内容小结

1. 偏导数的概念及有关结论

- 定义; 记号; 几何意义
- 函数在一点偏导数存在 $\nRightarrow$ 函数在此点连续
- 混合偏导数连续 $\Longrightarrow$ 与求导顺序无关

2. 偏导数的计算方法

- 求一点处偏导数的方法 $\begin{cases} \text{先代后求} \\ \text{先求后代} \\ \text{利用定义} \end{cases}$
- 求高阶偏导数的方法 —— 逐次求导法

(与求导顺序无关时, 应选择方便的求导顺序)

例 3　设$z=\arcsin\dfrac{x}{\sqrt{x^2+y^2}}$，求$\dfrac{\partial z}{\partial x}$，$\dfrac{\partial z}{\partial y}$.

解　$$\frac{\partial z}{\partial y}=\frac{1}{\sqrt{1-\dfrac{x^2}{x^2+y^2}}}\cdot\left(\frac{x}{\sqrt{x^2+y^2}}\right)'_y$$

$$=\frac{\sqrt{x^2+y^2}}{|y|}\cdot\frac{(-xy)}{\sqrt{(x^2+y^2)^3}}$$

$$=-\frac{x}{x^2+y^2}\operatorname{sgn}\frac{1}{y}\qquad(y\neq 0)$$

$\left.\dfrac{\partial z}{\partial y}\right|_{\substack{x\neq 0\\ y=0}}$　不存在.

二、高阶偏导数

函数$z = f(x, y)$的二阶偏导数为

$$\frac{\partial}{\partial x}\left(\frac{\partial z}{\partial x}\right) = \frac{\partial^2 z}{\partial x^2} = f_{xx}(x, y),\quad \frac{\partial}{\partial y}\left(\frac{\partial z}{\partial y}\right) = \frac{\partial^2 z}{\partial y^2} = f_{yy}(x, y)$$

纯偏导

$$\frac{\partial}{\partial y}\left(\frac{\partial z}{\partial x}\right) = \frac{\partial^2 z}{\partial x \partial y} = f_{xy}(x, y),\quad \frac{\partial}{\partial x}\left(\frac{\partial z}{\partial y}\right) = \frac{\partial^2 z}{\partial y \partial x} = f_{yx}(x, y)$$

混合偏导

定义：二阶及二阶以上的偏导数统称为高阶偏导数.

类似可以定义更高阶的偏导数.

例如， $z = f(x, y)$ 关于 x 的三阶偏导数为

$$\frac{\partial}{\partial x}\left(\frac{\partial^2 z}{\partial x^2}\right) = \frac{\partial^3 z}{\partial x^3}$$

$z = f(x, y)$ 关于 x 的 $n-1$ 阶偏导数，再关于 y 的一阶偏导数为

$$\frac{\partial}{\partial y}\left(\frac{\partial^{n-1} z}{\partial x^{n-1}}\right) = \frac{\partial^n z}{\partial x^{n-1}\partial y}$$

例5. 求函数 $z = e^{x+2y}$ 的二阶偏导数及 $\frac{\partial^3 z}{\partial y \partial x^2}$.

解： $\frac{\partial z}{\partial x} = e^{x+2y}$　　$\frac{\partial z}{\partial y} = 2e^{x+2y}$

$$\frac{\partial^2 z}{\partial x^2} = e^{x+2y} \qquad \frac{\partial^2 z}{\partial x \partial y} = 2e^{x+2y}$$

$$\frac{\partial^2 z}{\partial y \partial x} = 2e^{x+2y} \qquad \frac{\partial^2 z}{\partial y^2} = 4e^{x+2y}$$

$$\frac{\partial^3 z}{\partial y \partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial^2 z}{\partial y \partial x}\right) = 2e^{x+2y}$$

注意:此处 $\frac{\partial^2 z}{\partial x \partial y} = \frac{\partial^2 z}{\partial y \partial x}$，但这一结论并不总成立.

