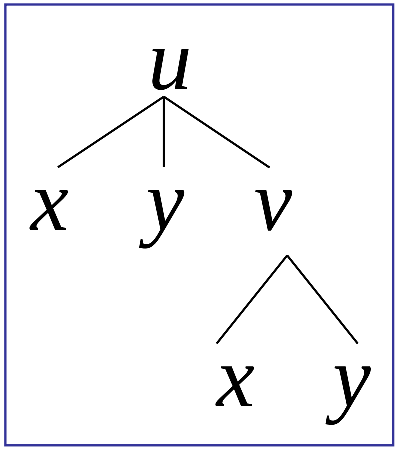
内容小结

1. 复合函数求导的链式法则

“分段用乘，分叉用加，单路全导，叉路偏导”

例如，$u = f(x, y, v),\ v = \varphi(x, y),$

$$\frac{\partial u}{\partial x} = f_1 + f_3 \cdot \varphi_1 ; \qquad \frac{\partial u}{\partial y} = f_2 + f_3 \cdot \varphi_2$$



2. 全微分形式不变性

对 $z = f(u, v)$，不论 u，v 是自变量还是因变量，

$$\mathrm{d}\, z = f_u(u, v)\,\mathrm{d}\, u + f_v(u, v)\,\mathrm{d}\, v$$

思考题 设 $z=f(u)$，方程 $u=\varphi(u)+\int_y^x p(t)\,dt$ 确定 u 是 x , y 的函数，其中 $f(u),\varphi(u)$ 可微，$p(t),\varphi'(u)$ 连续，且 $\varphi'(u)\neq 1$，求 $p(y)\frac{\partial z}{\partial x}+p(x)\frac{\partial z}{\partial y}$.

解: $\frac{\partial z}{\partial x}=f'(u)\frac{\partial u}{\partial x}$，$\frac{\partial z}{\partial y}=f'(u)\frac{\partial u}{\partial y}$

$$\left.\begin{array}{l}\frac{\partial u}{\partial x}=\varphi'(u)\frac{\partial u}{\partial x}+p(x)\\ \frac{\partial u}{\partial y}=\varphi'(u)\frac{\partial u}{\partial y}-p(y)\end{array}\right\}\Longrightarrow\left\{\begin{array}{l}\frac{\partial u}{\partial x}=\frac{p(x)}{1-\varphi'(u)}\\ \frac{\partial u}{\partial y}=\frac{-p(y)}{1-\varphi'(u)}\end{array}\right.$$

$$\therefore\ p(y)\frac{\partial z}{\partial x}+p(x)\frac{\partial z}{\partial y}=f'(u)\left[\,p(y)\frac{\partial u}{\partial x}+p(x)\frac{\partial u}{\partial y}\,\right]=0$$

思考题

1. 已知 $f(x,y)\Big|_{y=x^2}=1$， $f_1(x,y)\Big|_{y=x^2}=2x$， 求 $f_2(x,y)\Big|_{y=x^2}$.

解: 由 $f(x,x^2)=1$ 两边对 x 求导, 得

$$f_1(x,x^2)+f_2(x,x^2)\cdot 2x=0$$

$$\Big\downarrow\ f_1(x,x^2)=2x$$

$$f_2(x,x^2)=-1$$

2. 设函数 $z = f(x, y)$ 在点 $(1,1)$ 处可微，且

$$f(1,1)=1, \quad \left.\frac{\partial f}{\partial x}\right|_{(1,1)} = 2, \quad \left.\frac{\partial f}{\partial y}\right|_{(1,1)} = 3,$$

$\varphi(x) = f(x, f(x,x))$，求 $\left.\frac{\mathrm{d}}{\mathrm{d}x}\varphi^3(x)\right|_{x=1}$. **(2001考研)**

解: 由题设 $\varphi(1) = f(1, f(1,1)) = f(1,1) = 1$

$$\left.\frac{\mathrm{d}}{\mathrm{d}x}\varphi^3(x)\right|_{x=1} = 3\varphi^2(x)\left.\frac{\mathrm{d}\varphi}{\mathrm{d}x}\right|_{x=1}$$

