第4章 不定积分

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§2　换元积分法

2.1　第一类换元法(凑微分法)

2.2　第二类换元法

2.1 第一类换元法(凑微分法)

问题： 如何求 $\int \frac{\ln x}{x} dx$ ？

该问题可转化为：找一个函数F(x),使得 $d\,F(x) = \frac{\ln x}{x} dx.$

解决方法 利用一阶微分形式不变性

$$\ln x d(\ln x) = \frac{\ln x}{x} dx$$

$$\Downarrow$$

$$d\left(\frac{1}{2} \cdot \ln^2 x\right) = \frac{\ln x}{x} dx$$

$$\therefore \int \frac{\ln x}{x} dx = \int \ln x d(\ln x) = \frac{1}{2} \ln^2 x + C$$

与 $\int u du = \frac{1}{2} u^2 + C$ 做比较

定理1（第一类换元法）设 $\int f(u)du = F(u) + C$，

且 $u = \varphi(x)$ 可微，则

$$\int f[\varphi(x)]\varphi'(x)dx = F[\varphi(x)] + C$$

可形象地表述为

$$\int f[\varphi(x)]\varphi'(x)dx \overset{\text{凑微分}}{=} \int f[\varphi(x)]d\varphi(x) \overset{\varphi(x)=u}{=} \int f(u)du$$

$$= F(u) + C \overset{u=\varphi(x)}{=} F[\varphi(x)] + C$$

这种积分方法称为第一类换元法（**凑微分法**）

说明 使用此公式的关键在于将 $\int g(x)dx$ 化为 $\int f[\varphi(x)]\varphi'(x)dx$.

观察重点不同，所得结论不同.

例1 求 $\int \sin 2x dx.$

解（一） $\int \sin 2x dx = \frac{1}{2}\int \sin 2x d(2x)$

$= -\frac{1}{2}\cos 2x + C;$

解（二） $\int \sin 2x dx = 2\int \sin x \cos x dx$

$= 2\int \sin x d(\sin x) = (\sin x)^2 + C;$

解（三） $\int \sin 2x dx = 2\int \sin x \cos x dx$

$= -2\int \cos x d(\cos x) = -(\cos x)^2 + C.$

例2 求 $\int \frac{1}{3+2x}dx.$

解 $$\frac{1}{3+2x}=\frac{1}{2}\cdot\frac{1}{3+2x}\cdot(3+2x)',$$

$$\int\frac{1}{3+2x}dx=\frac{1}{2}\int\frac{1}{3+2x}\cdot(3+2x)'dx$$

$$=\frac{1}{2}\int\frac{1}{u}du=\frac{1}{2}\ln|u|+C=\frac{1}{2}\ln|3+2x|+C.$$

一般地 $$\int f(ax+b)dx=\frac{1}{a}\left[\int f(u)du\right]_{u=ax+b}$$

例3 求 $\int \cos 3x \cos 2x dx.$

解 $\cos A \cos B = \frac{1}{2}[\cos(A-B)+\cos(A+B)],$

$$\cos 3x \cos 2x = \frac{1}{2}(\cos x + \cos 5x),$$

$$\int \cos 3x \cos 2x dx = \frac{1}{2}\int(\cos x + \cos 5x)dx$$

$$= \frac{1}{2}\sin x + \frac{1}{10}\sin 5x + C.$$

例4 求积分 $\int \tan x dx$.

解 $\int \tan x dx = \int \frac{\sin x}{\cos x} dx$

$= -\int \frac{d \cos x}{\cos x} = -\ln|\cos x| + C.$

例5 求 $\displaystyle\int \frac{1}{x(1+2\ln x)}dx.$

解 $\displaystyle\int \frac{1}{x(1+2\ln x)}dx = \int \frac{1}{1+2\ln x}d(\ln x)$

$$= \frac{1}{2}\int \frac{1}{1+2\ln x}d(1+2\ln x)$$

$$u = 1+2\ln x$$

$$= \frac{1}{2}\int \frac{1}{u}du = \frac{1}{2}\ln|u| + C = \frac{1}{2}\ln|1+2\ln x| + C.$$

例6 求 $\displaystyle\int \frac{1}{a^2+x^2}dx.$

解 $\displaystyle\int \frac{1}{a^2+x^2}dx = \frac{1}{a^2}\int \frac{1}{1+\frac{x^2}{a^2}}dx$

$$= \frac{1}{a}\int \frac{1}{1+\left(\frac{x}{a}\right)^2}d\left(\frac{x}{a}\right) = \frac{1}{a}\arctan\frac{x}{a} + C.$$

