Computational Stochastic Processes - Assignment 1

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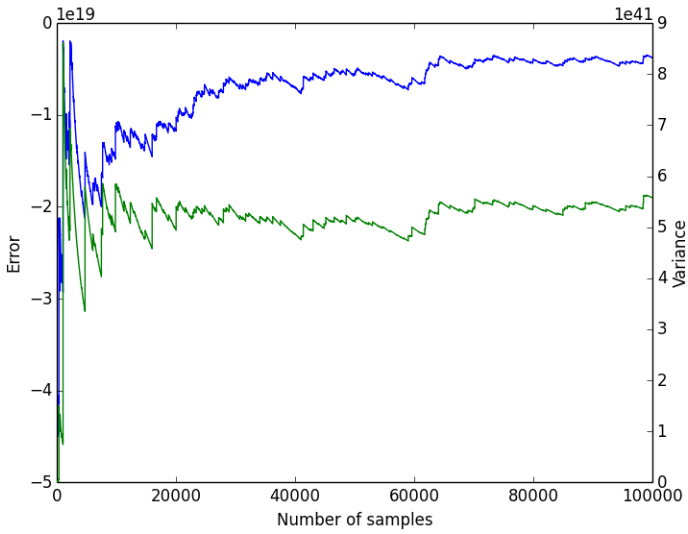
1 Question 1

1.1 q1.a.

Monte Carlo estimate with 100,000 samples: $Z = 4.2966847565923385e + 19$

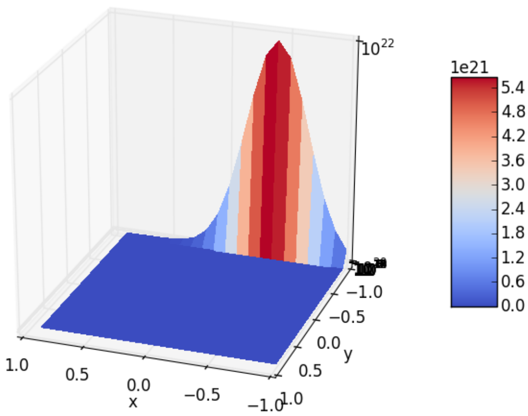
1.2 q1.b.

Numerical integration gives $Z = 4.667589495153099e + 19$ Graph of error (blue) and variance (green) for MC integration:

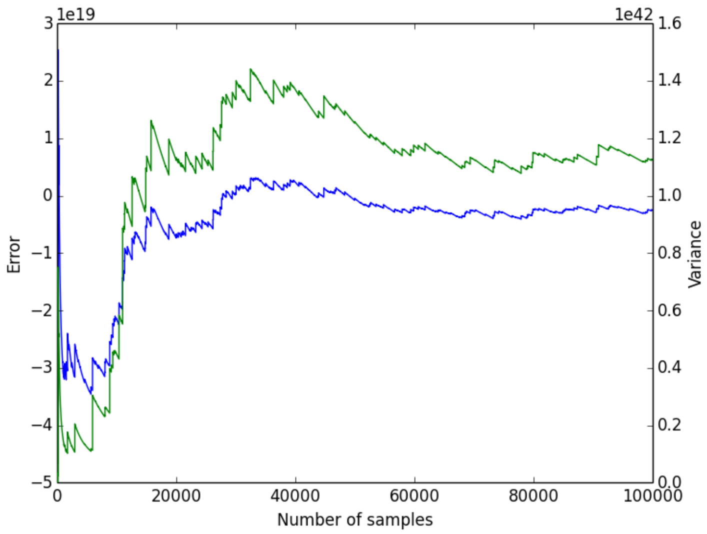


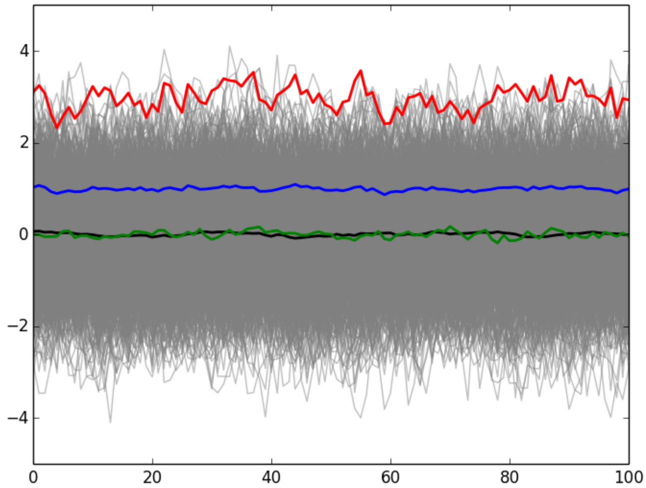
1.3 q1.c

Over $[-1,1] \times [-1,1]$, $f(x,y)$ looks like:



There is another peak around $(0.5, 1)$, but it is smaller. A mixture of Gaussians is the appropriate distribution for importance sampling. The error & variance performance is in the figure below:





2 Question 2

2.1 q2.a.

Generated sample paths and first four moments plotted below (mean is black, variance blue, 3rd moment green and 4th moment red):

Odd moments are zero, as we would expect.

Mean square displacement graph:

2.2 q2.b.

By Monte Carlo estimate with 100000 runs:

$$\mathbb{P}\left(sup_{t\in[0,2]}X(t) > 2\right) = 0.48992 \tag{1}$$

3 Q3

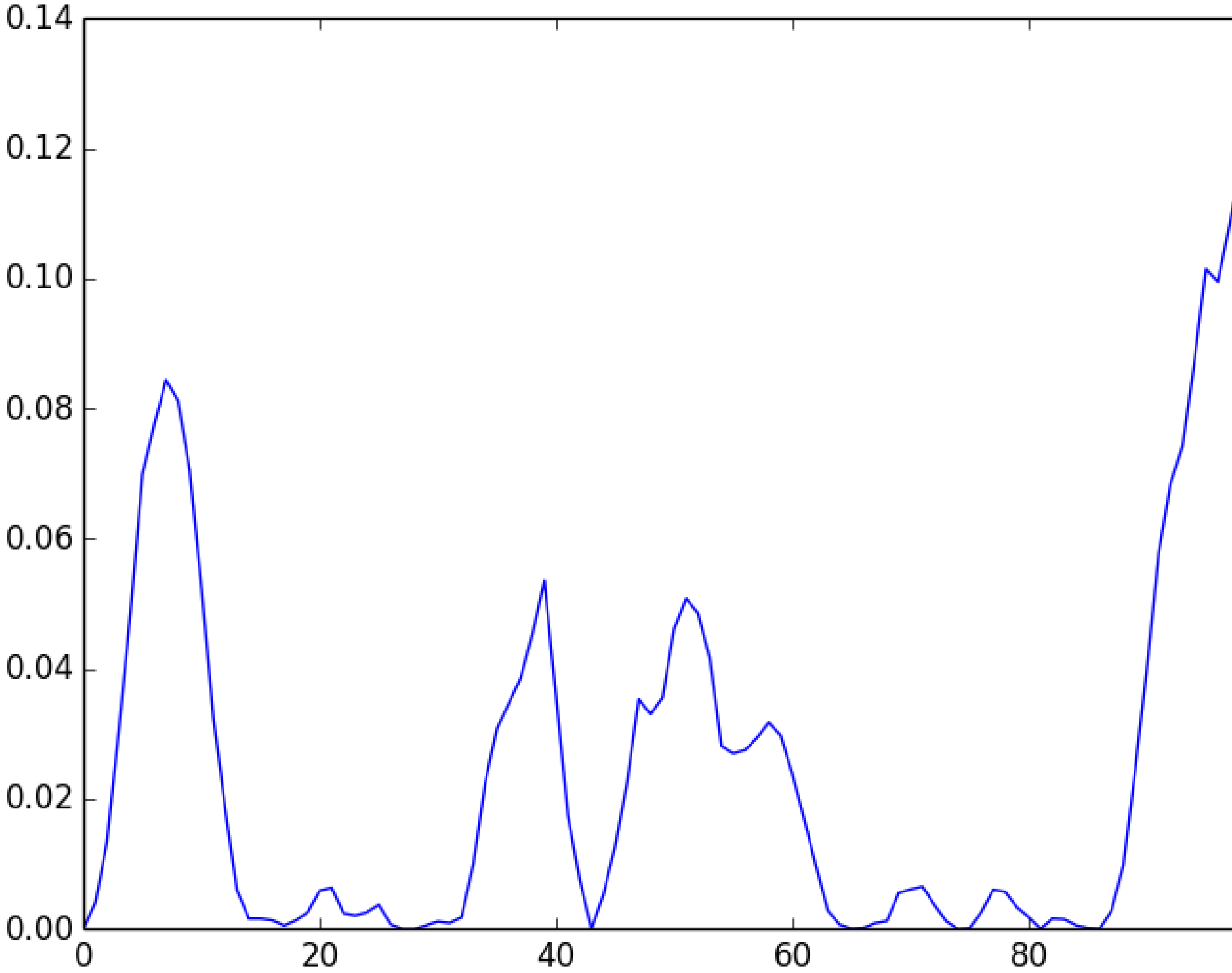
3.1 q3.a.

