THALES

LE SÉMINAIRE PALAISIEN

Statistics and learning on Lie groups: Information geometry by Jean-Louis Koszul and symplectic model of statistical physics by Jean-Marie Souriau

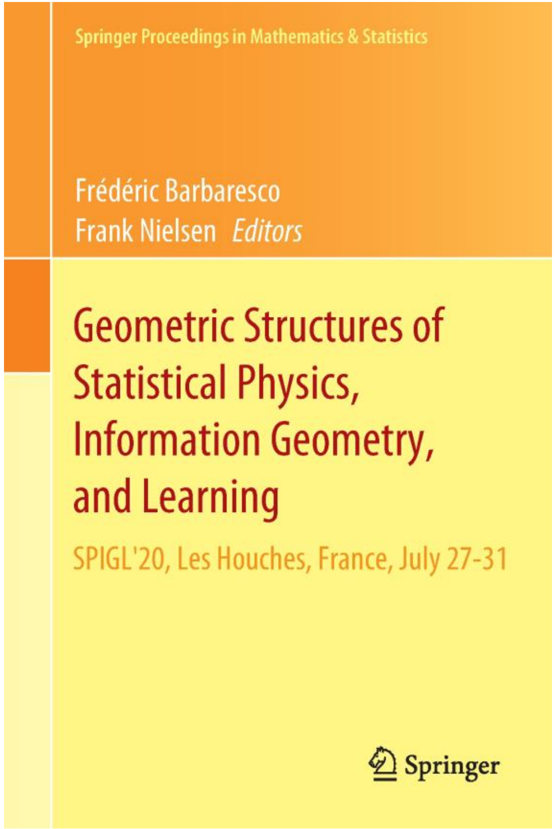
Frédéric BARBARESCO
THALES KTD PCC SENSING SEGMENT LEADER

06/07/2021


Jean-Marie Souriau


Jean-Louis Koszul

Ecole de Physique des Houches SPIGL'20, July 2020



https://www.springer.com/jp/book/9783030779566

SPIGL'20

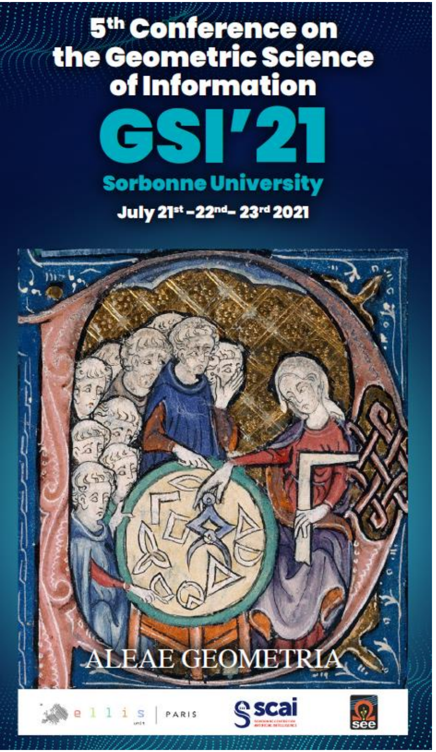
ÉCOLE DE PHYSIQUE DES HOUCHES

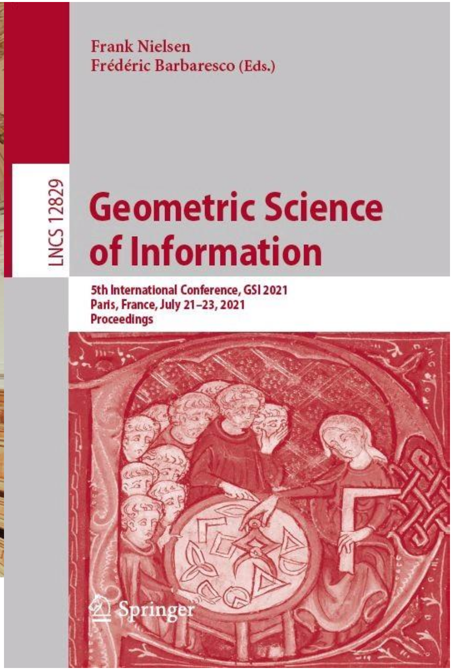
Joint Structures and Common Foundation of Statistical Physics, Information Geometry and Inference for Learning

https://franknielsen.github.io/SPIG-LesHouches2020/

https://www.youtube.com/playlist?list=PLo9ufcrEqwWExTBPgQPJwAJhoUChMbROr

GSI'21 LEARNING GEOMETRIC STRUCTURES – Sorbonne University – 21st-23rd July 2021 co-organized by SCAI Sorbonne & ELLIS Paris





www.gsi2021.org

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Talk structure

« There is nothing more in physical theories than symmetry groups except the mathematical construction which allows precisely to show that there is nothing more » - Jean-Marie Souriau

1. Préliminaries on JEAN-LOUIS KOSZUL and JEAN-MARIE SOURIAU
2. Statistical Physics, Information Geometry & Machine Learning
3. From Information Geometry Natural Gradient to Lie Groups Thermodynamics
4. ENTROPY Geometric Definition as Invariant Casimir Function in Coadjoint Representation
5. Gibbs Density on Poincaré Unit Disk from Souriau Lie Groups Thermodynamics and SU(1,1) Coadjoint Orbits
6. Gaussian density for SPD matrix via Gibbs Density for Sp(2n,R) in Siegel Upper-Half Plane and SU(n,n) in Siegel Disk
7. Gibbs density for SE(2) Lie group
8. Souriau Lie Groups Thermodynamics
9. Motivation for Lie Groups Machine Learning

Main Concepts behind Lie Group Machine Learning

GEOMETRIC SCIENCE OF INFORMATION

STRUCTURES

1. Geometric Structures
2. Lie Groups Structures
3. Poisson & Symplectic Structures
4. Information Structures

COORDINATE FREE

1. Fréchet Barycenter
2. Natural Gradient & Fisher Metric
3. Natural Langevin Dynamics
4. Geometric & Symplectic Integrators

INVARIANTS

1. Koszul-Souriau Fisher Metric
2. Souriau Gibbs Density
3. Souriau Moment Map
4. Representation Theory & Coadjoint Orbits

CALCULUS OF VARIATION

1. Poincaré-Cartan Integral Invariants
2. Euler-Poincaré Equation
3. Souriau Maxwell Principle

ENTROPY

1. Invariant Casimir Function in Coadjoint Representation
2. Maximum Entropy Probability Density
3. Legendre Transform of Laplace Transform

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PRELIMINARIES ON JEAN-LOUIS KOSZUL & JEAN-MARIE SOURIAU

Bedrock of Information Geometry

Jean-Marie Souriau (ENS 1942)

Jean-Louis Koszul (ENS 1940)

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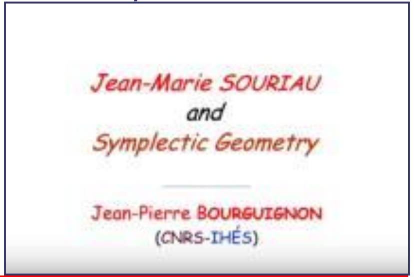
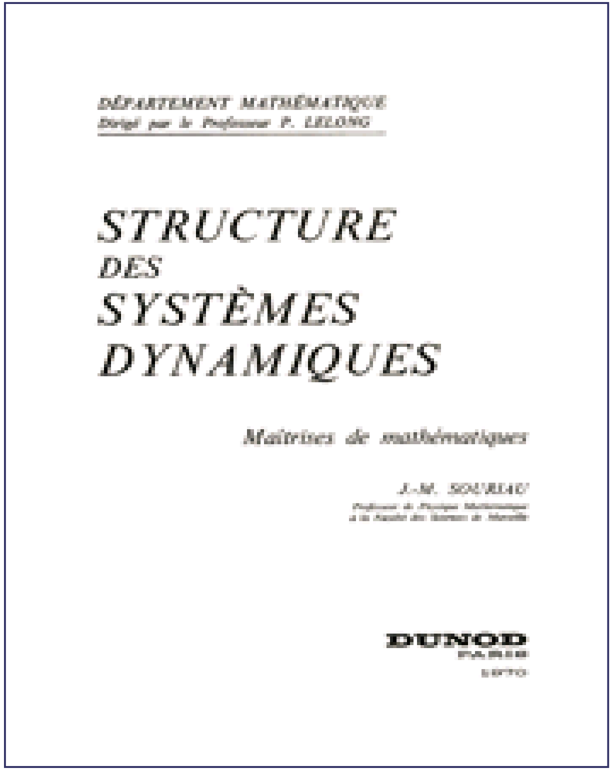
SOURIAU 2019

- Internet website : http://souriau2019.fr
- In 1969, 50 years ago, Jean-Marie Souriau published the book "**Structure des système dynamiques**", in which using the ideas of J.L. Lagrange, he formalized the "**Geometric Mechanics**" in its modern form based on **Symplectic Geometry**
- Chapter IV was dedicated to "Thermodynamics of Lie groups" (ref André Blanc-Lapierre)
- Testimony of **Jean-Pierre Bourguignon** at Souriau'19 (IHES, director of the European ERC)



https://www.youtube.com/watch?v=93hFolIBo0Q&t=3s

Souriau Book in French and in English
1969-2019 : 50th Birthday



http://www.jmsouriau.com/structure_des_systemes_dynamiques.htm
http://www.springer.com/us/book/9780817636951

Jean-Marie Souriau Seminal Paper - 1974

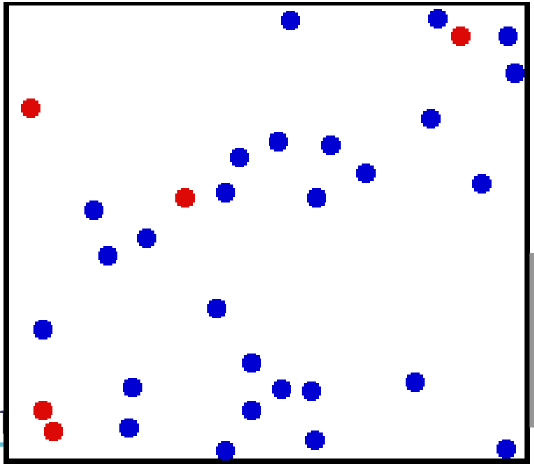
Statistical Mechanics, Lie Group and Cosmology - 1st part: Symplectic Model of Statistical Mechanics
Jean-Marie Souriau

Abstract: The classical notion of Gibbs' canonical ensemble is extended to the case of a symplectic manifold on which a Lie group has a symplectic action ("dynamic group"). The rigorous definition given here makes it possible to extend a certain number of classical thermodynamic properties (temperature is here an element of the Lie algebra of the group, heat an element of its dual), notably inequalities of convexity. In the case of non-commutative groups, particular properties appear: the symmetry is spontaneously broken, certain relations of cohomological type are verified in the Lie algebra of the group. Various applications are considered (rotating bodies, covariant or relativistic statistical Mechanics). [These results specify and complement a study published in an earlier work (*), which will be designated by the initials SSD].
(*) Souriau, J.-M., Structure des systèmes dynamique. Dunod, collection Dunod Université, Paris 1969. http://www.jmsouriau.com/structure_des_systemes_dynamiques.htm

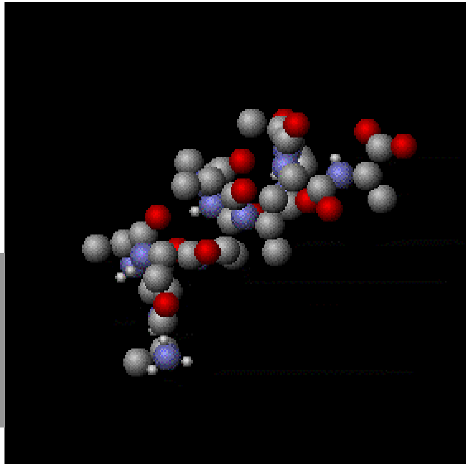
Souriau, J-M., Mécanique statistique, groupes de Lie et cosmologie, Colloques Internationaux C.N.R.S., n°237 – Géométrie symplectique et physique mathématique, pp.59-113, 1974
English translation by F. Barbaresco: https://link.springer.com/chapter/10.1007/978-3-030-77957-3_2
Barbaresco F. (2021) Jean-Marie Souriau's Symplectic Model of Statistical Physics: Seminal Papers on Lie Groups Thermodynamics - Quod Erat Demonstrandum. In: Barbaresco F., Nielsen F. (eds) Geometric Structures of Statistical Physics, Information Geometry, and Learning. SPIGL 2020. Springer Proceedings in Mathematics & Statistics, vol 361. Springer

Poly-Symplectic Model of Souriau Lie Groups Thermodynamics

- Souriau Geometric (Planck) Temperature is **an element of Lie Algebra** of Dynamical Group (Galileo/Poincaré groups) acting on the system
- Generalized Entropy is **Legendre Transform of minus logarithm of Laplace Transform**
- Fisher(-Souriau) Metric is a **Geometric Calorific Capacity (hessian of Massieu Potential)**
- Higher Order Souriau Lie Groups Thermodynamics is given by **Günther's Poly-Symplectic Model (vector-valued model in non-equivariant case)**



Souriau formalism is fully **covariant**, with no special coordinates (**covariance of Gibbs density wrt Dynamical Groups**)



Tribute to Jean-Louis Koszul

Jean-Louis Koszul and the elementary structures of Information Geometry

- Koszul has introduced fundamental tools to characterize the geometry of sharp convex cones, as Koszul-Vinberg characteristic Function, Koszul Forms, and affine representation of Lie Algebra and Lie Group.
- The 2nd Koszul form is linked to an extension of classical Fisher metric.
- Koszul theory of hessian structures and Koszul forms could be considered as main foundation and pillars of Information Geometry.

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Koszul Book on Souriau Work:
The Little Green Book

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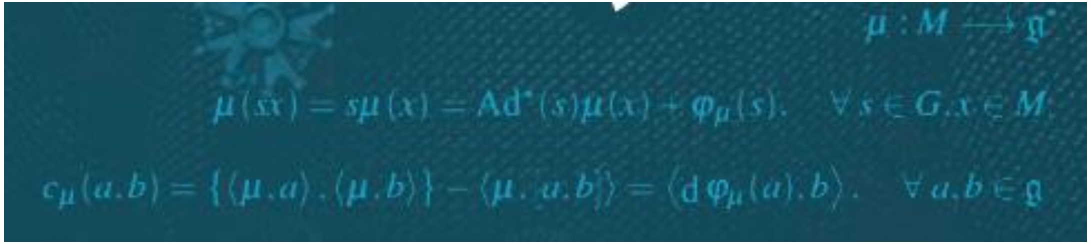
Koszul Book on Souriau Work: The Little Green Book

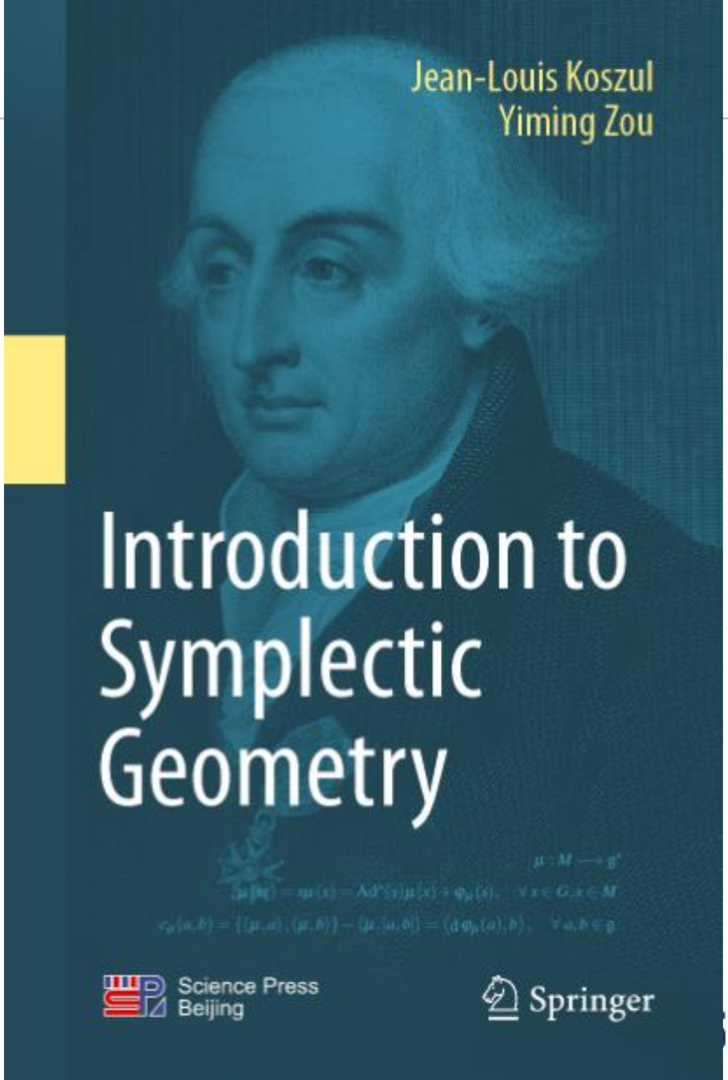
Jean-Louis Koszul · Yiming Zou

Introduction to Symplectic Geometry

Forewords by Michel Nguiffo Boyom, Frédéric Barbaresco and Charles-Michel Marle

This introductory book offers a unique and unified overview of symplectic geometry, highlighting the differential properties of symplectic manifolds. It consists of six chapters: Some Algebra Basics, Symplectic Manifolds, Cotangent Bundles, Symplectic G-spaces, Poisson Manifolds, and A Graded Case, concluding with a discussion of the differential properties of graded symplectic manifolds of dimensions (o,n). It is a useful reference resource for students and researchers interested in geometry, group theory, analysis and differential equations.



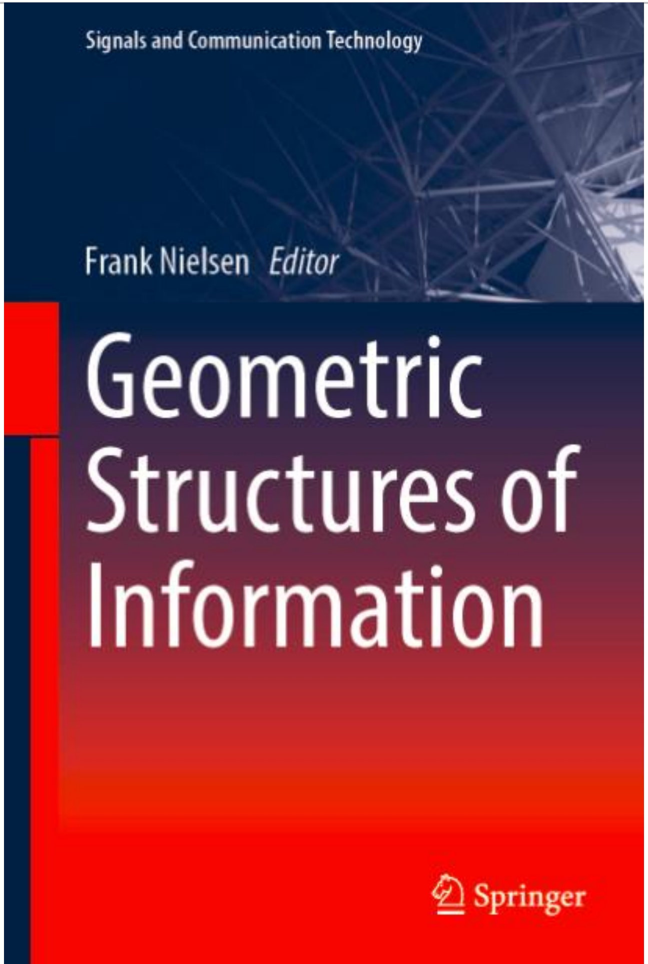


Koszul’s papers

Koszul’s paper at the foundation of the elementary structure of Information

- Koszul J.L., Sur la forme hermitienne canonique des espaces homogènes complexes. Can. J. Math., n°7, 562–576, 1955
- Koszul J.L., Exposés sur les Espaces Homogènes Symétriques; Publicação da Sociedade de Matematica de São Paulo: São Paulo, Brazil, 1959
- Koszul J.L., Domaines bornées homogènes et orbites de groupes de transformations affines, Bull. Soc. Math. France 89, pp. 515-533., 1961
- Koszul J.L., Ouverts convexes homogènes des espaces affines. Math. Z., n°79, 254–259, 1962
- Koszul J.L. Sous-groupes discrets des groupes de transformations affines admettant une trajectoire convexe, C.R. Acad. Sc. T.259, pp.3675-3677, 1964
- Koszul J.L., Variétés localement plates et convexité. Osaka. J. Math., n°2, 285–290, 1965
- Koszul J.L, Lectures on Groups of Transformations, Tata Institute of Fundamental Research, Bombay, 1965
- Koszul J.L., Déformations des variétés localement plates, .Ann Inst Fourier, n°18 , 103-114., 1968
- Koszul J.L., Trajectoires Convexes de Groupes Affines Unimodulaires. In Essays on Topology and Related Topics; Springer: Berlin, Germany, pp. 105–110, 1970
- Selected Papers of J L Koszul, Series in Pure Mathematics, Volume 17, World Scientific Publ, 1994

Geometric Structures of Information, SPRINGER



Geometric Structures of Information

- https://www.springer.com/us/book/9783030025199

Paper on Jean-Louis Koszul

- Barbaresco, F. , Jean-Louis Koszul and the Elementary Structures of Information Geometry, Geometric Structures of Information, pp 333-392, SPRINGER, 2018
- https://link.springer.com/chapter/10.1007%2F978-3-030-02520-5_12

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Statistical Physics, Information Geometry & Machine Learning

Preamble

- The geometric models of information, statistical physics and inference in machine learning share common structures as was illustrated at the **Ecole de Physique des Houches SPIGL'20** in July 2020 (https://franknielsen.github.io/SPIG-LesHouches2020/) and in the recent **SPRINGER book** on the subject [1].
- The geometry of information [2] uses **natural learning gradients** which have the remarkable property of being invariant to the coordinate systems encoding information through the so-called Fisher metric. These intrinsic schemes preserve symmetries and can extend to more abstract spaces when the data is encoded on differential manifolds or symplectic manifolds (case of encoding of information by Lie groups as in robotics or Guidance Navigation & Control). In these more structured coding spaces, the notion of **Fisher metric** extends to that of **Koszul-Souriau metric**, the definition of **Entropy** to that of an **invariant Casimir function in coadjoint representation**, by establishing the links with the systems integrable within the meaning of Liouville [3,4,5,6].
- These themes will be developed at the **5th edition of the SEE GSI'21** (Geometric Science of Information) conference in July 2021 (www.gsi2021.org) at Paris Sorbonne University, co-organized with **SCAI Sorbonne** (https://scai.sorbonne-universite.fr/) and **ELLIS Paris Unit** (https://ellis-paris.github.io/), and whose theme will be "**Learning Geometric Structures**".

References

- [1] Frédéric Barbaresco & Frank Nielsen, Geometric Structures of statistical Physics, Information Geometry and Learning, Actes de la Summer Week de l'Ecole de Physique des Houches SPIGL'20, SPRINGER Proceedings in Mathematics & Statistics, 2021; https://www.springer.com/jp/book/9783030779566
- [2] Frédéric Barbaresco & Frank Nielsen, La Géométrie de l'Information: hors-série n°73 du magazine TANGENTE ; https://www.decitre.fr/revues/tangente-hors-serie-n-73-mathematiques-et-emploi-9782848842387.html
- [3] Charles-Michel Marle, On Gibbs states of mechanical systems with symmetries, arXiv:2012.00582v1 [math.DG], https://arxiv.org/abs/2012.00582
- [4] Frédéric Barbaresco, Lie Group Statistics and Lie Group Machine Learning Based on Souriau Lie Groups Thermodynamics & Koszul-Souriau-Fisher Metric: New Entropy Definition as Generalized Casimir Invariant Function in Coadjoint Representation. Entropy 2020, 22, 642.
- [5] Frédéric Barbaresco, François Gay-Balmaz, Lie Group Cohomology and (Multi)Symplectic Integrators: New Geometric Tools for Lie Group Machine Learning Based on Souriau Geometric Statistical Mechanics. Entropy 2020, 22, 498.
- [6] Frédéric Barbaresco, Koszul lecture related to geometric and analytic mechanics, Souriau's Lie group thermodynamics and information geometry. SPRINGER Information Geometry, 2021

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Thermodynamics

Statistical Physics

Information Geometry

Lie Group Representation Theory

https://www.youtube.com/playlist?list=PLo9ufcrEqwWExTBPgQPJwAJhoUChMbROr

SPIGL'20

cnrs cea UNIVERSITÉ DE LYON ENS DE LYON ÉCOLE DE PHYSIQUE DES HOUCHES

Grenoble INP UGA — UGA Université Grenoble Alpes

Joint Structures and Common Foundation of Statistical Physics, Information Geometry and Inference for Learning

26th July to 31st July 2020

https://franknielsen.github.io/SPIG-LesHouches2020/

8 Lectures (90 min)

Langevin Dynamics: Old and News (x 2) – Eric Moulines

Computational Information Geometry

On statistical distances and information geometry for ML – **Frank Nielsen**

Information Manifold modeled with Orlicz Spaces – **Giovanni Pistone**

Non-Equilibrium Thermodynamic Geometry

A variational perspective of closed and open systems- **François Gay-Balmaz**

A Homogeneous Symplectic Approach - **Arjan van der Schaft**

Geometric Mechanics

Gallilean Mechanics & Thermodynamics of Continua - **Géry de Saxcé**

Souriau-Casimir Lie Groups Thermodynamics & Machine Learning – **F. Barbaresco**

17 Keynotes (60 min)

Learning with Few Labeled Data - **Pratik Chaudhari**

Sampling and statistical physics via symmetry - **Steve Huntsman**

The Bracket Geometry of Measure-Preserving Flows and Diffusions - **Alessandro Barp**

Exponential Family by Representation Theory - **Koichi Tojo**

Learning Physics from Data - **Francisco Chinesta**

Information Geometry and Integrable Hamiltonian - **Jean-Pierre Françoise**

Information Geometry and Quantum Fields - **Kevin Grosvenor**

Thermodynamic efficiency implies predictive inference- **Susanne Still**

Diffeological Fisher Metric - **Hông Vân Lê**

Deep Learning as Optimal Control - **Elena Celledoni**

Schroedinger's problem, Hamilton-Jacobi-Bellman equations and regularized Mass Transportation - **Jean-Claude Zambrini**

Mechanics of the probability simplex - **Luigi Malagò**

Dirac structures in nonequilibrium thermodynamics - **Hiroaki Yoshimura**

Port Thermodynamic Systems Control - **Bernhard Maschke**

Covariant Momentum Map Thermodynamics - **Goffredo Chirco**

Contact geometry and thermodynamical systems - **Manuel de León**

Computational dynamics of multibody-fluid system in Lie group setting- **Zdravko Terze**

Charles-Michel Marle (Corps des Mines, Sorbonne Université)

On Gibbs states of mechanical systems with symmetries
Charles-Michel Marle

Gibbs states for the Hamiltonian action of a Lie group on a symplectic manifold were studied, and their possible applications in Physics and Cosmology were considered, by the French mathematician and physicist Jean-Marie Souriau. They are presented here with detailed proofs of all the stated results. Using an adaptation of the cross product for pseudo-Euclidean three-dimensional vector spaces, we present several examples of such Gibbs states, together with the associated thermodynamic functions, for various two-dimensional symplectic manifolds, including the pseudo-spheres, the Poincaré disk and the Poincaré half-plane.