例7 求 $\int \frac{1}{\sqrt{a^2 - x^2}} dx.$

解 $\int \frac{1}{\sqrt{a^2 - x^2}} dx = \int \frac{1}{\sqrt{1 - \left(\frac{x}{a}\right)^2}} d\left(\frac{x}{a}\right) \quad a > 0$

$= \arcsin \frac{x}{a} + C.$

或 $= -\arccos \frac{x}{a} + C_1$

例8 求 $\int \frac{1}{x^2-8x+25}dx.$

解 $\int \frac{1}{x^2-8x+25}dx = \int \frac{1}{(x-4)^2+9}dx$

$$= \frac{1}{3^2}\int \frac{1}{\left(\frac{x-4}{3}\right)^2+1}dx = \frac{1}{3}\int \frac{1}{\left(\frac{x-4}{3}\right)^2+1}d\left(\frac{x-4}{3}\right)$$

$$= \frac{1}{3}\arctan\frac{x-4}{3}+C.$$

计算积分 $\int \frac{x+5}{x^2-8x+25}dx$

$$\int \frac{x+5}{x^2-8x+25}dx$$

$$=\int \frac{\frac{1}{2}(2x-8)+9}{x^2-8x+25}dx$$

$$=\frac{1}{2}\int \frac{d(x^2-8x+25)}{x^2-8x+25}+9\int \frac{1}{x^2-8x+25}dx$$

例9 求 $\int \frac{1}{1+e^x}dx.$

解 $\int \frac{1}{1+e^x}dx = \int \frac{1+e^x-e^x}{1+e^x}dx$

$$= \int \left(1-\frac{e^x}{1+e^x}\right)dx = \int dx - \int \frac{e^x}{1+e^x}dx$$

$$= \int dx - \int \frac{1}{1+e^x}d(1+e^x)$$

$$= x - \ln(1+e^x) + C.$$

例10 求 $\int\left(1-\frac{1}{x^{2}}\right)e^{x+\frac{1}{x}}dx.$

解 $\because \left(x+\frac{1}{x}\right)'=1-\frac{1}{x^{2}},$

$\therefore \int\left(1-\frac{1}{x^{2}}\right)e^{x+\frac{1}{x}}dx$

$=\int e^{x+\frac{1}{x}}d\left(x+\frac{1}{x}\right)=e^{x+\frac{1}{x}}+C.$

例11 求 $\int \frac{dx}{x^2-a^2}$

解：原式 $= \frac{1}{2a}\int(\frac{1}{x-a}-\frac{1}{x+a})dx$

$$= \frac{1}{2a}[\int \frac{d(x-a)}{x-a} - \int \frac{d(x+a)}{x+a}]$$

$$= \frac{1}{2a}\left[\ln|x-a| - \ln|x+a|\right] + c$$

$$= \frac{1}{2a}\ln\left|\frac{x-a}{x+a}\right| + c$$

$$\int \frac{1}{x^2-4x+1}dx$$

$$\int \frac{1}{x^2-4x+1}dx = \int \frac{1}{(x-2)^2-3}d(x-2)$$

$$= \frac{1}{2\sqrt{3}} \ln\left|\frac{x-2-\sqrt{3}}{x-2+\sqrt{3}}\right| + C$$

例12 求 $\int \frac{1}{1+\cos x} dx$.

解 $\int \frac{1}{1+\cos x} dx = \int \frac{1-\cos x}{(1+\cos x)(1-\cos x)} dx$

$$= \int \frac{1-\cos x}{1-\cos^2 x} dx = \int \frac{1-\cos x}{\sin^2 x} dx$$

$$= \int \frac{1}{\sin^2 x} dx - \int \frac{1}{\sin^2 x} d(\sin x)$$

$$= -\cot x + \frac{1}{\sin x} + C.$$

例13 求 $\int \csc x dx$.

