第三节（1）定积分的换元法

一、换元公式

定理 假设（1）$f(x)$在$[a,b]$上连续；

（2）函数$x=\varphi(t)$在$[\alpha,\beta]$上是单值的且有连续导数；

（3）当t在区间$[\alpha,\beta]$上变化时，$x=\varphi(t)$的值在$[a,b]$上变化，且$\varphi(\alpha)=a$、$\varphi(\beta)=b$，

则 有$\int_a^b f(x)dx=\int_\alpha^\beta f[\varphi(t)]\varphi'(t)dt.$

证 设$F(x)$是$f(x)$的一个原函数，

$$\int_a^b f(x)dx = F(b) - F(a),$$

令$\Phi(t) = F[\phi(t)]$,

$$\Phi'(t) = \frac{dF}{dx}\cdot\frac{dx}{dt} = f(x)\varphi'(t) = f[\varphi(t)]\varphi'(t),$$

$\therefore \Phi(t)$是$f[\varphi(t)]\varphi'(t)$的一个原函数.

$$\int_\alpha^\beta f[\phi(t)]\phi'(t)dt = \Phi(\beta) - \Phi(\alpha) = F[\varphi(\beta)] - F[\varphi(\alpha)]$$

$$= F(b) - F(a) = \int_a^b f(x)dx$$

注意 当$\alpha > \beta$时，换元公式仍成立.

应用换元公式时应注意:

（1）用 $x=\varphi(t)$ 把变量 x 换成新变量 t 时，积分限也相应的改变.

(2) 求出 $f[\varphi(t)]\varphi'(t)$ 的一个原函数 $\Phi(t)$ 后，不必还原成变量 x 的函数，只要把 t 的上、下限分别代入 $\Phi(t)$ 然后相减就行了.

（3）$x=\varphi(t)$ 要求是单值函数。

（4） 定积分换元法可以用来证明积分等式，

关键在于 $x=\varphi(t)$ 的构造。这与积分的上下限以及被积函数的形式有关。

(5) 定积分换元比不定积分换元的条件要弱，可能性更多。

例1　计算 $\int_0^{\frac{\pi}{2}} \cos^5 x \sin x dx.$

解　令 $t = \cos x, \quad dt = -\sin x dx,$

$$x = \frac{\pi}{2} \Rightarrow t = 0, \qquad x = 0 \Rightarrow t = 1,$$

$$\int_0^{\frac{\pi}{2}} \cos^5 x \sin x dx$$

$$= -\int_1^0 t^5 dt = \left.\frac{t^6}{6}\right|_0^1 = \frac{1}{6}.$$

换元要换限 凑元不换限

例2 计算 $\int_0^{\pi}\sqrt{\sin^3 x-\sin^5 x}\,dx.$

解 $\because f(x)=\sqrt{\sin^3 x-\sin^5 x}=|\cos x|(\sin x)^{\frac{3}{2}}$

$$\therefore \int_0^{\pi}\sqrt{\sin^3 x-\sin^5 x}\,dx=\int_0^{\pi}|\cos x|(\sin x)^{\frac{3}{2}}dx$$

$$=\int_0^{\frac{\pi}{2}}\cos x(\sin x)^{\frac{3}{2}}dx-\int_{\frac{\pi}{2}}^{\pi}\cos x(\sin x)^{\frac{3}{2}}dx$$

$$=\int_0^{\frac{\pi}{2}}(\sin x)^{\frac{3}{2}}d\sin x-\int_{\frac{\pi}{2}}^{\pi}(\sin x)^{\frac{3}{2}}d\sin x$$

$$=\frac{2}{5}(\sin x)^{\frac{5}{2}}\Big|_0^{\frac{\pi}{2}}-\frac{2}{5}(\sin x)^{\frac{5}{2}}\Big|_{\frac{\pi}{2}}^{\pi}=\frac{4}{5}.$$

$$\int_0^{\pi}\sqrt{\sin^3 x-\sin^5 x}dx.$$

令: $x=\pi-t$

$$=\int_0^{\pi/2}\sqrt{\sin^3 x-\sin^5 x}dx+\int_{\pi/2}^{\pi}\sqrt{\sin^3 x-\sin^5 x}dx$$

$$=2\int_0^{\pi/2}\sqrt{\sin^3 x-\sin^5 x}dx,$$

$$=2\int_0^{\frac{\pi}{2}}\cos x\left(\sin x\right)^{\frac{3}{2}}dx=2\int_0^{\frac{\pi}{2}}\left(\sin x\right)^{\frac{3}{2}}d\sin x$$

$$=\frac{4}{5}\left(\sin x\right)^{\frac{5}{2}}\Big|_0^{\frac{\pi}{2}}=\frac{4}{5}$$

例3　计算 $\int_{\sqrt{e}}^{e^{\frac{3}{4}}}\frac{dx}{x\sqrt{\ln x(1-\ln x)}}$.

