第三节（2）
定积分的分部积分法

一、分部积分公式

设函数$u(x)$、$v(x)$在区间$[a,b]$上具有连续导数，

则$\int_a^b uv'dx = [uv]_a^b - \int_a^b vu'dx$，或$\int_a^b udv = [uv]_a^b - \int_a^b vdu$.

定积分的分部积分公式

推导 $(uv)' = u'v + uv'$， $\int_a^b (uv)'dx = [uv]_a^b$，

$[uv]_a^b = \int_a^b u'vdx + \int_a^b uv'dx$，$\therefore \int_a^b uv'dx = [uv]_a^b - \int_a^b u'vdx$，

或 $\int_a^b udv = [uv]_a^b - \int_a^b vdu$.

例1　计算 $\int_0^{\frac{1}{2}} \arcsin x dx$.

解　令　$u = \arcsin x$,　$v' dx = dx$,

则　$du = \dfrac{dx}{\sqrt{1-x^2}}$,　$v = x$,

$$\int_0^{\frac{1}{2}} \arcsin x dx = \left[x \arcsin x\right]_0^{\frac{1}{2}} - \int_0^{\frac{1}{2}} \frac{x dx}{\sqrt{1-x^2}}$$

$$= \frac{1}{2} \cdot \frac{\pi}{6} + \frac{1}{2} \int_0^{\frac{1}{2}} \frac{1}{\sqrt{1-x^2}} d(1-x^2)$$

$$= \frac{\pi}{12} + \left[\sqrt{1-x^2}\right]_0^{\frac{1}{2}} = \frac{\pi}{12} + \frac{\sqrt{3}}{2} - 1.$$

例2　计算 $\int_0^{\frac{\pi}{4}} \frac{xdx}{1+\cos 2x}$.

解　$\because 1+\cos 2x = 2\cos^2 x,$

$$\therefore \int_0^{\frac{\pi}{4}} \frac{xdx}{1+\cos 2x} = \int_0^{\frac{\pi}{4}} \frac{xdx}{2\cos^2 x} = \int_0^{\frac{\pi}{4}} \frac{x}{2} d(\tan x)$$

$$= \frac{1}{2}\left[x\tan x\right]_0^{\frac{\pi}{4}} - \frac{1}{2}\int_0^{\frac{\pi}{4}} \tan x dx$$

$$= \frac{\pi}{8} - \frac{1}{2}\left[\ln\sec x\right]_0^{\frac{\pi}{4}} = \frac{\pi}{8} - \frac{\ln 2}{4}.$$

例3 计算 $\int_0^1 \frac{\ln(1+x)}{(2+x)^2}dx.$

解 $\int_0^1 \frac{\ln(1+x)}{(2+x)^2}dx = -\int_0^1 \ln(1+x)d\frac{1}{2+x}$

$$= -\left[\frac{\ln(1+x)}{2+x}\right]_0^1 + \int_0^1 \frac{1}{2+x}d\ln(1+x)$$

$$= -\frac{\ln 2}{3} + \int_0^1 \boxed{\frac{1}{2+x}\cdot\frac{1}{1+x}}dx \qquad \frac{1}{1+x} - \frac{1}{2+x}$$

$$= -\frac{\ln 2}{3} + [\ln(1+x) - \ln(2+x)]_0^1 = \frac{5}{3}\ln 2 - \ln 3.$$

例4 设 $f(x)=\int_1^{x^2}\frac{\sin t}{t}dt$，求 $\int_0^1 xf(x)dx$.