$$dX(t) = \mu(t)dt + \Sigma(t)dW(t), \quad X(0) = x \tag{2}$$

Integrating we obtain:

$$X(t) = X(0) + \int_0^t \mu(s)ds + \int_0^t \Sigma(s)dW(s) \tag{3}$$

We have to deal with the integral with respect to $W(s)$ - this is a martingale with quadratic variation:



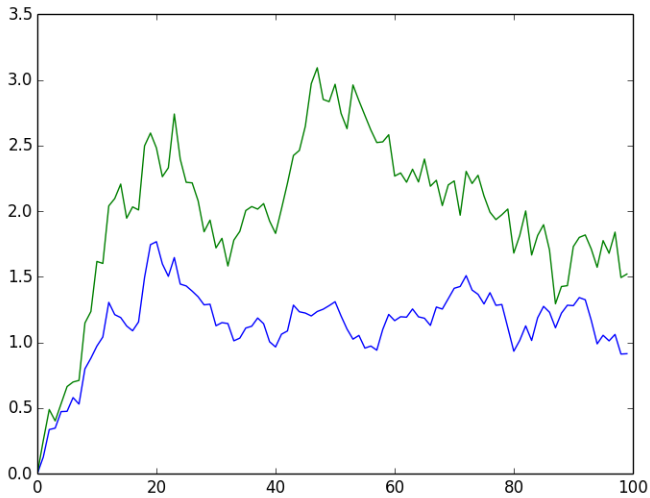
$$I(t) = \int_0^t \Sigma(s)ds \implies \langle I(t) \rangle = \int_0^t (\Sigma(s) \otimes \Sigma(s))ds = \int_0^t \Gamma(s)ds \tag{4}$$

Therefore we have:

$$X(t) = X(0) + \int_0^t \mu(s)ds + \int_0^t \Gamma(s)ds \tag{5}$$

The characteristic function of a random variable X is given by:

$$\phi(t) = \mathbb{E}(e^{itX}) \tag{6}$$



Which is Gaussian when:

$$\phi(t) = e^{\langle m,t\rangle - \frac{1}{2}\langle t,\Sigma t\rangle} \tag{7}$$

The $X(t)$ determined above satisfies this, and the equation for $X(t) - X(s)$ follows immediately.

3.2 q3.b.

By discretising time and calculating increments according to equation (4) on the problem sheet it is possible to generate sample paths for $X(t)$.