解　原式 $=\int_{\sqrt{e}}^{e^{\frac{3}{4}}}\frac{d(\ln x)}{\sqrt{\ln x(1-\ln x)}}$

$$=\int_{\sqrt{e}}^{e^{\frac{3}{4}}}\frac{d(\ln x)}{\sqrt{\ln x}\sqrt{(1-\ln x)}}=2\int_{\sqrt{e}}^{e^{\frac{3}{4}}}\frac{d\sqrt{\ln x}}{\sqrt{1-(\sqrt{\ln x})^2}}$$

$$=2\left[\arcsin(\sqrt{\ln x})\right]_{\sqrt{e}}^{e^{\frac{3}{4}}}=\frac{\pi}{6}.$$

例4 1) 计算 $\int_1^4 \frac{dx}{x(1+\sqrt{x})}$

解 令 $t=\sqrt{x}$, 则 $x=t^2, dx=2tdt,$

$$\text{原式}=\int_1^2 \frac{2dt}{t(1+t)} = 2\int_1^2 \left(\frac{1}{t}-\frac{1}{1+t}\right)dt$$

$$=2[\ln t-\ln(1+t)]_1^2 = 2\ln\frac{4}{3}$$

例4 2) 计算 $\int_0^1 x\sqrt{1-x}dx$

解：令 $t=\sqrt{1-x}, x=1-t^2, dx=-2tdt$

$$\text{原式}=\int_1^0 t(1-t^2)(-2t)dt=\int_0^1(2t^2-4t^4)dt=\frac{4}{15}$$

例5 计算 $\int_0^a \frac{1}{x+\sqrt{a^2-x^2}}dx. \quad (a>0)$

解 令 $x = a\sin t, \quad 0 \le t \le \frac{\pi}{2} \quad dx = a\cos t dt,$

$$x = a \Rightarrow t = \frac{\pi}{2}, \quad x = 0 \Rightarrow t = 0,$$

$$原式 = \int_0^{\frac{\pi}{2}} \frac{a\cos t}{a\sin t + \sqrt{a^2(1-\sin^2 t)}}dt$$

$$= \int_0^{\frac{\pi}{2}} \frac{\cos t}{\sin t + \cos t}dt = \frac{1}{2}\int_0^{\frac{\pi}{2}} \left(1 + \frac{\cos t - \sin t}{\sin t + \cos t}\right)dt$$

$$= \frac{1}{2}\cdot\frac{\pi}{2} + \frac{1}{2}\left[\ln\left|\sin t + \cos t\right|\right]_0^{\frac{\pi}{2}} = \frac{\pi}{4}.$$

例 6 当$f(x)$在$[-a,a]$上连续，则

(1) $\int_{-a}^{a} f(x)dx = \int_{0}^{a}\left[f(x)+f(-x)\right]dx$

(2) $f(x)$为偶函数，则

$$\int_{-a}^{a} f(x)dx = 2\int_{0}^{a} f(x)dx;$$

(3) $f(x)$为奇函数，则$\int_{-a}^{a} f(x)dx = 0.$

证 $\int_{-a}^{a} f(x)dx = \int_{-a}^{0} f(x)dx + \int_{0}^{a} f(x)dx,$

在 $\int_{-a}^{0} f(x)dx$ 中令 $x = -t,$

$$\int_{-a}^{0} f(x)dx = -\int_{a}^{0} f(-t)dt = \int_{0}^{a} f(-t)dt = \int_{0}^{a} f(-x)dx,$$