解（一） $$\int \csc x dx = \int \frac{1}{\sin x} dx = \int \frac{1}{2\sin\frac{x}{2}\cos\frac{x}{2}} dx$$

$$= \int \frac{1}{\tan\frac{x}{2}\left(\cos\frac{x}{2}\right)^2} d\left(\frac{x}{2}\right) = \int \frac{1}{\tan\frac{x}{2}} d\left(\tan\frac{x}{2}\right)$$

$$= \ln\left|\tan\frac{x}{2}\right| + C = \ln\left|\csc x - \cot x\right| + C.$$

注：$\tan\dfrac{x}{2}=\dfrac{\sin\dfrac{x}{2}}{\cos\dfrac{x}{2}}=\dfrac{2\sin\dfrac{x}{2}\sin\dfrac{x}{2}}{2\cos\dfrac{x}{2}\sin\dfrac{x}{2}}$

$$=\frac{1-\cos x}{\sin x}=\csc x-\cot x$$

$$\int \csc x dx=\ln\left|\csc x-\cot x\right|+C.$$

$$=-\ln\left|\csc x+\cot x\right|+C.$$

解（二）$\int \csc x dx = \int \frac{1}{\sin x} dx = \int \frac{\sin x}{\sin^2 x} dx$

$$= -\int \frac{1}{1-\cos^2 x} d(\cos x) \qquad u = \cos x$$

$$= -\int \frac{1}{1-u^2} du = -\frac{1}{2}\int \left(\frac{1}{1-u} + \frac{1}{1+u} \right) du$$

$$= \frac{1}{2}\ln\left|\frac{1-u}{1+u}\right| + C = \frac{1}{2}\ln\left|\frac{1-\cos x}{1+\cos x}\right| + C.$$

类似地可推出 $\int \sec x dx = \ln|\sec x + \tan x| + C.$

例11 $\int \sin^3 x \cos^4 x dx$

解 $\int \sin^3 x \cos^4 x dx = -\int (1-\cos^2 x)\cos^4 x d\cos x$

$= -\int \cos^4 x d\cos x + \int \cos^6 x d\cos x$

$= -\frac{1}{5}\cos^5 x + \frac{1}{7}\cos^7 x + C$

例12 $\int \sin^2 x \cos^2 x dx$

解 $\int \sin^2 x \cos^2 x dx = \int \frac{1-\cos 2x}{2} \cdot \frac{1+\cos 2x}{2} dx$

$= \frac{1}{4} \int \sin^2 2x dx = \frac{1}{8} \int (1-\cos 4x) dx$

$= \frac{1}{8} x - \frac{1}{32} \sin 4x + C$

$\int \sin^m x \cos^n x dx$类型积分方法：

1. m,n有一个为奇数时，凑微分

2. m,n均为偶数时，降幂，降到一次幂后凑微分

对于 $\int \sin^m x\cos^n x dx(m,n\in N)$

$$(1)\ \int \sin^{2k+1} x\cos^n x\mathrm{d}x = -\int (1-\cos^2 x)^k \cos^n x\mathrm{d}\cos x$$

$$(2)\int \sin^m x\cos^{2k+1} x\mathrm{d}x = \int \sin^m x(1-\sin^2 x)^k \mathrm{d}\sin x$$

$$(3)\int \sin^{2m} x\cos^{2n} x\mathrm{d}x = \int (\frac{1-\cos 2x}{2})^m (\frac{1+\cos 2x}{2})^n \mathrm{d}x$$

常用的几种凑微分形式:

$$(1)\ \int f(ax+b)\mathrm{d}x = \frac{1}{a}\int f(ax+b)\,\mathrm{d}(ax+b)$$

$$\left.\begin{aligned}
&(2)\ \int f(x^n)x^{n-1}\,\mathrm{d}x = \frac{1}{n}\int f(x^n)\,\mathrm{d}x^n \\
&(3)\ \int f(x^n)\frac{1}{x}\mathrm{d}x = \frac{1}{n}\int f(x^n)\,\frac{1}{x^n}\,\mathrm{d}x^n
\end{aligned}\right\}\text{万能凑幂法}$$

$$(4)\ \int f(\sin x)\cos x\,\mathrm{d}x = \int f(\sin x)\,\mathrm{d}\sin x$$

$$(5)\ \int f(\cos x)\sin x\,\mathrm{d}x = -\int f(\cos x)\,\mathrm{d}\cos x$$

(6) $\int f(\tan x)\sec^2 x\mathrm{d}x = \int f(\tan x)\ \mathrm{d}\tan x$

(7) $\int f(e^x)e^x\mathrm{d}x = \int f(e^x)\ \mathrm{d}e^x$

(8) $\int f(\ln x)\frac{1}{x}\mathrm{d}x = \int f(\ln x)\ \mathrm{d}\ln x$

二、第二类换元法

$$\int g(x)dx = \int f[\varphi(x)]\varphi'(x)dx = \int f[\varphi(x)]d\varphi(x)$$

$$\overset{\text{查基本积分表}}{=} F[\varphi(x)]+C$$ ----第一类换元法

问题 $\int x^5\sqrt{1-x^2}dx = ?$

解决方法 改变中间变量的设置方法.

