Distinguishability of Conductivities by Electric Current Computed Tomography

DAVID ISAACSON

Abstract—We give criteria for the distinguishability of two different conductivity distributions inside a body by electric current computed tomography (ECCT) systems with a specified precision.

It is shown in a special case how these criteria can be used to determine the measurement precision needed to distinguish between two different conductivity distributions.

It is also shown how to select the patterns of current to apply to the body in order to best distinguish given conductivity distributions with an ECCT system of finite precision.

I. Introduction

The problem of reconstructing the conductivity inside a body from low-frequency measurements on the body's surface has been discussed in [1]–[10], [13]–[20]. Descriptions of impedance cameras designed for medical imaging are given in [11], [16].

The first purpose of this paper is to give precise definitions of what it means for two different conductivity distributions to be distinguishable by finite precision measurements. The second purpose is to give a careful discussion of how to select the patterns of current to apply to the surface of a body to best distinguish between two conductivity distributions by an ECCT system of finite precision. We also give simple examples that illustrate the definitions and the discussion of current pattern selection.

Since the conductivities of blood, muscle, and lung differ considerably, an approximate reconstruction of the conductivity throughout the thorax would contain morphological information as well as electrophysiological information needed to make more accurate solutions in the forward and inverse problems of electrocardiography [11], [21].

II. Description of Forward and Inverse Problems

Throughout this paper we let B denote a body and S its surface. We assume that B is a linear conductor with scalar conductivity $\sigma = \sigma(p)$ where p is a point in the body.

The voltage or potential $U = U(p)$ is assumed to satisfy

$$\nabla \cdot \sigma \nabla U = 0$$

in B. The current density vector is denoted by $J = J(p)$ and is given by

$$J \equiv -\sigma \nabla U.$$

In the forward problem we assume that σ and the normal component of the current density vector on the surface S are given.

In other words, on S we are given

$$J \cdot n = -\sigma \frac{\partial U}{\partial n} = j$$

where n denotes the unit outward normal vector on S.

If we specify U at one point in B, and we choose a j for which

$$\int_S j dA = 0$$

then there is a unique solution $U(p)$ to the preceding equations. We denote its restriction to the boundary S by

$$V = V(p) \equiv U(p)$$

for p on s.

Thus, the forward problem is to find the voltage V on the surface, given the conductivity σ and current j. When we want to emphasize that the voltage V is a linear functional of j and a nonlinear functional of σ, we write

$$V = V(p; \sigma, j).$$

This forward problem can be solved numerically by finite difference or finite element methods.

The inverse problem is to find the conductivity σ in B from a knowledge of

$$V_k \equiv V(p; \sigma, j_k)$$

for a sequence of currents j_k, $k = 1, 2, 3, \cdots$.

Descriptions of attempts to solve this inverse problem are in [1]–[10], [13]–[20].

III. Distinguishability

We say that two conductivities σ_1 and σ_2 are distinguishable (in the mean square sense) by measurements of precision ϵ iff there is a current j for which $\|j\| = 1$ and

$$\|V(\cdot\,; \sigma_1, j) - V(\cdot\,; \sigma_2, j)\| > \epsilon.$$

Here if $f = f(p)$ and $g(p)$ are any functions on S

$$\langle f, g \rangle \equiv \int_S \bar{f}(p)\, g(p)\, dA$$

Manuscript received November 27, 1985; revised January 27, 1986.

The author is with the Department of Mathematical Sciences and the Department of Computer Science, Rensselaer Polytechnic Institute, Troy, NY 12181.

IEEE Log Number 8607881.

0278-0062/86/0600-0091$01.00 © 1986 IEEE

and

$$\|f\| \equiv \left[\int_S |f(p)|^2 \, dA \right]^{1/2}.$$

We also say that the conductivities σ_1 and σ_2 are not distinguishable by measurements of precision ϵ iff for all j with $\|j\| = 1$ we have that

$$\|V(\cdot\,; \sigma_1, j) - V(\cdot\,; \sigma_2, j)\| \le \epsilon.$$

We illustrate these two definitions with a simple two-dimensional example [14], [15].

