Basic Graph Algorithms

Graphs

- A graph $G = (V, E)$
  - V = set of vertices
  - E = set of edges = subset of $V \times V$
  - Thus $|E| = O(|V|^2)$

Graph Variations

- Variations:
  - A *connected graph* has a path from every vertex to every other
  - In an *undirected graph:*
    - Edge (u,v) = edge (v,u)
    - No self-loops
  - In a *directed* graph:
    - Edge (u,v) goes from vertex u to vertex v, notated u→v

Graph Variations

- More variations:
  - A *weighted graph* associates weights with either the edges or the vertices
    - E.g., a road map: edges might be weighted w/ distance
  - A *multigraph* allows multiple edges between the same vertices
    - E.g., the call graph in a program (a function can get called from multiple points in another function)

Graphs

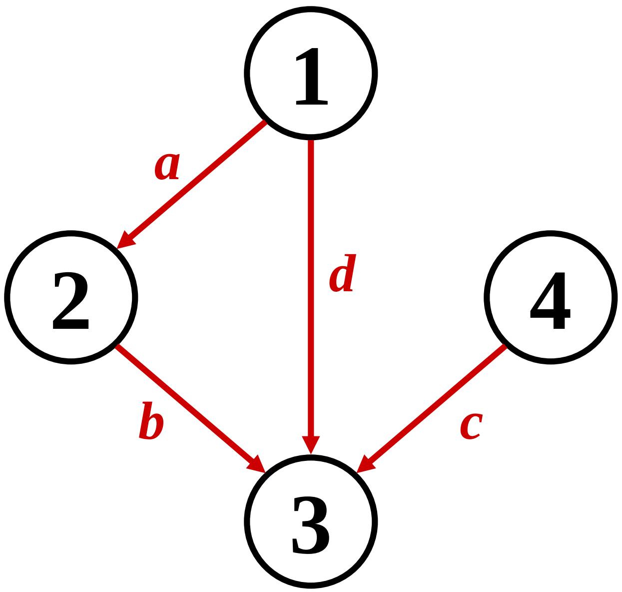
- We will typically express running times in terms of |E| and |V| (often dropping the |'s)
  - If $|E| \approx |V|^2$ the graph is *dense*
  - If $|E| \approx |V|$ the graph is *sparse*
- If you know you are dealing with dense or sparse graphs, different data structures may make sense

Representing Graphs

- Assume V = $\{1, 2, \ldots, n\}$
- An *adjacency matrix* represents the graph as a n x n matrix A:
  - A[i, j] = 1 if edge (i, j) $\in$ E (or weight of edge)
    = 0 if edge (i, j) $\notin$ E

Graphs: Adjacency Matrix

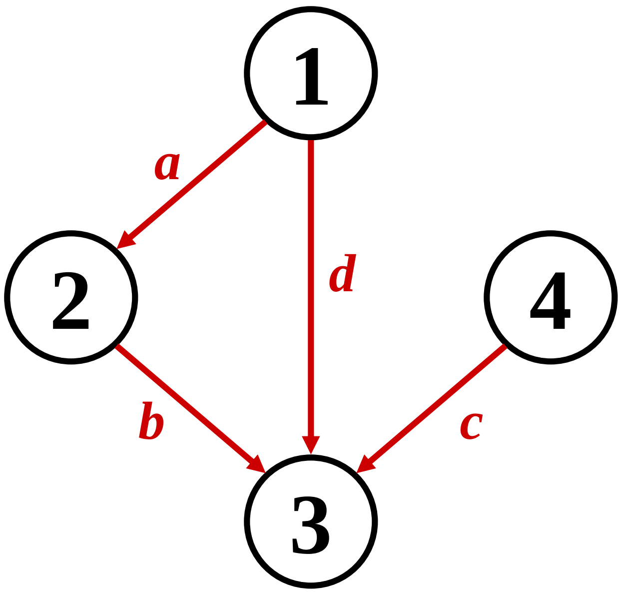
- Example:



A	1	2	3	4
1				
2				
3			??	
4				

Graphs: Adjacency Matrix

- Example:



A	1	2	3	4
1	0	1	1	0
2	0	0	1	0
3	0	0	0	0
4	0	0	1	0

Graphs: Adjacency Matrix

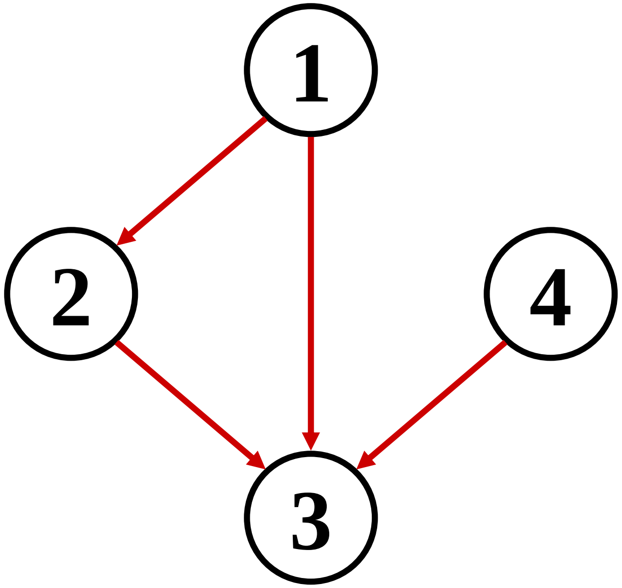
- *How much storage does the adjacency matrix require?*
- A: $O(V^2)$
- *What is the minimum amount of storage needed by an adjacency matrix representation of an undirected graph with 4 vertices?*
- A: 6 bits
  - Undirected graph → matrix is symmetric
  - No self-loops → don't need diagonal

Graphs: Adjacency Matrix

- The adjacency matrix is a dense representation
  - Usually too much storage for large graphs
  - But can be very efficient for small graphs
- Most large interesting graphs are sparse
  - E.g., planar graphs, in which no edges cross, have $|E| = O(|V|)$ by Euler's formula
  - For this reason the *adjacency list* is often a more appropriate respresentation

Graphs: Adjacency List

- Adjacency list: for each vertex $v \in V$, store a list of vertices adjacent to v
- Example:
  - Adj[1] = {2,3}
  - Adj[2] = {3}
  - Adj[3] = {}
  - Adj[4] = {3}



- Variation: can also keep a list of edges coming *into* vertex

Graphs: Adjacency List

- How much storage is required?
  - The *degree* of a vertex v = # incident edges
    - Directed graphs have in-degree, out-degree
  - For directed graphs, # of items in adjacency lists is
    $$\Sigma \text{ out-degree}(v) = |E|$$
    takes $\Theta(V + E)$ storage (*Why?*)
  - For undirected graphs, # items in adj lists is
    $$\Sigma \text{ degree}(v) = 2\,|E| \quad (\textit{handshaking lemma})$$
    also $\Theta(V + E)$ storage
- So: Adjacency lists take O(V+E) storage

Graph Searching

- Given: a graph G = (V, E), directed or undirected
- Goal: methodically explore every vertex and every edge
- Ultimately: build a tree on the graph
  - Pick a vertex as the root
  - Choose certain edges to produce a tree
  - Note: might also build a *forest* if graph is not connected

Breadth-First Search

- "Explore" a graph, turning it into a tree
  - One vertex at a time
  - Expand frontier of explored vertices across the *breadth* of the frontier
- Builds a tree over the graph
  - Pick a *source vertex* to be the root
  - Find ("discover") its children, then their children, etc.

Breadth-First Search

- Again will associate vertex “colors” to guide the algorithm
  - White vertices have not been discovered
    - All vertices start out white
  - Grey vertices are discovered but not fully explored
    - They may be adjacent to white vertices
  - Black vertices are discovered and fully explored
    - They are adjacent only to black and gray vertices
- Explore vertices by scanning adjacency list of grey vertices

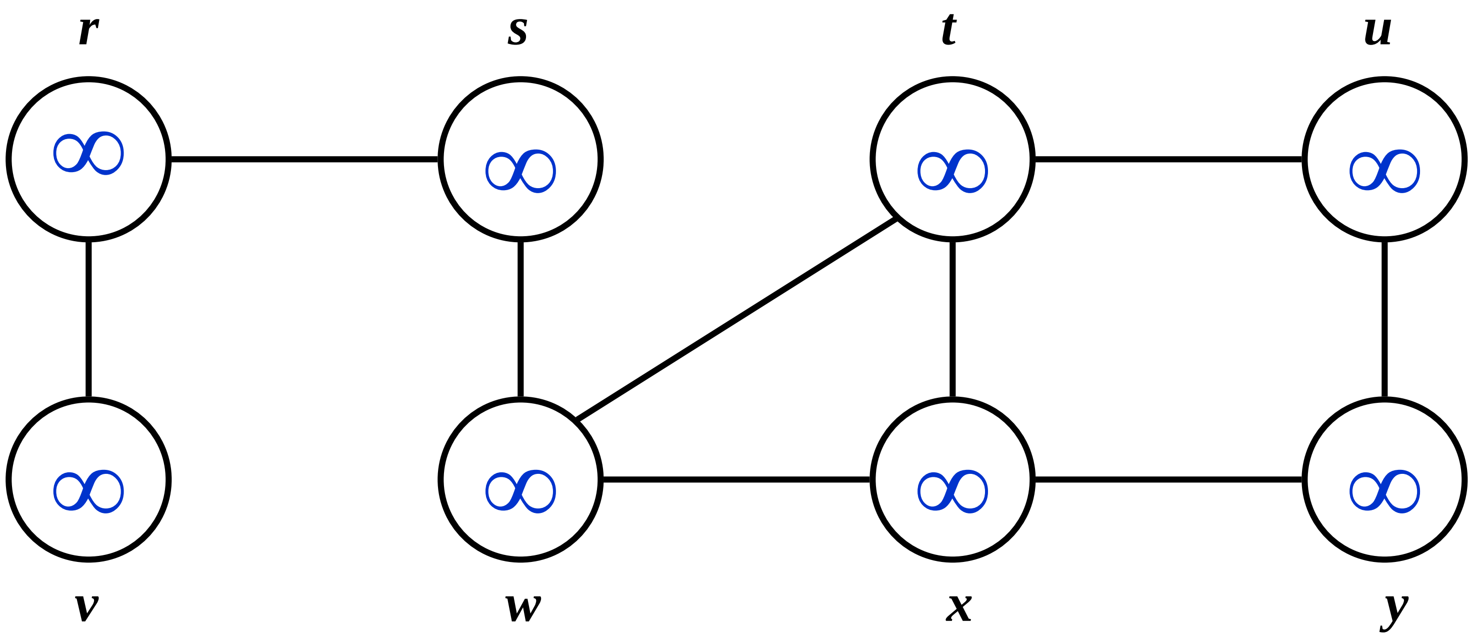
Breadth-First Search

```
BFS(G, s) {
    initialize vertices;
    Q = {s};          // Q is a queue (duh); initialize to s
    while (Q not empty) {
        u = RemoveTop(Q);
        for each v ∈ u->adj {
            if (v->color == WHITE)
                v->color = GREY;
                v->d = u->d + 1;
                v->p = u;
                Enqueue(Q, v);
        }
        u->color = BLACK;
    }
}
```

