Bachelier Ingénieur Civil

SYST0022-1 | Linear System Design

Lab 2
Cart-pole system : PID design via loop-shaping

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1) Block Diagram

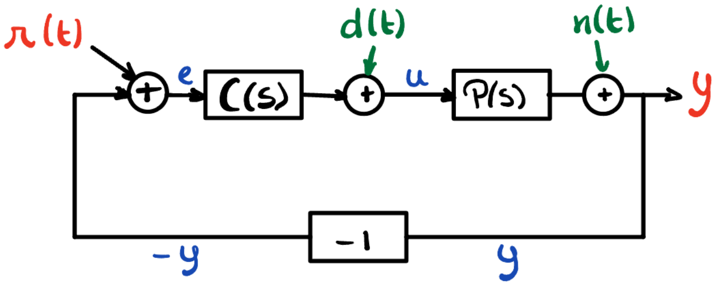

Figure 1: Block diagram of the cart-pole system, where $P(s)$ is the plant, $C(s)$ the controller, $u(t)$ controlled input, $y(t)$ the controlled output, $r(t)$ the reference, $d(t)$ the load disturbance, and $n(t)$ the measurement noise.

From the block diagram of the above Figure 1, we can write the expression of $Y(s)$ as

$$Y(s) = \frac{PC}{1+PC}R(s) + \frac{P}{1+PC}D(s) + \frac{1}{1+PC}N(s) \tag{1.1}$$

where R(s), D(s) and N(s) are the Laplace transforms of the reference r(t) , the load disturbance d(t) and the noise measurement n(t) respectively.

2) Gang of Four

The "Gang of Four" refers to the four transfer functions that govern the behavior of a closed-loop control system. They describe how the different inputs influence the outputs of the system.

- **Sensitivity Function**

$$S(s) = \frac{1}{1+PC} \tag{2.1}$$

Describes how the measurement noise $n(t)$ propagates to the controlled output $y(t)$, which we could write G_{yn}

- **Complementary Sensitivity Function**

$$T(s) = \frac{PC}{1+PC} \tag{2.2}$$

Describes how the reference $r(t)$ propagates to the controlled output $y(t)$, which we could write G_{yr}

- **Load Sensitivity Function**

$$PS(s) = \frac{P}{1+PC} \tag{2.3}$$

Describes how the load disturbance $d(t)$ propagates to the controlled output $y(t)$, which we could write G_{yd}

- **Noise Sensitivity Function**

$$CS(s) = \frac{PC}{1+PC} \tag{2.4}$$

Describes how the measurement noise $n(t)$ propagates to the control input $u(t)$, which we could write G_{un}

3) Plant transfer function poles and zeros

The plant transfer function $P(s)$ we are given is

$$P(s) = \frac{1}{l \cdot m_c \cdot s^2 - g \cdot (m_c + m_p)} \tag{3.1}$$

From this, we can compute the poles by looking at the roots of the denominator:

$$\begin{aligned} l \cdot m_c \cdot s^2 - g \cdot (m_c + m_p) &= 0 \\ s^2 &= \frac{g \cdot (m_c + m_p)}{l \cdot m_c} \\ s &= \pm\sqrt{\frac{g(m_c + m_p)}{lm_c}} \end{aligned} \tag{3.2}$$

Since the numerator is 1, this transfer function has no zeros.

We can not assess the stability of the system using the Nyquist plot of the plant transfer function, as the simplified Nyquist criterion requires the open-loop transfer function $L(s) = P(s)C(s)$ to be plotted, rather than only the plant transfer function $P(s)$.

4) Plant Transfer Function stability

Looking at the poles of $P(s)$ in the Equation 3.2, we notice there is one pole in the right-half-plane, meaning the system without control is unstable.

