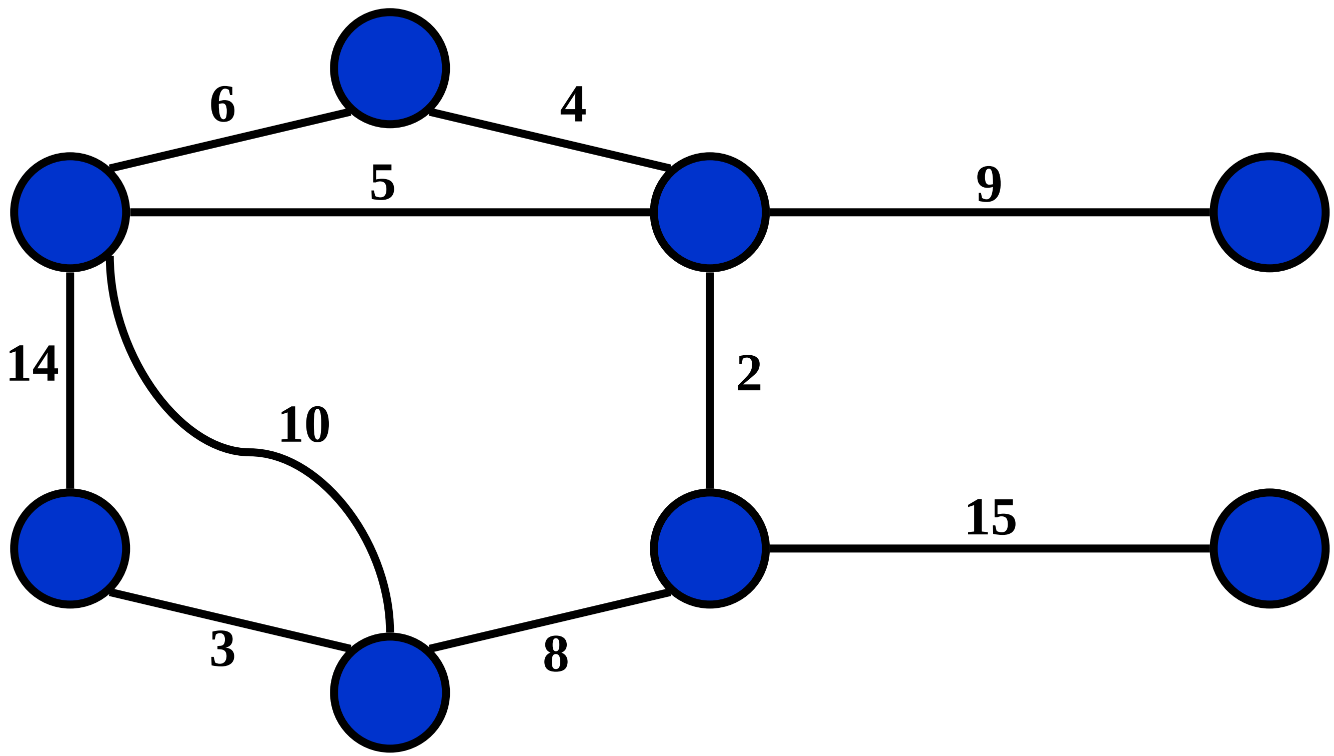
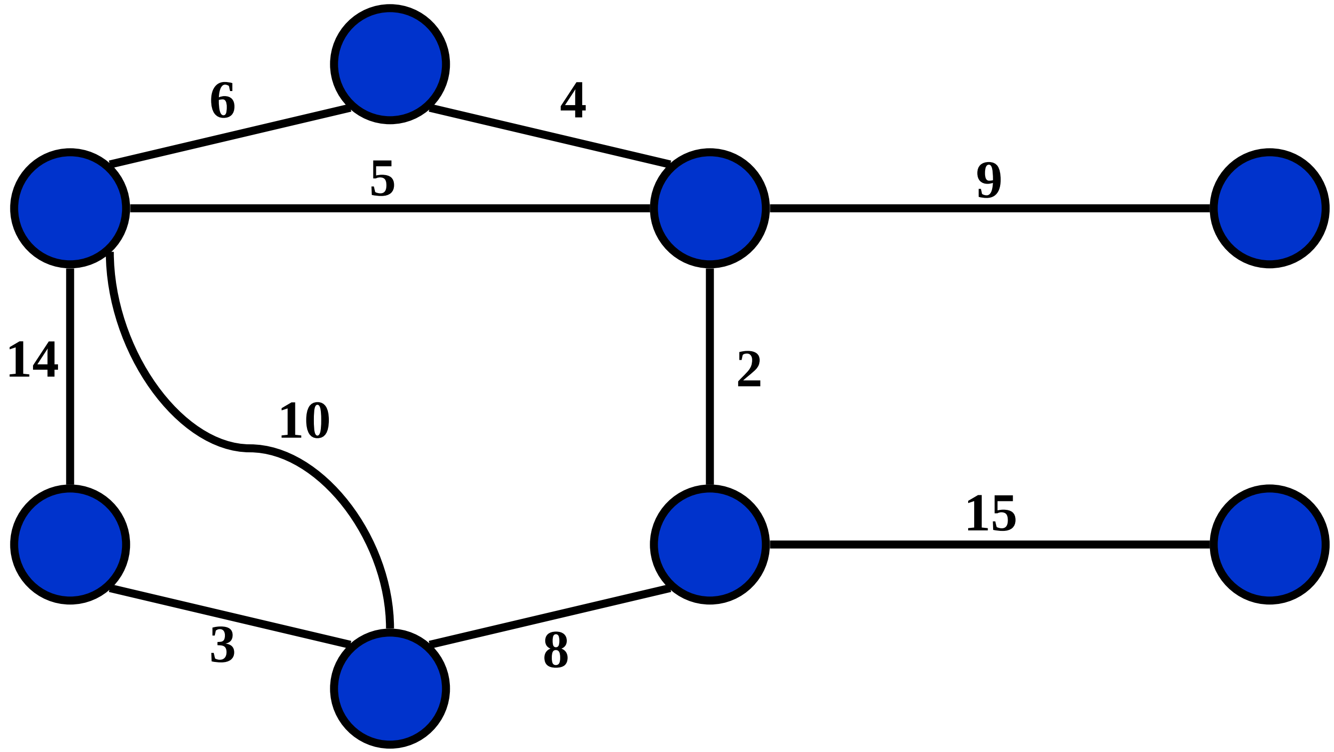
Minimum Spanning Tree

- Problem: given a connected, undirected, weighted graph:



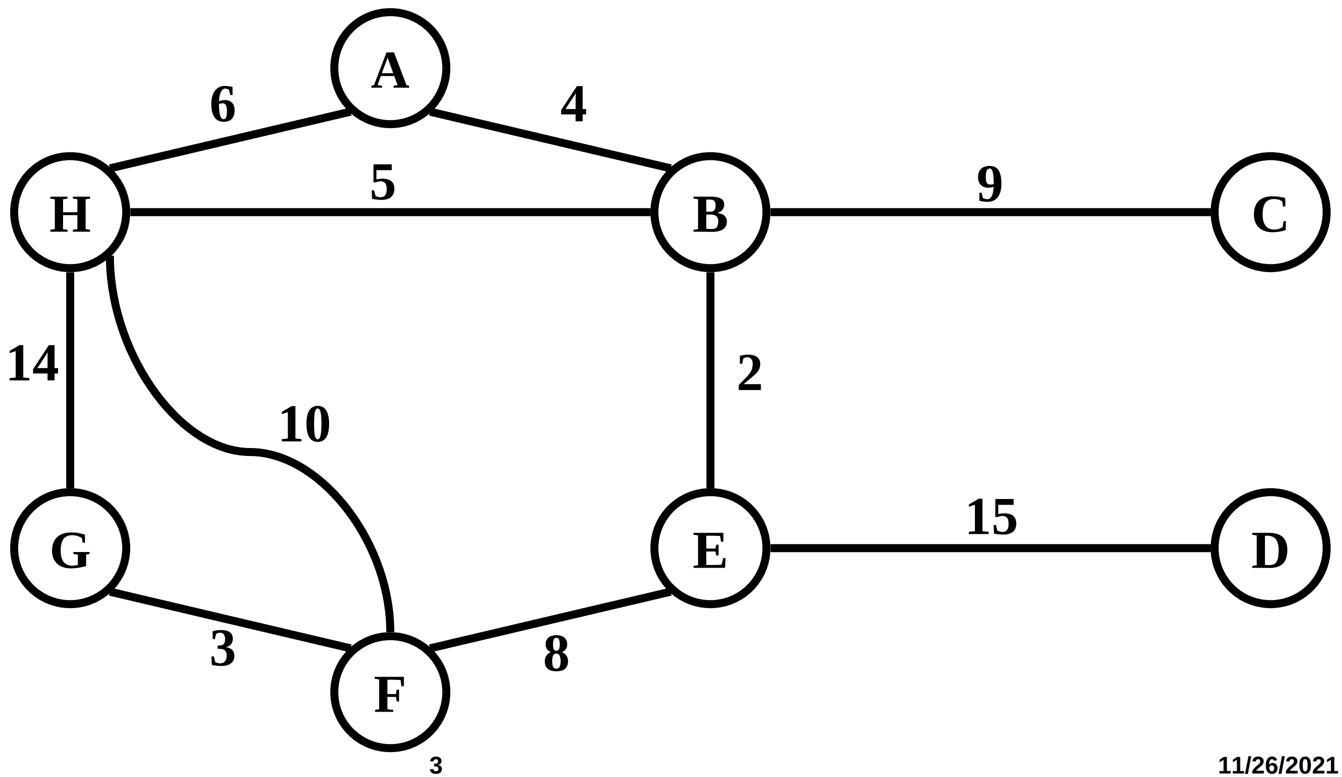
Minimum Spanning Tree

- Problem: given a connected, undirected, weighted graph, find a *spanning tree* using edges that minimize the total weight



