二、小结

二重积分在直角坐标下的计算公式

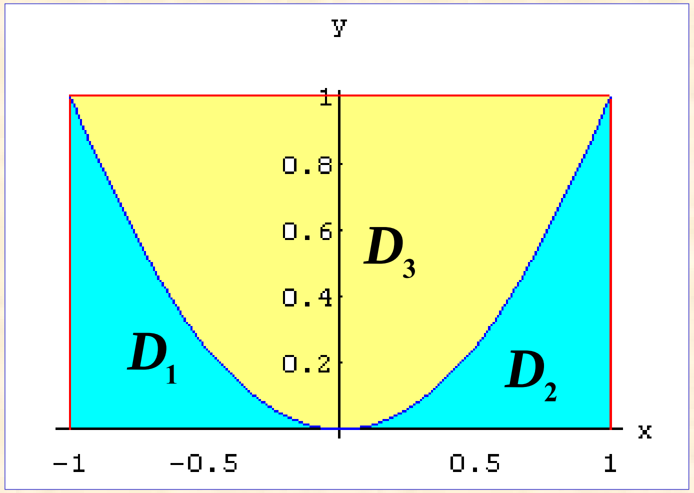
$$\iint_D f(x,y)d\sigma = \int_a^b dx \int_{\varphi_1(x)}^{\varphi_2(x)} f(x,y)dy. \quad [\text{X－型}]$$

$$\iint_D f(x,y)d\sigma = \int_c^d dy \int_{\varphi_1(y)}^{\varphi_2(y)} f(x,y)dx. [\text{Y－型}]$$

（在积分中要正确选择积分次序）

例9 计算 $\iint\limits_D |y-x^2|d\sigma$. 其中 $D:-1\le x\le 1, 0\le y\le 1$.

解 先去掉绝对值符号，如图



$$\iint\limits_D |y-x^2|d\sigma$$

$$=\iint\limits_{D_1+D_2}(x^2-y)d\sigma+\iint\limits_{D_3}(y-x^2)d\sigma$$

$$=\int_{-1}^{1}dx\int_0^{x^2}(x^2-y)dy+\int_{-1}^{1}dx\int_{x^2}^{1}(y-x^2)dy=\frac{11}{15}.$$

例10 证明

$$\int_0^x\left[\int_0^v\left(\int_0^u f(t)dt\right)du\right]dv=\frac{1}{2}\int_0^x(x-t)^2 f(t)dt.$$

证 思路：从改变积分次序入手.

$$\because \int_0^v du\int_0^u f(t)dt=\int_0^v dt\int_t^v f(t)du=\int_0^v(v-t)f(t)dt,$$

$$\therefore \int_0^x\left[\int_0^v\left(\int_0^u f(t)dt\right)du\right]dv=\int_0^x dv\int_0^v(v-t)f(t)dt$$

$$=\int_0^x dt\underline{\int_t^x(v-t)f(t)dv}=\frac{1}{2}\int_0^x(x-t)^2 f(t)dt.$$

例11. 求两个底圆半径为R的直交圆柱面所围的体积.

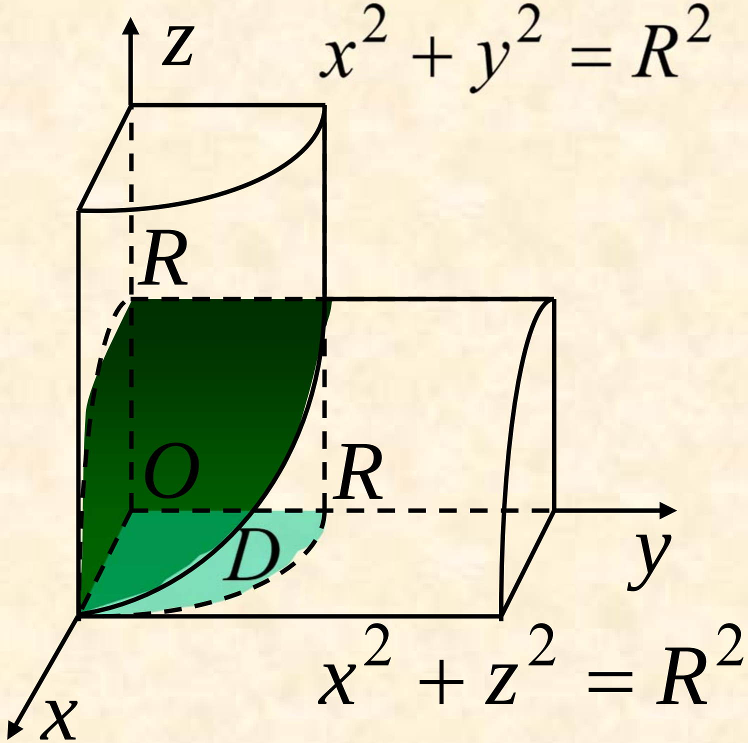
解: 设两个直圆柱方程为

$$x^2+y^2=R^2,\ \ x^2+z^2=R^2$$

利用对称性, 考虑第一卦限部分,

其曲顶柱体的顶为 $z=\sqrt{R^2-x^2}$

$$(x,y)\in D:\begin{cases}0\le y\le\sqrt{R^2-x^2}\\ 0\le x\le R\end{cases}$$



则所求体积为

$$V=8\iint_D\sqrt{R^2-x^2}\,\mathrm{d}x\,\mathrm{d}y=8\int_0^R\sqrt{R^2-x^2}\,\mathrm{d}x\int_0^{\sqrt{R^2-x^2}}\mathrm{d}y$$