Now looking at the Figure 2, the Nyquist plot and Bode plots do not show signs of instability, as there is no net encirclement of the $(-1, 0)$ point in the Nyquist plot, nor a peak in the Bode amplitude diagram.
However, looking at the time response graph of the system to a finite disturbance $d(t)$, we see that the response has an exponential and unbounded growth, as it takes values up to $10^{28}[rad]$. This means the open-loop system, without control, is unstable. This is because in a stable system, a bounded excitation should cause a bounded response.

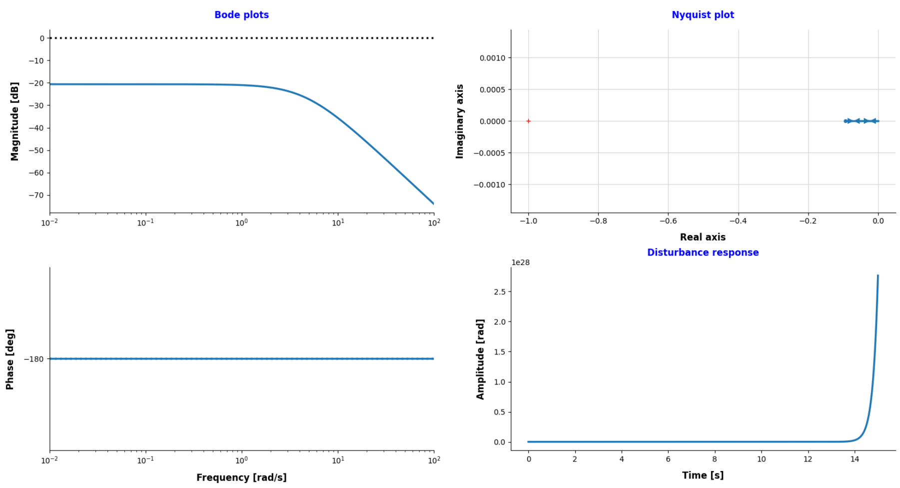

Figure 2: From top to bottom, left to right: the Bode plots, magnitude and phase, the Nyquist plot and the time response of the open-loop system to a disturbance.

5) Phase margin determination

The phase margin φ_m is a sense of how much delay is acceptable until the system becomes unstable due to that delay.
To be able to accept a delay $\tau_{delay} = 20ms$ without instability in the system, φ_m must be greater than the phase lag φ_{delay} it causes.

To compute the phase lag τ_{delay} creates, we use the Laplace transform property and get

$$\begin{aligned} y(t-\tau) &\overset{\mathcal{L}}{\Leftrightarrow} Y(s)e^{-s\tau} \\ s &= j\omega_c \\ \varphi_{delay} &= -\omega_c \tau_{delay} \end{aligned} \tag{5.1}$$

We thus want the phase margin $\varphi_m \geq |\varphi_{delay}|$ so that the system can handle this delay without becoming unstable.
Using the numerical data given, we get

$$\varphi_m \geq 2\omega^* \cdot \tau_{delay} = 8\pi \cdot 0.02 = 0.503\text{rad} \tag{5.2}$$

6) Determination of m for the lead compensator

We want a phase margin of φ_m at the cutoff frequency ω_c. We can combine the assumption we were given and the expression of the phase margin at a given frequency to give us the frequency boost we need with the lead compensator.

$$\begin{aligned} \varphi_m = \angle L(j\omega_c) + \pi = \angle P(j\omega_c)C(j\omega_c) + \pi &= \angle P(j\omega_c) + \angle C(j\omega_c) + \pi \\ \angle P(j\omega_c) &= -\pi \\ \varphi_m &= \angle C(j\omega_c) \end{aligned} \tag{6.1}$$

From this, we can compute the value of m to achieve a maximum phase boost of φ_m

$$\begin{aligned} \phi_{max} = \varphi_m = \sin^{-1}(\frac{m-1}{m+1}) \\ \Leftrightarrow m = \frac{1+\sin\varphi_m}{1-\sin\varphi_m} = 2.86 \end{aligned} \tag{6.2}$$

