Learning step sizes for unfolded sparse coding

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Solving sparse linear inverse problems

- **Objective:** Solve the Lasso for a fixed D and $x \sim \mathbb{P}$.

$$z^*(x) = \operatorname*{argmin}_z F_x(z) = \frac{1}{2}\|x - Dz\|_2^2 + \lambda\|z\|_1$$

- **Iterative algorithm:** Use ISTA for each x
- **Deep Learning:** Use a NN to learn a mapping

$$\Phi_\Theta^T : x \mapsto z^*(x); \qquad \text{for } x \sim \mathbb{P}$$

Learned ISTA [Gregor & Le Cun 2010]

ISTA: $z^{(t+1)} = \mathrm{ST}(z^{(t)} - \gamma D^\top(Dz^{(t)} - x), \gamma\lambda)$,
where γ is the step size, usually chosen as $1/L$.

LISTA: Let $W_z = I_m - \gamma D^\top D$; $W_x = \gamma D^\top$ and $\beta = \gamma$

$z^{(t+1)} = \mathrm{ST}(W_z z^{(t)} + W_x x, \lambda\beta)$

Re-parametrization: skjhfkjhdskjfdfh

$W_z = I_m - \alpha W^\top D$; $W_x = \alpha W^\top$

- Learn parameters $\Theta = \{W^{(t)}, \alpha^{(t)}, \beta^{(t)}\}$

Supervised learning	**Semi-supervised learning**	**Unsupervised learning**
Ground truth available $s_1, \dots, s_N$	Compute $s_i = \operatorname{argmin}_z F_{x_i}(z)$	Learn to solve the Lasso
$\sum_{i=1}^N (\Phi_\Theta^T(x_i) - s_i)^2$	$\sum_{i=1}^N (\Phi_\Theta^T(x_i) - s_i)^2$	$\sum_{i=1}^N F_{x_i}(\Phi_\Theta^T(x_i))$

Local smoothness constants

- $L = \max \|Dz\|_2^2$ subject to $\|z\|_2 = 1$

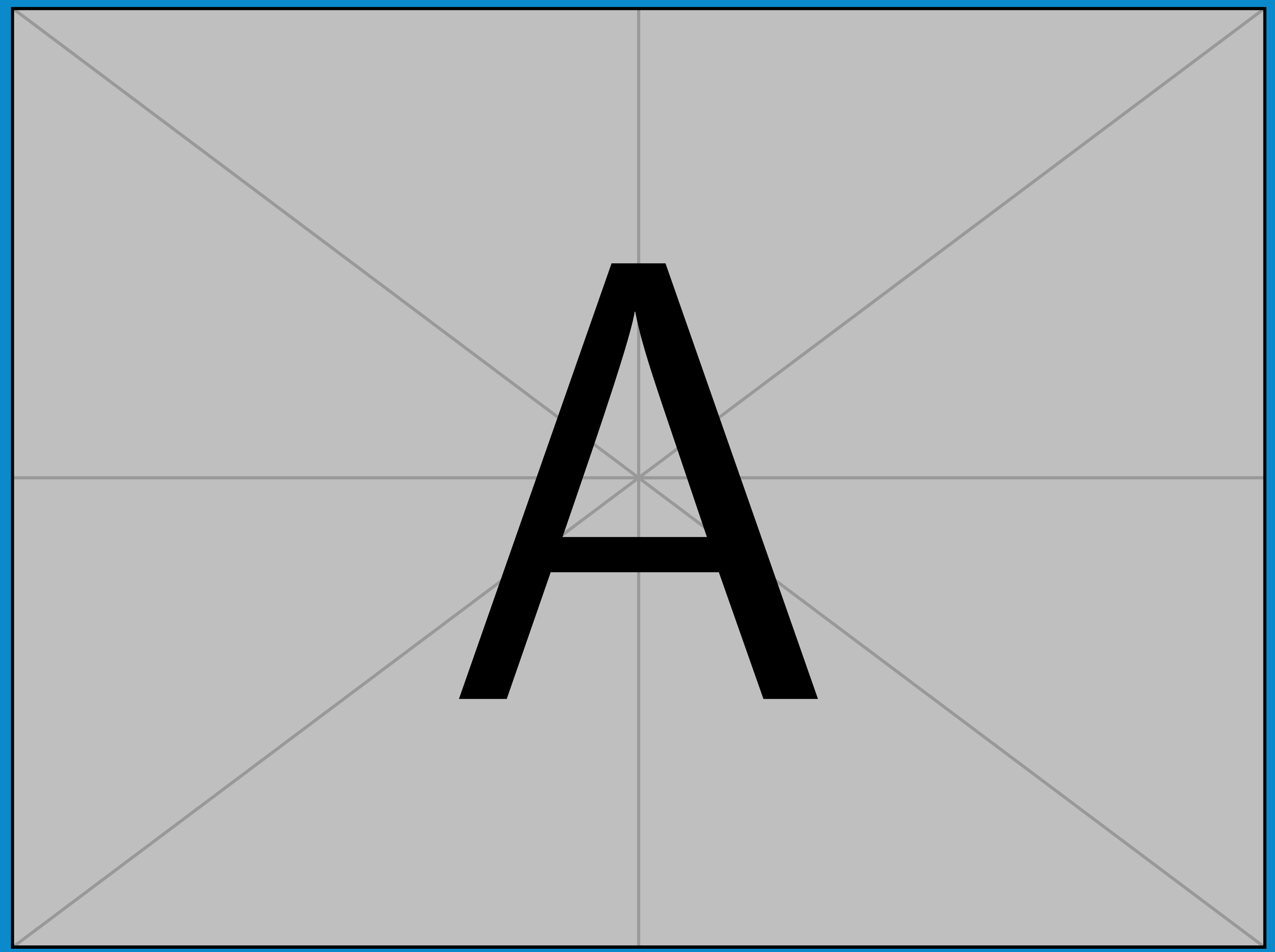
$$\begin{aligned} F_x(z) &= f_x(z^{(t)}) + \langle \nabla f_x(z^{(t)}), z - z^{(t)} \rangle + \frac{1}{2}\|D(z - z^{(t)})\|_2^2 + \lambda\|z\|_1 \\ &\leq f_x(z^{(t)}) + \langle \nabla f_x(z^{(t)}), z - z^{(t)} \rangle + \frac{L}{2}\|z - z^{(t)}\|_2^2 + \lambda\|z\|_1 \end{aligned}$$