$$=8\int_0^R(R^2-x^2)\,\mathrm{d}x=\frac{16}{3}R^3$$

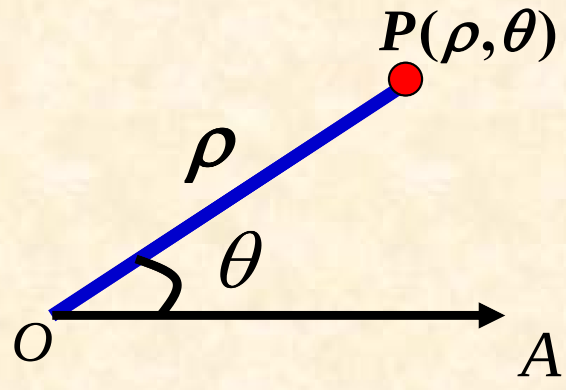
极坐标系是由极点O和极轴OA组成,

点P坐标 (ρ,θ) 其中ρ为点P到极点O的距离,

θ为OA到OP的夹角, $0 \le \rho < +\infty, 0 \le \theta \le 2\pi$

ρ =常数,(同心圆族)

θ =常数,(从O出发射线族)



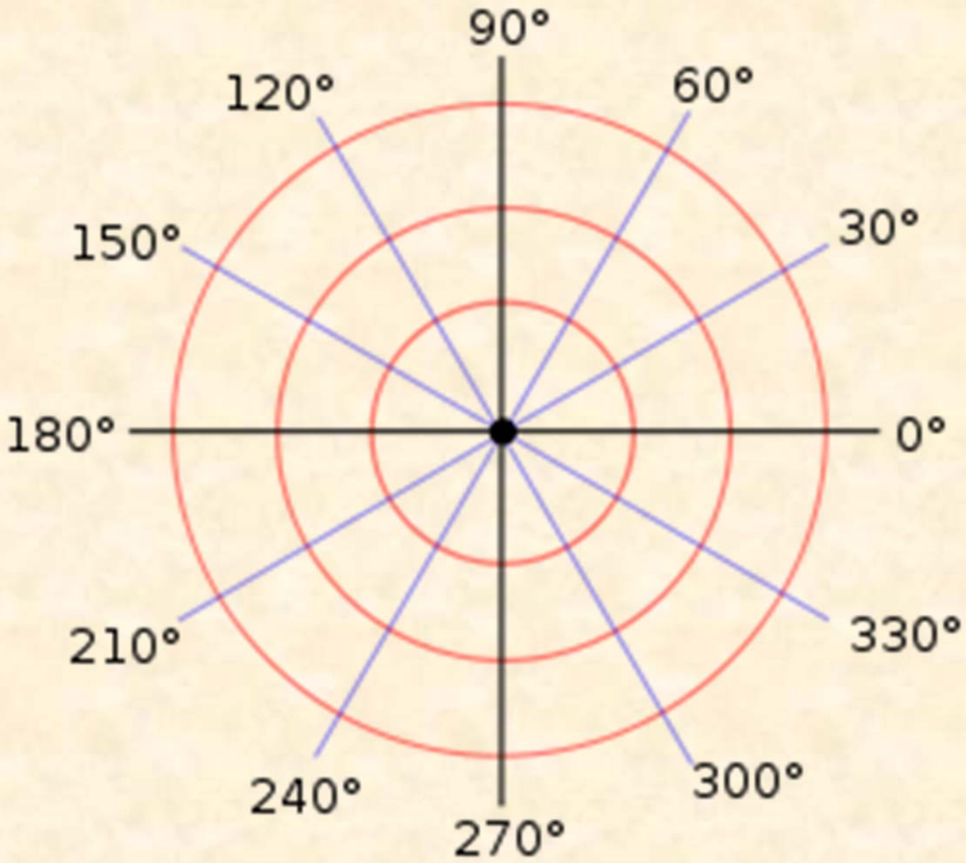
若令极点与xoy直角坐标系

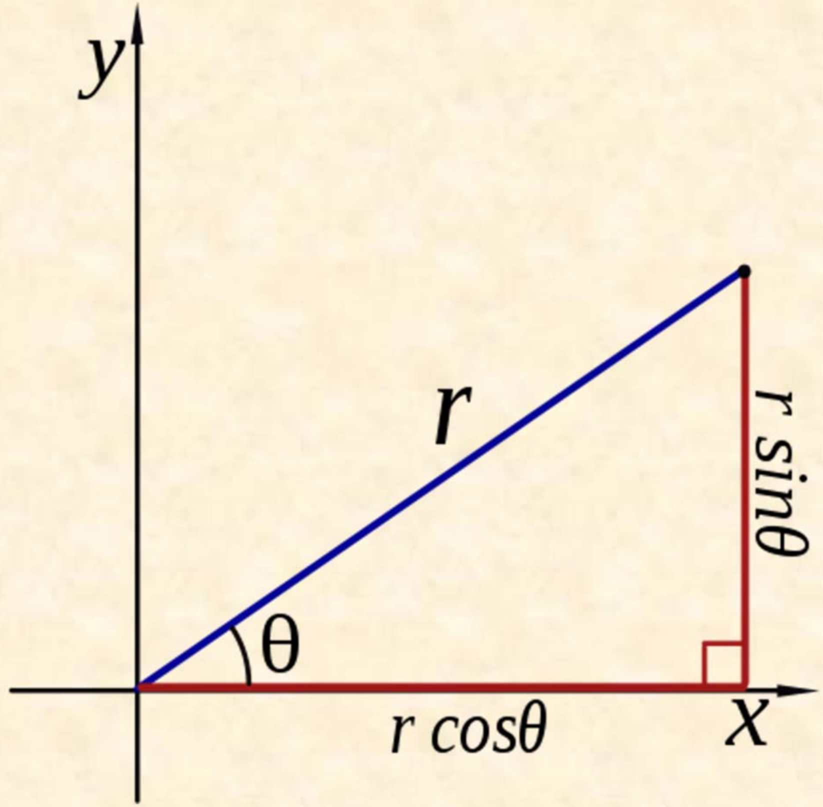
的原点重合, x轴取为极轴,则

直角坐标与极坐标的关系为:

$$\boxed{\begin{cases} x = \rho\cos\theta \\ y = \rho\sin\theta \end{cases}}$$

极坐标系与直角坐标系





坐标转化公式

$$\begin{cases} x = r\cos\theta \\ y = r\sin\theta \end{cases}$$

$$\begin{cases} r = \sqrt{y^2 + x^2} \\ \theta = \begin{cases} \arctan(\frac{y}{x}) & \text{if } x > 0 \\ \arctan(\frac{y}{x}) + \pi & \text{if } x < 0 \text{ and } y \ge 0 \\ \arctan(\frac{y}{x}) - \pi & \text{if } x < 0 \text{ and } y < 0 \\ \frac{\pi}{2} & \text{if } x = 0 \text{ and } y > 0 \\ -\frac{\pi}{2} & \text{if } x = 0 \text{ and } y < 0 \\ 0 & \text{if } x = 0 \text{ and } y = 0 \end{cases} \end{cases}$$

二、 用极坐标计算二重积分

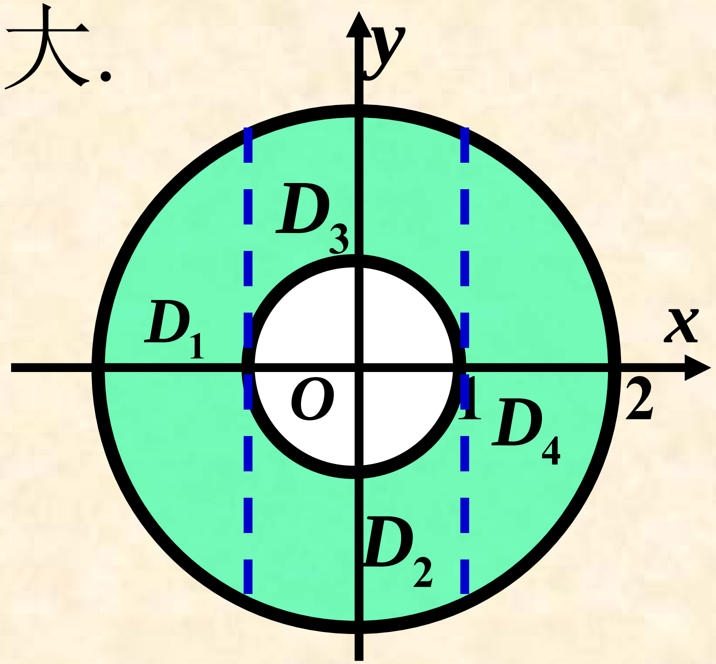
计算$\iint\limits_D f(x,y)d\sigma$ 其中$D:1\le x^2+y^2\le 4.$

在直角坐标系下，若把积分区域看作X型，

须划分为四个子域， 计算量较大.

注意到圆的极坐标表示，

考虑在极坐标下求二重积分.



极坐标下面积元素 $\boxed{d\sigma = ?\ \rho\,\mathrm{d}\rho\,\mathrm{d}\theta}$

用极坐标曲线网

ρ =常数，（同心圆族）

$\boxed{\iint_D f(x,y)d\sigma}$

θ =常数，（射线族）

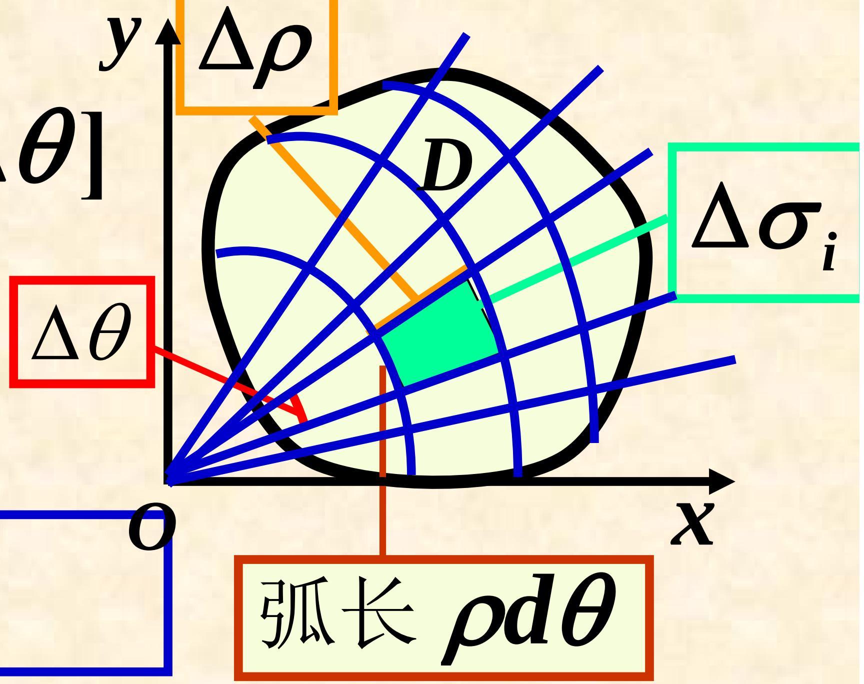
来划分积分域，规则的子域 $\Delta\sigma_i$ 的面积

$$\Delta\sigma_i = \frac{1}{2}[(\rho+\Delta\rho)^2\Delta\theta - \rho^2\Delta\theta]$$

$$= \rho\Delta\rho\Delta\theta + \boxed{\frac{1}{2}(\Delta\rho)^2\Delta\theta}$$

$$\approx \rho\Delta\rho\Delta\theta$$

高阶项 略去



由直角坐标和极坐标的对应关系，得到

二重积分在极坐标下的形式

$$\iint_D f(x,y)\,d\sigma = \iint_D f\left(\rho\cos\theta, \rho\sin\theta\right)\rho\,d\rho\,d\theta$$