$$\therefore \int_{-a}^{a} f(x)dx = \int_{0}^{a} [f(x) + f(-x)]dx$$

$$\int_{-a}^{a} f(x)dx = \int_{0}^{a} [f(x) + f(-x)]dx$$

①$f(x)$为偶函数，则 $f(-t) = f(t)$,

$$\int_{-a}^{a} f(x)dx = \int_{-a}^{0} f(x)dx + \int_{0}^{a} f(x)dx$$

$$= 2\int_{0}^{a} f(t)dt;$$

②$f(x)$为奇函数，则 $f(-t) = -f(t)$,

$$\int_{-a}^{a} f(x)dx = \int_{-a}^{0} f(x)dx + \int_{0}^{a} f(x)dx = 0.$$

例7 计算$\int_{-1}^{1}\frac{2x^2+x\cos x}{1+\sqrt{1-x^2}}dx.$

解 原式$=\int_{-1}^{1}\underbrace{\boxed{\frac{2x^2}{1+\sqrt{1-x^2}}}}_{\text{偶函数}}dx+\int_{-1}^{1}\underbrace{\boxed{\frac{x\cos x}{1+\sqrt{1-x^2}}}}_{\text{奇函数}}dx$

$$=4\int_0^1\frac{x^2}{1+\sqrt{1-x^2}}dx=4\int_0^1\frac{x^2(1-\sqrt{1-x^2})}{1-(1-x^2)}dx$$

$$=4\int_0^1(1-\sqrt{1-x^2})dx=4-\underbrace{\boxed{4\int_0^1\sqrt{1-x^2}dx}}_{\text{单位圆的面积}}$$

$$=4-\pi.$$

例 8 若 $f(x)$ 在 $[0,1]$ 上连续，证明

（1）$\int_0^{\frac{\pi}{2}} f(\sin x)dx = \int_0^{\frac{\pi}{2}} f(\cos x)dx$；

由此计算 $\int_0^{\frac{\pi}{2}} \frac{\sin x}{\sin x + \cos x} dx$

$$\sin\left(\frac{\pi}{2} - t\right) = \cos t$$

（2）$\int_0^{\pi} xf(\sin x)dx = \frac{\pi}{2}\int_0^{\pi} f(\sin x)dx$.

由此计算 $\int_0^{\pi} \frac{x\sin x}{1+\cos^2 x} dx$.

证 （1）设 $x=\frac{\pi}{2}-t \quad \Rightarrow dx=-dt$,

$$x=0\Rightarrow t=\frac{\pi}{2}, \qquad x=\frac{\pi}{2}\Rightarrow t=0,$$

$$\int_0^{\frac{\pi}{2}} f(\sin x)dx=-\int_{\frac{\pi}{2}}^{0} f\left[\sin\left(\frac{\pi}{2}-t\right)\right]dt$$

$$=\int_0^{\frac{\pi}{2}} f(\cos t)dt=\int_0^{\frac{\pi}{2}} f(\cos x)dx;$$

（2）设 $x=\pi-t \quad \Rightarrow dx=-dt$,

$$x=0 \Rightarrow t=\pi, \qquad x=\pi \Rightarrow t=0,$$

$$\int_0^{\pi} xf(\sin x)dx=-\int_{\pi}^{0}(\pi-t)f[\sin(\pi-t)]dt$$

$$=\int_0^{\pi}(\pi-t)f(\sin t)dt,$$

$$\int_0^{\pi} xf(\sin x)dx = \pi\int_0^{\pi} f(\sin t)dt - \int_0^{\pi} tf(\sin t)dt$$

$$= \pi\int_0^{\pi} f(\sin x)dx - \int_0^{\pi} xf(\sin x)dx,$$

$$\therefore \int_0^{\pi} xf(\sin x)dx = \frac{\pi}{2}\int_0^{\pi} f(\sin x)dx.$$

$$\int_0^{\pi} \frac{x\sin x}{1+\cos^2 x}dx = \frac{\pi}{2}\int_0^{\pi} \frac{\sin x}{1+\cos^2 x}dx$$

$$= -\frac{\pi}{2}\int_0^{\pi} \frac{1}{1+\cos^2 x}d(\cos x) = -\frac{\pi}{2}\left[\arctan(\cos x)\right]_0^{\pi}$$

$$= -\frac{\pi}{2}\left(-\frac{\pi}{4} - \frac{\pi}{4}\right) = \frac{\pi^2}{4}.$$

二、小结

定积分的换元法

$$\int_a^b f(x)dx = \int_\alpha^\beta f[\varphi(t)]\varphi'(t)dt$$

几个特殊积分、定积分的几个等式

思考题

指出求$\int_{-2}^{-\sqrt{2}}\frac{dx}{x\sqrt{x^2-1}}$的解法中的错误，并写出正确的解法.