$$= 3\Big[\, f_1(x, f(x,x))$$

$$\left. + f_2(x, f(x,x))\big(f_1(x,x) + f_2(x,x)\big) \,\Big]\right|_{x=1}$$

$$= 3\cdot\big[2 + 3\cdot(2+3)\big] = 51$$

二、多元复合函数的全微分

设函数 $z=f(u,v),u=\varphi(x,y),v=\psi(x,y)$ 都可微,

则复合函数 $z=f(\varphi(x,y),\psi(x,y))$ 的全微分为

$$
\begin{aligned}
\mathrm{d}z &= \frac{\partial z}{\partial x}\mathrm{d}x+\frac{\partial z}{\partial y}\mathrm{d}y \\
&= (\frac{\partial z}{\partial u}\cdot\frac{\partial u}{\partial x}+\frac{\partial z}{\partial v}\cdot\frac{\partial v}{\partial x})\,\mathrm{d}x+(\frac{\partial z}{\partial u}\cdot\frac{\partial u}{\partial y}+\frac{\partial z}{\partial v}\cdot\frac{\partial v}{\partial y})\,\mathrm{d}y \\
&= \frac{\partial z}{\partial u}\;(\frac{\partial u}{\partial x}\mathrm{d}x+\frac{\partial u}{\partial y}\mathrm{d}y)\;+\frac{\partial z}{\partial v}\;(\frac{\partial v}{\partial x}\mathrm{d}x+\frac{\partial v}{\partial y}\mathrm{d}y) \\
&= \frac{\partial z}{\partial u}\;\mathrm{d}u+\frac{\partial z}{\partial v}\,\mathrm{d}v
\end{aligned}
$$

可见无论 u , v 是自变量还是中间变量, 其全微分表达形式都一样, 这性质叫做**全微分形式不变性.**

例 6. 利用全微分形式不变性再解例1.

解: $\mathrm{d}z = \mathrm{d}(e^u \sin v)$

$$= e^u \sin v \,\mathrm{d}u + e^u \cos v \,\mathrm{d}v$$

$$= e^{xy}[\sin(x+y)\,\mathrm{d}(xy) + \cos(x+y)\,\mathrm{d}(x+y)]$$

$$= e^{xy}[\sin(x+y)(y\mathrm{d}x + x\mathrm{d}y) + \cos(x+y)(\mathrm{d}x + \mathrm{d}y)]$$

$$= e^{xy}[y\sin(x+y) + \cos(x+y)]\mathrm{d}x$$

$$+ e^{xy}[x\sin(x+y) + \cos(x+y)]\mathrm{d}y$$

所以 $$\frac{\partial z}{\partial x} = e^{xy}[y \cdot \sin(x+y) + \cos(x+y)]$$

$$\frac{\partial z}{\partial y} = e^{xy}[x \cdot \sin(x+y) + \cos(x+y)]$$

第五节

隐函数的求导方法

一、一个方程所确定的隐函数及其导数

二、方程组所确定的隐函数组及其导数

本节讨论：

1) 方程在什么条件下才能确定隐函数 .

例如, 方程 $x^2+\sqrt{y}+C=0$

当 $C<0$ 时, 能确定隐函数;

当 $C>0$ 时, 不能确定隐函数;

2) 在方程能确定隐函数时, 研究其连续性、可微性及求导方法问题 .

一、一个方程所确定的隐函数及其导数

定理1. 设函数 $F(x,y)$在点 $P(x_0,y_0)$的某一邻域内满足

① 具有连续的偏导数;

② $F(x_0\,,y_0)=0$;

③ $F_y(x_0\,,y_0)\neq 0$

则方程 $F(x,y)=0$ 在点 x_0 的**某邻域内**可唯一确定一个单值连续函数 $y=f(x)$, 满足条件 $y_0=f(x_0)$, 并有连续导数

$$\frac{\mathrm{d}y}{\mathrm{d}x}=-\frac{F_x}{F_y}$$ (隐函数求导公式)

定理证明从略，仅就求导公式推导如下：

设 $y = f(x)$ 为方程 $F(x, y) = 0$ 所确定的隐函数，则

$$F(x, f(x)) \equiv 0$$

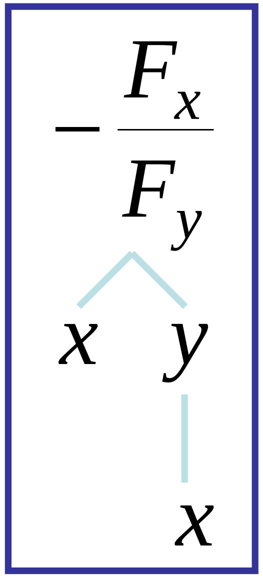
$$\Big\downarrow \text{两边对 } x \text{ 求导}$$

$$\frac{\partial F}{\partial x} + \frac{\partial F}{\partial y}\frac{\mathrm{d}y}{\mathrm{d}x} \equiv 0$$