例如，$$f(x,y)=\begin{cases} xy\dfrac{x^2-y^2}{x^2+y^2}, & x^2+y^2\neq 0 \\ 0, & x^2+y^2=0 \end{cases}$$

$$f_x(x,y)=\begin{cases} \boxed{y\dfrac{x^4+4x^2y^2-y^4}{(x^2+y^2)^2}}, & x^2+y^2\neq 0 \\ 0, & x^2+y^2=0 \end{cases}$$

$$f_y(x,y)=\begin{cases} \boxed{x\dfrac{x^4-4x^2y^2-y^4}{(x^2+y^2)^2}}, & x^2+y^2\neq 0 \\ 0, & x^2+y^2=0 \end{cases}$$

$$\left.\begin{aligned} f_{xy}(0,0)&=\lim_{\Delta y\to 0}\frac{\boxed{f_x(0,\Delta y)}-f_x(0,0)}{\Delta y}=\lim_{\Delta y\to 0}\frac{-\Delta y}{\Delta y}=-1 \\ f_{yx}(0,0)&=\lim_{\Delta x\to 0}\frac{\boxed{f_y(\Delta x,0)}-f_y(0,0)}{\Delta x}=\lim_{\Delta x\to 0}\frac{\Delta x}{\Delta x}=1 \end{aligned}\right\}\text{二者不等}$$

定理. 若 $f_{xy}(x,y)$ 和 $f_{yx}(x,y)$ 都在点 (x_0, y_0) 连续, 则

$$f_{xy}(x_0, y_0) = f_{yx}(x_0, y_0)$$ (证明略)

本定理对 n 元函数的高阶混合导数也成立.

例如, 对三元函数 $u = f(x, y, z)$, 当三阶混合偏导数在点 (x, y, z) **连续**时, 有

$$f_{xyz}(x, y, z) = f_{yzx}(x, y, z) = f_{zxy}(x, y, z)$$

$$= f_{xzy}(x, y, z) = f_{yxz}(x, y, z) = f_{zyx}(x, y, z)$$

说明: 因为初等函数的偏导数仍为初等函数, 而初等函数在其定义区域内是连续的, 故求初等函数的高阶导数可以选择方便的求导顺序.

定理. 若 $f_{xy}(x,y)$ 和 $f_{yx}(x,y)$ 都在点 (x_0, y_0) 连续,则

$$f_{xy}(x_0, y_0) = f_{yx}(x_0, y_0)$$

证:令 $F(\Delta x, \Delta y) = f(x_0 + \Delta x, y_0 + \Delta y) - f(x_0 + \Delta x, y_0)$

$$- f(x_0, y_0 + \Delta y) + f(x_0, y_0)$$

令 $\phi(x) = f(x, y_0 + \Delta y) - f(x, y_0)$

$$\psi(y) = f(x_0 + \Delta x, y) - f(x_0, y)$$

则 $F(\Delta x, \Delta y) = \phi(x_0 + \Delta x) - \phi(x_0)$

$$= \phi'(x_0 + \theta_1 \Delta x)\Delta x \qquad (0 < \theta_1 < 1)$$

$$= [\, f_x(x_0 + \theta_1 \Delta x, y_0 + \Delta y) - f_x(x_0 + \theta_1 \Delta x, y_0)\,]\Delta x$$

$$= f_{xy}(x_0 + \theta_1 \Delta x, y_0 + \theta_2 \Delta y)\Delta x \Delta y \quad (0 < \theta_1, \theta_2 < 1)$$

同样

$$F(\Delta x,\Delta y)= f(x_0+\Delta x, y_0+\Delta y)-f(x_0+\Delta x, y_0)$$
$$-f(x_0, y_0+\Delta y)+f(x_0, y_0)$$

$$=\psi(y_0+\Delta y)-\psi(y_0)$$

$$= f_{yx}(x_0+\theta_3\Delta x, y_0+\theta_4\Delta y)\,\Delta x\,\Delta y$$

$$(0<\theta_3, \theta_4<1)$$

$$\therefore\quad f_{xy}(x_0+\theta_1\Delta x, y_0+\theta_2\Delta y)$$
$$= f_{yx}(x_0+\theta_3\Delta x, y_0+\theta_4\Delta y)$$

因 $f_{xy}(x,y)$, $f_{yx}(x,y)$ 在点 (x_0, y_0) 连续, 故令 $\Delta x\to 0$, $\Delta y\to 0$ 得 $\quad f_{xy}(x_0, y_0)= f_{yx}(x_0, y_0)$

2 设$u=e^{ax}\cos by$，求二阶偏导数.

