Huffman Codes

- Widely used technique for data compression

- Assume the data to be a sequence of characters

- Looking for an effective way of storing the data

Huffman Codes

- Idea:
  - Use the frequencies of occurrence of characters to build an optimal way of representing each character

	a	b	c	d	e	f
Frequency (thousands)	45	13	12	16	9	5

- ***Binary character code***
  - Uniquely represents a character by a binary string

Fixed-Length Codes

E.g.: Data file containing 100,000 characters

	a	b	c	d	e	f
Frequency (thousands)	45	13	12	16	9	5

- 3 bits needed
- a = 000, b = 001, c = 010, d = 011, e = 100, f = 101
- Requires: 100,000 · 3 = 300,000 bits

Variable-Length Codes

E.g.: Data file containing 100,000 characters

	a	b	c	d	e	f
Frequency (thousands)	45	13	12	16	9	5

- Assign short codewords to frequent characters and long codewords to infrequent characters
- a = 0, b = 101, c = 100, d = 111, e = 1101, f = 1100
- $(45 \cdot 1 + 13 \cdot 3 + 12 \cdot 3 + 16 \cdot 3 + 9 \cdot 4 + 5 \cdot 4) \cdot 1{,}000$

  $= 224{,}000$ bits

Prefix Codes

- Prefix codes:
  - Codes for which no codeword is also a prefix of some other codeword
  - Better name would be “prefix-free codes”
- We can achieve optimal data compression using prefix codes
  - We will restrict our attention to prefix codes

Encoding with Binary Character Codes

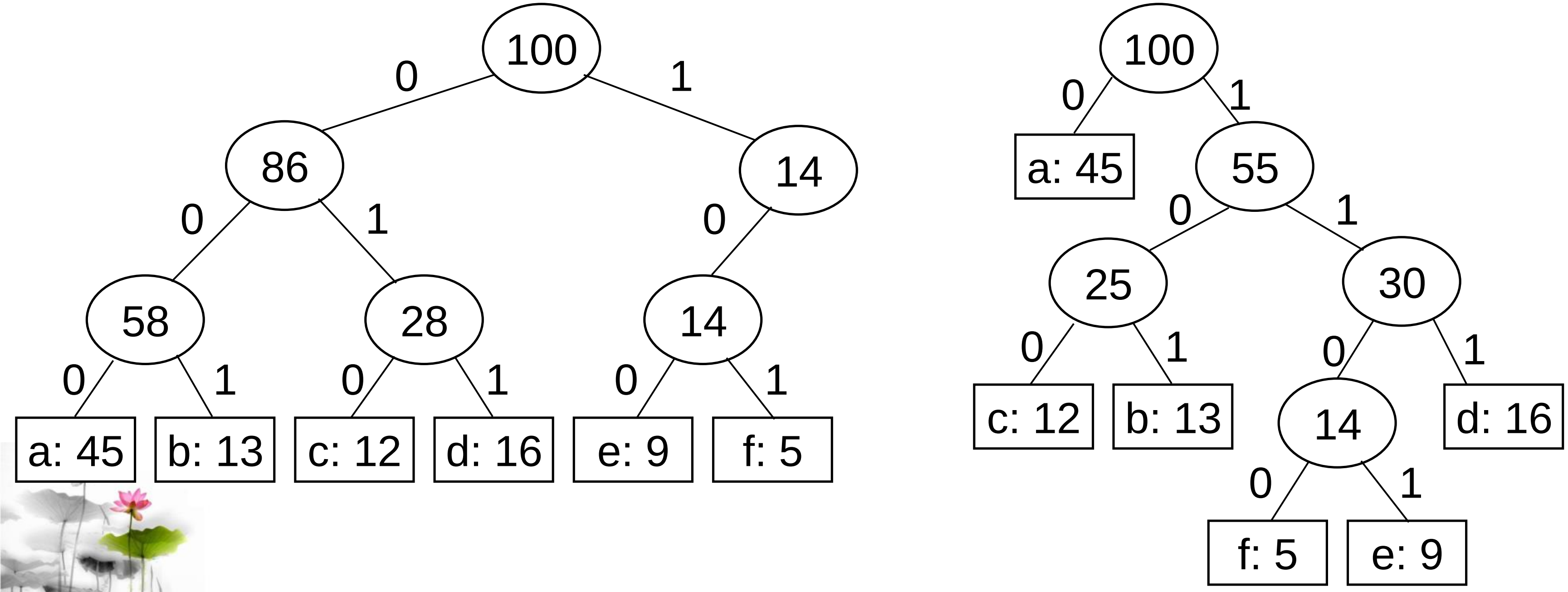
- Encoding
  - Concatenate the codewords representing each character in the file
- *E.g.*:
  - a = 0, b = 101, c = 100, d = 111, e = 1101, f = 1100
  - abc = 0 · 101 · 100 = 0101100

