内容小结

1. 隐函数(组) 存在定理

2. 隐函数 (组) 求导方法

   方法1. 代公式

   方法2. 利用复合函数求导法则直接计算 ;

   方法3. 利用微分形式不变性 ;

定理3. 设函数 $F(x,y,u,v)$, $G(x,y,u,v)$ 满足:

① 在点 $P(x_0,y_0,u_0,v_0)$ 的某一邻域内具有连续偏导数;

② $F(x_0,y_0,u_0,v_0)=0$, $G(x_0,y_0,u_0,v_0)=0$;

③ $J\Big|_P=\left.\dfrac{\partial(F,G)}{\partial(u,v)}\right|_P\neq 0$

则方程组 $F(x,y,u,v)=0$, $G(x,y,u,v)=0$ 在点 (x_0,y_0) 的某一邻域内可唯一确定一组满足条件 $u_0=u(x_0,y_0)$, $v_0=v(x_0,y_0)$ 的单值连续函数 $u=u(x,y)$, $v=v(x,y)$, 且有偏导数公式 :

$$\frac{\partial u}{\partial x}=-\frac{1}{J}\frac{\partial(F,G)}{\partial(\underline{x},v)}=-\frac{1}{\begin{vmatrix}F_u & F_v\\ G_u & G_v\end{vmatrix}}\begin{vmatrix}F_x & F_v\\ G_x & G_v\end{vmatrix}$$

$$\frac{\partial u}{\partial y}=-\frac{1}{J}\frac{\partial(F,G)}{\partial(\underline{y},v)}=-\frac{1}{\begin{vmatrix}F_u & F_v\\ G_u & G_v\end{vmatrix}}\begin{vmatrix}F_y & F_v\\ G_y & G_v\end{vmatrix}$$

$$\frac{\partial v}{\partial x}=-\frac{1}{J}\frac{\partial(F,G)}{\partial(u,\underline{x})}=-\frac{1}{\begin{vmatrix}F_u & F_v\\ G_u & G_v\end{vmatrix}}\begin{vmatrix}F_u & F_x\\ G_u & G_x\end{vmatrix}$$

定理证明略.
仅推导偏导
数公式如下:

$$\frac{\partial v}{\partial y}=-\frac{1}{J}\frac{\partial(F,G)}{\partial(u,\underline{y})}=-\frac{1}{\begin{vmatrix}F_u & F_v\\ G_u & G_v\end{vmatrix}}\begin{vmatrix}F_u & F_y\\ G_u & G_y\end{vmatrix}$$

设方程组 $\begin{cases} F(x,y,u,v)=0 \\ G(x,y,u,v)=0 \end{cases}$ 有隐函数组 $\begin{cases} u=u(x,y) \\ v=v(x,y) \end{cases}$，则

$$\begin{cases} F(x,y,u(x,y),v(x,y))\equiv 0 \\ G(x,y,u(x,y),v(x,y))\equiv 0 \end{cases}$$

两边对 x 求导得 $\begin{cases} F_x+F_u\cdot\dfrac{\partial u}{\partial x}+F_v\cdot\dfrac{\partial v}{\partial x}=0 \\ G_x+G_u\cdot\dfrac{\partial u}{\partial x}+G_v\cdot\dfrac{\partial v}{\partial x}=0 \end{cases}$

这是关于 $\dfrac{\partial u}{\partial x},\dfrac{\partial v}{\partial x}$ 的线性方程组, 在点P 的某邻域内

系数行列式 $J=\begin{vmatrix} F_u & F_v \\ G_u & G_v \end{vmatrix}\neq 0$, 故得

$$\frac{\partial u}{\partial x} = -\frac{1}{J}\frac{\partial(F,G)}{\partial(x,v)}$$

$$\frac{\partial v}{\partial x} = -\frac{1}{J}\frac{\partial(F,G)}{\partial(u,x)}$$

同样可得

$$\frac{\partial u}{\partial y} = -\frac{1}{J}\frac{\partial(F,G)}{\partial(y,v)}$$

$$\frac{\partial v}{\partial y} = -\frac{1}{J}\frac{\partial(F,G)}{\partial(u,y)}$$

例4. 设 $xu - yv = 0,\ yu + xv = 1$, 求 $\dfrac{\partial u}{\partial x}, \dfrac{\partial u}{\partial y}, \dfrac{\partial v}{\partial x}, \dfrac{\partial v}{\partial y}$.

