内容小结

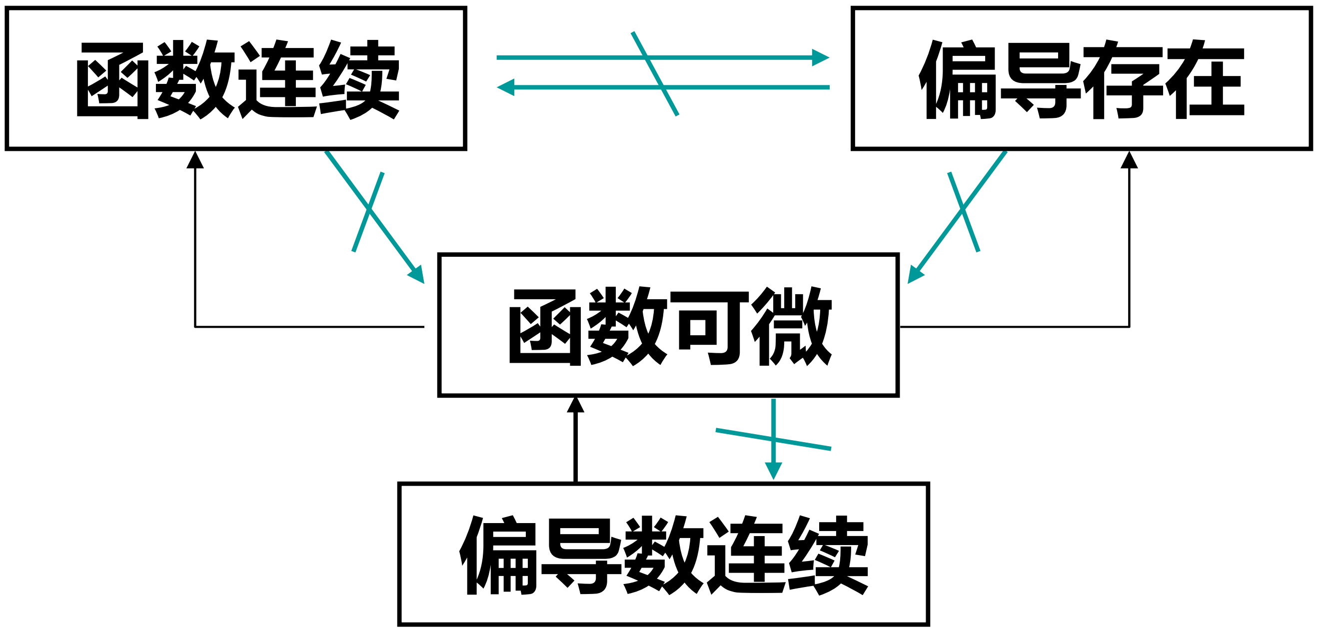
1. 微分定义: ($z = f(x, y)$)

$$\Delta z = f_x(x, y)\Delta x + f_y(x, y)\Delta y + o(\rho)$$

$$\rho = \sqrt{(\Delta x)^2 + (\Delta y)^2}$$

$$\mathrm{d}z = f_x(x, y)\mathrm{d}x + f_y(x, y)\mathrm{d}y$$

2. 重要关系:



思考与练习

1. 选择题

函数 $z=f(x,y)$在 (x_0,y_0)可微的充分条件是(D)

(A) $f(x,y)$ 在 (x_0,y_0) 连续;

(B) $f_x(x,y),\ f_y(x,y)$在 (x_0,y_0) 的某邻域内存在;

(C) $\Delta z - f_x(x,y)\Delta x - f_y(x,y)\Delta y$

当$\sqrt{(\Delta x)^2+(\Delta y)^2}\to 0$ 时是无穷小量;

(D) $\dfrac{\Delta z - f_x(x,y)\Delta x - f_y(x,y)\Delta y}{\sqrt{(\Delta x)^2+(\Delta y)^2}}$

当$\sqrt{(\Delta x)^2+(\Delta y)^2}\to 0$ 时是无穷小量.

2. 设 $f(x,y,z)=\dfrac{x\cos y+y\cos z+z\cos x}{1+\cos x+\cos y+\cos z}$，求 $\mathrm{d}f\big|_{(0,0,0)}$.

