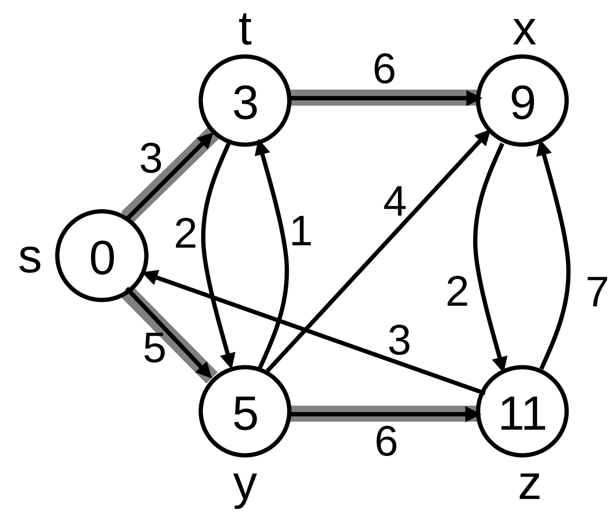
Chapter 24

Single-Source Shortest Paths

Shortest Path Problems



- **Input:**
  - Directed graph G = (V, E)
  - Weight function w : E → **R**
- **Weight of path** p = $\langle v_0, v_1, \ldots, v_k \rangle$

$$w(p) = \sum_{i=1}^{k} w(v_{i-1}, v_i)$$

- **Shortest-path weight** from u to v:

$$\delta(u, v) = \min \begin{cases} w(p) : u \overset{p}{\leadsto} v & \text{if there exists a path from } u \text{ to } v \\ \infty & \text{otherwise} \end{cases}$$

- Shortest path u to v is any path p such that w(p) = δ(u, v)

Variants of Shortest Paths

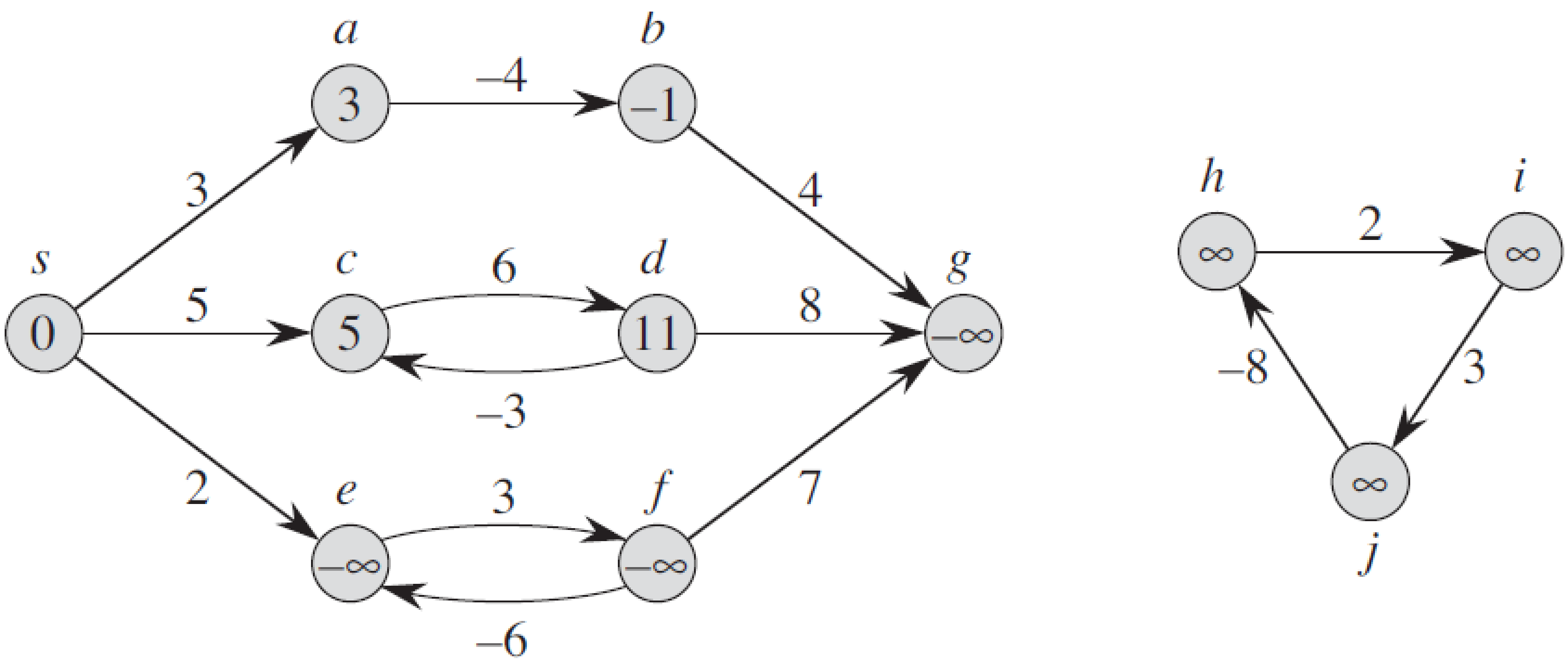
- **Single-source shortest path**
  - G = (V, E) $\Rightarrow$ find a shortest path from a given source vertex s to each vertex $v \in V$
- **Single-destination shortest path**
  - Find a shortest path to a given destination vertex t from each vertex v
  - Reverse the direction of each edge $\Rightarrow$ single-source
- **Single-pair shortest path**
  - Find a shortest path from u to v for given vertices u and v
  - Solve the single-source problem
- **All-pairs shortest-paths**
  - Find a shortest path from u to v for every pair of vertices u and v

Optimal substructure

Lemma 24.1 (Subpaths of shortest paths are shortest paths)
Given a weighted, directed graph $G = (V, E)$ with weight function $w : E \to \mathbb{R}$, let $p = \langle v_0, v_1, \ldots, v_k \rangle$ be a shortest path from vertex v_0 to vertex v_k and, for any i and j such that $0 \le i \le j \le k$, let $p_{ij} = \langle v_i, v_{i+1}, \ldots, v_j \rangle$ be the subpath of p from vertex v_i to vertex v_j. Then, p_{ij} is a shortest path from v_i to v_j.

