§ 2.3 极限的运算和两个重要极限

1. 极限的四则运算

用 $\lim$ 表示 $x \to \infty, +\infty, -\infty, x_0, x_0^+, x_0^-$ 之一.

定理1 设 $\lim f(x) = A, \lim g(x) = B,$ 则有

(1) $\lim\big(f(x) \pm g(x)\big) = \lim f(x) \pm \lim g(x) = A \pm B;$

(2) $\lim\big(f(x)g(x)\big) = \lim f(x) \cdot \lim g(x) = AB;$

(3) $\lim \dfrac{f(x)}{g(x)} = \dfrac{\lim f(x)}{\lim g(x)} = \dfrac{A}{B} \ (B \neq 0);$

(4) $\lim\big(c \cdot f(x)\big) = c \cdot \lim f(x),$ C 是常数;

(5) $\lim[f(x)]^k = [\lim f(x)]^k,$ k 是正整数.

证 由P51推论知 $\lim f(x)=A, \lim g(x)=B$ 等价于

$$f(x)=A+\alpha(x), g(x)=B+\beta(x),$$

其中 $\alpha(x), \beta(x)$ 是同过程无穷小量， 得

(1) $f(x)\pm g(x)=A\pm B+\alpha(x)\pm\beta(x),$

$$\Rightarrow \lim\big(f(x)\pm g(x)\big)=A\pm B.$$

(2) $f(x)g(x)=AB+A\beta(x)+B\alpha(x)+\alpha(x)\beta(x),$

$$\Rightarrow \lim\big(f(x)g(x)\big)=AB.$$

(3) $\dfrac{f(x)}{g(x)}-\dfrac{A}{B}=\dfrac{B(A+\alpha(x))-A(B+\beta(x))}{B(B+\beta(x))}=\dfrac{B\alpha(x)-A\beta(x)}{B(B+\beta(x))},$

$$\lim\frac{B\alpha(x)-A\beta(x)}{B(B+\beta(x))}=0, \qquad \Rightarrow \lim\frac{f(x)}{g(x)}=\frac{A}{B}.$$

求极限举例

例1. $P(x)=a_0x^n+a_1x^{n-1}+\cdots+a_n$ 是多项式，求 $\lim\limits_{x\to x_0}P(x)$.

解 $\lim\limits_{x\to x_0}P(x)=\lim\limits_{x\to x_0}\left(a_0x^n+a_1x^{n-1}+\cdots+a_n\right)$

$$=a_0(\lim_{x\to x_0}x)^n+a_1(\lim_{x\to x_0}x)^{n-1}+\cdots+(\lim_{x\to x_0}a_n)$$

$$=a_0x_0^n+a_1x_0^{n-1}+\cdots+a_n=P(x_0)$$

例2. $R(x)=\dfrac{P(x)}{Q(x)}=\dfrac{a_0x^n+a_1x^{n-1}+\cdots+a_n}{b_0x^m+b_1x^{m-1}+\cdots+b_m}$ 是有理分式函数，

$Q(x_0)\neq 0$, 求 $\lim\limits_{x\to x_0}R(x)$.

解 $\lim\limits_{x\to x_0}R(x)=\dfrac{\lim\limits_{x\to x_0}P(x)}{\lim\limits_{x\to x_0}Q(x)}=\dfrac{P(x_0)}{Q(x_0)}=R(x_0)$.

例 3. 设 $a_0 \neq 0, b_0 \neq 0, n, m$ 是正整数，求 $\lim\limits_{x\to\infty}\dfrac{a_0x^n + a_1x^{n-1} + \cdots + a_n}{b_0x^m + b_1x^{m-1} + \cdots + b_m}$.

解 当 $n = m$ 时，

$$\lim_{x\to\infty}\frac{a_0x^n + a_1x^{n-1} + \cdots + a_n}{b_0x^n + b_1x^{n-1} + \cdots + b_m} = \lim_{x\to\infty}\frac{a_0 + a_1\dfrac{1}{x} + \cdots + a_n\dfrac{1}{x^n}}{b_0 + b_1\dfrac{1}{x} + \cdots + b_m\dfrac{1}{x^m}} = \frac{a_0}{b_0}.$$

当 $n < m$ 时，

$$\lim_{x\to\infty}\frac{a_0x^n + a_1x^{n-1} + \cdots + a_n}{b_0x^m + b_1x^{m-1} + \cdots + b_m} = \lim_{x\to\infty}\frac{a_0\dfrac{1}{x^{m-n}} + a_1\dfrac{1}{x^{m-n+1}} + \cdots + a_n\dfrac{1}{x^m}}{b_0 + b_1\dfrac{1}{x} + \cdots + b_m\dfrac{1}{x^m}} = \frac{0}{b_0} = 0.$$