- $L_S = \max \|Dz\|_2^2$ subject to $\|z\|_2 = 1, Supp(z) \subset S$.

ISTA with **greater step-size:** $\gamma = 1/L_s$

<u>**Theorem:**</u> The weights of a neural network trained to solve the lasso asymptotically only learn a step size.

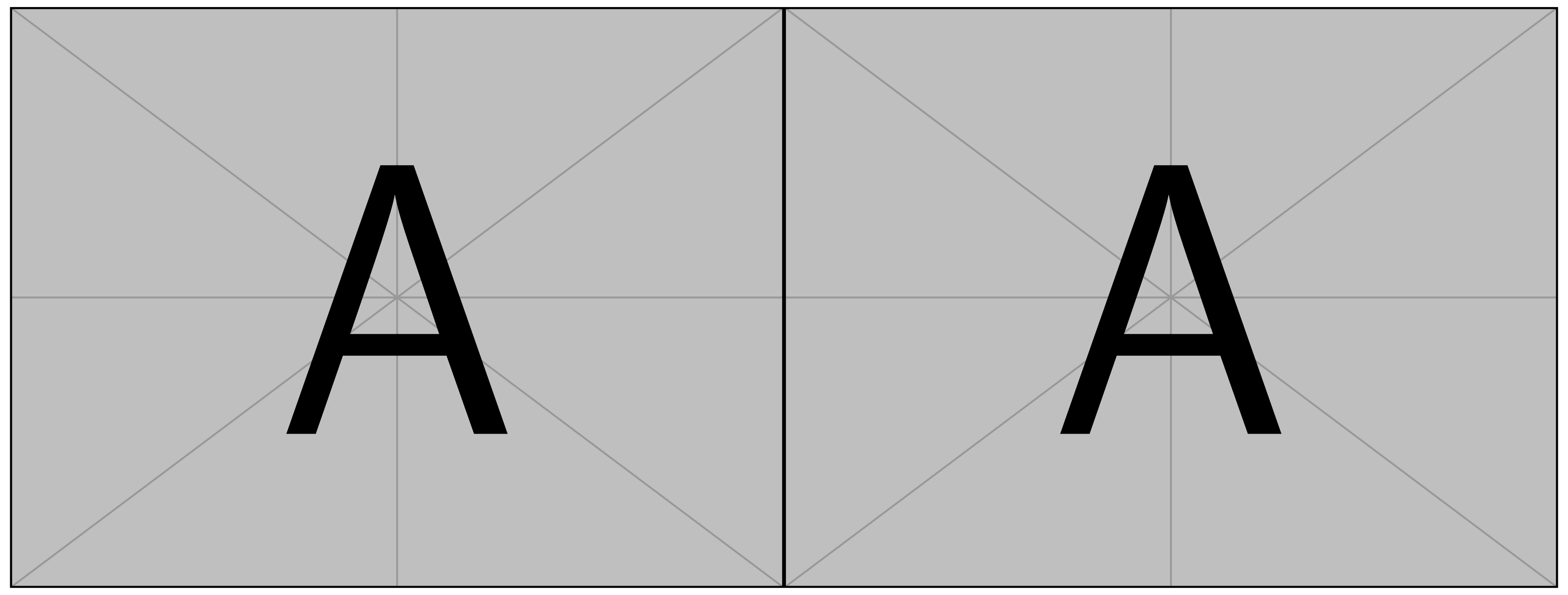
$$\frac{\alpha^{(t)}}{\beta^{(t)}} W^{(t)} \xrightarrow[t\to\infty]{} D$$