Proof If we decompose path p into $v_0 \overset{p_{0i}}{\leadsto} v_i \overset{p_{ij}}{\leadsto} v_j \overset{p_{jk}}{\leadsto} v_k$, then we have that $w(p) = w(p_{0i}) + w(p_{ij}) + w(p_{jk})$. Now, assume that there is a path p'_{ij} from v_i to v_j with weight $w(p'_{ij}) < w(p_{ij})$. Then, $v_0 \overset{p_{0i}}{\leadsto} v_i \overset{p'_{ij}}{\leadsto} v_j \overset{p_{jk}}{\leadsto} v_k$ is a path from v_0 to v_k whose weight $w(p_{0i}) + w(p'_{ij}) + w(p_{jk})$ is less than $w(p)$, which contradicts the assumption that p is a shortest path from v_0 to v_k. ■

Negative-weight edges & Cycles



Shortest-paths tree

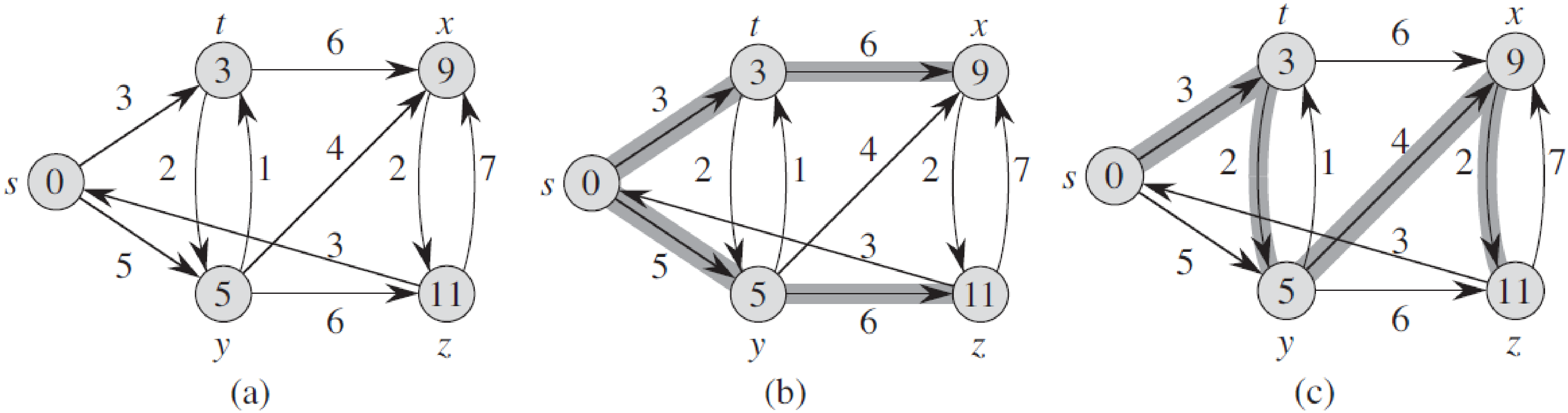
predecessor subgraph $G_\pi = (V_\pi, E_\pi)$

$$V_\pi = \{v \in V : v.\pi \neq \text{NIL}\} \cup \{s\}$$

$$E_\pi = \{(v.\pi, v) \in E : v \in V_\pi - \{s\}\}$$

shortest-paths tree rooted at s $\quad G' = (V', E')$

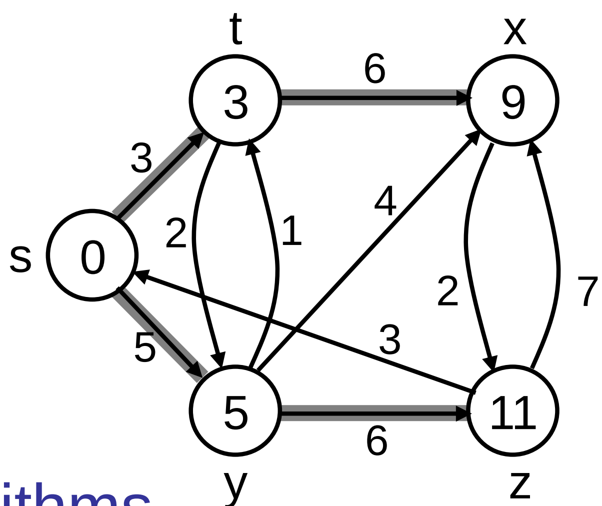
1. V' is the set of vertices reachable from s in G,
2. G' forms a rooted tree with root s, and
3. for all $v \in V'$, the unique simple path from s to v in G' is a shortest path



Computing Shortest-Paths

For each vertex v ∈ V maintain:

- d[v] = δ(s, v): a **shortest-path estimate**
  - Initially, d[v]=∞
  - Reduces as algorithms progress
- π[v] = **predecessor** of v on a shortest path from ***s***
- All the single-source shortest-paths algorithms
  - Initialize d[v] and π[v]
  - Then relax edges
- The algorithms differ in the order and how many times they relax each edge



Relaxation

- **Relaxing** an edge (u, v) = testing whether we can improve the shortest path to v found so far by going through u

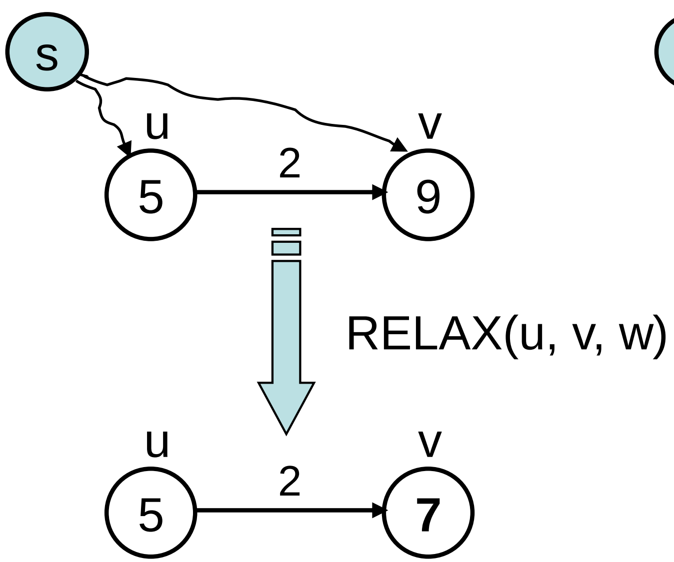
If d[v] > d[u] + w(u, v)