解: $\because\ f(x,0,0)=\dfrac{x}{3+\cos x}$

$$\therefore\ f_x(0,0,0)=\left(\frac{x}{3+\cos x}\right)'\Bigg|_{x=0}=\frac{1}{4}$$

利用轮换对称性 , 可得

$$f_y(0,0,0)=f_z(0,0,0)=\frac{1}{4}$$

$$\therefore \mathrm{d}f\big|_{(0,0,0)}=f_y(0,0,0)\,\mathrm{d}x+f_y(0,0,0)\,\mathrm{d}y+f_z(0,0,0)\,\mathrm{d}z$$

$$=\frac{1}{4}(\mathrm{d}x+\mathrm{d}y+\mathrm{d}z)$$

例　试证函数

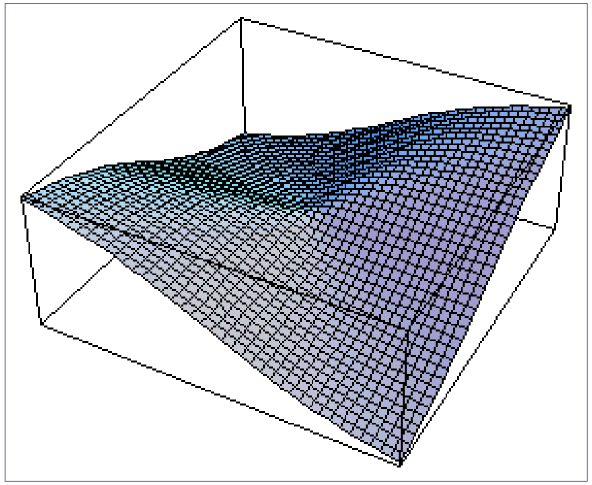
$$f(x,y)=\begin{cases} xy\sin\dfrac{1}{\sqrt{x^2+y^2}}, & (x,y)\neq(0,0) \\ 0, & (x,y)=(0,0) \end{cases}$$ 在

点$(0,0)$连续且偏导数存在，但偏导数在点$(0,0)$不连续，而 f 在点$(0,0)$可微.

思路：按有关定义讨论；对于偏导数需分

$(x,y)\neq(0,0)$，$(x,y)=(0,0)$讨论.

证　令 $x=\rho\cos\theta,\ y=\rho\sin\theta,$



则 $\lim\limits_{(x,y)\to(0,0)} xy\sin\dfrac{1}{\sqrt{x^2+y^2}}$

$$=\lim_{\rho\to 0}\rho^2\sin\theta\cos\theta\cdot\sin\frac{1}{\rho}=0=f(0,0),$$

故函数在点(0,0)连续,

$$f_x(0,0)=\lim_{\Delta x\to 0}\frac{f(\Delta x,0)-f(0,0)}{\Delta x}=\lim_{\Delta x\to 0}\frac{0-0}{\Delta x}=0,$$

同理 $f_y(0,0)=0.$

当$(x,y)\neq(0,0)$时，

$$f_x(x,y)=y\sin\frac{1}{\sqrt{x^2+y^2}}-\frac{x^2y}{\sqrt{(x^2+y^2)^3}}\cos\frac{1}{\sqrt{x^2+y^2}},$$

当点$P(x,y)$沿直线$y=x$趋于$(0,0)$时，

$$\lim_{(x,x)\to(0,0)}f_x(x,y)$$

$$=\lim_{x\to0}\left(x\sin\frac{1}{\sqrt{2}\,|x|}-\frac{x^3}{2\sqrt{2}\,|x|^3}\cos\frac{1}{\sqrt{2}\,|x|}\right),$$

不存在.

所以$f_x(x,y)$在$(0,0)$不连续.

