§2 换元积分法

1. 第一类换元积分法

若 $F'(u)=f(u)$, 则 $\int f(u)\mathrm{d}u=F(u)+C.$

对复合函数 $y=F(u(x))$, 链式法则为

$$\frac{d}{dx}F(u(x))=F'(u)u'(x)=f(u(x))u'(x),$$

所以 $$\int f(u(x))u'(x)\mathrm{d}x=F(u(x))+C.$$

实际解题时, $$\int\varphi(x)\mathrm{d}x=\cdots=\int f(u(x))u'(x)\mathrm{d}x=\int f(u(x))\mathrm{d}u(x)$$
$$=\int f(u)\mathrm{d}u=F(u)+C=F(u(x))+C.$$

这个方法也称“凑微分法”， 凑出“复合函数”.

积分举例

例1 计算 $\int \frac{1}{3+2x}\mathrm{d}x.$

解 令 $u=3+2x,$ 则 $\mathrm{d}u=2\mathrm{d}x,$

$$\int \frac{1}{3+2x}\mathrm{d}x=\frac{1}{2}\int \frac{1}{3+2x}(3+2x)'\mathrm{d}x$$

$$=\frac{1}{2}\int \frac{1}{u}\mathrm{d}u$$

$$=\frac{1}{2}\ln|u|+C$$

$$=\frac{1}{2}\ln|3+2x|+C.$$

积分举例

例2 计算 $\int 2x\mathrm{e}^{x^2+1}\mathrm{d}x.$

解 令 $u=x^2+1,$ 则 $\mathrm{d}u=2x\mathrm{d}x,$

$$\begin{aligned}\int 2x\mathrm{e}^{x^2+1}\mathrm{d}x &= \int \mathrm{e}^{x^2+1}(x^2+1)'\mathrm{d}x\\ &= \int \mathrm{e}^{x^2+1}\mathrm{d}(x^2+1)\\ &= \int \mathrm{e}^{u}\,\mathrm{d}u\\ &= \mathrm{e}^{u}+C=\mathrm{e}^{x^2+1}+C.\end{aligned}$$

在运用第一类换元积分法比较熟练后，变量 u 就不再写出来，直接计算.

积分举例

例3 计算 $\int \tan x \mathrm{d}x.$

解 $$\int \tan x \mathrm{d}x = \int \frac{\sin x}{\cos x} \mathrm{d}x = -\int \frac{\mathrm{d}\cos x}{\cos x}$$

$$= -\ln|\cos x| + C.$$

同理

$$\int \cot x \mathrm{d}x = \int \frac{\cos x}{\sin x} \mathrm{d}x = \int \frac{\mathrm{d}\sin x}{\sin x}$$

$$= \ln|\sin x| + C.$$

积分举例

例4 计算 $\int \frac{1}{(x-a)^m}\mathrm{d}x$ (m 为正整数).

解 当 $m=1$ 时,

$$\int \frac{1}{x-a}\mathrm{d}x=\int \frac{1}{x-a}\mathrm{d}(x-a)=\ln|x-a|+C.$$

当 $m>1$ 时,

$$\int \frac{1}{(x-a)^m}\mathrm{d}x=\int (x-a)^{-m}\mathrm{d}(x-a)$$

$$=\frac{1}{1-m}(x-a)^{1-m}+C.$$

积分举例

例5 计算 $\int \frac{\mathrm{d}x}{a^2+x^2} \quad (a>0).$

解 $\int \frac{\mathrm{d}x}{a^2+x^2} = \frac{1}{a}\int \frac{1}{1+(\frac{x}{a})^2}\mathrm{d}(\frac{x}{a}) = \frac{1}{a}\arctan\frac{x}{a}+C.$

例6 计算 $\int \frac{\mathrm{d}x}{\sqrt{a^2-x^2}} \quad (a>0).$

解 $\int \frac{\mathrm{d}x}{\sqrt{a^2-x^2}} = \int \frac{\mathrm{d}(\frac{x}{a})}{\sqrt{1-(\frac{x}{a})^2}} = \arcsin\frac{x}{a}+C.$