解 因为$\frac{\sin t}{t}$没有初等形式的原函数，

无法直接求出 $f(x)$，所以采用分部积分法

$$\int_0^1 xf(x)dx=\frac{1}{2}\int_0^1 f(x)d(x^2)$$

$$=\frac{1}{2}\left[x^2f(x)\right]_0^1-\frac{1}{2}\int_0^1 x^2df(x)$$

$$=\frac{1}{2}f(1)-\frac{1}{2}\int_0^1 x^2f'(x)dx$$

$$\because\ f(x)=\int_1^{x^2}\frac{\sin t}{t}dt,\qquad f(1)=\int_1^1\frac{\sin t}{t}dt=0,$$

$$f'(x)=\frac{\sin x^2}{x^2}\cdot 2x=\frac{2\sin x^2}{x},$$

$$\therefore \int_0^1 xf(x)dx=\frac{1}{2}f(1)-\frac{1}{2}\int_0^1 x^2 f'(x)dx$$

$$=-\frac{1}{2}\int_0^1 2x\sin x^2dx=-\frac{1}{2}\int_0^1\sin x^2dx^2$$

$$=\frac{1}{2}\left[\cos x^2\right]_0^1=\frac{1}{2}(\cos 1-1).$$

例5　证明定积分公式

$$I_n=\int_0^{\frac{\pi}{2}}\sin^n x\,dx=\int_0^{\frac{\pi}{2}}\cos^n x\,dx$$

$$=\begin{cases}\dfrac{n-1}{n}\cdot\dfrac{n-3}{n-2}\cdots\cdots\dfrac{3}{4}\cdot\dfrac{1}{2}\cdot\dfrac{\pi}{2}, & n\text{为正偶数}\\[2ex] \dfrac{n-1}{n}\cdot\dfrac{n-3}{n-2}\cdots\cdots\dfrac{4}{5}\cdot\dfrac{2}{3}, & n\text{为大于1的正奇数}\end{cases}$$

证

$$I_n = \int_0^{\frac{\pi}{2}} \sin^n x dx = -\int_0^{\frac{\pi}{2}} \sin^{n-1} x \cdot d\cos x \qquad n > 1$$

$$= \underbrace{\left[-\sin^{n-1} x \cos x\right]_0^{\frac{\pi}{2}}}_{0} + (n-1)\int_0^{\frac{\pi}{2}} \sin^{n-2} x \underbrace{\cos^2 x}_{1-\sin^2 x} dx$$

$$I_n = (n-1)\int_0^{\frac{\pi}{2}} \sin^{n-2} x dx - (n-1)\int_0^{\frac{\pi}{2}} \sin^n x dx$$

$$= (n-1)I_{n-2} - (n-1)I_n$$

$$I_n = \frac{n-1}{n} I_{n-2}$$ 积分I_n关于下标的递推公式

$$I_{n-2} = \frac{n-3}{n-2} I_{n-4}$$ ……，直到下标减到0或1为止

$$I_{2m}=\frac{2m-1}{2m}\cdot\frac{2m-3}{2m-2}\cdot\cdots\cdot\frac{5}{6}\cdot\frac{3}{4}\cdot\frac{1}{2}I_0,$$

$$(m=1,2,\cdots)$$

$$I_{2m+1}=\frac{2m}{2m+1}\cdot\frac{2m-2}{2m-1}\cdot\cdots\cdot\frac{6}{7}\cdot\frac{4}{5}\cdot\frac{2}{3}I_1,$$

$$I_0=\int_0^{\frac{\pi}{2}}dx=\frac{\pi}{2},\qquad I_1=\int_0^{\frac{\pi}{2}}\sin x\,dx=1,$$

于是 $$I_{2m}=\frac{2m-1}{2m}\cdot\frac{2m-3}{2m-2}\cdot\cdots\cdot\frac{5}{6}\cdot\frac{3}{4}\cdot\frac{1}{2}\cdot\frac{\pi}{2},$$

$$I_{2m+1}=\frac{2m}{2m+1}\cdot\frac{2m-2}{2m-1}\cdot\cdots\cdot\frac{6}{7}\cdot\frac{4}{5}\cdot\frac{2}{3}.$$

例6：求 $\int_0^{\pi} \sin^6 \frac{x}{2} dx$.