$$\Big\downarrow \text{在}(x_0, y_0)\text{的某邻域内 } F_y \neq 0$$

$$\frac{\mathrm{d}y}{\mathrm{d}x} = -\frac{F_x}{F_y}$$

若$F(x,y)$的二阶偏导数也都连续，则还有



二阶导数：

$$\frac{\mathrm{d}^2 y}{\mathrm{d}x^2} = \frac{\partial}{\partial x}\left(-\frac{F_x}{F_y}\right) + \frac{\partial}{\partial y}\left(-\frac{F_x}{F_y}\right)\frac{\mathrm{d}y}{\mathrm{d}x}$$

$$= -\frac{F_{xx}F_y - F_{yx}F_x}{F_y^2} - \frac{F_{xy}F_y - F_{yy}F_x}{F_y^2}\left(-\frac{F_x}{F_y}\right)$$

$$= -\frac{F_{xx}F_y^{\,2} - 2F_{xy}F_xF_y + F_{yy}F_x^{\,2}}{F_y^3}$$

例1. 验证方程$\sin y + e^x - xy - 1 = 0$在点(0,0)某邻域可确定一个单值可导隐函数$y = f(x)$,并求

$$\left.\frac{\mathrm{d}y}{\mathrm{d}x}\right|_{x=0}, \quad \left.\frac{\mathrm{d}^2 y}{\mathrm{d}x^2}\right|_{x=0}$$

解: 令$F(x, y) = \sin y + e^x - xy - 1$, 则

① $F_x = e^x - y,\ F_y = \cos y - x$ 连续,

② $F(0,0) = 0$,

③ $F_y(0,0) = 1 \neq 0$

由 定理1 可知, 在 $x = 0$ 的某邻域内方程存在单值可导的隐函数 $y = f(x)$, 且

$$\left.\frac{\mathrm{d}y}{\mathrm{d}x}\right|_{x=0} = -\left.\frac{F_x}{F_y}\right|_{x=0} = -\left.\frac{e^x - y}{\cos y - x}\right|_{x=0,\, y=0} = -1$$

$$\left.\frac{\mathrm{d}^2 y}{\mathrm{d}x^2}\right|_{x=0}$$

$$= -\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{e^x - y}{\cos y - x}\right)\Bigg|_{x=0,\ y=0,\ y'=-1}$$

$$= -\frac{(e^x - y')(\cos y - x) - (e^x - y)(-\sin y \cdot y' - 1)}{(\cos y - x)^2}\Bigg|_{\substack{x=0\\ y=0\\ y'=-1}}$$

$$= -3$$

导数的另一求法 — 利用隐函数求导

$$\sin y + e^x - xy - 1 = 0,\quad y = y(x)$$

两边对 x 求导

$$\cos y \cdot y' + e^x - y - xy' = 0$$

$$y'\Big|_{x=0} = -\frac{e^x - y}{\cos y - x}\Bigg|_{(0,0)} = -1$$

两边再对 x 求导

$$-\sin y \cdot (y')^2 + \cos y \cdot y'' + e^x - y' - y' - xy'' = 0$$

令 $x = 0$，注意此时 $y = 0$，$y' = -1$

$$\frac{\mathrm{d}^2 y}{\mathrm{d}x^2}\Bigg|_{x=0} = -3$$

定理2. 若函数 $F(x, y, z)$ 满足:

① 在点 $P(x_0, y_0, z_0)$ 的某邻域内具有**连续偏导数**,

② $F(x_0, y_0, z_0) = 0$

③ $F_z(x_0, y_0, z_0) \neq 0$

则方程 $F(x, y, z) = 0$ 在点 (x_0, y_0) 某一邻域内可唯一确定一个单值连续函数 $z = f(x, y)$, 满足 $z_0 = f(x_0, y_0)$, 并有连续偏导数

$$\frac{\partial z}{\partial x} = -\frac{F_x}{F_z}, \quad \frac{\partial z}{\partial y} = -\frac{F_y}{F_z}$$

定理证明从略, 仅就求导公式推导如下:

设 $z=f(x,y)$ 是方程 $F(x,y)=0$ 所确定的隐函数，则

$$F(x,y,f(x,y))\equiv 0$$

$\Big\downarrow$ 两边对 x 求偏导

$$F_x+F_z\frac{\partial z}{\partial x}\equiv 0$$

$\Big\downarrow$ 在 (x_0,y_0,z_0) 的某邻域内 $F_z\neq 0$

$$\frac{\partial z}{\partial x}=-\frac{F_x}{F_z}$$

同样可得 $$\frac{\partial z}{\partial y}=-\frac{F_y}{F_z}$$

例2. 设 $x^2+y^2+z^2-4z=0$, 求 $\dfrac{\partial^2 z}{\partial x^2}$.