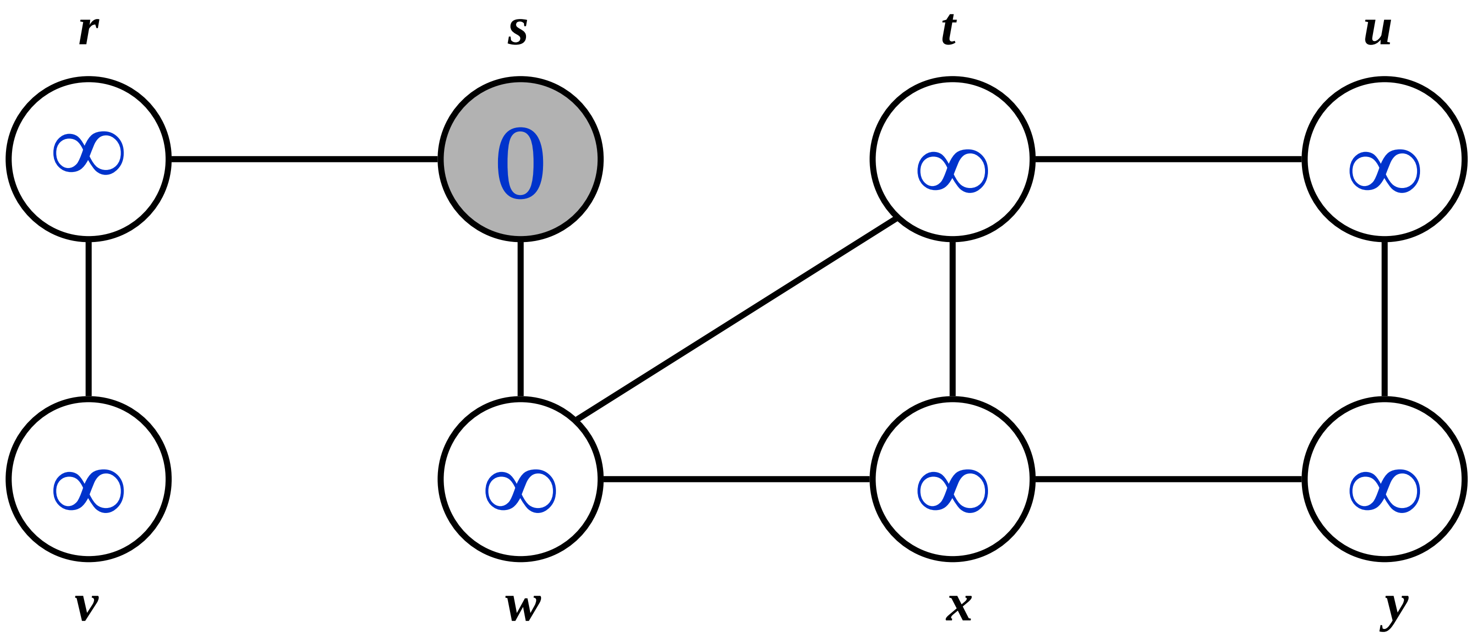
What does `v->d` *represent?*

What does `v->p` *represent?*

Breadth-First Search: Example

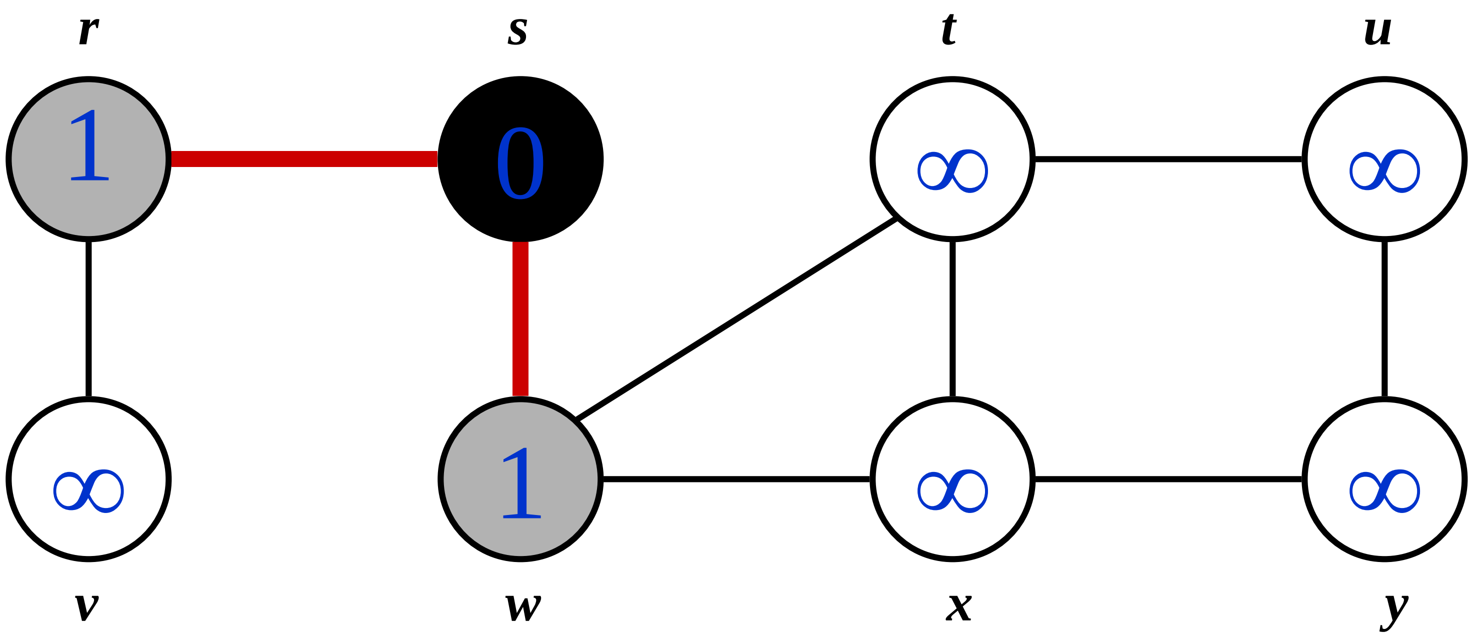


Breadth-First Search: Example



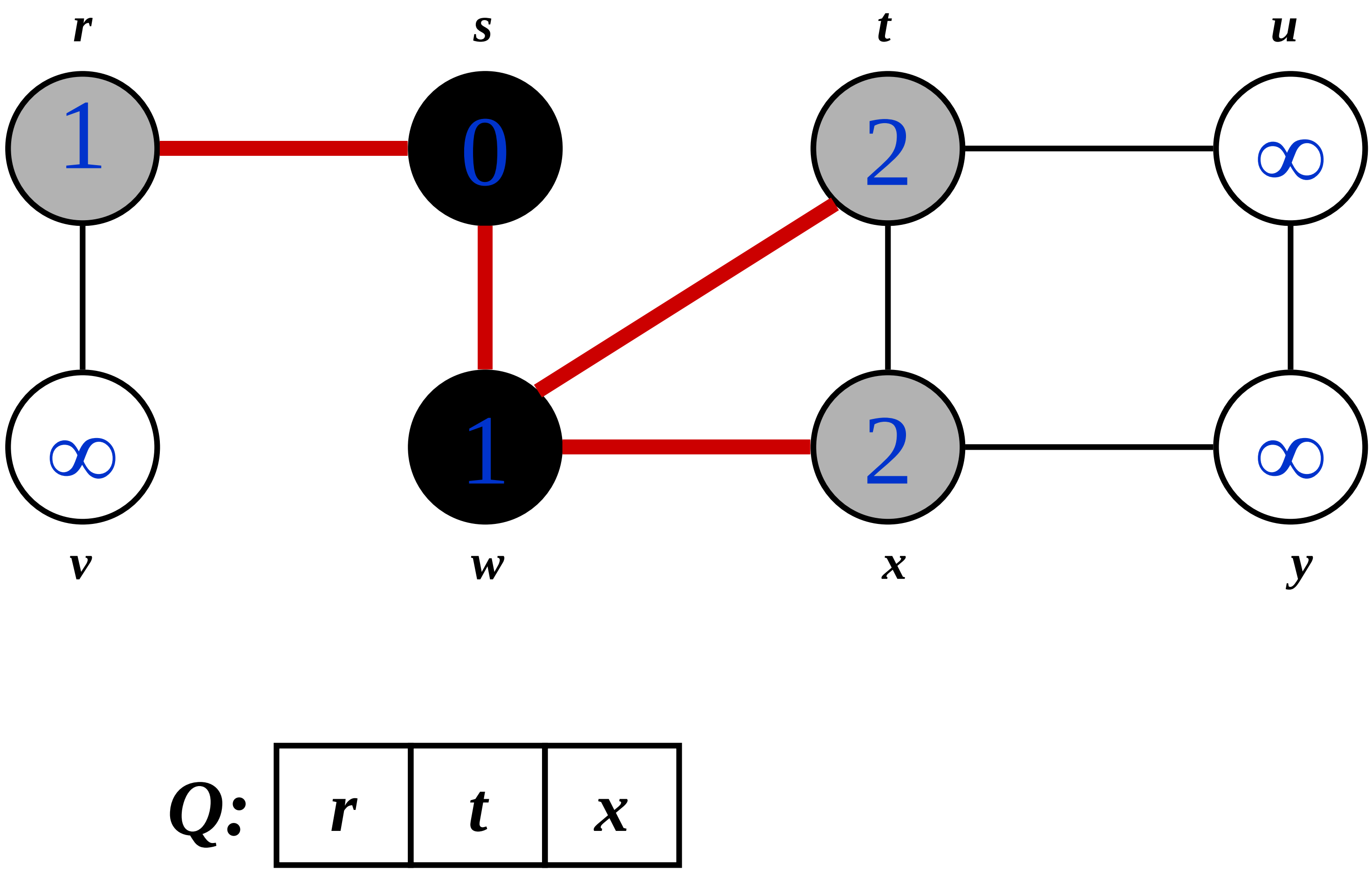
Q: | ***s*** |

Breadth-First Search: Example

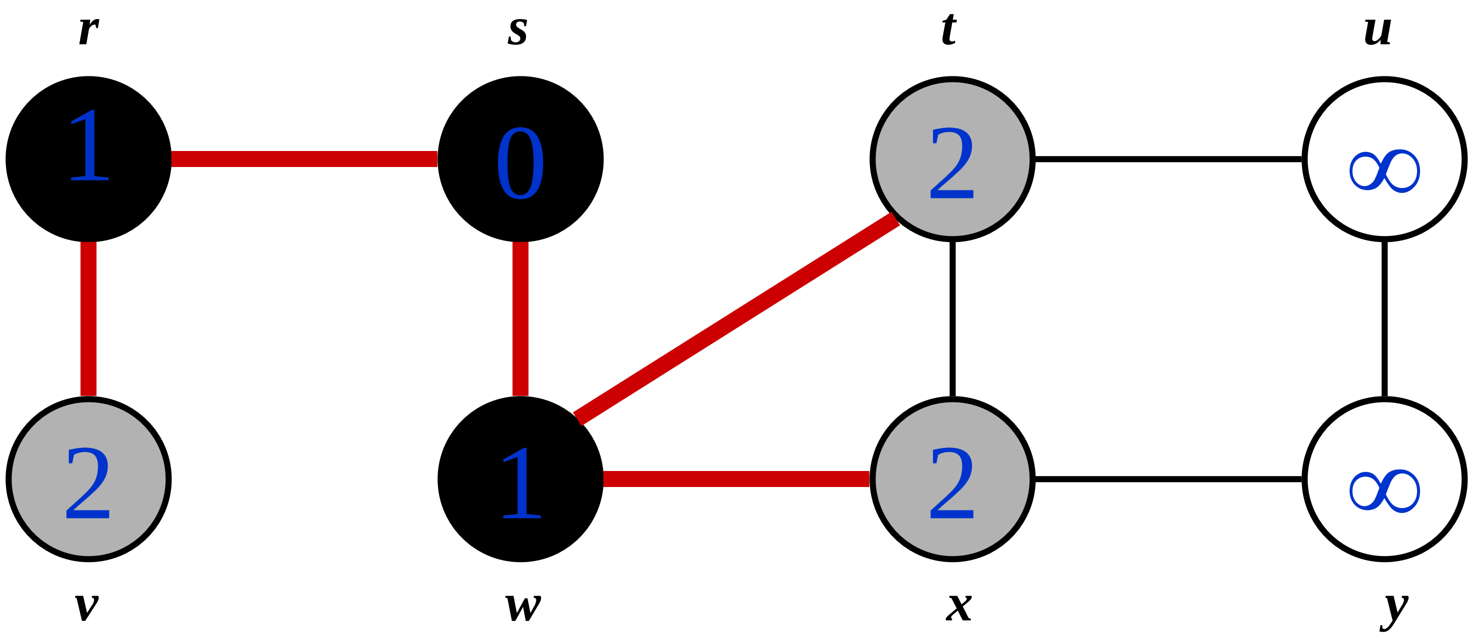


Q: | *w* | *r* |

Breadth-First Search: Example

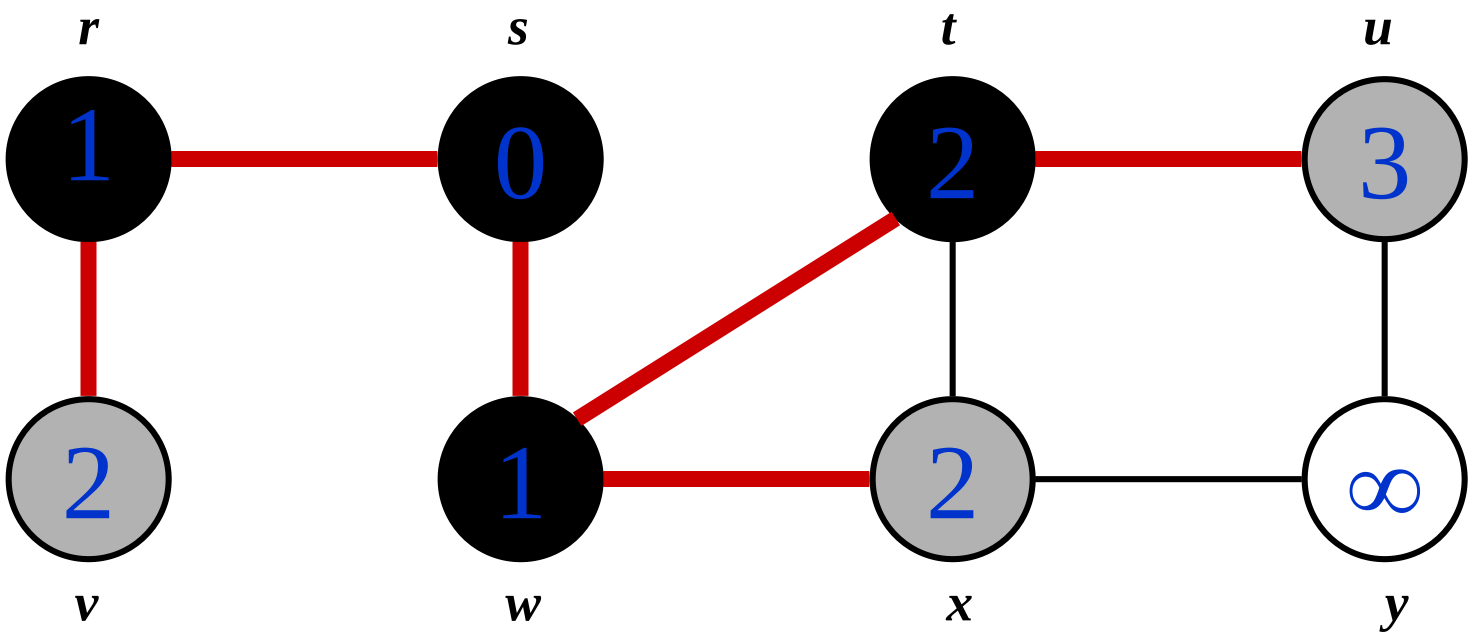


Breadth-First Search: Example



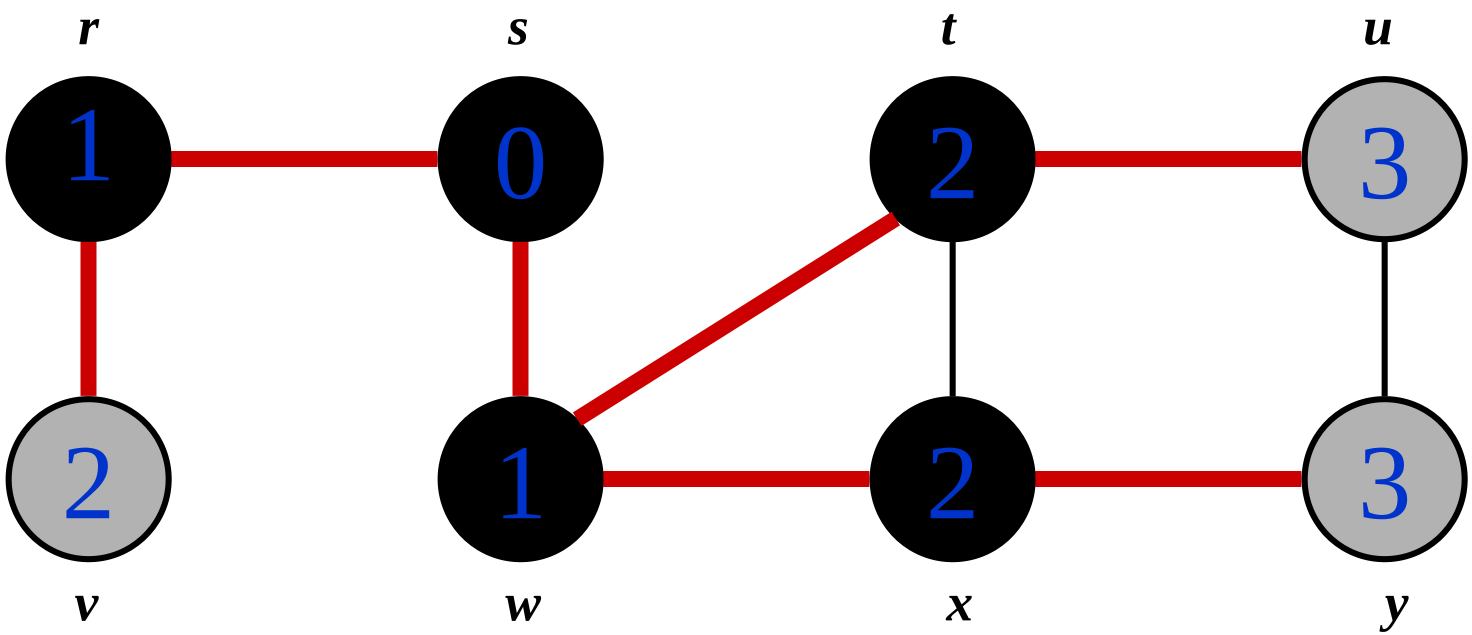
Q:			
	t	*x*	*v*

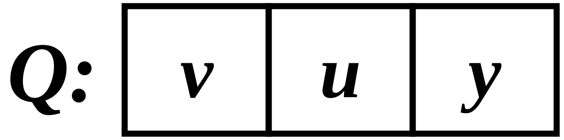
Breadth-First Search: Example



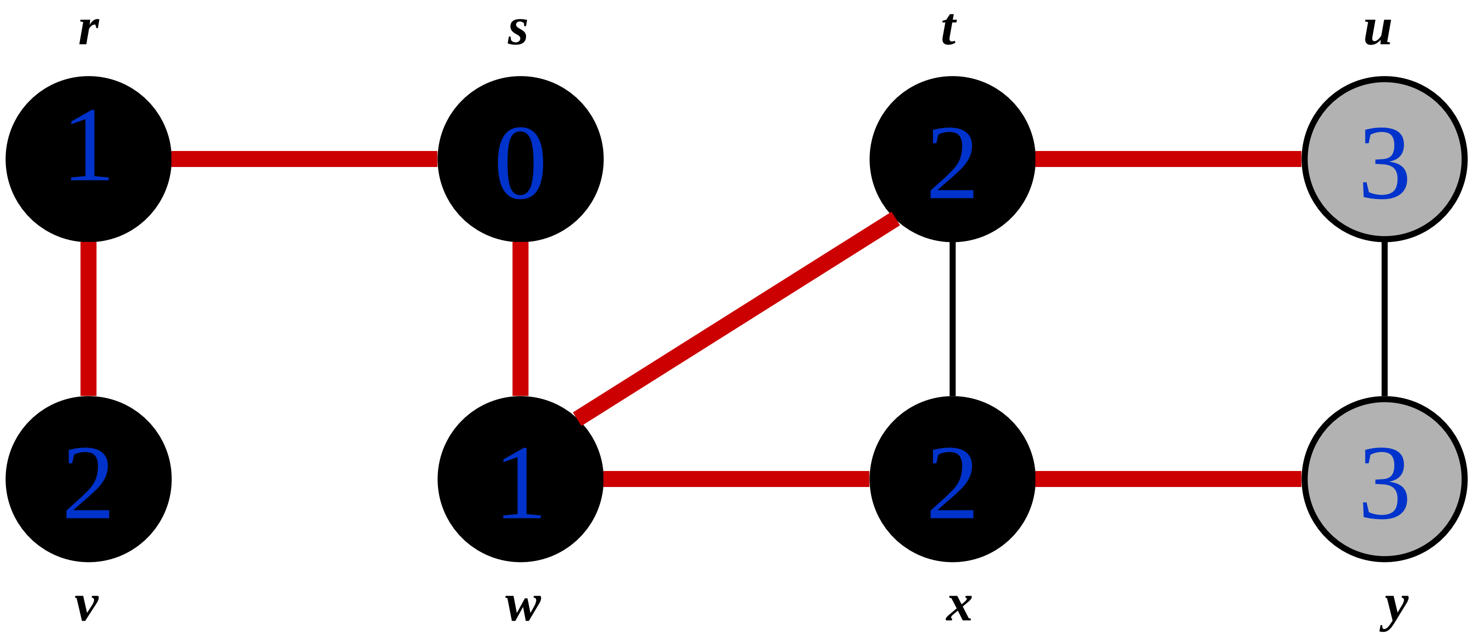
Q:	*x*	*v*	*u*

Breadth-First Search: Example



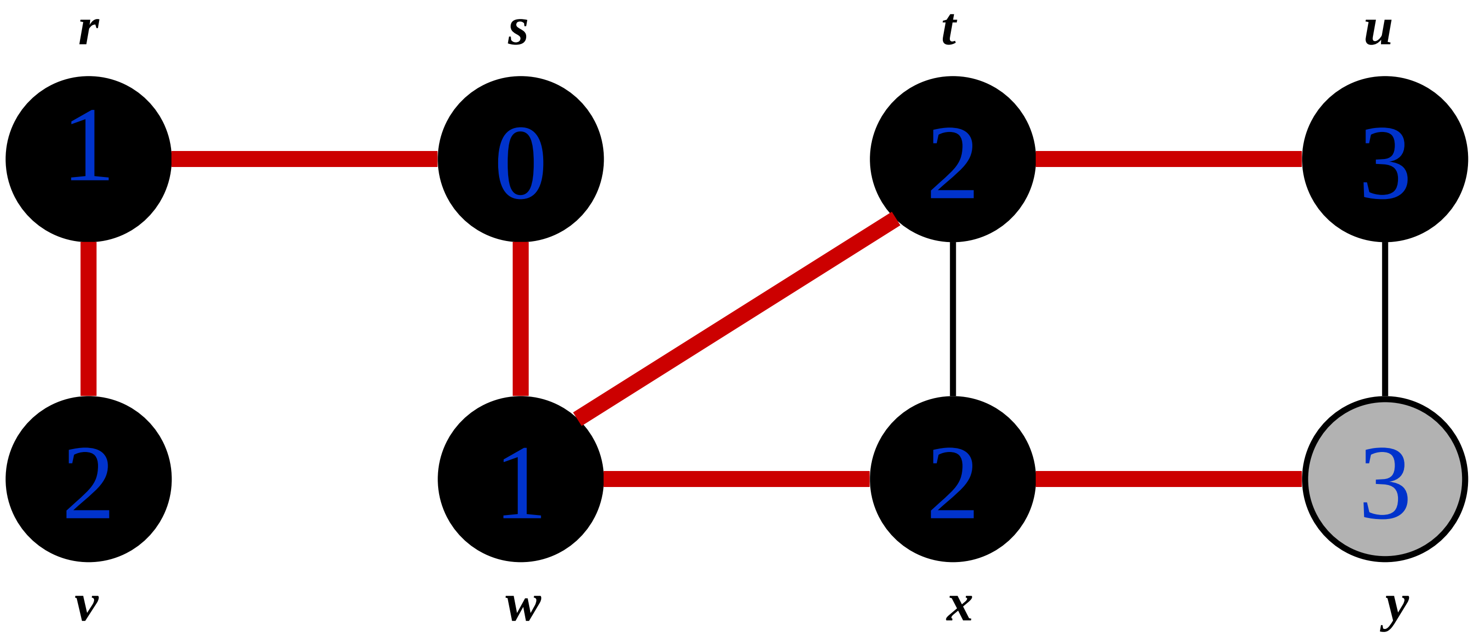


Breadth-First Search: Example



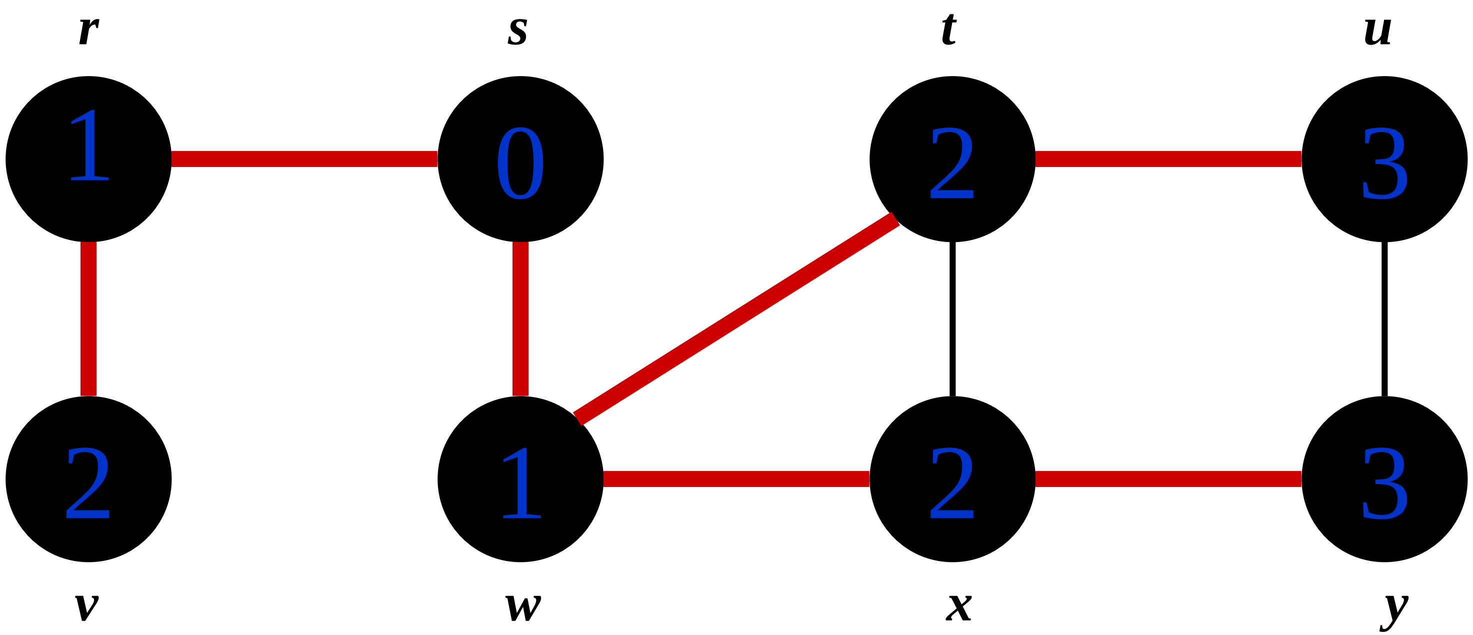
Q: | *u* | *y* |

Breadth-First Search: Example



Q: $\boxed{y}$

Breadth-First Search: Example



Q: **Ø**

BFS: The Code Again

```
BFS(G, s) {
    initialize vertices;
    Q = {s};
    while (Q not empty) {
        u = RemoveTop(Q);
        for each v ∈ u->adj {
            if (v->color == WHITE)
                v->color = GREY;
                v->d = u->d + 1;
                v->p = u;
                Enqueue(Q, v);
        }
        u->color = BLACK;
    }
}
```

Touch every vertex: O(V)

u = every vertex, but only once

(Why?)

So v = every vertex that appears in some other vert's adjacency list

What will be the running time?

Total running time: O(V+E)

BFS: The Code Again

```
BFS(G, s) {
    initialize vertices;
    Q = {s};
    while (Q not empty) {
        u = RemoveTop(Q);
        for each v ∈ u->adj {
            if (v->color == WHITE)
                v->color = GREY;
                v->d = u->d + 1;
                v->p = u;
                Enqueue(Q, v);
        }
        u->color = BLACK;
    }
}
```

What will be the storage cost in addition to storing the tree?

Total space used: O(max(degree(v))) = O(E)

Breadth-First Search: Properties

- BFS calculates the *shortest-path distance* to the source node
  - Shortest-path distance $\delta(s,v)$ = minimum number of edges from s to v, or ∞ if v not reachable from s
  - Proof given in the book (p. 472-5)
- BFS builds *breadth-first tree*, in which paths to root represent shortest paths in G
  - Thus can use BFS to calculate shortest path from one vertex to another in O(V+E) time

Depth-First Search

- *Depth-first search* is another strategy for exploring a graph
  - Explore "deeper" in the graph whenever possible
  - Edges are explored out of the most recently discovered vertex v that still has unexplored edges
  - When all of v's edges have been explored, backtrack to the vertex from which v was discovered

Depth-First Search

- Vertices initially colored white
- Then colored gray when discovered
- Then black when finished

Depth-First Search: The Code

```
DFS(G)
{
    for each vertex u ∈ G->V
    {
        u->color = WHITE;
    }
    time = 0;
    for each vertex u ∈ G->V
    {
        if (u->color == WHITE)
            DFS_Visit(u);
    }
}
```

```
DFS_Visit(u)
{
    u->color = GREY;
    time = time+1;
    u->d = time;
    for each v ∈ u->Adj[]
    {
        if (v->color == WHITE)
            DFS_Visit(v);
    }
    u->color = BLACK;
    time = time+1;
    u->f = time;
}
```

Depth-First Search: The Code

```
DFS(G)
{
    for each vertex u ∈ G->V
    {
        u->color = WHITE;
    }
    time = 0;
    for each vertex u ∈ G->V
    {
        if (u->color == WHITE)
            DFS_Visit(u);
    }
}
```

```
DFS_Visit(u)
{
    u->color = GREY;
    time = time+1;
    u->d = time;
    for each v ∈ u->Adj[]
    {
        if (v->color == WHITE)
            DFS_Visit(v);
    }
    u->color = BLACK;
    time = time+1;
    u->f = time;
}
```

