Karhunen–Loéve Expansion of Stochastic Processes with a Modified Exponential Covariance Kernel

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Abstract: The spectral representation of stationary stochastic processes via the Karhunen-Loéve (KL) expansion is examined from a numerical efficiency perspective. Attention is focused on processes which have commonly been characterized by covariance kernels decaying exponentially versus the position/time delay variable. By introducing a slight modification in the mathematical description of this covariance kernel, the nondifferentiability at its peak is eliminated, whereas most of its advantageous properties are retained. It is shown that compared to the common exponential model, the requisite number of terms for representing the process in context with the modified kernel is significantly smaller. The effect is demonstrated by means of a specific numerical example. This is done by first determining the eigenfunctions/eigenvalues associated with the KL expansion for the modified kernel model, and by afterwards estimating the approximation errors corresponding to the two kernels considered for specific numerical values. Clearly, the enhanced computational efficiency of the KL expansion associated with the modified kernel can significantly expedite its incorporation in stochastic finite elements and other areas of stochastic mechanics.

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Introduction

The numerically efficient representation of stochastic processes is of major importance in stochastic mechanics (Spanos and Ghanem 1989; Schuëller 2001; Jensen and Iwan 1991; Ghanem and Spanos 2003; Zhang and Ellingwood 1996; Gutierrez and Zaldivar 2000; Deb et al. 2001; Du et al. 2005). This aspect is particularly important in analyzing large systems (Schuëller et al. 2003). Among the available numerical models for representing stochastic processes, the Karhunen-Loéve (KL) expansion has attracted considerable attention (Spanos and Ghanem 1989; Ghanem and Spanos 2003; Iwan and Jensen 1993; Phoon et al. 2002; Schuëller et al. 2003). Its appeal is based on its straightforward and general applicability. Stationary as well as nonstationary multidimensional processes with arbitrary covariance kernels may be described; Gaussian and non-Gaussian random processes may also be treated.

The numerical efficiency of the representation depends on the number N of terms in the expansion needed to approximate adequately the process under consideration. In general, the approximation error decreases by increasing the number of these terms. Clearly, for an ordinary value of N, some approximation error remains in the numerical model. Thus, a compromising approach to this problem involves finding a process representation model that, on one hand, comprises only a few terms in the expansion so that it can be efficiently implemented in numerical computations and, on the other hand, a reasonable degree of accuracy is achieved.

Various studies relating to the aforementioned problem have been reported; see Spanos and Ghanem (1989) and Field and Grigoriu (2004). In assessing the accuracy of the KL expansion of a Gaussian random process with an exponentially decaying covariance function, concerns regarding the numerical efficiency have been reported. For the particular problem which was considered, even using $N=10$ terms in the expansion, a considerable approximation error has been reported.

In examining the results of the aforementioned studies, it is pointed out that the exponentially decaying covariance function is not differentiable at the origin, zero lag. This appears to be a non-natural property of this model, which, to the best of the writers' knowledge, is not necessarily dictated by experimental data.

Obviously, an expansion associated with a kernel which resembles the exponential one but requires a relatively small number of terms, for ordinary applications, is quite desirable in stochastic mechanics. In this paper, an attempt is made to formulate a KL expansion of improved numerical efficiency. In this context, a modification is introduced into this covariance model that eliminates the lack of nondifferentiability at its peak while retaining its advantageous properties. The solution of the associated Fredholm integral equation for the KL expansion is presented, and the resulting enhancement of the numerical efficiency, without a loss in approximation accuracy, is demonstrated by

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means of a numerical example. In this regard, it is appropriate to cite herein (Nashed and Wahba 1974) which came to the writers' attention after this paper was accepted for publication; the cited work provides a useful mathematical perspective on the effect of kernel smoothness on the rate of convergence of the associated series expansion.

Karhunen-Loéve Expansion with Modified Exponential Covariance Kernel

Modified Covariance Function

Let $w(x,\theta)$ be a one-dimensional, stationary process which is defined over the domain D (with real values) with θ being an element of Ω. Hereby, $[\Omega,\mathfrak{S},P]$ denotes a probability space, and x=element of D. The covariance function

$$C(x_1,x_2)=e^{-|x_1-x_2|a}=e^{-a|u|};\quad u=x_1-x_2 \tag{1}$$

is used for modeling various physical processes (Vanmarcke 1983). The parameter "a" is related to the "correlation length"

$$b=\frac{1}{a} \tag{2}$$

of the process. Considering the derivatives from the left and from the right of the function $C(u)$ at $u=0$, one finds

$$\frac{dC}{du}\rightarrow +a \quad \text{as } u\rightarrow 0^- \tag{3}$$

and

$$\frac{dC}{du}\rightarrow -a \quad \text{as } u\rightarrow 0^+ \tag{4}$$

Thus, considering the operator of mathematical expectation, E, it is found that the property

$$E[ww']=\left.\frac{dC}{du}\right|_{u=0}=0 \tag{5}$$

involving the process w and its derivative w', and which is commonly ascribed to stationary stochastic process models of several physical phenomena is not satisfied.