过程 令 $x=\sin t \Rightarrow dx=\cos t dt$,

$$\int x^5\sqrt{1-x^2}dx = \int(\sin t)^5\sqrt{1-\sin^2 t}\cos t dt$$

$$= \int \sin^5 t\cos^2 t dt = \cdots\cdots$$ 再用“凑微分”

$$\int f(x)dx \overset{x=\varphi(t)}{=} \int f[\varphi(t)]\varphi'(t)dt = F(t)+c \overset{t=\varphi^{-1}(x)}{=} F[\varphi^{-1}(x)]+c.$$

难

易

定理2 设 $x=\varphi(t)$ 是单调、可导函数，且 $\varphi'(t)\neq 0$

又设 $f[\varphi(t)]\varphi'(t)$ 具有原函数，即

$$\int f[\varphi(t)]\varphi'(t)dt=F(t)+c,\quad t=\varphi^{-1}(x)$$

则 $\int f(x)dx=F[\varphi^{-1}(x)]+c.$

证：只要证右端的导数等于左端的被积函数

由复合函数与反函数的导数，有

$$\frac{d}{dx}F[\varphi^{-1}(x)]=\frac{dF}{dt}\cdot\frac{dt}{dx}=F'(t)\frac{1}{\frac{dx}{dt}}=f[\varphi(t)]\varphi'(t)\cdot\frac{1}{\varphi'(t)}$$

$$=f[\varphi(t)]=f(x).$$

$$\int f(x)dx\overset{x=\varphi(t)}{=}\int f[\varphi(t)]\varphi'(t)dt=F(t)+c\overset{t=\varphi^{-1}(x)}{=}F[\varphi^{-1}(x)]+c.$$

$$\int f(x)dx \overset{x=\varphi(t)}{=} \int f[\varphi(t)]\varphi'(t)dt = F(t)+c \overset{t=\varphi^{-1}(x)}{=} F[\varphi^{-1}(x)]+c.$$

注：1）保证代换x=φ(t)的严格单调且可导（具有可导的反函数）；φ(t)应取得x的所有值！

2) $$\int f(x)dx = \left[\int f[\varphi(t)]\varphi'(t)dt\right]_{t=\varphi^{-1}(x)}$$

代换 x=φ(t)，被积函数和微分表达式一起换。

第二类换元积分公式

3）最后需将 $t=\varphi^{-1}(x)$ 反代回去。

两类换元法的区别与联系

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第一类换元法 $\longrightarrow$

$$\int g(x)dx = \int f[\varphi(x)]\varphi'(x)dx = \int f[\varphi(x)]d\varphi(x) \overset{u=\varphi(x)}{=} \int f(u)du$$

$\longleftarrow$ 第二类换元法

易 难

积分后需要将u反代回去

注意两类换元法对$u=\varphi(x)$的要求不同。

例9 求 $\int \frac{1}{\sqrt{x^2+a^2}}dx \quad (a>0).$

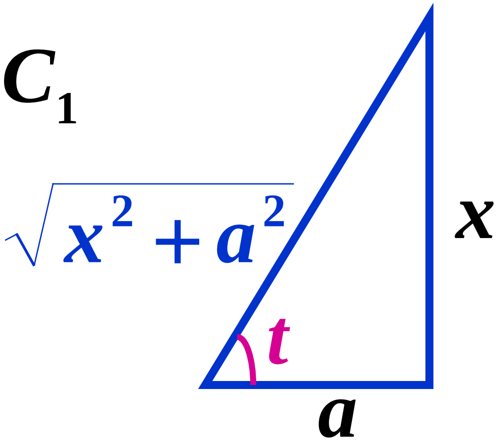
解 令 $x = a\tan t \Rightarrow dx = a\sec^2 t dt \quad t \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$

$$\int \frac{1}{\sqrt{x^2+a^2}}dx = \int \frac{1}{a\sec t} \cdot a\sec^2 t dt$$

$$= \int \sec t dt = \ln\left|\sec t + \tan t\right| + C_1$$

$$= \ln\left(\frac{x}{a} + \frac{\sqrt{x^2+a^2}}{a}\right) + C_1$$

$$= \ln\left(x + \sqrt{x^2+a^2}\right) + C.$$



例10 求 $\int x^3\sqrt{4-x^2}dx.$

解 令 $x=2\sin t \quad dx=2\cos t dt \quad t\in\left(-\frac{\pi}{2},\frac{\pi}{2}\right)$