解: 方程组两边对 x 求导，并移项得

$$\begin{cases} x\dfrac{\partial u}{\partial x} - y\dfrac{\partial v}{\partial x} = -u \\ y\dfrac{\partial u}{\partial x} + x\dfrac{\partial v}{\partial x} = -v \end{cases}$$

由题设 $J = \begin{vmatrix} x & -y \\ y & x \end{vmatrix} = x^2 + y^2 \neq 0$

故有 $$\begin{cases} \dfrac{\partial u}{\partial x} = \dfrac{1}{J}\begin{vmatrix} -u & -y \\ -v & x \end{vmatrix} = -\dfrac{xu + yv}{x^2 + y^2} \\ \dfrac{\partial v}{\partial x} = \dfrac{1}{J}\begin{vmatrix} x & -u \\ y & -v \end{vmatrix} = -\dfrac{xv - yu}{x^2 + y^2} \end{cases}$$

练习: 求 $\dfrac{\partial u}{\partial y}, \dfrac{\partial v}{\partial y}$

答案:

$$\begin{cases} \dfrac{\partial u}{\partial y} = -\dfrac{yu - xv}{x^2 + y^2} \\ \dfrac{\partial v}{\partial y} = -\dfrac{xu + yv}{x^2 + y^2} \end{cases}$$

例. 设 $y = y(x), z = z(x)$ 是由方程 $z = x f(x + y)$ 和 $F(x, y, z) = 0$ 所确定的函数，求 $\frac{\mathrm{d}z}{\mathrm{d}x}$. **(99考研)**

解法一： $\begin{cases} F(x, y, z) = 0 \\ G(x, y, z) = z - xf(x + y) = 0 \end{cases}$

$$\therefore \quad \frac{\mathrm{d}z}{\mathrm{d}x} = -\frac{\partial(F, G)}{\partial(y, x)} \Big/ \frac{\partial(F, G)}{\partial(y, z)}$$

$$-\frac{\begin{vmatrix} F_y & F_x \\ -xf' & -f - xf' \end{vmatrix}}{\begin{vmatrix} F_y & F_z \\ -xf' & 1 \end{vmatrix}} = \frac{(f + x f')F_y - x f' \cdot F_x}{F_y + x f' \cdot F_z}$$

解法二：　微分法.

$$z = xf(x+y),\ \ F(x,y,z)=0$$

对各方程两边分别求微分:

$$\begin{cases} \mathrm{d}z = f\,\mathrm{d}x + xf'\cdot(\mathrm{d}x+\mathrm{d}y) \\ F_1\mathrm{d}x + F_2\mathrm{d}y + F_3\mathrm{d}z = 0 \end{cases}$$

化简得

$$\begin{cases} (f+xf')\,\mathrm{d}x + xf'\mathrm{d}y - \mathrm{d}z = 0 \\ F_1\,\mathrm{d}x\ \ +F_2\,\mathrm{d}y\ \ +F_3\mathrm{d}z = 0 \end{cases}$$

消去$\mathrm{d}y$可得 $\dfrac{\mathrm{d}z}{\mathrm{d}x}$.

例5. 设函数 $x=x(u,v)$, $y=y(u,v)$在点(u,v) 的某一邻域内有连续的偏导数, 且 $\dfrac{\partial(x,y)}{\partial(u,v)}\neq 0$

1) 证明函数组 $\begin{cases} x=x(u,v) \\ y=y(u,v) \end{cases}$ 在与点 (u,v) 对应的点 (x,y) 的某一邻域内唯一确定一组单值、连续且具有连续偏导数的反函数 $u=u(x,y)$, $v=v(x,y)$.