解法1 利用隐函数求导

$$2x+2z\frac{\partial z}{\partial x}-4\frac{\partial z}{\partial x}=0 \longrightarrow \frac{\partial z}{\partial x}=\frac{x}{2-z}$$

再对 x 求导

$$2+2\left(\frac{\partial z}{\partial x}\right)^2+2z\frac{\partial^2 z}{\partial x^2}-4\frac{\partial^2 z}{\partial x^2}=0$$

$$\frac{\partial^2 z}{\partial x^2}=\frac{1+\left(\frac{\partial z}{\partial x}\right)^2}{2-z}=\frac{(2-z)^2+x^2}{(2-z)^3}$$

解法2 利用公式

设 $F(x,y,z)=x^2+y^2+z^2-4z$

则 $F_x=2x,\ F_z=2z-4$

$$\therefore \quad \frac{\partial z}{\partial x}=-\frac{F_x}{F_z}=-\frac{x}{z-2}=\frac{x}{2-z}$$

两边对 x 求偏导

$$\frac{\partial^2 z}{\partial x^2}=\frac{\partial}{\partial x}\left(\frac{x}{2-z}\right)=\frac{(2-z)+x\dfrac{\partial z}{\partial x}}{(2-z)^2}=\frac{(2-z)^2+x^2}{(2-z)^3}$$

例 3　已知$\ln\sqrt{x^2+y^2}=\arctan\dfrac{y}{x}$，求$\dfrac{dy}{dx}$.

解　令 $F(x,y)=\ln\sqrt{x^2+y^2}-\arctan\dfrac{y}{x}$,

则　$F_x(x,y)=\dfrac{x+y}{x^2+y^2}$，　$F_y(x,y)=\dfrac{y-x}{x^2+y^2}$,

$$\frac{dy}{dx}=-\frac{F_x}{F_y}=-\frac{x+y}{y-x}.$$

例4. 设$F(x,y)$具有连续偏导数,已知方程 $F(\frac{x}{z},\frac{y}{z})=0$,求 $\mathrm{d}z$.

解法1 利用偏导数公式. 设 $z=f(x,y)$ 是由方程 $F(\frac{x}{z},\frac{y}{z})=0$ 确定的隐函数,则

$$\frac{\partial z}{\partial x}=-\frac{F_1\cdot\frac{1}{z}}{F_1\cdot(-\frac{x}{z^2})+F_2\cdot(-\frac{y}{z^2})}=\frac{zF_1}{xF_1+yF_2}$$

$$\frac{\partial z}{\partial y}=-\frac{F_2\cdot\frac{1}{z}}{F_1\cdot(-\frac{x}{z^2})+F_2\cdot(-\frac{y}{z^2})}=\frac{zF_2}{xF_1+yF_2}$$

故 $$\mathrm{d}z=\frac{\partial z}{\partial x}\mathrm{d}x+\frac{\partial z}{\partial y}\mathrm{d}y=\frac{z}{xF_1+yF_2}(F_1\mathrm{d}x+F_2\mathrm{d}y)$$

解法2　微分法. 对方程两边求微分:

$$F(\frac{x}{z},\frac{y}{z})=0$$

$$F_1\cdot \mathrm{d}(\frac{x}{z}) + F_2\cdot \mathrm{d}(\frac{y}{z})=0$$

$$F_1\cdot(\frac{z\mathrm{d}x - x\mathrm{d}z}{z^2}) + F_2\cdot(\frac{z\mathrm{d}y - y\mathrm{d}z}{z^2})=0$$

$$\frac{xF_1 + yF_2}{z^2}\mathrm{d}z = \frac{F_1\mathrm{d}x + F_2\mathrm{d}y}{z}$$

$$\mathrm{d}z = \frac{z}{xF_1 + yF_2}(F_1\mathrm{d}x + F_2\mathrm{d}y)$$

二元线性代数方程组解的公式

$$\begin{cases} a_1 x + b_1 y = c_1 \\ a_2 x + b_2 y = c_2 \end{cases}$$

解:

$$x = \frac{1}{\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}} \begin{vmatrix} c_1 & b_1 \\ c_2 & b_2 \end{vmatrix}$$

$$y = \frac{1}{\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}} \begin{vmatrix} a_1 & c_1 \\ a_2 & c_2 \end{vmatrix}$$

二、方程组所确定的隐函数组及其导数

隐函数存在定理还可以推广到方程组的情形.

