M5A44 COMPUTATIONAL STOCHASTIC PROCESSES
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PROJECT 2
HANDED OUT: 21/02/2014
DEADLINE FOR HANDING IN: 07/03/2014
PLEASE MAKE SURE TO INCLUDE ALL THE CODE THAT YOU HAVE WRITTEN.
YOU MUST SUBMIT A SINGLE DOCUMENT WHICH SHOULD START WITH A DECLARATION THAT THIS IS ALL YOUR OWN WORK.

1. Consider the stochastic differential equation in $\mathbb{R}^2$

$$d\mathbf{X}_t = \mathbf{v}(t, \mathbf{X}_t)\, dt + \sqrt{2\kappa}\, dW_t, \quad \mathbf{X}_0 = \mathbf{x}, \tag{1}$$

where W_t denotes standard Brownian motion on $\mathbb{R}^2$. Let $\mathbf{X}_t = (X_t,\, Y_t)$ and $\mathbf{x} = (x, y)$.

   (a) Solve (1) for

$$\mathbf{v}(t, \mathbf{x}) = (0,\, \sin(x)\sin(\omega t)).$$

   Calculate

$$D(t) = \frac{\text{Var}(Y_t)}{2t}.$$

   Show numerically that it becomes constant, the **effective diffusion coefficient** D when t is sufficiently long. Study numerically the dependence of D on κ and ω. Generate plots of D as a function of κ for $\omega = 1$, for $\kappa \in [10^{-2},\, 10^2]$, and of D as a function of ω for $\kappa = 0.1$ for $\omega \in [10^{-2},\, 10^2]$. You can take the initial conditions for (1) to be either deterministic or random.

   (b) Repeat the same experiment for

$$\mathbf{v}(t, \mathbf{x}) = (0,\, \sin(x)\eta_t),$$

   where η_t is the solution of the SDE

$$d\eta_t = -\alpha\eta_t\, dt + \sqrt{2\alpha}\, dW_t, \quad \eta_0 \sim \mathcal{N}(0, 1).$$

   Study the dependence of the effective diffusion coefficient on κ and on α.

2. Consider the system of SDEs

$$\frac{dx_t}{dt} = \frac{x_t y_t}{\epsilon} - y_t^2 x_t^3, \quad x_0 = 1, \tag{2a}$$

$$dy_t = -\frac{1}{\epsilon^2} y_t \, dt + \sqrt{2}\frac{1}{\epsilon} \, dW_t, \quad y_0 \sim \mathcal{N}(0,1). \tag{2b}$$

In the limit as $\epsilon \to 0$ the solution of (2a), x_t converges to the solution of the SDE

$$dX_t = (X_t - X_t^3) \, dt + \sqrt{2} X_t \, dW_t, \quad X_0 = 1. \tag{2c}$$

Investigate this limit numerically:

(a) Solve (2) for ϵ sufficiently small to show that x_t is close to X_t. Solve the equations over $[0, T]$ with $T = 10$ with a sufficiently small stepsize and an appropriate numerical method. Discuss about the choice of the numerical method.

(b) Calculate $\text{err}(\epsilon) = \mathbb{E}|x_T - X_T|^2$ as a function of ϵ. Show numerically that $\text{err}(\epsilon) \approx C\epsilon^\gamma$ for ϵ sufficiently small the estimate the exponent γ.

(c) Investigate numerically the accuracy of the approximation of x_t by X_t as T increases.

3. Consider the scalar SDE

$$dX_t = b(X_t) \, dt + \sigma(X_t) \, dW_t, \quad X_0 = x.$$

The θ Milstein scheme is

$$X_{n+1} = X_n + \big(\theta \, b_{n+1} + (1-\theta) b_n\big) \, \Delta t + \sigma_n \, \Delta W_n + \frac{1}{2}\sigma_n \sigma_n' \big((\Delta W_n)^2 - \Delta t\big), \tag{3}$$

where $\theta \in [0,1]$ and $X_n = X(n\Delta t)$, $b_n = b(X_n)$, $\sigma_n = \sigma(X_n)$. Consider the SDE

$$dX_t = \lambda X_t \, dt + \sigma X_t \, dW_t, \tag{4}$$

where λ, $\sigma \in \mathbb{C}$.

(a) Obtain a formula for $\mathbb{E}|X_t|^2 = \mathbb{E} X_t \overline{X_t}$ where $\overline{X_t}$ denotes complex conjugate. Show that X_t is mean square stable provided that

$$2\,\text{Re}(\lambda) + |\sigma|^2 < 0.$$

(b) Apply the θ Milstein scheme to (4). Show that it can be written in the form

$$X_{n+1} = G(\Delta t, \lambda, \sigma, \theta) X_n$$

and obtain a formula for $G(\Delta t, \lambda, \sigma, \theta)$. Let $Z_n = \mathbb{E} X_n^2$. Use the previous calculation to obtain an equation of the form

$$Z_{n+1} = R(\Delta t, \lambda, \sigma, \theta) Z_n.$$

(c) Investigate the region of mean square stability for the θ Milstein scheme. Compare the stability region of the numerical scheme with the stability region of the SDE. For what values of θ is the θ Milstein scheme A-stable?

(d) Test your results for the SDEs

$$dX_t = -100X_t\,dt + 10X_t\,dW_t$$

and

$$dX_t = (-100 + 100i)X_t\,dt + 10X_t\,dW_t.$$

Comment on your results.

4. **MASTERY COMPONENT** The following SDE appears in population dynamics:

$$dX_t = -\lambda X_t(1 - X_t)\,dt - \mu X_t(1 - X_t)\,dW_t \tag{5}$$

(a) Show that $X_t = 1$ is a fixed point for (5) and that linearizing about this fixed point we obtain the SDE for geometric Brownian motion.

(b) Solve (5) numerically using the explicit Euler and Milstein schemes for $\lambda = -1$, $X_0 = 1.1$ and for $\mu = 0.5, 0.6, 0.7, 0.8, 0.9$. Calculate numerically $\mathbb{E}(X_t - 1)^2$. Comment on the mean square stability of the explicit Euler and Milstein schemes for the nonlinear SDE (5).

(c) Solve (5) using the theta scheme with $\theta = \frac{1}{2}$. Investigate the mean square stability of this numerical scheme when applied to (5).