Decoding with Binary Character Codes

- Prefix codes simplify decoding
  - No codeword is a prefix of another $\Rightarrow$ the codeword that begins an encoded file is unambiguous
- Approach
  - Identify the initial codeword
  - Translate it back to the original character
  - Repeat the process on the remainder of the file
- *E.g.*:
  - a = 0, b = 101, c = 100, d = 111, e = 1101, f = 1100
  - 001011101 = $0 \cdot 0 \cdot 101 \cdot 1101$ = aabe

Prefix Code Representation

- Binary tree whose leaves are the given characters
- Binary codeword
  - the path from the root to the character, where 0 means “go to the left child” and 1 means “go to the right child”
- Length of the codeword
  - Length of the path from root to the character leaf (depth of node)



Optimal Codes

- An optimal code is always represented by a **full binary tree**
  - Every non-leaf has two children
  - Fixed-length code is not optimal, variable-length is
- How many bits are required to encode a file?
  - Let C be the alphabet of characters
  - Let $f(c)$ be the frequency of character c
  - Let $d_T(c)$ be the depth of c's leaf in the tree T corresponding to a prefix code

$$B(T) = \sum_{c \in C} f(c) d_T(c)$$

the cost of tree T

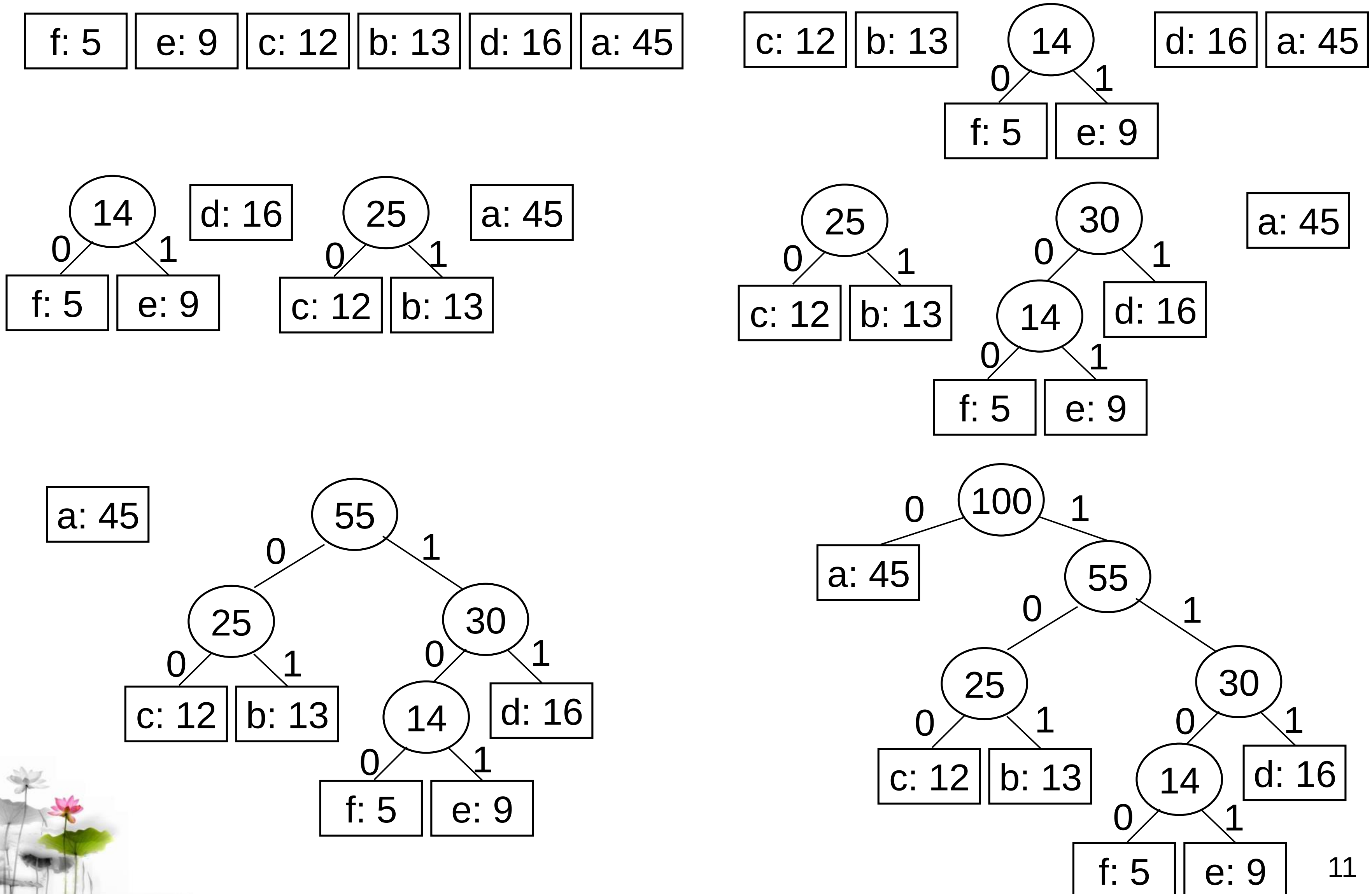
Constructing a Huffman Code