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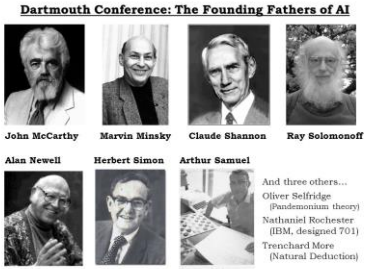
Towards Lie Group & Symplectic Machine Learning

Neural Networks

McCulloch & Pitts (networks of binary neurons can do logic), Donald Hebb (Hebbian synaptic plasticity) Norbert Wiener (cybernetics)

1 — <1950

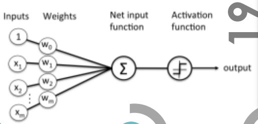
Dartmouth convention



2 — 1956

Perceptron

Rosenblatt computes learning capabilities



3 — 1957

1st winter 60's

Natural Gradient

In Information Geometry Framework, RIKEN introduces Natural Gradient based on Fisher Information Matrix (coordinate Free Gradient)

4 — 1990

2nd winter 90's

Deep Learning

Facebook's AI can perform facial recognition as accurate as human & other developments (Convolutional, Full Convolutional, …)

5 — 2010

Deep Natural Gradient

LRI ORSAY Lab introduces Robust Natural Gradient for Deep Learning (Yann Ollivier & Gaetan Marceau-Caron)

6 — 2016

Natural Langevin Dynamics

FACEBOOK & MILA propose Natural Gradient with Stochastic Langevin Gradient (Yann Ollivier & Gaetan Marceau-Caron)

7 — 2017

Deep Learning On Graph

FACEBOOK & UCLA extend Deep Learning on Graph. Convolutive Network on Graph (Xavier Bresson, …)

8 — 2017

Deep Learning on Lie Group

Deep network architecture to learn more appropriate Lie group features for 3D action recognition (ETH Zurich, KU Leuven)

9 — 2018

Lie Group Machine Learning

Semi-supervised and nuclear learning frameworks based on Lie group Theory (Fanzhang Li, Li Zhang, Zhao Zhang)

2019

Machine Learning on Homogeneous Symplectic Manifold of Lie Group co-adjoint Orbits (Jean-Marie Souriau model)

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AI/Machine Learning Evolution: ALGEBRA COMPUTATION STRUCTURES

Calcul formel pour les méthodes de Lie en mécanique hamiltonienne
P.V. Koseleff, X/CMLS PhD, 1993 (P. Cartier)

Souriau Exponential Map Algorithm for Machine Learning on Matrix Lie Groups
Frédéric Barbaresco, Springer GSI'19, 2019

Supervarieties, Sow. Math. Dokl. 16 (1975), 1218-1222.
F. A. Berzin and D. A. Leites

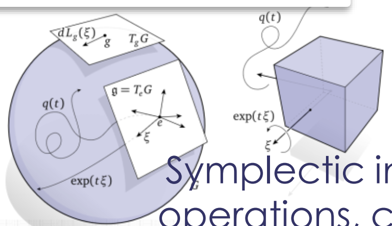
LIE SUPER ALGEBRA

$$X = \begin{bmatrix} A & B \\ C & D \end{bmatrix}$$

$$\mathrm{Ber}(X) = \det(A)\det(D - CA^{-1}B)^{-1}$$

Berezian Determinant

LIE ALGEBRA



Symplectic integrators, non-commutative operations, coadjoint orbits, moment map

Computer Algebra Group- Scratchpad, IBM,1971

axiom

GEOMSTATS

LINEAR ALGEBRA

$$\begin{bmatrix} A & B \\ C & D \\ E & F \end{bmatrix} \times \begin{bmatrix} G \\ H \end{bmatrix} = \begin{bmatrix} A\times G + B\times H \\ C\times G + D\times H \\ E\times G + F\times H \end{bmatrix}$$

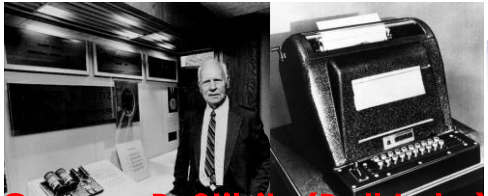
Vectors space, commutative matrix operations, eigen-analysis

GOOGLE TPU (Tensor Processing Unit)

x	y	x AND y	x OR y	NOT x	NOT y
0	0	0	0	1	1
0	1	0	1	1	0
1	0	0	1	0	1
1	1	1	1	0	0

BOOLE ALGEBRA

Boolean logic digital circuits using electromechanical relays as the switching element.



George R. Stibitz (Bell Labs)

ALGEBRA is the study of mathematical symbols and the rules for manipulating these symbols

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GEOMSTATS: PYTHON Library for Lie Group Machine Learning

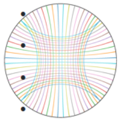
Geomstats

https://github.com/geomstats/geomstats

pypi package 2.1.0 | build passing | codecov 92% | codecov unknown | codecov unknown (Coverages for: numpy, tensorflow, pytorch)

Geomstats is an open-source Python package for computations and statistics on manifolds. The package is organized into two main modules: `geometry` and `learning`.

The module `geometry` implements concepts in differential geometry, and the module `learning` implements statistics and learning algorithms for data on manifolds.



To get started with `geomstats`, see the examples directory.

For more in-depth applications of `geomstats`, see the applications repository.

The documentation of `geomstats` can be found on the documentation website.

If you find `geomstats` useful, please kindly cite our paper.

Install geomstats via pip3

Video: https://m.youtube.com/watch?v=Ju-Wsd84uG0

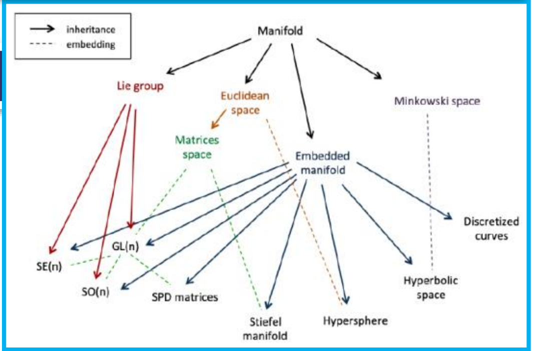
```
pip3 install geomstats
```

hal-02536154, version 1 | Pré-publication, Document de travail

https://hal.inria.fr/hal-02536154

Geomstats: A Python Package for Riemannian Geometry in Machine Learning

Nina Miolane [1], Alice Le Brigant, Johan Mathe [2], Benjamin Hou [3], Nicolas Guigui [4, 5], Yann Thanwerdas [4, 5], Stefan Heyder [6], Olivier Peltre, Niklas Koep, Hadi Zaatiti [7], Hatem Hajri [7], Yann Cabanes, Thomas Gerald, Paul Chauchat [8], Christian Shewmake, Bernhard Kainz, Claire Donnat [9], Susan Holmes [1], Xavier Pennec [4, 5] Détails



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PYTHON Library for Machine Learning on Manifold and Lie Group

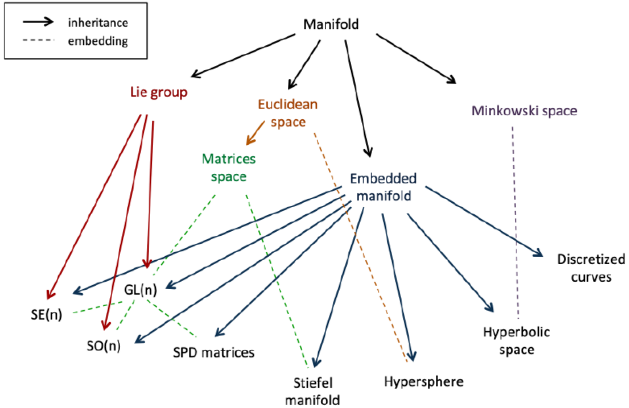
Computations and statistics on manifolds with geometric structures

- Initiated by INRIA & Stanford University
- Point of contact **Nina Miolane**
(Department of Statistics - Stanford Statistics)
- **PYTHON GEOMSTATS** Package:
  - https://github.com/geomstats/geomstats
  - Python Package for Riemannian Geometry in Machine Learning
  - Paper: https://arxiv.org/abs/1805.08308

Nina MIOLANE
L'Oréal – Unesco Prize 2016 « Woman in Science »

SCAI GEOMSTATS HACKATHON, GSI'21

- Chaired by **Nina Miolane**
- **https://www.see.asso.fr/en/wiki/369069_scai-geo**



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From Information Geometry Natural Gradient to Lie Groups Thermodynamics

Fisher Metric and Fréchet-Darmois (Cramer-Rao) Bound

- Cramer-Rao –Fréchet-Darmois Bound has been introduced by Fréchet in 1939 and by Rao in 1945 as inverse of the Fisher Information Matrix: $I(\theta)$

$$R_{\hat{\theta}} = E\left[\left(\theta-\hat{\theta}\right)\left(\theta-\hat{\theta}\right)^{+}\right] \geq I(\theta)^{-1} \qquad \left[I(\theta)\right]_{i,j} = -E\left[\frac{\partial^2 \log p_\theta(z)}{\partial\theta_i \partial\theta_j^*}\right]$$

- Rao has proposed to introduced an invariant metric in parameter space of density of probabilities (axiomatised by N. Chentsov):

$$ds_\theta^2 = Kullback\_Divergence\left(p_\theta(z), p_{\theta+d\theta}(z)\right)$$

$$ds_\theta^2 = -\int p_\theta(z)\log\frac{p_{\theta+d\theta}(z)}{p_\theta(z)}dz$$

$$w = W(\theta) \Rightarrow ds_w^2 = ds_\theta^2$$

$$ds_\theta^2 \underset{Taylor}{\approx} \sum_{i,j} g_{ij} d\theta_i d\theta_j^* = \sum_{i,j}\left[I(\theta)\right]_{i,j} d\theta_i d\theta_j^* = d\theta^+ . I(\theta) . d\theta$$

Distance Between Gaussian Density with Fisher Metric

Fisher Matrix for Gaussian Densities:

$$I(\theta)=\begin{bmatrix}\frac{1}{\sigma^2} & 0\\ 0 & \frac{2}{\sigma^2}\end{bmatrix} \quad \text{avec} \quad E\left[\left(\theta-\hat{\theta}\right)\left(\theta-\hat{\theta}\right)^T\right]\geq I(\theta)^{-1} \quad \text{et} \quad \theta=\begin{pmatrix} m\\ \sigma\end{pmatrix}$$

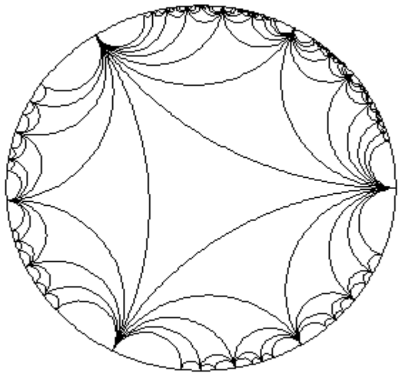
- Fisher matrix induced the following differential metric :

$$ds^2=d\theta^T.I(\theta).d\theta=\frac{dm^2}{\sigma^2}+2.\frac{d\sigma^2}{\sigma^2}=\frac{2}{\sigma^2}\left[\left(\frac{dm}{\sqrt{2}}\right)^2+(d\sigma)^2\right]$$

- Poincaré Model of upper half-plane and unit disk

$$z=\frac{m}{\sqrt{2}}+i.\sigma \qquad \omega=\frac{z-i}{z+i} \quad \left(|\omega|<1\right)$$

$$\Rightarrow ds^2=8.\frac{|d\omega|^2}{\left(1-|\omega|^2\right)^2}$$



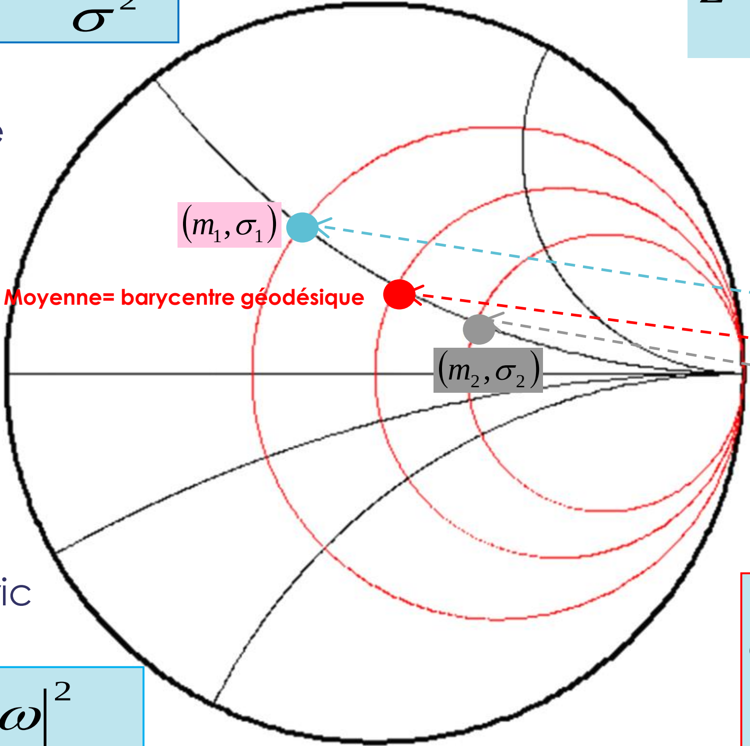
1 monovariate gaussian = 1 point in Poincaré unit disk

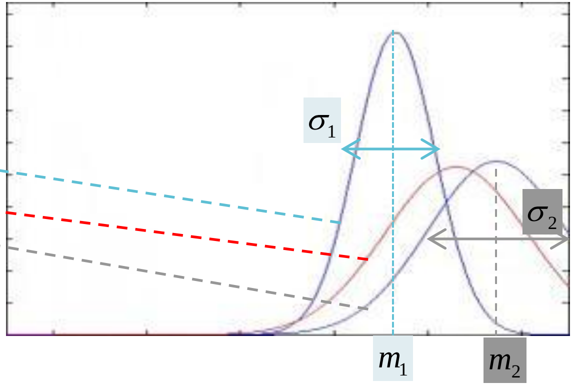
$$ds^2 = \frac{dm^2}{\sigma^2} + 2.\frac{d\sigma^2}{\sigma^2}$$

Fisher Metric in
Poincaré Half-Plane

$$z = \frac{m}{\sqrt{2}} + i.\sigma$$

$$\omega = \frac{z-i}{z+i} \quad (|\omega| < 1)$$





Poincaré-Fisher metric
In Unit Disk

$$ds^2 = 8.\frac{|d\omega|^2}{\left(1-|\omega|^2\right)^2}$$

$$d^2\left(\{m_1,\sigma_1\},\{m_2,\sigma_2\}\right) = 2.\left(\log\frac{1+\delta(\omega^{(1)},\omega^{(2)})}{1-\delta(\omega^{(1)},\omega^{(2)})}\right)^2$$

$$\text{with} \quad \delta(\omega^{(1)},\omega^{(2)}) = \left|\frac{\omega^{(1)}-\omega^{(2)}}{1-\omega^{(1)}\omega^{(2)*}}\right|$$

Machine Learning & Gradient descent

Gradient descent for Learning

- Information geometry has been derived from invariant geometrical structure involved in statistical inference. The Fisher metric defines a Riemannian metric as the Hessian of two dual potential functions, linked to dually coupled affine connections in a manifold of probability distributions. With the Souriau model, this structure is extended preserving the Legendre transform between two dual potential function parametrized in Lie algebra of the group acting transitively on the homogeneous manifold.
- Classically, to optimize the parameter θ of a probabilistic model, based on a sequence of observations y_t , is an online gradient descent with learning rate η_t , and the loss function $l_t = -\log p\left(y_t / \hat{y}_t\right)$:

$$\theta_t \leftarrow \theta_{t-1} - \eta_t \frac{\partial l_t\left(y_t\right)^T}{\partial \theta}$$

Information Geometry & Natural Gradient

- This simple gradient descent has a first drawback of using the same non-adaptive learning rate for all parameter components, and a second drawback of non invariance with respect to parameter re-encoding inducing different learning rates. **S.I. Amari** has introduced the **natural gradient** to preserve this invariance to be insensitive to the characteristic scale of each parameter direction. The gradient descent could be corrected by $I(\theta)^{-1}$ where I is the **Fisher information matrix** with respect to parameter θ , given by:

$$\theta_t \leftarrow \theta_{t-1} - \eta_t I(\theta_{t-1})^{-1} \frac{\partial l_t(y_t)^T}{\partial \theta}$$

$$I(\theta) = [g_{ij}]$$

$$\text{with } g_{ij} = \left[-E_{y \approx p(y/\theta)} \left[\frac{\partial^2 \log p(y/\theta)}{\partial \theta_i \partial \theta_j} \right] \right]_{ij} = \left[E_{y \approx p(y/\theta)} \left[\frac{\partial \log p(y/\theta)}{\partial \theta_i} \frac{\partial \log p(y/\theta)}{\partial \theta_j} \right] \right]_{ij}$$

Natural Gradient & Stochastic Gradient: Natural Langevin Dynamics

Natural Langevin Dynamics: Natural Gradient with Langevin Stochastics descent

- To regularize solution and avoid over-fitting, Stochastic gradient is used, as Langevin Stochastic Gradients
- **Yann Ollivier** (FACEBOOK FAIR, previously CNRS LRI Orsay) and **Gaëtan Marceau-Caron** (MILA, previously CNRS LRI Orsay and THALES LAS/ATM & TRT PhD) have proposed to coupled **Natural Gradient** with **Langevin Dynamics**: **Natural Langevin Dynamics (Best SMF/SEE GSI'17 paper)**

$$\theta_t \leftarrow \theta_{t-1} - \eta_t I(\theta_{t-1})^{-1} \frac{\partial\left(l_t(y_t)^T - \frac{1}{N}\log\alpha(\theta_{t-1}) \right)}{\partial\theta} + \sqrt{\frac{2\eta_t}{N}} I(\theta_{t-1})^{-1/2} N(0, I_d)$$

- The resulting natural Langevin dynamics combines the advantages of Amari's natural gradient descent and Fisher-preconditioned Langevin dynamics for large neural networks

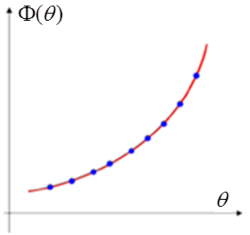
Legendre Transform, Dual Potentials & Fisher Metric

- **S.I. Amari** has proved that the Riemannian metric in an exponential family is the **Fisher information matrix** defined by:

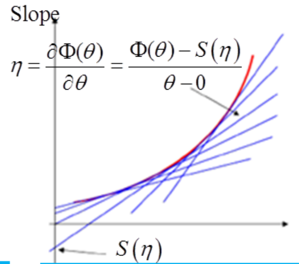
$$g_{ij} = -\left[\frac{\partial^2 \Phi}{\partial \theta_i \partial \theta_j}\right]_{ij} \quad \text{with} \quad \Phi(\theta) = -\log \int_R e^{-\langle \theta, y \rangle} dy$$

- and the dual potential, the **Shannon entropy**, is given by the **Legendre transform**:

$$S(\eta) = \langle \theta, \eta \rangle - \Phi(\theta) \quad \text{with} \quad \eta_i = \frac{\partial \Phi(\theta)}{\partial \theta_i} \quad \text{and} \quad \theta_i = \frac{\partial S(\eta)}{\partial \eta_i}$$

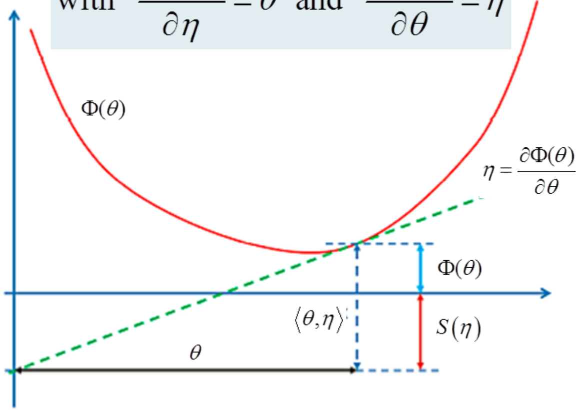


Classical Geometry
(curve is given by a continuum of points)



Plücker Geometry
(curve is given by the envelop of its tangents)

$$S(\eta) = \langle \theta, \eta \rangle - \Phi(\theta)$$
$$\text{with} \quad \frac{\partial S(\eta)}{\partial \eta} = \theta \quad \text{and} \quad \frac{\partial \Phi(\theta)}{\partial \theta} = \eta$$



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Information Geometry & Machine Learning : Legendre structure

Legendre Transform, Dual Potentials & Fisher Metric

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$$S(\eta) = \langle \theta, \eta \rangle - \Phi(\theta) \quad \text{with} \quad \eta_i = \frac{\partial \Phi(\theta)}{\partial \theta_i} \quad \text{and} \quad \theta_i = \frac{\partial S(\eta)}{\partial \eta_i}$$

Information Geometry Natural Gradient

$$\theta_t \leftarrow \theta_{t-1} - \eta_t I(\theta_{t-1})^{-1} \frac{\partial l_t(y_t)^T}{\partial \theta}$$

$$I(\theta) = [g_{ij}] \qquad l_t = -\log p(y_t / \hat{y}_t)$$

$$\text{with} \quad g_{ij} = \left[-E_{y \approx p(y/\theta)} \left[\frac{\partial^2 \log p(y/\theta)}{\partial\theta_i \partial\theta_j} \right] \right]_{ij} = \left[E_{y \approx p(y/\theta)} \left[\frac{\partial \log p(y/\theta)}{\partial\theta_i} \frac{\partial \log p(y/\theta)}{\partial\theta_j} \right] \right]_{ij}$$

Koszul-Vinberg Characteristic Function, Koszul Forms

- **J.L. Koszul** and **E. Vinberg** have introduced an affinely invariant Hessian metric on a sharp convex cone through its **characteristic function**

$$\Phi_\Omega(\theta) = -\log \int_{\Omega^*} e^{-\langle \theta, y \rangle} dy = -\log \psi_\Omega(\theta) \quad \text{with } \theta \in \Omega \text{ sharp convex cone}$$

$$\psi_\Omega(\theta) = \int_{\Omega^*} e^{-\langle \theta, y \rangle} dy \quad \text{with Koszul-Vinberg Characteristic function}$$


Jean-Louis Koszul

- **1st Koszul form** α: $\alpha = d\Phi_\Omega(\theta) = -d\log \psi_\Omega(\theta)$
- **2nd Koszul form** γ: $\gamma = D\alpha = Dd\log \psi_\Omega(\theta)$

$$\left(Dd\log \psi_\Omega(x)\right)(u) = \frac{1}{\psi_\Omega(u)^2}\left[\int_{\Omega^*} F(\xi)^2 d\xi . \int_{\Omega^*} G(\xi)^2 d\xi - \left(\int_{\Omega^*} F(\xi).G(\xi) d\xi\right)^2\right] > 0 \quad \text{with} \quad F(\xi) = e^{-\frac{1}{2}\langle x,\xi\rangle} \quad \text{and} \quad G(\xi) = e^{-\frac{1}{2}\langle x,\xi\rangle}\langle u,\xi\rangle$$

- Diffeomorphism: $\eta = \alpha = -d\log \psi_\Omega(\theta) = \int_{\Omega^*} \xi p_\theta(\xi) d\xi \quad \text{with } p_\theta(\xi) = \dfrac{e^{-\langle \xi,\theta\rangle}}{\int_{\Omega^*} e^{-\langle \xi,\theta\rangle} d\xi}$
- Legendre transform: $S_\Omega(\eta) = \langle \theta, \eta \rangle - \Phi_\Omega(\theta) \quad \text{with} \quad \eta = d\Phi_\Omega(\theta) \quad \text{and} \quad \theta = dS_\Omega(\eta)$

Koszul-Vinberg Characteristic Function

Koszul-Vinberg Characteristic function

- The name "characteristic function" come from the following link:
  Let Ω be a cone in U and Ω^* its dual, for any $\lambda > 0, H_\lambda(x) = \left\{ y \in U / \langle x, y \rangle = \lambda \right\}$

  and let $d^{(\lambda)}y$ denote the Lebesgue measure on $H_\lambda(x)$:

$$\psi_\Omega(x) = \int_{\Omega^*} e^{-\langle x,y\rangle} dy = \frac{(m-1)!}{\lambda^{m-1}} \int_{\Omega^* \cap H_\lambda(x)} d^{(\lambda)} y$$

- There exist a bijection $x \in \Omega \mapsto x^* \in \Omega^*$, satisfying the relation $(gx)^* = {}^t g^{-1} x^*$ for all $g \in G(\Omega) = \left\{ g \in GL(U) / g\Omega = \Omega \right\}$ the linear automorphism group of Ω and x^* is:

$$x^* = \int_{\Omega^* \cap H_\lambda(x)} y d^{(\lambda)} y \Big/ \int_{\Omega^* \cap H_\lambda(x)} d^{(\lambda)} y$$

- We can observe that x^* is the center of gravity of $\Omega^* \cap H_\lambda(x)$. We have the property that $\psi_\Omega(gx) = \left| \det(g) \right|^{-1} \psi_\Omega(x)$ for all $x \in \Omega, g \in G(\Omega)$ and then that $\psi_\Omega(x) dx$ is an invariant measure on Ω.