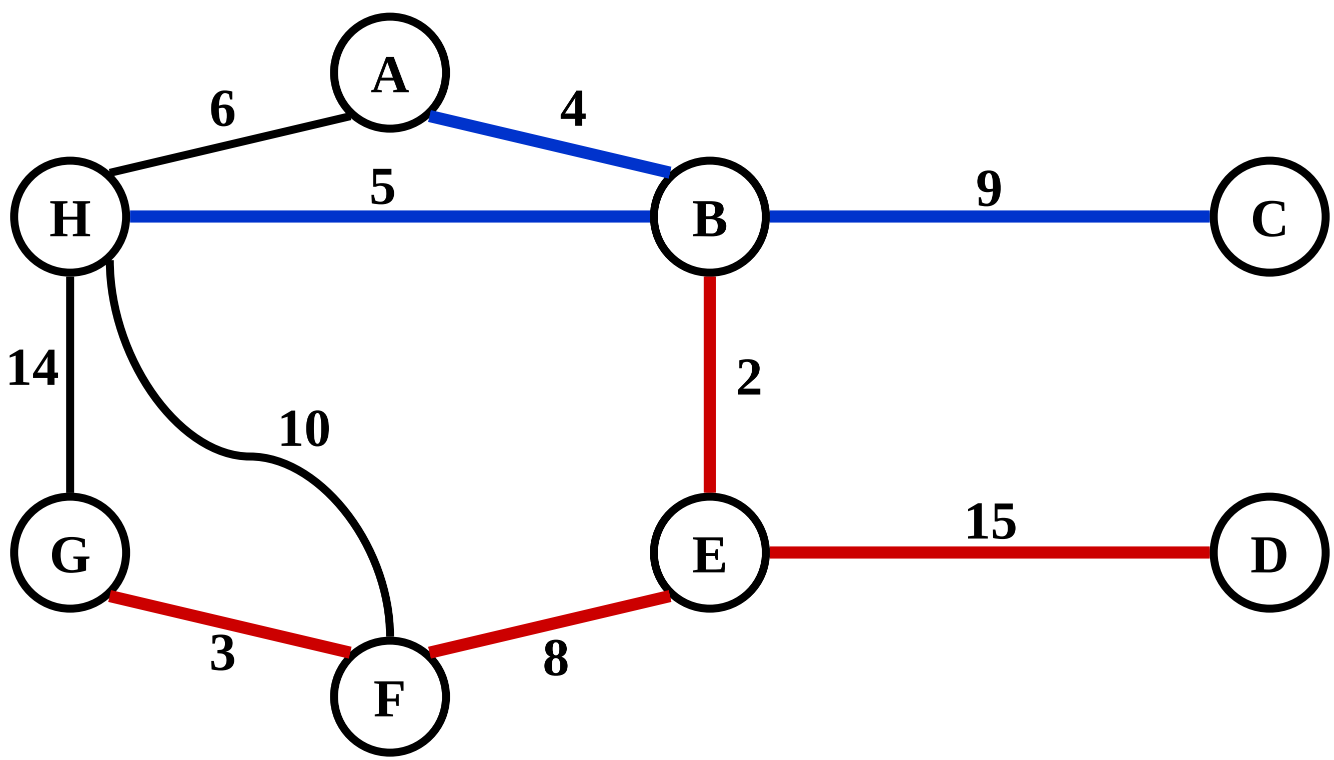
Minimum Spanning Tree

- *Which edges form the minimum spanning tree (MST) of the below graph?*



Minimum Spanning Tree

- Answer:



Minimum Spanning Tree

- MSTs satisfy the *optimal substructure* property: an optimal tree is composed of optimal subtrees
  - Let T be an MST of G with an edge (u,v) in the middle
  - Removing (u,v) partitions T into two trees T_1 and T_2
  - Claim: T_1 is an MST of $G_1 = (V_1,E_1)$, and T_2 is an MST of $G_2 = (V_2,E_2)$ (*Do V_1 and V_2 share vertices? Why?*)
  - Proof: $w(T) = w(u,v) + w(T_1) + w(T_2)$
    (There can't be a better tree than T_1 or T_2, or T would be suboptimal)

Minimum Spanning Tree

```
GENERIC-MST(G, w)
1  A = ∅
2  while A does not form a spanning tree
3      find an edge (u, v) that is safe for A
4      A = A ∪ {(u, v)}
5  return A
```

Initialization: After line 1, the set A trivially satisfies the loop invariant.

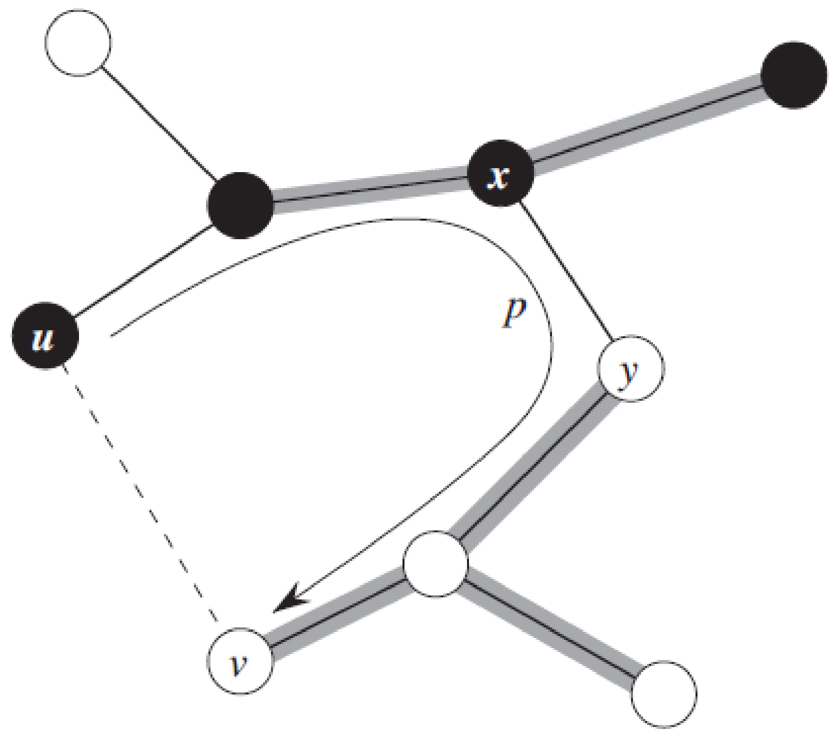
Maintenance: The loop in lines 2–4 maintains the invariant by adding only safe edges.

Termination: All edges added to A are in a minimum spanning tree, and so the set A returned in line 5 must be a minimum spanning tree.

Minimum Spanning Tree

- Thm 23.1 :
  - Let T be MST of G, and let A $\subseteq$ T be subtree of T
  - Let (u,v) be min-weight edge connecting A to V-A
  - Then $(u,v) \in$ T
- Proof:

  By exchange..