解：$\int_0^{\pi} \sin^6 \frac{x}{2} dx = \int_0^{\frac{\pi}{2}} \sin^6 t d2t$

$= 2\int_0^{\frac{\pi}{2}} \sin^6 t dt \quad = 2\frac{5\cdot 3\cdot 1}{6\cdot 4\cdot 2}\frac{\pi}{2} = \frac{5\pi}{16}.$

练习：求 $\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} (\cos^5 x + \sin^9 x) dx.$

解：$\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} (\cos^5 x + \sin^9 x) dx = 2\int_0^{\frac{\pi}{2}} \cos^5 dx$

$= 2\frac{4\cdot 2}{5\cdot 3} = \frac{16}{15}.$

二、小结

定积分的分部积分公式

$$\int_a^b u\,dv = \left[uv\right]_a^b - \int_a^b v\,du.$$

（注意与不定积分分部积分法的区别）

思考题

设 $f''(x)$ 在 $[0,1]$ 上连续，且 $f(0)=1$，$f(2)=3$，$f'(2)=5$，求$\int_0^1 xf''(2x)dx$.

解: $\int_0^1 xf''(2x)dx = \frac{1}{2}\int_0^1 xdf'(2x)$

$$= \frac{1}{2}\left[xf'(2x)\right]_0^1 - \frac{1}{2}\int_0^1 f'(2x)dx = \frac{1}{2}f'(2) - \frac{1}{4}\left[f(2x)\right]_0^1$$

$$= \frac{5}{2} - \frac{1}{4}\left[f(2) - f(0)\right] = 2.$$

习题:

1、求 $I=\int_{-\sqrt{3}}^{\sqrt{3}}|\arctan x|dx$

解: $I=2\int_{0}^{\sqrt{3}}|\arctan x|dx=2\int_{0}^{\sqrt{3}}\arctan x dx$

$$=2[x\arctan x]|_{0}^{\sqrt{3}}-2\int_{0}^{\sqrt{3}}xd(\arctan x)$$

$$=2\sqrt{3}\arctan\sqrt{3}-2\int_{0}^{\sqrt{3}}\frac{xdx}{1+x^2}$$

$$=\frac{2\sqrt{3}\pi}{3}-\ln(1+x^2)|_{0}^{\sqrt{3}}=\frac{2\sqrt{3}\pi}{3}-\ln 4$$

2、计算定积分 $\int_{-2}^{2}(|x|+x)e^{-|x|}dx$

解： $|x|e^{-|x|}$为偶函数 $xe^{-|x|}$为奇函数

$$\text{原式}=2\int_0^2 xe^{-x}dx=-2\int_0^2 x\,de^{-x}=-2xe^{-x}\Big|_0^2+2\int_0^2 e^{-x}dx$$

$$=(-2xe^{-x}-2e^{-x})\Big|_0^2$$

$$=2-\frac{6}{e^2}$$

3、求 $\int_0^{\frac{\pi}{2}} e^x \sin x dx$

解：$I=\int_0^{\frac{\pi}{2}} \sin x de^x = e^x \sin x \Big|_0^{\frac{\pi}{2}} - \int_0^{\frac{\pi}{2}} e^x d \sin x$

$$= e^{\frac{\pi}{2}} - \int_0^{\frac{\pi}{2}} e^x \cos x dx = e^{\frac{\pi}{2}} - \int_0^{\frac{\pi}{2}} \cos x de^x$$

$$= e^{\frac{\pi}{2}} - (e^x \cos x \Big|_0^{\frac{\pi}{2}} - \int_0^{\frac{\pi}{2}} e^x d \cos x)$$

$$= e^{\frac{\pi}{2}} - (0 - 1 + \int_0^{\frac{\pi}{2}} e^x \sin x dx) = e^{\frac{\pi}{2}} + 1 - I$$

$$\therefore I = \frac{1}{2}(e^{\frac{\pi}{2}} + 1)$$

4、设f (x)在[0,a]上存在连续导数f(a)=0，$\max\limits_{[0,a]}|f'(x)|=M$

求证：$|\int_0^a f(x)dx|\le a^2M$ 。

5、设f(x)在[0,1]上连续，在(0,1)内可导，且f(0)=0，

0≤f ′(x)≤1，　求证：　$(\int_0^1 f(x)dx)^2\ge\int_0^1 f^3(x)dx$.

