§3 分部积分法

由莱布尼茨法则 $\big(u(x)v(x)\big)' = u'(x)v(x)+u(x)v'(x)$ 得

$$u(x)v'(x) = \big(u(x)v(x)\big)' - u'(x)v(x),$$

两边积分得

$$\int u(x)v'(x)\mathrm{d}x = u(x)v(x) - \int v(x)u'(x)\mathrm{d}x.$$

这个公式称为分部积分公式.

实际解题时，先凑微分，后分部积分.

$$\int u(x)v'(x)\mathrm{d}x = \int u(x)\mathrm{d}v(x) = u(x)v(x) - \int v(x)u'(x)\mathrm{d}x.$$

分部积分举例

例 1　求 $\int x\cos x\mathrm{d}x.$

解　取 $u=x,\ dv=\cos x dx,$　则 $du=dx,\ v=\sin x.$

$$\int x\cos x\mathrm{d}x=\int x\mathrm{d}\sin x$$

$$=x\sin x-\int\sin x\mathrm{d}x$$

$$=x\sin x+\cos x+C.$$

熟练后，不再指出具体的 $u,v.$

归纳　$\int P_n(x)\sin x\mathrm{d}x,$　$\int P_n(x)\cos x\mathrm{d}x.$

分部积分举例

例 2　求 $\int \arctan x \mathrm{d}x.$

解
$$\int \arctan x \mathrm{d}x = x\arctan x - \int \frac{x}{1+x^2}\mathrm{d}x$$

u　　v

$$= x\arctan x - \frac{1}{2}\int \frac{1}{1+x^2}\mathrm{d}(1+x^2)$$

$$= x\arctan x - \frac{1}{2}\ln(1+x^2) + C.$$

归纳　$\int P_n(x)\arctan x \mathrm{d}x$

分部积分举例

例 3　求　$\int x^2 \ln x \mathrm{d}x.$

解
$$\int x^2 \ln x \mathrm{d}x = \frac{1}{3}\int \ln x \mathrm{d}x^3$$
$$= \frac{1}{3}x^3 \ln x - \frac{1}{3}\int x^3 \frac{1}{x}\mathrm{d}x$$
$$= \frac{1}{3}x^3 \ln x - \frac{1}{3}\int x^2 \mathrm{d}x$$
$$= \frac{1}{3}x^3 \ln x - \frac{1}{9}x^3 + C.$$

归纳　$\int P_n(x) \ln x \mathrm{d}x$

分部积分举例

例 4　求 $\int x^2 \mathrm{e}^x \mathrm{d}x.$

解 $\int x^2 \mathrm{e}^x \mathrm{d}x = \int x^2 d\,\mathrm{e}^x = x^2 \mathrm{e}^x - 2\int \mathrm{e}^x\, x dx$

$$= x^2 \mathrm{e}^x - 2\int x d\,\mathrm{e}^x$$

$$= x^2 \mathrm{e}^x - 2xe^x + 2\int \mathrm{e}^x dx$$

$$= x^2 \mathrm{e}^x - 2xe^x + 2e^x + C.$$

归纳 $\int P_n(x)\mathrm{e}^{\alpha x}\,\mathrm{d}x$

分部积分举例

例 5　求 $\int e^x \sin x \mathrm{d}x.$

解 $$\int e^x \sin x \mathrm{d}x = \int \sin x \mathrm{d}e^x = e^x \sin x - \int e^x \cos x \mathrm{d}x$$

$$= e^x \sin x - \int \cos x \mathrm{d}e^x$$

$$= e^x \sin x - (e^x \cos x - \int e^x (-\sin x)\mathrm{d}x)$$

$$= e^x \sin x - e^x \cos x - \int e^x \sin x \mathrm{d}x,$$

所以 $$\int e^x \sin x \mathrm{d}x = \frac{1}{2} e^x (\sin x - \cos x) + C.$$

分部积分举例

例 6　求 $\int \sec^3 x\mathrm{d}x.$

解
$$\begin{aligned}\int \sec^3 x\mathrm{d}x &= \int \sec x\mathrm{d}(\tan x)\\ &= \sec x\tan x - \int \tan x\sec x\tan x d\\ &= \sec x\tan x - \int (\sec^2 x - 1)\sec x\mathrm{d}x\\ &= \sec x\tan x + \int \sec x\mathrm{d}x - \int \sec^3 x\mathrm{d}x\\ &= \sec x\tan x + \ln|\sec x + \tan x| - \int \sec^3 x\mathrm{d}x,\end{aligned}$$

