x}{a}+C$$

(23) $$\int \frac{1}{\sqrt{x^2+a^2}}\,\mathrm{d}\,x = \ln(x+\sqrt{x^2+a^2})+C$$

(24) $$\int \frac{1}{\sqrt{x^2-a^2}}\,\mathrm{d}\,x = \ln\left|\,x+\sqrt{x^2-a^2}\,\right|+C$$

例11. 求 $I=\int\frac{\mathrm{d}x}{\sqrt{4x^2+9}}$.

解: $I=\frac{1}{2}\int\frac{\mathrm{d}(2x)}{\sqrt{(2x)^2+3^2}}=\frac{1}{2}\ln\left|2x+\sqrt{4x^2+9}\right|+C$

例12. 求 $\int \frac{\mathrm{d}x}{x^2\sqrt{x^2+a^2}}$.

解: 令 $x=\frac{1}{t}$, 得

$$\text{原式} = -\int \frac{t}{\sqrt{a^2t^2+1}}\,\mathrm{d}\,t$$

$$= -\frac{1}{2a^2}\int \frac{\mathrm{d}\,(a^2t^2+1)}{\sqrt{a^2t^2+1}} = -\frac{1}{a^2}\sqrt{a^2t^2+1}+C$$

$$= -\frac{\sqrt{x^2+a^2}}{a^2x}+C$$

备用题 1. 求下列积分:

1) $$\int x^2 \frac{1}{\sqrt{x^3+1}}\,\mathrm{d}x = \frac{1}{3}\int \frac{1}{\sqrt{x^3+1}}\,\mathrm{d}(x^3+1)$$

$$= \frac{2}{3}\sqrt{x^3+1}+C$$

2) $$\int \frac{2x+3}{\sqrt{1+2x-x^2}}\,\mathrm{d}x = \int \frac{-(2-2x)+5}{\sqrt{1+2x-x^2}}\,\mathrm{d}x$$

$$= -\int \frac{\mathrm{d}(1+2x-x^2)}{\sqrt{1+2x-x^2}} + 5\int \frac{\mathrm{d}(x-1)}{\sqrt{2-(x-1)^2}}$$

$$= -2\sqrt{1+2x-x^2} + 5\arcsin\frac{x-1}{\sqrt{2}} + C$$