What does* `u->d` *represent?

Depth-First Search: The Code

```
DFS(G)
{
    for each vertex u ∈ G->V
    {
        u->color = WHITE;
    }
    time = 0;
    for each vertex u ∈ G->V
    {
        if (u->color == WHITE)
            DFS_Visit(u);
    }
}
```

```
DFS_Visit(u)
{
    u->color = GREY;
    time = time+1;
    u->d = time;
    for each v ∈ u->Adj[]
    {
        if (v->color == WHITE)
            DFS_Visit(v);
    }
    u->color = BLACK;
    time = time+1;
    u->f = time;
}
```

What does `u->f` represent?

Depth-First Search: The Code

```
DFS(G)
{
    for each vertex u ∈ G->V
    {
        u->color = WHITE;
    }
    time = 0;
    for each vertex u ∈ G->V
    {
        if (u->color == WHITE)
            DFS_Visit(u);
    }
}
```

```
DFS_Visit(u)
{
    u->color = GREY;
    time = time+1;
    u->d = time;
    for each v ∈ u->Adj[]
    {
        if (v->color == WHITE)
            DFS_Visit(v);
    }
    u->color = BLACK;
    time = time+1;
    u->f = time;
}
```

Will all vertices eventually be colored black?

Depth-First Search: The Code

```
DFS(G)
{
    for each vertex u ∈ G->V
    {
        u->color = WHITE;
    }
    time = 0;
    for each vertex u ∈ G->V
    {
        if (u->color == WHITE)
            DFS_Visit(u);
    }
}
```

```
DFS_Visit(u)
{
    u->color = GREY;
    time = time+1;
    u->d = time;
    for each v ∈ u->Adj[]
    {
        if (v->color == WHITE)
            DFS_Visit(v);
    }
    u->color = BLACK;
    time = time+1;
    u->f = time;
}
```

What will be the running time?

Depth-First Search: The Code

```
DFS(G)
{
    for each vertex u ∈ G->V
    {
        u->color = WHITE;
    }
    time = 0;
    for each vertex u ∈ G->V
    {
        if (u->color == WHITE)
            DFS_Visit(u);
    }
}
```

```
DFS_Visit(u)
{
    u->color = GREY;
    time = time+1;
    u->d = time;
    for each v ∈ u->Adj[]
    {
        if (v->color == WHITE)
            DFS_Visit(v);
    }
    u->color = BLACK;
    time = time+1;
    u->f = time;
}
```

Running time: $O(n^2)$ because call DFS_Visit on each vertex, and the loop over Adj[] can run as many as |V| times

Depth-First Search: The Code

```
DFS(G)
{
    for each vertex u ∈ G->V
    {
        u->color = WHITE;
    }
    time = 0;
    for each vertex u ∈ G->V
    {
        if (u->color == WHITE)
            DFS_Visit(u);
    }
}
```

```
DFS_Visit(u)
{
    u->color = GREY;
    time = time+1;
    u->d = time;
    for each v ∈ u->Adj[]
    {
        if (v->color == WHITE)
            DFS_Visit(v);
    }
    u->color = BLACK;
    time = time+1;
    u->f = time;
}
```

BUT, there is actually a tighter bound.

How many times will DFS_Visit() actually be called?

Depth-First Search: The Code

```
DFS(G)
{
    for each vertex u ∈ G->V
    {
        u->color = WHITE;
        u->π = NIL;
    }
    time = 0;
    for each vertex u ∈ G->V
    {
        if (u->color == WHITE)
            DFS_Visit(u);
    }
}
```

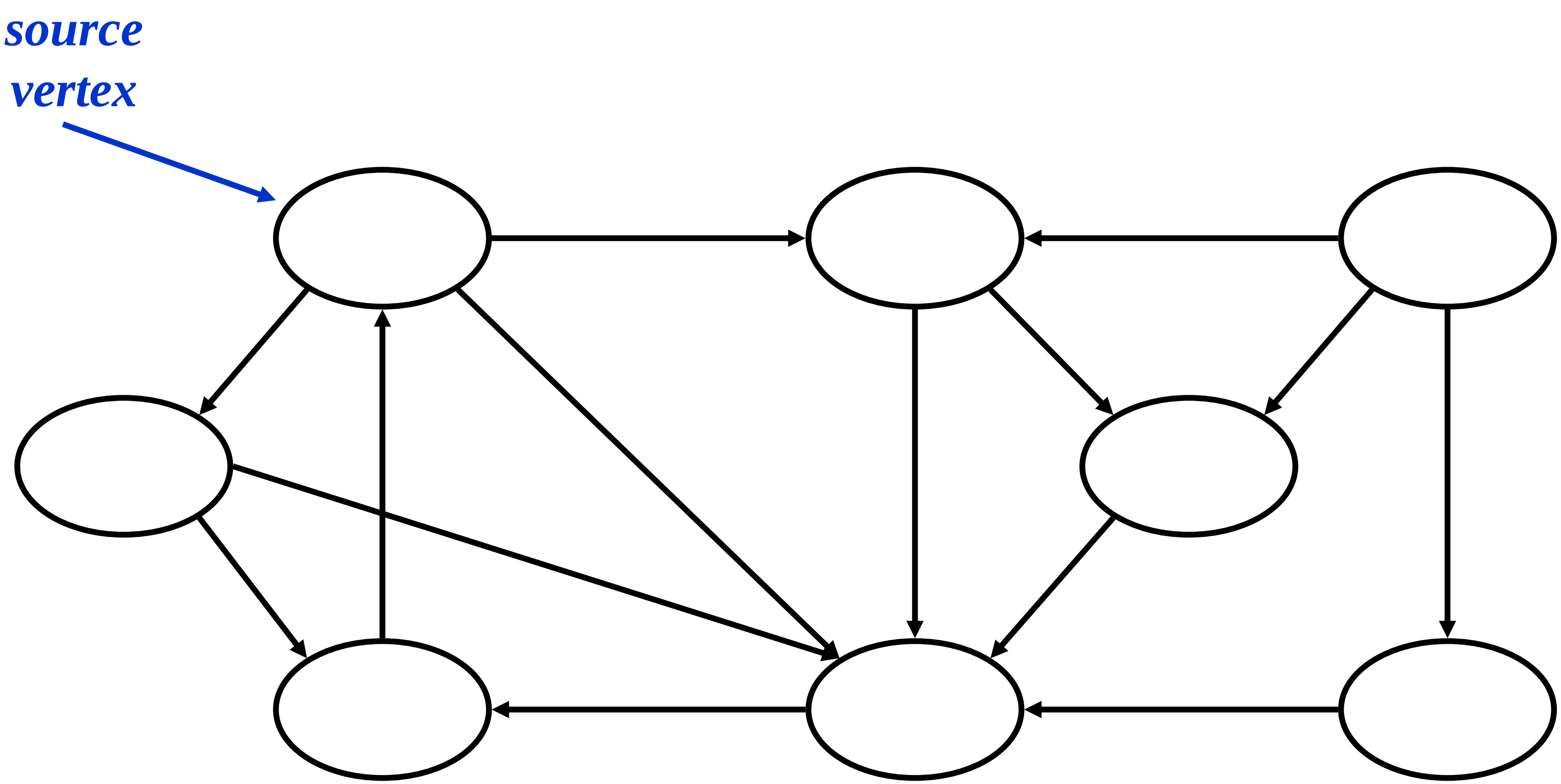
```
DFS_Visit(u)
{
    u->color = GREY;
    time = time+1;
    u->d = time;
    for each v ∈ u->Adj[]
    {
        if (v->color == WHITE)
            v->π = u;
            DFS_Visit(v);
    }
    u->color = BLACK;
    time = time+1;
    u->f = time;
}
```

So, running time of DFS = O(V+E)

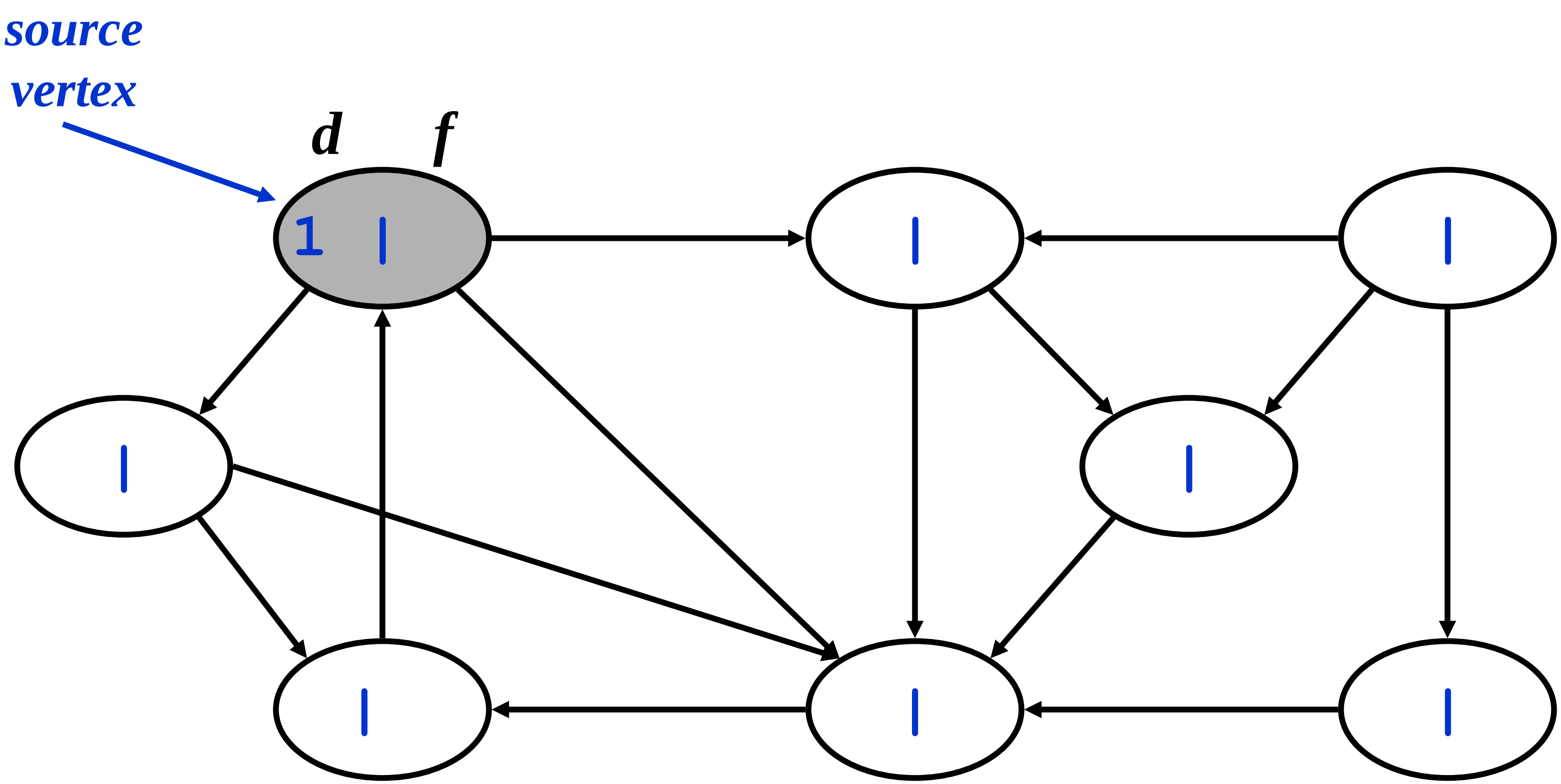
Depth-First Sort Analysis

- This running time argument is an informal example of *amortized analysis*
  - "Charge" the exploration of edge to the edge:
    - Each loop in DFS_Visit can be attributed to an edge in the graph
    - Runs once/edge if directed graph, twice if undirected
    - Thus loop will run in O(E) time, algorithm O(V+E)
      - Considered linear for graph, b/c adj list requires O(V+E) storage
  - Important to be comfortable with this kind of reasoning and analysis