Prim's Algorithm

```
MST-Prim(G, w, r)
    Q = V[G];
    for each u ∈ Q
        key[u] = ∞;
    key[r] = 0;
    p[r] = NULL;
    while (Q not empty)
        u = ExtractMin(Q);
        for each v ∈ Adj[u]
            if (v ∈ Q and w(u,v) < key[v])
                p[v] = u;
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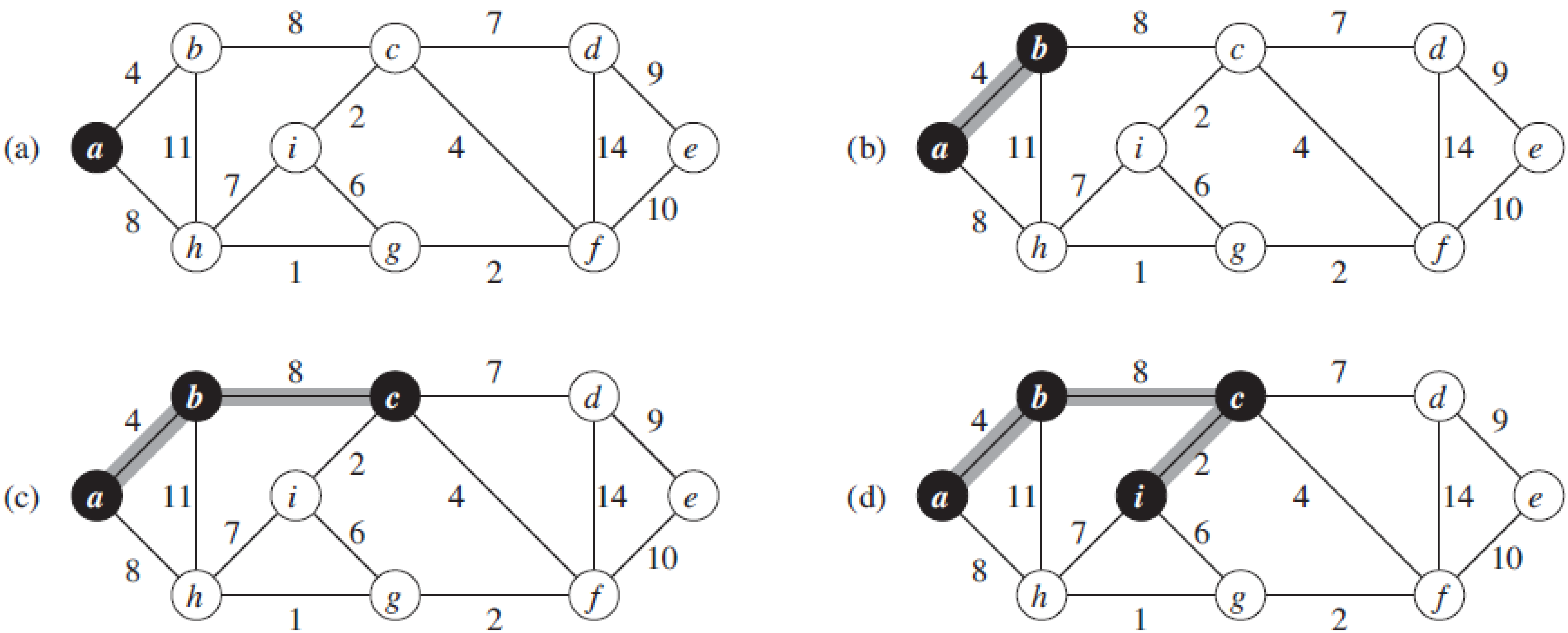
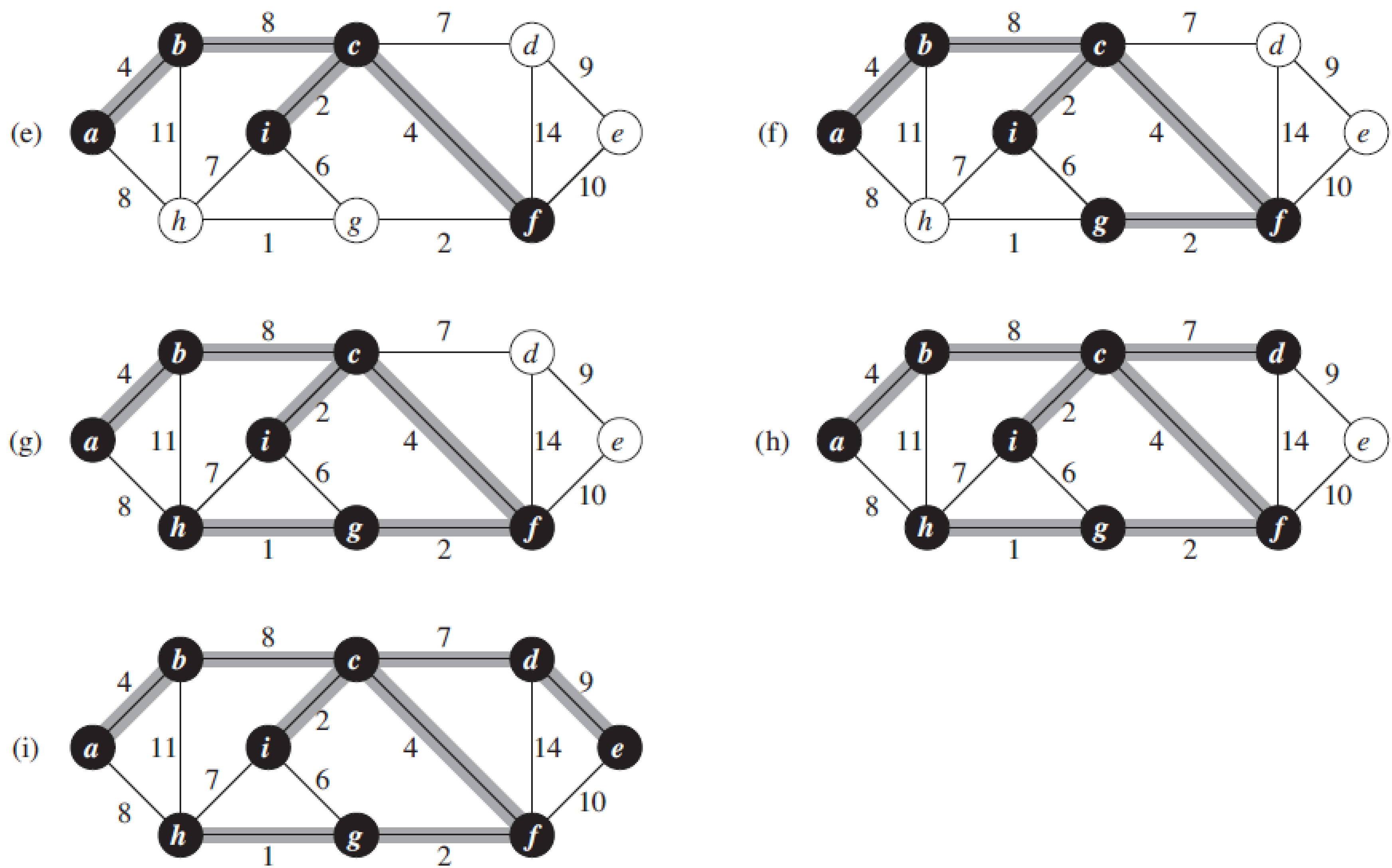
Prim’s Algorithm




(e)
(f)
(g)
(h)
(i)
a b c d e f g h i
4 8 7 9 11 2 4 14 7 6 8 10 1 2

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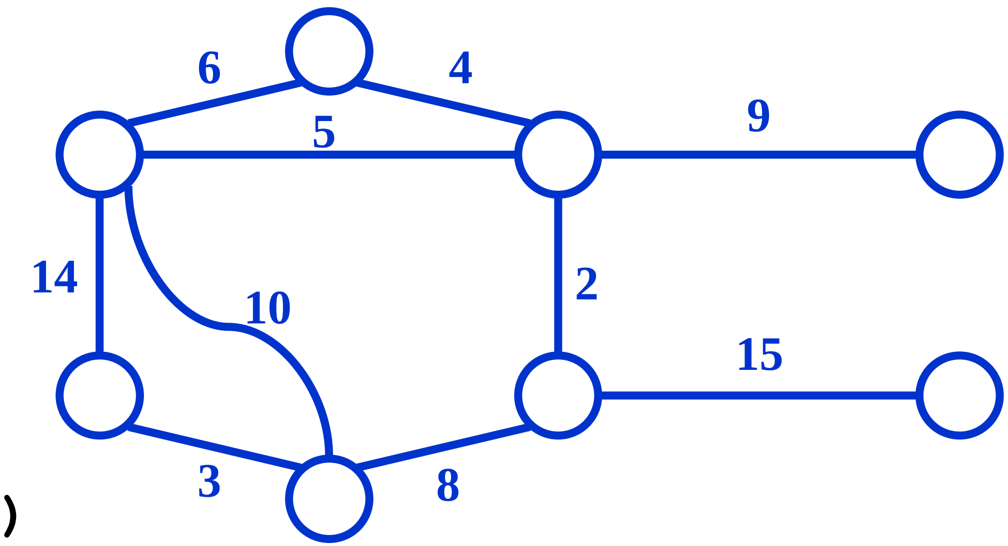

Run on example graph

Prim's Algorithm

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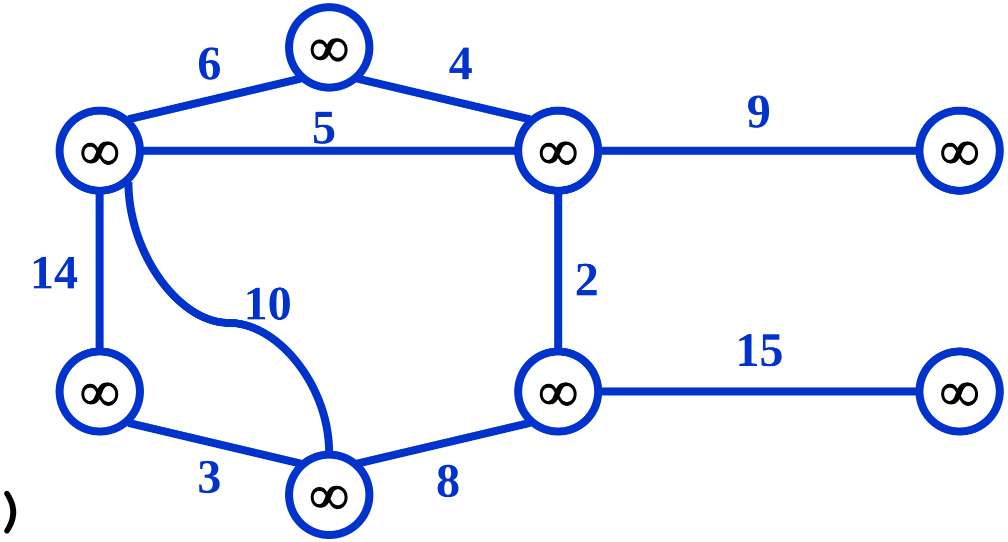

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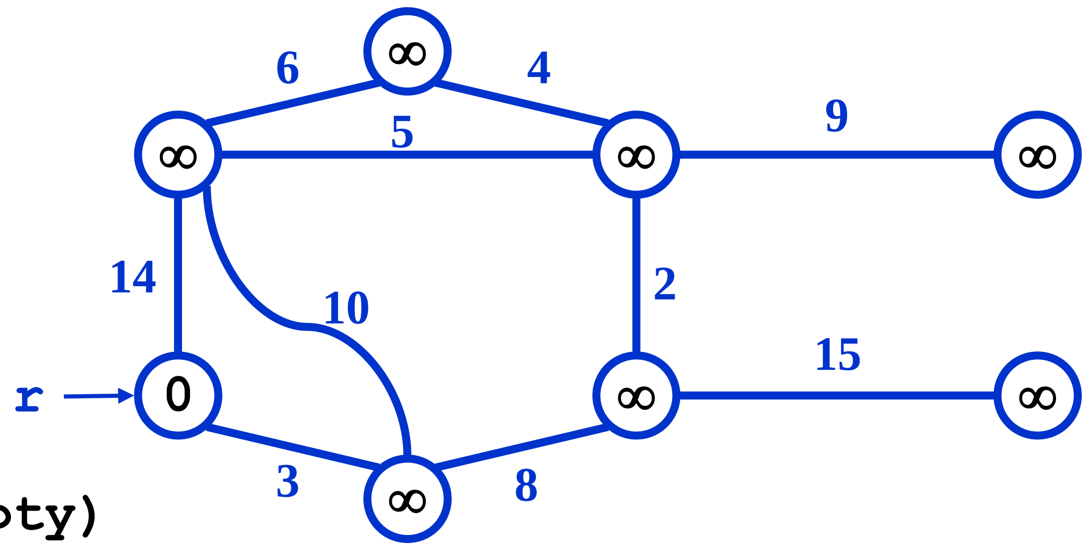

Pick a start vertex r

Prim's Algorithm

```
MST-Prim(G, w, r)
    Q = V[G];
    for each u ∈ Q
        key[u] = ∞;
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    while (Q not empty)
        u = ExtractMin(Q);
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            if (v ∈ Q and w(u,v) < key[v])
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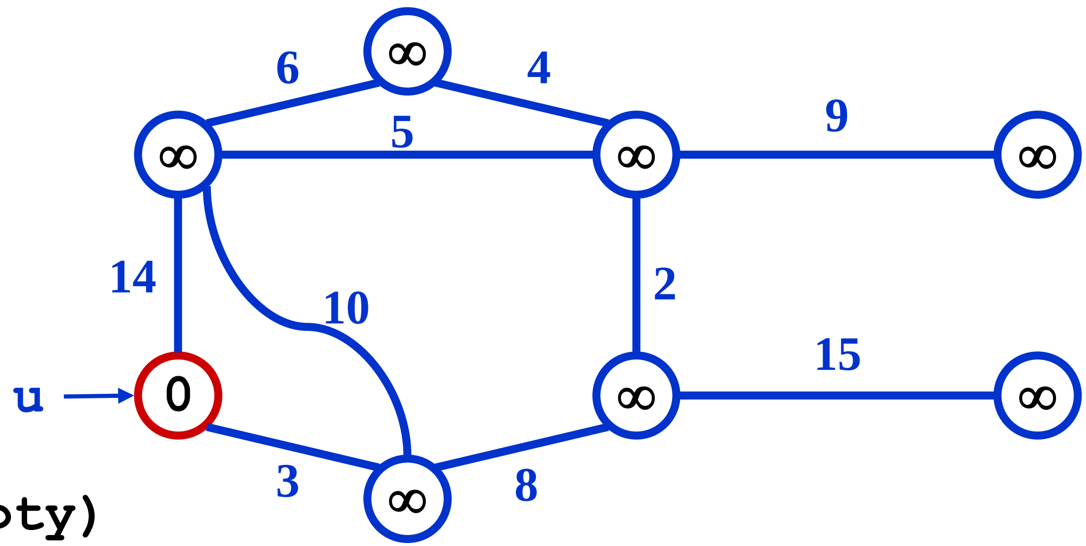