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Koszul-Vinberg Characteristic Function

Koszul-Vinberg Characteristic function

- We can observe that x^* is the center of gravity of $\Omega^* \cap H_\lambda(x)$. We have the property that $\psi_\Omega(gx) = \left|\det(g)\right|^{-1} \psi_\Omega(x)$ for all $x \in \Omega, g \in G(\Omega)$ and then that $\psi_\Omega(x)dx$ is an invariant measure on Ω.
- Writing $\partial_a = \sum_{i=1}^{m} a^i \frac{\partial}{\partial x^i}$, one can write:

$$\partial_a \Phi_\Omega(x) = \partial_a\left(-\log \psi_\Omega(x)\right) \quad , \quad a \in U, x \in \Omega$$

$$\partial_a \Phi_\Omega(x) == \psi_\Omega(x)^{-1} \int_{\Omega^*} \langle a, y\rangle e^{-\langle x,y\rangle} dy = \left\langle a, x^*\right\rangle$$

- Then, the tangent space to the hypersurface $\left\{y \in U / \psi_\Omega(y) = \psi_\Omega(x)\right\}$ at $x \in \Omega$ is given by $\left\{y \in U / \left\langle x^*, y\right\rangle = m\right\}$. For $x \in \Omega, a, b \in U$, the bilinear form $\partial_a \partial_b \log \psi_\Omega(x)$ is symmetric and positive definite, so that it defines an invariant Riemannian metric on Ω .

Fisher Metric and Souriau 2-form: Lie Groups Thermodyamics

Statistical Mechanics, Dual Potentials & Fisher Metric

- In geometric statistical mechanics, **J.M. Souriau** has developed a "**Lie groups thermodynamics**" of dynamical systems where the (maximum entropy) **Gibbs density is covariant** with respect to the action of the Lie group. In the Souriau model, previous structures of information geometry are preserved:

$$I(\beta) = -\frac{\partial^2 \Phi}{\partial \beta^2} \text{ with } \Phi(\beta) = -\log \int_M e^{-\langle \beta, U(\xi) \rangle} d\lambda \qquad U : M \to \mathfrak{g}^*$$

$$S(Q) = \langle \beta, Q \rangle - \Phi(\beta) \text{ with } Q = \frac{\partial \Phi(\beta)}{\partial \beta} \in \mathfrak{g}^* \text{ and } \beta = \frac{\partial S(Q)}{\partial Q} \in \mathfrak{g}$$

Jean-Marie Souriau

- In the Souriau **Lie groups thermodynamics** model, β is a "geometric" (Planck) temperature, element of Lie algebra $\mathfrak{g}$ of the group, and Q is a "geometric" heat, element of dual Lie algebra $\mathfrak{g}^*$ of the group.

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Fisher-Souriau Metric and its invariance

Statistical Mechanics & Invariant Souriau-Fisher Metric

- In Souriau's **Lie groups thermodynamics**, the invariance by re-parameterization in information geometry has been replaced by invariance with respect to the action of the group. When an element of the group g acts on the element $\beta \in \mathfrak{g}$ of the Lie algebra, given by adjoint operator Ad_g. Under the action of the group, $Ad_g(\beta)$, **the entropy $S(Q)$ and the Fisher metric $I(\beta)$ are invariant**:

$$\beta \in \mathfrak{g} \to Ad_g(\beta) \Rightarrow \begin{cases} S\left[Q\left(Ad_g(\beta)\right)\right] = S(Q) \\ I\left[Ad_g(\beta)\right] = I(\beta) \end{cases}$$

$$I(\beta) = -\frac{\partial^2 \Phi}{\partial \beta^2} \text{ with } \Phi(\beta) = -\log \int_M e^{-\langle \beta, U(\xi) \rangle} d\lambda$$

$$S(Q) = \langle \beta, Q \rangle - \Phi(\beta) \text{ with } Q = \frac{\partial \Phi(\beta)}{\partial \beta} \in \mathfrak{g}^* \text{ and } \beta = \frac{\partial S(Q)}{\partial Q} \in \mathfrak{g}$$

Fisher-Souriau Metric Definition by Souriau Cocycle & Moment Map

Statistical Mechanics & Fisher Metric

- Souriau has proposed a Riemannian metric that we have identified as a generalization of the Fisher metric:

$$I(\beta)=\left[g_\beta\right] \text{ with } g_\beta\left([\beta,Z_1],[\beta,Z_2]\right)=\tilde{\Theta}_\beta\left(Z_1,[\beta,Z_2]\right)$$

$$\text{with } \tilde{\Theta}_\beta\left(Z_1,Z_2\right)=\tilde{\Theta}\left(Z_1,Z_2\right)+\left\langle Q,ad_{Z_1}(Z_2)\right\rangle \text{ where } ad_{Z_1}(Z_2)=\left[Z_1,Z_2\right]$$

- The tensor $\tilde{\Theta}$ used to define this extended Fisher metric is defined by the moment map $J(x)$, from M (homogeneous symplectic manifold) to the dual Lie algebra $\mathfrak{g}^*$, given by:

$$\tilde{\Theta}(X,Y)=J_{[X,Y]}-\left\{J_X,J_Y\right\} \text{ with } J(x):M\rightarrow\mathfrak{g}^* \text{ such that } J_X(x)=\left\langle J(x),X\right\rangle, X\in\mathfrak{g}$$

- This tensor $\tilde{\Theta}$ is also defined in tangent space of the cocycle $\theta(g)\in\mathfrak{g}^*$ (this cocycle appears due to the non-equivariance of the coadjoint operator Ad_g^* , action of the group on the dual lie algebra): $Q\left(Ad_g(\beta)\right)=Ad_g^*(Q)+\theta(g)$

$$\tilde{\Theta}\left(X,Y\right):\mathfrak{g}\times\mathfrak{g}\rightarrow\Re \qquad \text{with } \Theta(X)=T_e\theta\left(X(e)\right)$$

$$X,Y\mapsto\left\langle\Theta(X),Y\right\rangle$$

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Fisher-Souriau Metric as a non-null Cohomology extension of KKS 2 form (Kirillov-Kostant-Souriau 2 form)

Souriau definition of Fisher Metric is related to the extension of KKS 2-form (Kostant-Kirillov-Souriau) in case of non-null Cohomogy:

Souriau-Fisher Metric

$$I(\beta)=\left[g_\beta\right] \text{ with } g_\beta\left([\beta,Z_1],[\beta,Z_2]\right)=\tilde{\Theta}_\beta\left(Z_1,[\beta,Z_2]\right)$$

$$\text{with } \tilde{\Theta}_\beta\left(Z_1,Z_2\right)=\tilde{\Theta}\left(Z_1,Z_2\right)+\left\langle Q,[Z_1,Z_2]\right\rangle$$

Non-null cohomology: aditional term from Souriau Cocycle

Equivariant KKS 2 form

$$\tilde{\Theta}(X,Y)=J_{[X,Y]}-\left\{J_X,J_Y\right\} \text{ with } J(x):M\to\mathfrak{g}^* \text{ such that } J_X(x)=\left\langle J(x),X\right\rangle,\ X\in\mathfrak{g}$$

$$\tilde{\Theta}(X,Y):\mathfrak{g}\times\mathfrak{g}\to\Re$$

$$\mathrm{X,Y}\mapsto\left\langle\Theta(X),Y\right\rangle$$

$$\text{with } \Theta(X)=T_e\theta\left(X(e)\right)$$

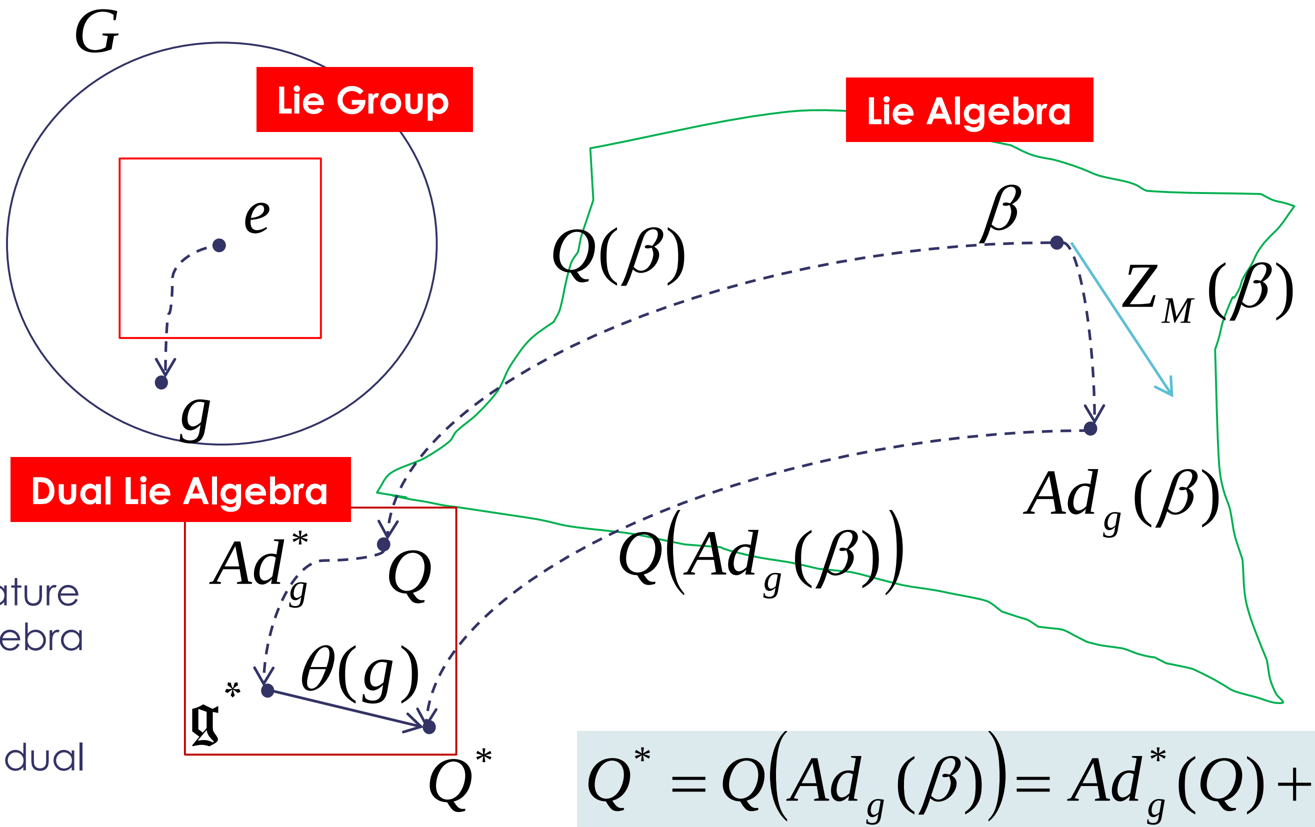
$$\tilde{\Theta}\left(\beta,Z\right)+\left\langle Q,[\beta,Z]\right\rangle=0$$

$$\beta\in Ker\,\tilde{\Theta}_\beta$$

Souriau Fundamental Equation of Lie Group Thermodynamics

$$Q\left(Ad_g(\beta)\right)=Ad_g^*(Q)+\theta(g)$$

Fundamental Souriau Theorem



β : (Planck) température element of Lie algebra

Q : Heat, element of dual Lie Algebra

$$Q^* = Q\left(Ad_g(\beta)\right) = Ad_g^*(Q) + \theta(g)$$

Non-equivariance of Coadjoint operator

- Non-equivariance of Coadjoint operator:

$$Q\left(Ad_g(\beta)\right) = Ad_g^*(Q) + \theta\left(g\right)$$

- This is the action of Lie Group on Moment map:

$$J\left(\Phi_g(x)\right) = a(g, J(x)) = Ad_g^*\left(J(x)\right) + \theta(g)$$

- By noting the action of the group on the dual space of the Lie algebra:

$$G \times \mathfrak{g}^* \to \mathfrak{g}^*, (s,\xi) \mapsto s\xi = Ad_s^*\xi + \theta(s)$$

- Associativity is given by:

$$(s_1 s_2)\xi = Ad_{s_1 s_2}^*\xi + \theta(s_1 s_2) = Ad_{s_1}^* Ad_{s_2}^*\xi + \theta(s_1) + Ad_{s_1}^*\theta(s_2)$$

$$(s_1 s_2)\xi = Ad_{s_1}^*\left(Ad_{s_2}^*\xi + \theta(s_2)\right) + \theta(s_1) = s_1\left(s_2\xi\right) \quad , \quad \forall s_1, s_2 \in G, \xi \in \mathfrak{g}^*$$

Souriau Cocycle

- $\theta(g) \in \mathfrak{g}^*$ is called nonequivariance one-cocycle, and it is a measure of the lack of equivariance of the moment map.

$$\theta(st) = J((st).x) - Ad_{st}^* J(x)$$

$$\theta(st) = \left[J(s.(t.x)) - Ad_s^* J(t.x) \right] + \left[Ad_s^* J(t.x) - Ad_s^* Ad_t^* J(x) \right]$$

$$\theta(st) = \theta(s) + Ad_s^* \left[J(t.x) - Ad_t^* J(x) \right]$$

$$\theta(st) = \theta(s) + Ad_s^* \theta(t)$$

Souriau one-cocycle and compute 2-cocycle

$$\tilde{\Theta}(X,Y):\mathfrak{g}\times\mathfrak{g}\to\Re \qquad \text{with} \quad \Theta(X)=T_e\theta\left(X(e)\right)$$

$$\mathrm{X,Y}\mapsto\langle\Theta(X),Y\rangle$$

- We can also compute tangent of one-cocycle θ at neutral element, to compute 2-cocycle Θ :

$$\zeta\in\mathfrak{g} \quad , \quad \theta_\zeta(s)=\langle\theta(s),\zeta\rangle=\langle J(s.x),\zeta\rangle-\langle Ad_s^* J(x),\zeta\rangle$$

$$\theta_\zeta(s)=\langle J(s.x),\zeta\rangle-\langle J(x),Ad_{s^{-1}}\zeta\rangle$$

$$T_e\theta_\zeta(\xi)=\langle T_xJ.\xi_p(x),\zeta\rangle+\langle J(x),ad_\xi\zeta\rangle \quad \text{with} \quad \xi_p=X_{\langle J,\xi\rangle}$$

$$T_e\theta_\zeta(\xi)=X_{\langle J(x),\xi\rangle}\left[\langle J(x),\zeta\rangle\right]+\langle J(x),[\xi,\zeta]\rangle$$

$$T_e\theta_\zeta(\xi)=-\left\{\langle J,\xi\rangle,\langle J,\zeta\rangle\right\}+\langle J(x),[\xi,\zeta]\rangle=\Theta(\xi)$$

Souriau Tensor

$$\tilde{\Theta}(X,Y) = J_{[X,Y]} - \{J_X, J_Y\} = -\langle d\theta(X), Y\rangle \quad , \quad X,Y \in \mathfrak{g}$$

$$\tilde{\Theta}\left([X,Y],Z\right) + \tilde{\Theta}\left([Y,Z],X\right) + \tilde{\Theta}\left([Z,X],Y\right) = 0 \quad , \quad X,Y,Z \in \mathfrak{g}$$

- By differentiating the equation on affine action, we have:

$$T_x J\left(\xi_p(x)\right) = -ad_{\xi}^{*} J(x) + \Theta(\xi,.)$$

$$dJ(Xx) = ad_X J(x) + d\theta(X) \quad , \quad x \in M, X \in \mathfrak{g}$$

$$\langle dJ(Xx), Y\rangle = \langle ad_X J(x), Y\rangle + \langle \mathrm{d}\theta(X), Y\rangle, \quad x \in M, X, Y \in \mathfrak{g}$$

$$\langle dJ(Xx), Y\rangle = \langle J(x), [X,Y]\rangle + \langle \mathrm{d}\theta(X), Y\rangle \quad = \{\langle J, X\rangle, \langle J, Y\rangle\}(x)$$

$$\langle J(x), [X,Y]\rangle - \{\langle J, X\rangle, \langle J, Y\rangle\}(x) = -\langle \mathrm{d}\theta(X), Y\rangle$$

Souriau Riemannan Metric and Fisher Metric

- For $\beta \in \Omega$, let g_β be the Hessian form on $T_\beta \Omega \equiv \mathfrak{g}$ with the potential $\Phi(\beta) = -\log \psi_\Omega(\beta)$. For $X, Y \in \mathfrak{g}$, we define:

$$g_\beta(X,Y) = -\frac{\partial^2 \Phi}{\partial \beta^2}(X,Y) = \left(\frac{\partial^2}{\partial s \partial t}\right)_{s=t=0} \log \psi_\Omega(\beta + sX + tY)$$

- The positive definitiveness is given by Cauchy-Schwarz inequality:

$$g_\beta(X,Y) = \frac{1}{\psi_\Omega(\beta)^2}\left\{\begin{array}{l} \int\limits_M e^{-\langle U(\xi),\beta\rangle} d\lambda(\xi) . \int\limits_M \langle U(\xi), X\rangle^2 e^{-\langle U(\xi),\beta\rangle} d\lambda(\xi) \\ -\left(\int\limits_M \langle U(\xi), X\rangle e^{-\langle U(\xi),\beta\rangle} d\lambda(\xi)\right)^2 \end{array}\right\}$$

$$= \frac{1}{\psi_\Omega(\beta)^2}\left\{\begin{array}{l} \int\limits_M \left(e^{-\langle U(\xi),\beta\rangle/2}\right)^2 d\lambda(\xi) . \int\limits_M \left(\langle U(\xi), X\rangle e^{-\langle U(\xi),\beta\rangle/2}\right)^2 d\lambda(\xi) \\ -\left(\int\limits_M e^{-\langle U(\xi),\beta\rangle/2} . \langle U(\xi), X\rangle e^{-\langle U(\xi),\beta\rangle/2} d\lambda(\xi)\right)^2 \end{array}\right\} \geq 0$$

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Souriau Riemannan Metric and Fisher Metric

$$g_\beta(X,Y) = \left\langle -\frac{\partial Q}{\partial \beta}(X),Y \right\rangle \quad \text{for} \quad X,Y \in \mathfrak{g}$$

- we have for any $\beta \in \Omega, g \in G$ and $Y \in \mathfrak{g}$:

$$\langle Q(Ad_g \beta),Y \rangle = \langle Q(\beta), Ad_{g^{-1}} Y \rangle + \langle \theta(g),Y \rangle$$

- Let us differentiate the above expression with respect to g. Namely, we substitute $g = \exp(tZ_1), t \in R$ and differentiate at t = 0. Then the left-hand side becomes:

$$\left(\frac{d}{dt}\right)_{t=0} \langle Q(\beta + t[Z_1,\beta] + o(t^2)),Y \rangle = \left\langle \frac{\partial Q}{\partial \beta}([Z_1,\beta]),Y \right\rangle$$

- and the right-hand side of is calculated as:

$$\left(\frac{d}{dt}\right)_{t=0} \langle Q(\beta),Y - t[Z_1,Y] + o(t^2) \rangle + \langle \theta(I + tZ_1 + o(t^2)),Y \rangle$$

$$= -\langle Q(\beta),[Z_1,Y] \rangle + \langle d\theta(Z_1),Y \rangle$$

Souriau Riemannan Metric and Fisher Metric

- Therefore,

$$\left\langle \frac{\partial Q}{\partial \beta}([Z_1,\beta]),Y \right\rangle = \langle d\theta(Z_1),Y\rangle - \langle Q(\beta),[Z_1,Y]\rangle$$

- Substituting $Y=-[\beta,Z_2]$ to the above expression:

$$g_\beta([\beta,Z_1],[\beta,Z_2]) = \left\langle -\frac{\partial Q}{\partial \beta}([Z_1,\beta]),[\beta,Z_2] \right\rangle$$

$$g_\beta([\beta,Z_1],[\beta,Z_2]) = \langle d\theta(Z_1),[\beta,Z_2]\rangle + \langle Q(\beta),[Z_1,[\beta,Z_2]]\rangle$$

- We define then symplectic 2-cocycle and the tensor:

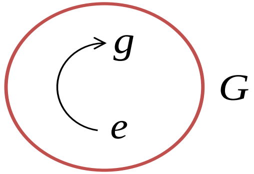
$$\Theta(Z_1) = d\theta(Z_1)$$

$$\tilde{\Theta}(Z_1,Z_2) = \langle \Theta(Z_1),Z_2\rangle = J_{[Z_1,Z_2]} - \{J_{Z_1},J_{Z_2}\}$$

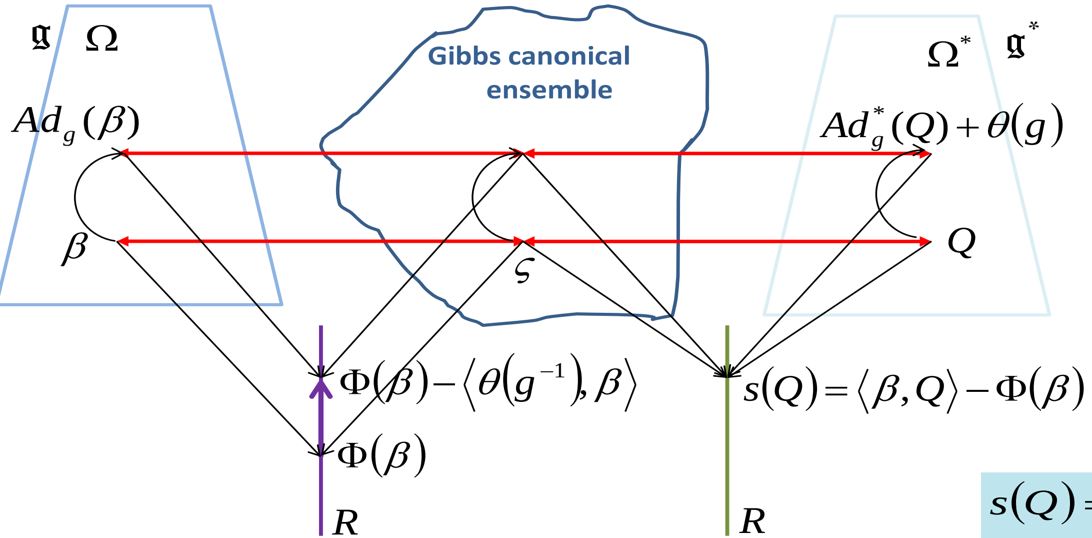
$$\tilde{\Theta}_\beta(Z_1,Z_2) = \langle Q(\beta),[Z_1,Z_2]\rangle + \tilde{\Theta}(Z_1,Z_2) \Rightarrow g_\beta([\beta,Z_1],[\beta,Z_2]) = \tilde{\Theta}_\beta(Z_1,[\beta,Z_2])$$

Souriau-Fisher Metric & Souriau Lie Groups Thermodynamics: Bedrock for Lie Group Machine Learning

TEMPERATURE
In Lie Algebra



HEAT
In Dual Lie Algebra



Logarithm of Partition Function
(Massieu Characteristic Function)

Entropy

Fisher Metric

$$g_\beta([\beta,Z_1],[\beta,Z_2])=\tilde{\Theta}_\beta(Z_1,[\beta,Z_2])\geq 0$$

$$I(\beta)=I(Ad_g(\beta))=-\frac{\partial^2\Phi}{\partial\beta^2}=\frac{\partial^2\log\int_M e^{-\langle\beta,U(\xi)\rangle}d\lambda}{\partial\beta^2}$$