Let B be the disk $x^2 + y^2 < 1$ and S the circle $x^2 + y^2 = 1$. Using the polar coordinates $x = r \cos \theta$ and $y = r \sin \theta$, we define

$$\sigma_1 = \sigma_1(r, \theta) \equiv \begin{Bmatrix} \sigma & \text{if } 0 \le r \le R \\ 1 & \text{if } R < r \le 1 \end{Bmatrix}$$

and

$$\sigma_2 = \sigma_2(r, \theta) \equiv 1 \qquad \text{for } 0 \le r \le 1.$$

We next ask which inhomogeneous distributions σ_1 are distinguishable from the homogeneous distribution σ_2 by measurements of precision ϵ?

For $\sigma_2 \equiv 1$, we have that

$$\nabla \cdot \nabla U = 0.$$

If

$$j(\theta) = \sum_{n=1}^{\infty} C_n \cos n\theta + S_n \sin n\theta$$

where

$$C_n = \frac{1}{\pi} \int_0^{2\pi} j(\theta) \cos n\theta \, d\theta$$

$$S_n = \frac{1}{\pi} \int_0^{2\pi} j(\theta) \sin n\theta \, d\theta$$

and

$$U(r = 0, \theta) = 0,$$

then by separation of variables

$$U(r, \theta) = -\sum_{n=1}^{\infty} (r^n/n) \, [C_n \cos n\theta + S_n \sin n\theta]$$

and

$$V = V(\theta; \sigma_2, j) = -\sum_{n=1}^{\infty} n^{-1} [C_n \cos n\theta + S_n \sin n\theta].$$

The solution to the inhomogeneous problem is again found by separation of variables and by imposing the conditions

$$U(R^-, \theta) = U(R^+, \theta)$$

and

$$-\sigma \frac{\partial U}{\partial r} (R^-, \theta) = -1 \frac{\partial U}{\partial r} (R^+, \theta).$$

It is

$$V(\theta; \sigma_1, j) = -\sum_{n=1}^{\infty} \frac{1}{n} \frac{1 - \mu R^{2n}}{1 + \mu R^{2n}} \cdot \{C_n \cos n\theta + S_n \sin n\theta\}$$

where

$$\mu \equiv (\sigma - 1)/(\sigma + 1).$$

Note that if $\sigma > 0$, then $|\mu| < 1$.

To determine the distinguishability of σ_2 from σ_1, we compute

$$\|V(\cdot\,; \sigma_2, j) - V(\cdot\,; \sigma_1, j)\| = \sqrt{\pi}\, 2|\mu| \left[\sum_{n=1}^{\infty} (R^{2n}/n(1 + \mu R^{2n}))^2 \, (C_n^2 + S_n^2) \right]^{1/2}.$$

Since $R^{2n}/n(1 + \mu R^{2n})$ decreases as n increases, and since

$$1 = \|j\| = \sqrt{\pi} \sum_{n=1}^{\infty} (C_n^2 + S_n^2),$$

we have that

$$\|V(\cdot\,; \sigma_2, j) - V(\cdot\,; \sigma_1, j)\| \le 2|\mu| \, R^2/(1 + \mu R^2).$$

Thus, if

$$2|\mu| \, R^2/(1 + \mu R^2) \le \epsilon$$

then σ_1 and σ_2 are not distinguishable.

We note also that the currents $j(\theta) = (\cos \theta)/\sqrt{\pi}$ or $(\sin \theta)/\sqrt{\pi}$ will distinguish between σ_1 and σ_2 if

$$2|\mu| \, R^2/(1 + \mu R^2) > \epsilon.$$

Thus, the curve in the $\sigma - R$ plane given by

$$2|\mu| \, R^2/(1 + \mu R^2) = \epsilon$$

divides the plane into regions in which σ_1 and σ_2 are distinguishable by measurements of precision ϵ from those regions in which they are not distinguishable.