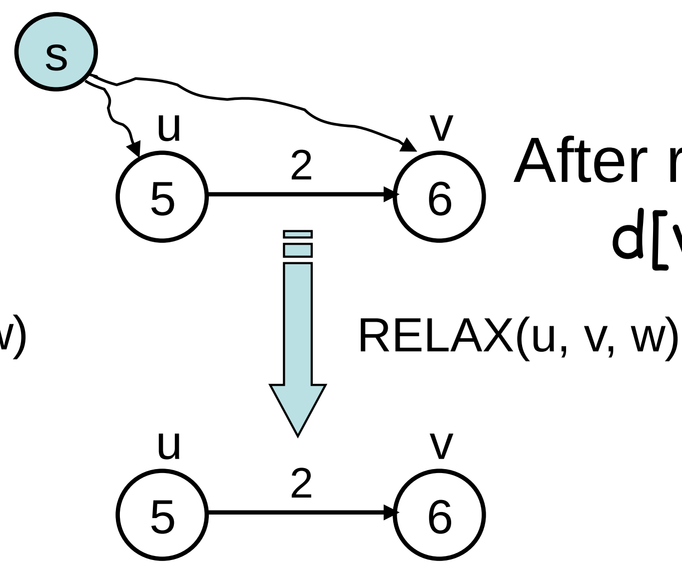
we can improve the shortest path to v

$\Rightarrow$ update d[v] and π[v]

RELAX(u, v, w)

if $v.d > u.d + w(u, v)$
    $v.d = u.d + w(u, v)$
    $v.\pi = u$





After relaxation:

d[v] $\leq$ d[u] + w(u, v)

Properties

Triangle inequality (Lemma 24.10)

For any edge $(u, v) \in E$, we have $\delta(s, v) \le \delta(s, u) + w(u, v)$.

Upper-bound property (Lemma 24.11)

We always have $v.d \ge \delta(s, v)$ for all vertices $v \in V$, and once $v.d$ achieves the value $\delta(s, v)$, it never changes.

No-path property (Corollary 24.12)

If there is no path from s to v, then we always have $v.d = \delta(s, v) = \infty$.

Convergence property (Lemma 24.14)

If $s \leadsto u \to v$ is a shortest path in G for some $u, v \in V$, and if $u.d = \delta(s, u)$ at any time prior to relaxing edge (u, v), then $v.d = \delta(s, v)$ at all times afterward.

Path-relaxation property (Lemma 24.15)

If $p = \langle v_0, v_1, \ldots, v_k \rangle$ is a shortest path from $s = v_0$ to v_k, and we relax the edges of p in the order $(v_0, v_1), (v_1, v_2), \ldots, (v_{k-1}, v_k)$, then $v_k.d = \delta(s, v_k)$. This property holds regardless of any other relaxation steps that occur, even if they are intermixed with relaxations of the edges of p.

Predecessor-subgraph property (Lemma 24.17)

Once $v.d = \delta(s, v)$ for all $v \in V$, the predecessor subgraph is a shortest-paths tree rooted at s.

Bellman-Ford Algorithm

- Single-source shortest paths problem
  - Computes d[v] and π[v] for all v $\in$ V
- Allows negative edge weights
- Returns:
  - TRUE if no negative-weight cycles are reachable from the source s
  - FALSE otherwise $\Rightarrow$ no solution exists
- Idea:
  - Traverse all the edges |V – 1| times, every time performing a relaxation step of each edge

Bellman-Ford Algorithm

```
BELLMAN-FORD(G, w, s)
INITIALIZE-SINGLE-SOURCE(G, s)
for i = 1 to |G.V| - 1                          Θ(V)
    for each edge (u, v) ∈ G.E                    O(E)
        RELAX(u, v, w)
for each edge (u, v) ∈ G.E                      O(E)
    if v.d > u.d + w(u, v)
        return FALSE
return TRUE
```

$$O(VE)$$

Correctness

Lemma 24.2

Let $G = (V, E)$ be a weighted, directed graph with source s and weight function $w : E \to \mathbb{R}$, and assume that G contains no negative-weight cycles that are reachable from s. Then, after the $|V| - 1$ iterations of the **for** loop of lines 2–4 of BELLMAN-FORD, we have $v.d = \delta(s, v)$ for all vertices v that are reachable from s.

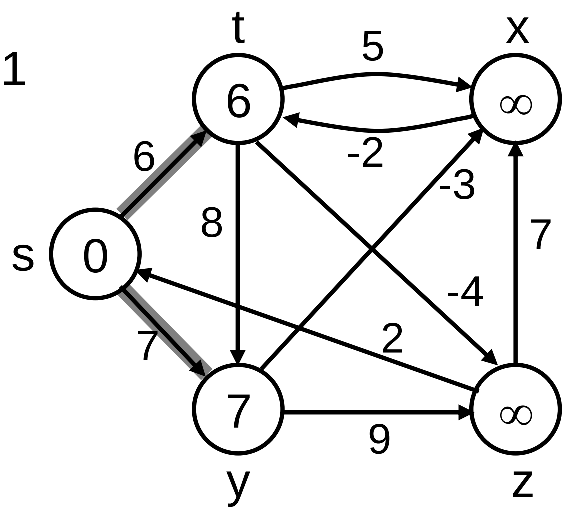
Theorem 24.4 (Correctness of the Bellman-Ford algorithm)

Let BELLMAN-FORD be run on a weighted, directed graph $G = (V, E)$ with source s and weight function $w : E \to \mathbb{R}$. If G contains no negative-weight cycles that are reachable from s, then the algorithm returns TRUE, we have $v.d = \delta(s, v)$ for all vertices $v \in V$, and the predecessor subgraph G_π is a shortest-paths tree rooted at s. If G does contain a negative-weight cycle reachable from s, then the algorithm returns FALSE.

