

Matrix multiplication

Input: $A = [a_{ij}], B = [b_{ij}].$ } $i, j = 1, 2, \dots, n.$
Output: $C = [c_{ij}] = A \cdot B.$

$$\begin{bmatrix} c_{11} & c_{12} & \cdots & c_{1n} \\ c_{21} & c_{22} & \cdots & c_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ c_{n1} & c_{n2} & \cdots & c_{nn} \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix} \cdot \begin{bmatrix} b_{11} & b_{12} & \cdots & b_{1n} \\ b_{21} & b_{22} & \cdots & b_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ b_{n1} & b_{n2} & \cdots & b_{nn} \end{bmatrix}$$

$$c_{ij} = \sum_{k=1}^n a_{ik} \cdot b_{kj}$$

Standard algorithm

```
for  $i \leftarrow 1$  to  $n$ 
  do for  $j \leftarrow 1$  to  $n$ 
    do  $c_{ij} \leftarrow 0$ 
      for  $k \leftarrow 1$  to  $n$ 
        do  $c_{ij} \leftarrow c_{ij} + a_{ik} \cdot b_{kj}$ 
```

Running time = $\Theta(n^3)$

Divide-and-conquer algorithm

IDEA:

$n \times n$ matrix = 2×2 matrix of $(n/2) \times (n/2)$ submatrices:

$$\begin{bmatrix} r & s \\ t & u \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \cdot \begin{bmatrix} e & f \\ g & h \end{bmatrix}$$

$$C = A \cdot B$$

$$r = ae + bg$$

$$s = af + bh$$

$$t = ce + dh$$

$$u = cf + dg$$

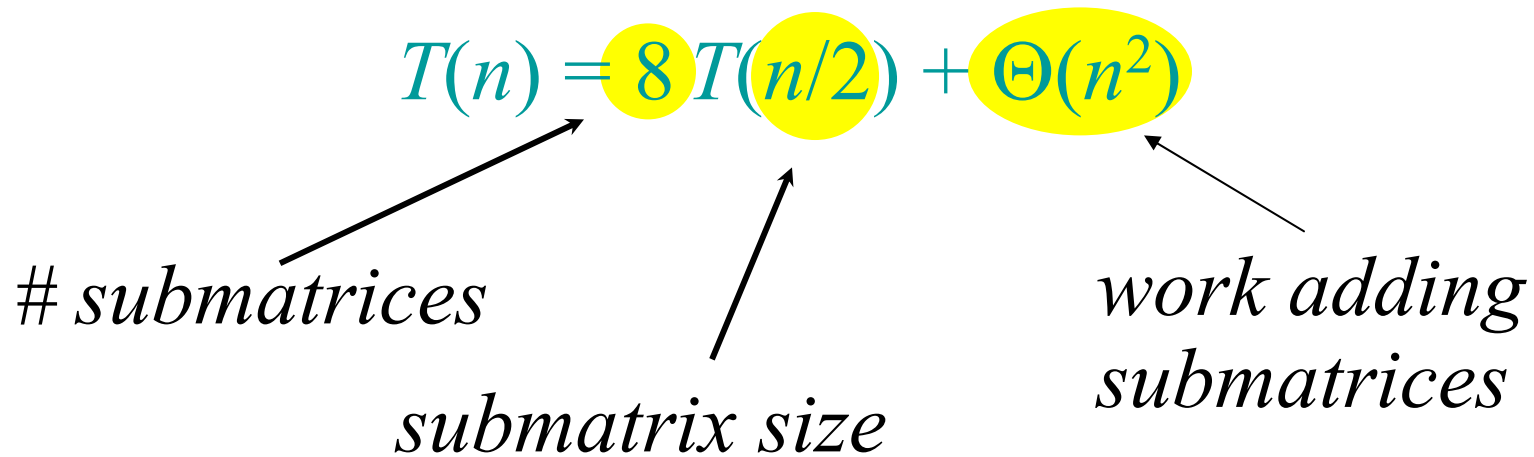
8 mults of $(n/2) \times (n/2)$ submatrices

4 adds of $(n/2) \times (n/2)$ submatrices

Analysis of D&C algorithm

$$T(n) = 8T(n/2) + \Theta(n^2)$$

submatrices *submatrix size* *work adding submatrices*



$$n^{\log_b a} = n^{\log_2 8} = n^3 \Rightarrow \text{CASE 1} \Rightarrow T(n) = \Theta(n^3).$$

No better than the ordinary algorithm.

Strassen's idea

- Multiply 2×2 matrices with only 7 recursive mults.

$$P_1 = a \cdot (f - h)$$

$$P_2 = (a + b) \cdot h$$

$$P_3 = (c + d) \cdot e$$

$$P_4 = d \cdot (g - e)$$

$$P_5 = (a + d) \cdot (e + h)$$

$$P_6 = (b - d) \cdot (g + h)$$

$$P_7 = (a - c) \cdot (e + f)$$

$$r = P_5 + P_4 - P_2 + P_6$$

$$s = P_1 + P_2$$

$$t = P_3 + P_4$$

$$u = P_5 + P_1 - P_3 - P_7$$

7 mults, 18 adds/subs.

Note: No reliance on commutativity of mult!

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$$P_5 = (a + d) \cdot (e + h)$$

$$P_6 = (b - d) \cdot (g + h)$$

$$P_7 = (a - c) \cdot (e + f)$$

$$r = P_5 + P_4 - P_2 + P_6$$

$$= (a + d)(e + h)$$

$$+ d(g - e) - (a + b)h$$

$$+ (b - d)(g + h)$$

$$= ae + ah + de + dh$$

$$+ dg - de - ah - bh$$

$$+ bg + bh - dg - dh$$

$$= ae + bg$$

Strassen's algorithm

- 1. *Divide*:** Partition A and B into $(n/2) \times (n/2)$ submatrices. Form terms to be multiplied using $+$ and $-$.
- 2. *Conquer*:** Perform 7 multiplications of $(n/2) \times (n/2)$ submatrices recursively.
- 3. *Combine*:** Form C using $+$ and $-$ on $(n/2) \times (n/2)$ submatrices.

$$T(n) = 7 T(n/2) + \Theta(n^2)$$

Analysis of Strassen

$$T(n) = 7 T(n/2) + \Theta(n^2)$$

$$n^{\log_b a} = n^{\log_2 7} \approx n^{2.81} \Rightarrow \text{CASE 1} \Rightarrow T(n) = \Theta(n^{\lg 7}).$$

The number 2.81 may not seem much smaller than 3, but because the difference is in the exponent, the impact on running time is significant. In fact, Strassen's algorithm beats the ordinary algorithm on today's machines for $n \geq 30$ or so.

Best to date (of theoretical interest only): $\Theta(n^{2.376\dots})$.