Entropy invariant under the action of the group

Legendre Clairaut

$$s(Q)=\langle\beta,Q\rangle-\Phi(\beta)=\langle\Theta^{-1}(Q),Q\rangle-\Phi(\Theta^{-1}(Q))$$

$$Q=\Theta(\beta)=\frac{\partial\Phi}{\partial\beta}\in\mathfrak{g}^*$$

$$\beta=\Theta^{-1}(Q)\in\mathfrak{g}$$

THALES

Link with Classical Thermodynamics

We have the reciprocal formula:

$$Q = \frac{\partial \Phi}{\partial \beta} \qquad \beta = \frac{\partial s}{\partial Q}$$

$$s(Q) = \left\langle \frac{\partial \Phi}{\partial \beta}, \beta \right\rangle - \Phi \qquad \Phi(\beta) = \left\langle Q, \frac{\partial s}{\partial Q} \right\rangle - s$$

For Classical Thermodynamics (Time translation only), we recover the definition of Boltzmann-Clausius Entropy:

$$\begin{cases} \beta = \dfrac{\partial s}{\partial Q} \\ \beta = \dfrac{1}{T} \end{cases} \Rightarrow ds = \frac{dQ}{T}$$

OPEN

THALES

Souriau Model of Covariant Gibbs Density

Covariant Souriau-Gibbs density

- Souriau has then defined a Gibbs density that is covariant under the action of the group:

$$p_{Gibbs}(\xi) = e^{\Phi(\beta) - \langle U(\xi), \beta \rangle} = \frac{e^{-\langle U(\xi), \beta \rangle}}{\int\limits_M e^{-\langle U(\xi), \beta \rangle} d\lambda_\omega}$$

$$\text{with } \Phi(\beta) = -\log \int\limits_M e^{-\langle U(\xi), \beta \rangle} d\lambda_\omega$$

$$Q = \frac{\partial \Phi(\beta)}{\partial \beta} = \frac{\int\limits_M U(\xi) e^{-\langle U(\xi), \beta \rangle} d\lambda_\omega}{\int\limits_M e^{-\langle U(\xi), \beta \rangle} d\lambda_\omega} = \int\limits_M U(\xi) p(\xi) d\lambda_\omega$$

- We can express the Gibbs density with respect to Q by inverting the relation $Q = \frac{\partial \Phi(\beta)}{\partial \beta} = \Theta(\beta)$. Then $p_{Gibbs,Q}(\xi) = e^{\Phi(\beta) - \langle U(\xi), \Theta^{-1}(Q) \rangle}$ with $\beta = \Theta^{-1}(Q)$

Multivariate Gaussian Density as 1st order Maximum Entropy in Souriau Book (Chapter IV)

Exemple : (*loi normale*) :

Prenons le cas $V = R^n$, $\lambda =$ mesure de Lebesgue, $\Psi(x) \equiv \begin{pmatrix} x \\ x \otimes x \end{pmatrix}$;
un élément Z du dual de E peut se définir par la formule

$$Z(\Psi(x)) \equiv \bar{a}.x + \tfrac{1}{2}\bar{x}.H.x$$

[$a \in R^n$; $H =$ matrice symétrique]. On vérifie que la convergence de l'intégrale I_0 a lieu si la matrice H est positive (¹) ; dans ce cas la loi de Gibbs s'appelle *loi normale de Gauss* ; on calcule facilement I_0 en faisant le changement de variable $x^* = H^{1/2}\,x + H^{-1/2}\,a$ (²) ; il vient

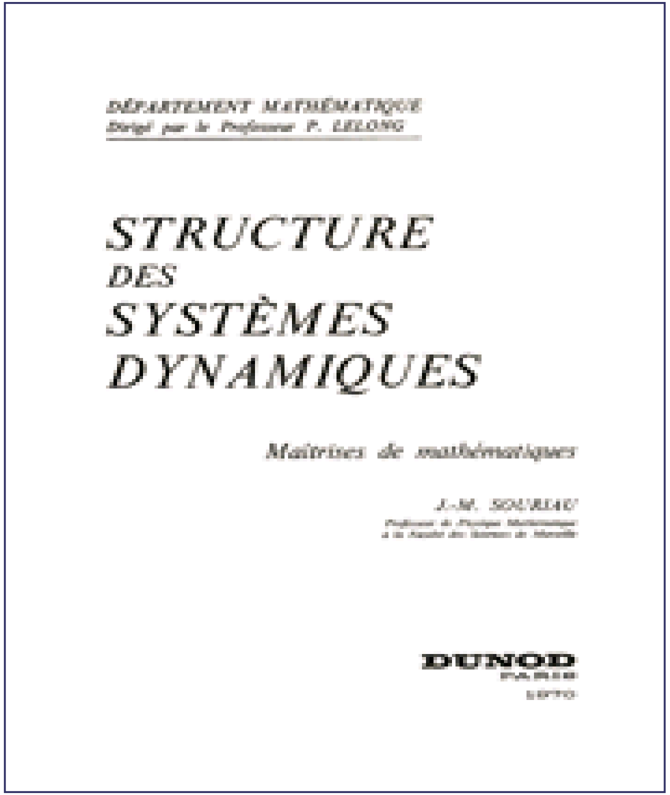
$$\boxed{z = \tfrac{1}{2}\left[\bar{a}.H^{-1}.a - \log(\text{dét}(H)) + n\log(2\pi)\right]}$$

alors la convergence de I_1 a lieu également ; on peut donc calculer M, qui est défini par les moments du premier et du second ordre de la loi (16.196) ; le calcul montre que le moment du premier ordre est égal à $-H^{-1}.a$ et que les composantes du tenseur *variance* (16.196) sont égales aux éléments de la matrice H^{-1} ; le moment du second ordre s'en déduit immédiatement.

La formule (16.200) donne l'*entropie* :

$$\boxed{s = \frac{n}{2}\log(2\pi e) - \frac{1}{2}\log(\text{dét}(H))} \; ;$$

(¹) Voir *Calcul linéaire*, tome II.
(²) C'est-à-dire en recherchant l'*image* de la loi par l'application $x \mapsto x^*$.



http://www.jmsouriau.com/structure_des_systemes_dynamiques.htm

Example of Multivariate Gaussian Law (real case)

Multivariate Gaussian law parameterized by moments

$$p_{\hat{\xi}}(\xi) = \frac{1}{(2\pi)^{n/2}\det(R)^{1/2}} e^{-\frac{1}{2}(z-m)^T R^{-1}(z-m)}$$

$$\frac{1}{2}(z-m)^T R^{-1}(z-m) = \frac{1}{2}\left[z^T R^{-1}z - m^T R^{-1}z - z^T R^{-1}m + m^T R^{-1}m\right]$$

$$= \frac{1}{2}z^T R^{-1}z - m^T R^{-1}z + \frac{1}{2}m^T R^{-1}m$$

$$p_{\hat{\xi}}(\xi) = \frac{1}{(2\pi)^{n/2}\det(R)^{1/2} e^{\frac{1}{2}m^T R^{-1}m}} e^{-\left[-m^T R^{-1}z + \frac{1}{2}z^T R^{-1}z\right]} = \boxed{\frac{1}{Z}e^{-\langle \xi,\beta\rangle}}$$

Gaussian Density is a 1st order Maximum Entropy Density !

$$\xi = \begin{bmatrix} z \\ zz^T \end{bmatrix} \text{ and } \beta = \begin{bmatrix} -R^{-1}m \\ \frac{1}{2}R^{-1} \end{bmatrix} = \begin{bmatrix} a \\ H \end{bmatrix} \text{ with } \langle \xi,\beta\rangle = a^T z + z^T H z = Tr\left[za^T + H^T zz^T\right]$$

Souriau Gibbs states for Hamiltonian actions of subgroups of the Galilean group

Souriau Gibbs states for one-parameter subgroups of the Galilean group

- **Souriau Result:** Action of the full Galilean group on the space of motions of an isolated mechanical system is not related to any Equilibrium Gibbs state (the open subset of the Lie algebra, associated to this Gibbs state, is empty)
- The **1-parameter subgroup of the Galilean group** generated by β element of Lie Algebra, is the set of matrices

$$\exp(\tau\beta) = \begin{pmatrix} A(\tau) & \vec{b}(\tau) & \vec{d}(\tau) \\ 0 & 1 & \tau\varepsilon \\ 0 & 0 & 1 \end{pmatrix} \text{ with } \begin{cases} A(\tau) = \exp(\tau j(\vec{\omega})) \text{ and } \vec{b}(\tau) = \left(\sum_{i=1}^{\infty} \frac{\tau^i}{i!} \left(j(\vec{\omega}) \right)^{i-1} \right) \vec{\alpha} \\ \vec{d}(\tau) = \left(\sum_{i=1}^{\infty} \frac{\tau^i}{i!} \left(j(\vec{\omega}) \right)^{i-1} \right) \vec{\delta} + \varepsilon \left(\sum_{i=2}^{\infty} \frac{\tau^i}{i!} \left(j(\vec{\omega}) \right)^{i-2} \right) \vec{\alpha} \end{cases}$$

$$\beta = \begin{pmatrix} j(\vec{\omega}) & \vec{\alpha} & \vec{\delta} \\ 0 & 1 & \varepsilon \\ 0 & 0 & 0 \end{pmatrix} \in \mathfrak{g}$$

Souriau Thermodynamics of butter churn (device used to convert cream into butter) or “La Thermodynamique de la crémière”

If we consider the case of the centrifuge

$$\vec{\omega} = \omega \vec{e}_z \ , \vec{\alpha} = 0 \ \text{ and } \ \vec{\delta} = 0$$

$$Rotation\ speed : \frac{\omega}{\varepsilon}$$

$$f_i(\vec{r}_{i0}) = -\frac{\omega^2}{2\varepsilon^2} \left\| \vec{e}_z \times \vec{r}_{i0} \right\|^2$$

$$\text{with } \ \Delta = \left\| \vec{e}_z \times \vec{r}_{i0} \right\| \ \text{ distance to axis z}$$

> “The angular momentum is imparted to the gas when the molecules collide with the rotating walls, which changes the Maxwell distribution at every point, shifting its origin. The walls play the role of an angular momentum reservoir. Their motion is characterized by a certain angular velocity, and the angular velocities of the fluid and of the walls become equal at equilibrium, exactly like the equalization of the temperature through energy exchanges”. – Roger Balian

$$\rho_i(\beta) = \frac{1}{P_i(\beta)} \exp\left(-\langle J_i, \beta \rangle\right) = cst.\exp\left(-\frac{1}{2m_i \kappa T} \left\| \vec{p}_{i0} \right\|^2 + \frac{m_i}{2\kappa T} \left(\frac{\omega}{\varepsilon} \right)^2 \Delta^2 \right)$$

- the behaviour of a gas made of point particles of various masses in a centrifuge rotating at a constant angular velocity (the heavier particles concentrate farther from the rotation axis than the lighter ones)

$$\frac{\omega}{\varepsilon}$$

THALES

Roger Balian Computation of Gibbs density for centrifuge

- Balian has computed the Boltzmann-Gibbs distribution without knowing Souriau equations. Exercice 7b of :
  - Balian, R. From Microphysics to Macrophysics, 2nd ed.; Springer: Berlin, Germany, 2007; Volume I
- Balian started by considering the constants of motion that are the energy and the component J_z of the total angular momentum: $J = \sum_i (r_i \times p_i)$
- Balian observed that he must add to the Lagrangian parameter, given by (Planck) temperature β for energy, an additional one associated with J_z. He identifies this additional multiplier with $-\beta\omega$ by evaluating the mean velocity at each point.
- He then introduced the same results also by changing the frame of reference, the Lagrangian and the Hamiltonian in the rotating frame and by writing down the canonical equilibrium in that frame. He uses the resulting distribution to find, through integration, over the momenta, an expression for the particles density as the function of the distance from the cylinder axis.

THALES

Roger Balian Computation of Gibbs density for centrifuge

- The fluid carried along by the walls of the rotating vessel acquires a non-vanishing average angular momentum $\langle J_z \rangle$ around the axis of rotation, that is a constant of motion. In order to be able to assign to it a definite value, Balian proposed to associate with it a Lagrangian multiplier λ , in exactly the same way as we classically associate the multiplier β with the energy in canonical equilibrium. The average $\langle J_z \rangle$ will be a function of λ . The Gibbs density for rotating gas is given by Balian as:

$$D = \frac{1}{Z} e^{-\beta H - \lambda J_z} = \frac{1}{Z} \exp\left\{ \sum_i \left[\frac{\beta p_i^2}{2m} + \lambda \left(x_i p_{y_i} - y_i p_{x_i} \right) \right] \right\}$$

- With the energy and the average angular momentum given by:

$$U = -\frac{\partial \ln Z}{\partial \beta} = \frac{1}{kT} \qquad \langle J_z \rangle = -\frac{\partial \ln Z}{\partial \lambda}$$

Roger Balian Computation of Gibbs density for centrifuge

- The Lagrangian parameter λ has a mechanical nature. To identify this parameter, Balian compared microscopic and macroscopy descriptions of fluid mechanics. He described the single-particle reduced density by:

$$f(r,p) \propto \exp\left\{-\frac{\beta p^2}{2m} - \lambda\left(xp_y - yp_x\right)\right\} = \exp\left\{-\frac{\beta}{2m}\left(p + \frac{m}{\beta}[\lambda \times r]\right)^2 + \frac{m\lambda^2}{2\beta}\left(x^2 + y^2\right)\right\}$$

- Whence Balian finds the velocity distribution at a point r to be proportional to:

$$\exp\left\{-\frac{m}{2kT}\left(v + \frac{1}{\beta}[\lambda \times r]\right)^2\right\}$$

- The mean velocity of the fluid at the point r is equal to: $\langle v \rangle = -\frac{1}{\beta}[\lambda \times r]$

- and can be identified with the velocity $[\omega \times r]$ in an uniform rotation with angular velocity ω . By comparison, Balian put : $\omega = -\lambda / \beta$

Supervised & Non-Supervised Learning on Lie Groups

Geodesic Natural Gradient on Lie Algebra

Extension of Neural Network Natural Gradient from Information Geometry on Lie Algebra for Lie Groups Machine Learning

Souriau-Fisher Metric on Coadjoint Orbits

Extension of Fisher Metric for Lie Group through homogeneous Symplectic Manifolds on Lie Group Co-Adjoint Orbits

Souriau Exponential Map on Lie Algebra

Exponential Map for Geodesic Natural Gradient on Lie Algebra based on Souriau Algorithm for Matrix Characteristic Polynomial

Souriau Maximum Entropy Density on Co-Adjoint Orbits

Covariant Maximum Entropy Probability Density for Lie Groups defined with Souriau Moment Map, Co-Adjoint Orbits Method & Kirillov Representation Theory

Fréchet Geodesic Barycenter by Hermann Karcher Flow

Extension of Mean/Median on Lie Group by Fréchet Definition of Geodesic Barycenter on Souriau-Fisher Metric Space, solved by Karcher Flow

Symplectic Integrator preserving Moment Map

Extension of Neural Network Natural Gradient to Geometric Integrators as Symplectic integrators that preserve moment map

Mean-Shift on Lie Groups with Souriau-Fisher Distance

Extension of Mean-Shift for Homogeneous Symplectic Manifold and Souriau-Fisher Metric Space

LIE GROUP SUPERVISED LEARNING

LIE GROUP NON-SUPERVISED LEARNING

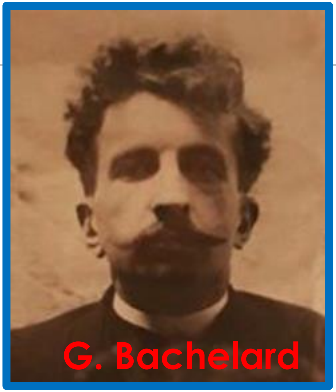
THALES

ENTROPY GEOMETRIC DEFINITION as Invariant Casimir Function in Coadjoint Representation

Gaston Bachelard – Le nouvel esprit scientifique

« *La Physique mathématique, en incorporant à sa base la notion de groupe, marque la suprématie rationnelle… Chaque géométrie – et sans doute plus généralement chaque organisation mathématique de l'expérience – est caractérisée par un groupe spécial de transformations…. **Le groupe apporte la preuve d'une mathématique fermée sur elle-même. Sa découverte clôt l'ère des conventions, plus ou moins indépendantes, plus ou moins cohérentes*** » - **Gaston Bachelard**, Le nouvel esprit scientifique, 1934

« *Sous cette aspiration, la physique qui était d'abord une science des "agents" doit devenir une science des "milieux". C'est en s'adressant à des milieux nouveaux que l'on peut espérer pousser la diversification et l'analyse des phénomènes jusqu'à en provoquer la géométrisation fine et complexe, vraiment intrinsèque… Sans doute, la réalité ne nous a pas encore livré tous ses modèles, mais nous savons déjà qu'elle ne peut en posséder un plus grand nombre que celui qui lui est assigné par la théorie mathématique des groupes.* » – ***Gaston Bachelard***, Etude sur l'Evolution d'un problème de Physique
La propagation thermique dans les solides, 1928



http://www.vrin.fr/book.php?title_url=Etude_sur_l_evolution_d_un_probleme_de_physique_La_propagation_thermique_dans_les_solides_9782711600434&search_back=&editor_back=%&page=2

Souriau Entropy Invariance

Casimir Invariant Function in coadjoint representation

- We conclude the paper by a deeper study of Souriau model structure. We observe that Souriau Entropy $S(Q)$ defined on coadjoint orbit of the group has a property of invariance :

$$S\left(Ad_g^{\#}(Q)\right) = S(Q)$$

- with respect to Souriau affine definition of coadjoint action:

$$Ad_g^{\#}(Q) = Ad_g^{*}(Q) + \theta(g)$$

- where $\theta(g)$ is called the Souriau cocyle.

$$Q\left(Ad_g(\beta)\right) = Ad_g^{*}(Q) + \theta(g)$$

$$S\left(Q\left(Ad_g(\beta)\right)\right) = S(Q)$$

New Entropy Definition: Function in Coadjoint Representation Invariant under the action of the Group

Hendrik Casimir (Thesis supervised by Niels Bohr & Paul Ehrenfest)

H.B.G. Casimir, On the Rotation of a Rigid Body in Quantum Mechanics, Doctoral Thesis, Leiden, 1931.

Entropy as Invariant Casimir Function in Coadjoint Representation

NEW GEOMETRIC DEFINITION OF ENTROPY

$$\{S,H\}_{\tilde{\Theta}}(Q)=0$$

$$ad^{*}_{\frac{\partial S}{\partial Q}}Q+\Theta\left(\frac{\partial S}{\partial Q}\right)=0$$

$$\{S,H\}(Q)=\left\langle Q,\left[\frac{\partial S}{\partial Q},\frac{\partial H}{\partial Q}\right]\right\rangle=-C_{ij}^{k}Q_k\frac{\partial S}{\partial Q_i}\cdot\frac{\partial H}{\partial Q_j}$$

$$\left[e_i,e_j\right]=C_{ij}^{k}e_k\quad,\quad C_{ij}^{k}\ \text{ structure coefficients}$$

$$\{S,H\}_{\tilde{\Theta}}(Q)=\left\langle Q,\left[\frac{\partial S}{\partial Q},\frac{\partial H}{\partial Q}\right]\right\rangle+\tilde{\Theta}\left(\frac{\partial S}{\partial Q},\frac{\partial H}{\partial Q}\right)=0\quad,\quad\forall H:\mathfrak{g}^{*}\to R,\ \ Q\in\mathfrak{g}^{*}$$

$$\tilde{\Theta}(X,Y)=J_{[X,Y]}-\{J_X,J_Y\}\quad\text{where}\ \ J_X(x)=\langle J(x),X\rangle$$

$$\tilde{\Theta}(X,Y)=\langle\Theta(X),Y\rangle\ \text{ with }\ \Theta(X)=T_e\theta(X(e))$$

$$\theta(g)=Q\left(Ad_g(\beta)\right)-Ad_g^{*}(Q)$$

THALES

Fundamental Equation of Geometric Thermodynamic: Entropy Function is an Invariant Casimir Function in Coadjoint Representation

Entropy S

$$S:\mathfrak{g}^* \to R$$
$$Q \mapsto S(Q)$$

Heat Q, (Planck) Temperature β and Φ Massieu Characteristic Function

$$S(Q)=\langle \beta, Q\rangle - \Phi(\beta)\ ,\ Q=\frac{\partial \Phi(\beta)}{\partial \beta}\in \mathfrak{g}^*\ ,\ \beta=\frac{\partial S(Q)}{\partial Q}\in \mathfrak{g}$$

Invariance of Entropy S Under the action of the Group

$$Q\left(Ad_g(\beta)\right)=Ad_g^*(Q)+\theta(g)$$

$$S\left(Q\left(Ad_g(\beta)\right)\right)=S(Q)$$

New Definition of Entropy S as Invariant Casimir Function in Coadjoint Representation

$$ad^*_{\frac{\partial S}{\partial Q}}Q+\Theta\left(\frac{\partial S}{\partial Q}\right)=0$$

$$\{S,H\}_{\tilde{\Theta}}(Q)=\left\langle Q,\left[\frac{\partial S}{\partial Q},\frac{\partial H}{\partial Q}\right]\right\rangle+\tilde{\Theta}\left(\frac{\partial S}{\partial Q},\frac{\partial H}{\partial Q}\right)=0$$

$$\Theta(X)=T_e\theta\left(X(e)\right)$$

$$\tilde{\Theta}(X,Y)=\langle \Theta(X),Y\rangle=J_{[X,Y]}-\{J_X,J_Y\}=-\langle d\theta(X),Y\rangle$$

Moment Map J

Lie Groups Thermodynamic Equations and its extension (1/3)

Q : Heat, element of dual Lie Algebra

β : (Planck) température element of Lie algebra

$$Q\left(Ad_g(\beta)\right) = Ad_g^*(Q) + \theta(g)$$

$\theta(g)$: Souriau Cocyle

Φ : Massieu Characteristic Function

J : Souriau Moment Map

$$\Phi(\beta) = -\log \int_M e^{-\langle \beta, J(\xi) \rangle} d\lambda$$

$$J : M \to \mathfrak{g}^*$$

S : Entropy

Legendre Transform

$$S(Q) = \langle \beta, Q \rangle - \Phi(\beta) \ \text{ with } \ Q = \frac{\partial \Phi(\beta)}{\partial \beta} \in \mathfrak{g}^* \ \text{ and } \ \beta = \frac{\partial S(Q)}{\partial Q} \in \mathfrak{g}$$

I : Fisher Information Metric

$$I(\beta) = -\frac{\partial^2 \Phi}{\partial \beta^2}$$

Lie Groups Thermodynamic Equations and its extension (2/3)

Entropy Invariance under the action of the Group !

$$S\left(Ad_g^{\#}(Q)\right)=S(Q)$$

$$Ad_g^{\#}(Q)=Ad_g^{*}(Q)+\theta(g)$$

Souriau characteristic of the foliation

$$\langle Q,[\beta,Z]\rangle+\tilde{\Theta}(\beta,Z)=0$$

Entropy Solution of Casimir Equation

$$ad^{*}_{\frac{\partial S}{\partial Q}}Q+\Theta\left(\frac{\partial S}{\partial Q}\right)=0$$

Souriau cocycle

$$\Theta(X)=T_e\theta(X(e))$$

$$\theta(g)=Q\left(Ad_g(\beta)\right)-Ad_g^{*}(Q)$$

Entropy & Poisson Bracket

$$\{S,H\}_{\tilde{\Theta}}(Q)=\left\langle Q,\left[\frac{\partial S}{\partial Q},\frac{\partial H}{\partial Q}\right]\right\rangle+\tilde{\Theta}\left(\frac{\partial S}{\partial Q},\frac{\partial H}{\partial Q}\right)=0$$

$$\tilde{\Theta}(X,Y)=\langle\Theta(X),Y\rangle=J_{[X,Y]}-\{J_X,J_Y\}=-\langle d\theta(X),Y\rangle$$

THALES

Lie Groups Thermodynamic Equations and its extension (3/3)

Entropy Production

$$dS = \tilde{\Theta}_\beta\left(\frac{\partial H}{\partial Q}, \beta\right)dt$$

2nd principle is related to positivity of Fisher tensor

$$\frac{dS}{dt} = \tilde{\Theta}_\beta\left(\frac{\partial H}{\partial Q}, \beta\right) \geq 0$$

Metric Tensor related to Fisher Metric

$$\tilde{\Theta}_\beta\left(\frac{\partial H}{\partial Q}, \beta\right) = \tilde{\Theta}\left(\frac{\partial H}{\partial Q}, \beta\right) + \left\langle Q, \left[\frac{\partial H}{\partial Q}, \beta\right]\right\rangle$$

Time Evolution of Heat wrt to Hamiltonian H

$$\frac{dQ}{dt} = \{Q, H\}_{\tilde{\Theta}} = ad^*_{\frac{\partial H}{\partial Q}} Q + \Theta\left(\frac{\partial H}{\partial Q}\right)$$

Stochastic Equation

$$dQ + \left[ad^*_{\frac{\partial H}{\partial Q}} Q + \Theta\left(\frac{\partial H}{\partial Q}\right)\right]dt + \sum_{i=1}^{N}\left[ad^*_{\frac{\partial H_i}{\partial Q}} Q + \Theta\left(\frac{\partial H_i}{\partial Q}\right)\right] \circ dW_i(t) = 0$$

Euler-Poincaré Equation in case of Non-Null Cohomology

$$\frac{dQ}{dt} = ad^*_{\frac{\partial H}{\partial Q}} Q + \Theta\left(\frac{\partial H}{\partial Q}\right)$$

$$Q = \frac{\partial \Phi}{\partial \beta}$$

$$\frac{d}{dt}\frac{\partial \Phi}{\partial \beta} = ad^*_{\frac{\partial H}{\partial Q}} \frac{\partial \Phi}{\partial \beta} + \Theta\left(\frac{\partial H}{\partial Q}\right)$$

$$\left(ad^*_{\frac{\partial H}{\partial Q}} \frac{\partial \Phi}{\partial \beta}\right)_j + \Theta\left(\frac{\partial H}{\partial Q}\right)_j = C^k_{ij}\, ad^*_{\left(\frac{\partial H}{\partial Q}\right)^i} \left(\frac{\partial \Phi}{\partial \beta}\right)_k + \Theta_j$$

« Ayant eu l'occasion de m'occuper du mouvement de rotation d'un corps solide creux, dont la cavité est remplie de liquide, j'ai été conduit à mettre les équations générales de la mécanique sous une forme que je crois nouvelle et qu'il peut être intéressant de faire connaître » - Henri Poincaré, CRAS, 18 Février 1901

SÉANCE DU LUNDI 18 FÉVRIER 1901,

PRÉSIDENCE DE M. FOUQUÉ.