We double the value obtained above to get a delay margin, and get the final value for m

$$m_{final} = 2 \cdot m = 5.72 \tag{6.3}$$

7) Lead compensator pole and zero determination

Starting from the general expression of the lead compensator,

$$C_{\text{lead}}(s) = K\frac{s+z}{s+p} \tag{7.1}$$

We get that the pole is located in $s_{\text{pole}} = -p$ and the zero in $s_{\text{zero}} = -z$. The maximum phase boost occurs in ω_{max} given by

$$\omega_{max} = \sqrt{|z||p|} = \sqrt{zp} \tag{7.2}$$

So, setting the maximum phase boost occuring at the gain crossover frequency ω_{gc}, and using $|p| = m|z|$ we get

$$\omega_{max} = \omega_{gc} = \sqrt{mz^2} \tag{7.3}$$

And end up with

$$z = \frac{\omega_{gc}}{\sqrt{m}} \quad ; \quad p = \omega_{gc}\sqrt{m} \tag{7.4}$$

8) Determination of the Lead Compensator Gain K

We want the gain crossover frequency ω_{gc} to be at the cutoff frequency $\omega_c = 8\pi$rad/s, with $|P(j\omega_c) = -50.2808$dB, so we have

$$\begin{aligned} |P(j\omega_c)C(j\omega_c)| = |P(j\omega_c)||C(j\omega_c)| = 10^{\frac{-50.2828}{20}}|C(j\omega_c)| = 1 \\ \text{with} \quad |C(j\omega_c)| = K\left|\frac{j\omega_c+z}{j\omega_c+p}\right| \end{aligned} \tag{8.1}$$

Using the expressions of p and z from Equation 7.4, in which we replace ω_{gc} with ω_c computed earlier, we get

$$K = 10^{\frac{50.2828}{20}}\left|\frac{j\omega_c+p}{j\omega_c+z}\right| = 10^{\frac{50.2828}{20}} \cdot 2.39 = 781.336 \tag{8.2}$$

with $z = 10.51$ and $p = 60.11$.

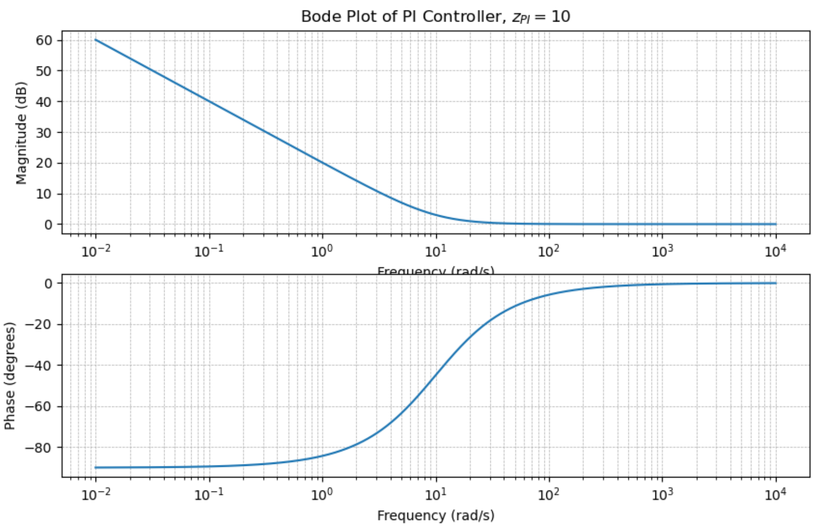

Figure 3: Bode plots of a PI controller, with $z_{PI} = 10$

9) Determination of the PI parameter z_{PI}

A PI controller writes

$$C_{PI}(s) = \frac{s + z_{PI}}{s} \tag{9.1}$$

The Bode plots of this transfer function are shown in Figure 3.
To tune a good value of z_{PI}, we have to look at the effects of introducing a PI controller to the system :

- **Low frequency rejection** : at low frequencies (under z_{PI}), the PI controller amplifies the response. To reject most effectively the disturbances z_{PI} has to be chosen large enough so that the disturbance frequency ω^* is in the amplification region.
- **Phase lag** : a PI controller introduces a phase lag of -90° at low frequencies, which vanishes at high frequencies. To maintain the stability, z_{PI} has to be significantly smaller than the disturbance frequency ω^* so that the phase lag caused at ω^* does not degrade the phase margin.