To rectify this difficulty, it is proposed herein to modify the function $C(u)$ by multiplying it by the function

$$g(x_1-x_2)=1-r|x_1-x_2|=1-r|u| \tag{6}$$

so that the first derivative of the modified covariance function becomes null at $u=0$. That is, consider the modified covariance function

$$\hat{C}(u)=e^{-\hat{a}|u|}(1-r|u|) \tag{7}$$

which is differentiable at all points in the domain of definition. Specifically, select the parameter r so that

$$\left.\frac{d\hat{C}}{du}\right|_{u=0}=0 \tag{8}$$

This leads to

$$r=-\hat{a} \tag{9}$$

and the modified covariance function becomes

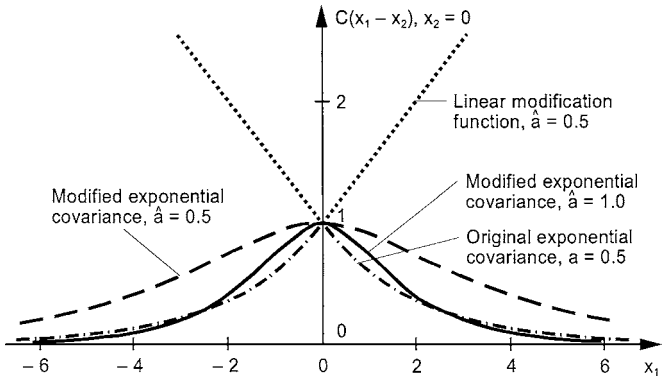

Fig. 1. Modification of the exponential covariance kernel

$$\hat{C}(u)=e^{-\hat{a}|u|}(1+\hat{a}|u|) \tag{10}$$

Due to the decreasing branches of the function in Eq. (1) the influence of the linear term decreases (in absolute terms) with increasing distance from the original peak at $u=0$. Fig. 1 shows this modification for $x_2=0$ and $\hat{a}=0.5$. For the correlation length b one finds

$$b=\frac{\int_0^\infty u\hat{C}(u)}{\int_0^\infty \hat{C}(u)}=\frac{1.5}{\hat{a}} \tag{11}$$

For a good matching between the original exponential curve and the modified version, Eq. (10), the parameter $\hat{a}$ may be selected accordingly. For example, $\hat{a}=1.0$ yields a good agreement as seen in Fig. 1.

Fredholm Integral Equation Solution

Consider next the random process $\bar{w}(x,\theta)$ which has the modified covariance function $\hat{C}(u)$ and is represented by its KL expansion (Loéve 1977; Van Trees 1968)

$$\bar{w}(x,\theta)=\sum_{n=1}^{\infty}\sqrt{\lambda_n}\xi_n(\theta)f_n(x) \tag{12}$$

For numerical implementation, the expansion is truncated at $n=N$. In this spectral process, representation $\{\xi_n(\theta)\}$ denotes a set of random variables; λ_n=constant positive real numbers; $\{f_n(x)\}$=orthonormal set of deterministic functions in the Hilbert space of functions defined over D; and x=spatial (or temporal) coordinate. The associated spectral decomposition of the covariance function is

$$\hat{C}(x_1-x_2)=\sum_{n=1}^{\infty}\lambda_n f_n(x_1)f_n(x_2) \tag{13}$$

with λ_n and $f_n(x)$ denoting the eigenvalues and the eigenfunctions of the modified covariance kernel $\hat{C}(x_1-x_2)$, respectively. The associated eigenvalue problem is described by a homogeneous Fredholm integral equation of the second kind (Mikhlin 1957; Kanwal 1971). That is

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$$\int_D \hat{C}(x_1 - x_2) f_n(x_2) dx_2 = \lambda_n f_n(x_1) \tag{14}$$

Eq. (14) must be solved for the specific covariance function Eq. (10) to obtain the KL expansion of the random process with modified exponential covariance. The process is assumed to be defined over the one-dimensional interval $[-k, +k]$. Thus, Eq. (14), after abandoning, for convenience, indexing λ and $f(x)$ becomes

$$\int_{-k}^{+k} e^{-a|x_1 - x_2|}(1 + a|x_1 - x_2|) f(x_2) dx_2 = \lambda f(x_1) \tag{15}$$