当 $n > m$ 时，

$$\lim_{x\to\infty}\frac{b_0x^m + b_1x^{m-1} + \cdots + b_m}{a_0x^n + a_1x^{n-1} + \cdots + a_n} = 0,\ \text{所以} \lim_{x\to\infty}\frac{a_0x^n + a_1x^{n-1} + \cdots + a_n}{b_0x^m + b_1x^{m-1} + \cdots + b_m} = \infty.$$

例 4. 求 $\lim\limits_{x\to 1}\left(\dfrac{1}{x-1}-\dfrac{3}{x^3-1}\right)$.

解 $x^3-1=(x-1)(x^2+x+1)$,

$$\frac{1}{x-1}-\frac{3}{x^3-1}=\frac{x^2+x+1-3}{(x-1)(x^2+x+1)}=\frac{x^2+x-2}{(x-1)(x^2+x+1)}$$

$$=\frac{(x-1)(x+2)}{(x-1)(x^2+x+1)}=\frac{x+2}{x^2+x+1},$$

所以 $$\lim_{x\to 1}\left(\frac{1}{x-1}-\frac{3}{x^3-1}\right)=\lim_{x\to 1}\frac{x+2}{x^2+x+1}=1.$$

求极限举例

例 5. 求 $\lim\limits_{x\to+\infty}\left(\sqrt{x^2+1}-x\right)$.

解 $$\lim_{x\to+\infty}\left(\sqrt{x^2+1}-x\right)=\lim_{x\to+\infty}\frac{(\sqrt{x^2+1}-x)(\sqrt{x^2+1}+x)}{\sqrt{x^2+1}+x}$$

$$=\lim_{x\to+\infty}\frac{x^2+1-x^2}{\sqrt{x^2+1}+x}$$

$$=\lim_{x\to+\infty}\frac{\frac{1}{x}}{\sqrt{1+\frac{1}{x^2}}+1}=0.$$

2. 两个重要的极限

(1) $\lim\limits_{x\to0}\dfrac{\sin x}{x}=1.$

证　当 $0<x<\dfrac{\pi}{2}$ 时，有 $\sin x<x<\tan x,$

$$\Rightarrow 1<\frac{x}{\sin x}<\frac{1}{\cos x}, \qquad \Rightarrow \cos x<\frac{\sin x}{x}<1.$$

由于 $\cos x,\dfrac{\sin x}{x},1$ 都是偶函数，当 $-\dfrac{\pi}{2}<x<0$ 时，上式也成立.

由 $\lim\limits_{x\to0}\cos x=1$ 及追敛性知 $\lim\limits_{x\to0}\dfrac{\sin x}{x}=1.$

用 $\varepsilon-\delta$ 估计式，对无穷小量 $\alpha(x)$ 有 $\lim\dfrac{\sin\alpha(x)}{\alpha(x)}=1.$

例 **6.** 求 $\lim\limits_{x\to 0}\dfrac{\tan x}{x}$.

解 $$\lim_{x\to 0}\frac{\tan x}{x}=\lim_{x\to 0}\frac{\sin x}{x}\cdot\frac{1}{\cos x}=\lim_{x\to 0}\frac{\sin x}{x}\cdot\lim_{x\to 0}\frac{1}{\cos x}=1.$$

例 **7.** 求 $\lim\limits_{x\to 0}\dfrac{1-\cos x}{x^2}$.

解 $$\lim_{x\to 0}\frac{1-\cos x}{x^2}=\lim_{x\to 0}\frac{2\sin^2\frac{x}{2}}{x^2}=\frac{1}{2}\lim_{x\to 0}\left(\frac{\sin\frac{x}{2}}{\frac{x}{2}}\right)^2=\frac{1}{2}.$$

例 8. 求 $\lim\limits_{x\to 0}\dfrac{\sin 5x}{\sin 2x}$.

解 $$\lim_{x\to 0}\frac{\sin 5x}{\sin 2x}=\frac{5}{2}\lim_{x\to 0}\frac{\dfrac{\sin 5x}{5x}}{\dfrac{\sin 2x}{2x}}=\frac{5}{2}\frac{\lim\limits_{x\to 0}\dfrac{\sin 5x}{5x}}{\lim\limits_{x\to 0}\dfrac{\sin 2x}{2x}}=\frac{5}{2}.$$

例 9. 求 $\lim\limits_{x\to a}\dfrac{\sin x-\sin a}{x-a}$.