It follows that the disk of radius R and conductivity σ is indistinguishable from the homogeneous distribution with conductivity 1 if σ and R lie in the region

$$U \equiv \{(\sigma, R) | \sigma_m(R) \le \sigma \le \sigma_M(R), \text{ and } 0 \le R \le 1\}$$

where

$$\sigma_m(R) \equiv \max \{0, (R^2 - \epsilon/(2 + \epsilon))/(R^2 + \epsilon/(2 + \epsilon))\}$$

and

$$\sigma_M(R) \equiv \begin{cases} 1 + (2\epsilon/(2 - \epsilon))/(R^2 - \epsilon/(2 - \epsilon)), & \text{for } \dfrac{\epsilon}{2 - \epsilon} < R^2 \le 1 \\ +\infty, & \text{for } R^2 \le \dfrac{\epsilon}{2 - \epsilon} \end{cases}.$$

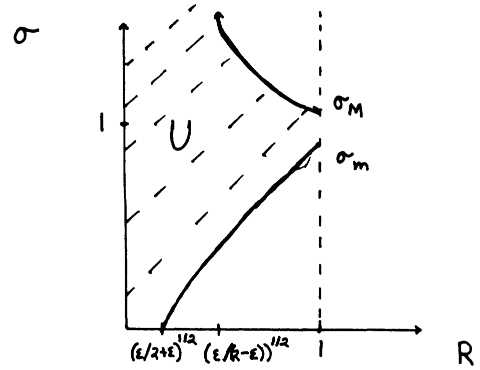

Fig. 1.

It also follows that disks which lie in the subset of $\{0 < \sigma;\ 0 \le R \le 1\}$ that is outside of U are distinguishable from the homogeneous distribution by measurements of precision ϵ. See Fig. 1 for a diagram of these regions.

This simple calculation tells us that we cannot distinguish any conductivity variations at all for disks of radius R_m or less where

$$R_m = [\epsilon/(2 + \epsilon)]^{1/2}.$$

Thus, if we want to see conductivity variations in disks of radius R, we need a precision ϵ smaller than

$$2R^2/(1 - R^2).$$

If $R = 0.1$, this expression says ϵ must be smaller than 0.025.

We suspect that to distinguish conductivity variations in regions of the same size but nearer the boundary, the same estimates will yield sufficient precision.

IV. Selection of Currents to Apply

In a certain sense we give an answer to the question: which currents j_k, $k = 1, 2, \cdots$, should we apply to S to best distinguish between two conductivities σ_1 and σ_2?

The current j_1 with $\|j_1\| = 1$ is said to be the current that best distinguishes between σ_1 and σ_2 iff

$$\|V(\cdot\,; \sigma_2, j_1) - V(\cdot\,; \sigma_1, j_1)\| = \max_{\|j\|=1} \|V(\cdot\,; \sigma_2, j) - V(\cdot\,; \sigma_1, j)\|.$$

Let $A(\sigma)$ denote the linear operator defined by

$$A(\sigma)\, j(p) \equiv V(p; \sigma, j).$$

The best current j is then the current j that maximizes

$$\|A(\sigma_2)\, j - A(\sigma_1)\, j\|^2$$

and has $\|j\| = 1$. Since $A(\sigma)$ is self-adjoint with respect to the inner product previously defined, we have that j maximizes the expression

$$\langle j, [A(\sigma_2) - A(\sigma_1)]^2 j\rangle / \langle j, j\rangle.$$

Let $D \equiv [A(\sigma_2) - A(\sigma_1)]$. In general D^2 is a compact self-adjoint nonnegative operator with a complete set of orthonormal eigenfunctions $j_1, j_2, j_3, \cdots$, and eigenvalues $\lambda_1^2 \ge \lambda_2^2 \ge \cdots \ge 0$. In other words,

$$Dj_k \equiv \lambda_k j_k$$

and

$$D^2 j_k = \lambda_k^2 j_k$$

for $k = 1, 2, 3, \cdots$.