同理可证$f_y(x,y)$在$(0,0)$不连续.

$$\Delta f = f(\Delta x, \Delta y) - f(0,0)$$

$$= \Delta x \cdot \Delta y \cdot \sin\frac{1}{\sqrt{(\Delta x)^2 + (\Delta y)^2}}$$

$$= o(\sqrt{(\Delta x)^2 + (\Delta y)^2})$$

故$f(x,y)$在点$(0,0)$可微 $\quad df\big|_{(0,0)} = 0.$

*二、全微分在数值计算中的应用

1. 近似计算

由全微分定义

$$\Delta z = \underbrace{f_x(x,y)\Delta x + f_y(x,y)\Delta y}_{\mathrm{d}\, z} + o(\rho)$$

可知当 $|\Delta x|$ 及 $|\Delta y|$ 较小时, 有近似等式:

$$\Delta z \approx \mathrm{d}\, z = f_x(x,y)\Delta x + f_y(x,y)\Delta y$$

(可用于近似计算; 误差分析)

$$f(x+\Delta x, y+\Delta y) \approx f(x,y) + f_x(x,y)\Delta x + f_y(x,y)\Delta y$$

(可用于近似计算)

例3.计算 $1.04^{2.02}$ 的近似值.

解: 设 $f(x,y)=x^y$,则

$$f_x(x,y)=yx^{y-1},\quad f_y(x,y)=x^y\ln x$$

取 $x=1,\ y=2,\ \Delta x=0.04,\ \Delta y=0.02$

则 $$1.04^{2.02}=f(1.04,\ 2.02)$$

$$\approx f(1,2)+f_x(1,2)\Delta x+f_y(1,2)\Delta y$$

$$=1+2\times 0.04+0\times 0.02=1.08$$

2. 误差估计

利用 $\Delta z \approx f_x(x,y)\Delta x + f_y(x,y)\Delta y$

令 $\delta_x, \delta_y, \delta_z$ 分别表示 x, y, z 的绝对误差界，则

z 的绝对误差界约为

$$\delta_z = |f_x(x,y)|\delta_x + |f_y(x,y)|\delta_y$$

z 的相对误差界约为

$$\frac{\delta_z}{|z|} = \left|\frac{f_x(x,y)}{f(x,y)}\right|\delta_x + \left|\frac{f_y(x,y)}{f(x,y)}\right|\delta_y$$

第四节

多元复合函数的求导法则

一元复合函数 $y = f(u),\ u = \varphi(x)$

求导法则 $$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\mathrm{d}y}{\mathrm{d}u} \cdot \frac{\mathrm{d}u}{\mathrm{d}x}$$

微分法则 $\mathrm{d}y = f'(u)\,\mathrm{d}u = f'(u)\varphi'(x)\,\mathrm{d}x$

本节内容:

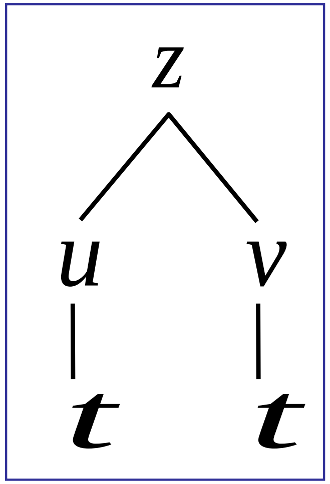
一、多元复合函数求导的链式法则

二、多元复合函数的全微分

一、多元复合函数求导的链式法则

定理. 若函数 $u=\varphi(t), v=\psi(t)$ 在点 t 可导, $z=f(u,v)$ 在点 (u,v) 处可微，则复合函数 $z=f(\varphi(t),\psi(t))$ 在点 t 可导, 且有链式法则

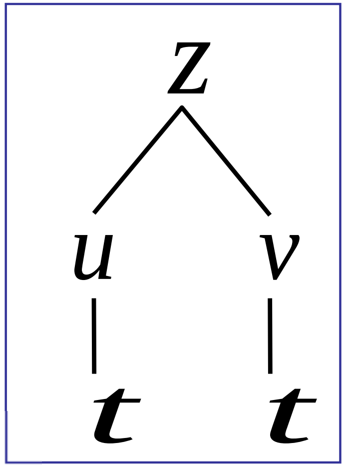
$$\frac{\mathrm{d}z}{\mathrm{d}t}=\frac{\partial z}{\partial u}\cdot\frac{\mathrm{d}u}{\mathrm{d}t}+\frac{\partial z}{\partial v}\cdot\frac{\mathrm{d}v}{\mathrm{d}t}$$



