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Abstract. We introduce a definition of 'best' currents to apply to an electrode array on the surface of a body in order to distinguish between the conductivity inside the body and a conjectured conductivity. Using these 'best' currents, we illustrate with a simple example the general fact that a single current applied between a pair of electrodes, loses its ability to distinguish between different conductivities as the size of the region over which the current is applied goes to zero. We next introduce approximations to the best currents on systems having L electrodes, and calculate the ability of these systems to distinguish between conductivities as L goes to infinity and the electrode size goes to zero. We conclude with a simple example that illustrates a process for producing the 'best' currents without a previous knowledge of what is inside the body.

1. Distinguishability and best currents

Seagar *et al* (1984) and Seagar and Bates (1985) introduced the notions of visibilities and sensitivities in order to have quantitative measures of the ability of different currents to distinguish between conductivities. We introduce a single number δ, referred to as distinguishability, that measures the ability of a pattern of currents to distinguish between two conductivities. The set of currents that maximise the distinguishability are called the 'best' currents and are shown to be the eigenfunctions with largest eigenvalue of a certain linear operator (Isaacson 1986). The best currents are used in sections 2 and 3 to study the distinguishability of currents applied by a pair of electrodes of size Δ or by many electrodes of size Δ as Δ goes to zero. In section 4 we describe an experimental process to produce the best currents.

We illustrate all the notions in this paper with the simple problem, studied by Seagar and co-workers and by Isaacson of trying to distinguish a circular region of radius r with $0 < r < 1$, and conductivity ω centred in a circle of radius 1 and conductivity 1 from a circular region of radius 1 whose conductivity is identically 1.

The conclusions of this paper have been proven for arbitrary geometries, conductivities, and more general notions of distinguishability (Gisser *et al* 1986).

Let B be a body and S its surface. Denote an arbitrary point in B by p. The conductivity throughout B is denoted by the function $\sigma = \sigma(p)$. Assume that the voltage $U(p)$ throughout B satisfies

$$\nabla \cdot \sigma \nabla U = 0 \tag{1}$$

in B. If the outward normal component of the current density vector on S is $j(p)$ then

$$-\sigma\, \partial U/\partial n = j(p) \tag{2}$$

on S. Here n is the unit outward normal to B on S.

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When $\int_S j\,\mathrm{d}A = 0$ and U is specified at one point in B then there is a unique solution $U(p)$ to equations (1) and (2). The voltage $V = V(p)$ that results at the surface S due to the current j is then given by solving equations (1) and (2). Thus for p on S

$$V(p) = U(p) = [R(\sigma)\, j](p)$$

Here $R(\sigma)$ denotes the linear operator that solves equations (1) and (2), i.e. $R(\sigma)\, j = V$. Note that $R(\sigma)$ depends non-linearly on the conductivity σ.

Two examples that will be used throughout this paper are given by taking B to be a disc of radius 1 so that S, its boundary, is a circle of radius 1. Denote two conductivities by σ_0 and σ_1 where $\sigma_1(p) = \omega > 0$ for $|p| < r$, $\sigma_1(p) = 1$ for $r \le |p| \le 1$, and $\sigma_0(p)$ denotes a conductivity identically 1 throughout B.

If we denote the current j on S by

$$j(\theta) = \sum_{n=1}^{\infty} C_n \cos n\theta + S_n \sin n\theta$$

where

$$C_n = \frac{1}{\pi} \int_0^{2\pi} j(\theta) \cos n\theta \,\mathrm{d}\theta$$

and

$$S_n = \frac{1}{\pi} \int_0^{2\pi} j(\theta) \sin n\theta \,\mathrm{d}\theta$$

then

$$R(\sigma_0)\, j = -\sum_{n=1}^{\infty} \frac{1}{n} (C_n \cos n\theta + S_n \sin n\theta)$$

and

$$R(\sigma_1)\, j = -\sum_{n=1}^{\infty} \frac{1}{n} \beta_n (C_n \cos n\theta + S_n \sin n\theta)$$

where

$$\beta_n = \frac{1 - \mu r^{2n}}{1 + \mu r^{2n}}$$

and

$$\mu = (\omega - 1)/(\omega + 1).$$

More generally if $F(p)$ and $G(p)$ are functions on any surface S let

$$\langle F, G \rangle = \int_S F(p) G(p) \,\mathrm{d}A$$

and

$$\|F\| = [\langle F, F \rangle]^{1/2} = [\int_S |F(p)|^2 \,\mathrm{d}A]^{1/2}$$

where $\mathrm{d}A$ denotes the element of area on S.