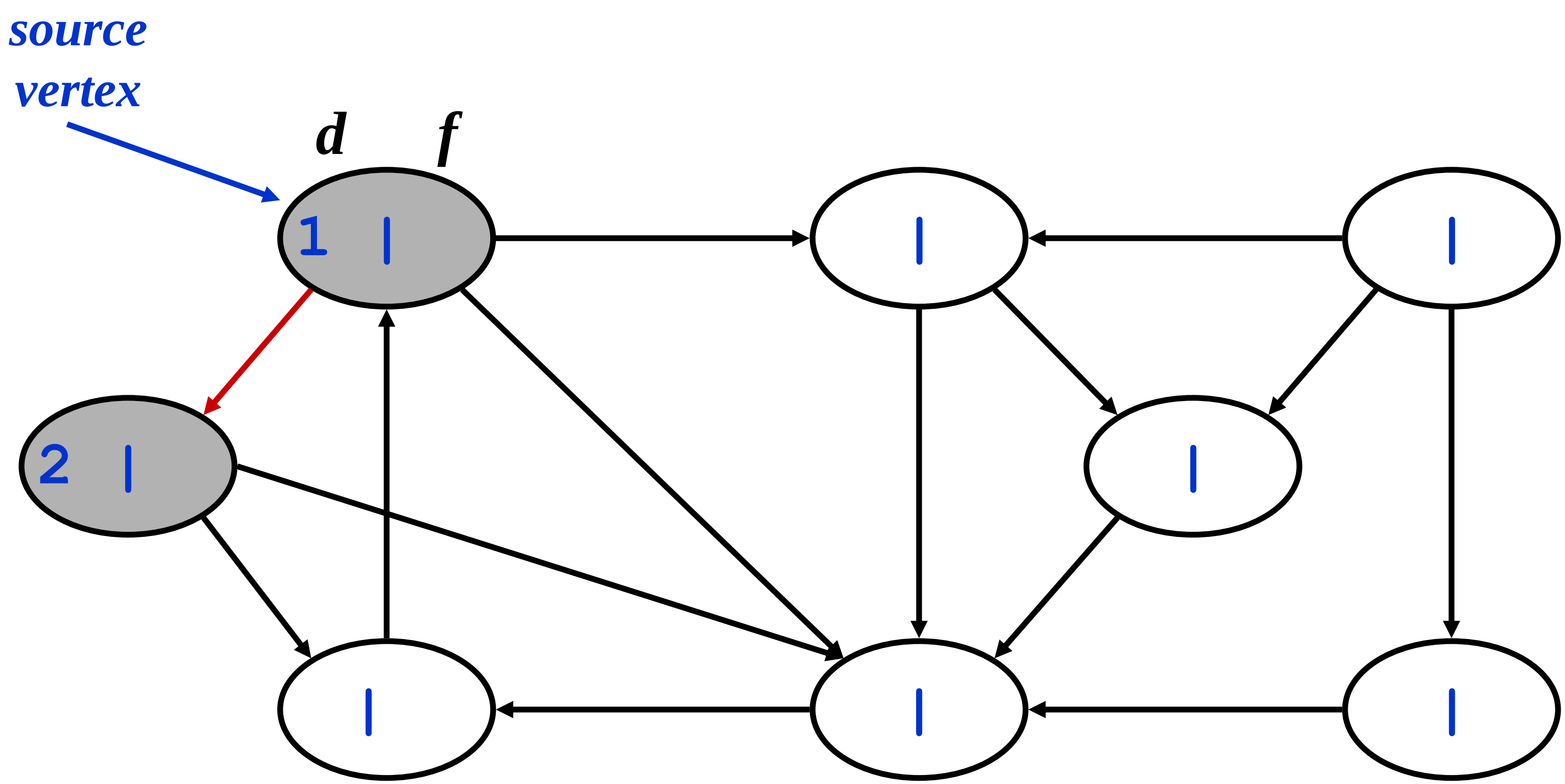
DFS Example



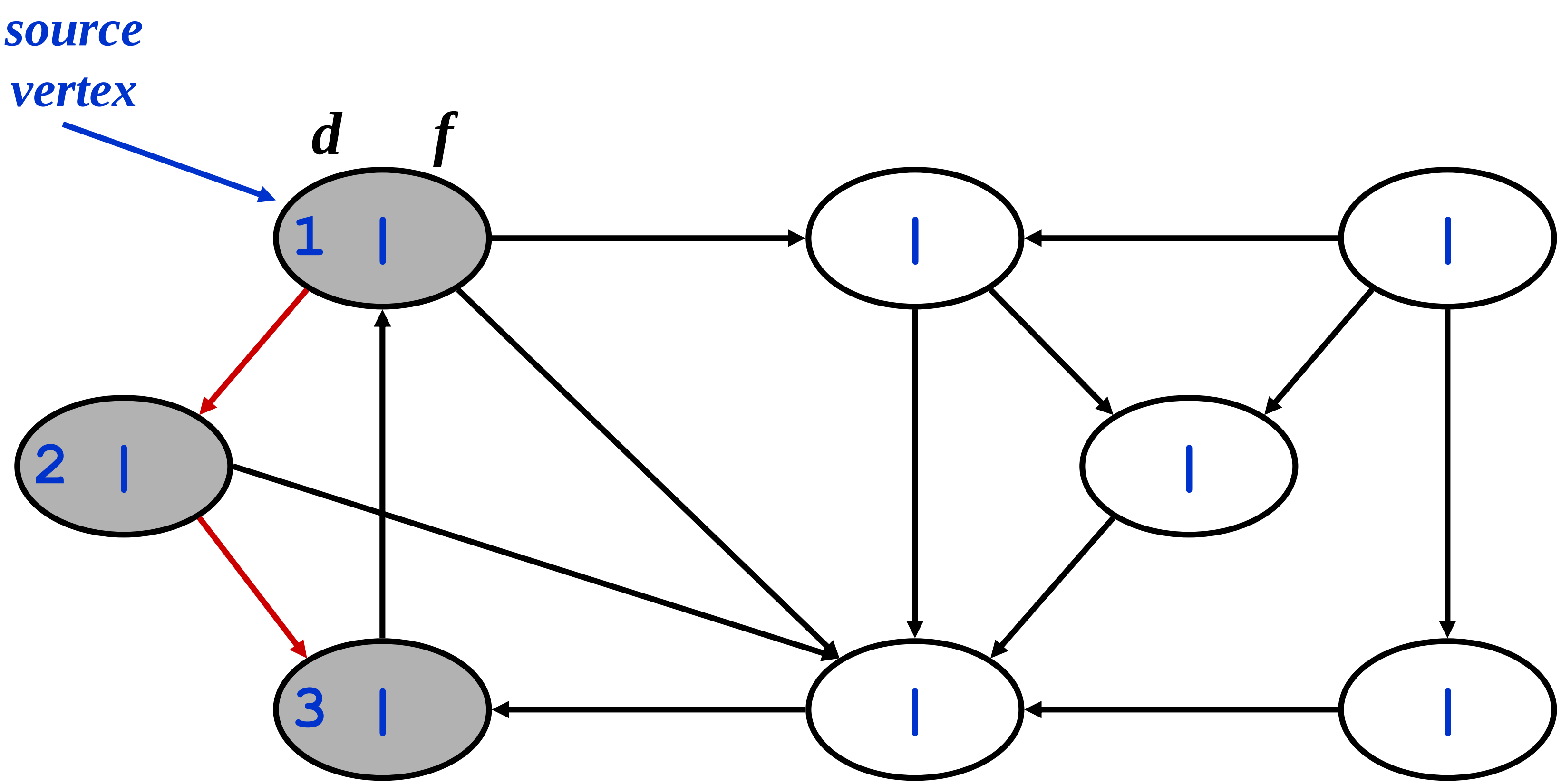
DFS Example



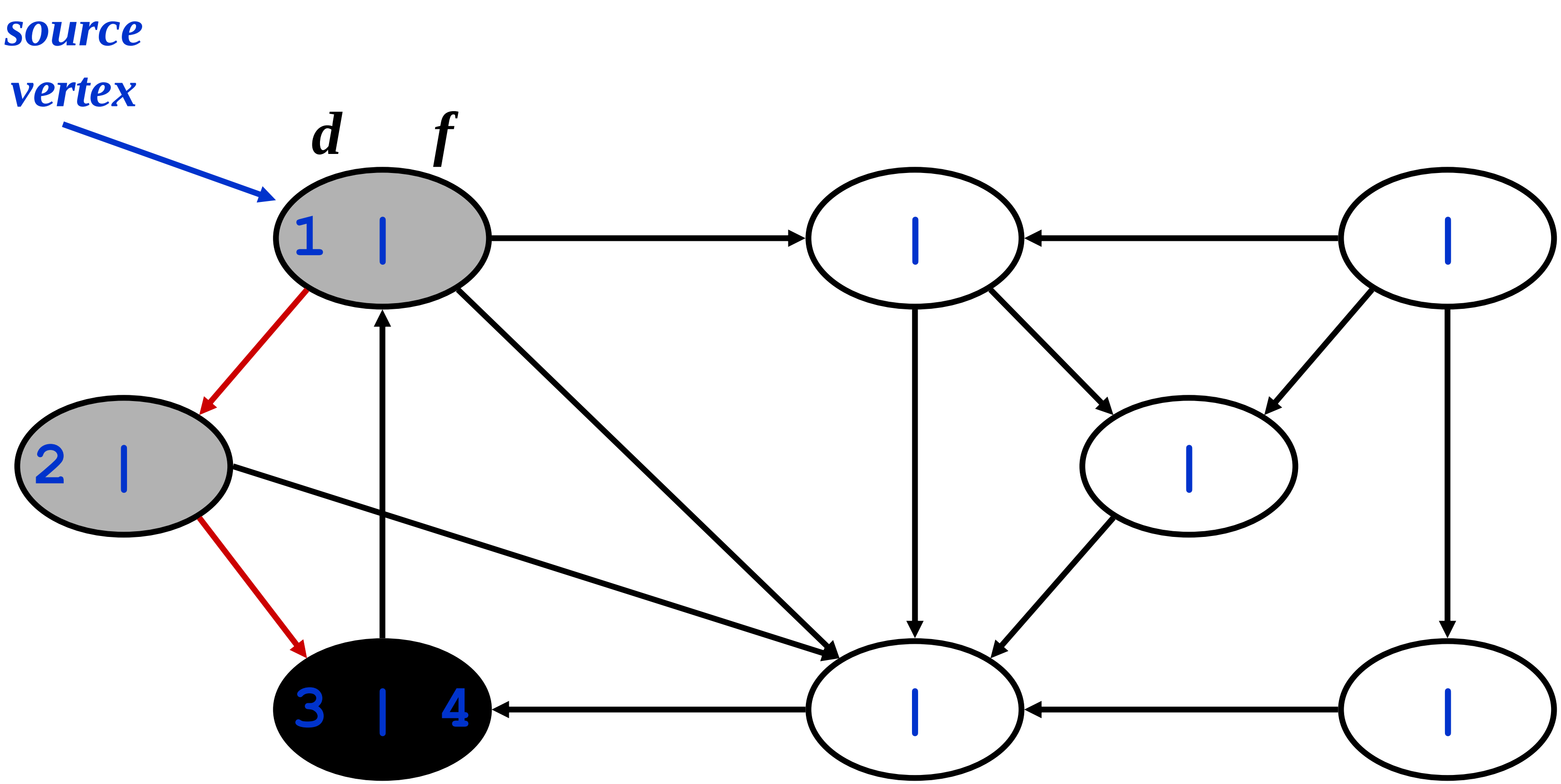
DFS Example



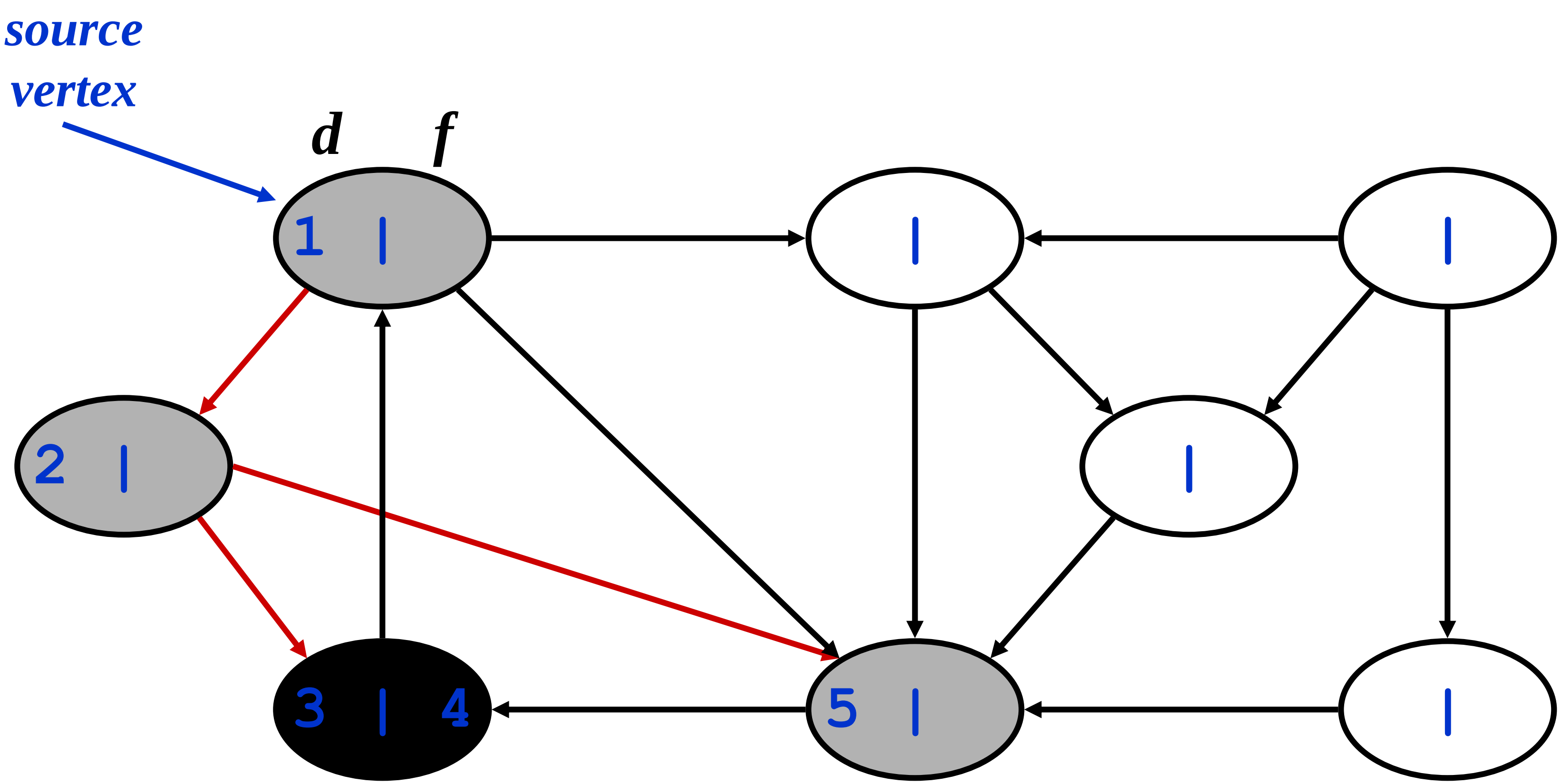
DFS Example



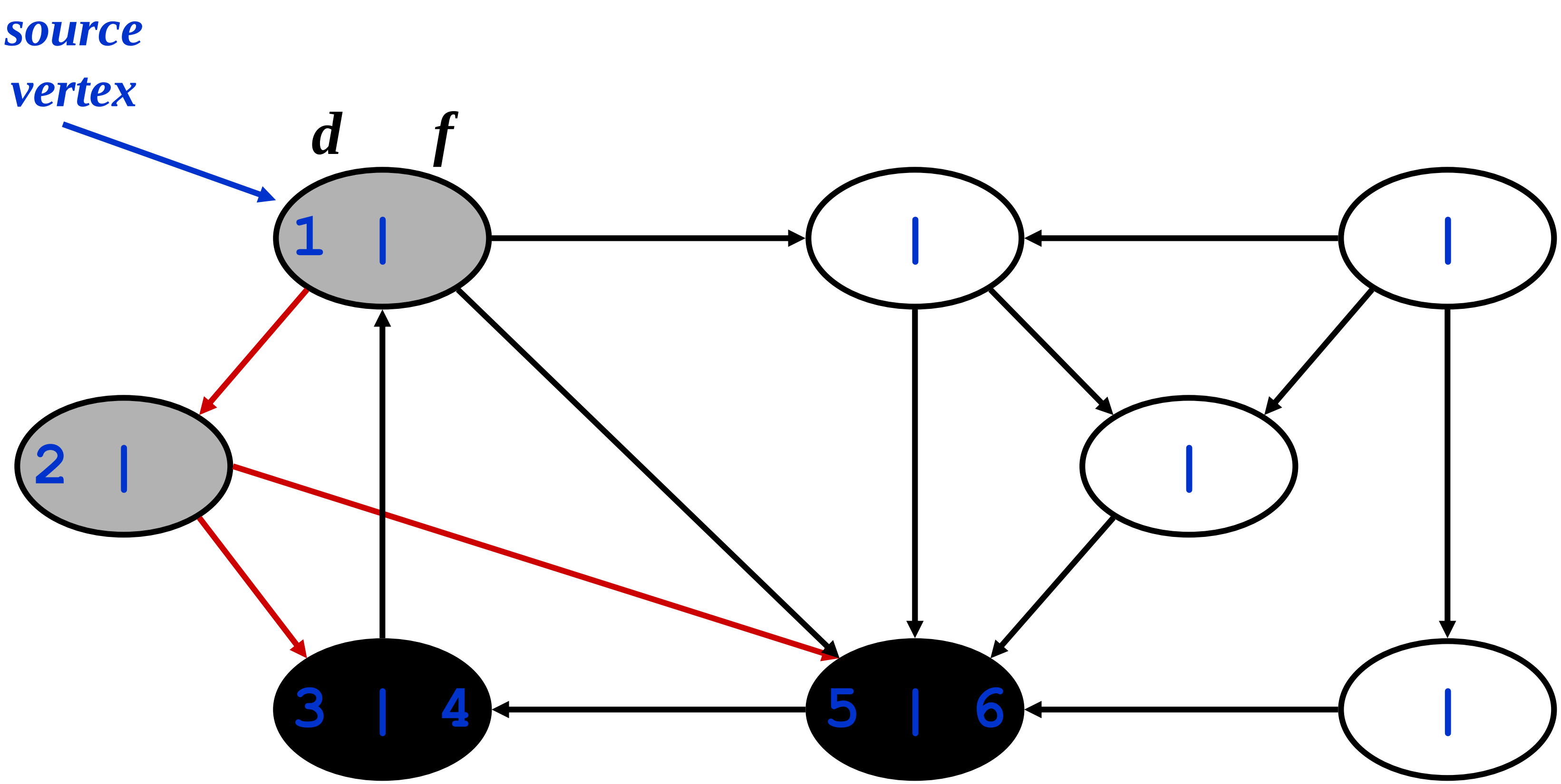
DFS Example



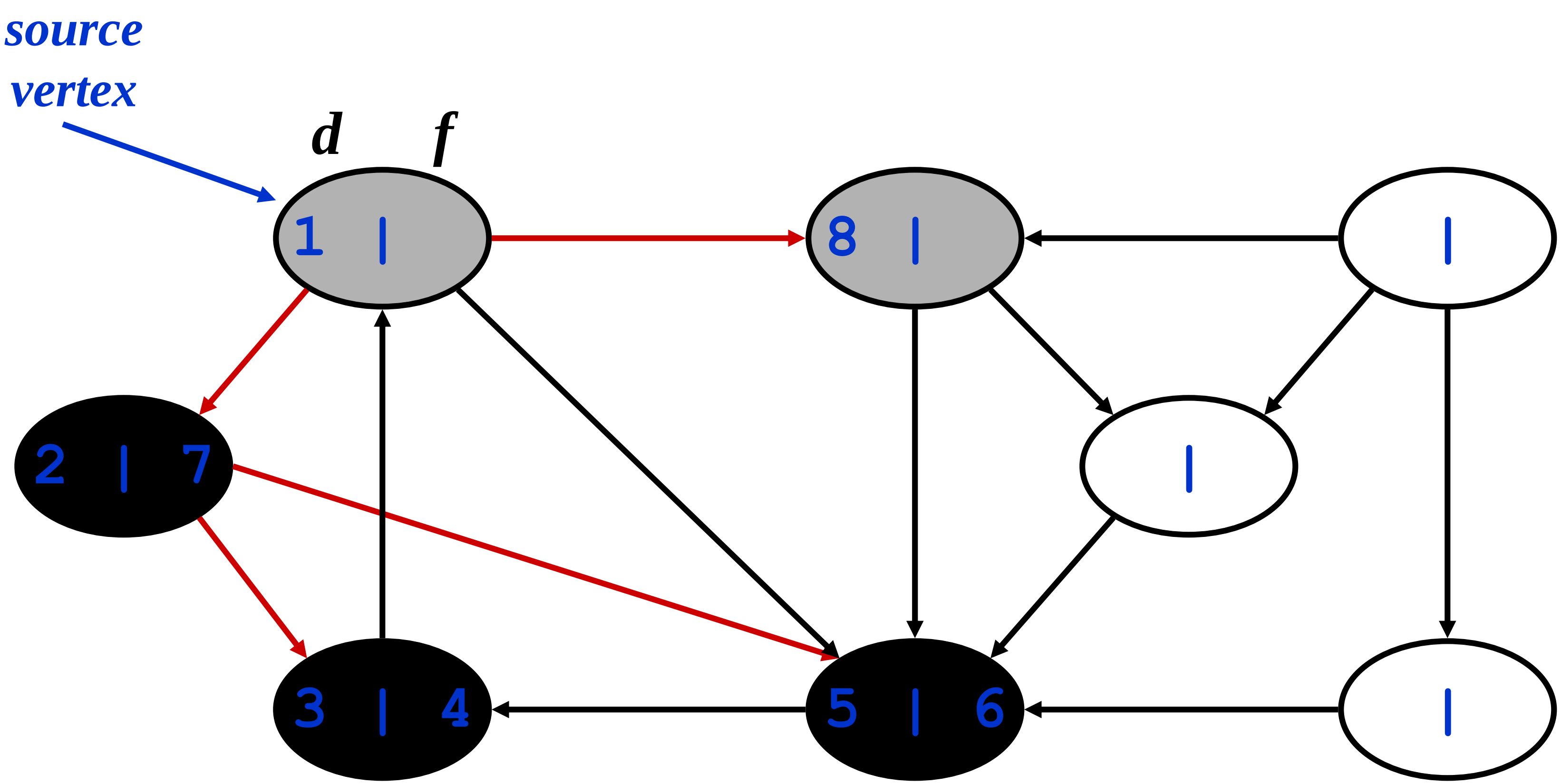
DFS Example