Eq. (15) is recast in the form

$$\int_{-k}^{x_1} e^{-a(x_1 - x_2)}(1 + a(x_1 - x_2)) f(x_2) dx_2 + \int_{x_1}^{+k} e^{a(x_1 - x_2)}(1 - a(x_1 - x_2)) f(x_2) dx_2 = \lambda f(x_1) \tag{16}$$

Further, differentiating Eq. (16) twice with respect to x_1 yields

$$-a^2\left(\int_{-k}^{x_1} e^{-a(x_1 - x_2)}(1 - a(x_1 - x_2)) f(x_2) dx_2 + \int_{x_1}^{+k} e^{a(x_1 - x_2)}(1 + a(x_1 - x_2)) f(x_2) dx_2 \right) = \lambda f''(x_1) \tag{17}$$

Eq. (17) may be rearranged in the form

$$a^2\left(\int_{-k}^{x_1} e^{-a(x_1 - x_2)}(1 + a(x_1 - x_2)) f(x_2) dx_2 + \int_{x_1}^{+k} e^{a(x_1 - x_2)}(1 - a(x_1 - x_2)) f(x_2) dx_2 \right) - 2a^2\left(\int_{-k}^{x_1} e^{-a(x_1 - x_2)} f(x_2) dx_2 + \int_{x_1}^{+k} e^{a(x_1 - x_2)} f(x_2) dx_2 \right) = \lambda f''(x_1) \tag{18}$$

The first part of Eq. (18) is next replaced using Eq. (16) to derive

$$\int_{-k}^{x_1} e^{-a(x_1 - x_2)} f(x_2) dx_2 + \int_{x_1}^{+k} e^{a(x_1 - x_2)} f(x_2) dx_2 = \frac{1}{2}\lambda f(x_1) - \frac{1}{2a^2}\lambda f''(x_1) \tag{19}$$

Eq. (19) is again differentiated twice with respect to x_1, yielding

$$a^2\left(\int_{-k}^{x_1} e^{-a(x_1 - x_2)} f(x_2) dx_2 + \int_{x_1}^{+k} e^{a(x_1 - x_2)} f(x_2) dx_2 \right) - 2af(x_1) = \frac{1}{2}\lambda f''(x_1) - \frac{1}{2a^2}\lambda f''''(x_1) \tag{20}$$

Substituting the expression in parentheses in Eq. (20) and taking into consideration Eq. (19) yield

$$f''''(x_1) - 2a^2 f''(x_1) + \left(a^4 - 4\frac{a^3}{\lambda}\right) f(x_1) = 0 \tag{21}$$

This is a fourth-order homogeneous linear differential equation with constant coefficients. The boundary conditions associated with this differential equation are found by evaluating integral equation (16) and its first three derivatives at $x_1 = k$ and $x_1 = -k$. For $x_1 = k$, the following set of equations is obtained:

$$\lambda f(k) = e^{-ka} \int_{-k}^{+k} e^{ax_2}(1 + a(k - x_2)) f(x_2) dx_2 \tag{22}$$

$$\lambda f'(k) = a \cdot e^{-ka} \int_{-k}^{+k} e^{ax_2}(-a(k - x_2)) f(x_2) dx_2 \tag{23}$$

$$\lambda f''(k) = a^2 \cdot e^{-ka} \int_{-k}^{+k} e^{ax_2}(-1 + a(k - x_2)) f(x_2) dx_2 \tag{24}$$

$$\lambda f'''(k) = a^3 \cdot e^{-ka} \int_{-k}^{+k} e^{ax_2}(2 - a(k - x_2)) f(x_2) dx_2 \tag{25}$$

Eqs. (22)–(24) are combined to obtain

$$f''(k) + 2af'(k) + a^2 f(k) = 0 \tag{26}$$

Combining Eqs. (22), (23), and (25), leads to

$$f'''(k) - 3a^2 f'(k) - 2a^3 f(k) = 0 \tag{27}$$

Further, setting $x_1 = -k$ yields

$$\lambda f(-k) = e^{-ka} \int_{-k}^{+k} e^{-ax_2}(1 + a(k + x_2)) f(x_2) dx_2 \tag{28}$$

$$\lambda f'(-k) = a e^{-ka} \int_{-k}^{+k} e^{-ax_2}(a(k + x_2)) f(x_2) dx_2 \tag{29}$$

$$\lambda f''(-k) = a^2 e^{-ka} \int_{-k}^{+k} e^{-ax_2}(-1 + a(k + x_2)) f(x_2) dx_2 \tag{30}$$

$$\lambda f'''(-k) = a^3 e^{-ka} \int_{-k}^{+k} e^{-ax_2}(-2 + a(k + x_2)) f(x_2) dx_2 \tag{31}$$