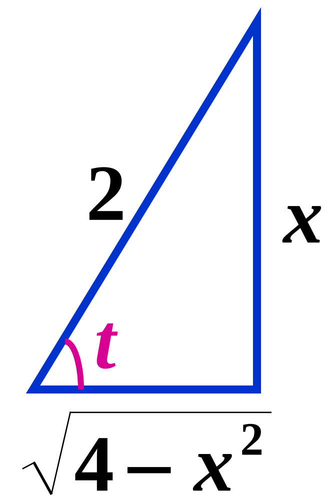
$$\int x^3\sqrt{4-x^2}dx=\int(2\sin t)^3\sqrt{4-4\sin^2 t}\cdot 2\cos t dt$$

$$=32\int \sin^3 t\cos^2 t dt=32\int \sin t(1-\cos^2 t)\cos^2 t dt$$

$$=-32\int(\cos^2 t-\cos^4 t)d\cos t$$

$$=-32\left(\frac{1}{3}\cos^3 t-\frac{1}{5}\cos^5 t\right)+C$$

$$=-\frac{4}{3}\left(\sqrt{4-x^2}\right)^3+\frac{1}{5}\left(\sqrt{4-x^2}\right)^5+C.$$



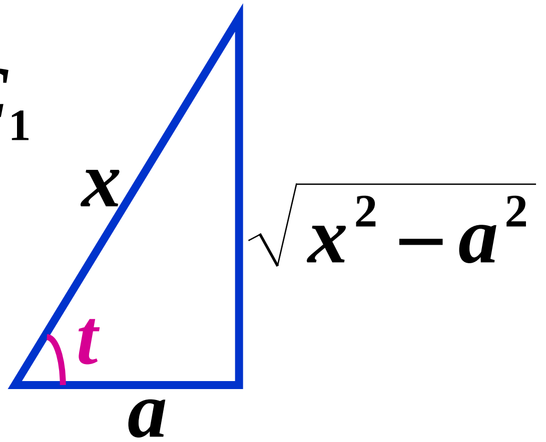
例11 求 $\int \frac{1}{\sqrt{x^2-a^2}}dx \quad (a>0).$

解 令 $x = a\sec t \quad dx = a\sec t\tan t dt \quad t\in\left(0,\frac{\pi}{2}\right)$

$$\int \frac{1}{\sqrt{x^2-a^2}}dx = \int \frac{a\sec t\cdot\tan t}{a\tan t}dt$$

$x>a$的情形！

$$=\int \sec t dt = \ln\left|\sec t+\tan t\right|+C_1$$



$$=\ln\left|\frac{x}{a}+\frac{\sqrt{x^2-a^2}}{a}\right|+C_1$$

$$=\ln\left|x+\sqrt{x^2-a^2}\right|+C.$$

$x < -a$时

令 $x = a\sec t$　　$dx = a\sec t\tan t dt$　　$t \in \left(\frac{\pi}{2}, \pi\right)$

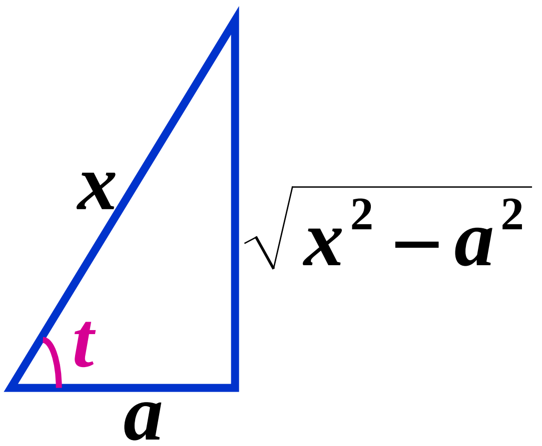
$$\int \frac{1}{\sqrt{x^2 - a^2}} dx = \int \frac{a\sec t \cdot \tan t}{-a\tan t} dt$$

$$= -\int \sec t dt = -\ln\left|\sec t + \tan t\right| + C_1$$

$$= -\ln\left|\frac{x}{a} - \frac{\sqrt{x^2 - a^2}}{a}\right| + C_1$$

进行分子有理化

$$= \ln\left|x + \sqrt{x^2 - a^2}\right| + C.$$



习题4.2 第33题

$$\int \frac{dx}{x\sqrt{x^2-1}} \quad 令 x=\sec t, t\in(0,\frac{\pi}{2})\cup(\frac{\pi}{2},\pi)$$