- A greedy algorithm that constructs an optimal prefix code called a **Huffman code**
- Assume that:
  - C is a set of n characters
  - Each character has a frequency $f(c)$
  - The tree T is built in a bottom up manner
- Idea:

f: 5	e: 9	c: 12	b: 13	d: 16	a: 45

  - Start with a set of $|C|$ leaves
  - At each step, merge the two least frequent objects: the frequency of the new node = sum of two frequencies
  - Use a min-priority queue Q, keyed on f to identify the two least frequent objects

Example



f: 5
e: 9
c: 12
b: 13
d: 16
a: 45
c: 12
b: 13
14
0
1
f: 5
e: 9
d: 16
a: 45
14
0
1
f: 5
e: 9
d: 16
25
0
1
c: 12
b: 13
a: 45
25
0
1
c: 12
b: 13
30
0
1
14
0
1
f: 5
e: 9
d: 16
a: 45
a: 45
55
0
1
25
0
1
c: 12
b: 13
30
0
1
14
0
1
f: 5
e: 9
d: 16
100
0
1
a: 45
55
0
1
25
0
1
c: 12
b: 13
30
0
1
14
0
1
d: 16
f: 5
e: 9

Building a Huffman Code

Alg.: HUFFMAN(C)

Running time: $O(n\lg n)$

```
n ← |C|
Q ← C                                   O(n)
for i ← 1 to n - 1
    do allocate a new node z
        left[z] ← x ← EXTRACT-MIN(Q)
        right[z] ← y ← EXTRACT-MIN(Q)   O(nlgn)
        f[z] ← f[x] + f[y]
        INSERT (Q, z)
return EXTRACT-MIN(Q)
```

Greedy Choice Property

Lemma: Let C be an alphabet in which each character $c \in C$ has frequency $f[c]$. Let x and y be two characters in C having the lowest frequencies.