积分举例

例7 计算 $\int \frac{x^3}{\sqrt{1+x^2}}\mathrm{d}x.$

解 $\int \frac{x^3}{\sqrt{1+x^2}}\mathrm{d}x = \frac{1}{2}\int \frac{x^2}{\sqrt{1+x^2}}\mathrm{d}(1+x^2)$

$$= \frac{1}{2}\int \frac{1+x^2-1}{\sqrt{1+x^2}}\mathrm{d}(1+x^2)$$

$$= \frac{1}{2}\int [(1+x^2)^{\frac{1}{2}} - (1+x^2)^{-\frac{1}{2}}]d(1+x^2)$$

$$= \frac{1}{2}[\frac{2}{3}(1+x^2)^{\frac{3}{2}} - 2(1+x^2)^{\frac{1}{2}}] + C$$

$$= \frac{1}{3}(1+x^2)^{\frac{3}{2}} - (1+x^2)^{\frac{1}{2}} + C.$$

积分举例

例8 计算 $\int \frac{\mathrm{d}x}{a^2 - x^2} (a > 0).$

解 $\int \frac{\mathrm{d}x}{a^2 - x^2} = \frac{1}{2a} \int (\frac{1}{a + x} + \frac{1}{a - x}) \mathrm{d}x$

$$= \frac{1}{2a} [\int \frac{1}{a + x} \mathrm{d}(a + x) - \int \frac{1}{a - x} \mathrm{d}(a - x)]$$

$$= \frac{1}{2a} [\ln |a + x| - \ln |a - x|] + C$$

$$= \frac{1}{2a} \ln |\frac{a + x}{a - x}| + C.$$

积分举例

例9 计算 $\int \sec x \mathrm{d}x.$

解 $$\int \sec x \mathrm{d}x = \int \frac{\cos x}{\cos^2 x}\mathrm{d}x = \int \frac{\mathrm{d}(\sin x)}{1-\sin^2 x}$$

$$= \frac{1}{2}\ln\left|\frac{1+\sin x}{1-\sin x}\right| + C$$

$$= \frac{1}{2}\ln\left|\frac{(1+\sin x)^2}{1-\sin^2 x}\right| + C = \ln\left|\frac{1+\sin x}{\cos x}\right| + C$$

$$= \ln|\sec x + \tan x| + C.$$

同理 $$\int \csc x \mathrm{d}x = \ln|\csc x - \cot x| + C.$$

积分举例

例10 计算 $\int \cos^2 x\mathrm{d}x.$

解 $$\int \cos^2 x\mathrm{d}x = \frac{1}{2}\int (1+\cos 2x)\mathrm{d}x = \frac{1}{2}\int \mathrm{d}x + \frac{1}{4}\int \cos 2x\mathrm{d}(2x)$$

$$= \frac{1}{2}x + \frac{1}{4}\sin 2x + C.$$

例11 计算 $\int \cos^3 x\mathrm{d}x.$

解 $$\int \cos^3 x\mathrm{d}x = \int \cos^2 x\mathrm{d}\sin x = \int (1-\sin^2 x)\mathrm{d}\sin x$$

$$= \sin x - \frac{1}{3}\sin^3 x + C.$$

积分举例

例12 计算 $\int \sin 3x \sin 2x \mathrm{d}x.$

解 $$\int \sin 3x \sin 2x \mathrm{d}x = -\int \frac{1}{2}(\cos 5x - \cos x)\mathrm{d}x$$

$$= -\frac{1}{10}\int \cos 5x \mathrm{d}(5x) + \frac{1}{2}\int \cos x \mathrm{d}x$$

$$= -\frac{1}{10}\sin 5x + \frac{1}{2}\sin x + C.$$

例13 计算 $\int \frac{1}{x}\ln x \mathrm{d}x.$

解 $$\int \frac{1}{x}\ln x \mathrm{d}x = \int \ln x \mathrm{d}\ln x = \frac{1}{2}(\ln x)^2 + C.$$

积分举例

例14　计算 $\int \frac{3}{x^2-4x+5}\mathrm{d}x.$

解　$$\int \frac{3}{x^2-4x+5}\mathrm{d}x = \int \frac{3\mathrm{d}(x-2)}{(x-2)^2+1}$$

$$= 3\arctan(x-2)+C.$$

例15　计算 $\int \frac{x-2}{x^2-4x+5}\mathrm{d}x.$

解　$$\int \frac{x-2}{x^2-4x+5}\mathrm{d}x = \int \frac{\frac{1}{2}(2x-4)}{x^2-4x+5}\mathrm{d}x = \frac{1}{2}\int \frac{\mathrm{d}(x^2-4x+5)}{x^2-4x+5}$$

