M5A44 COMPUTATIONAL STOCHASTIC PROCESSES
Dr G.A. PAVLIOTIS
PROJECT 1
HANDED OUT: 03/30/2014
DEADLINE FOR HANDING IN: 14/02/2014
PLEASE MAKE SURE TO INCLUDE ALL THE CODE THAT YOU HAVE WRITTEN

1. Consider the function
$$f(x,y) = e^{-\beta V_1(x,y)} + e^{-\beta V_2(x,y)}$$
where $\beta > 0$ and
$$V_1(x,y) = (x-0.5)^2 - \frac{1}{2}(y+0.1)^4, \quad V_2(x,y) = 0.75(x+0.4)^2 - (y-0.5)^4.$$

   (a) Use a Monte-Carlo algorithm to estimate the integral
   $$Z = \int_{-1}^{1}\int_{-1}^{1} f(x,y)\,dxdy, \tag{1}$$
   for $\beta = 10$.

   (b) Plot the variance of the estimator as a function of the number of random samples that you are generating. Use a deterministic numerical method to estimate Z and calculate the error as a function of the number of random samples.

   (c) Choose an appropriate distribution $\psi(x,y)$ and estimate Z by using importance sampling. Justify the choice of $\psi(x,y)$ and plot the variance and the error of the estimator as a function of the number of samples.

   (d) Define
   $$\pi(x,y) = \frac{1}{Z} f(x,y) \tag{2}$$
   with $x,\ y \in [-1,1]$. Write a Monte-Carlo routine with importance sampling for estimating the expectation $\mathbb{E}_\pi h(x,y)$. Test your code–recording the variance and the error as a function of the number of samples–for the functions
   $$h(x,y) = V_1(x,y), \quad h(x,y) = V_2(x,y).$$

2. (a) The autocorrelation function of the velocity $Y(t)$ a Brownian particle moving in a harmonic potential $V(x) = \frac{1}{2}\omega_0^2 x^2$ is a mean zero stationary Gaussian process with autocorrelation function

$$R(t) = e^{-\gamma|t|}\Big(cos(\delta|t|) - \frac{1}{\delta}\sin(\delta|t|)\Big),$$

where γ is the friction coefficient and $\delta = \sqrt{\omega_0^2 - \gamma^2}$.

i. Simulate $Y(t)$: generate sample paths and estimate the first four moments of the process.

ii. Consider the position of the Brownian particle $X(t) = \int_0^t Y(s)\,ds$. Estimate the mean square displacement $\mathbb{E}(X(t))^2$. Study the limit $t \to +\infty$. Investigate this problem numerically for $\omega_0^2 = 2,\ \gamma = 1$.

(b) Let $X(t)$ be a mean zero stationary Gaussian process with covariance function

$$\gamma(t) = \begin{cases} (2+|s|)(1-|s|)^2 & -1 < s < 1, \\ 0 & |s| \geq 1. \end{cases}$$

Simulate $X(t)$ in order to get a Monte Carlo estimate of

$$\mathbb{P}\left(\sup_{t\in[0,2]} X(t) > 2\right).$$

3. Let $W(t)$ be a standard Brownian motion in $\mathbb{R}^d$ and $\mu(t) : \mathbb{R}^+ \mapsto \mathbb{R}^d,\ \Sigma(t) : \mathbb{R}^+ \mapsto \mathbb{R}^{d\times d}$ smooth vector and matrix valued functions of time, respectively. Define the vector valued stochastic process $X(t)$ to be the solution of the equation

$$dX(t) = \mu(t)\,dt + \Sigma(t)\,dW(t), \quad X(0) = x, \tag{3}$$

where $x \in \mathbb{R}^d$ a constant.

(a) Show that $X(t)$ is a Gaussian process with independent increments and

$$X(t) - X(s) \sim \mathcal{N}\left(\int_s^t \mu(u)\,du,\ \int_s^t \Gamma(u)\,du\right), \tag{4}$$

where $\Gamma(t) = \Sigma(t)\Sigma^T(t)$.

(b) Use the above result or the general algorithm for simulating Gaussian processes to develop an algorithm for simulating $X(t)$ for given $\mu(t),\ \Sigma(t), x$.

(c) Use your algorithm to simulate $X(t),\ t \in [0,T]$ for

$$\mu(t) = \begin{bmatrix} 2t \\ 0.1t \end{bmatrix}, \quad \Sigma(t) = \begin{bmatrix} 1 & 0.5 + 0.1\cos^2(t) \\ 0.3 & 2 \end{bmatrix}, \quad x = \begin{bmatrix} 1 \\ 0.5 \end{bmatrix}.$$

Generate sample paths and calculate the mean and variance. Compare with the theoretical results.

4. **MASTERY COMPONENT** Consider the Gaussian random field $X(x)$ in $\mathbb{R}$ with covariance function

$$\gamma(x, y) = e^{-a|x-y|} \tag{5}$$

where $a > 0$.

(a) Simulate this field: generate samples and calculate the first four moments.

(b) Consider $X(x)$ for $x \in [-L, L]$. Calculate analytically the eigenvalues and eigenfunctions of the integral operator $\mathcal{K}$ with kernel $\gamma(x, y)$,

$$\mathcal{K}f(x) = \int_{-L}^{L} \gamma(x, y) f(y)\, dy.$$

Use this in order to obtain the Karhunen-Loéve expansion for X. Plot the first five eigenfunctions when $a = 1,\ L = -0.5$. Investigate (either analytically or by means of numerical experiments) the accuracy of the KL expansion as a function of the number of modes kept.

(c) Develop a numerical method for calculating the first few eigenvalues/eigenfunctions of $\mathcal{K}$ with $a = 1,\ L = -0.5$. Use the numerically calculated eigenvalues and eigenfunctions to simulate $X(x)$ using the KL expansion. Compare with the analytical results and comment on the accuracy of the calculation of the eigenvalues and eigenfunctions and on the computational cost.