Red vertices have been removed from Q

Prim's Algorithm

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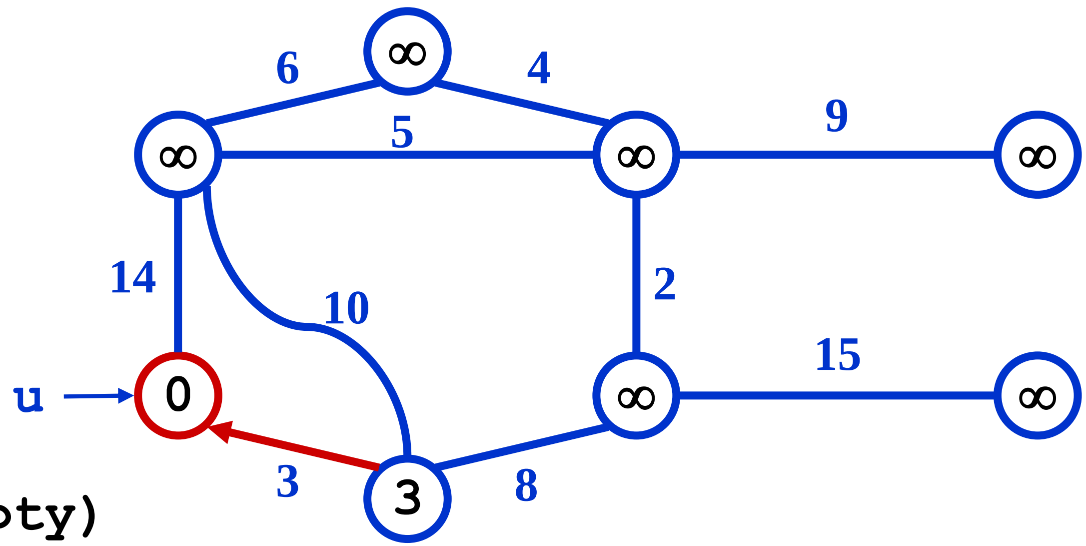

Red arrows indicate parent pointers

Prim's Algorithm

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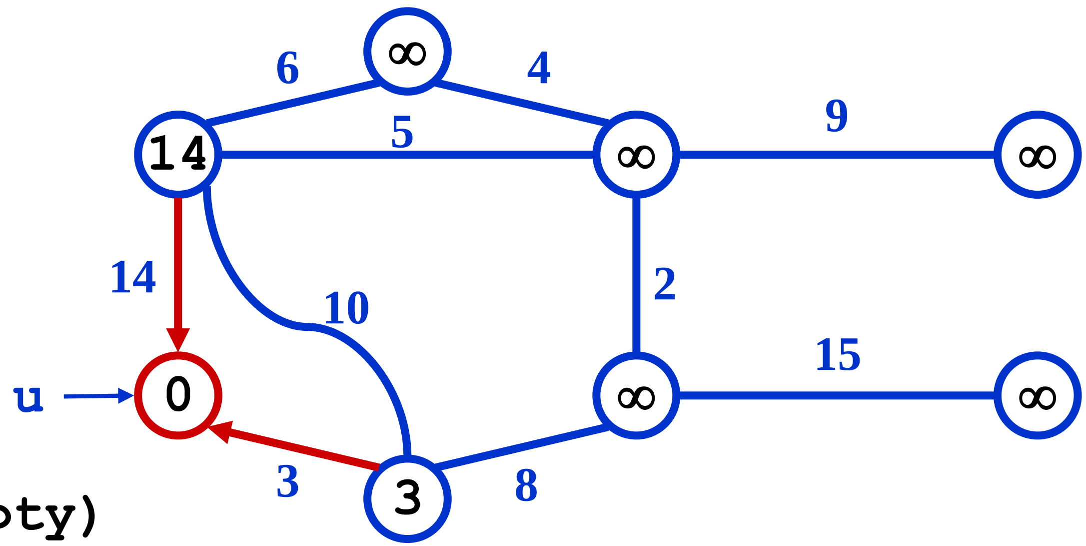

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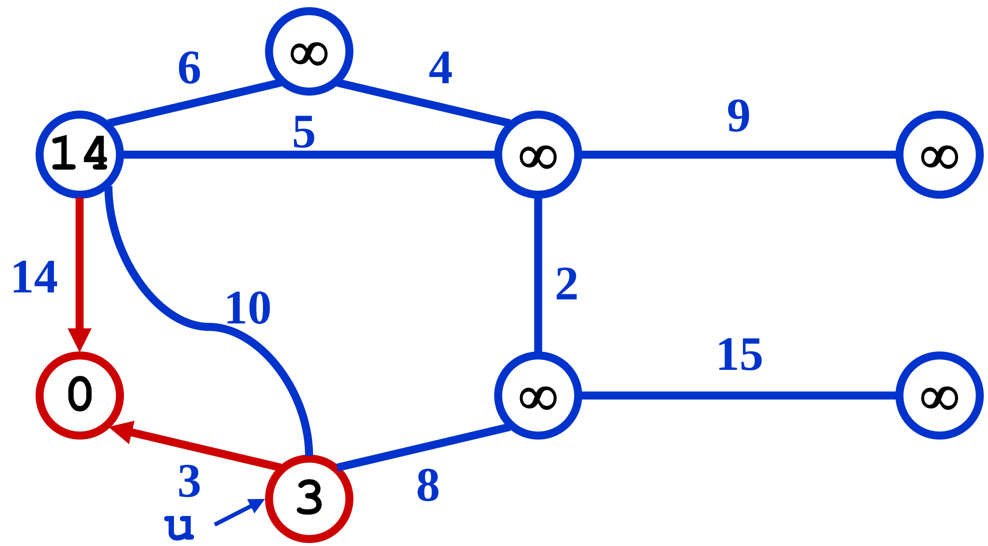

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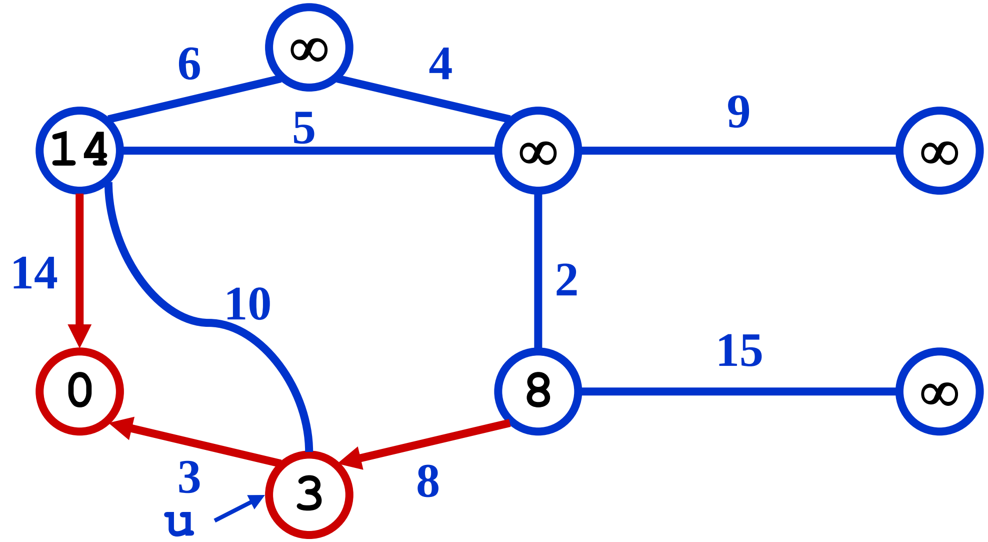

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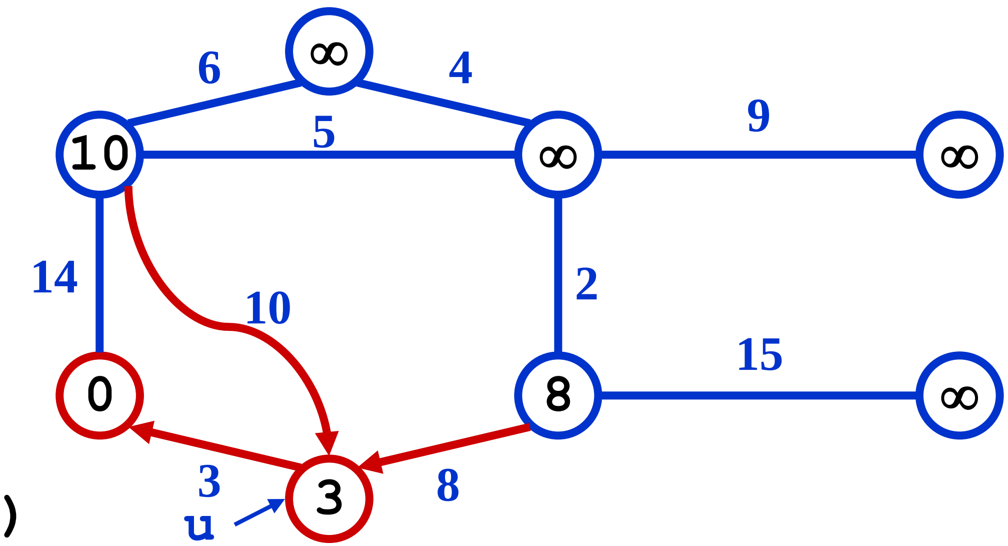