MEMOIRES ET COMMUNICATIONS

DES MEMBRES ET DES CORRESPONDANTS DE L'ACADÉMIE.

MÉCANIQUE RATIONNELLE. — *Sur une forme nouvelle des équations de la Mécanique*. Note de M. H. POINCARÉ.

« Ayant eu l'occasion de m'occuper du mouvement de rotation d'un corps solide creux, dont la cavité est remplie de liquide, j'ai été conduit à mettre les équations générales de la Mécanique sous une forme que je crois nouvelle et qu'il peut être intéressant de faire connaître.

$$\frac{d}{dt}\frac{dT}{d\eta_s} = \sum c_{ski}\frac{dT}{d\eta_i}\eta_k + \Omega_s.$$

« Elles sont surtout intéressantes dans le cas où U étant nul, T ne dépend que des η » - Henri Poincaré

de Saxcé, G. Euler-Poincaré equation for Lie groups with non null symplectic cohomology. Application to the mechanics. In GSI 2019. LNCS; Nielsen, F., Barbaresco, F., Eds.; Springer: Berlin, Germany, 2019; Volume 11712

Souriau Model Variational Principle : Poincaré-Cartan Integral Invariant on Massieu Characteristic Function

Extension of Poincaré-Cartan Integral Invariant for Souriau Model

$$\omega = \left\langle Q, \left(\beta.dt\right)\right\rangle - S.dt = \left(\left\langle Q, \beta\right\rangle - S\right).dt = \Phi(\beta).dt$$

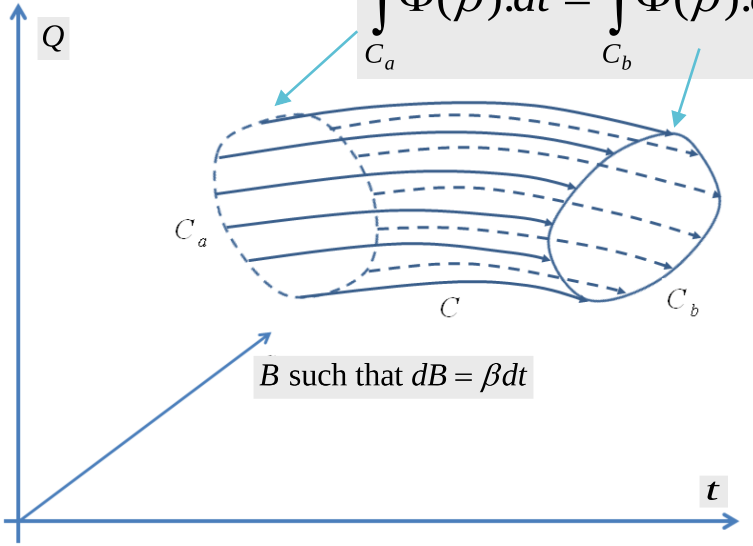
$$g(t) \in G \qquad \beta(t) = g(t)^{-1}\dot{g}(t) \in \mathfrak{g}$$

Variational Model for arbitrary path $\eta(t)$

$$\delta\beta = \dot{\eta} + \left[\beta, \eta\right]$$

$$\delta\int_{t_0}^{t_1} \Phi\left(\beta(t)\right).dt = 0$$

$$\int_{C_a} \Phi(\beta).dt = \int_{C_b} \Phi(\beta).dt$$



Souriau Entropy and Casimir Invariant Function

Geometric Definition of Entropy

- In the framework of Souriau Lie groups Thermodynamics, we can characterize the Entropy as a generalized Casimir invariant function in coadjoint representation,

Geometric Definition of Massieu Characteristic Function

- Massieu characteristic function (or log-partition function), dual of Entropy by Legendre transform, as a generalized Casimir function in adjoint representation.

Casimir Function Definition

- When M is a Poisson manifold, a function on M is a Casimir function if and only if this function is constant on each symplectic leaf (the non-empty open subsets of the symplectic leaves are the smallest embedded manifolds of M which are Poisson submanifolds)

Entropy Invariance under the action of the Group (1/2)

$$\beta \in \mathfrak{g} \rightarrow Ad_g(\beta) \Rightarrow \Psi(Ad_g(\beta)) = \int_M e^{-\langle U, Ad_g(\beta)\rangle} d\lambda_\omega$$

$$\Psi(Ad_g(\beta)) = \int_M e^{-\left\langle Ad^*_{g^{-1}}U, \beta\right\rangle} d\lambda_\omega = \int_M e^{-\left\langle U\left(Ad_{g^{-1}}\beta\right) - \theta\left(g^{-1}\right), \beta\right\rangle} d\lambda_\omega$$

$$\Psi(Ad_g(\beta)) = e^{\left\langle \theta\left(g^{-1}\right), \beta\right\rangle} \Psi(\beta)$$

$$\theta\left(g^{-1}\right) = -Ad^*_{g^{-1}}\theta(g) \Rightarrow \Psi(Ad_g(\beta)) = e^{-\left\langle Ad^*_{g^{-1}}\theta(g), \beta\right\rangle} \Psi(\beta)$$

$$\Phi(\beta) = -\log \Psi(\beta)$$

$$\Rightarrow \Phi\left(Ad_g(\beta)\right) = \Phi(\beta) - \left\langle \theta\left(g^{-1}\right), \beta\right\rangle = \Phi(\beta) + \left\langle Ad^*_{g^{-1}}\theta(g), \beta\right\rangle$$

Entropy Invariance under the action of the Group (2/2)

$$S(Q)=\langle Q,\beta\rangle-\Phi(\beta)\Rightarrow S\left(Q\left(Ad_g\beta\right)\right)=\left\langle Q\left(Ad_g\beta\right),Ad_g\beta\right\rangle-\Phi\left(Ad_g\beta\right)$$

$$\begin{cases} Q\left(Ad_g(\beta)\right)=Ad_g^*(Q)+\theta\left(g\right) \\ \Phi\left(Ad_g(\beta)\right)=-\log\Psi(Ad_g(\beta))=-\left\langle\theta\left(g^{-1}\right),\beta\right\rangle+\Phi\left(\beta\right) \end{cases}$$

$$\Rightarrow S\left(Q\left(Ad_g\beta\right)\right)=\left\langle Ad_g^*(Q)+\theta\left(g\right),Ad_g\beta\right\rangle+\left\langle\theta\left(g^{-1}\right),\beta\right\rangle-\Phi\left(\beta\right)$$

$$\Rightarrow S\left(Q\left(Ad_g\beta\right)\right)=\left\langle Ad_g^*(Q)+\theta\left(g\right),Ad_g\beta\right\rangle-\left\langle Ad_{g^{-1}}^*\theta(g),\beta\right\rangle-\Phi\left(\beta\right)$$

$$\Rightarrow S\left(Q\left(Ad_g\beta\right)\right)=\left\langle Ad_{g^{-1}}^*Ad_g^*(Q)+Ad_{g^{-1}}^*\theta\left(g\right),\beta\right\rangle-\left\langle Ad_{g^{-1}}^*\theta(g),\beta\right\rangle-\Phi\left(\beta\right)$$

$$Ad_{g^{-1}}^*Ad_g^*(Q)=Q\Rightarrow S\left(Q\left(Ad_g\beta\right)\right)=\langle Q,\beta\rangle-\Phi\left(\beta\right)=S\left(\beta\right)$$

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Souriau Entropy and Casimir Invariant Function

Casimir Function and Entropy

- Classically, the Entropy is defined axiomatically as Shannon or von Neumann Entropies without any geometric structures constraints.
- Entropy could be built by Casimir Function Equation:

$$\left(ad^*_{\frac{\partial S}{\partial Q}} Q \right)_j + \Theta\left(\frac{\partial S}{\partial Q} \right)_j = C^k_{ij} ad^*_{\left(\frac{\partial S}{\partial Q}\right)^i} Q_k + \Theta_j = 0$$

$$\tilde{\Theta}(X,Y) = \langle \Theta(X), Y \rangle = J_{[X,Y]} - \{ J_X, J_Y \} = -\langle d\theta(X), Y \rangle \quad , \ X,Y \in \mathfrak{g}$$

$$\Theta(X) = T_e\theta(X(e))$$

$$\theta(g) = Q\left(Ad_g(\beta)\right) - Ad^*_g(Q)$$

Souriau Entropy and Casimir Invariant Function

Demo

- if we consider the heat expression $Q=\frac{\partial \Phi}{\partial \beta}$, that we can write $\delta\Phi-\langle Q,\delta\beta\rangle=0$.
- For each $\delta\beta$ tangent to the orbit, and so generated by an element Z of the Lie algebra, if we consider the relation $\Phi\left(Ad_g(\beta)\right)=\Phi(\beta)-\left\langle \theta\left(g^{-1}\right),\beta\right\rangle$, we differentiate it at $g=e$ using the property that:

$$\tilde{\Theta}(X,Y)=-\langle d\theta(X),Y\rangle \ , \ X,Y\in\mathfrak{g}$$

- we obtain : $\langle Q,[\beta,Z]\rangle+\tilde{\Theta}(\beta,Z)=0$
- From last Souriau equation, if we use the identities $\beta=\frac{\partial S}{\partial Q}$, $ad_\beta Z=[\beta,Z]$ and $\tilde{\Theta}(\beta,Z)=\langle\Theta(\beta),Z\rangle$
- Then we can deduce that: $\left\langle ad^*_{\frac{\partial S}{\partial Q}}Q+\Theta\left(\frac{\partial S}{\partial Q}\right),Z\right\rangle=0,\ \ \forall Z$
- So, Entropy $S(Q)$ should verify:

$$ad^*_{\frac{\partial S}{\partial Q}}Q+\Theta\left(\frac{\partial S}{\partial Q}\right)=0 \qquad \{S,H\}_{\tilde{\Theta}}(Q)=0\quad \forall H:\mathfrak{g}^*\to R,\ \ Q\in\mathfrak{g}^*$$

$$\{S,H\}_{\tilde{\Theta}}(Q)=\left\langle Q,\left[\frac{\partial S}{\partial Q},\frac{\partial H}{\partial Q}\right]\right\rangle+\tilde{\Theta}\left(\frac{\partial S}{\partial Q},\frac{\partial H}{\partial Q}\right)=0$$

Souriau Entropy and Casimir Invariant Function

Demo

$$\{S,H\}_{\tilde{\Theta}}(Q)=\left\langle Q,\left[\frac{\partial S}{\partial Q},\frac{\partial H}{\partial Q}\right]\right\rangle+\left\langle \Theta\left(\frac{\partial S}{\partial Q}\right),\frac{\partial H}{\partial Q}\right\rangle=0$$

$$\{S,H\}_{\tilde{\Theta}}(Q)=\left\langle Q,ad_{\frac{\partial S}{\partial Q}}\frac{\partial H}{\partial Q}\right\rangle+\left\langle \Theta\left(\frac{\partial S}{\partial Q}\right),\frac{\partial H}{\partial Q}\right\rangle=0$$

$$\{S,H\}_{\tilde{\Theta}}(Q)=\left\langle ad^{*}_{\frac{\partial S}{\partial Q}}Q,\frac{\partial H}{\partial Q}\right\rangle+\left\langle \Theta\left(\frac{\partial S}{\partial Q}\right),\frac{\partial H}{\partial Q}\right\rangle=0$$

$$\forall H,\{S,H\}_{\tilde{\Theta}}(Q)=\left\langle ad^{*}_{\frac{\partial S}{\partial Q}}Q+\Theta\left(\frac{\partial S}{\partial Q}\right),\frac{\partial H}{\partial Q}\right\rangle=0\Rightarrow ad^{*}_{\frac{\partial S}{\partial Q}}Q+\Theta\left(\frac{\partial S}{\partial Q}\right)=0$$

Souriau Entropy and Casimir Invariant Function

Link with Souriau development

- Souriau property: $\beta \in Ker\tilde{\Theta}_\beta \Rightarrow \langle Q,[\beta,Z]\rangle + \tilde{\Theta}(\beta,Z) = 0$

$$\Rightarrow \langle Q, ad_\beta Z\rangle + \tilde{\Theta}(\beta,Z) = 0 \Rightarrow \langle ad_\beta^* Q, Z\rangle + \tilde{\Theta}(\beta,Z) = 0$$

$$\beta = \frac{\partial S}{\partial Q} \Rightarrow \left\langle ad^*_{\frac{\partial S}{\partial Q}} Q, Z \right\rangle + \tilde{\Theta}\left(\frac{\partial S}{\partial Q}, Z\right) = \left\langle ad^*_{\frac{\partial S}{\partial Q}} Q + \Theta\left(\frac{\partial S}{\partial Q}\right), Z \right\rangle = 0, \forall Z$$

$$\Rightarrow ad^*_{\frac{\partial S}{\partial Q}} Q + \Theta\left(\frac{\partial S}{\partial Q}\right) = 0$$

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Souriau relation on foliation

- In his 1974 paper, Jean-Marie Souriau has written (without proof):

  $$\left\langle Q,[\beta,Z]\right\rangle+\tilde{\Theta}(\beta,Z)=0$$

- To prove this equation, we have to consider the parametrized curve

  $t \mapsto Ad_{\exp(tZ)}\beta$ with $Z\in\mathfrak{g}$ and $t\in R$

- The parameterized curve $Ad_{\exp(tZ)}\beta$ passes, for $t=0$, through the point β, since $Ad_{\exp(0)}$ is the identical map of the Lie Algebra $\mathfrak{g}$. This curve is in the adjoint orbit of β. So by taking its derivative with respect to t, then for $t=0$, we obtain a tangent vector in β at the adjoint orbit of this point. When Z takes all possible values in $\mathfrak{g}$, the vectors thus obtained generate all the vector space tangent in β to the orbit of this point:

$$\left.\frac{d\Phi\left(Ad_{\exp(tZ)}\beta\right)}{dt}\right|_{t=0}=\left\langle\frac{d\Phi}{d\beta},\left(\left.\frac{d\left(Ad_{\exp(tZ)}\beta\right)}{dt}\right|_{t=0}\right)\right\rangle=\left\langle Q,ad_Z\beta\right\rangle=\left\langle Q,[Z,\beta]\right\rangle$$

Souriau relation on foliation

- As we have seen before $\Phi\left(Ad_g\beta\right)=\Phi(\beta)-\left\langle\theta\left(g^{-1}\right),\beta\right\rangle$. If we set $g=\exp(tZ)$, we obtain: $$\Phi\left(Ad_{\exp(tZ)}\beta\right)=\Phi(\beta)-\left\langle\theta\left(\exp(-tZ)\right),\beta\right\rangle$$
- By derivation with respect to t at $t=0$, we finally recover the equation given by Souriau :

$$\left.\frac{d\Phi\left(Ad_{\exp(tZ)}\beta\right)}{dt}\right|_{t=0}=\left\langle Q,[Z,\beta]\right\rangle=-\left\langle d\theta(-Z),\beta\right\rangle \text{ with } \tilde{\Theta}(X,Y)=-\left\langle d\theta(X),Y\right\rangle$$

Souriau Entropy and Casimir Invariant Function

Dynamic equation

- The dual space of the Lie algebra foliates into coadjoint orbits that are also the level sets on the entropy.
- The information manifold foliates into level sets of the entropy that could be interpreted in the framework of Thermodynamics by the fact that motion remaining on this complex surfaces is non-dissipative, whereas motion transversal to these surfaces is dissipative, where the dynamic is given by:

$$\frac{dQ}{dt} = \{Q,H\}_{\tilde{\Theta}} = ad^{*}_{\frac{\partial H}{\partial Q}} Q + \Theta\left(\frac{\partial H}{\partial Q}\right)$$

- with stable equilibrium when:

$$H = S \Rightarrow \frac{dQ}{dt} = \{Q,S\}_{\tilde{\Theta}} = ad^{*}_{\frac{\partial S}{\partial Q}} Q + \Theta\left(\frac{\partial S}{\partial Q}\right) = 0$$

Lie-Poisson variational principle

$$\frac{dQ}{dt} = ad^{*}_{\frac{\partial H}{\partial Q}} Q + \Theta\left(\frac{\partial H}{\partial Q}\right)$$

- This Lie-Poisson equation is equivalent to this **Lie-Poisson variational principle**:

$$\delta \int_{0}^{\tau} \left(\left\langle Q(t), \frac{\partial H}{\partial Q}(t) \right\rangle - H\left(Q(t)\right) \right) dt = 0 \quad \text{where} \quad \begin{cases} \frac{\partial H}{\partial Q} = g^{-1}\dot{g} \in \mathfrak{g}, g \in G \\ \frac{\partial^2 H}{\partial Q^2} \delta Q = \delta\left(\frac{\partial H}{\partial Q}\right), \eta = g^{-1}\delta g \\ \left\langle Q, \delta\left(\frac{\partial H}{\partial Q}\right)\right\rangle = \left\langle Q, \dot{\eta} + \left[\frac{\partial H}{\partial Q}, \eta\right]\right\rangle + \tilde{\Theta}\left(\frac{\partial H}{\partial Q}, \eta\right) \end{cases}$$

Lie-Poisson variational principle

- Proof of **Lie-Poisson variational principle**:

$$\delta\int_0^\tau \left(\left\langle Q(t), \frac{\partial H}{\partial Q}(t) \right\rangle - H\left(Q(t)\right) \right) dt = \int_0^\tau \left(\left\langle \delta Q, \frac{\partial H}{\partial Q} \right\rangle + \left\langle Q, \delta\left(\frac{\partial H}{\partial Q}\right) \right\rangle - \left\langle \delta Q, \frac{\partial H}{\partial Q} \right\rangle \right) dt \ = 0$$

$$= \int_0^\tau \left(\left\langle Q, \frac{d\eta}{dt} + \left[\frac{\partial H}{\partial Q}, \eta\right] \right\rangle + \tilde{\Theta}\left(\frac{\partial H}{\partial Q}, \eta\right) \right) dt = \int_0^\tau \left(\left\langle Q, \frac{d\eta}{dt} \right\rangle + \left\langle Q, ad_{\frac{dH}{dQ}}\eta \right\rangle + \left\langle \Theta\left(\frac{\partial H}{\partial Q}\right), \eta \right\rangle \right) dt$$

$$\underset{\substack{Int.\\ by\\ parts}}{=} \int_0^\tau \left\langle -\frac{dQ}{dt} + ad^*_{\frac{dH}{dQ}} Q + \Theta\left(\frac{\partial H}{\partial Q}\right), \eta \right\rangle dt + \left\langle Q, \eta \right\rangle \Big|_0^\tau = 0$$

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Entropy Production and 2nd Principle

2nd Principle

- We can observe that: $dS=\tilde{\Theta}_{\beta}\left(\frac{\partial H}{\partial Q},\beta\right)dt$
- Where:

$$\tilde{\Theta}_{\beta}\left(\frac{\partial H}{\partial Q},\beta\right)=\tilde{\Theta}\left(\frac{\partial H}{\partial Q},\beta\right)+\left\langle Q,\left[\frac{\partial H}{\partial Q},\beta\right]\right\rangle$$

- showing that Entropy production is linked with Souriau tensor related to Fisher metric: $\frac{dS}{dt}=\tilde{\Theta}_{\beta}\left(\frac{\partial H}{\partial Q},\beta\right)\geq 0$
- It allows to introduce the stochastic extension based on a Stratonovich differential equation for the stochastic process given by the following relation by mean of Souriau's symplectic cocycle

$$dQ+\left[ad^{*}_{\frac{\partial H}{\partial Q}}Q+\Theta\left(\frac{\partial H}{\partial Q}\right)\right]dt+\sum_{i=1}^{N}\left[ad^{*}_{\frac{\partial H_i}{\partial Q}}Q+\Theta\left(\frac{\partial H_i}{\partial Q}\right)\right]\circ dW_i(t)=0$$

Entropy Production and 2nd Principle

Demo

$$S(Q)=\langle Q,\beta\rangle-\Phi(\beta) \quad \text{with} \quad \frac{dQ}{dt}=ad^*_{\frac{\partial H}{\partial Q}}Q+\Theta\left(\frac{\partial H}{\partial Q}\right)$$

$$\frac{dS}{dt}=\left\langle Q,\frac{d\beta}{dt}\right\rangle+\left\langle ad^*_{\frac{\partial H}{\partial Q}}Q+\Theta\left(\frac{\partial H}{\partial Q}\right),\beta\right\rangle-\frac{d\Phi}{dt}=\left\langle Q,\frac{d\beta}{dt}\right\rangle+\left\langle ad^*_{\frac{\partial H}{\partial Q}}Q,\beta\right\rangle+\left\langle\Theta\left(\frac{\partial H}{\partial Q}\right),\beta\right\rangle-\frac{d\Phi}{dt}$$

$$\frac{dS}{dt}=\left\langle Q,\frac{d\beta}{dt}\right\rangle+\left\langle Q,\left[\frac{\partial H}{\partial Q},\beta\right]\right\rangle+\tilde{\Theta}\left(\frac{\partial H}{\partial Q},\beta\right)-\frac{d\Phi}{dt}=\left\langle Q,\frac{d\beta}{dt}\right\rangle+\tilde{\Theta}_\beta\left(\frac{\partial H}{\partial Q},\beta\right)-\left\langle\frac{\partial\Phi}{\partial\beta},\frac{d\beta}{dt}\right\rangle$$

$$\frac{dS}{dt}=\left\langle Q,\frac{d\beta}{dt}\right\rangle+\tilde{\Theta}_\beta\left(\frac{\partial H}{\partial Q},\beta\right)-\left\langle\frac{\partial\Phi}{\partial\beta},\frac{d\beta}{dt}\right\rangle \quad \text{with} \quad \frac{\partial\Phi}{\partial\beta}=Q$$

$$\frac{dS}{dt}=\tilde{\Theta}_\beta\left(\frac{\partial H}{\partial Q},\beta\right)\geq 0,\forall H \quad \text{(link to positivity of Fisher metric)}$$

$$\text{if } H=S \underset{\frac{\partial S}{\partial Q}=\beta}{\Rightarrow} \frac{dS}{dt}=\tilde{\Theta}_\beta(\beta,\beta)=0 \text{ because } \beta\in Ker\tilde{\Theta}_\beta$$

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THALES

Geometric Fourier Heat Equation

Fourier heat equation in Souriau Model

$$\frac{\partial Q}{\partial \beta}.\frac{\partial \beta}{\partial t}=ad^{*}_{\frac{\partial H}{\partial Q}}Q+\Theta\left(\frac{\partial H}{\partial Q}\right)=\left\{Q,H\right\}_{\tilde{\Theta}}$$

where $\frac{\partial Q}{\partial \beta}$ geometric heat capacity is given by $g_{\beta}\left(X,Y\right)=\left\langle -\frac{\partial Q}{\partial \beta}(X),Y\right\rangle$ for $X,Y\in\mathfrak{g}$

with $g_{\beta}\left(X,Y\right)=\tilde{\Theta}_{\beta}\left(X,Y\right)=\left\langle Q\left(\beta\right),\left[X,Y\right]\right\rangle+\tilde{\Theta}\left(X,Y\right)$ related to Souriau-Fisher tensor

We can recover the classical Fourier heat equation

$$\frac{\partial Q\left(\beta\right)}{\partial \beta}.\frac{\partial \beta}{\partial t}=\left\{Q\left(\beta\right),H\right\}_{\tilde{\Theta}}\underset{\beta=\frac{1}{T},Q=\frac{\partial \Phi}{\partial \beta}}{\Rightarrow}\frac{\partial T}{\partial t}=\Delta T=\frac{\partial^{2}T}{\partial t^{2}}$$

Group of time translation
in Euclidean space

Koszul Poisson Cohomology and Entropy Characterization

Poisson Cohomology was introduced by A. Lichnerowicz and J.L. Koszul.