证: 设 t 取增量 Δt， 则相应中间变量有增量 Δu , Δv ,

$$\Delta z=\frac{\partial z}{\partial u}\Delta u+\frac{\partial z}{\partial v}\Delta v+o(\rho)\quad\left(\rho=\sqrt{(\Delta u)^2+(\Delta v)^2}\right)$$

$$\frac{\Delta z}{\Delta t}=\frac{\partial z}{\partial u}\frac{\Delta u}{\Delta t}+\frac{\partial z}{\partial v}\frac{\Delta v}{\Delta t}+\frac{o(\rho)}{\Delta t}\quad (\rho=\sqrt{(\Delta u)^2+(\Delta v)^2}\,)$$



令 $\Delta t\to 0$, 则有 $\Delta u\to 0$, $\Delta v\to 0$,

$$\frac{\Delta u}{\Delta t}\to\frac{\mathrm{d}u}{\mathrm{d}t},\qquad \frac{\Delta v}{\Delta t}\to\frac{\mathrm{d}v}{\mathrm{d}t}$$

$$\frac{o(\rho)}{\Delta t}=\frac{o(\rho)}{\rho}\sqrt{(\frac{\Delta u}{\Delta t})^2+(\frac{\Delta v}{\Delta t})^2}\to 0$$

($\triangle$t $<$ 0 时,根式前加 “–”号)

$$\boxed{\frac{\mathrm{d}z}{\mathrm{d}t}=\frac{\partial z}{\partial u}\cdot\frac{\mathrm{d}u}{\mathrm{d}t}+\frac{\partial z}{\partial v}\cdot\frac{\mathrm{d}v}{\mathrm{d}t}}$$

(**全导数公式**)

说明：若定理中 $f(u,v)$ 在点 (u,v) **可微**减弱为

偏导数存在，则定理结论不一定成立.

例如：$z = f(u,v) = \begin{cases} \dfrac{u^2 v}{u^2+v^2}, & u^2+v^2 \neq 0 \\ 0, & u^2+v^2 = 0 \end{cases}$

$u = t, \quad v = t$

易知：$\left.\dfrac{\partial z}{\partial u}\right|_{(0,0)} = f_u(0,0) = 0$，$\left.\dfrac{\partial z}{\partial v}\right|_{(0,0)} = f_v(0,0) = 0$

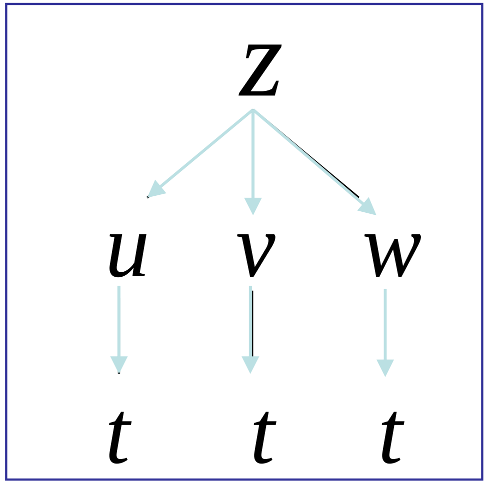
但复合函数 $z = f(t,t) = \dfrac{t}{2}$

$$\frac{\mathrm{d}z}{\mathrm{d}t} = \frac{1}{2} \neq \frac{\partial z}{\partial u}\cdot\frac{\mathrm{d}u}{\mathrm{d}t} + \frac{\partial z}{\partial v}\cdot\frac{\mathrm{d}v}{\mathrm{d}t} = 0\cdot 1 + 0\cdot 1 = 0$$

推广： 设下面所涉及的函数都可微 .

1) 中间变量多于两个的情形. 例如, $z = f(u, v, w)$,

$u = \varphi(t), v = \psi(t), w = \omega(t)$

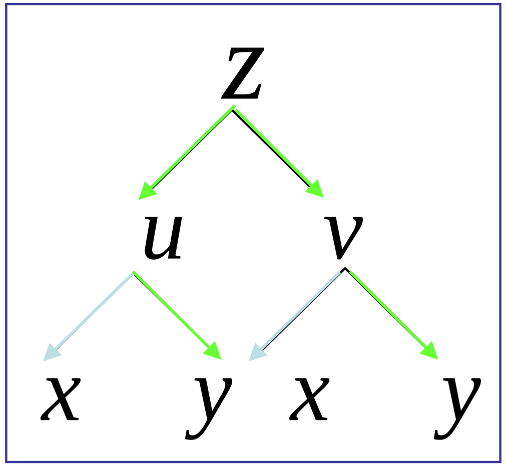


$$\frac{\mathrm{d}z}{\mathrm{d}t} = \frac{\partial z}{\partial u} \cdot \frac{\mathrm{d}u}{\mathrm{d}t} + \frac{\partial z}{\partial v} \cdot \frac{\mathrm{d}v}{\mathrm{d}t} + \frac{\partial z}{\partial w} \cdot \frac{\mathrm{d}w}{\mathrm{d}t}$$