3.3 q3.c.

Figure of sample path (x component in blue, y component in green):

Figure of mean and variance:

Figure of elements of covariance matrix:

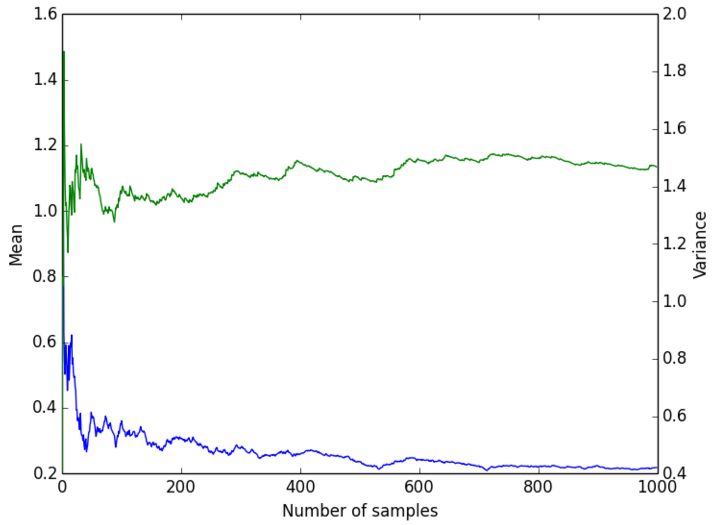
The results match theoretical predictions well. Even with a relatively low number of sample paths, a good agreement is still occurring.

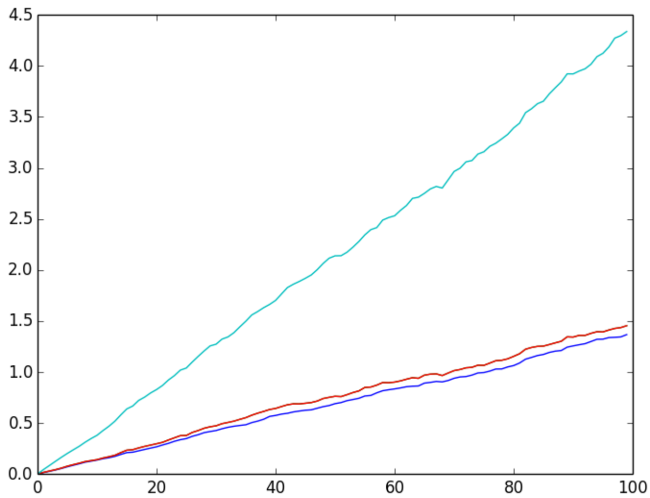
4 Q4

4.1 q4.a.

Using code from question 2 with the covariance function provided it is possible to generate samples from this field. Samples and first 4 moments are plotted below:

For 1000 samples, the first 4 moments are:





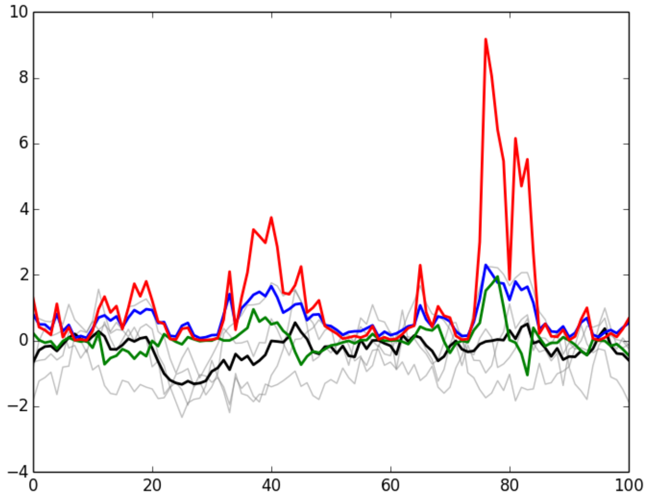
4.2 q4.b.