Koszul Cohomology and seminal work of Elie Cartan. Koszul made reference to seminal E. Cartan paper

- “*Elie Cartan does not explicitly mention $\Lambda(g')$ [the complex of alternate forms on a Lie algebra], because he treats groups as symmetrical spaces and is therefore interested in differential forms which are invariant to both by the translations to the left and the translations to the right, which corresponds to the elements of $\Lambda(g')$ invariant by the prolongation of the coadjoint representation. Nevertheless, it can be said that by 1929 an essential piece of the cohomological theory of Lie algebras was in place.*” – Jean-Louis Koszul

Koszul Poisson Cohomology and Entropy Characterization

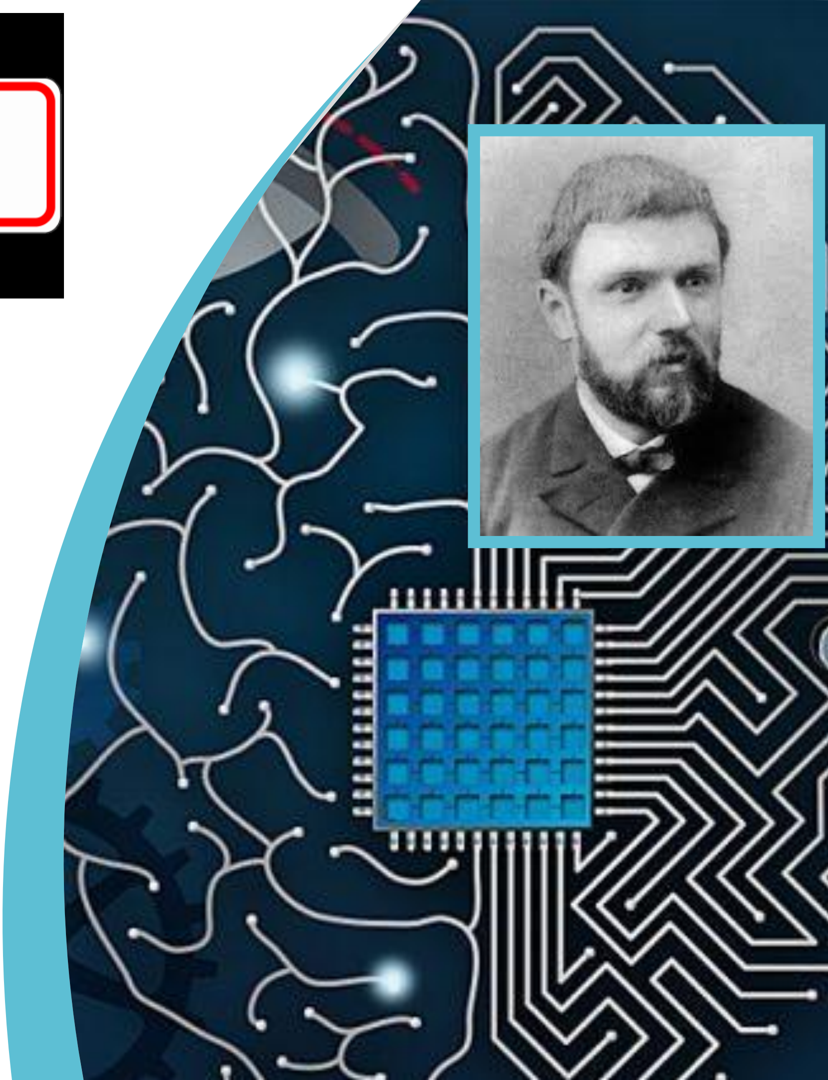
- Y. Vorob'ev and M.V. Karasev have suggested cohomology classification in terms of closed forms and de Rham Cohomology of coadjoint orbits Ω (called Euler orbits by authors), symplectic leaves of a Poisson manifold N.
- Let $Z^k(\Omega)$ and $H^k(\Omega)$ be the space of closed k-forms on Ω and their de Rham cohomology classes.
- Considering the base of the fibration of N by these orbits as N/Ω, they have introduced the smooth mapping

$$Z^k[\Omega] = C^\infty\left(N/\Omega \to Z^k(\Omega)\right) \text{and } H^k[\Omega] = C^\infty\left(N/\Omega \to H^k(\Omega)\right)$$

- The elements of $Z^k(\Omega)$ are closed forms on Ω, depending on coordinates on N/Ω
- Then $H^0[\Omega] = Casim(N)$ is the set of Casimir functions on N, of functions which are constant on all Euler orbits.
- **Entropy is then characterized by zero-dimensional de Rham Cohomology**.
- The center of Poisson algebra induced from the symplectic structure is the zero-dimensional de Rham cohomology group, the Casimir functions.

THALES

Gibbs Density on Poincaré Unit Disk from Souriau Lie Groups Thermodynamics and SU(1,1) Coadjoint Orbits



Poincaré Unit Disk and SU(1,1) Lie Group

- The group of complex unimodular pseudo-unitary matrices $SU(1,1)$:

$$G = SU(1,1) = \left\{ \begin{pmatrix} a & b \\ b^* & a^* \end{pmatrix} / |a|^2 - |b|^2 = 1,\ a,b \in C \right\}$$

- the Lie algebra $\mathfrak{g} = \mathfrak{su}(1,1)$ is given by:

$$\mathfrak{g} = \left\{ \begin{pmatrix} -ir & \eta \\ \eta^* & ir \end{pmatrix} / r \in R, \eta \in C \right\}$$

with the following bases $(u_1, u_2, u_3) \in \mathfrak{g}$:

$$u_1 = \frac{1}{2}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix},\ u_2 = \frac{1}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},\ u_3 = \frac{1}{2}\begin{pmatrix} -i & 0 \\ 0 & i \end{pmatrix}$$

with the commutation relation:

$$[u_3, u_2] = u_1, [u_3, u_1] = u_2, [u_2, u_1] = -u_3$$

Poincaré Unit Disk and SU(1,1) Lie Group

- Dual base on dual Lie algebra is named
  $\left(u_1^*, u_2^*, u_3^*\right) \in \mathfrak{g}^*$
- The dual vector space $\mathfrak{g}^* = \mathfrak{su}^*(1,1)$ can be identified with the subspace of $\mathfrak{sl}(2,C)$ of the form:

$$\mathfrak{g}^* = \left\{ \begin{pmatrix} z & x+iy \\ -x+iy & -z \end{pmatrix} = x\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} + y\begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix} + z\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} / \, x, y, z \in R \right\}$$

- Coadjoint action of $g \in G$ on dual Lie algebra $\xi \in \mathfrak{g}^*$ is written $g.\xi$

Coadjoint Orbit of SU(1,1) and Souriau Moment Map

- The torus $K=\left\{\begin{pmatrix} e^{i\theta} & 0 \\ 0 & e^{-i\theta}\end{pmatrix}, \theta\in R\right\}$ induces rotations of the unit disk
- K leaves 0 invariant. The stabilizer for the origin 0 of unit disk is maximal compact subgroup K of $SU(1,1)$.
- B. Cahen has observed that $O\left(ru_3^*\right)\approx G/K$ and is diffeomorphic to the unit disk $D=\left\{z\in C / |z|<1\right\}$
- The **moment map** is given by:

$$J:D\to O\left(ru_3^*\right)$$

$$z\mapsto J(z)=r\left(\frac{z+z^*}{\left(1-|z|^2\right)}u_1^*+\frac{z-z^*}{i\left(1-|z|^2\right)}u_2^*+\frac{1+|z|^2}{\left(1-|z|^2\right)}u_3^*\right)$$

Benjamin Cahen, Contraction de SU(1,1) vers le groupe de Heisenberg, Travaux mathématiques, Fascicule XV, pp.19-43, (2004)

Coadjoint Orbit of SU(1,1) and Souriau Moment Map

$$J : D \to \mathrm{O}_n$$

$$z \mapsto J(z) = \frac{n}{2}\left(\frac{z+z^*}{\left(1-|z|^2\right)} u_1^* + \frac{z-z^*}{i\left(1-|z|^2\right)} u_2^* + \frac{1+|z|^2}{\left(1-|z|^2\right)} u_3^* \right)$$

- Group G act on D by homography: $g.z = \begin{pmatrix} a & b \\ b^* & a^* \end{pmatrix} .z = \frac{az+b}{a^* z + b^*}$
- This action corresponds with coadjoint action of G on O_n .
- The Kirillov-Kostant-Souriau 2-form of O_n is given by:

$$\Omega_n(\zeta)\left(X(\zeta), Y(\zeta)\right) = \langle \zeta, [X,Y] \rangle \ , X,Y \in \mathfrak{g} \text{ and } \zeta \in \mathrm{O}_n$$

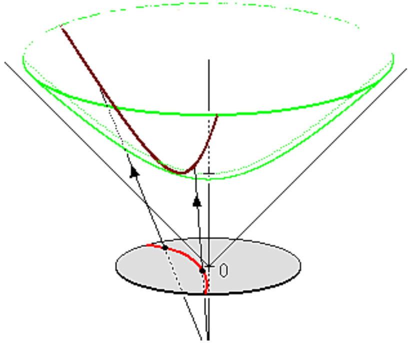
- and is associated in the frame by ψ_n with: $\omega_n = \frac{in}{\left(1-|z|^2\right)^2} dz \wedge dz^*$

Coadjoint Orbit of SU(1,1) and Souriau Moment Map

$$J(z)=r\left(\frac{z+z^*}{\left(1-|z|^2\right)}u_1^*+\frac{z-z^*}{i\left(1-|z|^2\right)}u_2^*+\frac{1+|z|^2}{\left(1-|z|^2\right)}u_3^*\right)\in O\left(ru_3^*\right), z\in D$$

- J is linked to the natural action of G on D (by fractional linear transforms) but also the coadjoint action of G on $O\left(ru_3^*\right)=G/K$
- J^{-1} could be interpreted as the stereographic projection from the two-sphere S^2 onto $C\cup\infty$:

The coadjoint action of $G=SU(1,1)$ is the upper sheet $x_3>0$ of the two-sheet hyperboloid



Charles-Michel Marle, Projection stéréographique et moments, hal-02157930, version 1, Juin 2019

$$\left\{\xi=x_1u_1^*+x_2u_2^*+x_3u_3^*:-x_1^2-x_2^2+x_3^2=r^2\right\}$$

THALES

Moment Map for SU(1,1)

Invariant Moment Map

- The associated moment map $J : D \rightarrow su^*(1,1)$ defined by $J(z).u_i = J_i(z,z^*)$, maps D into a coadjoint orbit in $su^*(1,1)$.
- Then, we can write the moment map as a matrix element of $su^*(1,1)$:

$$J(z) = J_1\left(z,z^*\right)u_1^* + J_2\left(z,z^*\right)u_2^* + J_3\left(z,z^*\right)u_3^*$$

$$J(z) = \rho \begin{pmatrix} \dfrac{1+|z|^2}{1-|z|^2} & -2\dfrac{z^*}{1-|z|^2} \\ 2\dfrac{z}{1-|z|^2} & -\dfrac{1+|z|^2}{1-|z|^2} \end{pmatrix} \in \mathfrak{g}^*$$

Gibbs State Equilibrium

One parameter subgroup

- We can then exponentiate β with exponential map to get :

$$g = \exp(\varepsilon\beta) = \sum_{k=0}^{\infty} \frac{(\varepsilon\beta)^k}{k!} = \begin{pmatrix} a_\varepsilon(\beta) & b_\varepsilon(\beta) \\ b_\varepsilon^*(\beta) & a_\varepsilon^*(\beta) \end{pmatrix}$$

- If we make the remark that we have the following relation

$$\beta^2 = \begin{pmatrix} ir & \eta \\ \eta^* & -ir \end{pmatrix}\begin{pmatrix} ir & \eta \\ \eta^* & -ir \end{pmatrix} = \left(|\eta|^2 - r^2\right) I$$

- we can developed the exponential map :

$$g = \exp(\varepsilon\beta) = \begin{pmatrix} \cosh(\varepsilon R) + ir\dfrac{\sinh(\varepsilon R)}{R} & \eta\dfrac{\sinh(\varepsilon R)}{R} \\ \eta^*\dfrac{\sinh(\varepsilon R)}{R} & \cosh(\varepsilon R) - ir\dfrac{\sinh(\varepsilon R)}{R} \end{pmatrix} \text{ with } R^2 = |\eta|^2 - r^2$$

Souriau Gibbs density for SU(1,1)

Covariant Gibbs density

- We can the write the covariant Gibbs density in the unit disk given by moment map of the Lie group $SU(1,1)$ and geometric temperature in its Lie algebra $\beta \in \Lambda_\beta$:

$$p_{Gibbs}(z) = \frac{e^{-\langle J(z),\beta\rangle}}{\int_D e^{-\langle J(z),\beta\rangle} d\lambda(z)} \text{ with } d\lambda(z) = 2i\rho \frac{dz \wedge dz^*}{\left(1-|z|^2\right)^2}$$

$$p_{Gibbs}(z) = \frac{e^{-\left\langle \rho\left(2\Im bb^+ - Tr\left(\Im bb^+\right)I\right),\beta\right\rangle}}{\int_D e^{-\langle J(z),\beta\rangle} d\lambda(z)} = \frac{e^{-\left\langle \rho\begin{pmatrix} \frac{1+|z|^2}{\left(1-|z|^2\right)} & \frac{-2z^*}{\left(1-|z|^2\right)} \\ \frac{2z}{\left(1-|z|^2\right)} & -\frac{1+|z|^2}{\left(1-|z|^2\right)} \end{pmatrix}, \begin{pmatrix} ir & \eta \\ \eta^* & -ir \end{pmatrix}\right\rangle}}{\int_D e^{-\langle J(z),\beta\rangle} d\lambda(z)}$$

$$J(z) == \rho\left(2Mbb^+ - Tr\left(Mbb^+\right)I\right) \text{ with } M = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \text{ and } b = \frac{1}{1-|z|^2}\begin{bmatrix} 1 \\ -z \end{bmatrix}$$

Souriau Gibbs density

Covariant Gibbs Density

$$p_{Gibbs}(z)=\frac{e^{-\left\langle \rho\begin{pmatrix}\frac{1+|z|^2}{(1-|z|^2)} & \frac{-2z^*}{(1-|z|^2)}\\ \frac{2z}{(1-|z|^2)} & -\frac{1+|z|^2}{(1-|z|^2)}\end{pmatrix},\begin{pmatrix} ir & \eta \\ \eta^* & -ir\end{pmatrix}\right\rangle}}{\int_D e^{-\langle J(z),\beta\rangle}d\lambda(z)}$$

- To write the Gibbs density with respect to its statistical moments, we have to express the density with respect to $Q=E\left[J(z)\right]$
- Then, we have to invert the relation between Q and β, to replace $\beta=\begin{pmatrix} ir & \eta \\ \eta^* & -ir\end{pmatrix}\in\Lambda_\beta$ by $\beta=\Theta^{-1}(Q)\in\mathfrak{g}$ where $Q=\frac{\partial\Phi(\beta)}{\partial\beta}=\Theta(\beta)\in\mathfrak{g}^*$ with $\Phi(\beta)=-\log\int_D e^{-\langle J(z),\beta\rangle}d\lambda(z)$ deduce from Legendre tranform. The mean moment map is given by:

$$Q=E\left[J(z)\right]=E\left[\rho\begin{pmatrix}\frac{1+|w|^2}{(1-|w|^2)} & \frac{-2w^*}{(1-|w|^2)}\\ \frac{2w}{(1-|w|^2)} & -\frac{1+|w|^2}{(1-|w|^2)}\end{pmatrix}\right]\quad \text{where } w\in D$$

THALES

Gaussian density for SPD matrix via Gibbs Density for SU(n,n) in Siegel Disk

Symplectic Group(Carl-Ludwig Siegel) : Siegel Upper half space SH_n

Siegel metric on the Siegel Upper Half Space:

- Siegel Upper half Space: $\boxed{SH_n = \{Z = X + iY \in Sym(n,C) / \operatorname{Im}(Z) = Y > 0\}}$
- Isometries of SH_n are given by quotient Group:

$PSp(n,R) \equiv Sp(n,R)/\{\pm I_{2n}\}$ with $Sp(n,F)$ Symplectic Group:

$$M = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \Rightarrow M(Z) = (AZ+B)(CZ+D)^{-1}$$

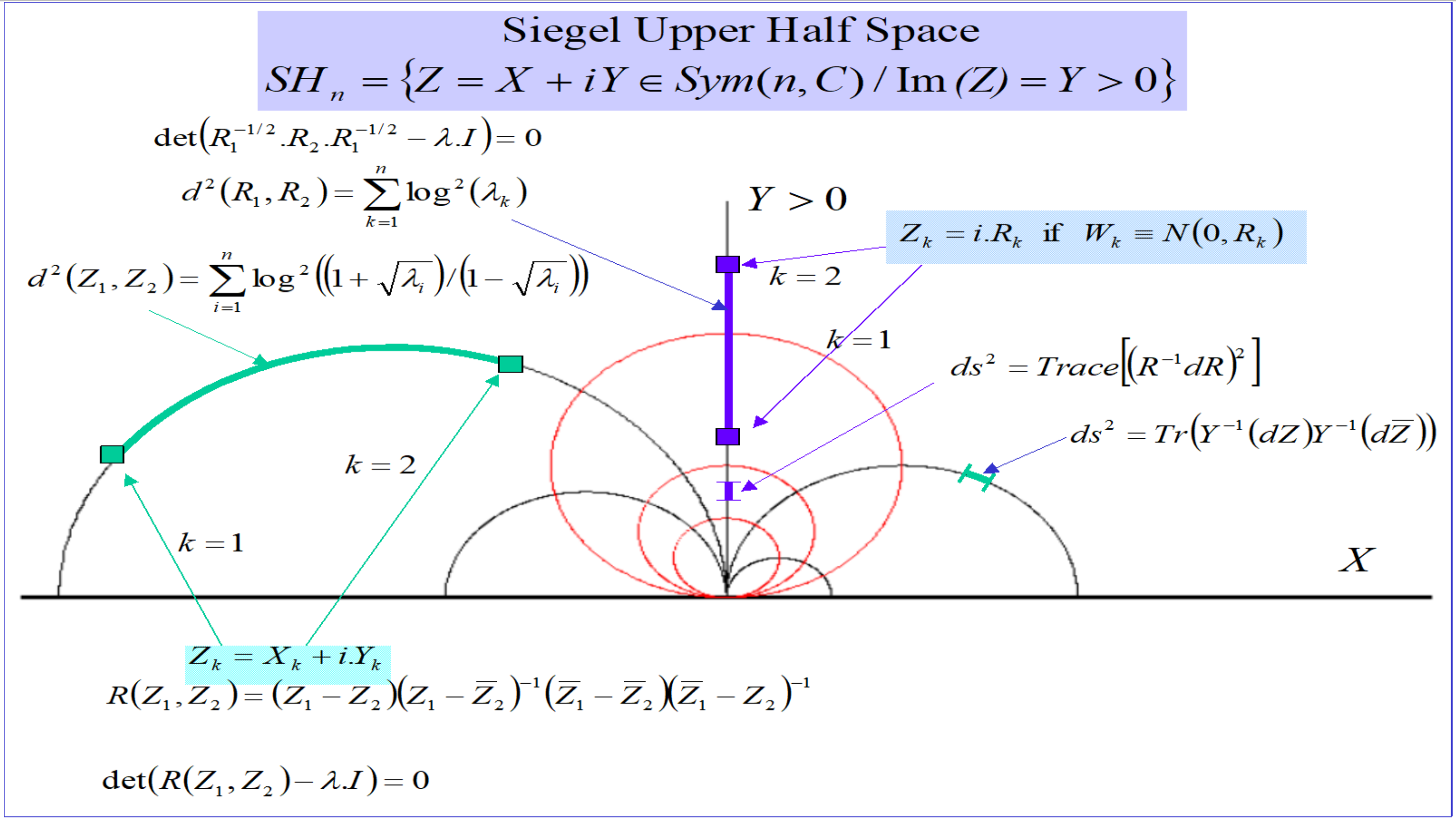
$$M = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in Sp(n,F) \Leftrightarrow \begin{cases} A^T C \text{ et } B^T D \quad \text{symmetric} \\ A^T D - C^T B = I_n \end{cases}$$

$$Sp(n,F) \equiv \{M \in GL(2n,F) / M^T J M = J\},\ J = \begin{pmatrix} 0 & I_n \\ -I_n & 0 \end{pmatrix} \in SL(2n,R)$$

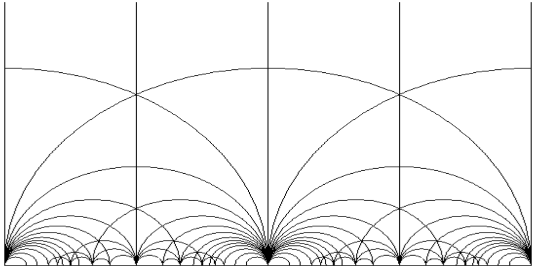
- Metric invariant by the automorpisms $M(Z)$:

$$ds^2_{Siegel} = Tr\left(Y^{-1}(dZ)Y^{-1}(d\bar{Z})\right) \qquad Z = X + iY$$

Extension of homogeneous Bounded symmetric domains: Siegel Upper half-space and Siegel disk



Extension of homogeneous bounded symmetric domains: Siegel Upper half-space and Siegel disk



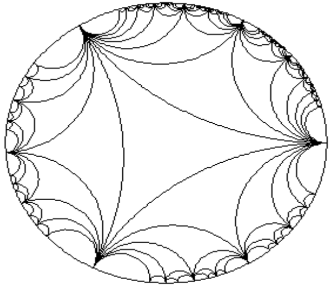
Poincaré Upper half Plane

$$ds^2 = \frac{dx^2 + dy^2}{y^2} = \frac{|dz|^2}{y^2}$$

$$ds^2 = y^{-1} dz y^{-1} dz^*$$

with $z = x + iy$ and $y > 0$

$$w = \frac{z-i}{z+i}$$



Poincaré Unit Disk

$$ds^2 = \frac{|dw|^2}{\left(1-|w|^2\right)^2}$$

$$ds^2 = \left(1 - ww^*\right)^{-1} dw \left(1 - ww^*\right)^{-1} dw^*$$

Siegel Upper Half Space

$$ds^2 = Tr\left(Y^{-1} dZ Y^{-1} d\bar{Z}\right)$$

with $Z = X + iY$

$X \in Herm(n,C)$ and $Y \in HPD(n,C)$

$$W = (Z - iI)(Z + iI)^{-1}$$

Siegel Unit Disk

$$ds^2 = Tr\left[\left(I - WW^+\right)^{-1} dW \left(I - W^+W\right)^{-1} dW^+\right]$$

THALES

Gauss Density on Siegel Unit Disk

Siegel Unit Disk as Homogeneous Symplectic Manifold associated to Coadjoint orbit of G/K

- To address computation of covariant Gibbs density for Siegel Unit Disk $SD_n = \left\{ Z \in Mat\left(n,\mathbb{C}\right) / I_n - ZZ^{+} > 0 \right\}$ we will consider in this section $SU(p,q)$ the Unitary Group:

$$G = SU(n,n) \ \text{ and } \ K = S\left(U(n)\times U(n)\right) = \left\{ \begin{pmatrix} A & 0 \\ 0 & D \end{pmatrix} / A \in U(n), D \in U(n), \det(A)\det(D) = 1 \right\}$$

- G/K acts transitively on SD_n
- We can use the following decomposition for $g \in G^{\mathbb{C}}$ (complexification of g), and consider its action on Siegel Unit Disk given by:

$$g = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in G^{\mathbb{C}}, g = \begin{pmatrix} I_n & BD^{-1} \\ 0 & I_n \end{pmatrix} \begin{pmatrix} A - BD^{-1}C & 0 \\ 0 & D \end{pmatrix} \begin{pmatrix} I_n & 0 \\ D^{-1}C & I_n \end{pmatrix}$$

Moment Map of SU(n,n)/S(U(n)xU(n))

- Benjamin Cahen has studied this case and introduced the moment map by identifing G-equivariantly $\mathfrak{g}^*$ with $\mathfrak{g}$ by means of the Killing form β on $\mathfrak{g}^{\mathbb{C}}$:

$\mathfrak{g}^*$ G-equivariant with $\mathfrak{g}$ by Killing form $\beta\left(X,Y\right)=2(p+q)Tr\left(XY\right)$

- The set of all elements of $\mathfrak{g}$ fixed by K is $\mathfrak{h}$:

$$\mathfrak{h}=\left\{\text{element of } G \text{ fixed by } K\right\} \quad , \xi_0 \in \mathfrak{h}, \xi_0 = i\lambda \begin{pmatrix} -nI_p & 0 \\ 0 & nI_q \end{pmatrix}$$

$$\Rightarrow \left\langle \xi_0, \left[Z, Z^+\right] \right\rangle = -2i\lambda\left(2n\right)^2 Tr\left(ZZ^+\right), \forall Z \in D$$

Gauss Density on Siegel Unit Disk

Moment Map of SU(n,n)/S(U(n)xU(n))

- Then, we the equivariant moment map is given by:

$$\forall X \in g^C \ , \ Z \in D, \quad \psi(Z) = Ad^*\left(\exp\left(-Z^+\right)\zeta\left(\exp Z^+ \exp Z\right)\right)\xi_0$$

$$\forall g \in G, Z \in D \ \text{ then } \psi(g.Z) = Ad_g^* \psi(Z)$$

ψ is a diffeomorphism from SD onto orbit $O(\xi_0)$

$$\psi(Z) = i\lambda \begin{pmatrix} \left(I_n - ZZ^+\right)^{-1}\left(-nZZ^+ - nI_n\right) & (2n)Z\left(I_n - Z^+Z\right)^{-1} \\ -(2n)\left(I_n - Z^+Z\right)^{-1}Z^+ & \left(nI_q + nZ^+Z\right)\left(I_n - Z^+Z\right)^{-1} \end{pmatrix}$$

$$\zeta\left(\exp Z^+ \exp Z\right) = \begin{pmatrix} I_p & Z\left(I_q - Z^+Z\right)^{-1} \\ 0 & I_q \end{pmatrix}$$

Gauss Density on Siegel Unit Disk

Moment Map of SU(n,n)/S(U(n)xU(n))

- The moment map for $SU(n,n)/S\left(U(n)\times U(n)\right)$ is then given by:

$$J(Z)=\rho n\begin{pmatrix} \left(I_n-ZZ^+\right)^{-1}\left(I_n+ZZ^+\right) & -2Z^+\left(I_n-ZZ^+\right)^{-1} \\ 2\left(I_n-ZZ^+\right)^{-1}Z & \left(I_n+ZZ^+\right)\left(I_n-ZZ^+\right)^{-1} \end{pmatrix}\in\mathfrak{g}^*$$

- The Souriau Gibbs density is then given with $\beta,M\in\mathfrak{g}$ and $Z\in SD_n$ by:

$$p_{Gibbs}(Z)=\frac{e^{-\left\langle \rho n\begin{pmatrix} \left(I_n-ZZ^+\right)^{-1}\left(I_n+ZZ^+\right) & -2Z^+\left(I_n-ZZ^+\right)^{-1} \\ 2\left(I_n-ZZ^+\right)^{-1}Z & \left(I_n+ZZ^+\right)\left(I_n-ZZ^+\right)^{-1} \end{pmatrix},\beta\right\rangle}}{\int\limits_{SD_n} e^{-\langle J(Z),\beta\rangle}d\lambda(Z)}$$

$$\beta=\Theta^{-1}(Q)\in\mathfrak{g}$$

$$Q=E\left[J(Z)\right]$$

$$Q=\frac{\partial\Phi(\beta)}{\partial\beta}=\Theta(\beta)\in\mathfrak{g}^*$$

- Gauss density of SPD matrix is given by Cayley Transform with:

$$Z=(Y-I)(Y+I)^{-1},\ Y\in Sym(n)^+$$

THALES

Gaussian density for SPD matrix via Gibbs Density for Sp(2n,R) in Siegel Upper-Half Plane and SU(n;n) in Siegel Disk

Gauss Density on Siegel Upper Half Plane

Siegel Upper Half Plane and Sympectic Group

- Considering $H_n = \{W = U + iV \,/\, U, V \in Sym(n), V > 0\}$ Siegel Upper Half Space, related to Siegel Unit Disk by Cayley transform $Z = (W + iI_n)^{-1}(W - iI_n)$, that we will analyze as an homogeneous space with respect to $Sp(2n,\mathbb{R})/U(n)$ group action where the Sympletic group is given by the definition:

$$Sp(2n,\mathbb{R}) = \left\{ \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in Mat(2n,\mathbb{R}) \,/\, \begin{matrix} A^T C = C^T A, B^T D = D^T B \\ A^T D - C^T B = I_n \end{matrix} \right\}$$

and the left transitive action:

$$\Phi : Sp(2n,\mathbb{R}) \times H_n \to H_n \ \text{ with } \ \Phi\left(\begin{pmatrix} A & B \\ C & D \end{pmatrix}, W \right) = (C + DW)(A + BW)^{-1}$$

$$\Phi : Sp(2n,\mathbb{R}) \times H_n \to H_n \ \text{ with } \ \Phi\left(\begin{pmatrix} V^{-1/2} & 0 \\ UV^{-1/2} & V^{1/2} \end{pmatrix}, iI_n \right) = U + iV$$

Gauss Density on Siegel Upper Half Plane

Siegel Upper Half Plane and Sympectic Group

- The isotropy subgroup of the element $iI_n \in H_n$ is:

$$Sp(2n,\mathbb{R})\cap O(2n)=\left\{\begin{bmatrix} X & Y \\ -Y & X \end{bmatrix}\in Mat(2n,\mathbb{R})\,/\,X^TX+Y^TY=I_n,X^TY=Y^TX\right\}$$

- that is identified with $U(n)$ by $Sp(2n,R)\cap O(2n)\to U(n);\begin{bmatrix} X & Y \\ -Y & X \end{bmatrix}\mapsto X+iY$

- Then $H_n \cong Sp(2n,\mathbb{R})/U(n)$ is also a symplectic manifold with symplectic form $\Omega_{H_n}=-d\Theta_{H_n}$ with one-form $\Theta_{H_n}=-tr\left(UdV^{-1}\right)$.