and

$$f''(-k) - 2af'(-k) + a^2 f(-k) = 0 \tag{32}$$

$$f'''(-k) - 3a^2 f'(-k) + 2a^3 f(-k) = 0 \tag{33}$$

The boundary conditions set by Eqs. (26), (27), (32), and (33) provide sufficient information to determine the solution of Eq. (21). Omitting, for clarity, the index of x_1 the function

$$f(x) = b_1 e^{\omega_1 x} + b_2 e^{-\omega_1 x} + b_3 e^{\omega_2 x} + b_4 e^{-\omega_2 x} \tag{34}$$

with

$$\omega_1 = \sqrt{a^2 + \sqrt{4\frac{a^3}{\lambda}}}, \quad \omega_2 = \sqrt{a^2 - \sqrt{4\frac{a^3}{\lambda}}} \tag{35}$$

satisfies Eq. (21).

Applying the condition $a > 0$ (to consider a positive correlation length $b = 1.5/a$) with $\lambda > 0$ [positive eigenvalues, see Ghanem and Spanos (2003)]

$$4\frac{a^3}{\lambda} > 0 \tag{36}$$

restricts ω_1 to be always a positive real value, and requires the following distinction of cases for ω_2.

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- Case a:

$$a^2 \geq \sqrt{4\frac{a^3}{\lambda}} \tag{37}$$

This leads to an always non-negative real value for ω_2. That is, there are no transcendental terms in the solution, which eventually cannot lead to rational results [countable infinite set of eigenvalues of the covariance kernel, see Ghanem and Spanos (2003)].

- Case b:

$$a^2 < \sqrt{4\frac{a^3}{\lambda}} \tag{38}$$

The relation in Eq. (38) causes ω_2 to be imaginary, which can be expressed using sine and cosine functions (via the Euler formula) leading to an infinite number of eigenvalues.

Case b is pursued further under the restriction

$$\lambda < \frac{4}{a} \tag{39}$$

which is derived from Eq. (38). Accordingly, the parameter ω_2 may be expressed as

$$\omega_2 = i\sqrt{-a^2 + \sqrt{4\frac{a^3}{\lambda}}} = i\bar{\omega}_2 \tag{40}$$

and the solution function Eq. (34) is rewritten as

$$f(x) = b_1 e^{\omega_1 x} + b_2 e^{-\omega_1 x} + q_1 \cos(\bar{\omega}_2 x) + q_2 \sin(\bar{\omega}_2 x) \tag{41}$$

Imposing the boundary conditions defined by Eqs. (26), (27), (32), and (33) on Eq. (41), the homogeneous system of transcendental equations in λ

$$\begin{bmatrix} A & B & C & D \\ B & A & C & -D \\ E & F & G & H \\ -F & -E & -G & H \end{bmatrix} \begin{bmatrix} b_1 \\ b_2 \\ q_1 \\ q_2 \end{bmatrix} = 0 \tag{42}$$

is obtained, with

$$A = (\omega_1^2 + 2a\omega_1 + a^2)e^{\omega_1 k} \tag{43}$$

$$B = (\omega_1^2 - 2a\omega_1 + a^2)e^{-\omega_1 k} \tag{44}$$

$$C = (a^2 - \bar{\omega}_2^2)\cos(\bar{\omega}_2 k) - 2a\bar{\omega}_2 \sin(\bar{\omega}_2 k) \tag{45}$$

$$D = (a^2 - \bar{\omega}_2^2)\sin(\bar{\omega}_2 k) + 2a\bar{\omega}_2 \cos(\bar{\omega}_2 k) \tag{46}$$

$$E = (\omega_1^3 - 3a^2\omega_1 - 2a^3)e^{\omega_1 k} \tag{47}$$

$$F = (-\omega_1^3 + 3a^2\omega_1 - 2a^3)e^{-\omega_1 k} \tag{48}$$

$$G = (\bar{\omega}_2^3 + 3a^2\bar{\omega}_2)\sin(\bar{\omega}_2 k) - 2a^3 \cos(\bar{\omega}_2 k) \tag{49}$$

and

$$H = (-\bar{\omega}_2^3 - 3a^2\bar{\omega}_2)\cos(\bar{\omega}_2 k) - 2a^3 \sin(\bar{\omega}_2 k) \tag{50}$$