$x>1$, $t\in(0,\frac{\pi}{2})$

$$\int \frac{dx}{x\sqrt{x^2-1}}=\int \frac{\tan t\sec t}{\sec t\tan t}dt=\int dt=t+C_1=\arccos\frac{1}{x}+C_1$$

$x<-1$, $t\in(\frac{\pi}{2},\pi)$

$$=-\arcsin\frac{1}{x}+C$$

$$\int \frac{dx}{x\sqrt{x^2-1}}=-\int \frac{\tan t\sec t}{\sec t\tan t}dt=-t+C_2=-\arccos\frac{1}{x}+C_2$$

$$=\arcsin\frac{1}{x}+C=-\arcsin\frac{1}{|x|}+C$$

$$\therefore \int \frac{dx}{x\sqrt{x^2-1}}=-\arcsin\frac{1}{|x|}+C=\arccos\frac{1}{|x|}+C$$

说明(1) 以上几例所使用的均为三角代换.

三角代换的目的是化掉根式.

一般规律如下：当被积函数中含有

(1) $\sqrt{a^2-x^2}$ 可令 $x=a\sin t$; $t\in\left(-\frac{\pi}{2},\frac{\pi}{2}\right)$

或 $x=a\cos t, t\in(0,\pi)$

(2) $\sqrt{a^2+x^2}$ 可令 $x=a\tan t$; $t\in\left(-\frac{\pi}{2},\frac{\pi}{2}\right)$

或 $x=a\cot t, t\in(0,\pi)$

(3) $\sqrt{x^2-a^2}$ 可令 $x=a\sec t$. $t\in(0,\frac{\pi}{2})\cup(\frac{\pi}{2},\pi)$

或 $x=a\csc t, t\in(-\frac{\pi}{2},0)\cup(0,\frac{\pi}{2})$

说明(2) 积分中为了化掉根式除采用三角代换外还可用双曲代换.

$$\because \operatorname{ch}^2 t - \operatorname{sh}^2 t = 1$$

$\therefore\ x = a\operatorname{sh} t,\ \ x = a\operatorname{ch} t$ 也可以化掉根式

在前面 **例** $\displaystyle\int \frac{1}{\sqrt{x^2+a^2}}dx$ 中, 令

$x = a\sinh t$ $dx = a\cosh t dt$

$$\int \frac{1}{\sqrt{x^2+a^2}}dx = \int \frac{a\cosh t}{a\cosh t}dt = \int dt = t + C$$

$$= \operatorname{arcsinh}\frac{x}{a} + C = \ln\left(\frac{x}{a} + \frac{\sqrt{x^2+a^2}}{a}\right) + C.$$

说明(3) 积分中为了化掉根式是否一定采用三角代换（或双曲代换）并不是绝对的，需根据被积函数的情况来定.

例12 求 $\int \frac{x^5}{\sqrt{1+x^2}}dx$ （三角代换很繁琐）

解 令 $t=\sqrt{1+x^2} \Rightarrow x^2=t^2-1, \quad xdx=tdt,$

$$\int \frac{x^5}{\sqrt{1+x^2}}dx=\int \frac{(t^2-1)^2}{t}tdt=\int (t^4-2t^2+1)dt$$

$$=\frac{1}{5}t^5-\frac{2}{3}t^3+t+C=\frac{1}{15}(8-4x^2+3x^4)\sqrt{1+x^2}+C.$$

例12 求 $\int \frac{1}{\sqrt{1+e^x}}dx.$

解 令 $t=\sqrt{1+e^x} \Rightarrow e^x=t^2-1,$

$$x=\ln\left(t^2-1\right), \quad dx=\frac{2t}{t^2-1}dt,$$

$$\int \frac{1}{\sqrt{1+e^x}}dx=\int \frac{2}{t^2-1}dt=\int\left(\frac{1}{t-1}-\frac{1}{t+1}\right)dt$$