$$= f_1 \phi' + f_2 \psi' + f_3 \omega'$$

2) 中间变量是多元函数的情形.例如,

$z = f(u, v)$, $\quad u = \varphi(x, y)$, $\quad v = \psi(x, y)$



$$\frac{\partial z}{\partial x} = \frac{\partial z}{\partial u} \cdot \frac{\partial u}{\partial x} + \frac{\partial z}{\partial v} \cdot \frac{\partial v}{\partial x} = f_1 \varphi_1 + f_2 \psi_1$$

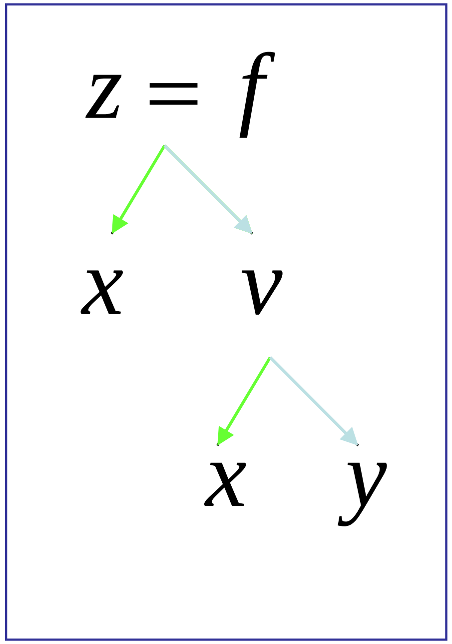
$$\frac{\partial z}{\partial y} = \frac{\partial z}{\partial u} \cdot \frac{\partial u}{\partial y} + \frac{\partial z}{\partial v} \cdot \frac{\partial v}{\partial y} = f_1 \varphi_2 + f_2 \psi_2$$

又如, $z = f(x,v),\ v = \psi(x,y)$

当它们都具有可微条件时, 有

$$\boxed{\frac{\partial z}{\partial x}} = \boxed{\frac{\partial f}{\partial x}} + \frac{\partial f}{\partial v}\cdot\frac{\partial v}{\partial x} = f_1 + f_2\psi_1$$

$$\frac{\partial z}{\partial y} = \frac{\partial f}{\partial v}\cdot\frac{\partial v}{\partial y} = f_2\psi_2$$



注意: 这里 $\frac{\partial z}{\partial x}$ 与 $\frac{\partial f}{\partial x}$ 不同,

$\frac{\partial z}{\partial x}$ 表示固定 y 对 x 求导, $\frac{\partial f}{\partial x}$ 表示固定 v 对 x 求导

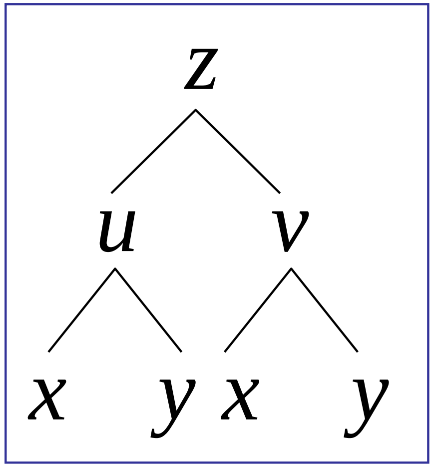
口诀: 分段用乘, 分叉用加, 单路全导, 叉路偏导

例1. 设 $z = e^u \sin v$， $u = xy$ ， $v = x + y$, 求 $\dfrac{\partial z}{\partial x}, \dfrac{\partial z}{\partial y}$.