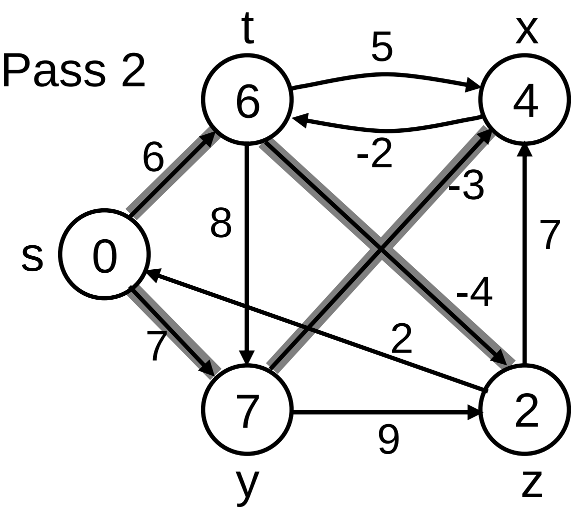
Example

(t, x), (t, y), (t, z), (x, t), (y, x), (y, z), (z, x), (z, s), (s, t), (s, y)

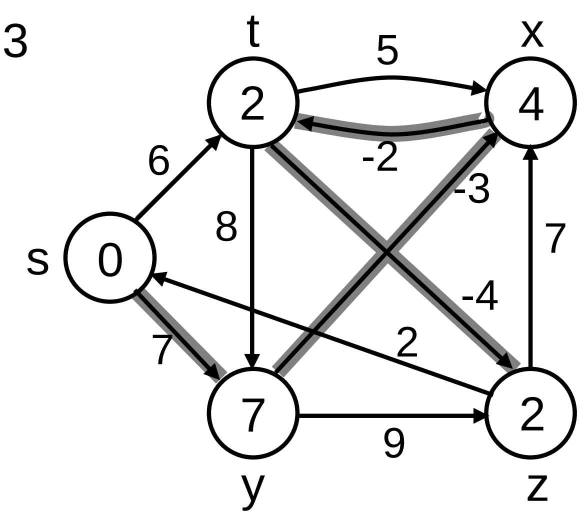
Pass 1



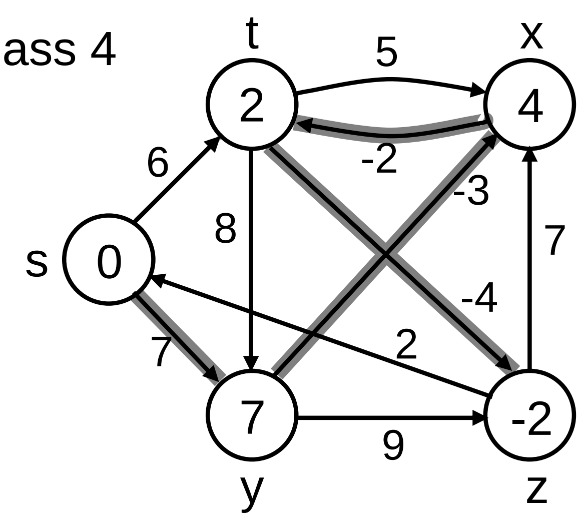
Pass 2



Pass 3

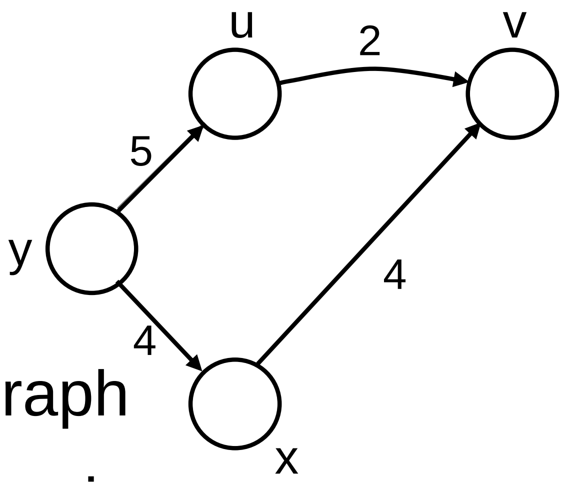


Pass 4



Single-Source Shortest Paths in DAGs

- Given a weighted DAG: $G = (V, E)$
  – solve the shortest path problem
- Idea:
  – Topologically sort the vertices of the graph
  – Relax the edges according to the order given by the topological sort
    - for each vertex, we relax each edge that starts from that vertex
- Are shortest-paths well defined in a DAG?
  – Yes, (negative-weight) cycles cannot exist



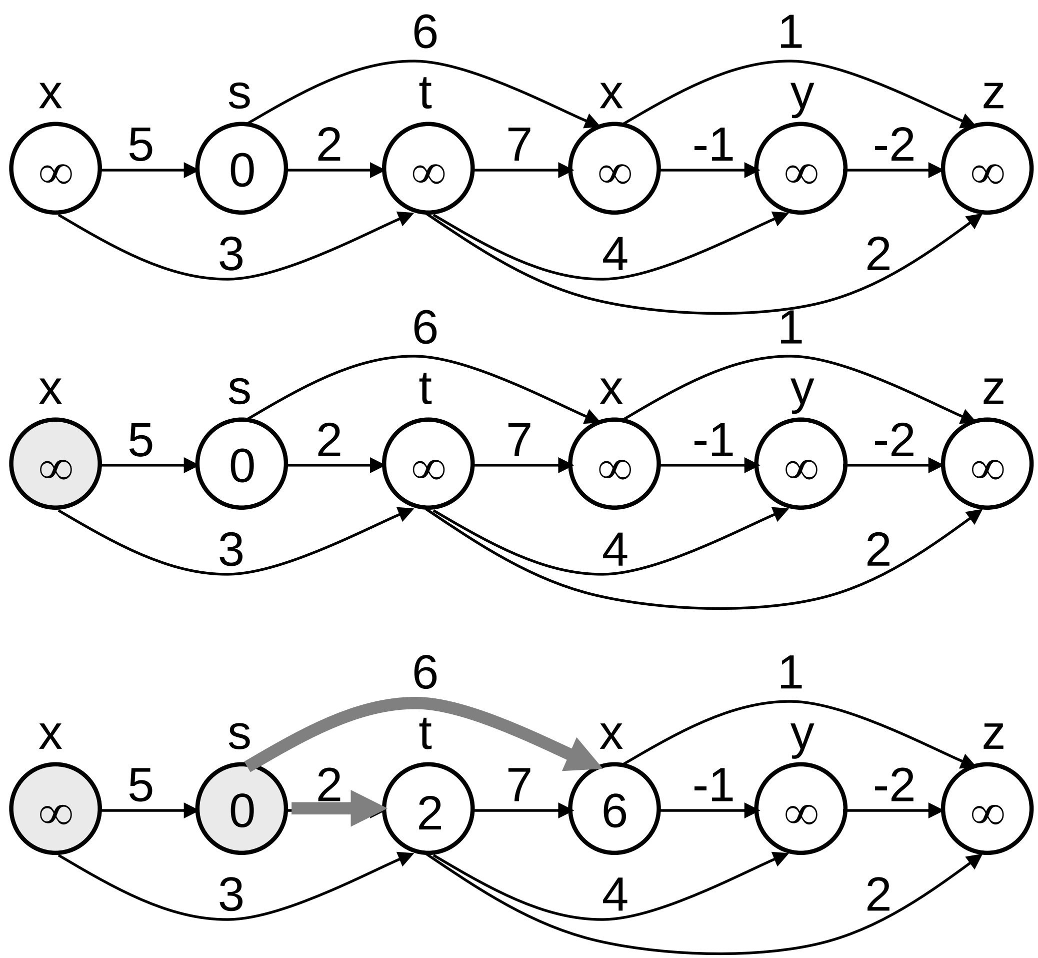
DAG-SHORTEST-PATHS(G, w, s)