Further, setting

$$b_1 + b_2 = p_1, \quad b_1 - b_2 = p_2 \tag{51}$$

the fourth-order system in Eq. (42) is split into two independent subsystems of order two

$$\begin{bmatrix} A+B & 2C & 0 & 0 \\ E+F & 2G & 0 & 0 \\ 0 & 0 & A-B & 2D \\ 0 & 0 & E-F & 2H \end{bmatrix} \begin{bmatrix} p_1 \\ q_1 \\ p_2 \\ q_2 \end{bmatrix} = 0 \tag{52}$$

where

$$A + B = 2(\omega_1^2 + a^2)\cosh(\omega_1 k) + 4a\omega_1 \sinh(\omega_1 k) \tag{53}$$

$$A - B = 2(\omega_1^2 + a^2)\sinh(\omega_1 k) + 4a\omega_1 \cosh(\omega_1 k) \tag{54}$$

$$E + F = 2(\omega_1^3 - 3a^2\omega_1)\sinh(\omega_1 k) - 4a^3 \cosh(\omega_1 k) \tag{55}$$

$$E - F = 2(\omega_1^3 - 3a^2\omega_1)\cosh(\omega_1 k) - 4a^3 \sinh(\omega_1 k) \tag{56}$$

The separated subsystems can now be readily solved. With $q_2=0$ and $p_2=b_1 - b_2=0$, Subsystem I

$$\det\begin{bmatrix} A+B & 2C \\ E+F & 2G \end{bmatrix} = 0 \tag{57}$$

yields the odd-numbered eigenvalues λ_n and eigenfunctions $f_n(x)$, $n=1,3,5,\ldots$, which correspond to the function

$$f_n(x) = 2b_{1n}\cosh(\omega_{1n}x) + q_{1n}\cos(\bar{\omega}_{2n}x) \tag{58}$$

derived from Eq. (41). Evaluating the first line of the homogeneous equation system Eq. (52), and incorporating the result in Eq. (58) leads to the eigenfunctions

$$f_n(x) = b_{1n}\left[2\cosh(\omega_{1n}x) - \frac{A+B}{C}\cos(\bar{\omega}_{2n}x)\right] \tag{59}$$

From the solution of Subsystem II

$$\det\begin{bmatrix} A-B & 2D \\ E-F & 2H \end{bmatrix} = 0 \tag{60}$$

the even-numbered eigenvalues λ_n and eigenfunctions $f_n(x)$, $n=2,4,6,\ldots$ are determined (marked by an asterisk). This is associated with the conditions $q_1^*=0$ and $p_1^*=b_1^*+b_2^*=0$ leading to the function

$$f_n^*(x) = 2b_{1n}^* \sinh(\omega_{1n}^* x) + q_{2n}^* \sin(\bar{\omega}_{2n}^* x) \tag{61}$$

from Eq. (41). Accounting for line three in Eq. (52), yields the eigenfunctions

$$f_n^*(x) = b_{1n}^*\left[2\sinh(\omega_{1n}^* x) - \frac{A^* - B^*}{D^*}\sin(\bar{\omega}_{2n}^* x)\right] \tag{62}$$

The scaling coefficients b_{1n} and b_{1n}^* in Eqs. (59) and (62), respectively, are determined by normalizing the eigenfunctions by the condition

$$\int_D f_m(x) f_n(x) dx = \delta_{mn} \tag{63}$$

where δ_{mn}=Kronecker delta. This yields

$$b_{1n} = \frac{1}{\sqrt{R_n}}, \quad b_{1n}^* = \frac{1}{\sqrt{R_n^*}}, \tag{64}$$

with

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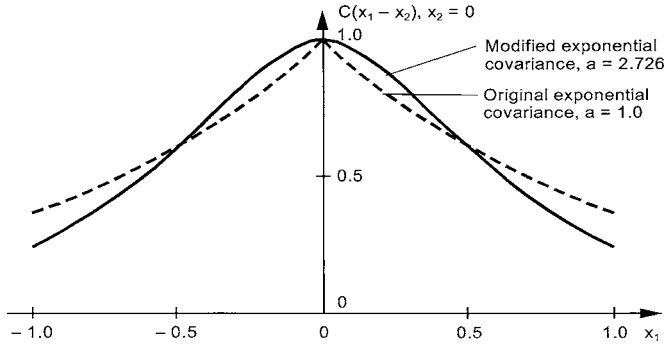

Fig. 2. Covariance kernels

$$R_n = \frac{2}{\omega_{1n}}\sinh(2k\omega_{1n}) + \frac{1}{2\bar{\omega}_{2n}}\left(\frac{A+B}{C}\right)^2 \sin(2k\bar{\omega}_{2n}) + k\left(\left(\frac{A+B}{C}\right)^2 + 4\right) - 2\frac{A+B}{C}\sqrt{\frac{\lambda}{a^3}}[\omega_{1n}\cos(k\bar{\omega}_{2n})\sinh(k\omega_{1n}) + \bar{\omega}_{2n}\sin(k\bar{\omega}_{2n})\cosh(k\omega_{1n})] \quad (65)$$

and

$$R_n^* = \frac{2}{\omega_{1n}^*}\sinh(2k\omega_{1n}^*) + \frac{1}{2\bar{\omega}_{2n}^*}\left(\frac{A^*-B^*}{D^*}\right)^2 \sin(2k\bar{\omega}_{2n}^*) + k\left(\left(\frac{A^*-B^*}{D^*}\right)^2 - 4\right) + 2\frac{A^*-B^*}{D^*}\sqrt{\frac{\lambda^*}{a^3}}\left[\bar{\omega}_{2n}^*\cos(k\bar{\omega}_{2n}^*)\sinh(k\omega_{1n}^*) - \omega_{1n}^*\sin(k\bar{\omega}_{2n}^*)\cosh(k\omega_{1n}^*)\right] \quad (66)$$