解:
$$\frac{\partial z}{\partial x} = \frac{\partial z}{\partial u}\cdot\frac{\partial u}{\partial x} + \frac{\partial z}{\partial v}\cdot\frac{\partial v}{\partial x}$$

$$= e^u \sin v \cdot y + e^u \cos v \cdot 1$$

$$= e^{xy}[y\cdot\sin(x+y) + \cos(x+y)]$$



$$\frac{\partial z}{\partial y} = \frac{\partial z}{\partial u}\cdot\frac{\partial u}{\partial y} + \frac{\partial z}{\partial v}\cdot\frac{\partial v}{\partial y}$$

$$= e^u \sin v \cdot x + e^u \cos v \cdot 1$$

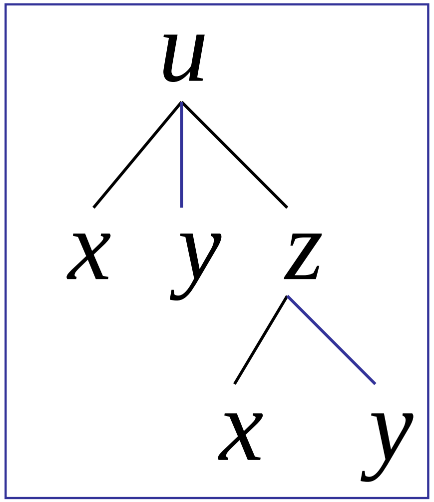
$$= e^{xy}[x\cdot\sin(x+y) + \cos(x+y)]$$

例2. $u = f(x, y, z) = e^{x^2+y^2+z^2}$, $z = x^2 \sin y$, 求 $\dfrac{\partial u}{\partial x}, \dfrac{\partial u}{\partial y}$

解:
$$\frac{\partial u}{\partial x} = \frac{\partial f}{\partial x} + \frac{\partial f}{\partial z} \cdot \frac{\partial z}{\partial x}$$

$$= 2xe^{x^2+y^2+z^2} + 2ze^{x^2+y^2+z^2} \cdot 2x \sin y$$

$$= 2x(1 + 2x^2 \sin^2 y) e^{x^2+y^2+x^4 \sin^2 y}$$



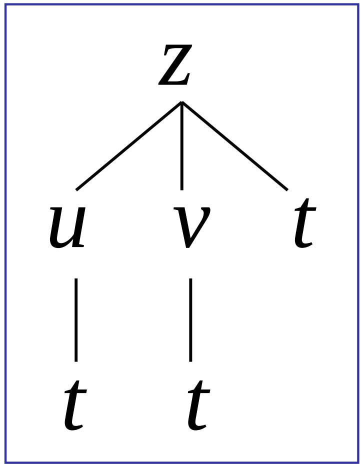
$$\frac{\partial u}{\partial y} = \frac{\partial f}{\partial y} + \frac{\partial f}{\partial z} \cdot \frac{\partial z}{\partial y}$$

$$= 2ye^{x^2+y^2+z^2} + 2ze^{x^2+y^2+z^2} \cdot x^2 \cos y$$

$$= 2(y + x^4 \sin y \cos y) e^{x^2+y^2+x^4 \sin^2 y}$$

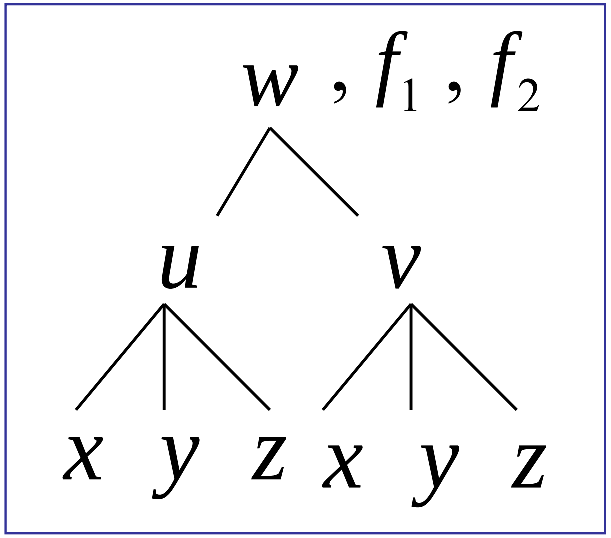
例3. 设 $z = uv + \sin t$, $u = e^t$, $v = \cos t$, 求全导数 $\dfrac{\mathrm{d}z}{\mathrm{d}t}$.