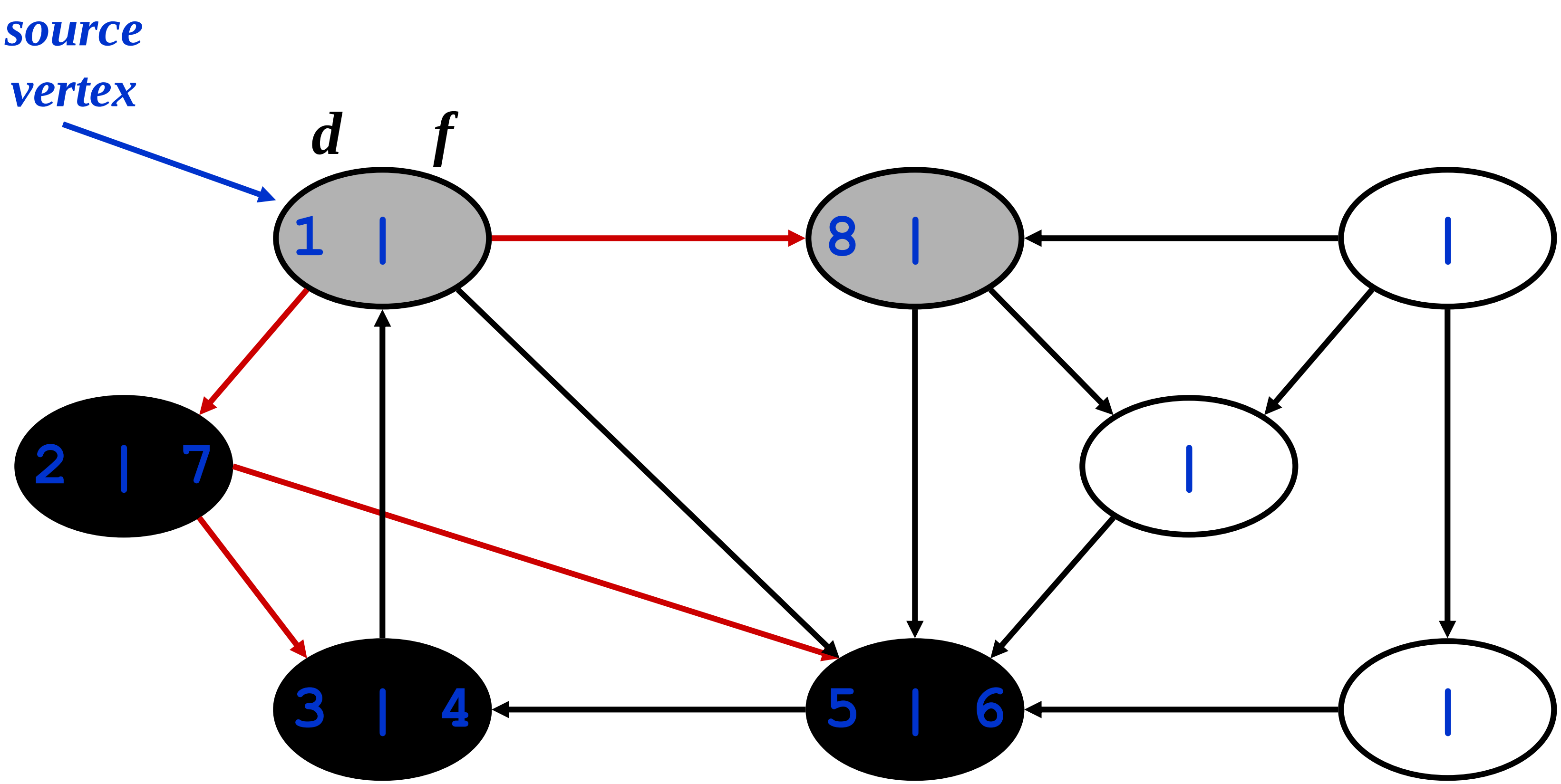
DFS Example



DFS Example



DFS Example



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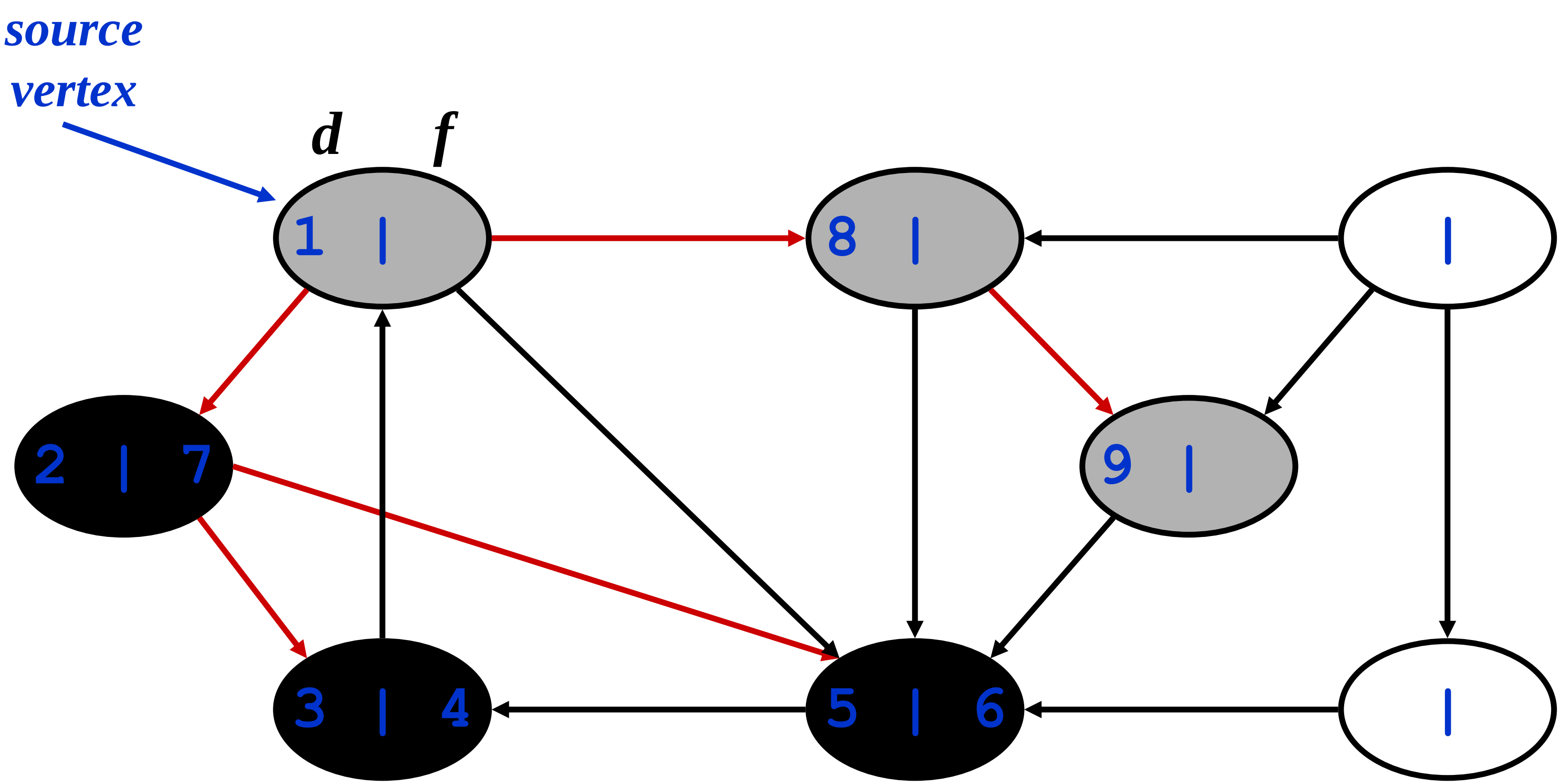
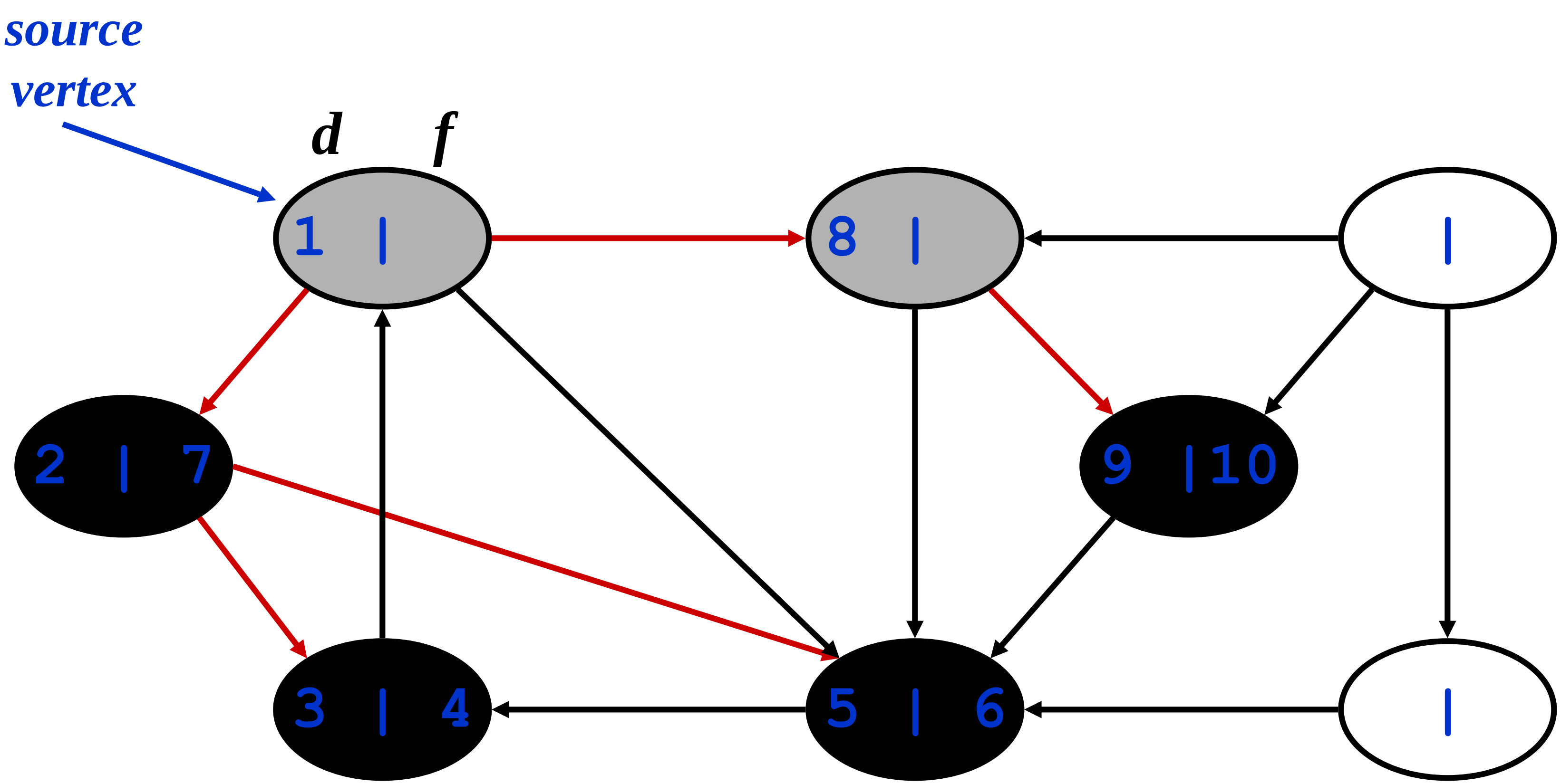
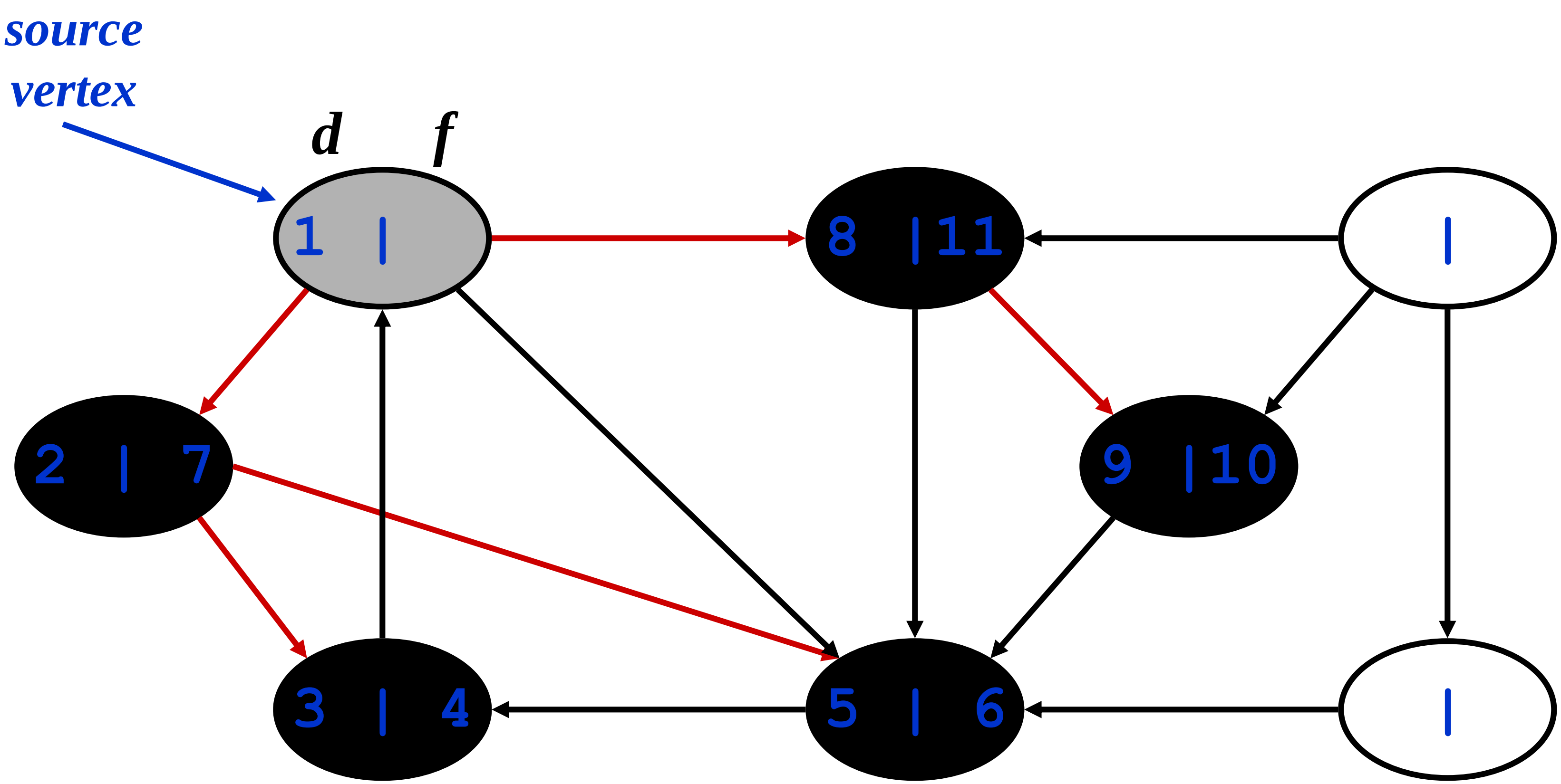


What is the structure of the grey vertices?
What do they represent?