Thus, Eq. (59) and Eq. (62) provide the requisite information for determining the KL representation of the process having the modified covariance kernel $\hat{C}(u)$.

Example and Approximation Error Assessment

For comparison purposes, a stationary process with exponential covariance is considered. To obtain results comparable to those presented in Ghanem and Spanos (2003) for the KL expansion of such a process, relevant parameter adjustments are made. In this context the values

$$a = 1 \quad (67)$$

and

$$k = 0.5 \quad (68)$$

are selected for the original exponential covariance kernel. Further, the modified exponential covariance kernel discussed in this paper is adjusted for that process. The parameter $\hat{a}$ of the modified covariance kernel is adjusted to yield a surface that closely resembles the traditional exponential covariance kernel. This is achieved by a mean squared error minimization. That is, require that the error

$$e = \int_{-k}^{+k}\int_{-k}^{+k}(e^{-\hat{a}|x_1-x_2|}(1+\hat{a}|x_1-x_2|) - e^{-|x_1-x_2|})^2 dx_2 dx_1 \quad (69)$$

becomes minimum. Using a standard least-squares algorithm, the value $\hat{a}=2.726$ is obtained for the modified covariance kernel in Eq. (10). Fig. 2 shows both covariance kernels with these adjustments for $x_2=0$. The modified covariance surface over the domain of definition $x_1, x_2 \in [-0.5, +0.5]$ is plotted in Fig. 3.

For the KL expansion of the process with the modified covariance kernel the ten first eigenvalues λ_n are determined from Eqs. (57) and (60); see Table 1. In view of the fast decreasing values λ_n, only the first four eigenvalues are used for further consideration. That is, $N=4$. The eigenfunctions associated with

Table 1. Eigenvalues of the Modified Covariance Kernel

Number n of eigenvalue	Eigenvalue λ_n
1	0.766773
2	0.177078
3	$3.83886\cdot10^{-2}$
4	$1.05690\cdot10^{-2}$
5	$3.70688\cdot10^{-3}$
6	$1.56687\cdot10^{-3}$
7	$7.60219\cdot10^{-4}$
8	$4.08892\cdot10^{-4}$
9	$2.37983\cdot10^{-4}$
10	$1.47352\cdot10^{-4}$