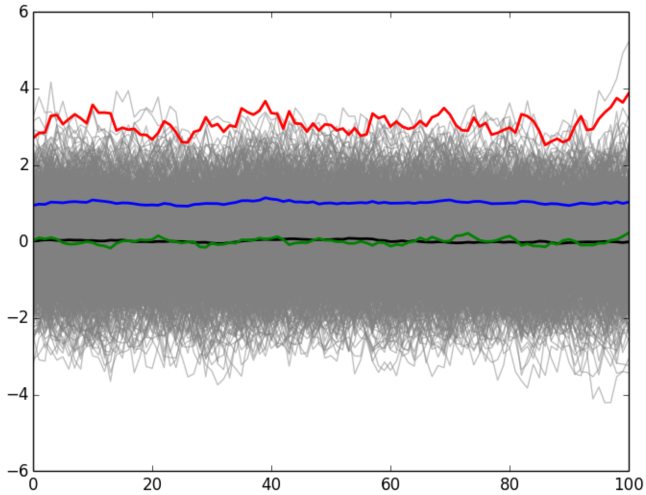
We have that:

$$\int_{-L}^{L} e^{-a|x-y|} f(y)dy = \lambda f(x) \tag{8}$$

Which is equivalent to

$$\int_{-L}^{x} e^{-a(x-y)} f(y)dy + \int_{x}^{L} e^{a(x-y)} f(y)dy = \lambda f(x) \tag{9}$$





Differentiating once with respect to t gives:

$$-a\int_{-L}^{x} e^{-a(x-y)}f(y)dy + a\int_{x}^{L} e^{a(x-y)}f(y)dy - e^{-ax}e^{ax} + e^{-ax}e^{ax} = \lambda f'(x) \tag{10}$$

The last 2 terms cancel. Differentiating again gives:

$$a^2\int_{-L}^{x} e^{-a(x-y)}f(y)dy + a^2\int_{x}^{L} e^{a(x-y)}f(y)dy - 2af(x) = \lambda f''(x) \tag{11}$$

The terms in a^2 are now the original integral multiplied by a factor of a^2, so we can write:

$$a^2\lambda f(x) - 2af(x) = \lambda f''(x) \tag{12}$$

We can obtain the boundary conditions by evaluating the undifferentiated expression and the first derivative at the boundaries $+L$ and $-L$ to give:

$$af(L) + f'(L) = 0 \tag{13}$$
$$af(-L) - f'(-L) = 0 \tag{14}$$

Now use the variation of constants method, with eigenfunctions in the form:

$$f(x) = c_1 \cos(\omega x) + c_2 \sin(\omega x) \tag{15}$$

Firstly we can recover the eigenvalues:

$$a^2\lambda f(x) - 2af(x) = \lambda f''(x) \implies a^2\lambda f(x) + 2af(x) = \omega^2\lambda f(x) \tag{16}$$

Rearranging gives:

$$(a^2 = \omega^2)\lambda = 2a \implies \lambda = \frac{2a}{a^2 + \omega^2} \tag{17}$$

Now inserting the eigenfunctions and their derivatives into the boundary conditions gives the following equations:

$$c_1(a - \omega \tan(\omega L)) + c_2(\omega + a \tan(\omega L)) = 0 \tag{18}$$
$$c_1(a - \omega \tan(\omega L)) - c_2(\omega + a \tan(\omega L)) = 0 \tag{19}$$

These have nontrivial solutions when either the term in $c1$ or the term in c_2 is zero, and gives solutions:

$$f(x) = \frac{\cos(\omega x)}{\sqrt{L + \frac{\sin(2\omega L)}{2\omega}}} \tag{20}$$

for the solutions given by $a - \omega \tan(\omega L) = 0$ and

$$f(x) = \frac{\sin(\omega x)}{\sqrt{L - \frac{\sin(2\omega L)}{2\omega}}} \tag{21}$$

for solutions given by $\omega + a \tan(\omega L) = L$