解 令 $x=\sec t$, $\quad t:\frac{2\pi}{3}\to\frac{3\pi}{4}$, $\quad dx=\tan t\sec t dt$,

$$\int_{-2}^{-\sqrt{2}}\frac{dx}{x\sqrt{x^2-1}}=\int_{\frac{2\pi}{3}}^{\frac{3\pi}{4}}\frac{1}{\sec t\cdot\boxed{\tan t}}\sec t\cdot\tan t dt$$

$$=\int_{\frac{2\pi}{3}}^{\frac{3\pi}{4}}dt=\frac{\pi}{12}.$$

思考题解答

计算中第二步是错误的. $\because x = \sec t$

$$t \in \left[\frac{2\pi}{3}, \frac{3\pi}{4}\right], \quad \tan t < 0, \quad \sqrt{x^2 - 1} = |\tan t| \neq \tan t.$$

正确解法是

$$\int_{-2}^{-\sqrt{2}} \frac{dx}{x\sqrt{x^2-1}} \overset{x=\sec t}{=\!=\!=} \int_{\frac{2\pi}{3}}^{\frac{3\pi}{4}} \frac{1}{\sec t \cdot |\tan t|} \sec t \cdot \tan t dt$$

$$= -\int_{\frac{2\pi}{3}}^{\frac{3\pi}{4}} dt = -\frac{\pi}{12}.$$

习题:

1、设f(x) 是连续函数，且 $\int_0^{x^3-1} f(t)dt = x$,

则 $f(7) = ?$.

解 $\left(\int_0^{x^3-1} f(t)dt\right)' = (x)' \Rightarrow f(x^3-1)\cdot 3x^2 = 1 \Rightarrow f(x^3-1) = \dfrac{1}{3x^2}$

令 $: x^3 - 1 = 7$ 解得x=2，所以 $f(7) = \dfrac{1}{3\times 2^2} = \dfrac{1}{12}$

2、设f(x)是连续函数，且 $f(x)=x+2\int_0^1 f(t)dt$，

求 $f(x)$

解1: 设$\int_0^1 f(t)dt=I$，　于是 $f(x)=x+2I$，

两边在[0，1]上积分

$$\int_0^1 f(x)dx=\int_0^1 xdx+2I\int_0^1 dx,$$

即: $I=\frac{1}{2}+2I, I=-\frac{1}{2}$　$\therefore f(x)=x-1.$

3、设 $f(x)=\begin{cases}1-x^2, & x\le 0,\\ e^{-x}, & x>0,\end{cases}$ 求 $\int_1^3 f(x-2)dx$

解 令 $x-2=t$,

原式 $=\int_{-1}^{1} f(t)dt=\int_{-1}^{0}(1-t^2)dt+\int_0^1 e^{-t}dt=\frac{7}{3}-\frac{1}{e}$

4、设 $f(x)$ 为连续函数，求

(1) $\frac{d}{dx}\int_0^{x^2}(x^2-t)f(t)dt$ $\qquad 2x\int_0^{x^2} f(t)dt$

(2) $\frac{d}{dx}\int_1^{2} f(x+t)dt$ $\qquad f(2+x)-f(1+x)$

5、设$f(x)$为连续函数，$I = t\int_0^{\frac{s}{t}} f(xt)dx$,

求 $\frac{dI}{dx}$ $\frac{dI}{dt}$ $\frac{dI}{ds}$

$\frac{dI}{dx} = 0$ $\frac{dI}{dt} = 0$ $\frac{dI}{ds} = f(s)$

关键在于换元:

x	0	$\frac{s}{t}$
u	0	s

$$xt = u, \quad t\int_0^{\frac{s}{t}} f(xt)dx = t\int_0^{s} f(u)\frac{1}{t}du = \int_0^{s} f(u)du$$

6、证明 $\int_1^a f(x^2+\frac{a^2}{x^2})\frac{dx}{x}=\int_1^a f(x+\frac{a^2}{x})\frac{dx}{x}$

证：设 $x^2=t$ 则 $xdx=\frac{1}{2}dt$，

x	1	a
t	1	a^2

左式

$$=\int_1^a f(x^2+\frac{a^2}{x^2})\frac{xdx}{x^2}=\frac{1}{2}\int_1^{a^2} f(t+\frac{a^2}{t})\frac{dt}{t}$$

$$=\frac{1}{2}\left[\int_1^a f(t+\frac{a^2}{t})\frac{dt}{t}+\int_a^{a^2} f(t+\frac{a^2}{t})\frac{dt}{t}\right]$$

$$\int_a^{a^2} f\left(t+\frac{a^2}{t}\right)\frac{dt}{t} \quad 令\ u=\frac{a^2}{t} \quad dt=-\frac{a^2}{u^2}du$$

t	a	a^2
u	a	1

$$\therefore I=-\int_a^1 f\left(\frac{a^2}{u}+u\right)\frac{\frac{a^2}{u^2}du}{\frac{a^2}{u}}=\int_1^a f\left(u+\frac{a^2}{u}\right)\frac{du}{u} \quad =\int_1^a f\left(t+\frac{a^2}{t}\right)\frac{dt}{x}$$