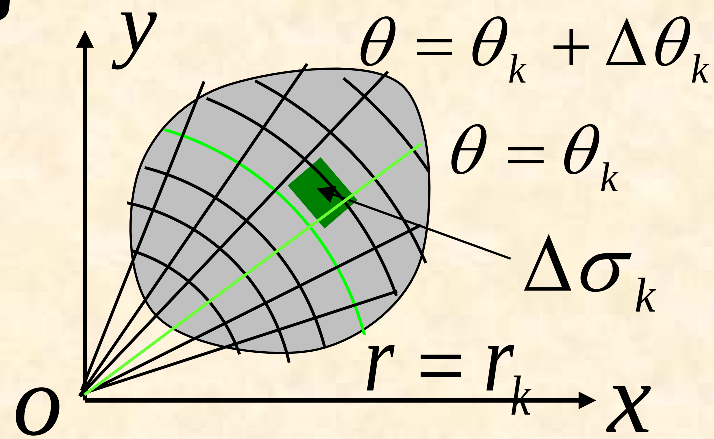
$$\boxed{\text{面积元素}\quad d\sigma = \rho\,d\rho\,d\theta}$$

二、利用极坐标计算二重积分

在极坐标系下, 用同心圆 $r=$常数

及射线 $\theta=$常数, 分划区域D 为

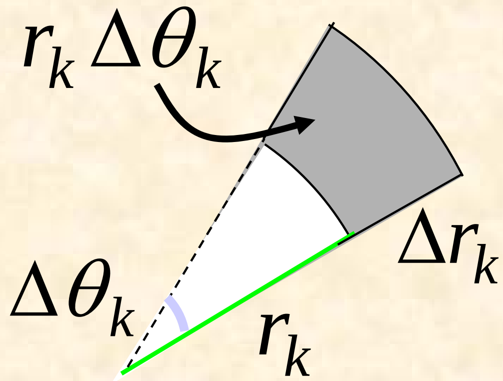
$$\Delta\sigma_k \quad (k=1,2,\cdots,n)$$



则除包含边界点的小区域外,小区域的面积

$$\Delta\sigma_k=\tfrac{1}{2}(r_k+\Delta r_k)^2\cdot\Delta\theta_k-\tfrac{1}{2}r_k{}^2\cdot\Delta\theta_k=\overline{r_k}\Delta r_k\cdot\Delta\theta_k$$

在 $\Delta\sigma_k$ 内取点$(\overline{r_k},\overline{\theta_k})$, 对应有



$$\xi_k=\overline{r_k}\cos\overline{\theta_k},\quad \eta_k=\overline{r_k}\sin\overline{\theta_k}$$

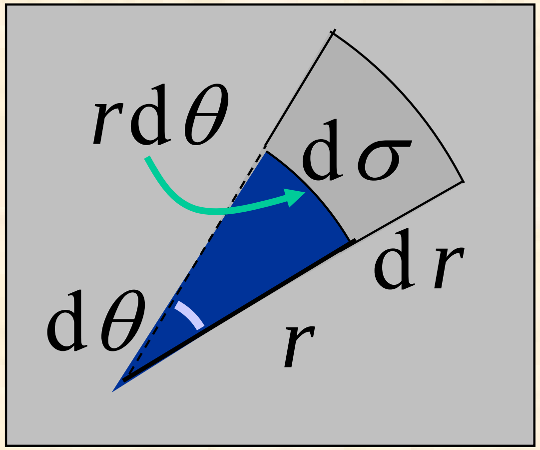
$$\lim_{\|\Delta\sigma\|\to 0}\sum_{k=1}^{n}f(\xi_k,\eta_k)\Delta\sigma_k$$

$$=\lim_{\|\Delta\sigma\|\to 0}\sum_{k=1}^{n}f(\overline{r_k}\cos\overline{\theta_k},\overline{r_k}\sin\overline{\theta_k})\overline{r_k}\Delta r_k\Delta\theta_k$$

$$\lim_{\|\Delta\sigma\|\to 0}\sum_{k=1}^{n} f(\xi_k,\eta_k)\Delta\sigma_k$$

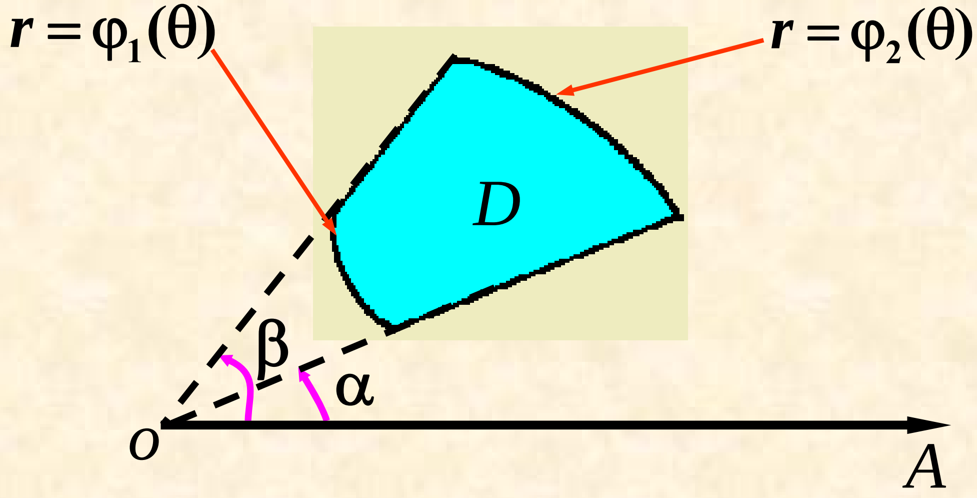
$$=\lim_{\|\Delta\sigma\|\to 0}\sum_{k=1}^{n} f(\overline{r_k}\cos\overline{\theta_k},\overline{r_k}\sin\overline{\theta_k})\overline{r_k}\Delta r_k\Delta\theta_k$$