DFS Example



DFS Example



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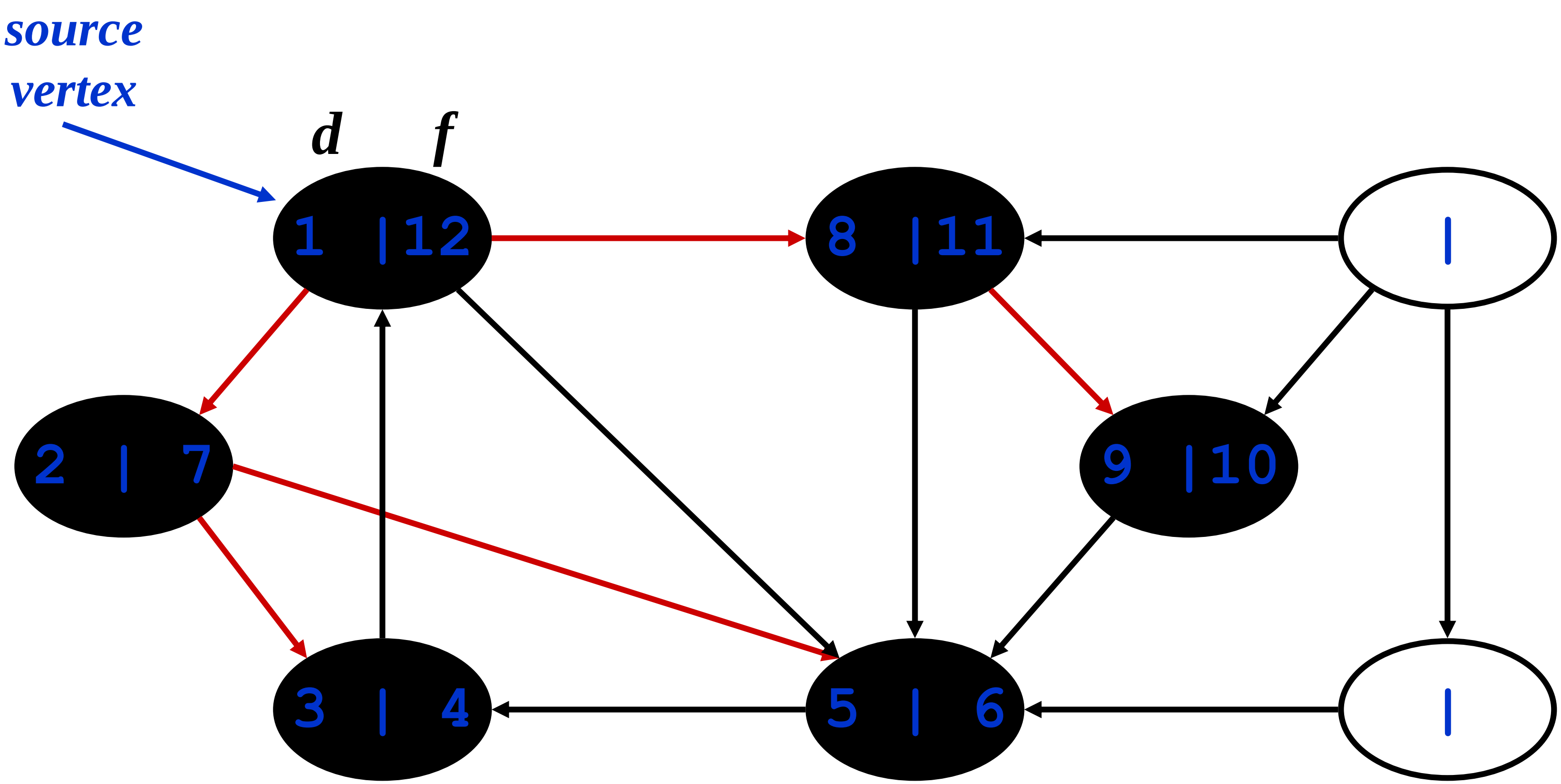
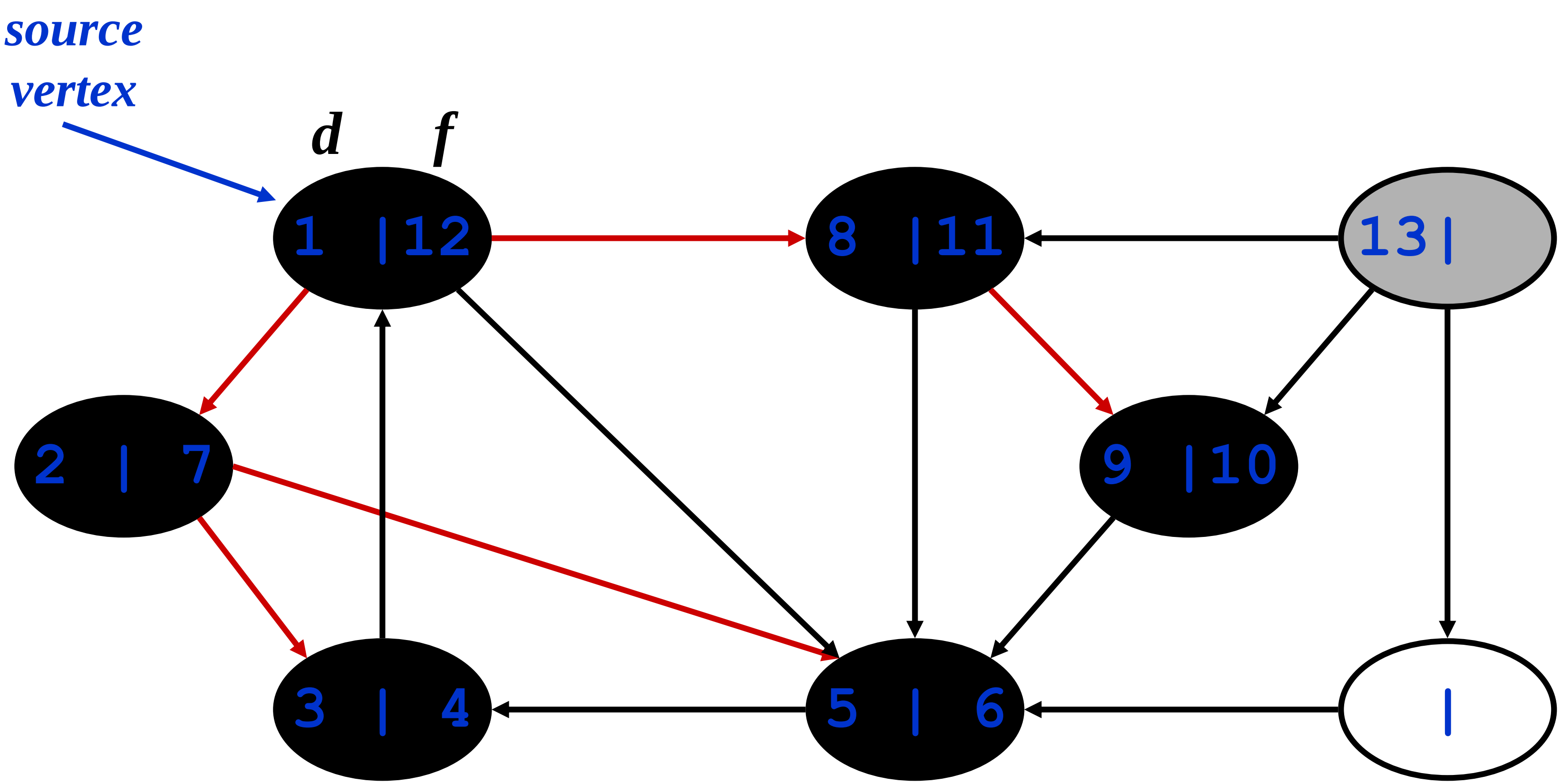
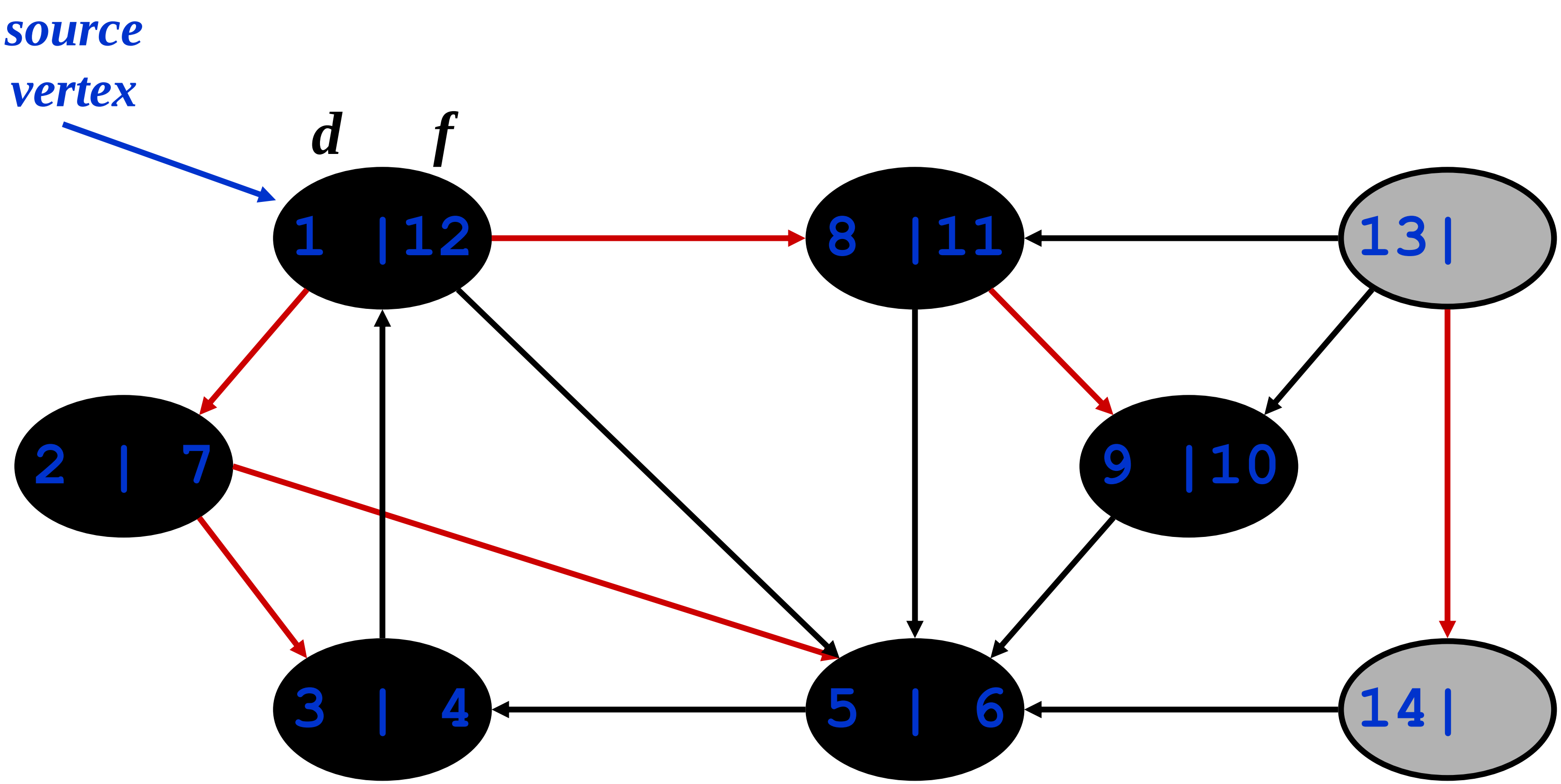