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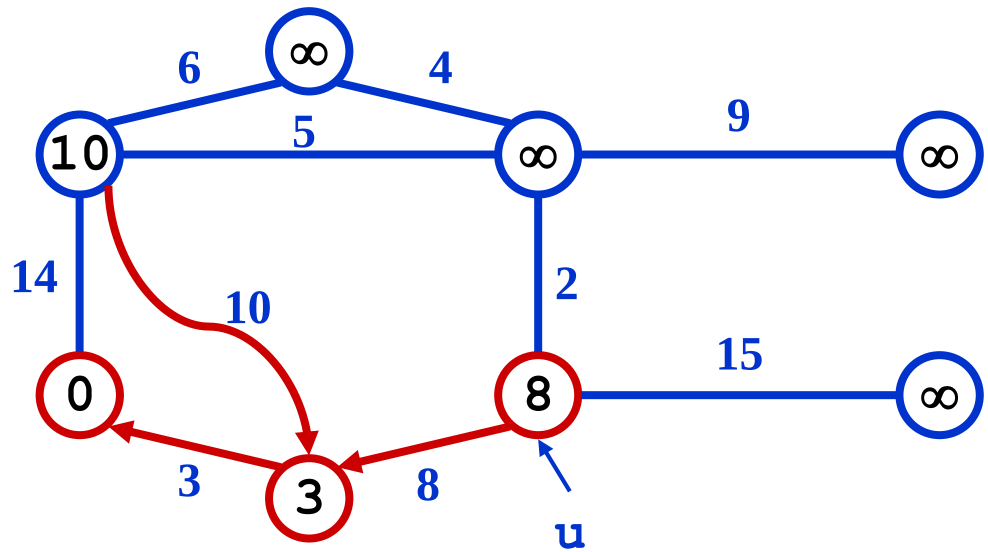

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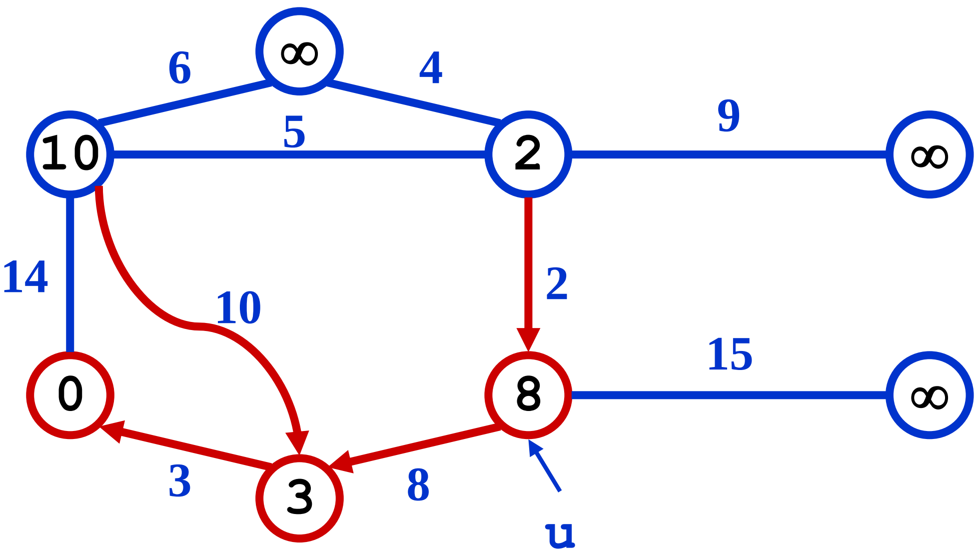

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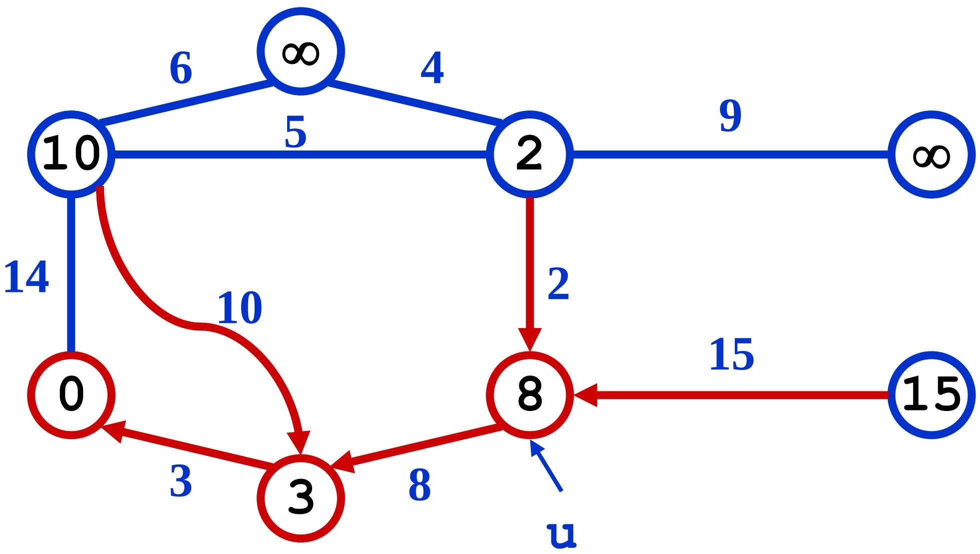

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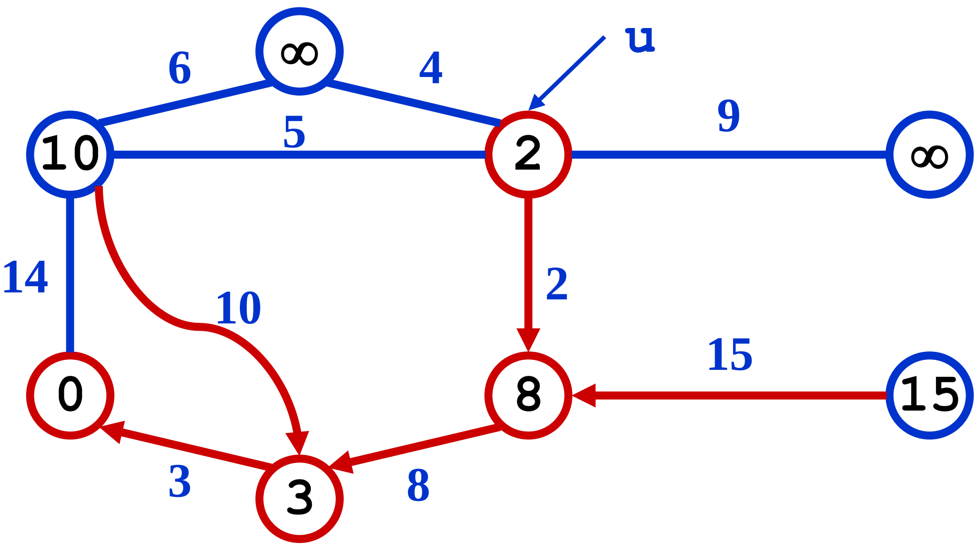

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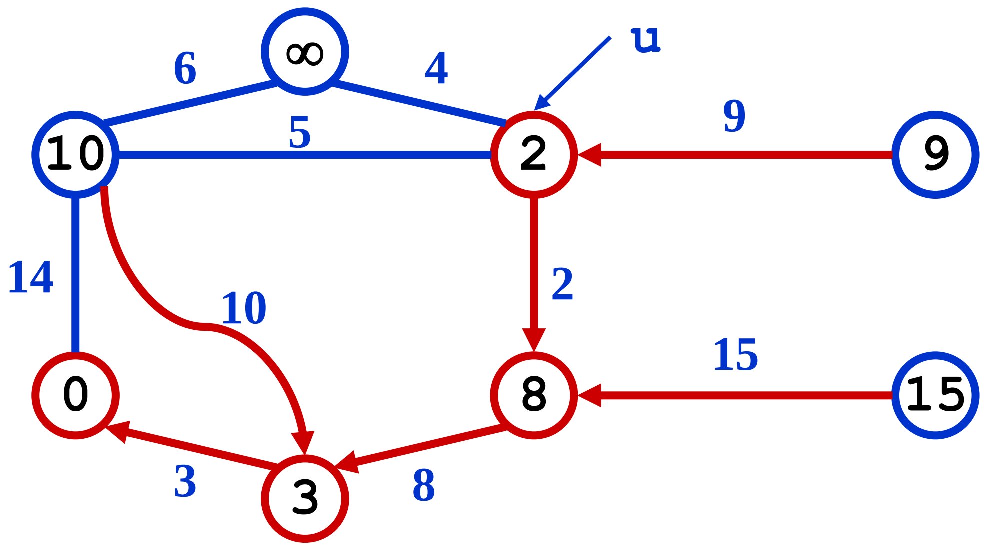

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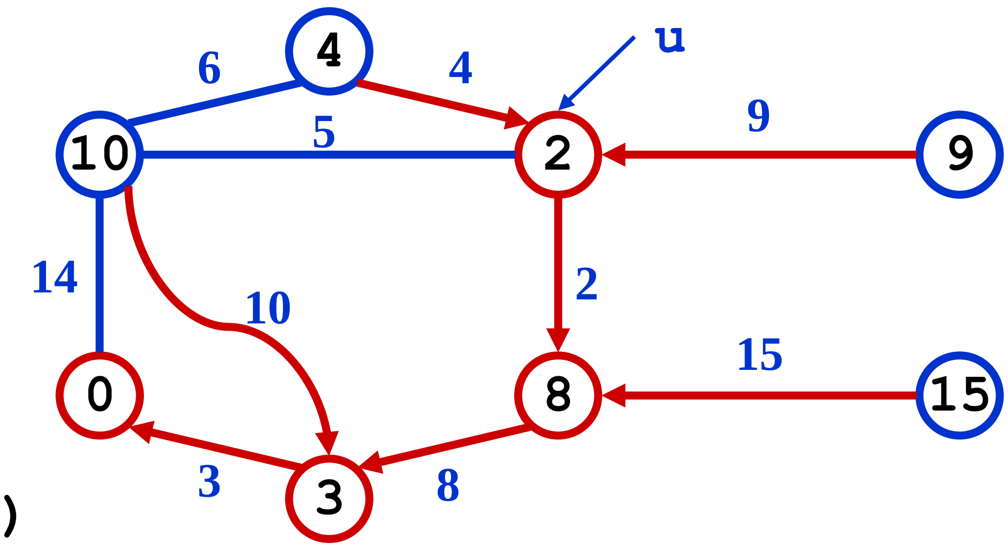