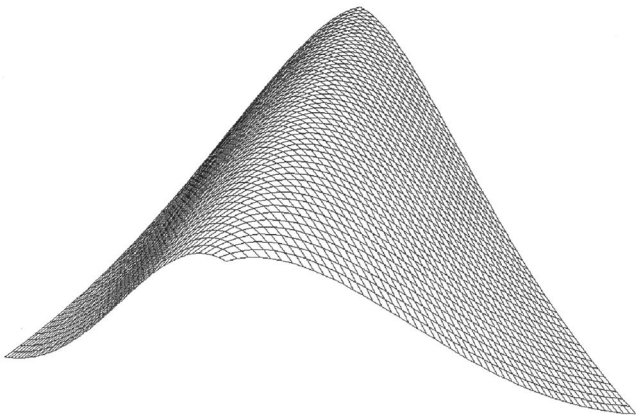

Fig. 3. Modified covariance surface over $x_1, x_2 \in [-0.5, +0.5]$

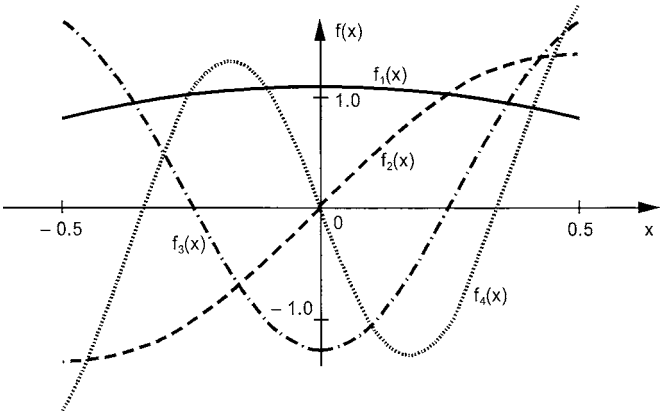

Fig. 4. Eigenfunctions

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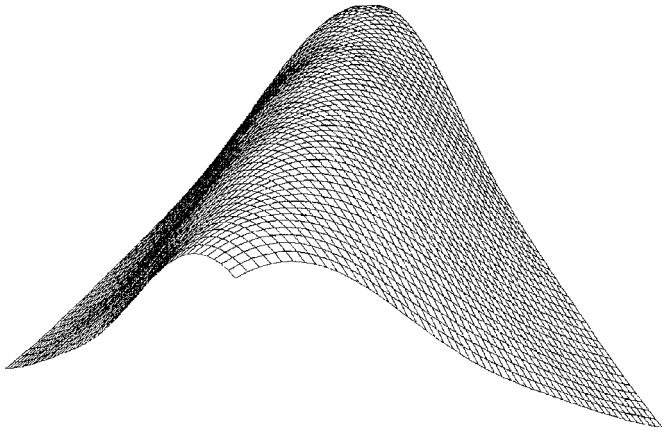

Fig. 5. Four-term approximation of the covariance kernel

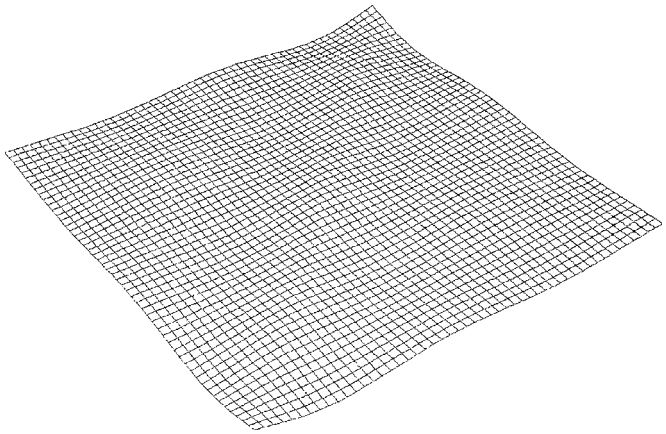

Fig. 7. Error surface, original scale

the first four eigenvalues are plotted in Fig. 4. The associated four-term approximation of the covariance kernel according to Eq. (13) is shown in Fig. 5 over $x_1, x_2 \in [-0.5, +0.5]$.

For $x_2=0$ the approximated covariance function is plotted in Fig. 6 versus the curve of the closed-form expression from Eq. (10). In the considered range between $x_1=-0.5$ and $x_1=+0.5$ there is almost no visible difference. Fig. 7 shows the complete (signed) error surface of the four-term approximation with the same scale as already used in Figs. 3 and 5 for the covariance surface plots. Again, no significant approximation error can be observed. The error surface approaches closely the zero plane. Stretching the error scale by a factor of 100 enables a proper visualization of the error distribution over $x_1, x_2 \in [-0.5, +0.5]$; see Fig. 8.

For a numerical evaluation of the approximation error the maximum error

$$\mathrm{err}_{\max} = \max\left[\left| C(x_1 - x_2) - \sum_{n=1}^{N=4} \lambda_n f_n(x_1) f_n(x_2) \right| | x_1, x_2 \in [-0.5, +0.5]\right] \tag{70}$$

as well as the average error

$$\mathrm{err}_{\mathrm{avg}} = \int_{-k}^{+k} \int_{-k}^{+k} \left| C(x_1 - x_2) - \sum_{n=1}^{N=4} \lambda_n f_n(x_1) f_n(x_2) \right| dx_2 dx_1 \tag{71}$$

of the covariance surface approximation is determined in absolute terms. The results are compared with the approximation errors obtained for the corresponding approximation of the original exponential covariance, see Tables 2 and 3. In both approximation cases, the absolute maximum deviation of the approximation surface from the closed-form surface is found at $x_1=x_2=+0.5$ and $x_1=x_2=-0.5$. That is, the maximum error is closely related to the approximation quality for $x_1=x_2$, which concerns the peak of the original exponential covariance surface. In conjunction with this observation, the results in Tables 2 and 3 clearly indicate that the modification of the exponential covariance kernel at and near $x_1=x_2$ which is proposed herein leads to a significant improvement in the approximation quality of the associated KL expansion. Specifically, a comparison of the corresponding four-term expansions shows that the approximation error is reduced by a factor of 4.26 for the maximum error and by a factor of 3.72 for the average error. Not before using at least ten terms in the KL expansion which is associated with the original exponential covariance kernel, the corresponding error becomes as low as that one induced by the four-term approximation associated with the modified exponential covariance kernel proposed herein.