$$= \frac{1}{2}\ln|x^2-4x+5|+C.$$

积分举例

例16 计算 $\int \frac{6x+1}{x^2-4x+5}\mathrm{d}x.$

解 $\int \frac{6x+1}{x^2-4x+5}\mathrm{d}x = \int \frac{3(2x-4)+13}{x^2-4x+5}\mathrm{d}x$

$$= 3\int \frac{2x-4}{x^2-4x+5}\mathrm{d}x + 1\int \frac{1}{x^2-4x+5}3\mathrm{d}x$$

$$= 3\int \frac{\mathrm{d}(x^2-4x+5)}{x^2-4x+5} + 13\int \frac{\mathrm{d}(x-2)}{1+(x-2)^2}$$

$$= 3\ln|x^2-4x+5| + 13\arctan(x-2) + C.$$

积分举例

例17 计算 $\int \frac{\ln(x+\sqrt{1+x^2})}{\sqrt{1+x^2}}\mathrm{d}x.$

解 $\int \frac{\ln(x+\sqrt{1+x^2})}{\sqrt{1+x^2}}\mathrm{d}x = \int \ln(x+\sqrt{1+x^2})d\ln(x+\sqrt{1+x^2})$

$$= \frac{1}{2}\ln^2(x+\sqrt{1+x^2})+C.$$

例18 计算 $\int \frac{x}{x-\sqrt{x^2-1}}\mathrm{d}x.$

解 $\int \frac{x}{x-\sqrt{x^2-1}}\mathrm{d}x = \int x(x+\sqrt{x^2-1})\mathrm{d}x = \int (x^2+x\sqrt{x^2-1})\mathrm{d}x$

$$= \frac{1}{3}x^3+\frac{1}{2}\int (x^2-1)^{\frac{1}{2}}\mathrm{d}(x^2-1) = \frac{1}{3}x^3+\frac{1}{3}(x^2-1)^{\frac{3}{2}}+C.$$

常见的凑微分

1 $\int f(ax+b)\mathrm{d}x=\frac{1}{a}\int f(ax+b)\mathrm{d}(ax+b).$

2 $\int f(ax^n+b)x^{n-1}\mathrm{d}x=\frac{1}{na}\int f(ax^n+b)\mathrm{d}(ax^n+b).$

3. $\int f(\mathrm{e}^x)\mathrm{e}^x\,\mathrm{d}x=\int f(\mathrm{e}^x)d(\mathrm{e}^x).$

4 $\int f(\frac{1}{x})\frac{1}{x^2}\mathrm{d}x=-\int f(\frac{1}{x})\mathrm{d}(\frac{1}{x}).$

5. $\int f(\ln x)\frac{\mathrm{d}x}{x}=\int f(\ln x)\mathrm{d}(\ln x).$

6 $\int f(\sqrt{x})\frac{\mathrm{d}x}{\sqrt{x}}=2\int f(\sqrt{x})\mathrm{d}(\sqrt{x}).$

常见的凑微分

7. $\int f(\sin x)\cos x\mathrm{d}x = \int f(\sin x)\mathrm{d}(\sin x).$

8. $\int f(\cos x)\sin x\mathrm{d}x = -\int f(\cos x)\mathrm{d}(\sin x).$

9. $\int f(\tan x)\sec^2 x\mathrm{d}x = \int f(\tan x)\mathrm{d}(\tan x).$

10. $\int f(\cot x)\csc^2 x\mathrm{d}x = -\int f(\cot x)\mathrm{d}(\cot x).$

11. $\int \frac{f(\arcsin x)}{\sqrt{1-x^2}}\mathrm{d}x = \int f(\arcsin x)\mathrm{d}(\arcsin x).$

12. $\int \frac{f(\arctan x)}{1+x^2}\mathrm{d}x = \int f(\arctan x)\mathrm{d}(\arctan x).$

积分举例

配合四则运算、代数恒等变形、三角恒等变形等积分.