- By identifying the symplectic algebra $sp(2n,\mathbb{R})$ with $sym(2n,\mathbb{R})$ [14] via

$$\varepsilon=\begin{bmatrix} \varepsilon_{11} & \varepsilon_{12} \\ \varepsilon_{12}^T & \varepsilon_{22} \end{bmatrix}\in sym(2n,\mathbb{R})\mapsto\tilde{\varepsilon}=J_n^{\,T}\varepsilon=\begin{bmatrix} -\varepsilon_{12}^T & -\varepsilon_{22} \\ \varepsilon_{11} & \varepsilon_{12} \end{bmatrix}\in sp(2n,\mathbb{R})$$

- with $\left[\varepsilon,\delta\right]_{sym}=\varepsilon J_n^{\,T}\delta-\delta J_n^{\,T}\varepsilon$ and $\left[\tilde{\varepsilon},\tilde{\delta}\right]_{sp}=\tilde{\varepsilon}\tilde{\delta}-\tilde{\delta}\tilde{\varepsilon}$, and the associated inner products $\left\langle\varepsilon,\delta\right\rangle_{sym}=tr\left(\varepsilon\delta\right)$ and $\left\langle\tilde{\varepsilon},\tilde{\delta}\right\rangle_{sp}=tr\left(\tilde{\varepsilon}^T\tilde{\delta}\right)$ that allow to identify their dual spaces with themselves $sym(2n,\mathbb{R})=sym(2n,\mathbb{R})^*, sp(2n,\mathbb{R})=sp(2n,\mathbb{R})^*$

Gauss Density on Siegel Upper Half Plane

Gibbs density on Siegel Upper Half Plane

- We define adjoint operator $Ad_G\varepsilon = \left(G^{-1}\right)^T \varepsilon G^{-1}, Ad_G\tilde{\varepsilon} = G\tilde{\varepsilon}G^{-1}$ with $G \in Sp(2n,\mathbb{R})$ and $ad_\varepsilon\delta = \varepsilon J_n^{\ T}\delta - \delta J_n^{\ T}\varepsilon = [\varepsilon,\delta]_{sym}$, and co-adjoint operators $Ad^*_{G^{-1}}\eta = G\eta G^T$ and $ad^*_\varepsilon\eta = J_n\varepsilon\eta - \eta\varepsilon J_n$. To coadjoint orbit $O = \left\{Ad^*_G\eta \in sym(2n,\mathbb{R})^* / G \in Sp(2n,\mathbb{R})\right\}$ for each $\eta \in sym(2n,\mathbb{R})^*$, we can associate a symplectic manifold with the KKS (Kirillov-Kostant-Souriau) 2-form $\Omega_O(\eta)\left(ad^*_\varepsilon\eta, ad^*_\delta\eta\right) = \left\langle \eta, [\varepsilon,\delta]_{sym}\right\rangle = tr\left(\eta[\varepsilon,\delta]_{sym}\right)$.
- We then compute the moment map $J : H_n \to sp(2n,\mathbb{R})^*$ such that $i_\varepsilon\Omega_{H_n} = d\left\langle J(.),\varepsilon\right\rangle$ given by $J(W) = \begin{bmatrix} V^{-1} & V^{-1}U \\ UV^{-1} & UV^{-1}U+V \end{bmatrix}$ for $W = U + iV \in H_n$. We then deduce:

$$p_{Gibbs}(W) = \frac{e^{-\langle J(W),\varepsilon\rangle}}{\int_{H_n} e^{-\langle J(W),\varepsilon\rangle} d\lambda(W)} \text{ with } d\lambda(W) = 2i\rho\left(V^{-1}dW \wedge V^{-1}dW^+\right)$$

$W = iV$ with $U = 0$

SPD matrix

$$\langle J(W),\varepsilon\rangle = Tr\left[V^{-1}\varepsilon_{11} + V\varepsilon_{22}\right]$$

$$\langle J(W),\varepsilon\rangle = Tr\left(J(W)\varepsilon\right) = Tr\left(\begin{bmatrix} V^{-1} & V^{-1}U \\ UV^{-1} & UV^{-1}U+V \end{bmatrix}\begin{bmatrix} \varepsilon_{11} & \varepsilon_{12} \\ \varepsilon_{12}^T & \varepsilon_{22} \end{bmatrix}\right) = Tr\left[V^{-1}\varepsilon_{11} + 2UV^{-1}\varepsilon_{12} + \left(UV^{-1}U+V\right)\varepsilon_{22}\right]$$

THALES

GIBBS DENSITY FOR SE(2) LIE GROUPS

Coadjoint action of SE(2)

- We will consider Souriau model for $SE(2)$ Lie group with non-null cohomology and then with introduction of Souriau one-cocycle.
- We consider $SE(2)=SO(2)\times R^2$:

$$SE(2)=\left\{\begin{bmatrix} R_\varphi & \tau \\ 0 & 1 \end{bmatrix} / R_\varphi \in SO(2), \tau \in R^2\right\}$$

- The Lie algebra $se(2)$ of $SE(2)$ has underlying vector space R^3 and Lie bracket:

$$(\xi,u)\in se(2)=R\times R^2, \begin{bmatrix} -\xi\Im & u \\ 0 & 0 \end{bmatrix} \in se(2) \text{ with } \Im=\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$$

- Coadjoint action of $SE(2)$ is given by:

$$Ad^*_{(R_\varphi,\tau)}(m,\rho)=\left(m+\Im R_\varphi\rho.\tau, R_\varphi\rho\right)$$

OPEN

THALES

GIBBS DENSITY FOR SE(2) LIE GROUP

Souriau Gibbs density for SE(2)

- Considering the symplectic form on R^2

$$\omega(\zeta,\upsilon)=\zeta.\Im\upsilon \quad \text{with} \quad \Im=\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$$

- the action of SE(2) is symplectic and admits the momentum map:

$$J(x)=-\left(\frac{1}{2}\|x\|^2,-\Im x\right) \quad , \quad x\in R^2$$

- For generalized temperature $\beta\in\Omega=\left\{(b,\mathrm{B})\in se(2)/b<0,\mathrm{B}\in R^2\right\}$, Souriau Gibbs density is given by :

$$p_{Gibbs}(x)=\frac{e^{-\langle J(x),\beta\rangle}}{\int\limits_{R^2} e^{-\langle J(x),\beta\rangle}d\lambda(x)}=\frac{e^{\frac{1}{2}b\|x\|^2-\mathrm{B}.\Im x}}{\int\limits_{R^2} e^{\frac{1}{2}b\|x\|^2-\mathrm{B}.\Im x}d\lambda(x)}$$

Gibbs density for SE(2)

- The Massieu Potential could be computed :

$$\Phi(\beta)=\log\int_{R^2} e^{\frac{1}{2}b\|x\|^2-\mathrm{B}.\Im x}\,d\lambda(x)=\log\left(-\frac{2\pi}{b}e^{-\frac{1}{2b}\|B\|^2}\right)$$

- By derivation of Massieu potential, we can deduce expression of Heat:

$$Q\in\Omega^*=\left\{(m,M)\in se^*(2)/m+\frac{\|M\|^2}{2}<0\right\};Q=\frac{\partial\Phi(\beta)}{\partial\beta}=\left(\frac{1}{b}-\frac{\|\mathrm{B}\|^2}{2b^2},\frac{1}{b}\mathrm{B}\right)=\Theta(\beta)$$

- We can the inverse this relation to express generalized temperature with respect to the heat:

$$\beta=\Theta^{-1}(Q)=\left(\left(m+\frac{1}{2}\|M\|^2\right)^{-1},\left(m+\frac{1}{2}\|M\|^2\right)^{-1}M\right)$$

- We can the express the Gibbs density with respect to the Heat Q which is the mean of moment map:

$$p_{Gibbs}(x)=\frac{e^{\frac{\frac{1}{2}\|x\|^2-M.\Im x}{\left(m+\frac{1}{2}\|M\|^2\right)}}}{\Gamma}\ \text{with}\ \Gamma=\int_{R^2}e^{\frac{\frac{1}{2}\|x\|^2-M.\Im x}{\left(m+\frac{1}{2}\|M\|^2\right)}}d\lambda(x)\ \text{with}\ (m,M)=E(J(x))=\left[-E\left(\|x\|^2\right),2\Im E(x)\right]$$

Gibbs density for SE(2)

Souriau Covariant Gibbs density for SE(2)

$$p_{Gibbs}(x) = \frac{e^{\frac{\frac{1}{2}\|x\|^2 + 2E(x).Ix}{\left(-E\left(\|x\|^2\right)+2\|E(x)\|^2\right)}}}{\int_{R^2} e^{\frac{\frac{1}{2}\|x\|^2 + 2E(x).Ix}{\left(-E\left(\|x\|^2\right)+2\|E(x)\|^2\right)}} d\lambda(x)}$$

Fisher Metric for SE(2)

- Entropy is given by:

$$S(Q)=\langle Q,\beta\rangle-\Phi(\beta)=1+\log(2\pi)+\log\left(-m-\frac{\|M\|^2}{2}\right)$$

- Fisher Metric is given by:

$$I_{Fisher}(Q)=\left(m+\frac{1}{2}\|M\|^2\right)^{-1}\begin{bmatrix} I & M^T \\ M^T & \frac{1}{2}\|M\|^2-m \end{bmatrix}$$

- With $(m,M)=E(J(x))=E\left[-2\left(\frac{1}{2}\|x\|^2,-\Im x\right)\right]=\left[-E\left(\|x\|^2\right),2\Im E(x)\right]$

- Fisher Metric with respect to moments

$$I_{Fisher}(Q)=\left(2\|E(x)\|^2-E\left(\|x\|^2\right)\right)^{-1}\begin{bmatrix} I & (2\Im E(x))^T \\ 2\Im E(x) & 2\|E(x)\|^2+E\left(\|x\|^2\right) \end{bmatrix}$$

THALES

Souriau Lie Groups Thermodynamics

Structuring Principles for Learning : Calculus of Variations

Pierre de Fermat

Fermat's principle of least time

Pierre Louis Maupertuis

Maupertuis's principle of least length

Joseph Louis Lagrange

(Euler) Lagrange Equation

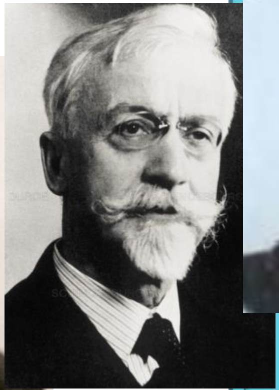
Simeon Denis Poisson

Poisson Bracket, Poisson Geometry Structure

Henri Poincaré

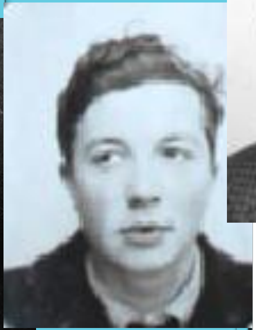
(Euler) Poincaré Equation

Elie Cartan



Poincaré Cartan Integral Invariant

Jean- Marie Souriau



Souriau Moment Map, Souriau Symplectic 2 Form, Lie Groups Thermodynamics

Jean-Michel Bismut

Random Mechanics

Souriau SSD Chapter IV: Gibbs Equilibrium is not covariant with respect to Dynamic Groups of Physics

MÉCANIQUE STATISTIQUE COVARIANTE

Le groupe des translations dans le temps (7.9) est un sous-groupe du groupe de Galilée ; mais *ce n'est pas un sous-groupe invariant*, ainsi que le montre un calcul trivial. Si un système dynamique est *conservatif* dans un repère d'inertie, il en résulte qu'il peut *ne plus être conservatif dans un autre*. La formulation (17.24) du principe de Gibbs doit donc être élargie, pour devenir compatible avec la relativité galiléenne.

Nous proposons donc le principe suivant :

(17.77) Si un système dynamique est invariant par un sous-groupe de Lie G' du groupe de Galilée, les équilibres naturels du système constituent l'ensemble de Gibbs du groupe dynamique G'.

Soit $\mathcal{G}'$ l'algèbre de Lie G' ; on sait que $\mathcal{G}'$ est une sous-algèbre de Lie de celle de G, notée $\mathcal{G}$; un équilibre du système sera caractérisé par un élément Z de $\mathcal{G}'$, donc de $\mathcal{G}$; on pourra écrire

(17.78)
$$Z = \begin{bmatrix} j(\omega) & \beta & \gamma \\ 0 & 0 & \varepsilon \\ 0 & 0 & 0 \end{bmatrix}$$

en utilisant les notations (13.4) ; Z parcourt l'ensemble Ω défini en (16.219) ; à chaque valeur de Z est associé un élément M du dual $\mathcal{G}'^*$ de $\mathcal{G}'$, valeur moyenne du moment μ ; on peut appliquer les formules (16.219), (16.220), qui généralisent les relations thermodynamiques (17.26), (17.27), (17.28).

(17.79) On voit que c'est Z (17.78) qui généralise la « température » ; le théorème d'isothermie (17.32) s'étend immédiatement : l'équilibre d'un système composé de plusieurs parties sans interactions s'obtient en attribuant à chaque composante un équilibre *correspondant à la même valeur de* Z ; l'entropie s, le potentiel de Planck z et le moment moyen M sont *additifs*. W

J.M. Souriau, Structure des systèmes dynamiques, Chapitre IV « Mécanique Statistique »

Trompette de Souriau

Lorsque le fait qu'on rencontre est en opposition avec une théorie régnante, il faut accepter le fait et abandonner la théorie, alors même que celle-ci, soutenue par de grands noms, est généralement adoptée
- Claude Bernard "Introduction à l'Étude de la Médecine Expérimentale"

Lagrange 2-form rediscovered by Jean-Marie Souriau

- Rewriting equations of classical mechanics in phase space $\begin{pmatrix} r \\ v \end{pmatrix}$

$$m\frac{d^2r}{dt^2}=F \quad \Longrightarrow \quad m\frac{dv}{dt}=F \quad \text{et} \quad v=\frac{dr}{dt}$$

- Souriau rediscovered that Lagrange had considered the evolution space: $y=\begin{pmatrix} t \\ r \\ v \end{pmatrix}\in V$

$$\begin{cases} m\delta v - F\delta t = 0 \\ \delta r - v\delta t = 0 \end{cases}$$

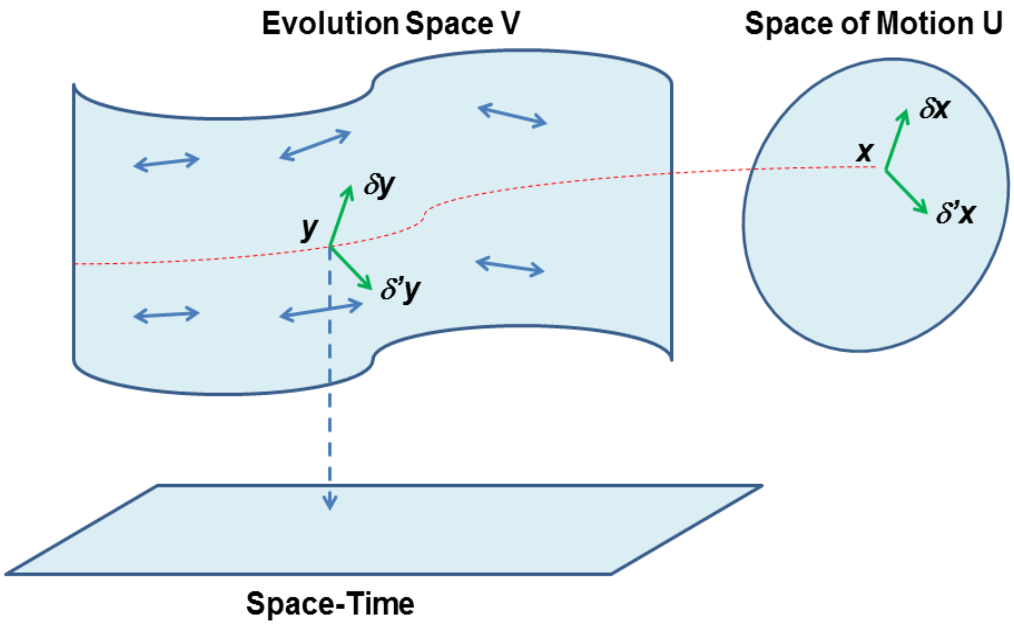
- A dynamic system is represented by a foliation. This foliation is determined by an antisymmetric covariant 2nd order tensor σ , called the Lagrange (-Souriau) form, a bilinear operator on the tangent vectors of V.

$$\sigma(\delta y)(\delta' y)=\langle m\delta v - F\delta t, \delta' r - v\delta' t\rangle - \langle m\delta' v - F\delta' t, \delta r - v\delta t\rangle \qquad \delta y=\begin{pmatrix} \delta t \\ \delta r \\ \delta v \end{pmatrix} \quad \text{et} \quad \delta' y=\begin{pmatrix} \delta' t \\ \delta' r \\ \delta' v \end{pmatrix}$$

- *In the Lagrange-Souriau model,* σ *is a 2-form on the evolution space V, and the differential equation of motion implies:* $\delta y\in\varepsilon$

$$\sigma(\delta y)(\delta' y)=0 \;,\; \forall \delta' y \qquad \sigma(\delta y)=0 \;\; \text{ou} \;\; \delta y\in\ker(\sigma)$$

Evolution space of Lagrange-Souriau



$$\begin{cases} m\delta v - F\delta t = 0 \\ \delta r - v\delta t = 0 \end{cases}$$

Gallileo Group & Algebra & V. Bargman Central extensions

- Symplectic cocycles of the Galilean group: V. Bargmann (Ann. Math. 59, 1954, pp 1–46) has proven that the symplectic cohomology space of the Galilean group is one-dimensional.

- Gallileo Lie Group & Algebra

$$\begin{cases} \vec{x}' = R.\vec{x} + \vec{u}.t + \vec{w} \\ t' = t + e \end{cases}$$

$$\begin{bmatrix} \vec{x}' \\ t' \\ 1 \end{bmatrix} = \begin{bmatrix} R & \vec{u} & \vec{w} \\ 0 & 1 & e \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \vec{x} \\ t \\ 1 \end{bmatrix}$$

$$\vec{x}, \vec{u} \text{ and } \vec{w} \in R^3, e \in R^+$$

$$R \in SO(3)$$

$$\begin{bmatrix} \vec{\omega} & \vec{\eta} & \vec{\gamma} \\ 0 & 0 & \varepsilon \\ 0 & 0 & 0 \end{bmatrix}, \begin{cases} \vec{\eta} \text{ and } \vec{\gamma} \in R^3, \varepsilon \in R^+ \\ \vec{\omega} \in so(3) : \vec{x} \mapsto \vec{\omega} \times \vec{x} \end{cases}$$

- Bargmann Central extension:

$$\begin{bmatrix} R & \vec{u} & 0 & \vec{w} \\ 0 & 1 & 0 & e \\ -\vec{u}^t R & -\frac{\|\vec{u}\|^2}{2} & 1 & f \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Souriau Work Roots: François Gallissot Theorem

- **Gallissot Theorem**: There are 3 types of differential forms generating the equations of a material point motion, **invariant by the action of the Galileo group**

$$A:\begin{cases} s=\dfrac{1}{2m}\sum_{i=1}^{3}\left(mdv_i-F_i dt\right)^2 \\ e=\dfrac{m}{2}\sum_{j=1}^{3}\left(dx_j-v_j dt\right)^2 \end{cases}$$

F. Gallissot, Les formes extérieures en Mécanique (*Thèse*), Durand, Chartres, 1954.

$$B: f=\sum_{1}^{3}\delta_{ij}\left(dx_i-v_i dt\right)\left(mdv_j-F_j dt\right) \text{ with } \delta_{ij} \text{ krönecker symbol}$$

$$C:\ \omega=\sum_{1}^{3}\delta_{ij}\left(mdv_i-F_i dt\right)\wedge\left(dx_j-v_j dt\right)$$

- $d\omega=0$ constrained the Pfaff form $\delta_{ij}F_i dx_j$ to be closed and to be reduced to the differential of U : $C\Rightarrow\omega=m\delta_{ij}dv_i\wedge dx_j-dH\wedge dt$ with $H=T-U$ and $T=1/2\sum_{i=1}^{3}m\left(v_i\right)^2$
- It proves that ω has an exterior differential $d\omega$ generating **Poincaré-Cartan Integral invariant:**

$$d\omega=\sum_{i=1}^{3}mv_i dx_j-Hdt$$

THALES

Souriau Moment Map (1/2)

- Let (X,σ) be a connected symplectic manifold.
- A vector field η on X is called symplectic if its flow preserves the 2-form :

$$L_\eta \sigma = 0$$

- If we use Elie Cartan's formula, we can deduce that :

$$L_\eta \sigma = di_\eta \sigma + i_\eta d\sigma = 0$$

- but as $d\sigma = 0$ then $di_\eta \sigma = 0$. We observe that the 1-form $i_\eta \sigma$ is closed.
- When this 1-form is exact, there is a smooth function $x \mapsto H$ on X with:

$$i_\eta \sigma = -dH$$

- This vector field η is called Hamiltonian and could be define as s symplectic gradient :

$$\eta = \nabla_{Symp} H$$

Souriau Moment Map (2/2)

$$di_\eta \sigma = 0 \qquad \boxed{i_\eta \sigma = -dH}$$

- We define the Poisson bracket of two functions H , H' by :

$$\{H, H'\} = \sigma(\eta, \eta') = \sigma\left(\nabla_{Symp} H', \nabla_{Symp} H\right)$$

with $i_\eta \sigma = -dH$ and $i_{\eta'} \sigma = -dH'$

- Let a Lie group G that acts on X and that also preserve σ .