We say that two conductivities $\sigma(p)$ and $\tau(p)$ are distinguishable (in the mean square sense) by measurements of precision ε if there is a current j for which

$$\delta = \delta(j) = \frac{\|R(\sigma)\, j - R(\tau)\, j\|}{\|j\|} > \varepsilon$$

If there is no current j with $\|j\| < \infty$ for which $\delta(j) > \varepsilon$ (i.e. $\delta(j) \leq \varepsilon$ for all j with $\|j\| < \infty$) then σ and τ are not distinguishable (in the mean square sense) by measurements of precision ε.

For the circular example j, with

$$1 = \|j\|^2 = \pi \sum_{n=1}^{\infty} C_n^2 + S_n^2$$

distinguishes between σ_0 and σ_1 if

$$\varepsilon^2 < \delta^2 = \|R(\sigma_0)\, j - R(\sigma_1)\, j\|^2 = \pi 4 \mu^2 \left[\sum_{n=1}^{\infty} \alpha_n^2 (C_n^2 + S_n^2) \right] \tag{3}$$

where

$$\alpha_n = \frac{r^{2n}}{n(1 + \mu r^{2n})}$$

In general the 'best' currents, in the mean square sense, to distinguish σ from τ are those that maximise $\delta(j)$, i.e. $\tilde{j}$ is a 'best' current to distinguish σ from τ if $\|\tilde{j}\| = 1$ and

$$\delta(\tilde{j}) = \max_{\|j\|=1} \delta(j) = \max_j \frac{\|R(\sigma)\, j - R(\tau)\, j\|}{\|j\|} = \max_j \left[\frac{\langle j, D^2 j \rangle}{\langle j, j \rangle} \right]^{1/2}$$

where D is the linear operator given by

$$D = |R(\sigma) - R(\tau)|$$

D has a complete orthonormal set of eigenfunctions $j_1(p), j_2(p), j_3(p), \ldots$ whose eigenvalues $\delta_1 \geq \delta_2 \geq \delta_3 \geq \ldots$ satisfy $\lim_{n \to \infty} \delta_n = 0$.

From $Dj_k = \delta_k j_k$ for $k = 1, 2, \ldots$ and the min-max principal (Courant and Hilbert 1953)

$$\delta_1 = \max_{\|j\|=1} \delta(j) = \delta(j_1)$$

Thus the eigenfunctions of D corresponding to its largest eigenvalue δ_1 are the 'best' currents to distinguish σ from τ in the mean square sense.

For the special example of the disc

$$Dj = |R(\sigma_0) - R(\sigma_1)|\, j = \sum_{n=1}^{\infty} 2|\mu|\, \alpha_n (C_n \cos n\theta + S_n \sin n\theta)$$

$\delta_n = 2|\mu|\, \alpha_n$ with eigenfunctions $j_n^c = (\cos n\theta)/\sqrt{\pi}$ and $j_n^s = (\sin n\theta)/\sqrt{\pi}$. The best currents in this case are the linear combinations of the above functions with $n = 1$, i.e. $\cos n(\theta - \theta_0)/\sqrt{\pi}$ for any θ_0 (Isaacson 1986).

We remark that the best currents to distinguish arbitrary σ and τ are usually not trigonometric functions. Since they depend on the unknown σ in the body they cannot be determined without experiment.

2. Distinguishability by the current between a pair of electrodes

It is shown here that the ability of a current applied to the circle in the previous example with a pair of electrodes to distinguish between σ_0 and σ_1 goes to zero as the size of the electrodes goes to zero.

This is an example of the general theorem that the distinguishability of arbitrary conductivities in arbitrary geometries by a current applied with a pair of electrodes goes to zero as the electrode size goes to zero (Gisser *et al* 1986).

The electrodes may be placed on the arcs Δ_l of the circle where

$$l = 1, 2, \ldots, L, \ \Delta \equiv 2\pi/L, \ 0 \leq f \leq 1, \ \theta_l = (l-1)\Delta$$

and

$$\Delta_l = \{\theta | \theta_l - f\Delta/2 < \theta < \theta_l + f\Delta/2\}$$

If Γ is a sub-set of the unit circle let

$$X_\Gamma(p) = \begin{Bmatrix} 1 & \text{if } p \text{ is in } \Gamma \\ 0 & \text{if } p \text{ is not in } \Gamma \end{Bmatrix}$$

Any current applied with the pair of electrodes Δ_1 and Δ_l is of the form

$$j_\Delta(\theta) = j_\Delta(\theta)[X_{\Delta_1}(\theta) + X_{\Delta_l}(\theta)]$$

The distinguishability $\delta(j_\Delta)$ of σ_0 from σ_1 by j_Δ, is given by formula (3) when j_Δ is normalised so that $\|j_\Delta\| = 1$. Thus

$$\delta^2 = \pi 4\mu^2 \sum_{n=1}^{\infty} \alpha_n^2 (C_n^2 + S_n^2)$$

where

$$C_n = \langle j_\Delta, \cos n\theta \rangle / \pi \quad \text{and} \quad S_n = \langle j_\Delta, \sin n\theta \rangle / \pi.$$

We show δ goes to zero as Δ goes to zero.