解:
$$\frac{\mathrm{d}z}{\mathrm{d}t} = \frac{\partial z}{\partial u}\cdot\frac{\mathrm{d}u}{\mathrm{d}t} + \frac{\partial z}{\partial v}\cdot\frac{\mathrm{d}v}{\mathrm{d}t} + \frac{\partial z}{\partial t}$$
$$= v e^t - u\sin t + \cos t$$
$$= e^t(\cos t - \sin t) + \cos t$$



注意：多元抽象复合函数求导在偏微分方程变形与验证解的问题中经常遇到, 下列两个例题有助于掌握这方面问题的求导技巧与常用导数符号.

例4. 设 $w = f(x+y+z, xyz)$，f 具有二阶连续偏导数，求 $\dfrac{\partial w}{\partial x}, \dfrac{\partial^2 w}{\partial x \partial z}$.



解: 令 $u = x+y+z$，$v = xyz$，则

$$w = f(u, v)$$

$$\frac{\partial w}{\partial x} = f_1 \cdot 1 + f_2 \cdot yz$$

$$= f_1(x+y+z, xyz) + \underline{yz\, f_2(x+y+z, xyz)}$$

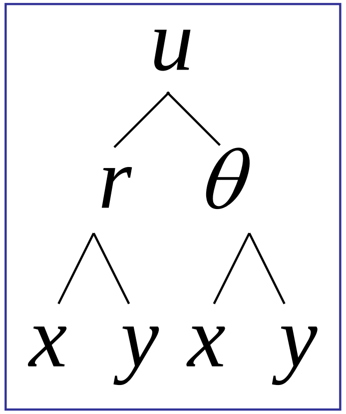
$$\frac{\partial^2 w}{\partial x \partial z} = f_{11} \cdot 1 + f_{12} \cdot xy + y\, f_2 + yz[\, f_{21} \cdot 1 + f_{22} \cdot xy \,]$$

$$= f_{11} + y(x+z)\, f_{12} + xy^2 z\, f_{22} + y\, f_2$$

例5. 设$u=f(x,y)$二阶偏导数连续,求下列表达式在极坐标系下的形式 (1) $(\frac{\partial u}{\partial x})^2+(\frac{\partial u}{\partial y})^2$, (2) $\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}$

解: 已知$x=r\cos\theta$, $y=r\sin\theta$, 则

$$r=\sqrt{x^2+y^2},\ \theta=\arctan\frac{y}{x}$$



(1) $$\frac{\partial u}{\partial x}=\frac{\partial u}{\partial r}\frac{\partial r}{\partial x}+\frac{\partial u}{\partial \theta}\frac{\partial \theta}{\partial x}$$

$$\frac{\partial r}{\partial x}=\frac{x}{r},\quad \frac{\partial \theta}{\partial x}=\frac{\frac{-y}{x^2}}{1+(\frac{y}{x})^2}=\frac{-y}{x^2+y^2}$$

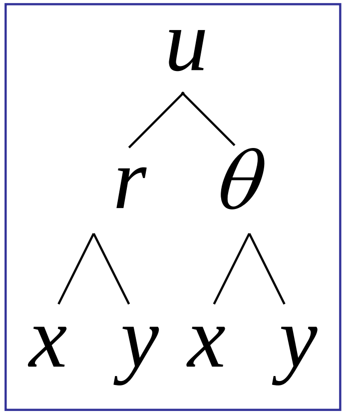
$$=\frac{\partial u}{\partial r}\frac{x}{r}-\frac{\partial u}{\partial \theta}\frac{y}{r^2}=\frac{\partial u}{\partial r}\cos\theta-\frac{\partial u}{\partial \theta}\frac{\sin\theta}{r}$$

$$\frac{\partial u}{\partial y}=\frac{\partial u}{\partial r}\frac{\partial r}{\partial y}+\frac{\partial u}{\partial \theta}\frac{\partial \theta}{\partial y}$$

$$\boxed{\frac{\partial r}{\partial y}=\frac{y}{r},\quad \frac{\partial \theta}{\partial y}=\frac{\frac{1}{x}}{1+(\frac{y}{x})^2}=\frac{x}{x^2+y^2}}$$