$$=\ln\left|\frac{t-1}{t+1}\right|+C=2\ln\left(\sqrt{1+e^x}-1\right)-x+C.$$

例15 求 $\int \frac{1}{\sqrt{x}(1+\sqrt[3]{x})}dx.$

解 令 $x = t^6 \Rightarrow dx = 6t^5 dt,$

$$\int \frac{1}{\sqrt{x}(1+\sqrt[3]{x})}dx = \int \frac{6t^5}{t^3(1+t^2)}dt = \int \frac{6t^2}{1+t^2}dt$$

$$= 6\int \frac{t^2+1-1}{1+t^2}dt \quad = 6\int \left(1 - \frac{1}{1+t^2}\right)dt$$

$$= 6[\sqrt[6]{x} - \arctan\sqrt[6]{x}] + C.$$

说明(4) 当被积函数含有两种或两种以上的根式 $\sqrt[k]{x},\cdots,\sqrt[l]{x}$ 时，可采用令 $x = t^n$（其中n为各根指数的最小公倍数）

说明(5) 当分母的阶较高时，可采用**倒代换** $x=\dfrac{1}{t}(\neq 0)$.

例13 求 $\displaystyle\int\frac{1}{x(x^7+2)}dx$

$$\int\frac{x^6}{x^7(x^7+2)}dx=\frac{1}{7}\int\frac{1}{x^7(x^7+2)}dx^7$$

解 令 $x=\dfrac{1}{t}\Rightarrow dx=-\dfrac{1}{t^2}dt$,

$$\int\frac{1}{x(x^7+2)}dx=\int\frac{t}{\left(\frac{1}{t}\right)^7+2}\cdot\left(-\frac{1}{t^2}\right)dt$$

$$=-\int\frac{t^6}{1+2t^7}dt\quad=-\frac{1}{14}\ln|1+2t^7|+C$$

$$=-\frac{1}{14}\ln|2+x^7|+\frac{1}{2}\ln|x|+C.$$

例14 求 $\int \frac{\sqrt{a^2-x^2}}{x^4}\,\mathrm{d}x$. （分母的阶较高）

倒代换：令 $t=\frac{1}{x}$, $\int \frac{\sqrt{a^2-x^2}}{x^4}\mathrm{d}x=\int t^4\sqrt{a^2-\frac{1}{t^2}}\left(-\frac{1}{t^2}\right)\mathrm{d}t$

$$=-\int t^2\frac{\sqrt{a^2t^2-1}}{|t|}\mathrm{d}t=\begin{cases}-\int t\sqrt{a^2t^2-1}\,\mathrm{d}t, t>0\\ \int t\sqrt{a^2t^2-1}\,\mathrm{d}t, t<0\end{cases}$$

$$=\begin{cases}-\frac{1}{2a^2}\int\sqrt{a^2t^2-1}\,\mathrm{d}(a^2t^2-1), t>0\\ \frac{1}{2a^2}\int\sqrt{a^2t^2-1}\,\mathrm{d}(a^2t^2-1), t<0\end{cases}=-\frac{1}{3a^2}\left(\frac{\sqrt{a^2-x^2}}{x}\right)^3+C$$

$$\int \frac{\sqrt{a^2 - x^2}}{x^4} \, \mathrm{d}x$$

方法2：令$x = a\sin t, t \in [-\frac{\pi}{2}, \frac{\pi}{2}]$

$$\int \frac{\sqrt{a^2 - x^2}}{x^4} \mathrm{d}x = \int \frac{a\cos t}{a^4 \sin^4 t} a\cos t \, \mathrm{d}t$$

$$= \frac{1}{a^2} \int \frac{\cos^2 t}{\sin^4 t} \mathrm{d}t = \frac{1}{a^2} \int \frac{\cos^2 t}{\sin^2 t} \frac{\mathrm{d}t}{\sin^2 t}$$

$$= -\frac{1}{a^2} \int \cot^2 t \quad d\cot t = -\frac{1}{3a^2} \cot^3 t + C$$

$$= -\frac{1}{3a^2} \left(\frac{\sqrt{a^2 - x^2}}{x}\right)^3 + C$$

1.2　基本积分公式

利用逆向思维

(0) $\int 0dx = C$

(1) $\int 1dx = x + C$

(2) $\int x^{\mu}dx = \frac{x^{\mu+1}}{\mu+1} + C \quad (\mu \neq -1);$

(3) $\int \frac{dx}{x} = \ln|x| + C;$

(4) $\int a^{x}dx = \frac{a^{x}}{\ln a} + C;$

(5) $\int e^{x}dx = e^{x} + C$

(6) $\int \cos x dx = \sin x + C$;

(7) $\int \sin x dx = -\cos x + C$;

(8) $\int \frac{dx}{\cos^2 x} = \int \sec^2 x dx = \tan x + C$;

(9) $\int \frac{dx}{\sin^2 x} = \int \csc^2 x dx = -\cot x + C$;

(10) $\int \sec x \tan x dx = \sec x + C$

(11) $\int \csc x \cot x dx = -\csc x + C$

(12) $\int \frac{1}{\sqrt{1-x^2}}dx = \arcsin x + C$; 或 $= -\arccos x + C$.