以两个方程确定两个隐函数的情况为例，即

$$\begin{cases} F(x,y,u,v)=0 \\ G(x,y,u,v)=0 \end{cases} \longrightarrow \begin{cases} u=u(x,y) \\ v=v(x,y) \end{cases}$$

由 F、G 的偏导数组成的行列式

$$J=\frac{\partial(F,G)}{\partial(u,v)}=\begin{vmatrix} F_u & F_v \\ G_u & G_v \end{vmatrix}$$

称为F、G 的**雅可比(Jacobi)**行列式.

定理3. 设函数 $F(x,y,u,v)$, $G(x,y,u,v)$ 满足:

① 在点 $P(x_0, y_0, u_0, v_0)$ 的某一邻域内具有连续偏导数;

② $F(x_0, y_0, u_0, v_0) = 0$, $G(x_0, y_0, u_0, v_0) = 0$;

③ $J\Big|_P = \left.\dfrac{\partial(F,G)}{\partial(u,v)}\right|_P \neq 0$

则方程组 $F(x,y,u,v)=0$, $G(x,y,u,v)=0$ 在点 (x_0, y_0) 的某一邻域内可**唯一**确定一组满足条件 $u_0 = u(x_0, y_0)$, $v_0 = v(x_0, y_0)$ 的**单值连续函数** $u = u(x,y), v = v(x,y)$, 且有偏导数公式：

$$\frac{\partial u}{\partial x}=-\frac{1}{J}\frac{\partial(F,G)}{\partial(\underline{x},v)}=-\frac{1}{\begin{vmatrix}F_u & F_v\\ G_u & G_v\end{vmatrix}}\begin{vmatrix}F_x & F_v\\ G_x & G_v\end{vmatrix}$$

$$\frac{\partial u}{\partial y}=-\frac{1}{J}\frac{\partial(F,G)}{\partial(\underline{y},v)}=-\frac{1}{\begin{vmatrix}F_u & F_v\\ G_u & G_v\end{vmatrix}}\begin{vmatrix}F_y & F_v\\ G_y & G_v\end{vmatrix}$$

$$\frac{\partial v}{\partial x}=-\frac{1}{J}\frac{\partial(F,G)}{\partial(u,\underline{x})}=-\frac{1}{\begin{vmatrix}F_u & F_v\\ G_u & G_v\end{vmatrix}}\begin{vmatrix}F_u & F_x\\ G_u & G_x\end{vmatrix}$$

定理证明略.
仅推导偏导
数公式如下:

$$\frac{\partial v}{\partial y}=-\frac{1}{J}\frac{\partial(F,G)}{\partial(u,\underline{y})}=-\frac{1}{\begin{vmatrix}F_u & F_v\\ G_u & G_v\end{vmatrix}}\begin{vmatrix}F_u & F_y\\ G_u & G_y\end{vmatrix}$$

设方程组 $\begin{cases} F(x,y,u,v)=0 \\ G(x,y,u,v)=0 \end{cases}$ 有隐函数组 $\begin{cases} u=u(x,y) \\ v=v(x,y) \end{cases}$,则

$$\begin{cases} F(x,y,u(x,y),v(x,y))\equiv 0 \\ G(x,y,u(x,y),v(x,y))\equiv 0 \end{cases}$$

两边对 x 求导得 $\begin{cases} F_x+F_u\cdot\dfrac{\partial u}{\partial x}+F_v\cdot\dfrac{\partial v}{\partial x}=0 \\ G_x+G_u\cdot\dfrac{\partial u}{\partial x}+G_v\cdot\dfrac{\partial v}{\partial x}=0 \end{cases}$

这是关于 $\dfrac{\partial u}{\partial x},\dfrac{\partial v}{\partial x}$ 的线性方程组,在点P 的某邻域内

系数行列式 $J=\begin{vmatrix} F_u & F_v \\ G_u & G_v \end{vmatrix}\neq 0$, 故得

$$\frac{\partial u}{\partial x} = -\frac{1}{J}\frac{\partial(F,G)}{\partial(x,v)}$$

$$\frac{\partial v}{\partial x} = -\frac{1}{J}\frac{\partial(F,G)}{\partial(u,x)}$$

同样可得

$$\frac{\partial u}{\partial y} = -\frac{1}{J}\frac{\partial(F,G)}{\partial(y,v)}$$

$$\frac{\partial v}{\partial y} = -\frac{1}{J}\frac{\partial(F,G)}{\partial(u,y)}$$