对 $\int_1^a f\left(x^2+\frac{a^2}{x^2}\right)\frac{dx}{x}=\frac{1}{2}\left[2\int_1^a f\left(t+\frac{a^2}{t}\right)\frac{dt}{t}\right] \quad =\int_1^a f\left(x+\frac{a^2}{x}\right)\frac{dx}{x}$

练 习 题

一、填空题：

1、$\int_{\frac{\pi}{3}}^{\pi}\sin(x+\frac{\pi}{3})dx=$______________________；

2、$\int_{0}^{\pi}(1-\sin^{3}\theta)d\theta=$____________________；

3、$\int_{0}^{\sqrt{2}}\sqrt{2-x^{2}}dx=$________________；

4、$\int_{-\frac{1}{2}}^{\frac{1}{2}}\frac{(\arcsin x)^{2}}{\sqrt{1-x^{2}}}dx=$______________；

5、$\int_{-5}^{5}\frac{x^{3}\sin^{2}x}{x^{4}+2x^{2}+1}dx=$______________.

二、计算下列定积分：

1、$\int_0^{\frac{\pi}{2}} \sin\varphi\cos^3\varphi d\varphi$；　　2、$\int_1^{\sqrt{3}} \frac{dx}{x^2\sqrt{1+x^2}}$；

3、$\int_{\frac{3}{4}}^{1} \frac{dx}{\sqrt{1-x}-1}$；　　4、$\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \sqrt{\cos x-\cos^3 x}dx$；

5、$\int_0^{\pi} \sqrt{1+\cos 2x}dx$；　　6、$\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} 4\cos^4\theta\, dx$；

7、$\int_{-1}^{1} (x^2\sqrt{1-x^2}+x^3\sqrt{1+x^2})dx$；

8、$\int_0^2 \max\{x, x^3\}dx$；

9、$\int_0^2 x|x-\lambda|dx$　（λ 为参数）.

三、设$f(x)=\begin{cases}\dfrac{1}{1+x}, & \text{当}x\geq 0\text{时}, \\ \dfrac{1}{1+e^{x}}, & \text{当}x<0\text{时},\end{cases}$求$\int_0^2 f(x-1)dx$.

四、设$f(x)$在$[a,b]$上连续,

证明 $\int_a^b f(x)dx=\int_a^b f(a+b-x)dx$.

五、证明:

$$\int_0^1 x^m(1-x)^n dx=\int_0^1 x^n(1-x)^m dx.$$

六、证明：

$$\int_{-a}^{a} f(x)dx = \int_{0}^{a} [f(x) + f(-x)]dx,$$

并求 $\displaystyle\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \frac{dx}{1+\sin x}$.

七、设 $f(x)$ 在 $[\,0\,,1\,]$ 上连续，

证明 $\displaystyle\int_{0}^{\frac{\pi}{2}} f(|\cos x|)dx = \frac{1}{4}\int_{0}^{2\pi} f(|\cos x|)dx$.

练习题答案

一、1、0；　2、$\pi-\dfrac{4}{3}$；　3、$\dfrac{\pi}{2}$；　4、$\dfrac{\pi^3}{32}$；　5、0.

二、1、$\dfrac{1}{4}$；　2、$\sqrt{2}-\dfrac{2\sqrt{3}}{3}$；　3、$1-2\ln 2$；　4、$\dfrac{4}{3}$；

5、$2\sqrt{2}$；　6、$\dfrac{3}{2}\pi$；　7、$\dfrac{\pi}{4}$；　8、$\dfrac{\pi}{8}$；

9、$\dfrac{17}{4}$；　10、当 $\lambda\le 0$ 时，$\dfrac{8}{3}-2\lambda$；　当 $0<\lambda\le 2$ 时，$\dfrac{8}{3}-2\lambda+\dfrac{\lambda^3}{3}$；　当 $\lambda>2$ 时，$-\dfrac{8}{3}+2\lambda$.

三、$1+\ln(1+e^{-1})$.

六、2.