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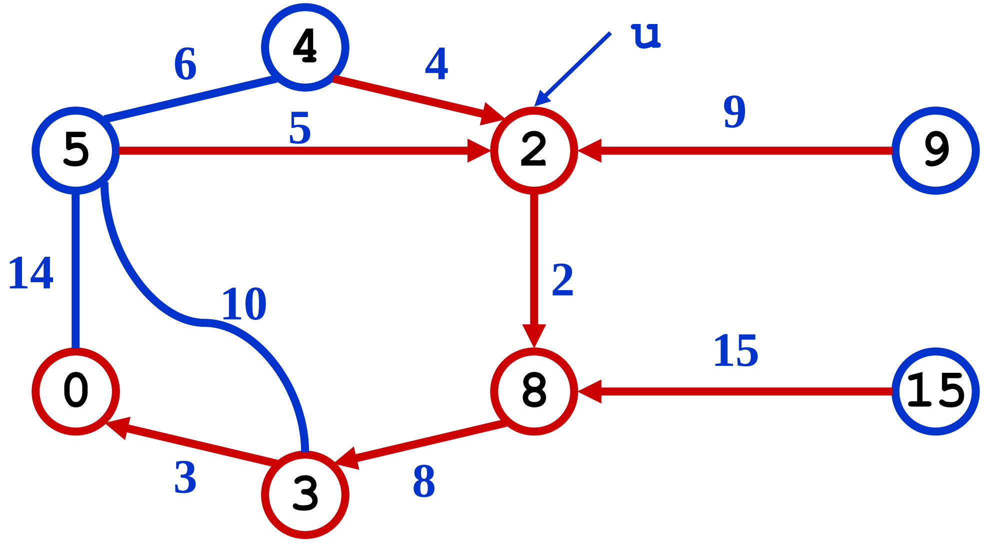

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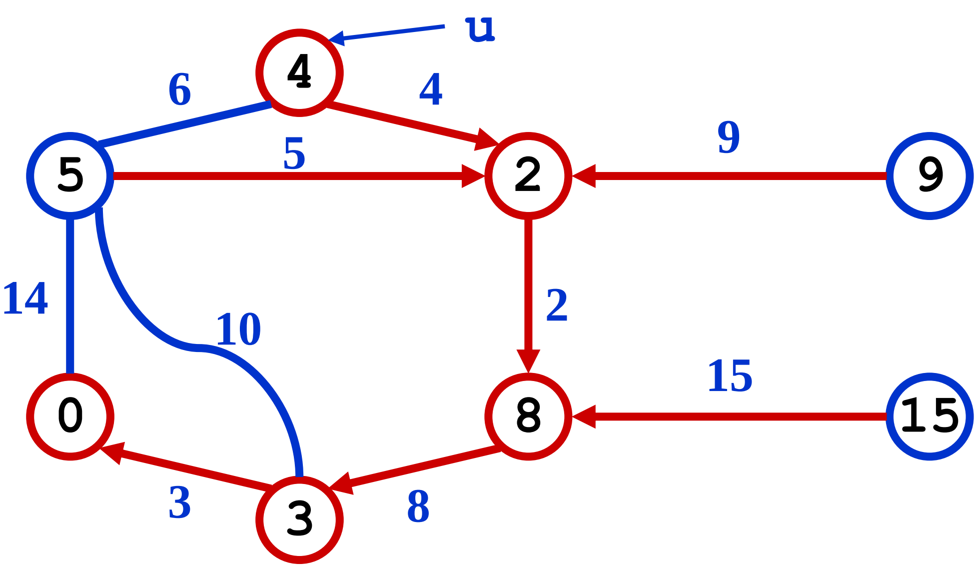

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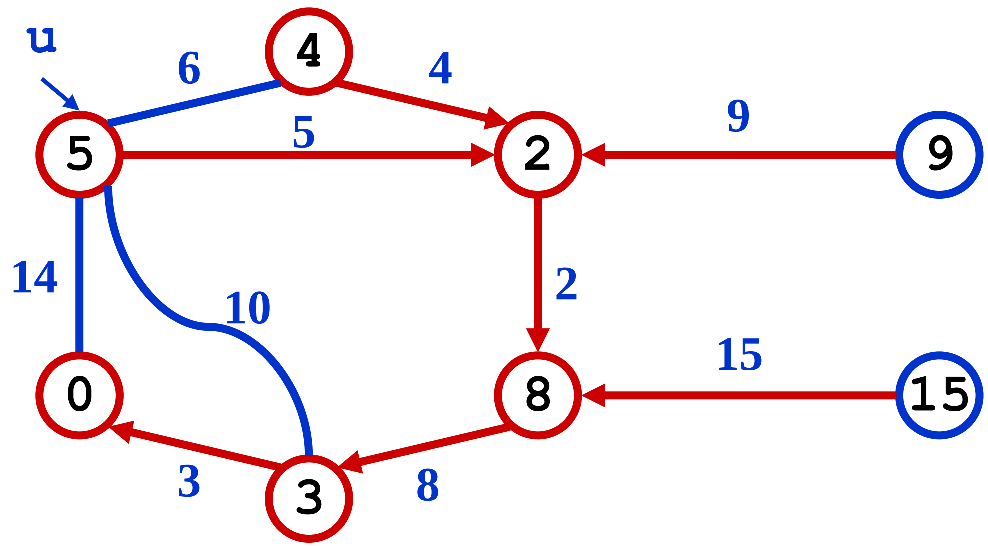

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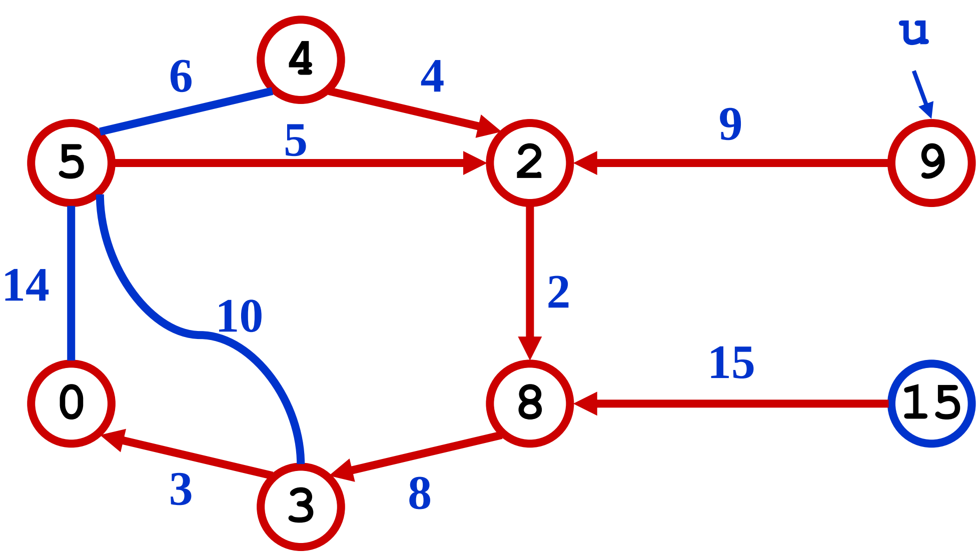

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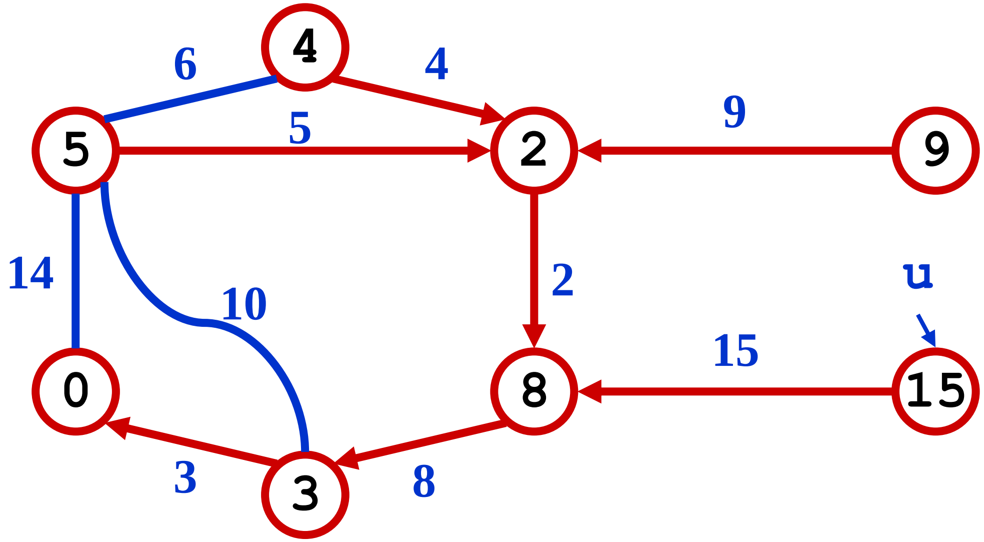

Analysis: Prim's Algorithm

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```

What is the hidden cost in this code?

Analysis: Prim's Algorithm

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                p[v] = u;
                DecreaseKey(v, w(u,v));
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                p[v] = u;
                DecreaseKey(v, w(u,v));
```

How often is ExtractMin() called?

How often is DecreaseKey() called?

Analysis: Prim's Algorithm

```
MST-Prim(G, w, r)
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```

What will be the running time?

A: Depends on queue

binary heap: O(E lg V)

Fibonacci heap: O(V lg V + E)

Kruskal's Algorithm

- Starts with each vertex in its own component.
- Repeatedly merges two components into one by choosing a light edge that connects them (i.e., a light edge crossing the cut between them).
- Scans the set of edges in monotonically increasing order by weight.
- Uses a disjoint-set data structure to determine whether an edge connects vertices in different components.