1. topologically sort the vertices of G $\leftarrow \Theta(V+E)$
2. INITIALIZE-SINGLE-SOURCE(V, s) $\leftarrow \Theta(V)$
3. **for** each vertex u, taken in topologically sorted order $\Theta(V)$
4. **do for** each vertex $v \in Adj[u]$
5. **do** RELAX(u, v, w)

$\Theta(E)$

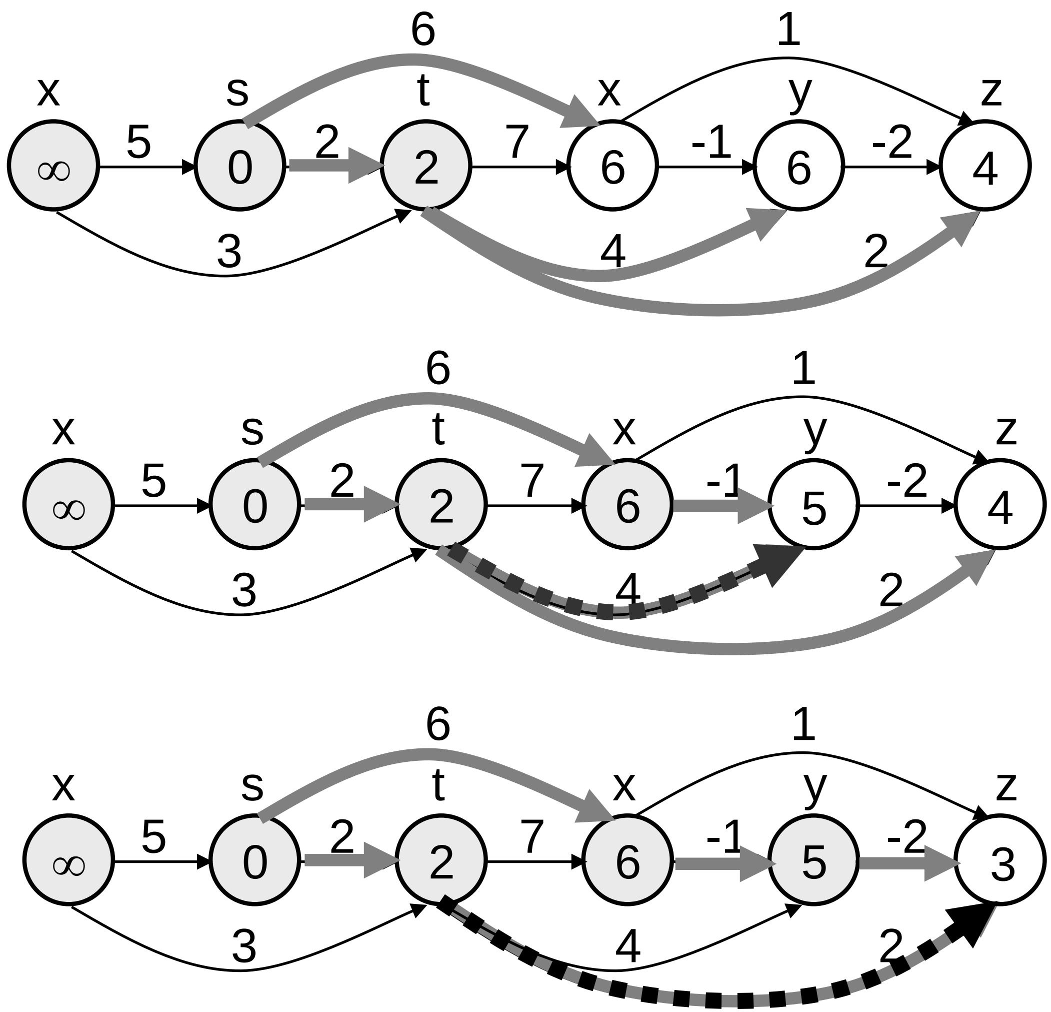
Running time: $\Theta(V+E)$

Example

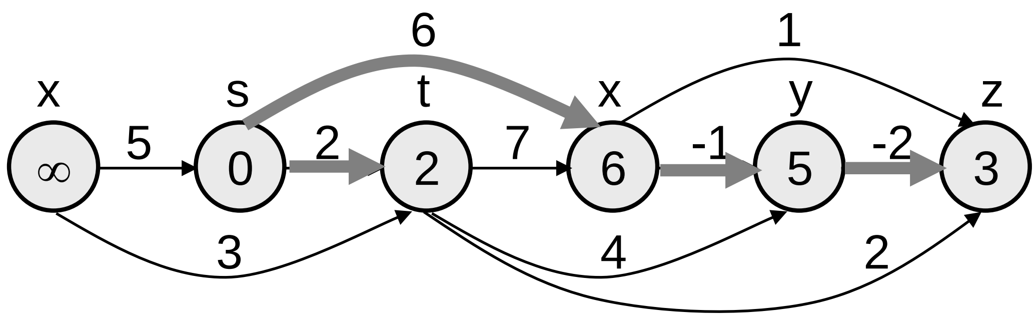


6
1
x
s
t
x
y
z
5
2
7
-1
-2
∞
0
∞
∞
∞
∞
3
4
2
6
1
x
s
t
x
y
z
5
2
7
-1
-2
∞
0
∞
∞
∞
∞
3
4
2
6
1
x
s
t
x
y
z
5
2
7
-1
-2
∞
0
2
6
∞
∞
3
4
2

Example (cont.)