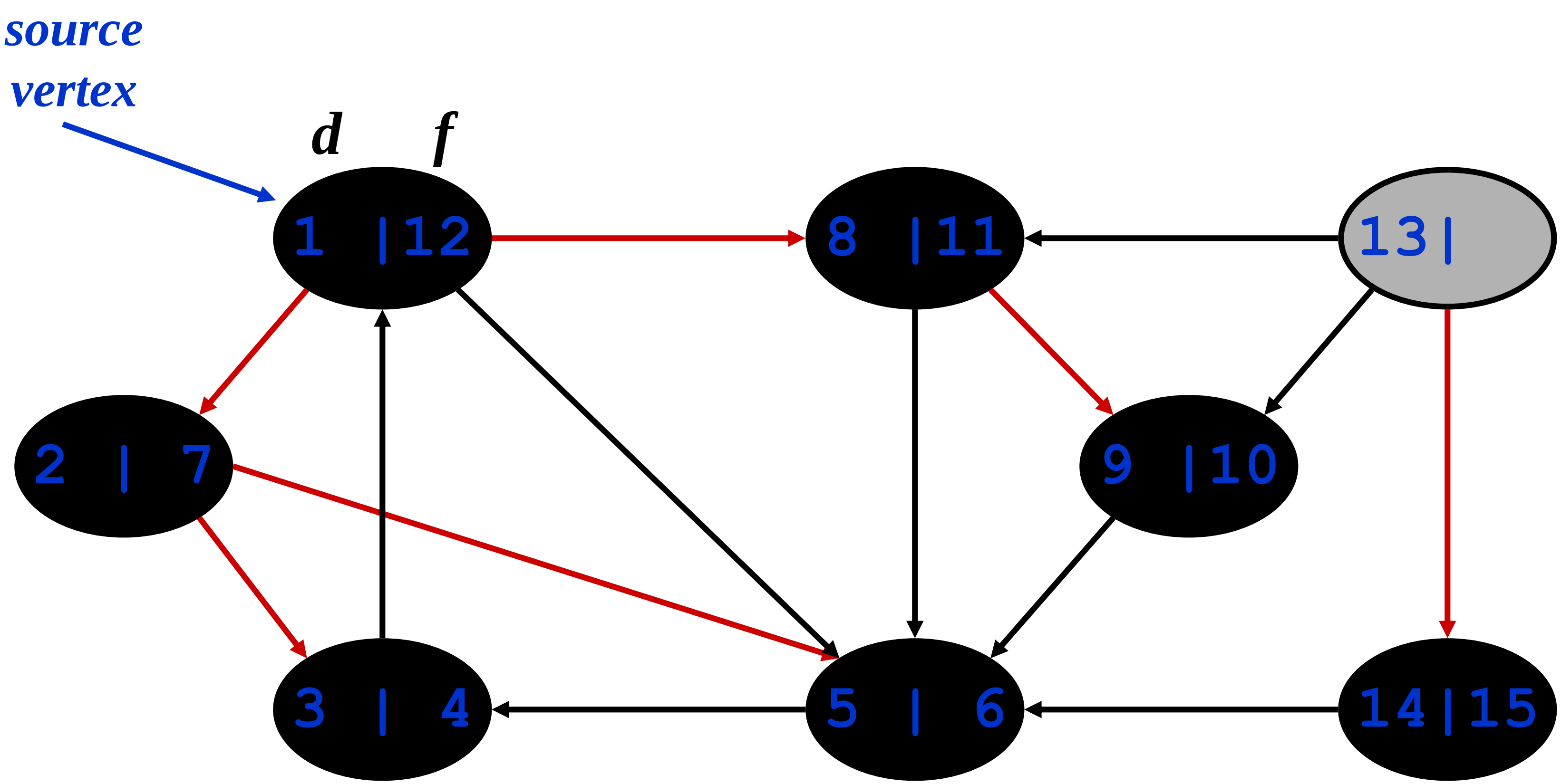
DFS Example



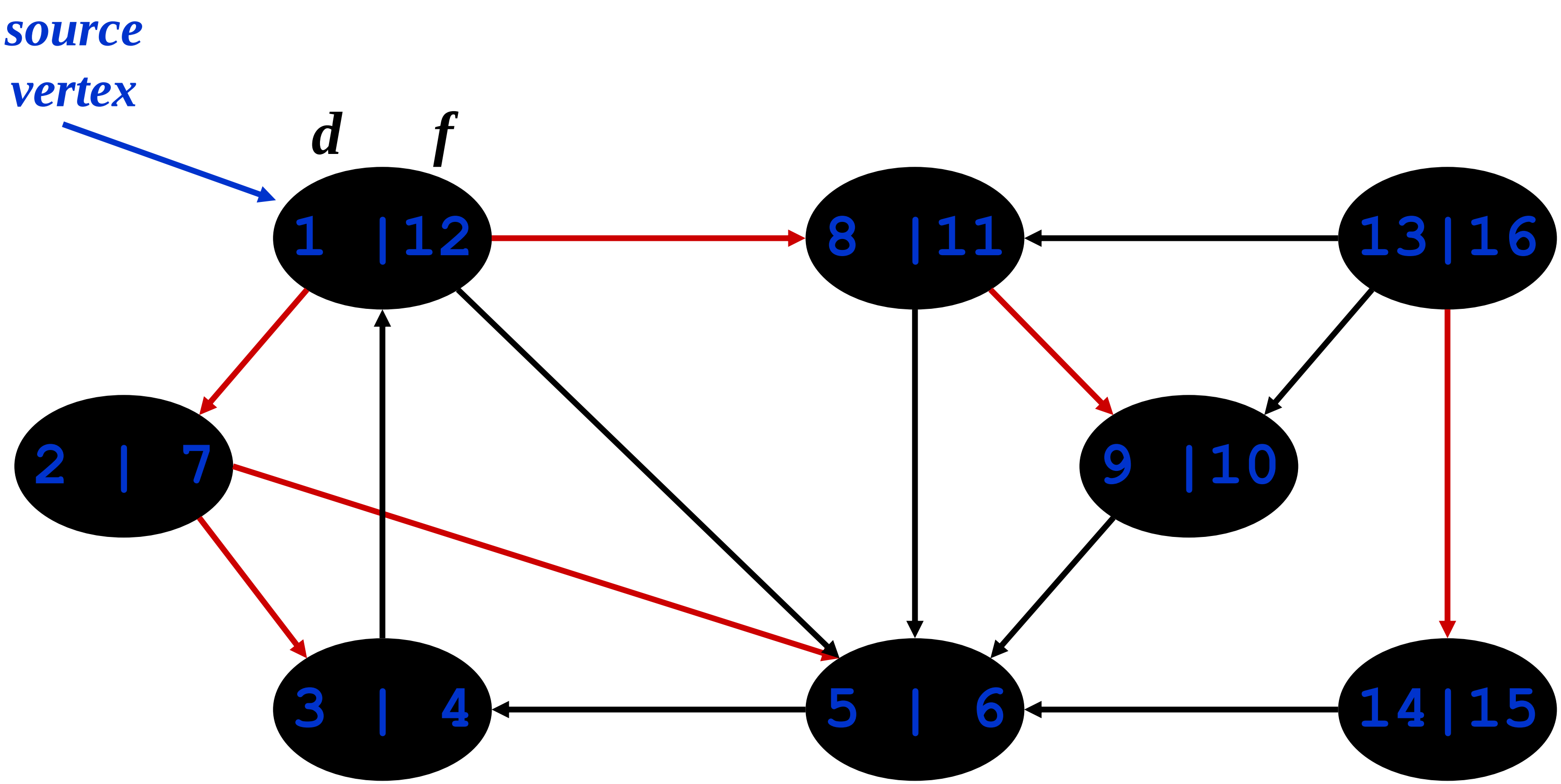
DFS Example



DFS Example



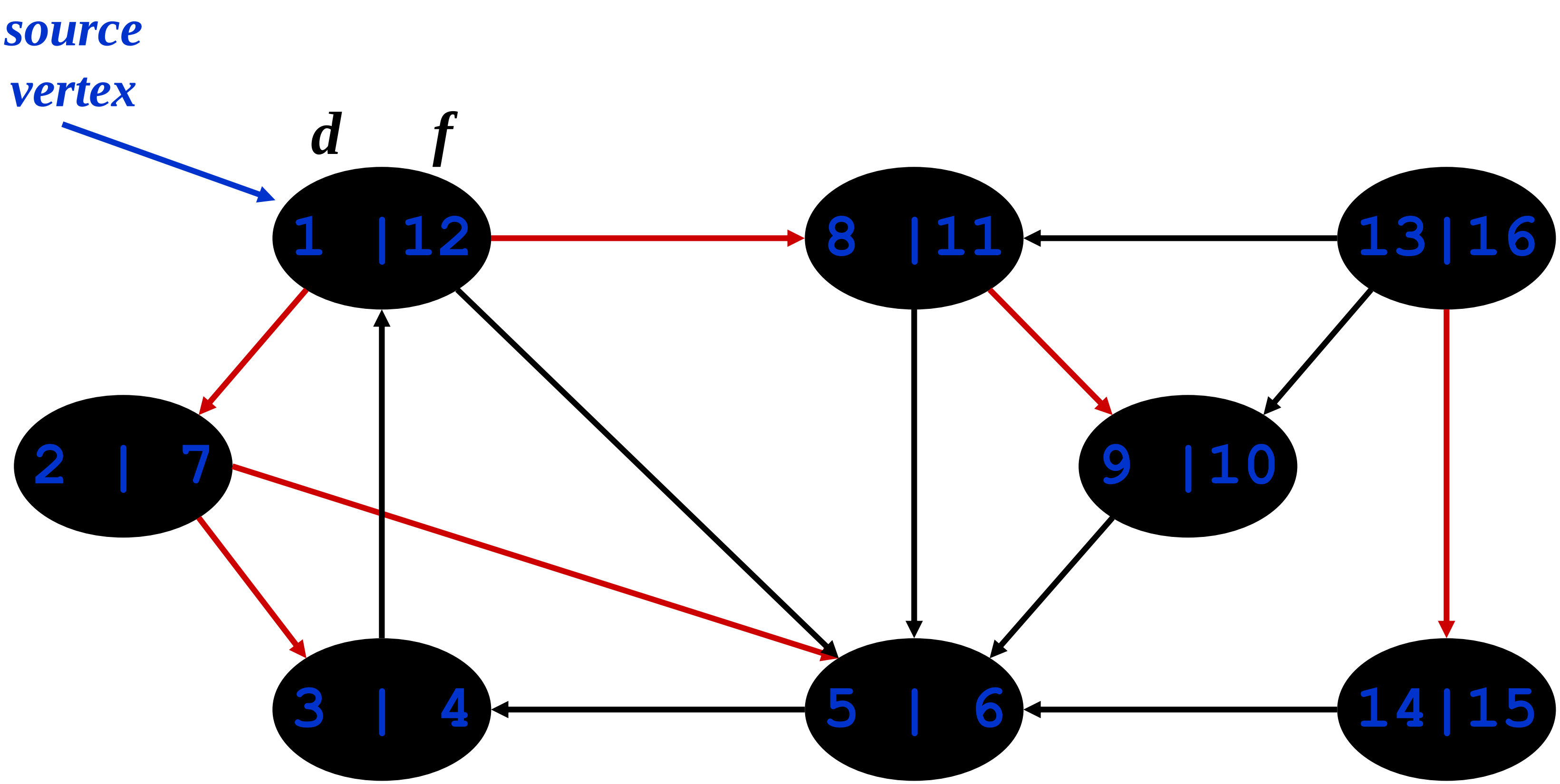
DFS Example



DFS: Kinds of edges

- DFS introduces an important distinction among edges in the original graph:
  - *Tree edge*: encounter new (white) vertex
    - The tree edges form a spanning forest
    - *Can tree edges form cycles? Why or why not?*

DFS Example

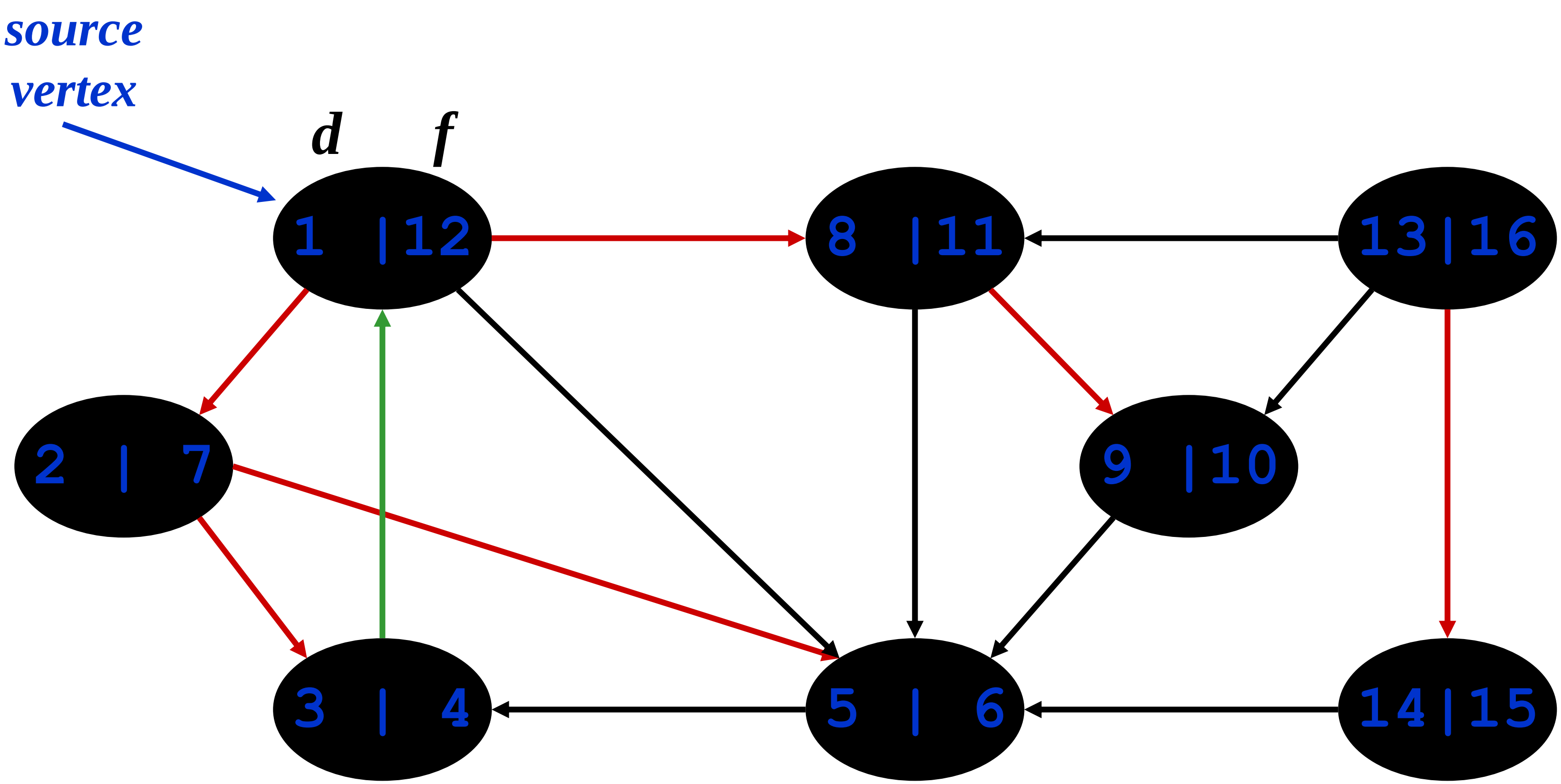


Tree edges

DFS: Kinds of edges

- DFS introduces an important distinction among edges in the original graph:
  - *Tree edge*: encounter new (white) vertex
  - *Back edge*: from descendent to ancestor
    - Encounter a grey vertex (grey to grey)

DFS Example

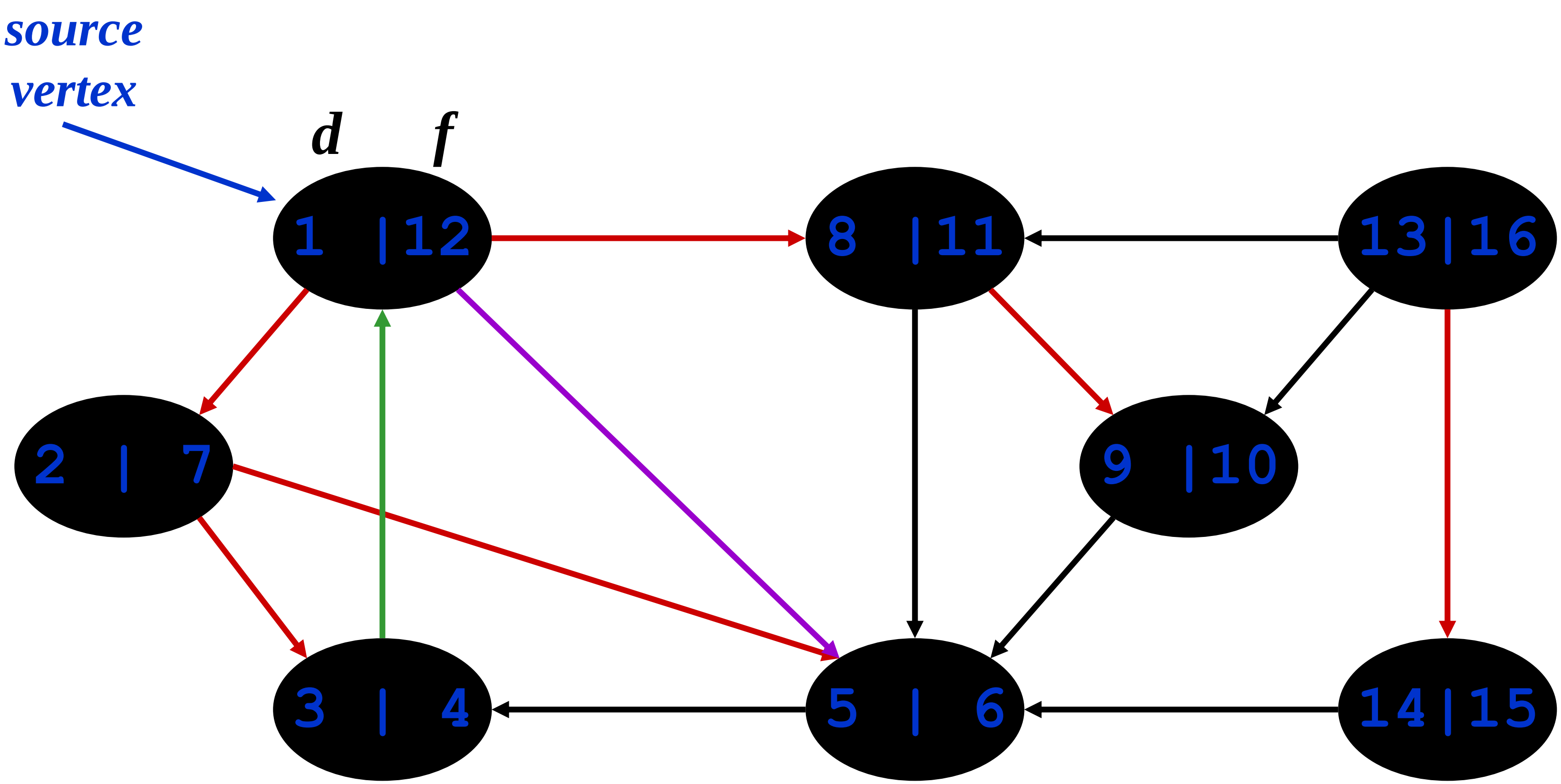


Tree edges *Back edges*

DFS: Kinds of edges

- DFS introduces an important distinction among edges in the original graph:
  - *Tree edge*: encounter new (white) vertex
  - *Back edge*: from descendent to ancestor
  - *Forward edge*: from ancestor to descendent
    - Not a tree edge, though
    - From grey node to black node

DFS Example

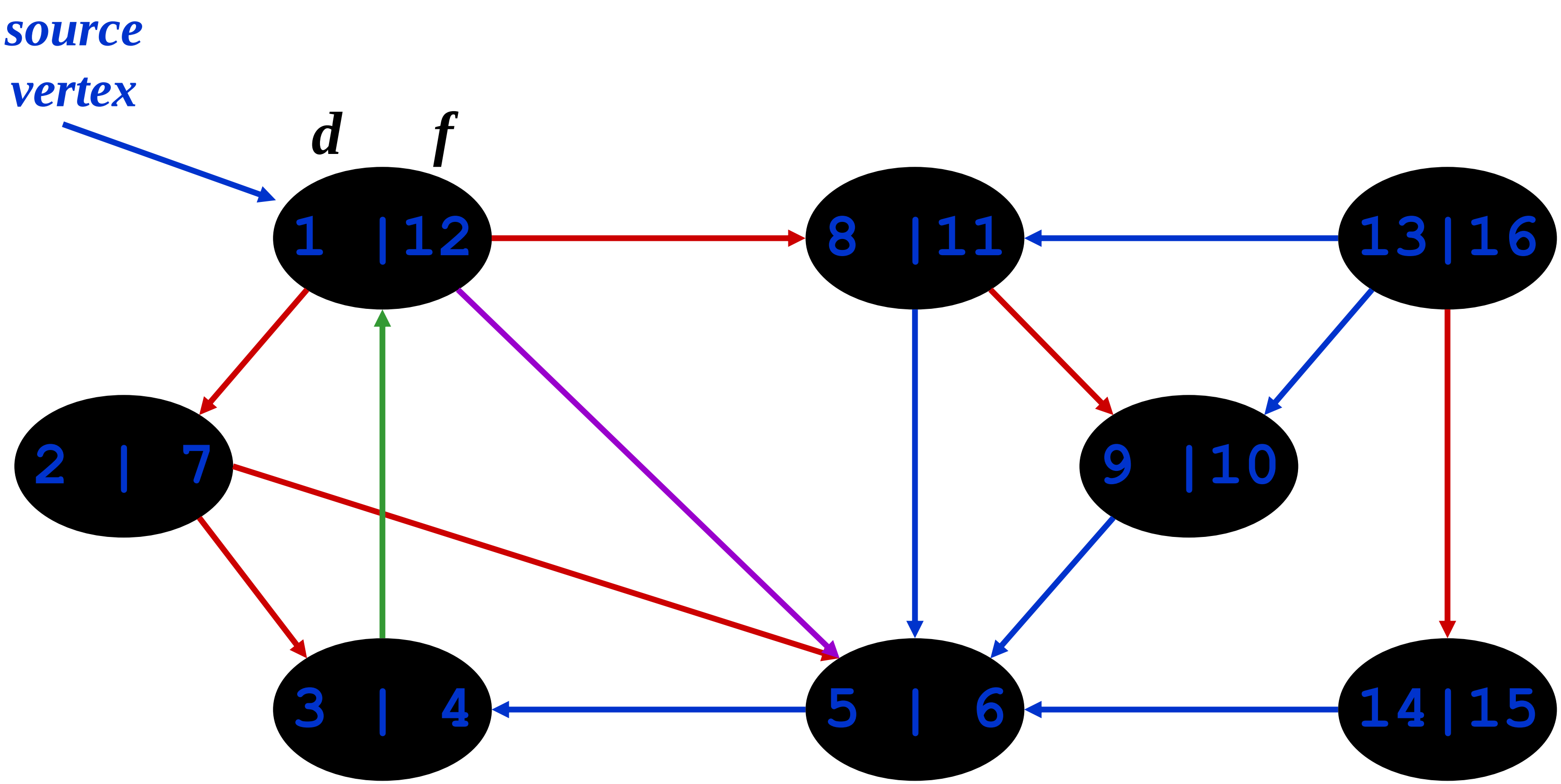


Tree edges ***Back edges*** ***Forward edges***

DFS: Kinds of edges

- DFS introduces an important distinction among edges in the original graph:
  - *Tree edge*: encounter new (white) vertex
  - *Back edge*: from descendent to ancestor
  - *Forward edge*: from ancestor to descendent
  - *Cross edge*: between a tree or subtrees
    - From a grey node to a black node