所以 $\int \sec^3 x\mathrm{d}x = \frac{1}{2}(\sec x\tan x + \ln|\sec x + \tan x|) + C.$

分部积分举例

例 7　求 $\int \sqrt{x^2+a^2}\,dx\ (a>0).$

解 $$\int \sqrt{x^2+a^2}\,dx = x\sqrt{x^2+a^2} - \int x\frac{2x}{2\sqrt{x^2+a^2}}dx$$

$$= x\sqrt{x^2+a^2} - \int \frac{x^2+a^2-a^2}{\sqrt{x^2+a^2}}dx$$

$$= x\sqrt{x^2+a^2} - \int \sqrt{x^2+a^2}\,d\,x + a^2\int \frac{dx}{\sqrt{x^2+a^2}}$$

$$= x\sqrt{x^2+a^2} + a^2\ln(x+\sqrt{x^2+a^2}) - \int \sqrt{x^2+a^2}\,d\,x,$$

所以 $$\int \sqrt{x^2+a^2}\,d\,x = \frac{1}{2}x\sqrt{x^2+a^2} + \frac{a^2}{2}\ln(x+\sqrt{x^2+a^2}) + C.$$

同理 $$\int \sqrt{x^2-a^2}\,d\,x = \frac{1}{2}x\sqrt{x^2-a^2} - \frac{a^2}{2}\ln(x+\sqrt{x^2-a^2}) + C.$$

分部积分举例

例 8　求 $\int \sqrt{a^2 - x^2}\,dx\ (a > 0).$

解 $\int \sqrt{a^2 - x^2}\,dx = x\sqrt{a^2 - x^2} - \int x \frac{-2x}{2\sqrt{a^2 - x^2}}\,dx$

$$= x\sqrt{a^2 - x^2} + \int \frac{x^2 - a^2 + a^2}{\sqrt{a^2 - x^2}}\,dx$$

$$= x\sqrt{a^2 - x^2} - \int \sqrt{a^2 - x^2}\,d\,x + a^2 \int \frac{dx}{\sqrt{a^2 - x^2}}$$

$$= x\sqrt{a^2 - x^2} + a^2 \arcsin\frac{x}{a} - \int \sqrt{a^2 - x^2}\,d\,x,$$

所以 $\int \sqrt{a^2 - x^2}\,d\,x = \frac{1}{2} x\sqrt{a^2 - x^2} + \frac{a^2}{2} \arcsin\frac{x}{a} + C.$

分部积分举例

例 9　求 $\int x\arcsin x\mathrm{d}x.$

解 $\int x\arcsin x\mathrm{d}x=\frac{1}{2}\int \arcsin x\mathrm{d}x^2=\frac{x^2}{2}\arcsin x-\frac{1}{2}\int\frac{x^2}{\sqrt{1-x^2}}\mathrm{d}x$

$$=\frac{x^2}{2}\arcsin x-\frac{1}{2}\int\frac{x^2-1+1}{\sqrt{1-x^2}}\mathrm{d}x$$

$$=\frac{x^2}{2}\arcsin x+\frac{1}{2}\int\sqrt{1-x^2}\mathrm{d}x-\frac{1}{2}\int\frac{1}{\sqrt{1-x^2}}\mathrm{d}x$$

$$=\frac{x^2}{2}\arcsin x+\frac{1}{4}(x\sqrt{1-x^2}+\arcsin x)-\frac{1}{2}\arcsin x+C$$

$$=\frac{1}{4}[(2x^2-1)\arcsin x+x\sqrt{1-x^2}]+C.$$

归纳　$\int P_n(x)\arcsin x\mathrm{d}x$

分部积分举例

例 10　求 $\int e^{\sqrt[3]{x}}\mathrm{d}x.$

解 $\int e^{\sqrt[3]{x}}\mathrm{d}x \overset{t=\sqrt[3]{x}}{=} \int e^t 3t^2 dt = 3\int t^2 de^t$