解 $$\lim_{x\to a}\frac{\sin x-\sin a}{x-a}=\lim_{x\to a}\frac{2\sin\dfrac{x-a}{2}\cos\dfrac{x+a}{2}}{x-a}$$

$$=\lim_{x\to a}\frac{\sin\dfrac{x-a}{2}}{\dfrac{x-a}{2}}\lim_{x\to a}\cos\frac{x+a}{2}=\cos a.$$

(2) $\lim\limits_{x\to\infty}\left(1+\dfrac{1}{x}\right)^{x}=\mathrm{e}.$

证 先证 $\lim\limits_{x\to+\infty}\left(1+\dfrac{1}{x}\right)^{x}=\mathrm{e}.$ 不妨设 $x>1$, 记 $n=[x]$,

则 $n\le x<n+1$, 而且 $x\to+\infty \Leftrightarrow n\to\infty.$

$$\left(1+\frac{1}{n+1}\right)^{n}\le\left(1+\frac{1}{x}\right)^{x}\le\left(1+\frac{1}{n}\right)^{n+1},$$

$$\lim_{n\to+\infty}\left(1+\frac{1}{n+1}\right)^{n}=\lim_{n\to+\infty}\left(1+\frac{1}{n+1}\right)^{n+1}\left(1+\frac{1}{n+1}\right)^{-1}=\mathrm{e},$$

$$\lim_{n\to+\infty}\left(1+\frac{1}{n}\right)^{n+1}=\lim_{n\to+\infty}\left(1+\frac{1}{n}\right)^{n}\left(1+\frac{1}{n}\right)=\mathrm{e},$$

由迫敛性, 得 $\lim\limits_{x\to+\infty}\left(1+\dfrac{1}{x}\right)^{x}=\mathrm{e}.$

重要极限

对 $x \to -\infty$ 时, 令 $t = -x$, 则当 $x \to -\infty$ 时, $t \to +\infty$.

$$\lim_{x\to-\infty}\left(1+\frac{1}{x}\right)^{x} = \lim_{t\to+\infty}\left(1-\frac{1}{t}\right)^{-t} = \lim_{t\to+\infty}\left(\frac{t-1}{t}\right)^{-t} = \lim_{t\to+\infty}\left(\frac{t}{t-1}\right)^{t}$$

$$= \lim_{t\to+\infty}\left(1+\frac{1}{t-1}\right)^{t} = \lim_{t\to+\infty}\left(1+\frac{1}{t-1}\right)^{t-1}\cdot\left(1+\frac{1}{t-1}\right) = \mathrm{e}$$

所以 $\lim_{x\to\infty}\left(1+\frac{1}{x}\right)^{x} = \mathrm{e}.$

等价于 $\lim_{x\to 0}(1+x)^{\frac{1}{x}} = \mathrm{e}.$

用估计式, 对无穷大量 $g(x)$ 有 $\lim\left(1+\frac{1}{g(x)}\right)^{g(x)} = e.$

对无穷小量 $\alpha(x)$ 有 $\lim(1+\alpha(x))^{\frac{1}{\alpha(x)}} = e.$

例 10. 求$\lim\limits_{x\to\infty}\left(1-\dfrac{1}{x}\right)^{x}$.

解 $\lim\limits_{x\to\infty}\left(1-\dfrac{1}{x}\right)^{x}=\lim\limits_{x\to\infty}\left[\left(1+\dfrac{1}{(-x)}\right)^{-x}\right]^{-1}=\left[\lim\limits_{x\to\infty}\left(1+\dfrac{1}{(-x)}\right)^{-x}\right]^{-1}=\dfrac{1}{e}$.

例 11. 求$\lim\limits_{x\to 0}\left(1+2x^{2}\right)^{\frac{1}{x^{2}}}$.

解 $\lim\limits_{x\to 0}\left(1+2x^{2}\right)^{\frac{1}{x^{2}}}=\lim\limits_{x\to 0}\left[\left(1+2x^{2}\right)^{\frac{1}{2x^{2}}}\right]^{2}=\left[\lim\limits_{x\to 0}\left(1+2x^{2}\right)^{\frac{1}{2x^{2}}}\right]^{2}=\mathrm{e}^{2}$.

例 12. 求$\lim\limits_{x\to\infty}\left(\dfrac{x+1}{x-1}\right)^{x}$.