例19 计算 $\int \frac{\mathrm{d}x}{\sqrt{x(1-x)}}.$

解 $$\int \frac{\mathrm{d}x}{\sqrt{x(1-x)}} = \int \frac{2\mathrm{d}\sqrt{x}}{\sqrt{1-\left(\sqrt{x}\right)^2}} = 2\arcsin\sqrt{x} + C.$$

例20 计算 $\int \mathrm{e}^{\mathrm{e}^x + x}\mathrm{d}x.$

解 $$\int e^{e^x+x}\mathrm{d}x = \int e^{e^x}\cdot e^x\mathrm{d}x = \int e^{e^x}\mathrm{d}e^x = e^{e^x} + C.$$

形式较复杂的不定积分

例21 计算 $\int (x\ln x)^{\frac{3}{2}}(\ln x+1)\mathrm{d}x.$

解 $$\int (x\ln x)^{\frac{3}{2}}(\ln x+1)\mathrm{d}x=\int (x\ln x)^{\frac{3}{2}}\mathrm{d}(x\ln x)$$

$$=\frac{2}{5}(x\ln x)^{\frac{5}{2}}+C.$$

例24 计算 $\int \frac{\arctan\frac{1}{x}}{1+x^2}\mathrm{d}x.$

解 $$\int \frac{\arctan\frac{1}{x}}{1+x^2}\mathrm{d}x=-\int \arctan\frac{1}{x}\mathrm{d}(\arctan\frac{1}{x})=-\frac{1}{2}\arctan^2\frac{1}{x}+C.$$

积分举例

例23 计算 $\int \frac{\sqrt{1+4\arctan x}}{1+x^2}\mathrm{d}x.$

解 $\int \frac{\sqrt{1+4\arctan x}}{1+x^2}\mathrm{d}x = \int \sqrt{1+4\arctan x}\,\mathrm{d}\arctan x$

$$= \frac{1}{4}\int (1+4\arctan x)^{\frac{1}{2}}\mathrm{d}(1+4\arctan x) = \frac{1}{6}(1+\arctan x)^{\frac{3}{2}} + C.$$

例22 计算 $\int \frac{\cos 2x}{1+\sin x\cos x}\mathrm{d}x.$

解 $\int \frac{\cos 2x}{1+\sin x\cos x}\mathrm{d}x = \int \frac{\frac{1}{2}\mathrm{d}(\sin 2x)}{1+\frac{1}{2}\sin 2x} = \int \frac{\mathrm{d}(2+\sin 2x)}{2+\sin 2x}$

$$= \ln(2+\sin 2x) + C.$$

形式较复杂的不定积分

例25 计算 $\int (1-\frac{1}{x^2})e^{x+\frac{1}{x}}\mathrm{d}x.$

解 $\int \left(1-\frac{1}{x^2}\right)e^{x+\frac{1}{x}}\mathrm{d}x = \int e^{x+\frac{1}{x}}\mathrm{d}\left(x+\frac{1}{x}\right) = e^{x+\frac{1}{x}}+C.$

例26 计算 $\int e^{e^x\cos x}(\cos x-\sin x)e^x\mathrm{d}x.$

解 $\int e^{e^x\cos x}(\cos x-\sin x)e^x\mathrm{d}x = \int e^{e^x\cos x}\mathrm{d}(e^x\cos x)$

$= e^{e^x\cos x}+C.$

通过拆项凑微分

例28　计算 $\int \frac{e^{2x}}{1+e^{x}}\mathrm{d}x.$

解 $\int \frac{e^{2x}}{1+e^{x}}\mathrm{d}x = \int \frac{e^{x}}{1+e^{x}}\mathrm{d}e^{x} = \int \frac{e^{x}+1-1}{1+e^{x}}\mathrm{d}e^{x}$

$$= \int \mathrm{d}e^{x} - \int \frac{1}{1+e^{x}}\mathrm{d}(1+e^{x})$$

$$= \mathrm{e}^{x} - \ln(1+e^{x}) + C.$$

2. 第二类换元积分法

对不定积分 $\int f(x)\mathrm{d}x,$ 作可导变换 $x=\varphi(t),$ 使得它的反函数 $t=\varphi^{-1}(x)$ 存在，且 $f(\varphi(t))\varphi'(t)$ 有原函数 $F(t),$ 则

$$\int f(x)\mathrm{d}x=\int f(\varphi(t))\varphi'(t)\mathrm{d}\,t=F(t)+C=F(\varphi^{-1}(x))+C.$$

这个积分方法称为第二类换元积分法.

只要对这个公式求导, 用复合函数和反函数求导公式就可得到结果.

这个方法的困难之处是反函数表示式 $t=\varphi^{-1}(x)$ 有时不容易求.