Example (cont.)

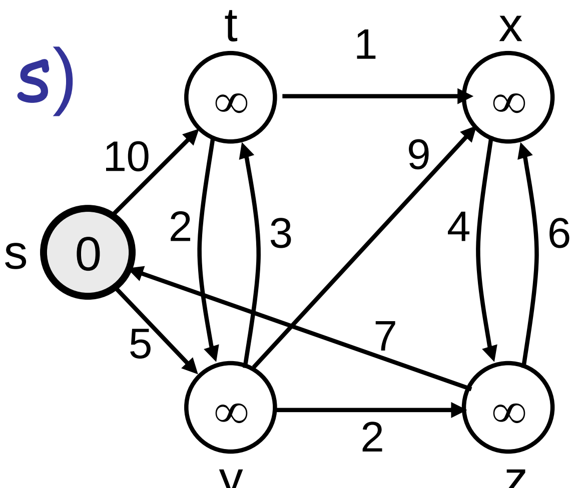


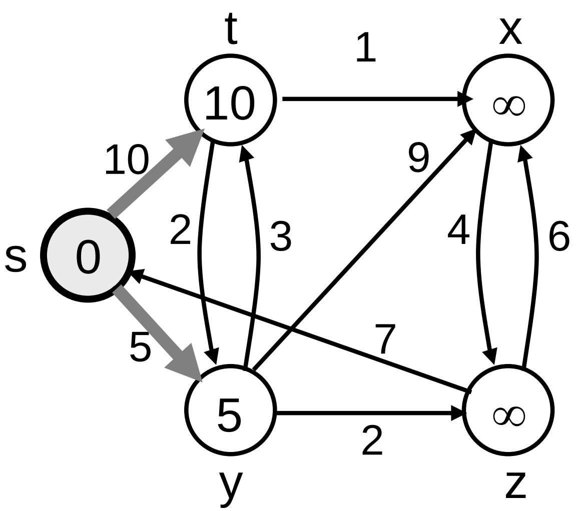
Dijkstra’s Algorithm

- Single-source shortest path problem:
  - No negative-weight edges: $w(u, v) > 0\ \forall\ (u, v) \in E$
- Maintains two sets of vertices:
  - S = vertices whose final shortest-path weights have already been determined
  - Q = vertices in V – S: min-priority queue
    - Keys in Q are estimates of shortest-path weights (d[v])
- Repeatedly select a vertex $u \in V - S$, with the minimum shortest-path estimate d[v]

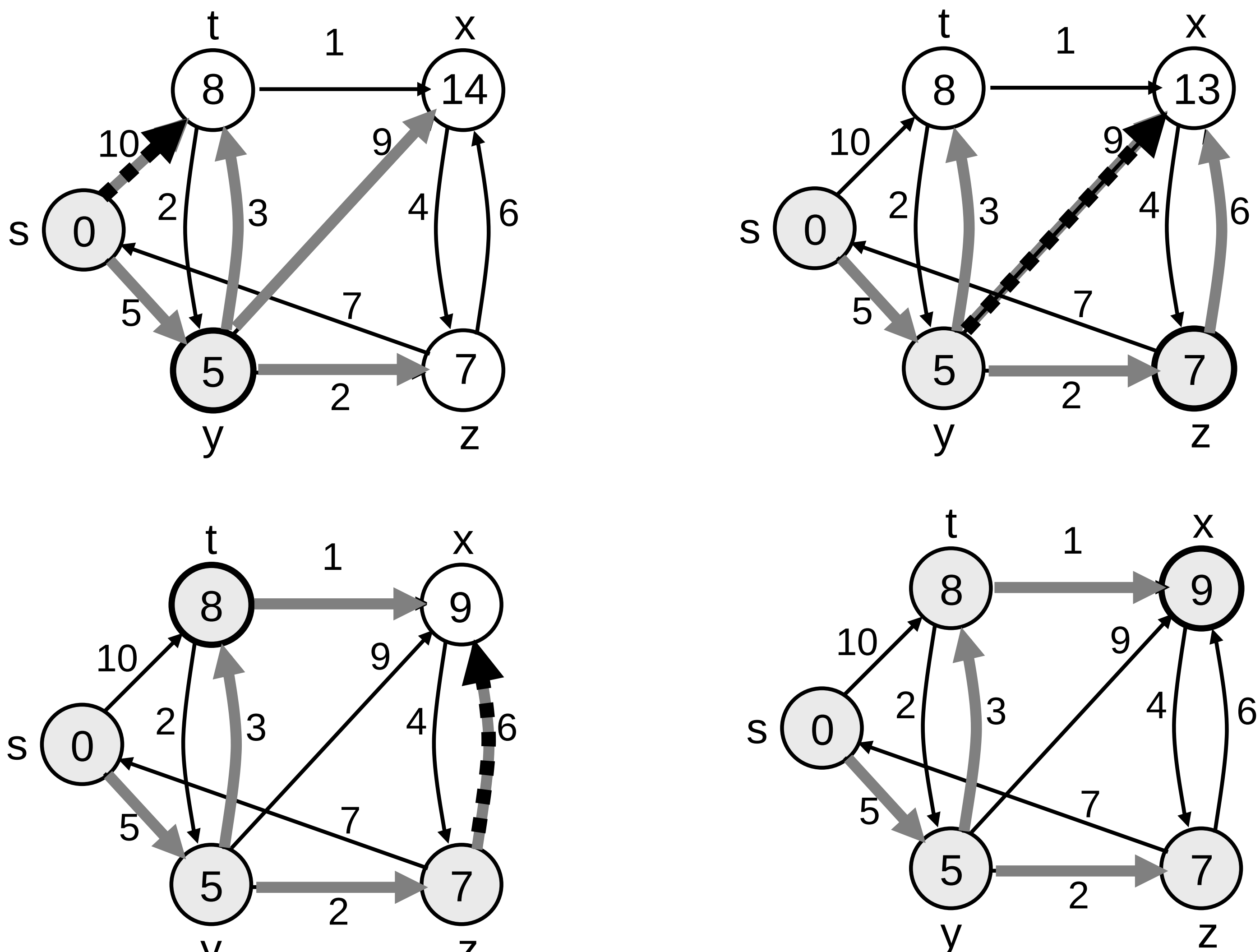
Dijkstra (G, w, s)

```
INITIALIZE-SINGLE-SOURCE(V, s)
S ← ∅
Q ← V[G]
while Q ≠ ∅
    do u ← EXTRACT-MIN(Q)
        S ← S ∪ {u}
        for each vertex v ∈ Adj[u]
            do RELAX(u, v, w)
```