解 $\frac{\partial u}{\partial x}=ae^{ax}\cos by,$ $\frac{\partial u}{\partial y}=-be^{ax}\sin by;$

$$\frac{\partial^2 u}{\partial x^2}=a^2e^{ax}\cos by,\qquad \frac{\partial^2 u}{\partial y^2}=-b^2e^{ax}\cos by,$$

$$\frac{\partial^2 u}{\partial x\partial y}=-abe^{ax}\sin by,\qquad \frac{\partial^2 u}{\partial y\partial x}=-abe^{ax}\sin by.$$

第三节 全微分

一元函数 $y=f(x)$ 的微分

$$\Delta y = \underline{A\Delta x} + o(\Delta x)$$

$$\mathrm{d}y = f'(x)\Delta x \xrightarrow{\text{应用}} \begin{cases}\text{近似计算}\\ \text{估计误差}\end{cases}$$

本节内容:

一、全微分的定义

***二、全微分在数值计算中的应用**

一、全微分的定义

由一元函数微分学中增量与微分的关系得

$$\boxed{\begin{aligned} &f(x+\Delta x, y) - f(x,y) \\ &f(x, y+\Delta y) - f(x,y) \end{aligned}} \begin{aligned} &\approx \\ &\approx \end{aligned} \boxed{\begin{aligned} &f_x(x,y)\Delta x \\ &f_y(x,y)\Delta y \end{aligned}}$$

二元函数　　　　　　二元函数

对 x 和对 y 的偏增量　　　　对 x 和对 y 的偏微分

全微分的定义

如果函数$z=f(x,y)$在点(x,y)的全增量$\Delta z=f(x+\Delta x,y+\Delta y)-f(x,y)$可以表示为$\Delta z=A\Delta x+B\Delta y+o(\rho)$，其中$A,B$不依赖于$\Delta x,\Delta y$而仅与$x,y$有关，$\rho=\sqrt{(\Delta x)^2+(\Delta y)^2}$，则称函数$z=f(x,y)$在点$(x,y)$可微分，$A\Delta x+B\Delta y$称为函数$z=f(x,y)$在点$(x,y)$的**全微分**，记为$dz$，即 $dz=A\Delta x+B\Delta y$.

函数若在某区域 D 内各点处处可微分，则称这函数在 D 内可微分.

如果函数$z = f(x, y)$在点(x, y)可微分, 则函数在该点连续.

事实上 $\Delta z = A\Delta x + B\Delta y + o(\rho)$, $\lim\limits_{\rho \to 0} \Delta z = 0$,

$$\lim_{\substack{\Delta x \to 0 \\ \Delta y \to 0}} f(x + \Delta x, y + \Delta y) = \lim_{\rho \to 0} [f(x, y) + \Delta z]$$

$$= f(x, y)$$

故函数$z = f(x, y)$在点(x, y)处连续.

即

$$\boxed{\begin{array}{l}\text{函数 } z = f(x, y) \text{ 在点 } (x, y) \text{ 可微} \\ \qquad\qquad\qquad\qquad \Longrightarrow \text{函数在该点连续}\end{array}}$$

一元函数在某点的导数存在 $\Longleftrightarrow$ 微分存在.

多元函数的各偏导数存在 $\overset{?}{\Longleftrightarrow}$ 全微分存在.