即 $$\iint_D f(x,y)\,\mathrm{d}\sigma=\iint_D f(r\cos\theta,r\sin\theta)\,r\,\mathrm{d}r\,\mathrm{d}\theta$$



二重积分化为二次积分的公式（1）

区域特征如图



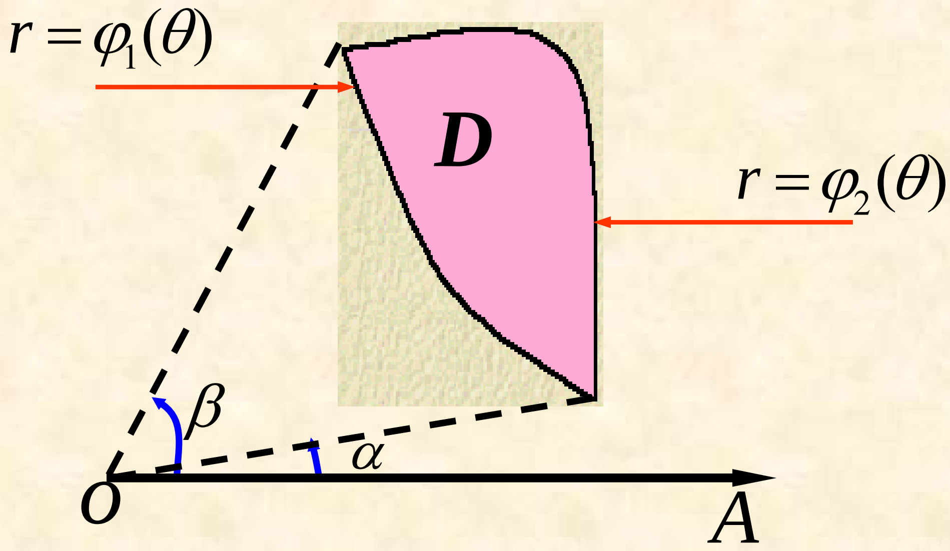
$$\alpha \le \theta \le \beta,$$

$$\varphi_1(\theta) \le r \le \varphi_2(\theta).$$

$$\iint_D f(r\cos\theta, r\sin\theta) r dr d\theta$$

$$= \int_\alpha^\beta d\theta \int_{\varphi_1(\theta)}^{\varphi_2(\theta)} f(r\cos\theta, r\sin\theta) r dr.$$

区域特征如图



$$\alpha \le \theta \le \beta,$$

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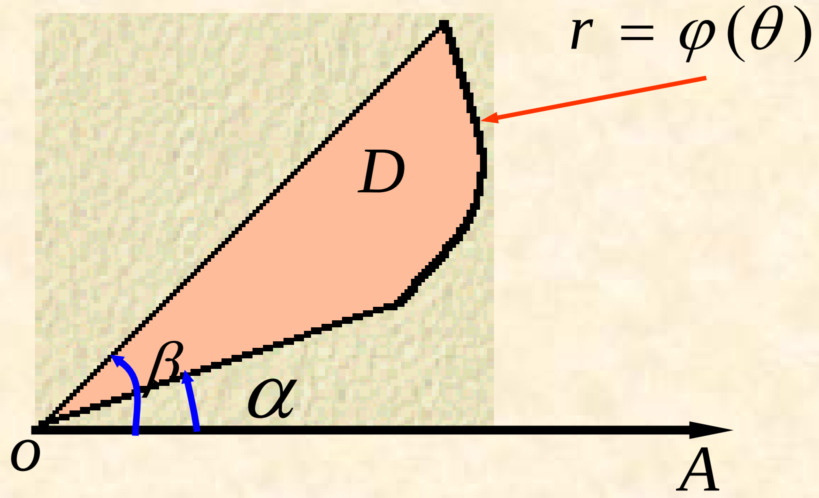
$$\iint_D f(r\cos\theta, r\sin\theta) r dr d\theta$$

$$= \int_\alpha^\beta d\theta \int_{\varphi_1(\theta)}^{\varphi_2(\theta)} f(r\cos\theta, r\sin\theta) r dr.$$

区域特征如图

$$\alpha \le \theta \le \beta,$$

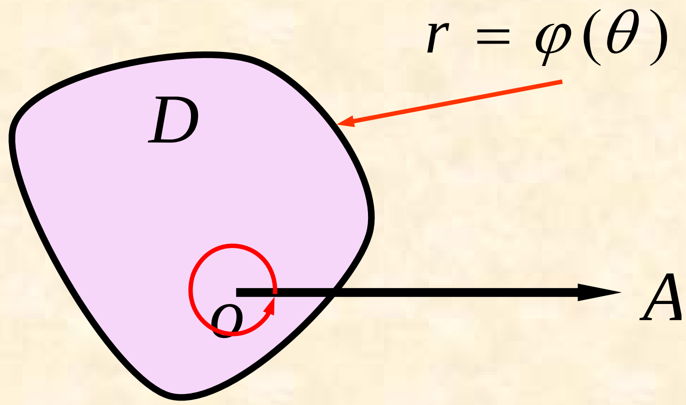
$$0 \le r \le \varphi(\theta).$$



$$\iint_D f(r\cos\theta, r\sin\theta) r dr d\theta$$

$$= \int_\alpha^\beta d\theta \int_0^{\varphi(\theta)} f(r\cos\theta, r\sin\theta) r dr.$$