2) 求 $u=u(x,y)$, $v=v(x,y)$对 x, y 的偏导数并证明

$$\frac{\partial(u,v)}{\partial(x,y)}\cdot\frac{\partial(x,y)}{\partial(u,v)}=1$$

解： 1) 令 $F(x,y,u,v)\equiv x-x(u,v)=0$

$$G(x,y,u,v)\equiv y-y(u,v)=0$$

则有 $$J=\frac{\partial(F,G)}{\partial(u,v)}=\frac{\partial(x,y)}{\partial(u,v)}\neq 0,$$

由定理 3 可知结论 1) 成立.

2) 求反函数的偏导数.

$$\frac{\partial u}{\partial x}=-\frac{1}{J}\frac{\partial(F,G)}{\partial(x,v)}=-\frac{1}{J}\begin{vmatrix}1 & -\frac{\partial x}{\partial v}\\ 0 & -\frac{\partial y}{\partial v}\end{vmatrix}=\frac{1}{J}\frac{\partial y}{\partial v}$$

$$\frac{\partial u}{\partial y}=-\frac{1}{J}\frac{\partial(F,G)}{\partial(y,v)}=-\frac{1}{J}\begin{vmatrix}0 & -\frac{\partial x}{\partial v}\\ 1 & -\frac{\partial y}{\partial v}\end{vmatrix}=-\frac{1}{J}\frac{\partial x}{\partial v}$$

同理，

$$\frac{\partial v}{\partial x}=-\frac{1}{J}\frac{\partial y}{\partial u},\frac{\partial v}{\partial y}=\frac{1}{J}\frac{\partial x}{\partial u}$$

$$\frac{\partial(u,v)}{\partial(x,y)}=\begin{vmatrix}\frac{1}{J}\frac{\partial y}{\partial v} & -\frac{1}{J}\frac{\partial x}{\partial v}\\ -\frac{1}{J}\frac{\partial y}{\partial u} & \frac{1}{J}\frac{\partial x}{\partial u}\end{vmatrix}=\frac{1}{J}$$

$$\therefore \frac{\partial(u,v)}{\partial(x,y)}\cdot\frac{\partial(x,y)}{\partial(u,v)}=1$$

内容小结

2. 隐函数 (组) 求导方法

方法1. 代公式

方法2. 利用复合函数求导法则直接计算；

方法3. 利用微分形式不变性；

思考与练习

设 $z = f(x+y+z, xyz)$，求 $\dfrac{\partial z}{\partial x}, \dfrac{\partial x}{\partial z}, \dfrac{\partial x}{\partial y}$.

解法一: $$z = f(x+y+z, xyz)$$

令$F(x,y,z) = z - f(x+y+z, xyz)$

$$F_x = -f_1 - yzf_2$$

$$F_y = -f_1 - xzf_2$$

$$F_z = 1 - f_1 - xyf_3$$

$$\Longrightarrow \quad \frac{\partial z}{\partial x} = \frac{f_1 + yz\,f_2}{1 - f_1 - xy\,f_2} \qquad \frac{\partial x}{\partial z} = \frac{1 - f_1 - xy\,f_2}{f_1 + yz\,f_2}$$

$$\frac{\partial x}{\partial y} = -\frac{f_1 + xz\,f_2}{f_1 + yz\,f_2}$$

解法二: $z = f(x+y+z, xyz)$

- $\frac{\partial z}{\partial x} = f_1 \cdot (1 + \frac{\partial z}{\partial x}) + f_2 \cdot (yz + xy\frac{\partial z}{\partial x})$

$$\Longrightarrow \frac{\partial z}{\partial x} = \frac{f_1 + yz f_2}{1 - f_1 - xy f_2}$$