下面两个定理给出了可微与偏导数的关系:

(1) 函数可微 $\overset{\longrightarrow}{\not\longleftarrow}$ 偏导数存在

(2) 偏导数连续 $\overset{\longrightarrow}{\not\longleftarrow}$ 函数可微

定理1(必要条件) 若函数 $z=f(x,y)$ 在点(x,y) **可微**，

则该函数在该点偏导数 $\dfrac{\partial z}{\partial x}, \dfrac{\partial z}{\partial y}$ 必存在,且有

$$\mathrm{d}\,z=\frac{\partial z}{\partial x}\Delta x+\frac{\partial z}{\partial y}\Delta y$$

证: 由全增量公式 $\Delta z=A\Delta x+B\Delta y+o(\rho)$, 令 $\Delta y=0$,

得到对 x 的偏增量

$$\Delta_x z=f(x+\Delta x,y)-f(x,y)=A\Delta x+o(|\Delta x|)$$

$$\therefore\quad \frac{\partial z}{\partial x}=\lim_{\Delta x\to 0}\frac{\Delta_x z}{\Delta x}=A$$

同样可证 $\dfrac{\partial z}{\partial y}=B$, 因此有 $\mathrm{d}\,z=\dfrac{\partial z}{\partial x}\Delta x+\dfrac{\partial z}{\partial y}\Delta y$

注意： 定理1 的逆定理不成立．即:

偏导数存在函数 不一定可微 ！

反例： 函数 $f(x,y)=\begin{cases}\dfrac{xy}{\sqrt{x^2+y^2}}, & x^2+y^2\neq 0\\ 0, & x^2+y^2=0\end{cases}$

易知 $f_x(0,0)=f_y(0,0)=0$，但

$$\Delta z-[f_x(0,0)\Delta x+f_y(0,0)\Delta y]=\frac{\Delta x\,\Delta y}{\sqrt{(\Delta x)^2+(\Delta y)^2}}$$

$$\frac{\Delta x\,\Delta y}{\sqrt{(\Delta x)^2+(\Delta y)^2}}\Big/\rho=\frac{\Delta x\,\Delta y}{(\Delta x)^2+(\Delta y)^2}\not\longrightarrow 0$$

$\neq o(\rho)$ 因此,函数在点 $(0,0)$ 不可微．

定理2 (充分条件) 若函数 $z = f(x, y)$ 的偏导数 $\frac{\partial z}{\partial x}, \frac{\partial z}{\partial y}$ 在点 (x, y) 连续, 则函数在该点可微分.

证: $\Delta z = f(x+\Delta x, y+\Delta y) - f(x, y)$

$$= [f(x+\Delta x, y+\Delta y) - f(x, y+\Delta y)]$$

$$+ [f(x, y+\Delta y) - f(x, y)]$$

$$= f_x(x+\theta_1 \Delta x, y+\Delta y)\Delta x + f_y(x, y+\theta_2 \Delta y)\Delta y$$

$$\boxed{(0 < \theta_1, \theta_2 < 1)}$$

$$= [f_x(x, y) + \alpha]\Delta x + [f_y(x, y) + \beta]\Delta y$$

$$\left(\lim_{\substack{\Delta x \to 0 \\ \Delta y \to 0}} \alpha = 0, \quad \lim_{\substack{\Delta x \to 0 \\ \Delta y \to 0}} \beta = 0 \right)$$

$$\Delta z = \cdots$$

$$= f_x(x, y)\Delta x + f_y(x, y)\Delta y + \alpha\,\Delta x + \beta\,\Delta y$$

$$\left(\lim_{\substack{\Delta x \to 0 \\ \Delta y \to 0}} \alpha = 0, \quad \lim_{\substack{\Delta x \to 0 \\ \Delta y \to 0}} \beta = 0 \right)$$

注意到 $\left| \dfrac{\alpha\,\Delta x + \beta\,\Delta y}{\rho} \right| \le |\alpha| + |\beta|$，故有

$$\Delta z = f_x(x, y)\Delta x + f_y(x, y)\Delta y + o(\rho)$$

所以函数 $z = f(x, y)$ 在点 (x, y) 可微.