区域特征如图



$$0\le\theta\le 2\pi,\qquad 0\le r\le\varphi(\theta).$$

$$\iint_D f(r\cos\theta, r\sin\theta)r\,dr\,d\theta$$

$$=\int_0^{2\pi}d\theta\int_0^{\varphi(\theta)}f(r\cos\theta, r\sin\theta)r\,dr.$$

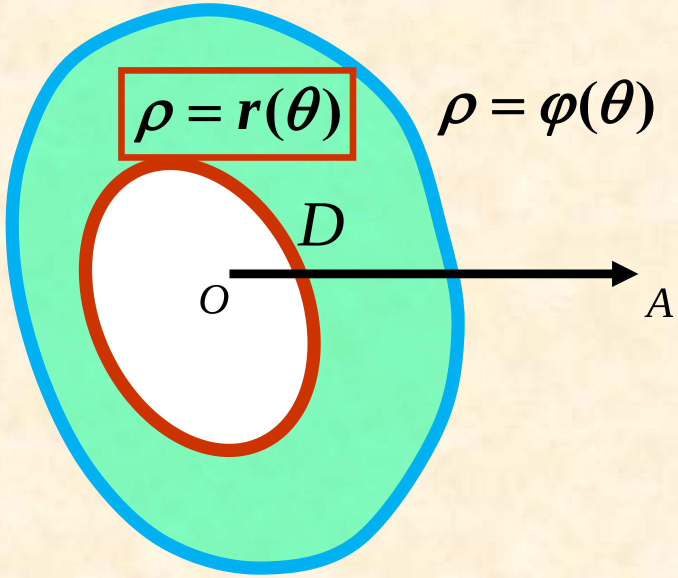
极坐标系下区域的面积 $\sigma=\iint_D r\,dr\,d\theta.$

若极点在D的内部

则D可以用不等式表示:

$$0 \le \rho \le \varphi(\theta), \quad 0 \le \theta \le 2\pi$$

这时有



$$\iint_D f(x,y)d\sigma = \int_0^{2\pi} d\theta \int_0^{\varphi(\theta)} f(\rho\cos\theta, \rho\sin\theta)\rho d\rho$$

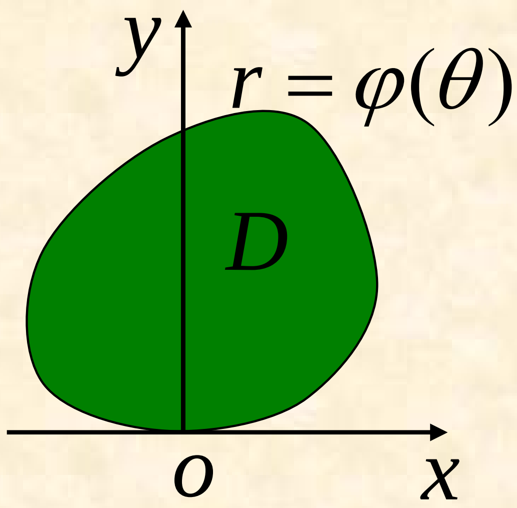
若D由两条封闭曲线围成（如图），则

$$\iint_D f(x,y)d\sigma = \int_0^{2\pi} d\theta \int_{\boxed{r(\theta)}}^{\varphi(\theta)} f(\rho\cos\theta, \rho\sin\theta)\rho d\rho$$

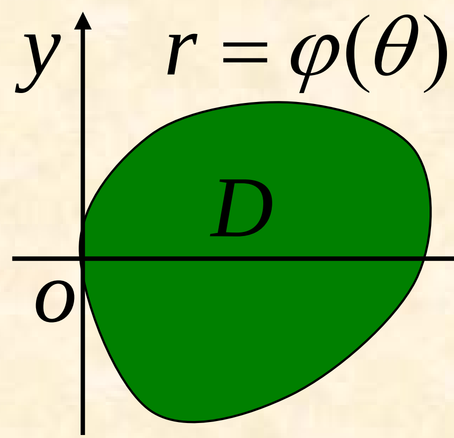
思考: 下列各图中域 D 分别与 x , y 轴相切于原点,试

问 θ 的变化范围是什么?

(1)



(2)



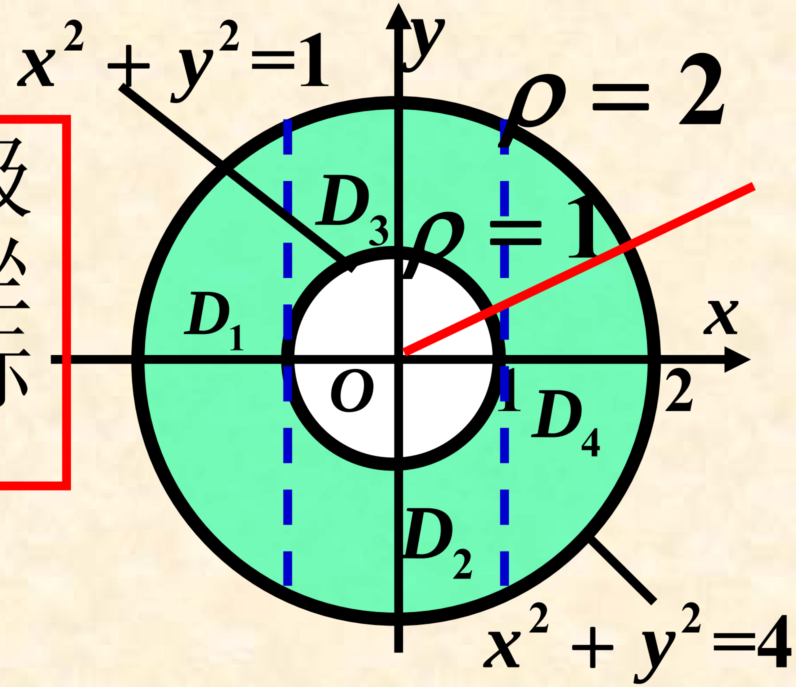
答: (1) $0 \le \theta \le \pi$;　　(2) $-\dfrac{\pi}{2} \le \theta \le \dfrac{\pi}{2}$