$$= 3t^2e^t - 3\int e^t 2t dt$$

$$= 3t^2e^t - 6\int t de^t$$

$$= 3t^2e^t - 6te^t + 6\int e^t dt$$

$$= 3(t^2 - 2t + 2)e^t + C$$

$$= 3(\sqrt[3]{x^2} - 2\sqrt[3]{x} + 2)e^{\sqrt[3]{x}} + C.$$

分部积分举例

例 11　求 $\int \frac{x\cos x}{\sin^3 x}\mathrm{d}x.$

解 $\int \frac{x\cos x}{\sin^3 x}\mathrm{d}x = \int \frac{x}{\sin^3 x}\mathrm{d}\sin x$

$$= \int x\mathrm{d}\left(-\frac{1}{2\sin^2 x}\right)$$

$$= -\frac{x}{2\sin^2 x} + \int \frac{1}{2\sin^2 x}dx$$

$$= -\frac{x}{2\sin^2 x} + \frac{1}{2}\int \csc^2 x dx$$

$$= -\frac{x}{2\sin^2 x} - \frac{1}{2}\cot x + C.$$

分部积分举例

例 12　求 $\int \frac{\ln x}{(1-x)^2}\mathrm{d}x.$

解 $\int \frac{\ln x}{(1-x)^2}\mathrm{d}x = \int \ln x \mathrm{d}\frac{1}{1-x}$

$$= \frac{\ln x}{1-x} - \int \frac{\mathrm{d}x}{x(1-x)}$$

$$= \frac{\ln x}{1-x} - \int (\frac{1}{x} + \frac{1}{1-x})\mathrm{d}x$$

$$= \frac{\ln x}{1-x} - \ln|x| + \ln|1-x| + C.$$

分部积分举例

例 13　求 $\int \frac{x^2 e^x}{(x+2)^2}\mathrm{d}x.$

解
$$\int \frac{x^2 e^x}{(x+2)^2}\mathrm{d}x = -\int x^2 e^x \mathrm{d}\left(\frac{1}{x+2}\right)$$
$$= -\frac{x^2 e^x}{x+2} + \int \frac{1}{x+2}(2xe^x + x^2 e^x)\mathrm{d}x$$
$$= -\frac{x^2 e^x}{x+2} + \int x de^x$$
$$= -\frac{x^2 e^x}{x+2} + xe^x - \int e^x dx$$
$$= -\frac{x^2 e^x}{x+2} + xe^x - e^x + C.$$

递推公式

例 13　求 $I_n = \int x^n \mathrm{e}^x \mathrm{d}x \ (n \geq 1)$ 的递推公式.

解　$I_n = \int x^n \mathrm{e}^x \mathrm{d}x = \int x^n de^x$

$= x^n e^x - n\int x^{n-1} e^x dx.$

递推公式为

$$I_n = x^n e^x - nI_{n-1}.$$

$$I_0 = \int \mathrm{e}^x \mathrm{d}x = e^x + C.$$

递推公式

例 14　求 $I_n=\int\frac{1}{(x^2+a^2)^n}\mathrm{d}x\ (n>1)$　的递推公式.

解 $I_n=\int\frac{1}{(x^2+a^2)^n}\mathrm{d}x=\frac{x}{(x^2+a^2)^n}-\int\frac{x(-n)2x}{(x^2+a^2)^{n+1}}\mathrm{d}x$

$$=\frac{x}{(x^2+a^2)^n}+2n\int\frac{x^2+a^2-a^2}{(x^2+a^2)^{n+1}}\mathrm{d}x$$

$$=\frac{x}{(x^2+a^2)^n}+2n\int\frac{1}{(x^2+a^2)^n}\mathrm{d}x-2na^2\int\frac{1}{(x^2+a^2)^{n+1}}\mathrm{d}x$$

$$=\frac{x}{(x^2+a^2)^n}+2nI_n-2na^2I_{n+1},$$

递推公式为　$I_n=\frac{x}{2(n-1)a^2(x^2+a^2)^{n-1}}-\frac{2n-3}{2(n-1)a^2}I_{n-1}.$

$$I_1=\int\frac{1}{x^2+a^2}\mathrm{d}x=\frac{1}{a}\arctan\frac{x}{a}+C.$$