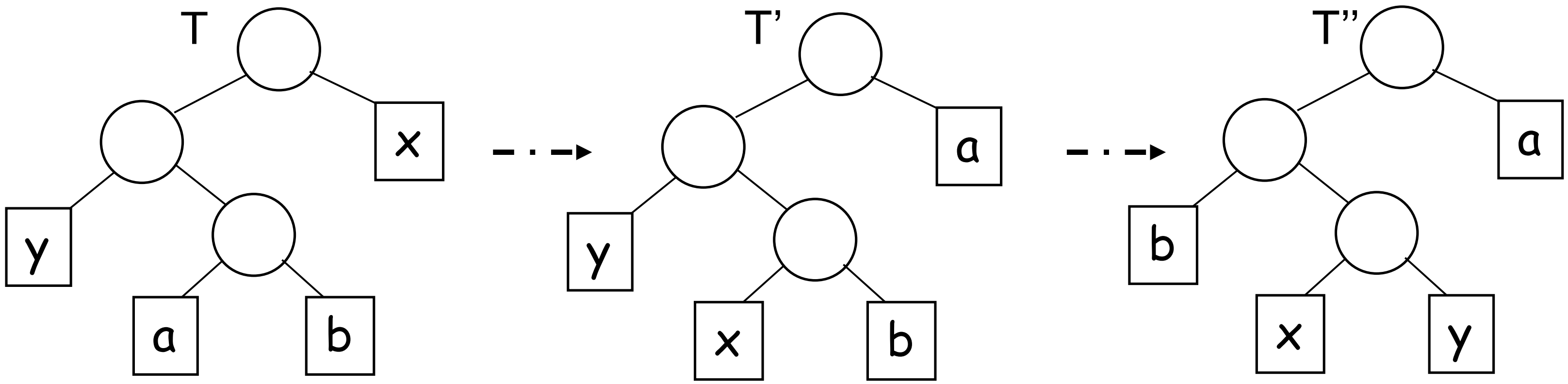
Then, there exists an optimal prefix code for C in which the codewords for x and y have the same length and differ only in the last bit.

Proof of the Greedy Choice

- Idea:
  - Consider a tree T representing an arbitrary optimal prefix code
  - Modify T to make a tree representing another optimal prefix code in which x and y will appear as sibling leaves of maximum depth

$\Rightarrow$ The codes of x and y will have the same length and differ only in the last bit

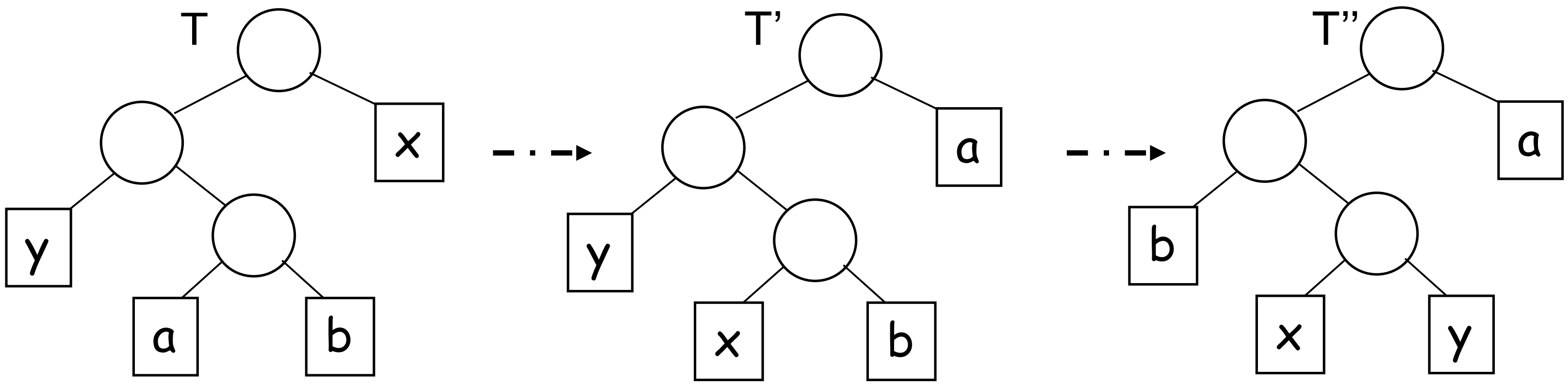
Proof of the Greedy Choice (cont.)