解 $$\lim_{x\to\infty}\left(\frac{x+1}{x-1}\right)^{x}=\lim_{x\to\infty}\left(1+\frac{2}{x-1}\right)^{x}=\lim_{x\to\infty}\left[\left(1+\frac{2}{x-1}\right)^{\frac{x-1}{2}}\right]^{2}\left(1+\frac{2}{x-1}\right)$$

$$=\lim_{x\to\infty}\left[\left(1+\frac{2}{x-1}\right)^{\frac{x-1}{2}}\right]^{2}\cdot\lim_{x\to\infty}\left(1+\frac{2}{x-1}\right)=\mathrm{e}^{2}.$$

或 $$\lim_{x\to\infty}\left(\frac{x+1}{x-1}\right)^{x}=\lim_{x\to\infty}\left(\frac{1+\frac{1}{x}}{1-\frac{1}{x}}\right)^{x}=\frac{\lim\limits_{x\to\infty}\left(1+\frac{1}{x}\right)^{x}}{\lim\limits_{x\to\infty}\left(1-\frac{1}{x}\right)^{x}}=\frac{\mathrm{e}}{\mathrm{e}^{-1}}=\mathrm{e}^{2}.$$

3. 无穷小量的比较

考察 $x, \sin 2x, \sqrt[3]{x}, x^2$ 都是$x \to 0$ 时的无穷小量.

但是 $\lim\limits_{x\to 0}\dfrac{\sin 2x}{x}=2, \lim\limits_{x\to 0}\dfrac{x^2}{x}=0, \lim\limits_{x\to 0}\dfrac{\sqrt[3]{x}}{x}=\infty$, 它们趋于0的速度有快慢.

定义1 设$\alpha(x), \beta(x)$ 自变量同一趋势下的无穷小量,

(1) 若$\lim\dfrac{\alpha(x)}{\beta(x)}=0$, 则称$\alpha(x)$是$\beta(x)$ 的高阶无穷小量,

记作 $\alpha(x)=o(\beta(x))$.

(2) 若$\lim\dfrac{\alpha(x)}{\beta(x)}=l\neq 0$, 则称$\alpha(x)$是$\beta(x)$ 的同阶无穷小量,

记作 $\alpha(x)=O(\beta(x))$.

当 $l=1$ 时, 则称$\alpha(x)$ 是$\beta(x)$的等价无穷小量, 记作 $\alpha(x)\sim\beta(x)$.

(3) 若 $\lim\dfrac{\alpha(x)}{\beta^k(x)}=l\neq 0(k>0)$, 则称 $\alpha(x)$是$\beta(x)$ 的 k 阶无穷小量.

牢记 当 $x \to 0$时,

$$\sin x \sim x, \tan x \sim x, 1-\cos x = 2\sin^2(\frac{x}{2}) \sim \frac{1}{2}x^2.$$

定理3 设 $\alpha, \alpha_1, \beta, \beta_1$ 自变量同一趋势下的无穷小量，且

$$\alpha \sim \alpha_1, \beta \sim \beta_1, \lim\frac{\alpha_1}{\beta_1} \text{ 存在, 则 } \lim\frac{\alpha}{\beta} = \lim\frac{\alpha_1}{\beta_1}.$$

证 由 $\dfrac{\alpha}{\beta} = \dfrac{\alpha}{\alpha_1}\cdot\dfrac{\alpha_1}{\beta_1}\cdot\dfrac{\beta_1}{\beta}$， 得

$$\lim\frac{\alpha}{\beta} = \lim\frac{\alpha}{\alpha_1}\cdot\lim\frac{\alpha_1}{\beta_1}\cdot\lim\frac{\beta_1}{\beta} = \lim\frac{\alpha_1}{\beta_1}.$$

例 13. 求 $\lim\limits_{x\to 0^+}\dfrac{(x^3+x^{\frac{5}{2}})\sqrt{\sin 2x}}{\tan^3 x}$.

解 $\lim\limits_{x\to 0^+}\dfrac{(x^3+x^{\frac{5}{2}})\sqrt{\sin 2x}}{\tan^3 x}=\lim\limits_{x\to 0^+}\dfrac{(x^3+x^{\frac{5}{2}})\sqrt{2x}}{x^3}$

$$=\lim_{x\to 0^+}\frac{x^3\sqrt{2x}}{x^3}+\lim_{x\to 0^+}\sqrt{2}\frac{x^3}{x^3}=\sqrt{2}.$$

例 14. 求 $\lim\limits_{x\to 0}\dfrac{\tan x-\sin x}{x^3}$.

解 $\lim\limits_{x\to 0}\dfrac{\tan x-\sin x}{x^3}=\lim\limits_{x\to 0}\dfrac{\sin x(1-\cos x)}{x^3\cos x}$

$$=\lim_{x\to 0}\frac{x\cdot\frac{1}{2}x^2}{x^3\cos x}=\frac{1}{2}.$$