积分举例

例29 计算 $\int \sqrt{a^2 - x^2}\,\mathrm{d}x \ (a > 0).$

解 令 $x = a\sin t, |t| < \frac{\pi}{2}$, 则 $\sqrt{a^2 - x^2} = a|\cos t| = a\cos t,$

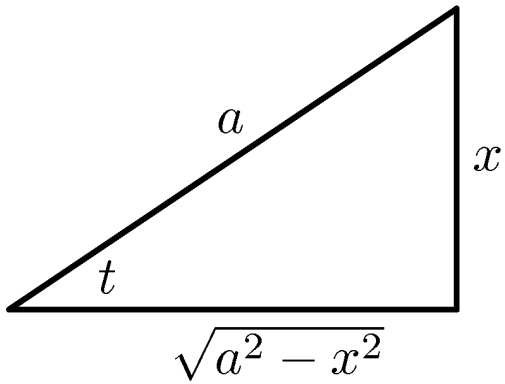
$\mathrm{d}x = a\cos t\mathrm{d}t.$

$$\int \sqrt{a^2 - x^2}\,\mathrm{d}x = \int a\cos t a\cos t\mathrm{d}t = a^2 \int \cos^2 t\mathrm{d}t$$

$$= \frac{a^2}{2}\int (1+\cos 2t)dt = \frac{1}{2}a^2(t + \frac{1}{2}\sin 2t) + C$$

$$= \frac{a^2}{2}(t + \sin t\cos t) + C$$

$$= \frac{a^2}{2}\arcsin\frac{x}{a} + \frac{1}{2}x\sqrt{a^2 - x^2} + C.$$



积分举例

例30 计算 $\int \frac{\mathrm{d}x}{\sqrt{x^2+a^2}} (a>0).$

解 令 $x = a\tan t, |t| < \frac{\pi}{2}$, 则 $\sqrt{x^2+a^2} = a|\sec t| = a\sec t,$

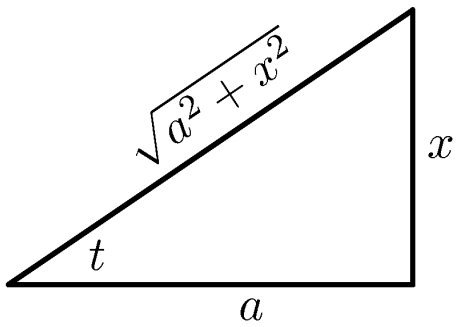
$\mathrm{d}x = a\sec^2 t\mathrm{d}t.$

$$\int \frac{\mathrm{d}x}{\sqrt{x^2+a^2}} = \int \frac{a\sec^2 t}{a\sec t}\mathrm{d}t = \int \sec t\mathrm{d}t$$

$$= \ln|\sec t + \tan t| + C$$

$$= \ln\left|\frac{\sqrt{x^2+a^2}}{a} + \frac{x}{a}\right| + C'$$

$$= \ln(x + \sqrt{x^2+a^2}) + C.$$



积分举例

例31 计算 $\int \frac{\mathrm{d}x}{\sqrt{x^2-a^2}}\ (a>0).$

解 当 $x>a$ 时, 令 $x=a\sec t, 0<t<\frac{\pi}{2}$, 则

$$\sqrt{x^2-a^2}=a\tan t, \qquad \mathrm{d}x=a\sec t\tan t\mathrm{d}t.$$

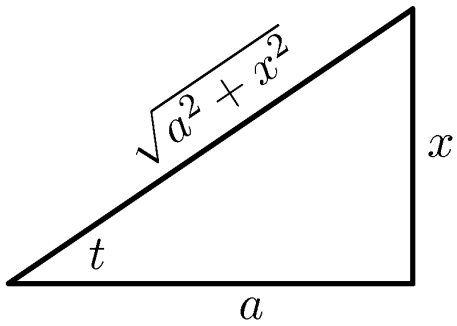
$$\int \frac{\mathrm{d}x}{\sqrt{x^2-a^2}}=\int \frac{a\sec t\tan t}{a\tan t}\mathrm{d}t=\int \sec t\mathrm{d}t$$

$$=\ln|\sec t+\tan t|+C$$

$$=\ln\left|\frac{x}{a}+\frac{\sqrt{x^2-a^2}}{a}\right|+C'$$

$$=\ln(x+\sqrt{x^2-a^2})+C.$$

同理可得 $x<-a$ 的情形.