6、已知 $f(x)=\tan^2 x$, 求 $\int_0^{\frac{\pi}{4}} f'(x)f''(x)dx$.　　8

7、若 $f''(x)$在$[0,\pi]$连续，$f(0)=2$，$f(\pi)=1$，

证明：$\int_0^{\pi}[f(x)+f''(x)]\sin x dx=3$.

4、设f (x)在[0,a]上存在连续导数f(a)=0，$\max\limits_{[0,a]}|f'(x)|=M$

求证：$|\int_0^a f(x)dx|\le a^2M$ 。

证明：

方法1: $$|\int_0^a f(x)dx|=|\int_0^a[f(x)-f(a)]dx|=|\int_0^a f'(\xi)(x-a)dx|$$

$$\le M\int_0^a|x-a|dx=M\int_0^a(a-x)dx=\frac{a^2M}{2}$$

方法2: 分部积分 $$|\int_0^a f(x)dx|=|xf(x)\Big|_0^a-\int_0^a xf'(x)dx|$$

$$\le M\int_0^a xdx=\frac{a^2M}{2}$$

5、设f(x)在[0,1]上连续，在(0,1)内可导，且f(0)=0，

0≤f ′(x)≤1， 求证： $(\int_0^1 f(x)dx)^2 \geq \int_0^1 f^3(x)dx$.

证明：构造： $F(x)=(\int_0^x f(x)dx)^2-\int_0^x f^3(x)dx\ , F(0)=0$

$$F'(x)=2f(x)\int_0^x f(x)dx-f^3(x)=f(x)[2\int_0^x f(x)dx-f^2(x)],$$

$F'(0)=0$，令 $G(x)=2\int_0^x f(x)dx-f^2(x), G(0)=0,$

$$G'(x)=2f(x)-2f(x)f'(x)=2f(x)[1-f'(x)]\geq 0,$$

$\Rightarrow G(x)\geq G(0)=0, \Rightarrow F'(x)\geq 0,$ 即： $F(x)\uparrow$

$\Rightarrow F(x)\geq F(0)=0, \Rightarrow F(1)\geq 0,$ 即: $(\int_0^1 f(x)dx)^2 \geq \int_0^1 f^3(x)dx.$

6、已知 $f(x)=\tan^2 x$，求 $\int_0^{\frac{\pi}{4}} f'(x)f''(x)dx$.

$$\int_0^{\frac{\pi}{4}} f'(x)f''(x)dx=\int_0^{\frac{\pi}{4}} f'(x)df'(x)=\frac{1}{2}[f'(x)]^2\Big|_0^{\frac{\pi}{4}},$$

而 $f'(x)=2\tan t\sec^2 t$,

$$\int_0^{\frac{\pi}{4}} f'(x)f''(x)dx=2\tan^2 t\sec^4 t\Big|_0^{\frac{\pi}{4}}=2\times1\times(1+1)^2=8$$

7、若 $f''(x)$ 在 $[0,\pi]$ 连续，$f(0)=2$，$f(\pi)=1$，

证明：$\int_0^{\pi}[f(x)+f''(x)]\sin x dx = 3$.

证明：$\int_0^{\pi} f''(x)\sin x dx$

$$= \int_0^{\pi}\sin x df'(x) = \sin x f'(x)\Big|_0^{\pi} - \int_0^{\pi} f'(x)\cos x dx$$

$$= -\int_0^{\pi}\cos x df(x) = -\cos x f(x)\Big|_0^{\pi} - \int_0^{\pi} f(x)\sin x dx$$

$$= f(\pi)+f(0)-\int_0^{\pi} f(x)\sin x dx,$$

$$\Rightarrow \int_0^{\pi}[f(x)+f''(x)]\sin x dx = f(\pi)+f(0)=3.$$