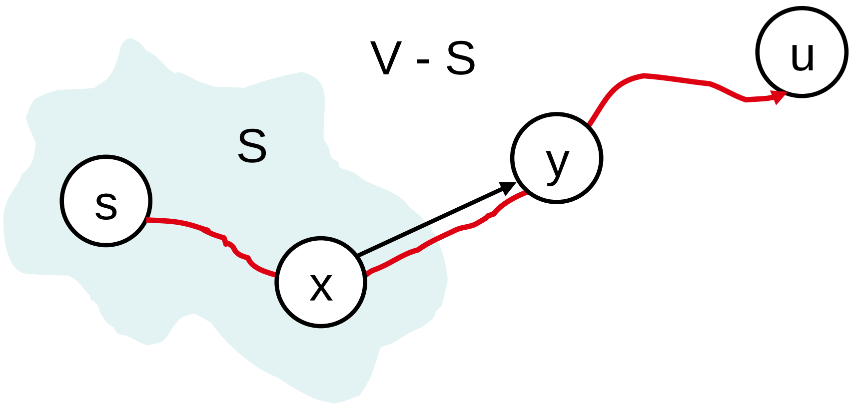


Example


t
1
x
8
14
10
9
2
3
4
6
s
0
7
5
5
7
2
y
z
t
1
x
8
13
10
9
2
3
4
6
s
0
7
5
5
7
2
y
z
t
1
x
8
9
10
9
2
3
4
6
s
0
7
5
5
7
2
y
z
t
1
x
8
9
10
9
2
3
4
6
s
0
7
5
5
7
2
y
z

Correctness of Dijskstra's Algorithm

- For each vertex $u \in V$, we have $d[u] = \delta(s, u)$ at the time when u is added to S.



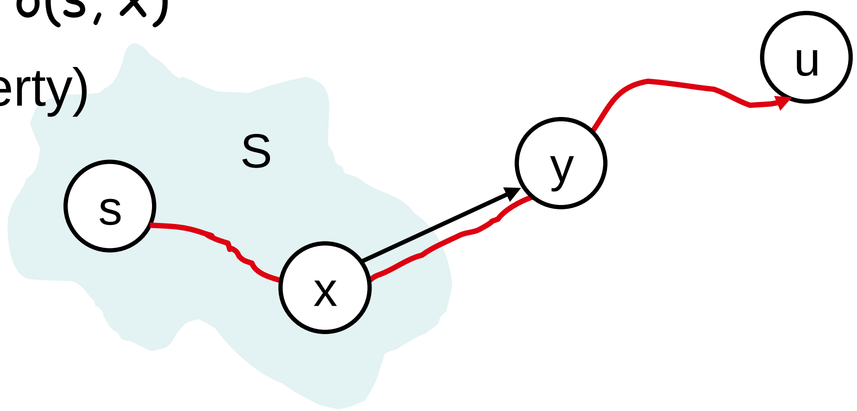
Proof:

- Let u be the first vertex for which $d[u] \neq \delta(s, u)$ when added to S
- Let's look at a true shortest path from s to u (p):
  - Before adding u to S, path p connects a vertex in S (s) with one in V - S (u)
  - Let (x, y) be the first edge crossing (S, V – S)

Correctness of Dijskstra's Algorithm

- Claim: $d[y] = \delta(s, y)$ at the time when u is added to S
  - u is the first vertex with $d[u] \neq \delta(s, u)$
  - $x \in S$ and we must have $d[x] = \delta(s, x)$
  - Relax (x, y) (convergence property)

    $\Rightarrow d[y] = \delta(s, y)$



- y is before u on the shortest path from s to u:

  $$\Rightarrow d[y] = \delta(s, y) \leq \delta(s, u) \leq d[u]$$

- Both u and $y \in V - S$ when u was chosen and since we chose u: $d[u] \leq d[y]$ $\Rightarrow$ relation above is an equality:

  $$d[y] = \delta(s, y) = \delta(s, u) = d[u]$$ contradiction with choice of u!

Time complexity

```
INITIALIZE-SINGLE-SOURCE(V, s)  ← Θ(V)
S ← ∅
Q ← V[G]  ← O(V) build min-heap
while Q ≠ ∅  ← Executed O(V) times
    do u ← EXTRACT-MIN(Q)  ← O(lgV)
        S ← S ∪ {u}
        for each vertex v ∈ Adj[u]
            do RELAX(u, v, w)  ← O(E) times; O(lgV)
```