积分举例

例32 计算 $\int \frac{\mathrm{d}x}{\sqrt{x}+\sqrt[3]{x}}.$

解 $\int \frac{\mathrm{d}x}{\sqrt{x}+\sqrt[3]{x}} \overset{t=\sqrt[6]{x}}{=} \int \frac{6t^5}{t^3+t^2}\mathrm{d}t = 6\int \frac{t^3}{t+1}\mathrm{d}t = 6\int \frac{t^3+1-1}{t+1}\mathrm{d}t$

$$= 6\int (t^2 - t + 1 - \frac{1}{t+1})\mathrm{d}t$$

$$= 6(\frac{t^3}{3} - \frac{t^2}{2} + t - \ln|t+1|) + C$$

$$= 2\sqrt{x} - 3\sqrt[3]{x} + 6\sqrt[6]{x} - 6\ln\sqrt[6]{x} + (1) + C.$$

积分举例

例33　计算 $\int \frac{x\mathrm{d}x}{\sqrt{x-3}}.$

解 $\int \frac{x\mathrm{d}x}{\sqrt{x-3}} \overset{t=\sqrt{x-3}}{=} \int \frac{t^2+3}{t} 2t\mathrm{d}t$

$$= 2\int (t^2+3)\mathrm{d}t$$

$$= 2\left(\frac{t^3}{3}+3t\right)+C$$

$$= \frac{2}{3}(x+6)\sqrt{x-3}+C.$$

积分举例—倒数变换

例34 计算 $\int \frac{\sqrt{a^2-x^2}}{x^4}dx\ (a>0).$

解 $\int \frac{\sqrt{a^2-x^2}}{x^4}dx \overset{t=\frac{1}{x}}{=} \int \frac{\sqrt{a^2-(\frac{1}{t})^2}}{(\frac{1}{t})^4}(-\frac{1}{t^2})\mathrm{d}t$

$$=-\int \sqrt{a^2t^2-1}tdt=-\frac{1}{2a^2}\int (a^2t^2-1)^{\frac{1}{2}}d(a^2t^2-1)$$

$$=-\frac{1}{3a^2}(a^2t^2-1)^{\frac{3}{2}}+C$$

$$=-\frac{1}{3a^2x^2}(a^2-x^2)^{\frac{3}{2}}+C.$$

积分举例

例35 计算 $\int \sqrt{1+e^x}\,dx.$

解 $\int \sqrt{1+e^x}\,dx. \overset{t=\sqrt{1+e^x}}{=} \int t\frac{2t}{t^2-1}\mathrm{d}t$

$$= 2\int \frac{t^2-1+1}{t^2-1}\mathrm{d}t = 2\int (1-\frac{1}{1-t^2})\mathrm{d}t$$

$$= 2\left(t-\frac{1}{2}\ln\left|\frac{1+t}{1-t}\right|\right)+C$$

$$= 2\sqrt{1+e^x} - \ln\left|\frac{1+\sqrt{1+e^x}}{1-\sqrt{1+e^x}}\right| + C.$$

基本积分公式

15. $\int \tan x \mathrm{d}x = -\ln|\cos x| + C.$

16. $\int \cot x \mathrm{d}x = \ln|\sin x| + C.$

17. $\int \sec x \mathrm{d}x = \ln|\sec x + \tan x| + C.$

18. $\int \csc x \mathrm{d}x = \ln|\csc x - \cot x| + C.$

19. $\int \frac{1}{a^2 + x^2} \mathrm{d}x = \frac{1}{a} \arctan \frac{x}{a} + C.$

20. $\int \frac{1}{a^2 - x^2} \mathrm{d}x = \frac{1}{2a} \ln\left|\frac{a + x}{a - x}\right| + C.$

基本积分公式

21. $\int \frac{1}{\sqrt{a^2 - x^2}} \mathrm{d}x = \arcsin \frac{x}{a} + C..$

22. $\int \frac{1}{\sqrt{x^2 \pm a^2}} \mathrm{d}x = \ln | x + \sqrt{x^2 \pm a^2} + C.$

23. $\int \sqrt{a^2 - x^2} \mathrm{d}x = \frac{1}{2} x \sqrt{a^2 - x^2} + \frac{a^2}{2} \arcsin \frac{x}{a} + C.$

24. $\int \sqrt{x^2 \pm a^2} \mathrm{d}x = \frac{1}{2} x \sqrt{x^2 \pm a^2} \pm \frac{a^2}{2} \ln | x + \sqrt{x^2 \pm a^2} | + C.$