Disjoint Set Data Structure

- Want to maintain a collection $S = \{S_1, \ldots, S_k\}$ of **disjoint dynamic sets**.
- Each set has a **representative member**.
- **Operations:**
  - Make-Set(x): Make new singleton set containing object x (x is representative).
  - Union(x, y): Unite sets containing x and y into a new set that is a union of the two sets. (Representative is any member of the union.)
  - Find-Set(x): Returns pointer to representative of set containing x.
- **Complexity:** $O(m\,\alpha(n))$
  - n = no. of Make-Set operations, m = total no. of operations.
  - Note: $m \geq n$.
  - Suffices to know $\alpha(n) = O(\lg^* n) = O(\lg n)$.

$$lg^*(n) = \min\{i \geq 0 : lg^{(i)} n \leq 1\}$$

Kruskal's Algorithm

```
MST-KRUSKAL(G, w)
A = ∅
for each vertex v ∈ G.V
    MAKE-SET(v)
sort the edges of G.E into nondecreasing order by weight w
for each edge (u, v) ∈ G.E, taken in nondecreasing order by weight
    if FIND-SET(u) ≠ FIND-SET(v)
        A = A ∪ {(u, v)}
        UNION(u, v)
return A
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        A = A ∪ {(u, v)}
        UNION(u, v)
return A
```
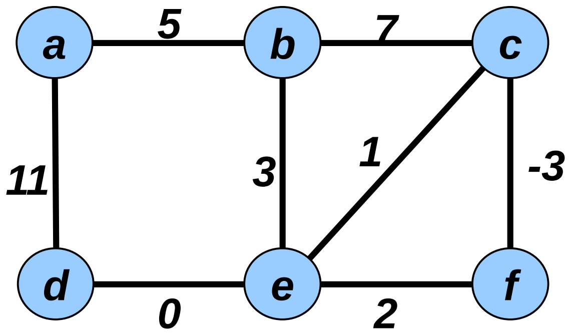

- *Initialization -- $O(V)$.*
- *Sort Edges -- $O(E \lg E)$.*
- *Set Operations -- $O((V+E)\alpha(V))$*
- *Total is $O(E \lg E)$.*

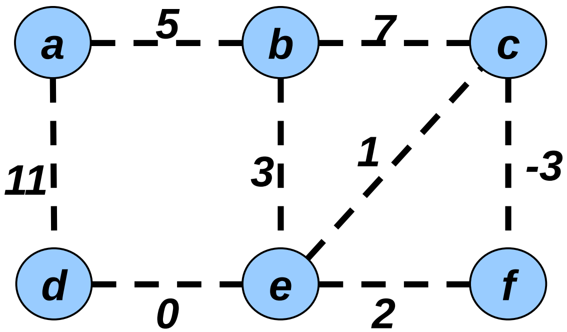
- *$E = O(V^2)$*
- *Therefore, $\lg E = O(\lg V)$*
- *Hence, complexity is* ***$O(E \lg V)$.***