Observe that since $j_\Delta = j_\Delta(X_{\Delta_1} + X_{\Delta_l})$

$$|C_n| = |\langle j_\Delta, (X_{\Delta_1} + X_{\Delta_l}) \cos n\theta \rangle| / \pi \leq \|j_\Delta\| \|(X_{\Delta_1} + X_{\Delta_l}) \cos n\theta\| / \pi \leq \sqrt{(2\Delta f)} / \pi$$

and similarly $|S_n| \leq \sqrt{(2\Delta f)}/\pi$.

Since $\alpha_1 \geq \alpha_2 \geq \ldots$, and $\lim_{n\to\infty} \alpha_n = 0$

$$\begin{aligned} \delta^2 &\leq \pi 4\mu^2 \sum_{n=1}^{N} \alpha_n^2 (C_n^2 + S_n^2) + \pi 4\mu^2 \alpha_{N+1}^2 \sum_{n=N+1}^{\infty} C_n^2 + S_n^2 \\ &\leq 16\mu^2 \alpha_1^2 N \cdot \Delta f/\pi + 4\mu^2 \alpha_{N+1}^2 \end{aligned}$$

This can be made less than any given ε by first choosing N so large that $4\mu^2\alpha_{N+1}^2 \leqslant \varepsilon/2$ and then choosing Δ so small that the first term is less than $\varepsilon/2$.

Thus $\lim_{\Delta\to 0} \delta(j_\Delta) = 0$ as claimed.

3. Distinguishability of approximations to the best currents on L electrodes

We introduce current densities j_L on L electrode systems that approximate the 'best' current densities for distinguishing σ_0 from σ_1 in the preceeding example.

We show that as the electrodes, of size $f \cdot 2\pi/L$, become smaller and more numerous the ability of j_L to distinguish σ_0 from σ_1 approaches $\sqrt{f}$ times the best currents ability to distinguish between these two conductivities.

In other words we show that $\lim_{L\to\infty} \delta(j_L) = \delta_1\sqrt{f}$. The same result holds for more general geometries and conductivities.

It was shown in section 1 that the best currents to distinguish σ_0 from σ_1 are of the form $\cos(\theta - \theta_0)/\sqrt{\pi}$. We next introduce L electrode approximations, j_L, to the 'best' current $j_B \equiv \cos\theta/\sqrt{\pi}$.

Let $P_L(\theta) = \sum_{l=1}^{L} X_{\Delta_l}(\theta)$. An 'L electrode' approximation to $j_B(\theta)$ is any current $j_L(\theta)$ for which

$$j_L(\theta) = \frac{j_B(\theta)\,P_L(\theta)}{\|j_B(\theta)\,P_L(\theta)\|} + \rho_L(\theta)$$

where $\int_0^{2\pi} j_L(\theta)\,\mathrm{d}\theta = 0$, $\|j_L\| = 1$, and $\rho_L(\theta)$ denotes a function for which

$$\lim_{L\to\infty} \|\rho_L\| = 0$$

One example of such a function results from taking j_l to be the average of $j_B(\theta)$ over the arc of length Δ centred at θ_l, and defining

$$j_L(\theta) = \left[\sum_{l=1}^{L} j_l X_{\Delta_l}(\theta)\right] \Big/ \left[\left\|\sum_{l=1}^{L} j_l X_{\Delta_l}\right\|\right]$$

We next prove that

$$\lim_{L\to\infty} \delta(j_L) = \sqrt{f}\,\delta(j_B)$$

This will follow from equation (3) by showing that

$$\lim_{L\to\infty} C_n = \lim_{L\to\infty} \langle j_L, \cos n\theta\rangle/\pi = \begin{cases} \sqrt{f}/\sqrt{\pi} & \text{if } n=1 \\ 0 & \text{if } n>1 \end{cases}$$

and $\lim_{n\to\infty} S_n = 0$.