It follows from the mini-max principal [12] that the best current (or currents) to distinguish between σ_1 and σ_2 are the eigenfunctions of D^2 having the largest eigenvalue. If, for example, λ_1^2 is nondegenerate, then j_1 would be the best current. It also follows that

$$\max_{\|j\|=1} \|[A(\sigma_2) - A(\sigma_1)]\, j\| = \|Dj_1\| = |\lambda_1|.$$

Thus, σ_1 is distinguishable from σ_2 by measurements of precision ϵ (using j_1) if $|\lambda_1| > \epsilon$, and they are not distinguishable if $|\lambda_1| \le \epsilon$.

The next best current we can apply to S to distinguish σ_1 from σ_2 is j_2 where

$$\|Dj_2\| = \max_{\substack{\|j\|=1 \\ \langle j, j_1\rangle = 0}} \|Dj\| = |\lambda_2|$$

(assuming λ's are nondegenerate).

This next eigenfunction will also distinguish between σ_1 and σ_2 to precision ϵ iff $|\lambda_2| > \epsilon$. By continuing this process we see from the mini-max principal [12] that the eigenfunctions $j_1, j_2, \cdots, j_k$ of D for which $|\lambda_k| > \epsilon$ and $|\lambda_{k+1}| \le \epsilon$ are in the least squares sense the best currents to apply to S in order to distinguish σ_1 from σ_2 by measurements of precision ϵ. It is pointless to use the eigenfunctions j_l with $l > k$ because their use would result in measurements of voltages whose differences would be smaller than our given precision.

We illustrate these ideas with a simple example. Let B, S, σ_1, and σ_2 be the same as before. Then the best currents to distinguish between σ_1 and σ_2 are the eigenfunctions of D^2 corresponding to its largest eigenvalue λ_1^2, where

$$\begin{aligned} D^2 j &= [A(\sigma_1) - A(\sigma_2)]^2 j \\ &= \sum_{n=1}^{\infty} [(-2\mu R^{2n})/n(1 + \mu R^{2n})]^2 \\ &\quad \cdot (C_n \cos n\theta + S_n \sin n\theta). \end{aligned}$$

and

$$j = \sum_{n=1}^{\infty} C_n \cos n\theta + S_n \sin n\theta.$$

The eigenvalues λ_n^2 are doubly degenerate and are given by

$$\lambda_n^2 = \left(\frac{1}{n}\right)^2 \left[\frac{2|\mu|\, R^{2n}}{1 + \mu R^{2n}}\right]^2$$

for $n = 1, 2, 3, \cdots$. The eigenfunctions corresponding to λ_n^2 are

$$j_n^c \equiv \cos n\theta/\sqrt{\pi}$$

and

$$j_n^s \equiv \sin n\theta/\sqrt{\pi}.$$

The largest eigenvalue is $\lambda_1^2 = [2|\mu| R^2/(1 + \mu R^2)]^2$ and there is a two-dimensional space of best eigenfunctions spanned by $\cos \theta/\sqrt{\pi}$ and $\sin \theta/\sqrt{\pi}$.

If our measurements are made with precision ϵ we only need the eigenfunctions with $|\lambda_n| > \epsilon$, i.e., those with

$$\frac{1}{n} \frac{2|\mu| R^{2n}}{1 + \mu R^{2n}} > \epsilon.$$

Since, for small R, this expression goes to zero rapidly as n increases, only a few functions will be effective. For example if $R = 0.1$, $\sigma = 2$, and $\epsilon = 10^{-3}$ only $\cos \theta/\sqrt{\pi}$ and $\sin \theta/\sqrt{\pi}$ will yield data of significance.