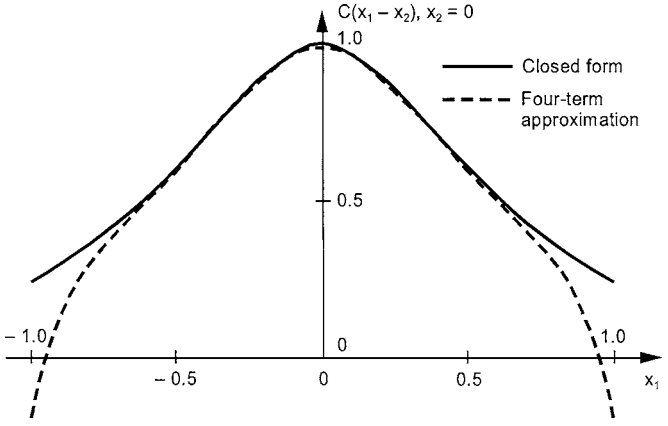

Fig. 6. Approximated covariance kernel versus closed-form curve

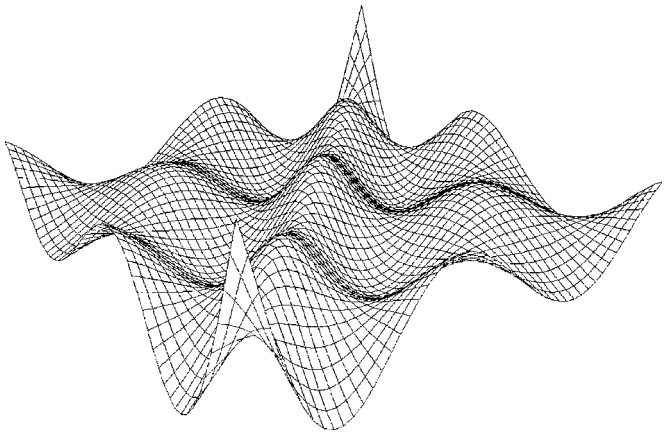

Fig. 8. Error surface, error scale stretched by factor of 100

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Table 2. Maximum Approximation Errors $\mathrm{err}_{\max}$ in the Covariance Surface

Modified exponential covariance kernel	Original exponential covariance kernel	
Four-term approximation	Four-term approximation	Ten-term approximation
0.0264	0.1126	0.0425

Table 3. Average Approximation Errors $\mathrm{err}_{\mathrm{avg}}$ in the Covariance Surface

Modified exponential covariance kernel	Original exponential covariance kernel	
Four-term approximation	Four-term approximation	Ten-term approximation
0.00346	0.012882	0.003115

Concluding Remarks

In this paper, an attempt has been made to elucidate the issues of numerical efficiency of the representation of stationary stochastic processes via the well known KL expansion. Specifically, the commonly used in physical applications model of a covariance kernel which decays exponentially away from the origin has been considered. It has been pointed out that this popular kernel model is not differentiable at the origin. In this regard, it has been shown that an adopted modified model which is differentiable at the origin and not necessarily a less natural choice than the popular one for many mechanics and other engineering disciplines, leads to KL expansions which themselves exhibit enhanced computational efficiency. For this purpose, a least-squares algorithm has been used first to fit the modified model to the original, exponentially decaying one. Next, the associated Fredholm integral equation has been solved to determine the Karhunen-Loéve expansion associated with the modified—exponentially decaying—covariance kernel. Finally, the analytical solution for the modified covariance function has been used to obtain numerical results demonstrating the enhanced efficiency of the associated numerical KL expansion.

In view of the preceding findings, it is perhaps appropriate to adopt as kernels of stochastic processes which represent physical phenomena, functions which through smoothness or other properties lend themselves to KL, or otherwise, expansions of enhanced numerical efficiency. This comment is especially applicable in dealing with large dimension problems and the associated polynomial chaos expansions of structural system responses.

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Notation

The following symbols are used in this paper:

a = real-valued parameter;
$C(x_1,x_2)$ = covariance kernel;
D = real valued domain of definition of a stochastic process;
$\mathrm{err}_{\mathrm{avg}}$ = average approximation error in the covariance surface;
$\mathrm{err}_{\max}$ = maximum approximation error in the covariance surface;
$f_n(x)$ = eigenfunction of a covariance kernel;
$\mathbb{G}'$ = derivative;
P = probability measure;
$\mathfrak{S}$ = σ algebra;
$w(x,\theta)$ = stochastic process;
$x \in D$ = position (or time) coordinate;
$\theta \in \Omega$ = elementary event;
λ_n = eigenvalue of a covariance kernel;
$\xi_n(\theta)$ = random variable;
Ω = space of elementary events; and
$[\Omega,\mathfrak{S},P]$ = probability space.

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