DFS Example



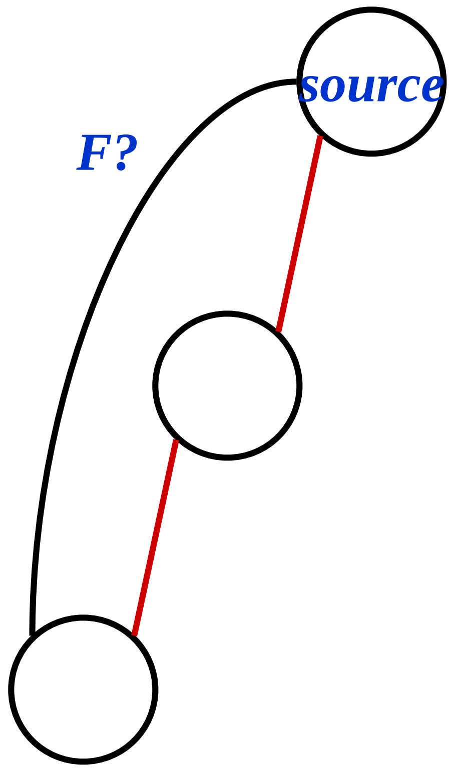
Tree edges *Back edges* *Forward edges* *Cross edges*

DFS: Kinds of edges

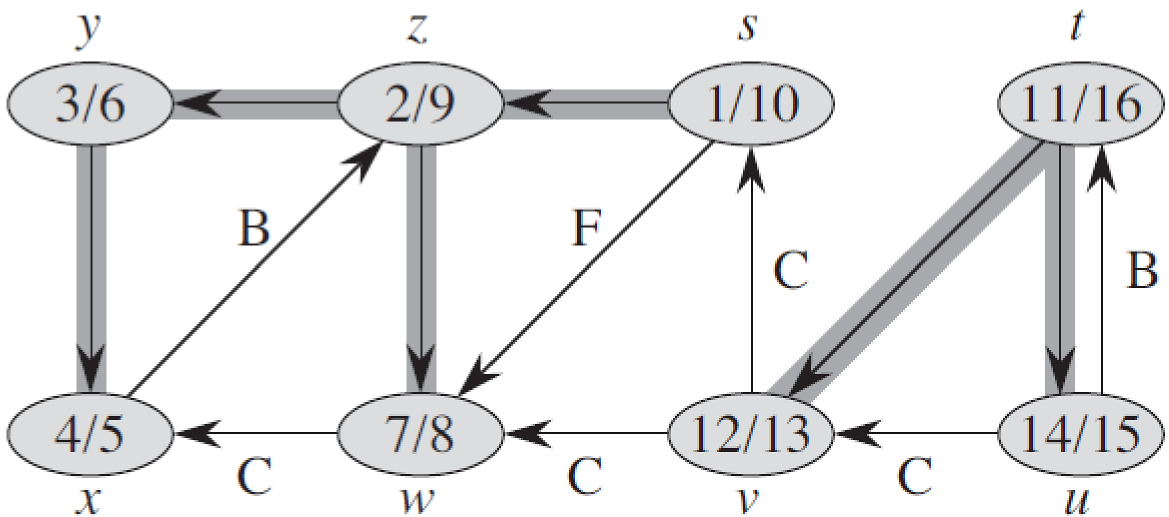
- DFS introduces an important distinction among edges in the original graph:
  - *Tree edge*: encounter new (white) vertex
  - *Back edge*: from descendent to ancestor
  - *Forward edge*: from ancestor to descendent
  - *Cross edge*: between a tree or subtrees
- Note: tree & back edges are important; most algorithms don't distinguish forward & cross

DFS: Kinds Of Edges

- Thm 23.9: If G is undirected, a DFS produces only tree and back edges
- Proof by contradiction:
  - Assume there's a forward edge
    - But F? edge must actually be a back edge (*why?*)



(a)


y
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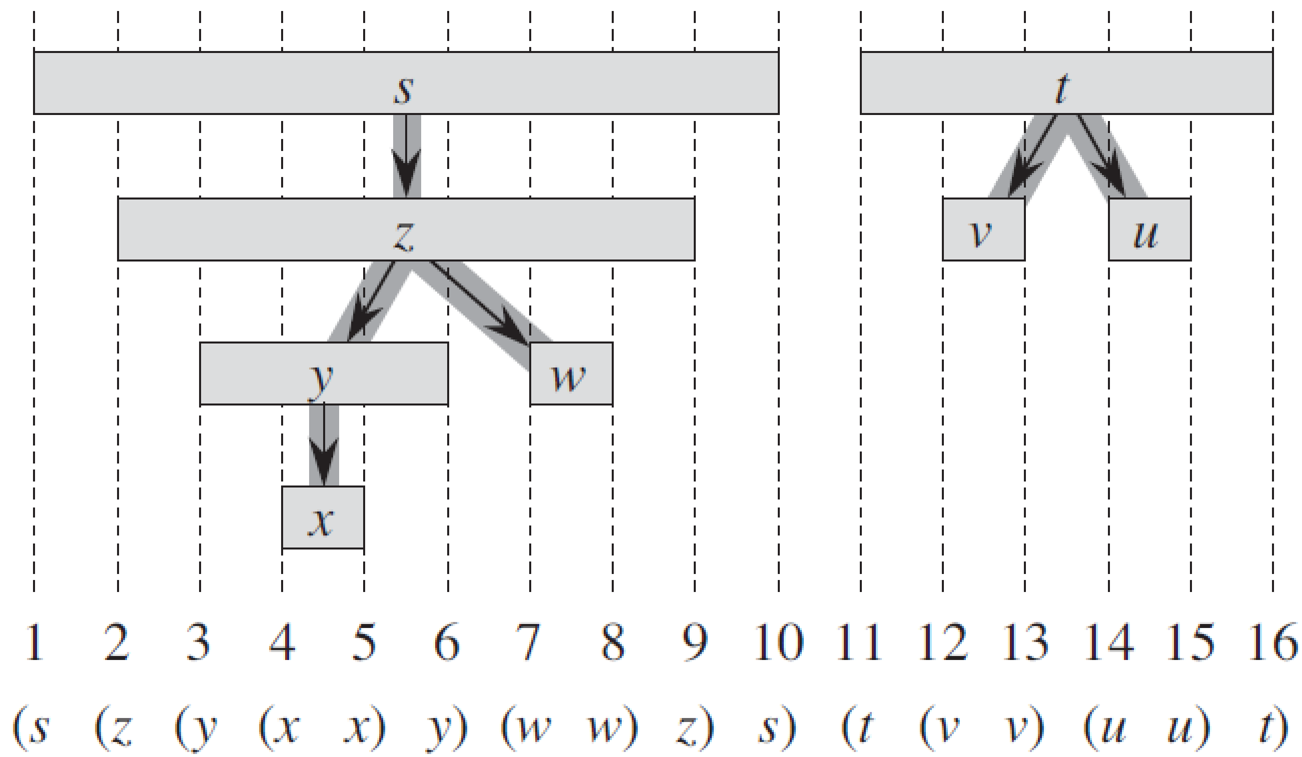
C
C
x
w
v
u

(b)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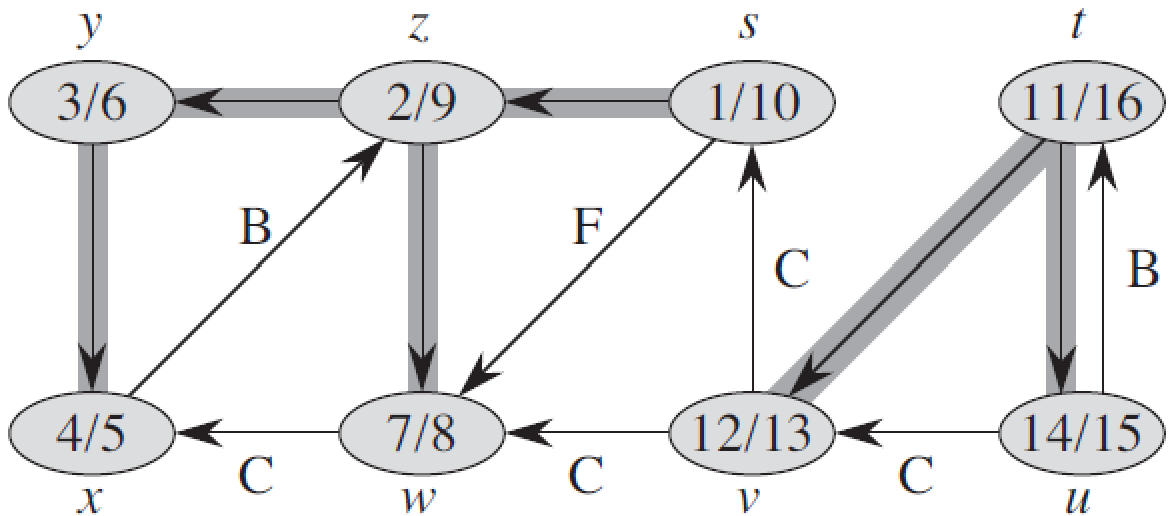
(a)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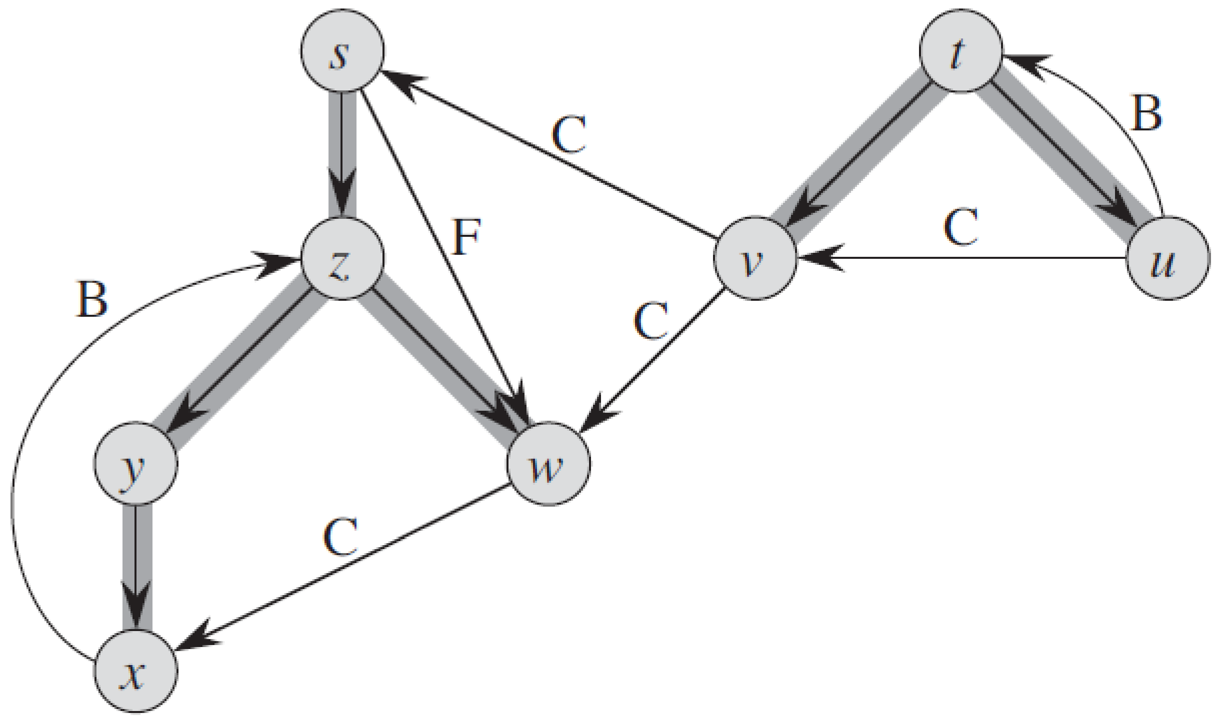
(c)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Theorem 22.7 (Parenthesis theorem)

In any depth-first search of a (directed or undirected) graph $G = (V, E)$, for any two vertices u and v, exactly one of the following three conditions holds:

- the intervals $[u.d, u.f]$ and $[v.d, v.f]$ are entirely disjoint, and neither u nor v is a descendant of the other in the depth-first forest,
- the interval $[u.d, u.f]$ is contained entirely within the interval $[v.d, v.f]$, and u is a descendant of v in a depth-first tree, or
- the interval $[v.d, v.f]$ is contained entirely within the interval $[u.d, u.f]$, and v is a descendant of u in a depth-first tree.

Corollary 22.8 (Nesting of descendants' intervals)

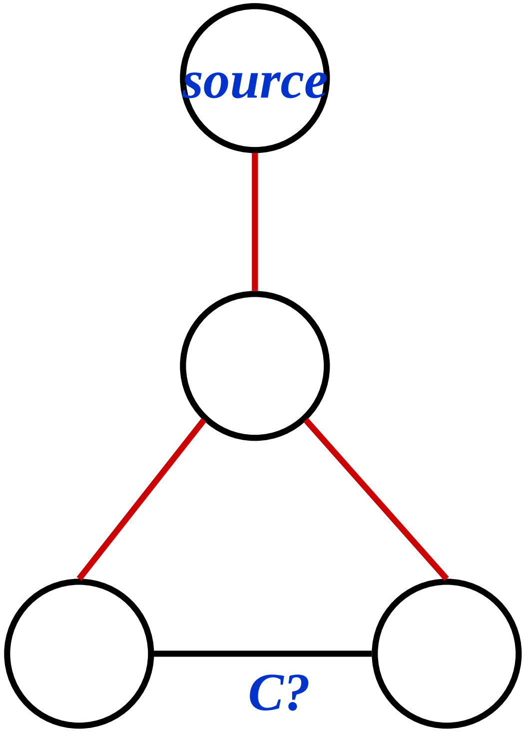
Vertex v is a proper descendant of vertex u in the depth-first forest for a (directed or undirected) graph G if and only if $u.d < v.d < v.f < u.f$.

Theorem 22.9 (White-path theorem)

In a depth-first forest of a (directed or undirected) graph $G = (V, E)$, vertex v is a descendant of vertex u if and only if at the time $u.d$ that the search discovers u, there is a path from u to v consisting entirely of white vertices.

DFS: Kinds Of Edges

- Thm 23.9: If G is undirected, a DFS produces only tree and back edges
- Proof by contradiction:
  - Assume there's a cross edge
    - But C? edge cannot be cross:
    - must be explored from one of the vertices it connects, becoming a tree vertex, before other vertex is explored
    - So in fact the picture is wrong…both lower tree edges cannot in fact be tree edges



DFS And Graph Cycles

- Thm: An undirected graph is *acyclic* iff a DFS yields no back edges
  - If acyclic, no back edges (because a back edge implies a cycle
  - If no back edges, acyclic
    - No back edges implies only tree edges (*Why?*)
    - Only tree edges implies we have a tree or a forest
    - Which by definition is acyclic
- Thus, can run DFS to find whether a graph has a cycle

DFS And Cycles

- *How would you modify the code to detect cycles?*

```
DFS(G)
{
   for each vertex u ∈ G->V
   {
      u->color = WHITE;
   }
   time = 0;
   for each vertex u ∈ G->V
   {
      if (u->color == WHITE)
         DFS_Visit(u);
   }
}
```

```
DFS_Visit(u)
{
   u->color = GREY;
   time = time+1;
   u->d = time;
   for each v ∈ u->Adj[]
   {
      if (v->color == WHITE)
         DFS_Visit(v);
   }
   u->color = BLACK;
   time = time+1;
   u->f = time;
}
```

DFS And Cycles

- *What will be the running time?*

```
DFS(G)
{
    for each vertex u ∈ G->V
    {
        u->color = WHITE;
    }
    time = 0;
    for each vertex u ∈ G->V
    {
        if (u->color == WHITE)
            DFS_Visit(u);
    }
}
```

```
DFS_Visit(u)
{
    u->color = GREY;
    time = time+1;
    u->d = time;
    for each v ∈ u->Adj[]
    {
        if (v->color == WHITE)
            DFS_Visit(v);
    }
    u->color = BLACK;
    time = time+1;
    u->f = time;
}
```

DFS And Cycles

- *What will be the running time?*
- A: O(V+E)
- We can actually determine if cycles exist in O(V) time:
  - In an undirected acyclic forest, $|E| \leq |V| - 1$
  - So count the edges: if ever see $|V|$ distinct edges, must have seen a back edge along the way

Homework

22.1-6

Most graph algorithms that take an adjacency-matrix representation as input require time $\Omega(V^2)$, but there are some exceptions. Show how to determine whether a directed graph G contains a ***universal sink***—a vertex with in-degree $|V| - 1$ and out-degree 0—in time $O(V)$, given an adjacency matrix for G.

22.2-6

Give an example of a directed graph $G = (V, E)$, a source vertex $s \in V$, and a set of tree edges $E_\pi \subseteq E$ such that for each vertex $v \in V$, the unique simple path in the graph (V, E_π) from s to v is a shortest path in G, yet the set of edges E_π cannot be produced by running BFS on G, no matter how the vertices are ordered in each adjacency list.

22.3-9

Give a counterexample to the conjecture that if a directed graph G contains a path from u to v, then any depth-first search must result in $v.d \leq u.f$.

22.4-4

Prove or disprove: If a directed graph G contains cycles, then TOPOLOGICAL-SORT(G) produces a vertex ordering that minimizes the number of "bad" edges that are inconsistent with the ordering produced.