前例：计算$\iint\limits_D f(x,y)d\sigma$其中$D:1\le x^2+y^2\le 4.$

解 把$\iint\limits_D f(x,y)d\sigma$化为极坐标下的二次积分，

直角坐标 $x^2+y^2=1 \longrightarrow \rho=1$ 极坐标

$x^2+y^2=4 \longrightarrow \rho=2$



$$D:1\le\rho\le 2, 0\le\theta\le 2\pi$$

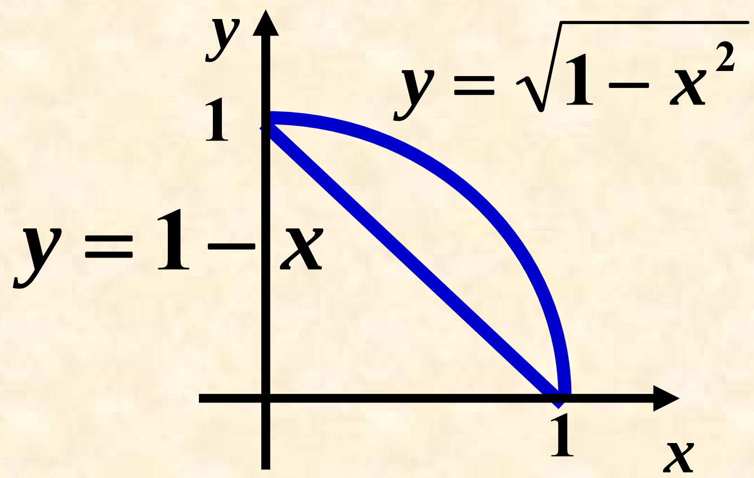
$$\iint\limits_D f(x,y)d\sigma=\int_0^{2\pi}d\theta\int_1^2 f(\rho\cos\theta,\rho\sin\theta)\rho d\rho$$

例1 将$\iint\limits_D f(x,y)d\sigma, D: 1-x \le y \le \sqrt{1-x^2}$,

$0 \le x \le 1$, 化为极坐标下的二次积分.

解 利用$\begin{cases} x=\rho\cos\theta \\ y=\rho\sin\theta \end{cases}$把积分区域的边界曲

线化为极坐标形式:



$$\iint\limits_D f(x,y)d\sigma$$

$$=\int_0^{\frac{\pi}{2}} d\theta \int_{\frac{1}{\sin\theta+\cos\theta}}^{1} f(\rho\cos\theta,\rho\sin\theta)\rho d\rho$$

例2 计算 $\iint\limits_D e^{-x^2-y^2}dxdy$， 其中$D$是以原点为圆心，半径为$a$的圆域.

解 D可以表示成 $0 \le \rho \le a, 0 \le \theta \le 2\pi$

$$\iint\limits_D e^{-x^2-y^2}dxdy = \iint\limits_D e^{-\rho^2}\rho d\rho d\theta$$

$$= \int_0^{2\pi} d\theta \int_0^a e^{-\rho^2}\rho d\rho = \int_0^{2\pi} [-\frac{1}{2}e^{-\rho^2}]_0^a d\theta$$

$$= \frac{1}{2}(1-e^{-a^2})\int_0^{2\pi} d\theta = \pi(1-e^{-a^2})$$

注:利用例2可得到一个在概率论与数理统计及工程上非常有用的反常积分公式

$$\int_0^{+\infty} e^{-x^2}\,\mathrm{d}x = \frac{\sqrt{\pi}}{2} \qquad ①$$

事实上, 当D为 R^2 时,

$$\iint_D e^{-x^2-y^2}\,\mathrm{d}x\mathrm{d}y = \int_{-\infty}^{+\infty} e^{-x^2}\,\mathrm{d}x \int_{-\infty}^{+\infty} e^{-y^2}\,\mathrm{d}y$$

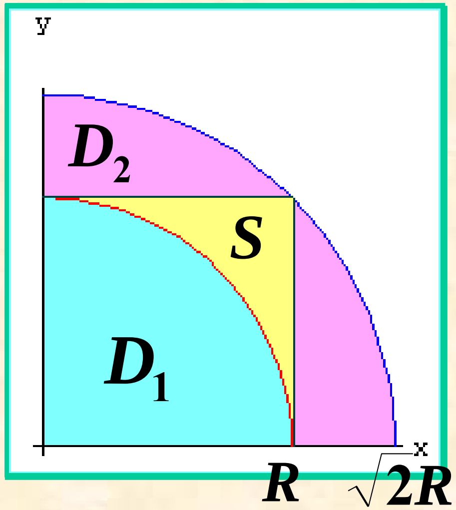
$$= 4\left(\int_0^{+\infty} e^{-x^2}\,\mathrm{d}x\right)^2$$

利用例2的结果, 得

$$4\left(\int_0^{+\infty} e^{-x^2}\,\mathrm{d}x\right)^2 = \lim_{a\to+\infty} \pi(1-e^{-a^2}) = \pi$$

故①式成立 .

例 10　求广义积分$\int_0^{\infty} e^{-x^2} dx$.