Improving ISTA step-size

Better step-sizes for ISTA

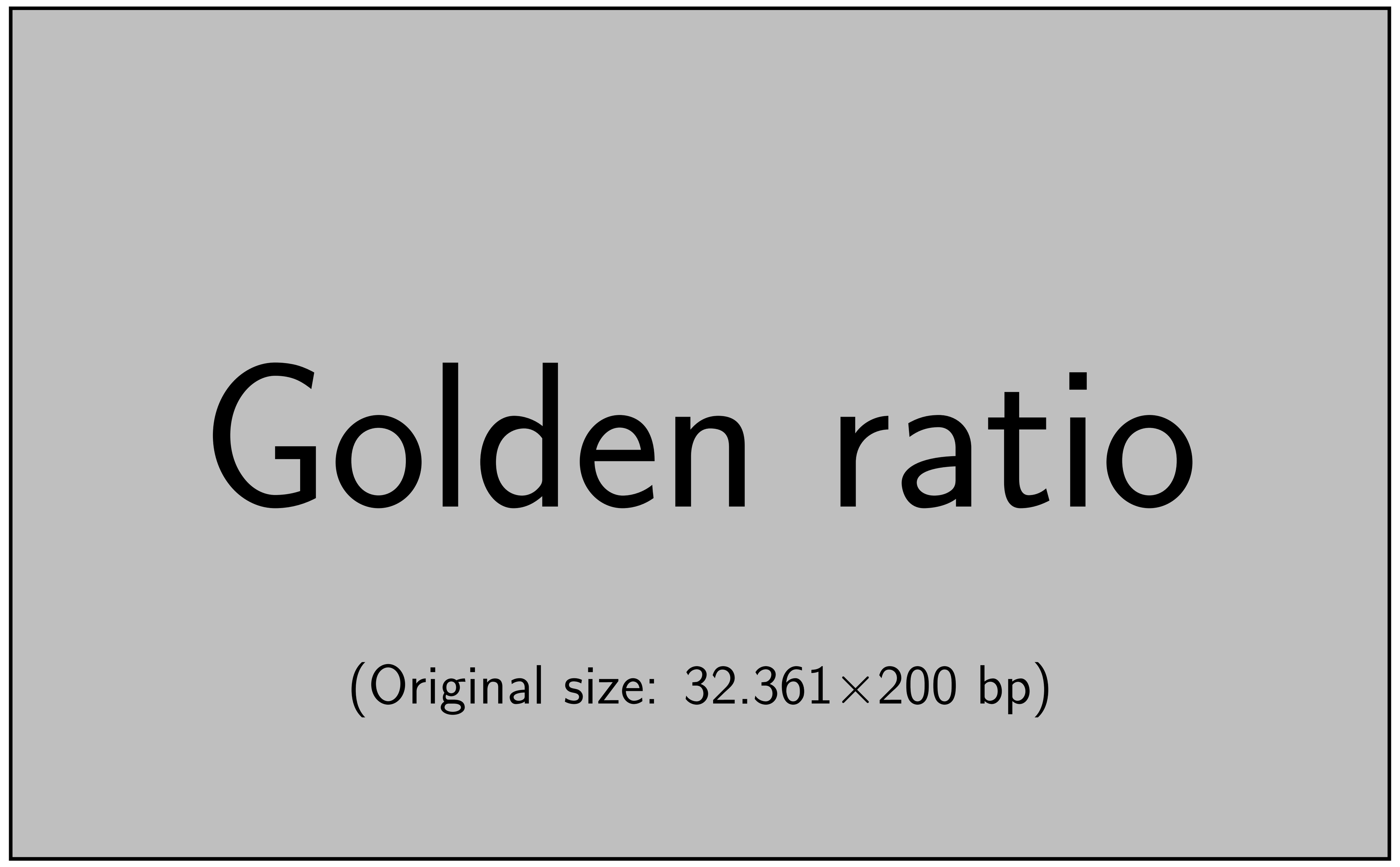
- Back-tracking line-search
- **OISTA:** Adapt step-sizes to **Local smoothness constants** L_S
- **SLISTA:** Learn only step-sizes with LISTA.



The step-sizes learned by SLISTA tend to be in $[\frac{1}{L_s}, \frac{2}{L_S}]$.

Varying the sparsity

SLISTA works better when z^* is sparse as this reduces L_S.



References

- Gregor, K. & Le Cun, Y. (2010) Learning Fast Approximations of Sparse Coding. ICML.
- Chen, X., Liu, J., Wang, Z. & Yin, W. (2018) Theoretical Linear Con