推广：类似可讨论三元及三元以上函数的可微性问题.

例如，三元函数 $u = f(x, y, z)$ 的全微分为

$$\mathrm{d}u = \frac{\partial u}{\partial x}\Delta x + \frac{\partial u}{\partial y}\Delta y + \frac{\partial u}{\partial z}\Delta z$$

习惯上把自变量的增量用微分表示，于是

$$\mathrm{d}u = \underbrace{\frac{\partial u}{\partial x}\mathrm{d}x}_{\text{记作 } \mathrm{d}_x u} + \underbrace{\frac{\partial u}{\partial y}\mathrm{d}y}_{\mathrm{d}_y u} + \underbrace{\frac{\partial u}{\partial z}\mathrm{d}z}_{\mathrm{d}_z u}$$

$\mathrm{d}_x u, \mathrm{d}_y u, \mathrm{d}_z u$ 称为**偏微分**. 故有下述叠加原理

$$\mathrm{d}u = \mathrm{d}_x u + \mathrm{d}_y u + \mathrm{d}_z u$$

例1. 计算函数 $z = e^{xy}$ 在点 $(2,1)$ 处的全微分.

解: $\dfrac{\partial z}{\partial x} = ye^{xy}$, $\dfrac{\partial z}{\partial y} = xe^{xy}$

$$\left.\frac{\partial z}{\partial x}\right|_{(2,1)} = e^2, \quad \left.\frac{\partial z}{\partial y}\right|_{(2,1)} = 2e^2$$

$$\therefore \quad \left.\mathrm{d}\, z\right|_{(2,1)} = e^2\,\mathrm{d}\, x + 2e^2\,\mathrm{d}\, y = e^2(\mathrm{d}\, x + 2\mathrm{d}\, y)$$

例2. 计算函数 $u = x + \sin\dfrac{y}{2} + e^{yz}$ 的全微分.

解: $\mathrm{d}\, u = 1 \cdot \mathrm{d}x + (\frac{1}{2}\cos\frac{y}{2} + z\, e^{yz})\,\mathrm{d}\, y + y\, e^{yz}\,\mathrm{d}z$

例 2 求函数$z = y\cos(x-2y)$，当$x=\frac{\pi}{4}$， $y=\pi$，$dx=\frac{\pi}{4}$，$dy=\pi$时的全微分.

解
$$\frac{\partial z}{\partial x} = -y\sin(x-2y),$$

$$\frac{\partial z}{\partial y} = \cos(x-2y)+2y\sin(x-2y),$$

$$dz\Big|_{(\frac{\pi}{4},\pi)} = \frac{\partial z}{\partial x}\Big|_{(\frac{\pi}{4},\pi)} dx + \frac{\partial z}{\partial y}\Big|_{(\frac{\pi}{4},\pi)} dy = \frac{\sqrt{2}}{8}\pi(4+7\pi).$$

例 3 计算函数$u = x + \sin\frac{y}{2} + e^{yz}$的全微分.

解 $\frac{\partial u}{\partial x} = 1$, $\frac{\partial u}{\partial y} = \frac{1}{2}\cos\frac{y}{2} + ze^{yz}$,

$\frac{\partial u}{\partial z} = ye^{yz}$,

所求全微分

$$du = dx + \left(\frac{1}{2}\cos\frac{y}{2} + ze^{yz}\right)dy + ye^{yz}dz.$$

例 4　试证函数

$$f(x,y)=\begin{cases} xy\sin\dfrac{1}{\sqrt{x^2+y^2}}, & (x,y)\neq(0,0) \\ 0, & (x,y)=(0,0) \end{cases}$$ 在

点$(0,0)$连续且偏导数存在，但偏导数在点$(0,0)$不连续，而 f 在点$(0,0)$可微.

思路：按有关定义讨论；对于偏导数需分

$(x,y)\neq(0,0)$，$(x,y)=(0,0)$讨论.