From the definition of j_L

$$\pi C_n = \langle j_B P_L/(\|j_B P_L\|) + \rho_L, \cos n\theta\rangle = \frac{\langle j_B P_L, \cos n\theta\rangle}{[\langle j_B P_L, j_B P_L\rangle]^{1/2}} + \langle \rho_L, \cos n\theta\rangle$$

Since $|\langle \rho_L, \cos n\theta\rangle| \le \|\rho_L\|\sqrt{\pi} \to 0$ as $L\to\infty$ we only have to show

$$\lim_{L\to\infty} \frac{\langle \cos\theta P_L, \cos n\theta\rangle}{[\langle \cos\theta, P_L\cos\theta\rangle]^{1/2}} = \begin{cases} \sqrt{(f\pi)} & \text{if } n=1 \\ 0 & \text{if } n>1 \end{cases}$$

From the mean value theorem

$$\langle \cos\theta, P_L \cos n\theta\rangle = \sum_{l=1}^{L} \int_{\Delta_l} \cos\theta\,\cos n\theta\,\mathrm{d}\theta = f\sum_{l=1}^{L} \cos\xi_l \cos n\xi_l \Delta$$

where ξ_l is some point in Δ_l, $\Delta = 2\pi/L$, and

$$\langle \cos\theta, P_L\cos\theta\rangle = f\sum_{l=1}^{L} \cos^2 \eta_l \Delta$$

Thus

$$\lim_{L\to\infty} \pi C_n = \sqrt{f}\cdot\int_0^{2\pi} \cos\theta\,\cos n\theta\,\mathrm{d}\theta/[\int_0^{2\pi}\cos^2\theta\,\mathrm{d}\theta]^{1/2} = \begin{cases} \sqrt{f}\cdot\sqrt{\pi} & \text{if } n=1 \\ 0 & \text{if } n>1 \end{cases}.$$

A similar calculation shows $\lim_{L\to\infty} S_n = 0$ and so by the argument in section 2 $\lim_{L\to\infty} \delta(j_L) = \sqrt{f}\,\delta_1$.

4. An experimental process for producing approximations to the 'best' currents

We describe a simple version of a process for experimentally producing an approximation to one of the 'best' currents, j_B, that distinguish between σ_0 and σ_1:

(1) Guess any current density $j_0(\theta)$ for which $\int_0^{2\pi} j_0(\theta)\,\mathrm{d}\theta = 0$ and $\|j_0\| = 1$. Set $k = 0$.

(2) *Measure* the voltage

$$V_k^1(\theta) = R(\sigma_1)\, j_k(\theta)$$

that results from applying $j_k(\theta)$ to the body whose conductivity is $\sigma_1(p)$.
(3) *Compute* the voltage

$$V_k^0(\theta) = R(\sigma_0)\, j_k(\theta)$$

that would result from applying $j_k(\theta)$ to a body with conductivity $\sigma_0(p)$.
(4) Compute the new estimate, $j_{k+1}(\theta)$, to the 'best' current to distinguish σ_0 from σ_1 by

$$j_{k+1}(\theta) = \frac{V_k^1(\theta) - V_k^0(\theta)}{\|V_k^1(\theta) - V_k^0(\theta)\|}$$

(5) If the changes in j_k are less than the measurement precision ε, i.e.

$$\|j_{k+1} - j_k\| < \varepsilon$$

stop, otherwise increment k and go to 2.

In this simple case we show by explicit computation that (when $\langle j_0, j_B \rangle \neq 0$)

$$\lim_{k \to \infty} j_k(\theta) = j_B$$

Take $D = R(\sigma_1) - R(\sigma_0)$. Then

$$j_1 = Dj_0/\|Dj_0\|$$

$$j_2 = Dj_1/\|Dj_1\| = D^2 j_0/\|D^2 j_0\|$$

$$j_3 = D^3 j_0/\|D^3 j_0\|$$

$$\vdots$$

$$j_k = D^k j_0/\|D^k j_0\|$$

$$= \frac{\sum_{n=1}^{\infty} (2\mu\alpha_n)^k [C_n \cos n\theta + S_n \sin n\theta]}{\left[\pi \sum_{n=1}^{\infty} (2\mu\alpha_n)^{2k} (C_n^2 + S_n^2)\right]^{1/2}} \tag{4}$$

where $\pi C_n = \langle j_0, \cos n\theta \rangle$ and $\pi S_n = \langle j_0, \sin n\theta \rangle$.

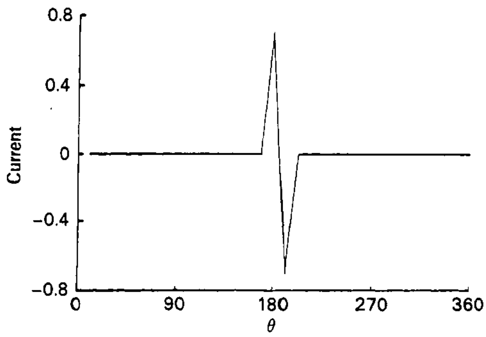

Figure 1. Graph of j_0, a single current applied between a pair of adjacent electrodes.