This creates a trade-off: z_{PI} must be high enough for disturbance rejection but low enough to prevent excessive phase lag and instability.
However, as ω^* is only half of the ω_c, it is too difficult to get an amplification of the response without introducing an important phase lag.
I thus chose a value of $z_{PI} = \frac{\omega_c}{10} = 2.5$rad/s so that no phase lag is introduced, while still allowing disturbance rejection over a wide range of frequencies (although not rejecting at the disturbance frequency ω^*).

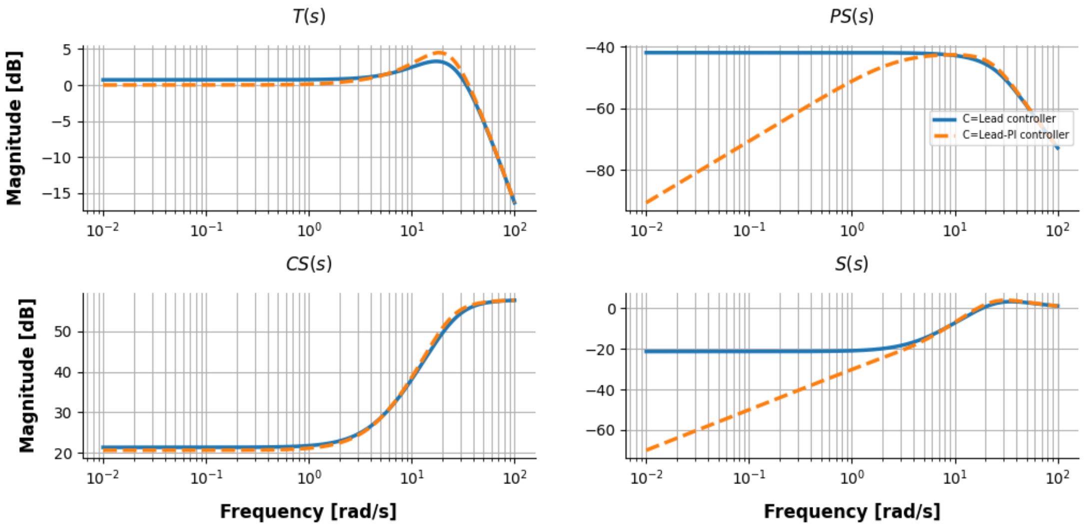

Figure 4: Bode magnitude plots of the "Gang of 4" transfer functions of the controller, assessing the performance of the introduction of a PI block

10) Effect of the introduction of a PI block on system response

Looking at the Bode magnitude plots of the Gang of Four of the controlled system, shown in Figure 4, we can assess the performance of the introduction of a PI block in our system.

- **Noise and Complementary Sensitivity Functions (T(s) & CS(s))**: These 2 transfer functions are nearly unchanged by the introduction of a PI block. This means the system ability to track the reference and to reject noise at the control input is not greatly impacted.
- **Load Sensitivity Function PS(s)**: Introducing a PI block in the system further reduces the response to disturbances at low frequencies ($\leq$ 4rad/s). The system with a PI block thus rejects better the low frequency disturbances and is almost unsensitive to them. However, since disturbances were already attenuated by 40 dB, the PI block does not noticeably impact the overall load sensitivity. The behavior at higher frequencies is not affected by the PI block.
- **Sensitivity Function S(s)**: Introducing a PI block also reduces the response to measurement noise at low frequencies ($\leq$ 4rad/s), although such noise is quite rare. This means that if there were to be some low-frequency noise, it would be better rejected and impact less the closed-loop system with a PI block. The behavior at higher frequencies is not affected by the PI block.