解　$D_1 = \{(x, y) \mid x^2 + y^2 \le R^2\}$

$D_2 = \{(x, y) \mid x^2 + y^2 \le 2R^2\}$

$S = \{(x, y) \mid 0 \le x \le R, 0 \le y \le R\}$

$\{x \ge 0, y \ge 0\}$　显然有 $D_1 \subset S \subset D_2$

$\because \quad e^{-x^2-y^2} > 0,$

$$\therefore \quad \iint_{D_1} e^{-x^2-y^2} dxdy \le \iint_{S} e^{-x^2-y^2} dxdy \le \iint_{D_2} e^{-x^2-y^2} dxdy.$$

又∵ $I=\iint\limits_{S} e^{-x^2-y^2}dxdy$

$$=\int_0^R e^{-x^2}dx\int_0^R e^{-y^2}dy \ =(\int_0^R e^{-x^2}dx)^2;$$

$$I_1=\iint\limits_{D_1} e^{-x^2-y^2}dxdy$$

$$=\int_0^{\frac{\pi}{2}}d\theta\int_0^R e^{-r^2}rdr \ =\frac{\pi}{4}(1-e^{-R^2});$$

同理$I_2=\iint\limits_{D_2} e^{-x^2-y^2}dxdy=\frac{\pi}{4}(1-e^{-2R^2});$

$$\because I_1 < I < I_2,$$

$$\therefore \frac{\pi}{4}(1-e^{-R^2}) < \left(\int_0^R e^{-x^2}dx\right)^2 < \frac{\pi}{4}(1-e^{-2R^2});$$

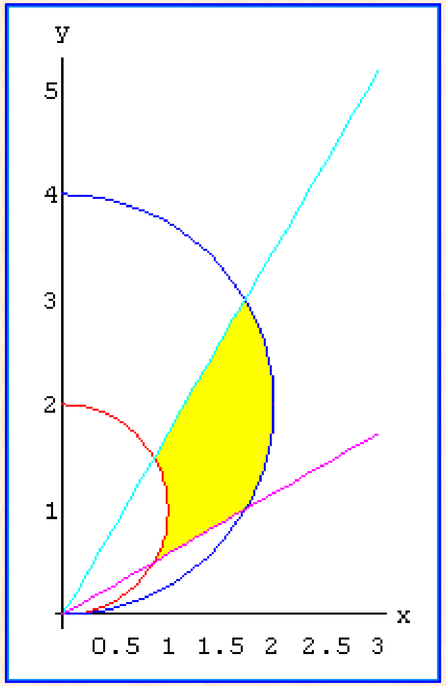
当$R\to\infty$时，$I_1\to\frac{\pi}{4}$，$I_2\to\frac{\pi}{4}$，

故当$R\to\infty$时，$I\to\frac{\pi}{4}$，即$\left(\int_0^\infty e^{-x^2}dx\right)^2=\frac{\pi}{4}$，

所求广义积分$\int_0^\infty e^{-x^2}dx=\frac{\sqrt{\pi}}{2}$.

例 3　计算$\iint\limits_{D}(x^2+y^2)dxdy$，其 D 为由圆 $x^2+y^2=2y$，$x^2+y^2=4y$及直线$x-\sqrt{3}y=0$，$y-\sqrt{3}x=0$ 所围成的平面闭区域.

解



$$y-\sqrt{3}x=0 \Rightarrow \theta_2=\frac{\pi}{3}$$

$$x^2+y^2=4y \Rightarrow r=4\sin\theta$$

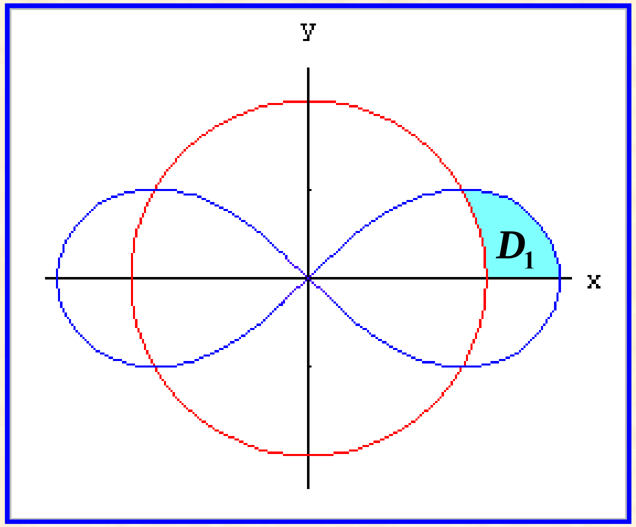
$$x-\sqrt{3}y=0 \Rightarrow \theta_1=\frac{\pi}{6}$$

$$x^2+y^2=2y \Rightarrow r=2\sin\theta$$

$$\iint\limits_{D}(x^2+y^2)dxdy=\int_{\frac{\pi}{6}}^{\frac{\pi}{3}}d\theta\int_{2\sin\theta}^{4\sin\theta}r^2\cdot rdr=15\left(\frac{\pi}{2}-\sqrt{3}\right).$$

例 4 求曲线 $(x^2+y^2)^2=2a^2(x^2-y^2)$ 和 $x^2+y^2\geq a^2$ 所围成的图形的面积.

解 根据对称性有 $D=4D_1$



在极坐标下

$$x^2+y^2=a^2\Rightarrow r=a,$$

$$(x^2+y^2)^2=2a^2(x^2-y^2)\Rightarrow\quad r=a\sqrt{2\cos 2\theta},$$

由$\begin{cases} r = a\sqrt{2\cos 2\theta} \\ r = a \end{cases}$，　得交点$A = (a, \dfrac{\pi}{6})$,

所求面积$\sigma = \iint\limits_{D} dxdy = 4\iint\limits_{D_1} dxdy$

$$= 4\int_0^{\frac{\pi}{6}} d\theta \int_a^{a\sqrt{2\cos 2\theta}} rdr$$

$$= a^2(\sqrt{3} - \frac{\pi}{3}).$$