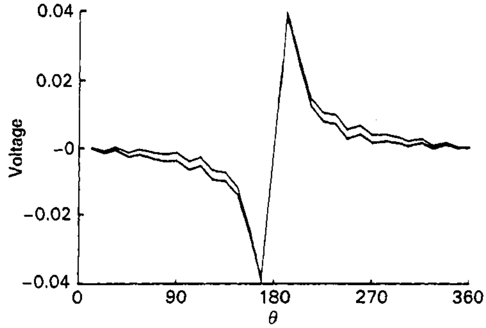

Figure 2. Graph of the voltages V_0^1 and V_0^0 that result from applying j_0 to the disc B with conductivities σ_1 or σ_0 respectively.

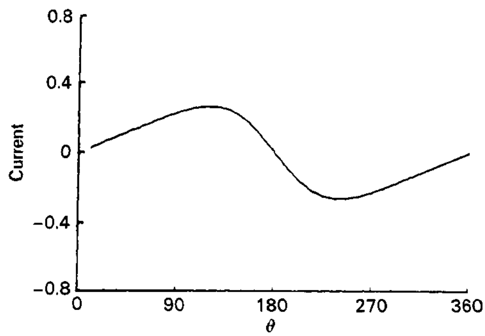

Figure 3. Graph of the first iterate j_1.

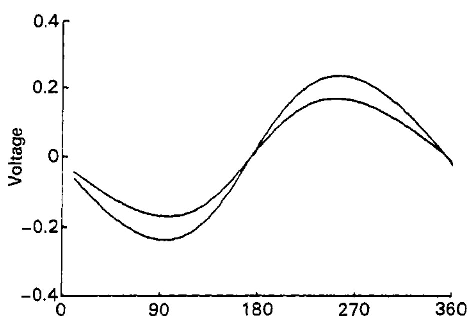

Figure 4. Graph of the voltages V_1^1 and V_1^0 it gives rise to on S. Note difference in scale from Fig. 2.

Since $\alpha_1 > \alpha_2 > \ldots$, the leading terms, as $k \to \infty$, in the numerator and denominator are the $n = 1$ terms. It follows from equation (4) that

$$\|j_k - \operatorname{sgn}(\mu)[C_1 \cos\theta + S_1 \sin\theta]/\sqrt{\pi(C_1^2 + S_1^2)}\| = 0((\alpha_2/\alpha_1)^k)$$

Thus

$$\lim_{k \to \infty} \|j_k - j_B\| = 0$$

The convergence of this process is illustrated in figures 1–6 and the table for the examples with $\omega = 5$, $r = 0.5$ (in figures 1–5) or 0.9 (figure 6), and $L = 32$.

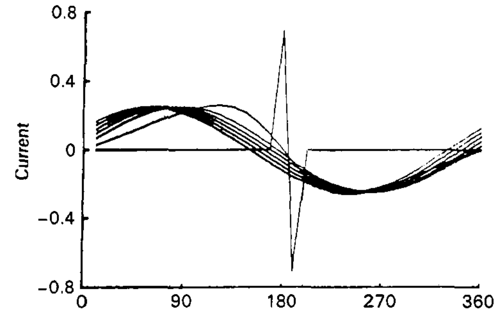

Figure 5. Graph of j_0, j_1, j_2, j_3, j_4, j_5 for $r = 0.5$.

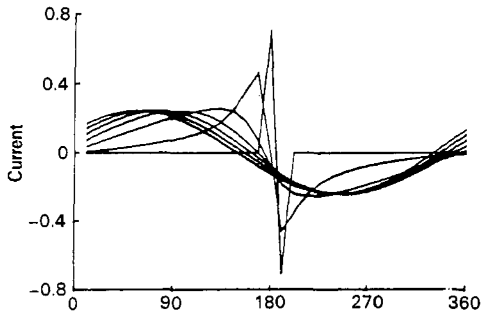

Figure 6. As figure 5 for $r = 0.9$.

Table 1. Shows how the distinguishabilities approximate their maximum possible values in the column labeled ∞.

Current j_k for $k =$	0	1	2	3	4	5	∞
$r = 0.5$, $\omega = 5$, $\delta(j_k) =$	0.011	0.273	0.281	0.282	0.282	0.283	0.286
$r = 0.9$, $\omega = 5$, $\delta(j_k) =$	0.046	0.415	0.651	0.687	0.693	0.695	0.701

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