- $1 = f_1 \cdot (\frac{\partial x}{\partial z} + 1) + f_2 \cdot (yz\frac{\partial x}{\partial z} + xy)$

$$\Longrightarrow \frac{\partial x}{\partial z} = \frac{1 - f_1 - xy f_2}{f_1 + yz f_2}$$

- $0 = f_1 \cdot (\frac{\partial x}{\partial y} + 1) + f_2 \cdot (yz\frac{\partial x}{\partial y} + xz)$

$$\Longrightarrow \frac{\partial x}{\partial y} = -\frac{f_1 + xz f_2}{f_1 + yz f_2}$$

解法三. 利用全微分形式不变性同时求出各偏导数.

$$z = f(x+y+z,\, xyz)$$

$$\mathrm{d}z = f_1 \cdot (\,\mathrm{d}x + \mathrm{d}y + \mathrm{d}z\,) + f_2 \cdot (\,yz\,\mathrm{d}x + xz\,\mathrm{d}y + xy\,\mathrm{d}z\,)$$

解出 $\mathrm{d}x$:

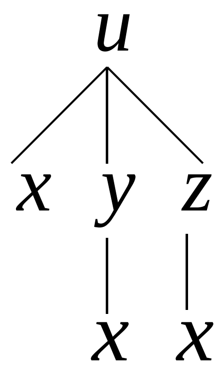
$$\mathrm{d}x = \frac{-(\,f_1 + xz f_2\,)\mathrm{d}y + (\,1 - f_1 - xy f_2\,)\mathrm{d}z}{f_1 + yz f_2}$$

由 $\mathrm{d}y, \mathrm{d}z$ 的系数即可得 $\dfrac{\partial x}{\partial y}, \dfrac{\partial x}{\partial z}$.

备用题 **1.** 设 $u = f(x,y,z)$ 有连续的一阶偏导数，又函数 $y = y(x)$ 及 $z = z(x)$ 分别由下列两式确定：

$\underline{e^{xy} - xy = 2}$，$\underline{e^{x} = \int_0^{x-z} \frac{\sin t}{t}\,\mathrm{d}t}$，求 $\frac{\mathrm{d}u}{\mathrm{d}x}$. **(2001考研)**

解： 两个隐函数方程两边对 x 求导，得



$$\begin{cases} e^{xy}(y + xy') - (y + xy') = 0 \\ e^{x} = \dfrac{\sin(x-z)}{x-z}(1 - z') \end{cases}$$

解得 $$y' = -\frac{y}{x}, \quad z' = 1 - \frac{e^{x}(x-z)}{\sin(x-z)}$$

因此 $$\frac{\mathrm{d}u}{\mathrm{d}x} = f_1 - \frac{y}{x} f_2 + \left[1 - \frac{e^{x}(x-z)}{\sin(x-z)} \right] f_3$$

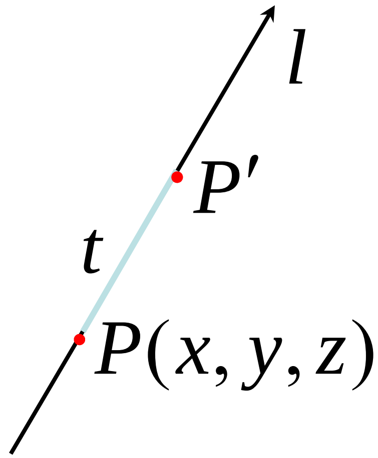
第六节
方向导数与梯度

一、方向导数

二、梯度

一、方向导数

定义: 若函数 $f(x,y,z)$ 在点 $P(x,y,z)$ 处沿方向 l (方向角为 α,β,γ) 存在下列极限:



$$\lim_{t\to 0^+}\frac{\Delta f}{t}$$

$$=\lim_{t\to 0^+}\frac{f(x+t\cos\alpha,y+t\cos\beta,z+t\cos\gamma)-f(x,y,z)}{t}$$

$$\overset{\text{记作}}{=}\frac{\partial f}{\partial l}$$

则称 $\dfrac{\partial f}{\partial l}$ 为函数在点 P 处沿方向 l 的**方向导数**.