If B is the unit disk, $\sigma_1 = \sigma_1(r)$ and $\sigma_2 = \sigma_2(r)$, i.e., they are both independent of θ, then the eigenfunctions of D are again $\cos n\theta/\sqrt{\pi}$ and $\sin n\theta/\sqrt{\pi}$ for $n = 1, 2, \cdots$. For arbitrary functions, $\sigma_i = \sigma_i(r, \theta)$, $i = 1, 2$, the best currents are, in general, not trigonometric functions.

As a general rule, low (spatial) frequency currents yield voltages that are the most sensitive to changes in conductivity far from the boundary, while high-frequency currents yield voltages sensitive mostly to changes near the boundary.

We next point out that it is possible to use a set of currents j^+ and j^- that span the same space as the best currents j_1 and j_2 but whose use results in meaningless measurements.

Suppose $\|Dj_1\| > \epsilon$ and $\|Dj_2\| < \epsilon$. Choose an $\alpha > 0$ but small and define

$$j^{\pm} \equiv (\alpha j_1 \pm (1 - \alpha) j_2)/(\alpha^2 + (1 - \alpha)^2)^{1/2}.$$

Clearly, $j^{\pm}$ span the same space as j_1 and j_2 but

$$\|Dj^{\pm}\| = [((\alpha\lambda_1)^2 + ((1 - \alpha) \lambda_2)^2)/(\alpha^2 + (1 - \alpha)^2)]^{1/2} < \epsilon$$

if α is chosen small enough.

Voltage measurements $V(p;\, \sigma_1,\, j^{\pm})$ and $V(p;\, \sigma_2,\, j^{\pm})$ differ from each other by less than the precision ϵ and are therefore meaningless while the use of j_1 alone would have yielded a meaningful measurement in attempting to distinguish between conductivity distributions.

In practice, one applies currents to K electrodes. Thus, there are $K - 1$ independent currents. If the measurements were done with perfect precision ($\epsilon = 0$) any $K - 1$ independent currents would have the same ability to distinguish between conductivities. This example shows that when the precision is finite ($\epsilon > 0$) different $K - 1$ independent currents have different abilities to distinguish between conductivities. Thus, the use of the best currents is necessary in order to decide whether the conductivity inside the body is distinguishable from a conjectured conductivity by a system of finite precision.

V. Summary

A conductivity distribution σ is identifiable in a given class of functions Σ by measurements of precision ϵ iff it is distinguishable from each member of Σ by measurements of precision ϵ.

We point out that it is impossible to identify a conductivity distribution σ in Σ, much less image it, by an ECCT system if this system cannot distinguish σ from each member of Σ. Hence, a systematic manner in which one may go about designing an ECCT system is to first select the resolution that one would like by choosing the class Σ that is to be identifiable. Second, determine the precision of the instrument that is needed by first estimating the smallest size and conductivity variations possible in a member of Σ. Then use estimates similar to those given in Section III to determine the precision needed. Finally, use the best patterns of current described in Section IV.

Acknowledgment

The author would like to thank R. Kohn, M. Vogelius, J. Sylvester, G. Uhlmann, T. J. Yorkey, J. G. Webster, and W. J. Tompkins for informing him of their work and making some of their work available before publication.

He would like to thank the referees for informing him of the references [14]–[17] and for pointing out the similarity between the definition of distinguishability given here and the notion of visibility in [14], [15].

He would especially like to thank D. Angwin and J. Newell for their collaboration in designing and carrying out the experimental studies in conductivity imaging that inspired this work.

References

[1] R. E. Langer, "An inverse problem in differential equations," *Amer. Math. Soc. Bull. Ser 2*, vol. 29, pp. 814–820, 1933.