(13) $\int \frac{1}{1+x^2}dx = \arctan x + C$; 或 $= -\operatorname{arccot} x + C$.

(14) $\int \tan x dx = -\ln|\cos x| + C$;

(15) $\int \cot x dx = \ln|\sin x| + C$;

(16) $\int \sec x dx = \ln|\sec x + \tan x| + C$;

(17) $\int \csc x dx = \ln|\csc x - \cot x| + C$;

$$(18)\quad \int \frac{1}{a^2+x^2}dx=\frac{1}{a}\arctan\frac{x}{a}+C;$$

$$(19)\quad \int \frac{1}{x^2-a^2}dx=\frac{1}{2a}\ln\left|\frac{x-a}{x+a}\right|+C;$$

$$(20)\quad \int \frac{1}{a^2-x^2}dx=\frac{1}{2a}\ln\left|\frac{a+x}{a-x}\right|+C;$$

$$(21)\quad \int \frac{1}{\sqrt{a^2-x^2}}dx=\arcsin\frac{x}{a}+C;$$

$$(22)\quad \int \frac{1}{\sqrt{x^2\pm a^2}}dx=\ln\left|x+\sqrt{x^2\pm a^2}\right|+C.$$

例22. $\int \frac{x+1}{\sqrt{x^2-2x-3}}dx$

$$\int \frac{1}{\sqrt{x^2 \pm a^2}}dx = \ln\left|x+\sqrt{x^2 \pm a^2}\right| + C$$

$$\int \frac{x+1}{\sqrt{x^2-2x-3}}dx = \int \frac{\frac{1}{2}(2x-2)+2}{\sqrt{x^2-2x-3}}dx$$

$$= \frac{1}{2}\int \frac{d(x^2-2x-3)}{\sqrt{x^2-2x-3}} + 2\int \frac{1}{\sqrt{x^2-2x-3}}dx$$

$$= \sqrt{x^2-2x-3} + 2\int \frac{1}{\sqrt{(x-1)^2-4}}d(x-1)$$

$$= \sqrt{x^2-2x-3} + 2\ln|x-1+\sqrt{(x-1)^2-4}| + C$$

$$= \sqrt{x^2-2x-3} + 2\ln|x-1+\sqrt{x^2-2x-3}| + C$$

思考：以下几种形式的积分，如何积分：

$$\int \frac{cx+d}{ax^2+bx+c}dx$$

$$\int \frac{Ax+B}{\sqrt{ax^2+bx+c}}dx$$

$$\int \frac{1}{a^2\cos^2 x+b^2\sin^2 x}dx$$

小结

两类积分换元法：

（一） 凑微分

（二） 三角代换、倒代换、根式代换

基本积分公式(续)

思考题

求积分 $\int (x\ln x)^p(\ln x+1)dx.$

求积分 $\int \frac{x^4+1}{x^6+1}dx.$

思考题解答

$$\because d(x\ln x)=(1+\ln x)dx$$

$$\therefore \int (x\ln x)^p(\ln x+1)dx=\int (x\ln x)^p d(x\ln x)$$

$$=\begin{cases}\dfrac{(x\ln x)^{p+1}}{p+1}+C, & p\neq -1\\ \ln\left|x\ln x\right|+C, & p=-1\end{cases}$$

提示：$x^6+1=(x^2+1)(x^4-x^2+1)$

$$\int\frac{x^4+1}{x^6+1}dx=\int\frac{(x^4-x^2+1)+x^2}{x^6+1}dx$$

$$=\int\frac{x^4-x^2+1}{x^6+1}dx+\int\frac{x^2}{x^6+1}dx$$

$$=\int\frac{1}{x^2+1}dx+\frac{1}{3}\int\frac{dx^3}{x^6+1}$$

$$=\arctan x+\frac{1}{3}\arctan x^3+C.$$