对于二元函数 $f(x,y)$, 在点$P(x,y)$处沿方向l (方向角为α, β) 的方向导数定义为

$$\frac{\partial f}{\partial l}=\lim_{t\to 0^+}\frac{f(x+t\cos\alpha, y+t\cos\beta)-f(x,y)}{t}$$

注意方向导数是单侧极限，与偏导数有所区别。

当偏导数存在时,

- 当 l 与 x 轴同向 $(\alpha=0, \beta=\frac{\pi}{2})$时,有 $\frac{\partial f}{\partial l}=\frac{\partial f}{\partial x}$
- 当 l 与 x 轴反向 $(\alpha=\pi, \beta=\frac{\pi}{2})$时,有 $\frac{\partial f}{\partial l}=-\frac{\partial f}{\partial x}$

同样，沿y轴正方向的方向导数为$\frac{\partial f}{\partial y}$,负方向为$-\frac{\partial f}{\partial y}$.

- 方向导数存在 $\underset{\not\Leftarrow}{\not\Rightarrow}$ 偏导数存在

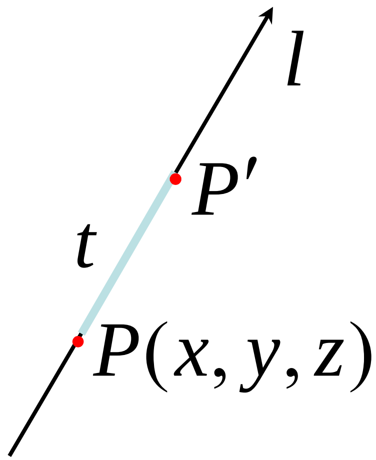
反例(1) $z = \sqrt{x^2 + y^2}$

反例(2) $z = f(x, y) = \begin{cases} \dfrac{xy}{x^2 + y^2}, & x^2 + y^2 \neq 0 \\ 0, & x^2 + y^2 = 0 \end{cases}$

定理: 若函数 $f(x,y,z)$ 在点 $P(x,y,z)$ 处可微，则函数在该点**沿任意方向** l 的方向导数存在，且有

$$\frac{\partial f}{\partial l}=\frac{\partial f}{\partial x}\cos\alpha+\frac{\partial f}{\partial y}\cos\beta+\frac{\partial f}{\partial z}\cos\gamma$$

其中 α,β,γ 为 l 的方向角.



证明: 由函数 $f(x,y,z)$ 在点 P 可微，得

$$\Delta f=\frac{\partial f}{\partial x}\Delta x+\frac{\partial f}{\partial y}\Delta y+\frac{\partial f}{\partial z}\Delta z+o(\rho)$$

$$=t\left(\frac{\partial f}{\partial x}\cos\alpha+\frac{\partial f}{\partial y}\cos\beta+\frac{\partial f}{\partial z}\cos\gamma\right)+o(\rho)$$

故 $$\frac{\partial f}{\partial l}=\lim_{t\to 0^+}\frac{\Delta f}{t}=\frac{\partial f}{\partial x}\cos\alpha+\frac{\partial f}{\partial y}\cos\beta+\frac{\partial f}{\partial z}\cos\gamma$$

类似地，对于二元函数的情形，若函数$f(x,y)$在点$P(x,y)$可微，则在该点处沿方向l(方向角为α,β)的方向导数为

$$\frac{\partial f}{\partial l} = f_x(x,y)\cos\alpha + f_y(x,y)\cos\beta$$

- 可微 $\begin{array}{c}\Longrightarrow\\ \not\Longleftarrow\end{array}$ 方向导数存在

反例 $z = \sqrt{x^2 + y^2}$