[2] R. J. Lytle and K. A. Dines, "An impedance camera: A system for determining the spatial variation of electrical conductivity," Lawrence Livermore Laboratory, Livermore, CA, Rep. UCRL-52413, 1978.

[3] R. H. Bates, G. C. McKinnon, and A. D. Seager, "A limitation on systems for imaging electrical conductivity distributions," *IEEE Trans. Biomed. Eng.*, vol. BME-27, pp. 418–420, July 1980.

[4] R. L. Parker, "The inverse problem of resistivity sounding," *Geophys.*, vol. 42, no. 12, pp. 2143–2158, Dec. 1984.

[5] H. Schomberg, "Nonlinear image reconstruction from projections of ultrasonic travel times and electric current densities," in *Mathematical Aspects of Computerized Tomography: Proceedings*, G. T. Herman and F. Natterer, Eds. New York: Springer-Verlag, 1981, pp. 270–291.

[6] A. P. Calderon, "On an inverse boundary problem," in *Seminar on Numerical Analysis and its Applications*, W. H. Meyer and M. A. Raupp, Eds. Rio de Janeiro, Brazil: Brazilian Math Society, 1980.

[7] R. V. Kohn and M. Vogelius, "Determining conductivity by boundary measurements," *Commun. Pure Appl. Math.*, vol. 37, pp. 289–298, 1984.

[8] ——, "Determining conductivity by boundary measurements II. Interior results," Univ. Maryland, Tech. Note BN-1028, 1984.

[9] J. Sylvester and G. Uhlmann, "A uniqueness theorem for an inverse boundary value problem in electrical prospection," preprint 1985.

[10] T. Muari and Y. Kagawa, "Electrical impedance computed tomography based on a finite element model," *IEEE Trans. Biomed Eng.*, vol. BME-32, no. 3, pp. 177–184, Mar. 1985.

[11] R. P. Henderson and J. G. Webster, "An impedance camera for spatially specific measurements of the thorax," *IEEE Trans. Biomed. Eng.*, vol. BME-25, no. 3, pp. 250–254, May 1978.

[12] R. Courant and D. Hilbert, *Methods of Mathematical Physics. Vol. I.* New York: Wiley, 1953, pp. 405–406.

[13] R. H. T. Bates, "Full-wave computed tomography, Part 1: Fundamental theory," *IEE Proc.*, vol. 131, A, no. 8, pp. 610–615, 1984.

[14] A. D. Seagar, T. S. Yeo, R. H. T. Bates, "Full-wave computed tomography, Part 2: Resolution limits," *IEE Proc.*, vol. 131, A, no. 8, pp. 616–622, 1984.

[15] A. D. Seagar and R. H. T. Bates, "Full-wave computed tomography, Part 4: Low frequency electric current CT," *IEE Proc.*, vol. 132, A, no. 7, pp. 455–466, 1985.

[16] D. C. Barber and B. H. Brown, "Applied potential tomography," *J. Phys. E: Sci. Instrum.*, vol. 17, pp. 723–733, 1984.

[17] Y. Kim, J. G. Webster, and W. J. Tompkins, "Electrical impedance imaging of the thorax," *J. Microwave Power*, vol. 18, no. 3, pp. 245–257, 1983.

[18] T. J. Yorkey, J. G. Webster, and W. J. Tompkins, "Errors caused by contact impedance in impedance imaging," in *Proc. IEEE 7th Conf. Eng. Medicine and Biol.*, 1985, pp. 632–637.

[19] K. A. Dines and R. J. Lytle, "Analysis of electrical conductivity imaging," in *Geophys.*, vol. 46, no. 7, pp. 1025–1036, 1981.

[20] T. J. Yorkey, "A quantitative comparison of impedance tomographic systems for a new instrument," Ph.D. proposal, Univ. Wisconsin, 1985.

[21] T. C. Pilkington and R. Polnsey, *Engineering Contributions to Biophysical Electrocardiography.* New York: IEEE Press, 1982.