Example of Kruskal's Algorithm

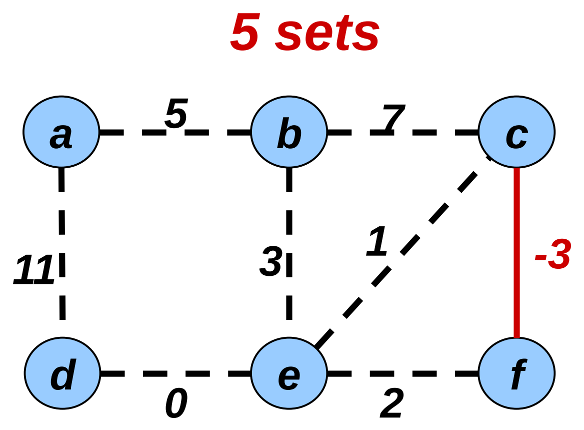


Example of Kruskal's Algorithm

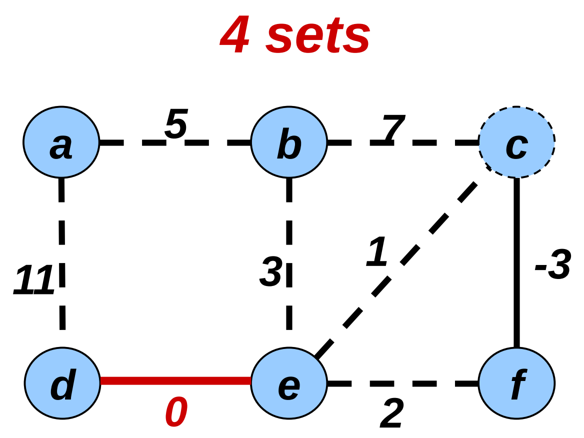
6 sets



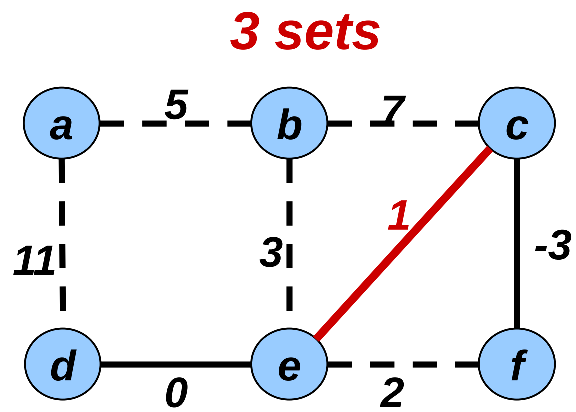
Example of Kruskal's Algorithm



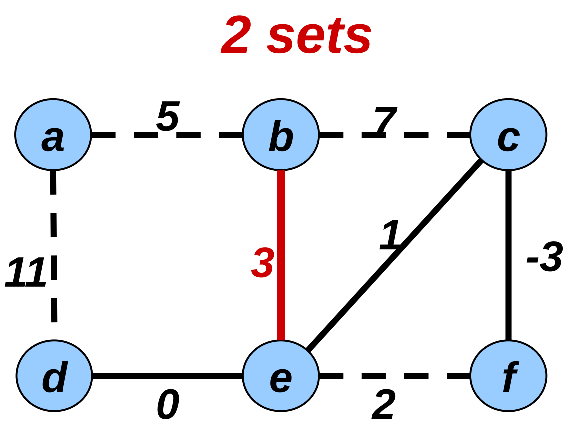
Example of Kruskal's Algorithm



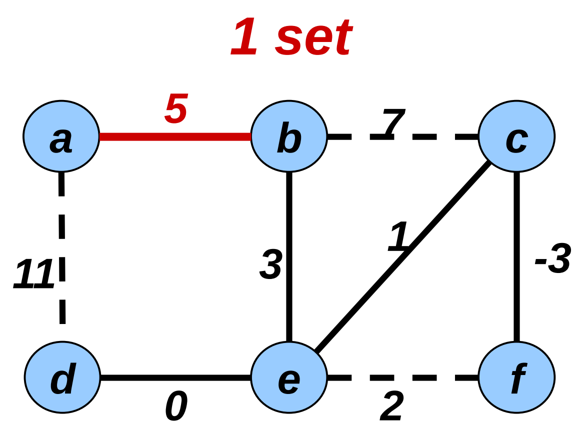
Example of Kruskal's Algorithm



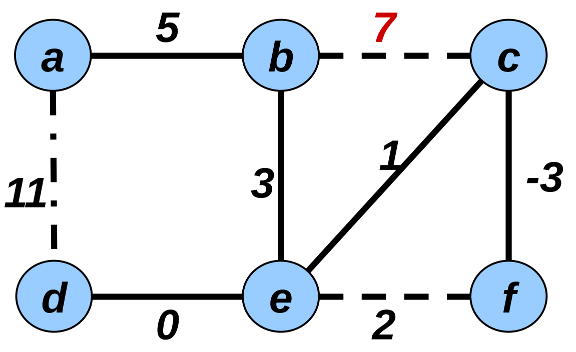
Example of Kruskal's Algorithm



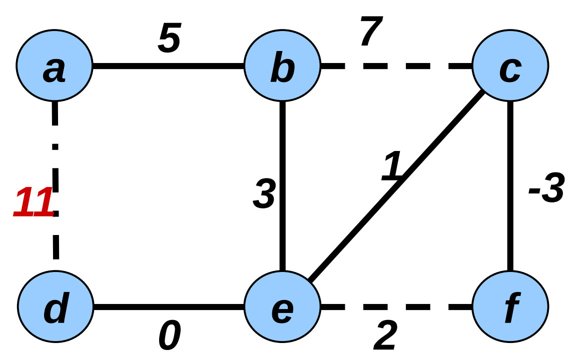
Example of Kruskal's Algorithm



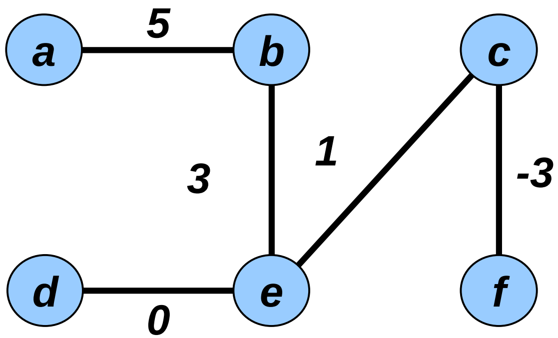
Example of Kruskal's Algorithm

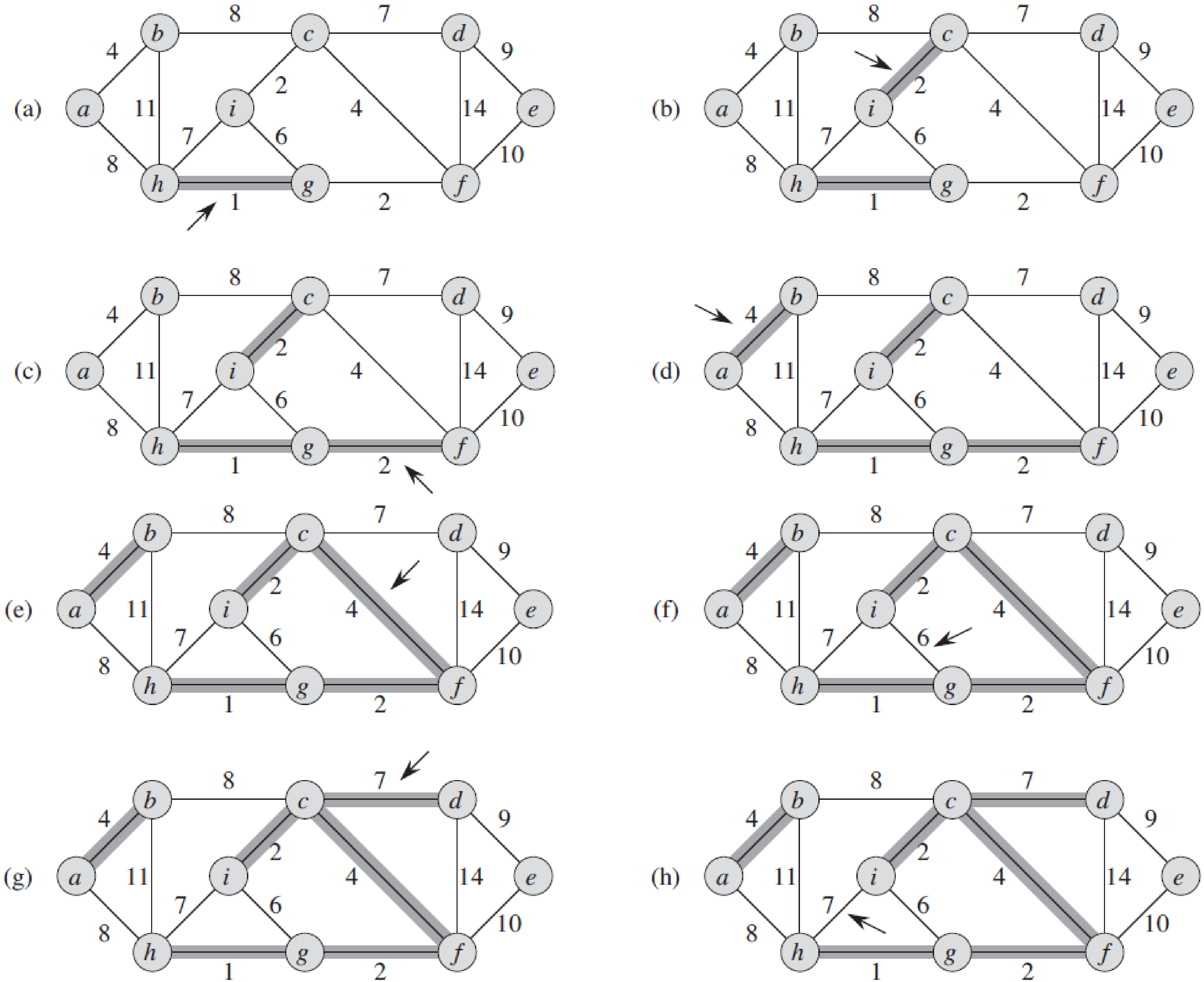


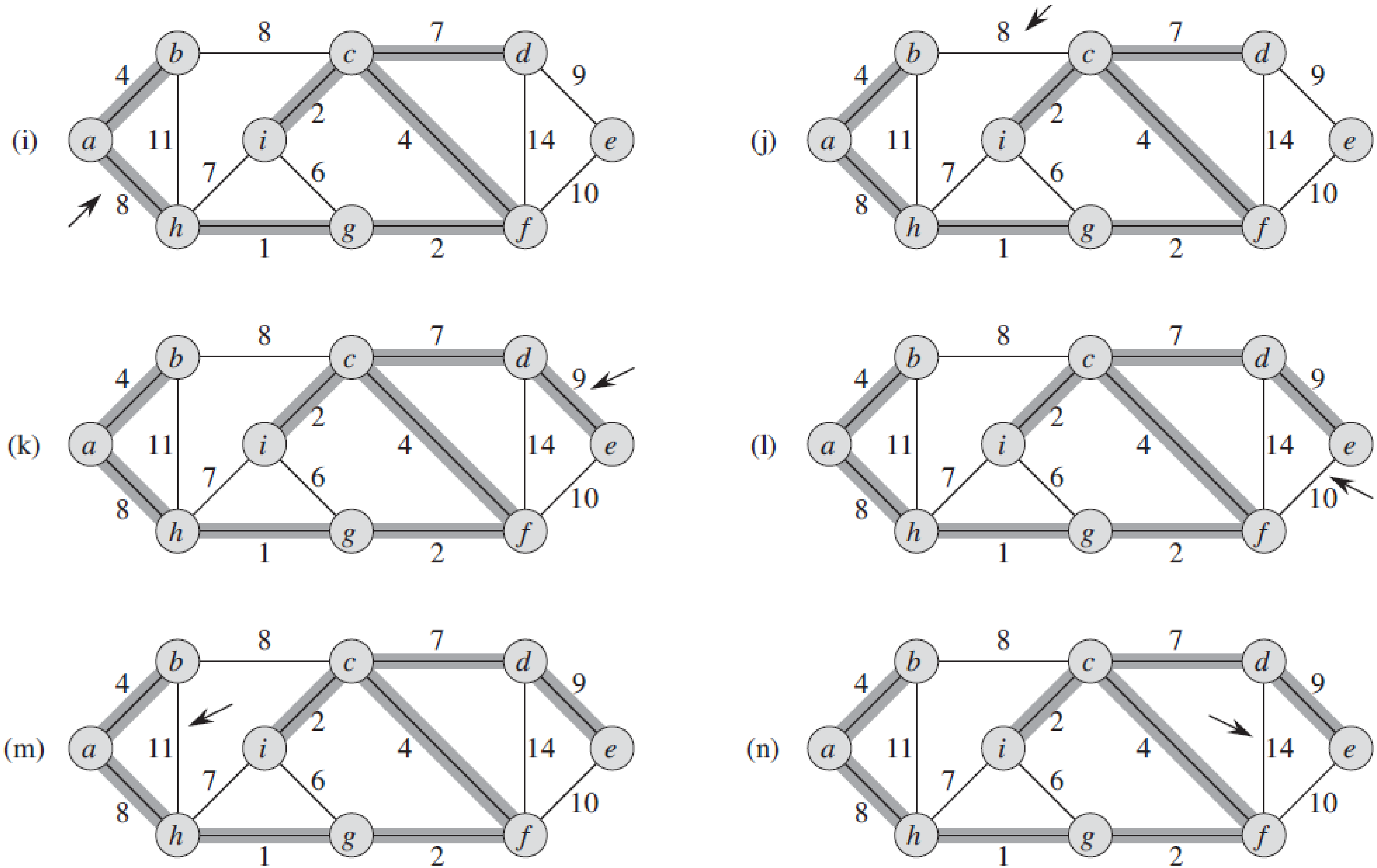
Example of Kruskal’s Algorithm

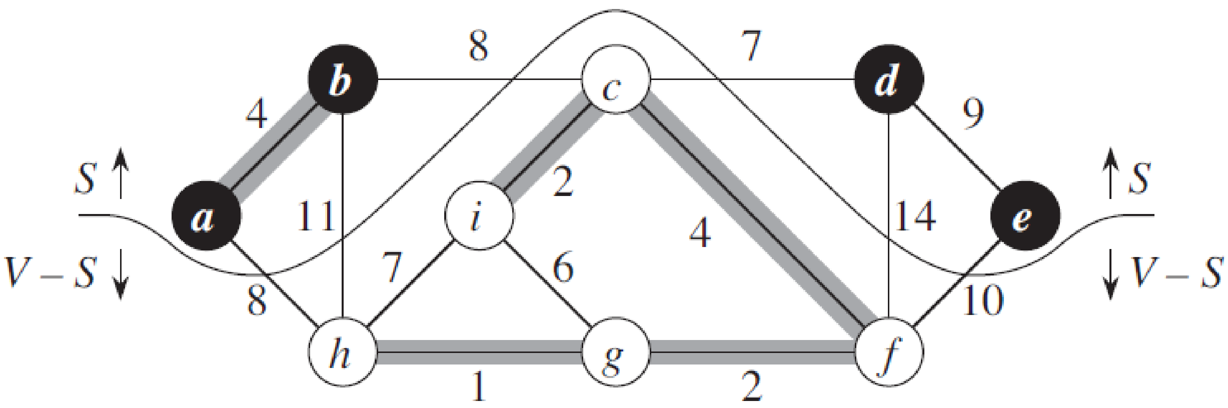


Example of Kruskal's Algorithm



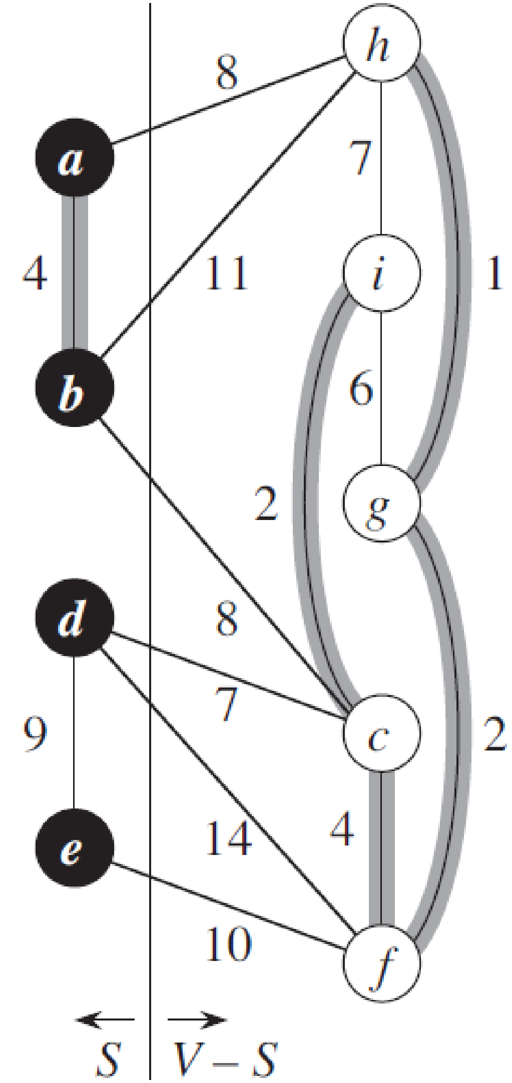

(a)
(b)
(c)
(d)
(e)
(f)
(g)
(h)
a b c d e f g h i
4 8 7 9 11 2 4 14 7 6 10 8 1 2


(i)
(j)
(k)
(l)
(m)
(n)



S
V – S
a
b
c
d
e
f
g
h
i
4
8
11
7
2
6
1
2
4
7
14
9
10
S
V – S

(a)



a
b
d
e
h
i
g
c
f
8
4
11
7
1
6
2
8
7
2
9
14
4
10
S
V – S

(b)