- a, b – two characters, sibling leaves of maximum depth in T
- Assume: f[a] ≤ f[b] and f[x] ≤ f[y]
- f[x] and f[y] are the two lowest leaf frequencies, in order

  ⇒ f[x] ≤ f[a] and f[y] ≤ f[b]
- Exchange the positions of a and x (T') and of b and y (T'')

Proof of the Greedy Choice (cont.)



$$B(T) - B(T') = \sum_{c\in C} f(c)d_T(c) - \sum_{c\in C} f(c)d_{T'}(c)$$

$$= f[x]d_T(x) + f[a]d_T(a) - f[x]d_{T'}(x) - f[a]d_{T'}(a)$$

$$= f[x]d_T(x) + f[a]d_T(a) - f[x]d_T(a) - f[a]d_T(x)$$

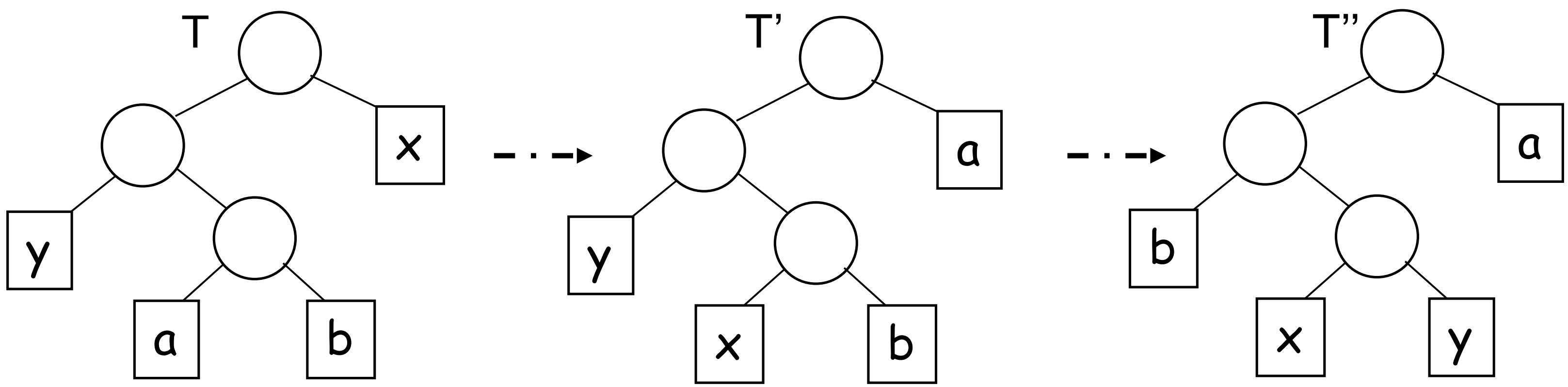
$$= \underbrace{(f[a] - f[x])}_{\geq 0} \underbrace{(d_T(a) - d_T(x))}_{\geq 0}$$

x is a minimum frequency leaf

a is a leaf of maximum depth

≥ 0

Proof of the Greedy Choice (cont.)



$B(T) - B(T') \geq 0$

Similarly, exchanging y and b does not increase the cost

$\Rightarrow B(T') - B(T'') \geq 0$

$\Rightarrow B(T'') \leq B(T)$ and since T is optimal $\Rightarrow B(T) \leq B(T'')$

$\Rightarrow B(T) = B(T'') \Rightarrow$ T'' is an optimal tree, in which x and y are sibling leaves of maximum depth

Discussion

- Greedy choice property:
  - Building an optimal tree by mergers can begin with the greedy choice: merging the two characters with the lowest frequencies
  - The cost of each merger is the sum of frequencies of the two items being merged
  - Of all possible mergers, HUFFMAN chooses the one that incurs the least cost