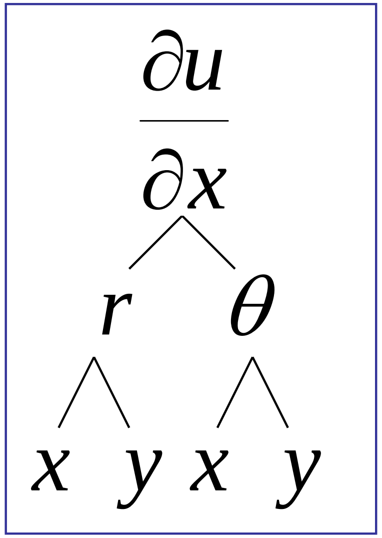
$$=\frac{\partial u}{\partial r}\frac{y}{r}+\frac{\partial u}{\partial \theta}\frac{x}{r^2}$$

$$=\frac{\partial u}{\partial r}\sin\theta+\frac{\partial u}{\partial \theta}\frac{\cos\theta}{r}$$



$$\therefore \quad (\frac{\partial u}{\partial x})^2+(\frac{\partial u}{\partial y})^2=(\frac{\partial u}{\partial r})^2+\frac{1}{r^2}(\frac{\partial u}{\partial \theta})^2$$

已知 $\dfrac{\partial u}{\partial x}=\boxed{\dfrac{\partial u}{\partial r}\cos\theta-\dfrac{\partial u}{\partial \theta}\dfrac{\sin\theta}{r}}$



注意利用已有公式

$$(2)\quad \frac{\partial^2 u}{\partial x^2}=\frac{\partial}{\partial r}\left(\frac{\partial u}{\partial x}\right)\cdot\cos\theta-\frac{\partial}{\partial \theta}\left(\frac{\partial u}{\partial x}\right)\frac{\sin\theta}{r}$$

$$=\frac{\partial}{\partial r}\left(\frac{\partial u}{\partial r}\cos\theta-\frac{\partial u}{\partial \theta}\frac{\sin\theta}{r}\right)\cdot\cos\theta$$

$$-\frac{\partial}{\partial \theta}\left(\frac{\partial u}{\partial r}\cos\theta-\frac{\partial u}{\partial \theta}\frac{\sin\theta}{r}\right)\cdot\frac{\sin\theta}{r}$$

$$=\frac{\partial^2 u}{\partial r^2}\cos^2\theta-2\frac{\partial^2 u}{\partial r\partial\theta}\frac{\sin\theta\cos\theta}{r}+\frac{\partial^2 u}{\partial \theta^2}\frac{\sin^2\theta}{r^2}$$

$$+\frac{\partial u}{\partial \theta}\frac{2\sin\theta\cos\theta}{r^2}+\frac{\partial u}{\partial r}\frac{\sin^2\theta}{r}$$

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 u}{\partial r^2}\cos^2\theta - 2\cancel{\frac{\partial^2 u}{\partial r \partial \theta}\frac{\sin\theta\cos\theta}{r}} + \frac{\partial^2 u}{\partial \theta^2}\frac{\sin^2\theta}{r^2}$$

$$+\cancel{\frac{\partial u}{\partial \theta}\frac{2\sin\theta\cos\theta}{r^2}} + \frac{\partial u}{\partial r}\frac{\sin^2\theta}{r}$$

同理可得

$$\frac{\partial^2 u}{\partial y^2} = \frac{\partial^2 u}{\partial r^2}\sin^2\theta + 2\cancel{\frac{\partial^2 u}{\partial r \partial \theta}\frac{\sin\theta\cos\theta}{r}} + \frac{\partial^2 u}{\partial \theta^2}\frac{\cos^2\theta}{r^2}$$

$$-\cancel{\frac{\partial u}{\partial \theta}\frac{2\sin\theta\cos\theta}{r^2}} + \frac{\partial u}{\partial r}\frac{\cos^2\theta}{r}$$

$$\therefore \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = \frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2}$$

$$= \frac{1}{r^2}[r\frac{\partial}{\partial r}(r\frac{\partial u}{\partial r}) + \frac{\partial^2 u}{\partial \theta^2}]$$