Running time: $O(V\lg V + E\lg V) = O(E\lg V)$

Time complexity

Min-priority queue：

1. An array. $O(V^2)$
2. Binary min-heap. $O(E \lg V)$
3. Fibonacci heap $O(V \lg V + E)$

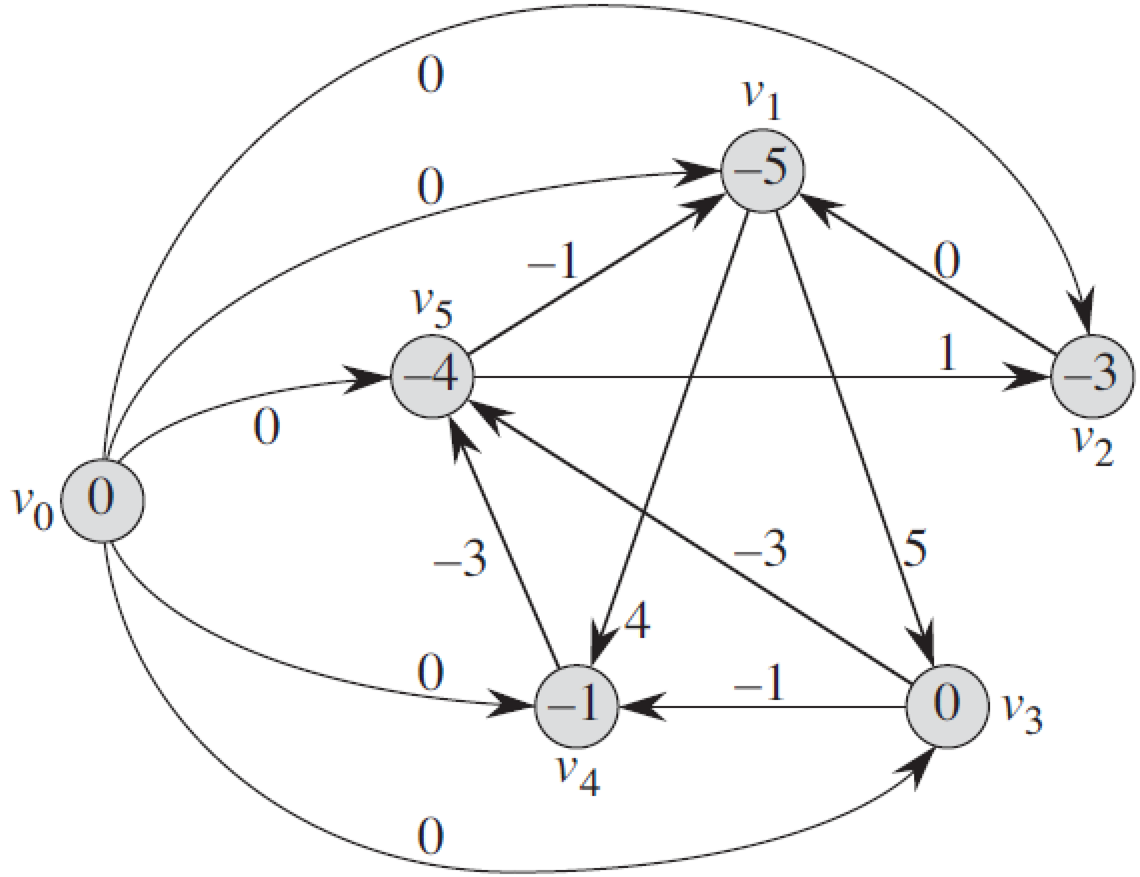
Difference constraints by shortest paths

maximization linear program

$$\begin{aligned}
\text{maximize} \quad & x_1 + x_2 && (29.11)\\
\text{subject to} \quad & \\
& 4x_1 - x_2 \le 8 && (29.12)\\
& 2x_1 + x_2 \le 10 && (29.13)\\
& 5x_1 - 2x_2 \ge -2 && (29.14)\\
& x_1, x_2 \ge 0 \; . && (29.15)
\end{aligned}$$

$$\begin{pmatrix} 1 & -1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & -1 \\ 0 & 1 & 0 & 0 & -1 \\ -1 & 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 1 & 0 \\ 0 & 0 & -1 & 1 & 0 \\ 0 & 0 & -1 & 0 & 1 \\ 0 & 0 & 0 & -1 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{pmatrix} \le \begin{pmatrix} 0 \\ -1 \\ 1 \\ 5 \\ 4 \\ -1 \\ -3 \\ -3 \end{pmatrix} .$$

$$
\begin{aligned}
x_1 - x_2 &\leq 0\,, \\
x_1 - x_5 &\leq -1\,, \\
x_2 - x_5 &\leq 1\,, \\
x_3 - x_1 &\leq 5\,, \\
x_4 - x_1 &\leq 4\,, \\
x_4 - x_3 &\leq -1\,, \\
x_5 - x_3 &\leq -3\,, \\
x_5 - x_4 &\leq -3\,.
\end{aligned}
$$



$$
E = \{(v_i, v_j) : x_j - x_i \leq b_k \text{ is a constraint}\} \cup \{(v_0, v_1), (v_0, v_2), (v_0, v_3), \ldots, (v_0, v_n)\}\ .
$$

Theorem 24.9
Given a system $Ax \leq b$ of difference constraints, let $G = (V, E)$ be the corresponding constraint graph. If G contains no negative-weight cycles, then

$$x = (\delta(v_0, v_1), \delta(v_0, v_2), \delta(v_0, v_3), \ldots, \delta(v_0, v_n)) \tag{24.11}$$

is a feasible solution for the system. If G contains a negative-weight cycle, then there is no feasible solution for the system.

Proof We first show that if the constraint graph contains no negative-weight cycles, then equation (24.11) gives a feasible solution. Consider any edge $(v_i, v_j) \in E$. By the triangle inequality, $\delta(v_0, v_j) \leq \delta(v_0, v_i) + w(v_i, v_j)$ or, equivalently, $\delta(v_0, v_j) - \delta(v_0, v_i) \leq w(v_i, v_j)$. Thus, letting $x_i = \delta(v_0, v_i)$ and

$x_j = \delta(v_0, v_j)$ satisfies the difference constraint $x_j - x_i \leq w(v_i, v_j)$ that corresponds to edge (v_i, v_j).

Now we show that if the constraint graph contains a negative-weight cycle, then the system of difference constraints has no feasible solution. Without loss of generality, let the negative-weight cycle be $c = \langle v_1, v_2, \ldots, v_k \rangle$, where $v_1 = v_k$. (The vertex v_0 cannot be on cycle c, because it has no entering edges.) Cycle c corresponds to the following difference constraints:

$$
\begin{aligned}
x_2 - x_1 &\leq w(v_1, v_2) , \\
x_3 - x_2 &\leq w(v_2, v_3) , \\
&\vdots \\
x_{k-1} - x_{k-2} &\leq w(v_{k-2}, v_{k-1}) , \\
x_k - x_{k-1} &\leq w(v_{k-1}, v_k) .
\end{aligned}
$$

We will assume that x has a solution satisfying each of these k inequalities and then derive a contradiction. The solution must also satisfy the inequality that results when we sum the k inequalities together. If we sum the left-hand sides, each unknown x_i is added in once and subtracted out once (remember that $v_1 = v_k$ implies $x_1 = x_k$), so that the left-hand side of the sum is 0. The right-hand side sums to $w(c)$, and thus we obtain $0 \leq w(c)$. But since c is a negative-weight cycle, $w(c) < 0$, and we obtain the contradiction that $0 \leq w(c) < 0$. ■