- A moment map exists if these infinitesimal generators are actually hamiltonian, so that a map exists:

$\boxed{\Phi : X \to \mathfrak{g}^*}$ with $\boxed{i_{Z_X} \sigma = -dH_Z}$ where $\boxed{H_Z = \langle \Phi(x), Z \rangle}$

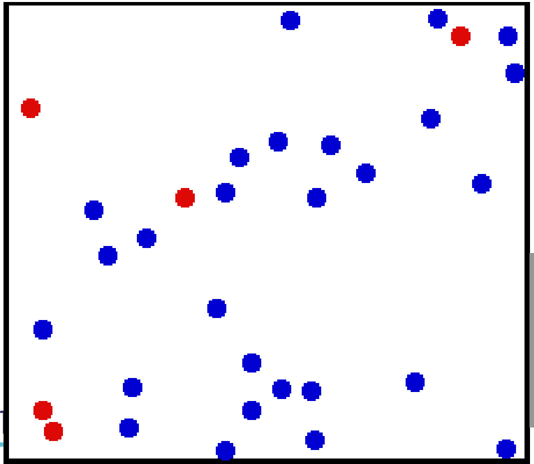
Souriau Model of Lie Groups Thermodynamics

- Souriau Geometric (Planck) Temperature is **an element of Lie Algebra** of Dynamical Group (Galileo/Poincaré groups) acting on the system

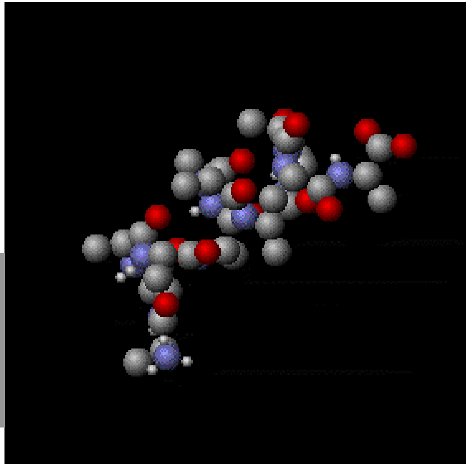
- Generalized Entropy is **Legendre Transform of minus logarithm of Laplace Transform**

- Fisher(-Souriau) Metric is a **Geometric Calorific Capacity (hessian of Massieu Potential)**

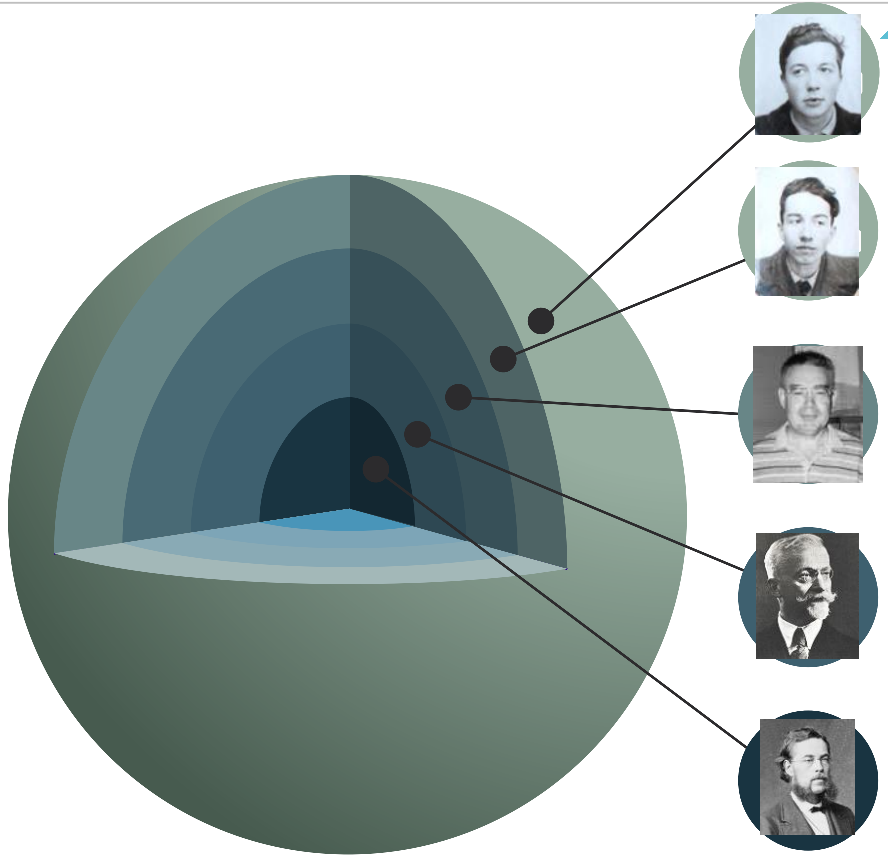
- Higher Order Souriau Lie Groups Thermodynamics is given by **Günther's Poly-Symplectic Model (vector-valued model in non-equivariant case)**



Souriau formalism is fully **covariant**, with no special coordinates (**covariance of Gibbs density wrt Dynamical Groups**)



Lie Groups Tools Development: From Group to Co-adjoint Orbits



Lie Group & Statistical Physics
Jean-Michel Bismut – Random Mechanics
Jean-Marie Souriau – Lie Group Thermodynamics, Souriau Metric
Jean-Louis Koszul – Affine Lie Group & Algebra representation

Harmonic Analysis on Lie Group & Orbits Method
Pierre Torasso & Michèle Vergne – Poisson-Plancherel Formula
Michel Duflo – Extension of Orbits Method, Plancherel & Character
Alexandre Kirillov – Coadjoint Orbits, Kirillov Character
Jacques Dixmier – Unitary representation of nilpotent Group

Lie Group Representation
Bertram Kostant – KKS 2-form, Geometric Quantization
Alexandre Kirillov – Representation Theory, KKS 2-form
Jean-Marie Souriau – Moment Map, KKS 2-form, Souriau Cocycle
Valentine Bargmann – Unitary representation, Central extension

Lie Group Classification
Carl-Ludwig Siegel – Symplectic Group
Hermann Weyl – Conformal Geometry, Symplectic Group
Elie Cartan – Lie algebra classification, Symmetric Spaces
Willem Killing – Cartan-Killing form, Killing Vectors

Group/Lie Group Foundation
Henri Poincaré – Fuchsian Groups
Felix Klein – Erlangen Program (Homogeneous Manifold)
Sophus Lie – Lie Group
Evariste Galois/Louis Joseph Lagange – Substitution Group

Lie Group

GROUP (Mathematics)

A set equipped with a binary operation with 4 axioms:

- Closure $\forall a,b \in G \text{ then } a \bullet b \in G$
- Associativity $\forall a,b,c \in G \text{ then } (a \bullet b) \bullet c = a \bullet (b \bullet c)$
- Identity $\exists e \in G \text{ such that } e \bullet a = a \bullet e = a$
- invertibility $\forall a \in G, \exists b \in G \text{ such that } b \bullet a = a \bullet b = e$

LIE GROUP

- A group that is a differentiable manifold, with the property that the group operations of multiplication and inversion are smooth maps:

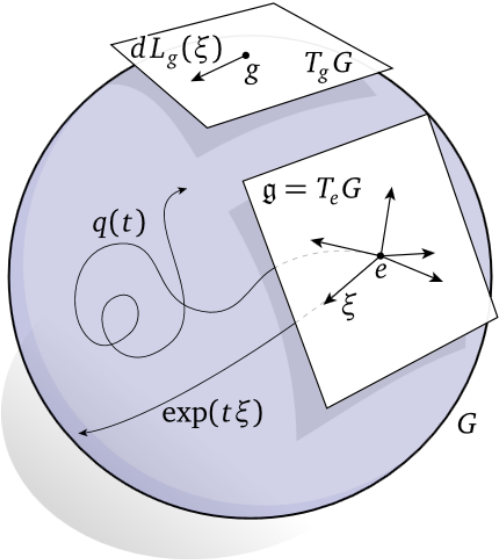
$$\forall x,y \in G \text{ then } \phi : G \times G \to G \text{ then } \phi(x,y) = x^{-1}y \text{ is smooth}$$

- A Lie algebra $\mathfrak{g} = T_e G$ is a vector space with a binary operation called the Lie bracket $[.,.] : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$ that satisfies axioms:

$$[ax+by,z] = a[x,z] + b[y,z] \ ; \ [x,x] = 0 \ ; \ [x,y] = -[y,x]$$

$$\text{Jacobi Identity: } [x,[y,z]] + [z,[x,y]] + [y,[z,x]] = 0$$

$$[x,y] = xy - yx \text{ for Matrix Lie Group}$$



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Coadjoint operator and Coadjoint Orbits (Kirillov Representation)

Lie Group Adjoint Representation

- the adjoint representation of a Lie group Ad_g is a way of representing its elements as linear transformations of the Lie algebra, considered as a vector space

$$Ad_g = \left(d\Psi_g\right)_e : \mathfrak{g} \to \mathfrak{g}$$
$$X \mapsto Ad_g(X) = gXg^{-1}$$

$$\Psi : G \to Aut(G)$$
$$g \mapsto \Psi_g(h) = ghg^{-1}$$
$$ad = T_e Ad : T_e G \to End\left(T_e G\right)$$
$$X, Y \in T_e G \mapsto ad_X(Y) = [X, Y]$$

Lie Group Co-Adjoint Representation

- the coadjoint representation of a Lie group Ad_g^*, is the dual of the adjoint representation ($\mathfrak{g}^*$ denotes the dual space to $\mathfrak{g}$):

$$\forall g \in G, Y \in \mathfrak{g}, F \in \mathfrak{g}^*, \text{ then } \left\langle Ad_g^* F, Y \right\rangle = \left\langle F, Ad_{g^{-1}} Y \right\rangle$$

$$K = Ad_g^* = \left(Ad_{g^{-1}}\right)^* \quad \text{and} \quad K_*(X) = -\left(ad_X\right)^*$$

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Coadjoint operator and Coadjoint Orbits (Kirillov Representation)

Co-adjoint Orbits as Homogeneous Symplectic Manifold by KKS 2-form

- A coadjoint orbit: $O_F = \left\{ Ad_g^* F, g \in G \right\}$ subset of $\mathfrak{g}^*, F \in \mathfrak{g}^*$

  carry a natural homogeneous symplectic structure by a closed G-invariant 2-form:

$$\sigma_\Omega\left(K_{*X}F, K_{*Y}F\right) = B_F\left(X,Y\right) = \left\langle F, [X,Y] \right\rangle, X,Y \in \mathfrak{g}$$

- The coadjoint action on O_F is a Hamiltonian G-action with moment map $\Omega \to \mathfrak{g}^*$

Souriau Foundamental Theorem « *Every symplectic manifold is a coadjoint orbit* » is based on classification of symplectic homogeneous Lie group actions by Souriau, Kostant and Kirillov

$g \in G$ → $O_F = \left\{ Ad_g^* F, g \in G, F \in \mathfrak{g}^* \right\}$ → $\sigma_\Omega\left(ad_F X, ad_F Y\right) = \left\langle F, [X,Y] \right\rangle$, $X,Y \in \mathfrak{g}, F \in \mathfrak{g}^*$

Lie Group

Coadjoint Orbit
(action of Lie Group on dual Lie algebra)

Homogeneous Symplectic Manifold
(a smooth manifold with a closed differential 2-form σ, such that dσ=0, where the Lie Group acts transitively)

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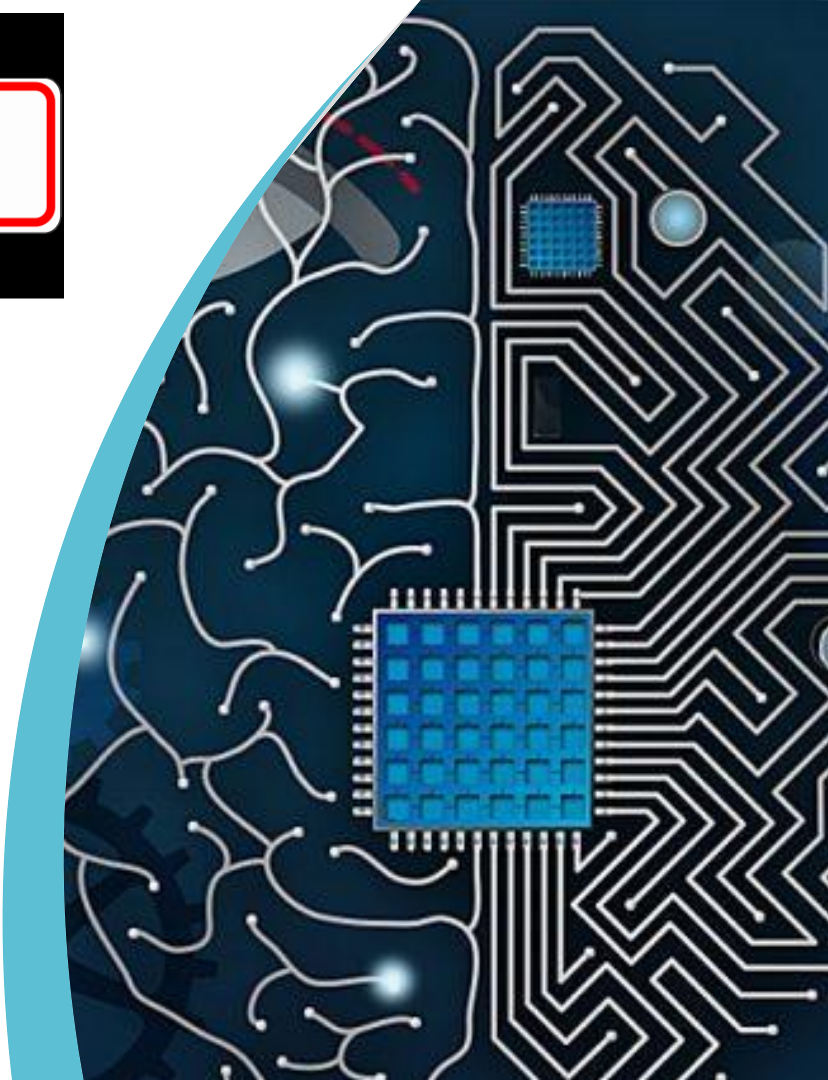
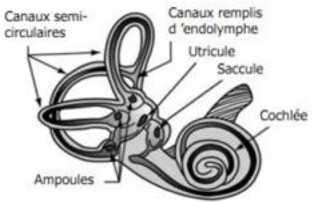
Motivation for Lie Group Machine Learning

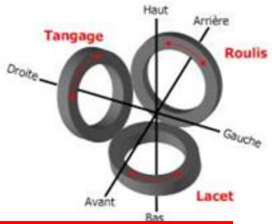


Motivation for Lie Group Machine Learning : Data as Lie Groups

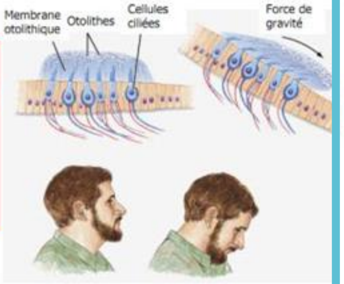
https://github.com/HTLife/VINet

Geolocalization and Navigation : Visio-Inertial SLAM: Visio-Vestibular Brain System

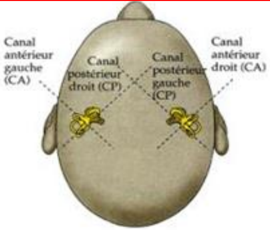


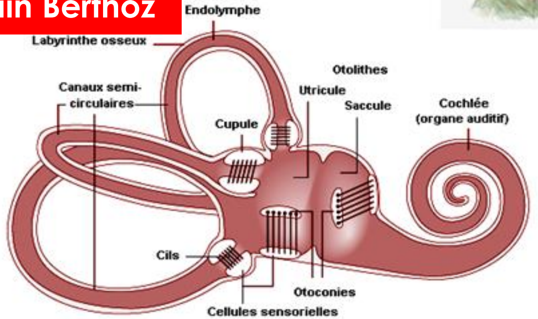


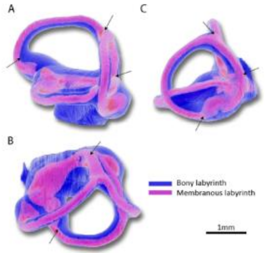
Coding of Homogeneous Galileo Group By Vestibular System and Otolithes



Works of Daniel Bennequin & Alain Berthoz

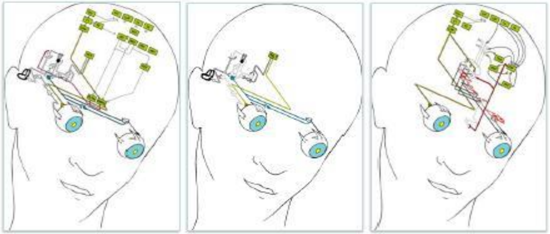






$$\begin{bmatrix} Z' \\ 1 \end{bmatrix} = \begin{bmatrix} \Omega & t \\ 0 & 1 \end{bmatrix} \begin{bmatrix} Z \\ 1 \end{bmatrix}$$

$$\Omega \in SO(3), t \in R^3$$



VINet: Visual-Inertial Odometry as a Sequence-to-Sequence Learning Problem

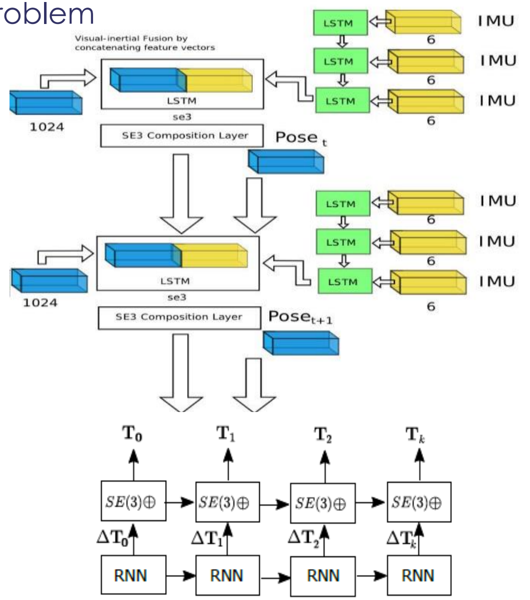


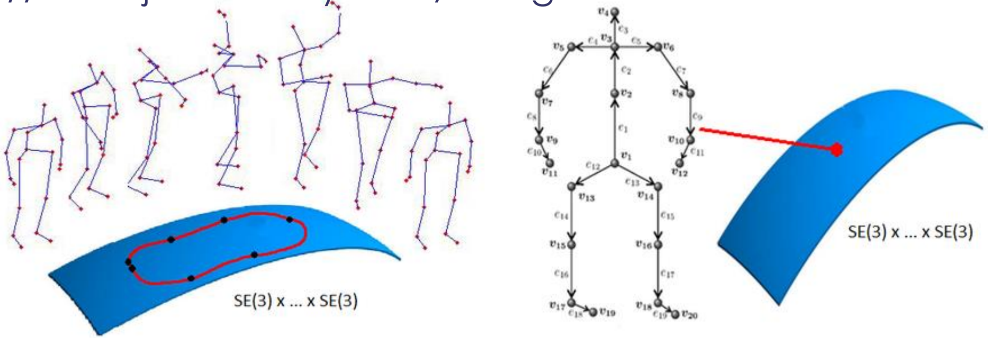
Illustration of the SE(3) composition layer - a parameter-free layer which concatenates transformations between frames on SE(3).

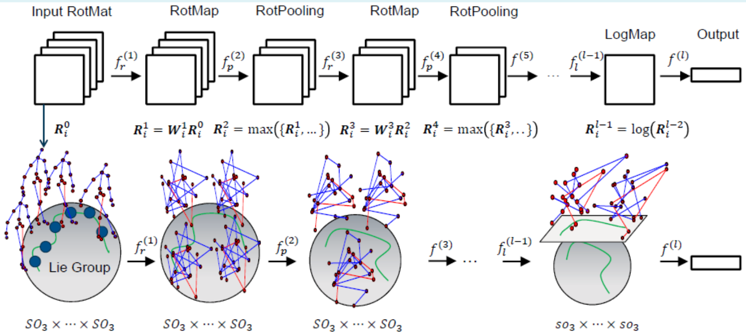
Motivation for Lie Group Machine Learning: Data as Lie Groups

Articulated 3D Movment/Posture Learning

http://ravitejav.weebly.com/rolling.html

Zhiwu Huang, Chengde Wan, Thomas Probst, Luc Van Gool, Deep Learning on Lie Groups for Skeleton-based Action Recognition, Computer Vision and Pattern Recognition, CVPR 2017





$$SO(3)=\left\{\Omega / \Omega^T\Omega=\Omega\Omega^T=I, \det{}^2\Omega=1\right\}$$

$$\begin{bmatrix} Z' \\ 1 \end{bmatrix}=\begin{bmatrix} \Omega & t \\ 0 & 1 \end{bmatrix}\begin{bmatrix} Z \\ 1 \end{bmatrix}, \quad \begin{cases} \Omega \in SO(3) \\ t \in R^3 \end{cases}$$

$$\begin{bmatrix} \Omega & t \\ 0 & 1 \end{bmatrix} \in SE(3)$$

$$\begin{bmatrix} \Omega_1 \\ \Omega_2 \\ \vdots \\ \Omega_p \end{bmatrix} \in SO(3)\times\ldots\times SO(3)$$

Vectors of $SE(3)$: $\begin{bmatrix} \begin{bmatrix} \Omega_1 & t_1 \\ 0 & 1 \end{bmatrix} \\ \begin{bmatrix} \Omega_2 & t_2 \\ 0 & 1 \end{bmatrix} \\ \vdots \\ \begin{bmatrix} \Omega_m & t_m \\ 0 & 1 \end{bmatrix} \end{bmatrix} \in SE(3)\times\ldots\times SE(3)$

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Motivation for Lie Group Machine Learning: Data as Lie Groups

Spatio-temporal attention on manifold space for 3D human action recognition

Chongyang Ding[1] · Kai Liu[1] · Fei Cheng[1] · Evgeny Belyaev[2]

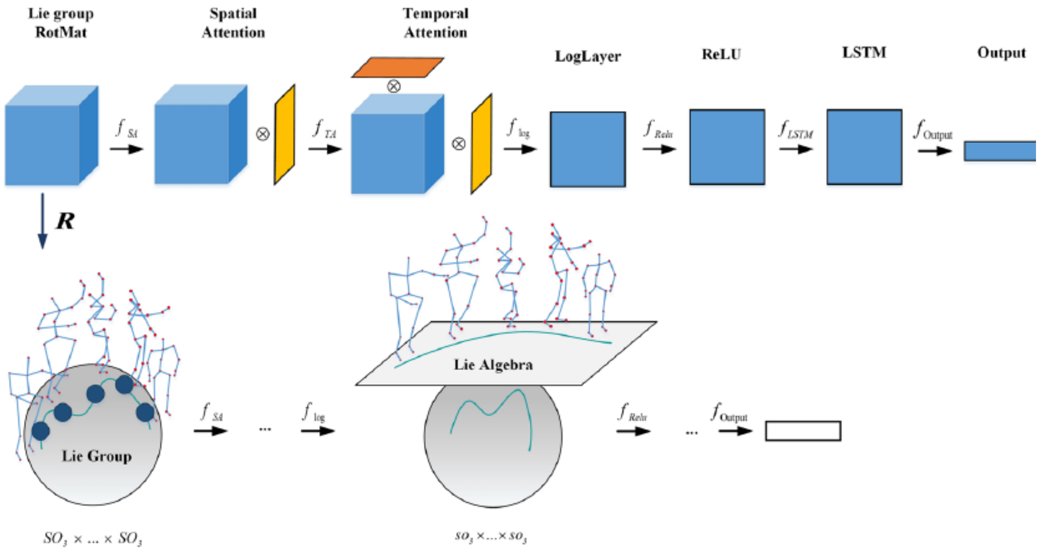

Table 3 Comparison of recognition results on the G3D-Gaming dataset

Method	G3D-Gaming
SE [39]	87.23%
SO [40]	87.95%
LieNet-3Blocks [21]	89.10%
RBM + HMM [33]	86.40%
DeepLG	86.06%
SA-DeepLG	87.88%
TA-DeepLG	88.18%
STA-DeepLG	90.30%

Path Signatures on Lie Groups

Path Signatures on Lie Groups

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Editor:

Abstract

Path signatures are powerful nonparametric tools for time series analysis, shown to form a universal and characteristic feature map for Euclidean valued time series data. We lift the theory of path signatures to the setting of Lie group valued time series, adapting these tools for time series with underlying geometric constraints. We prove that this generalized path signature is universal and characteristic. To demonstrate universality, we analyze the human action recognition problem in computer vision, using $SO(3)$ representations for the time series, providing comparable performance to other shallow learning approaches, while offering an easily interpretable feature set. We also provide a two-sample hypothesis test for Lie group-valued random walks to illustrate its characteristic property. Finally we provide algorithms and a Julia implementation of these methods.

Keywords: path signature, Lie groups, universal and characteristic kernels

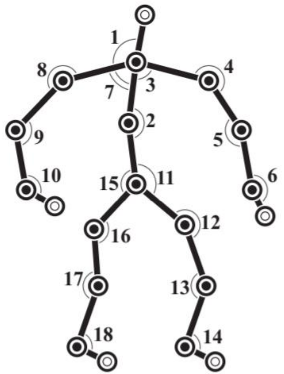
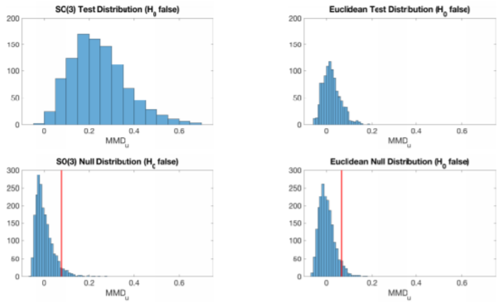


Figure 5: Numbering of the primary pairs of body parts



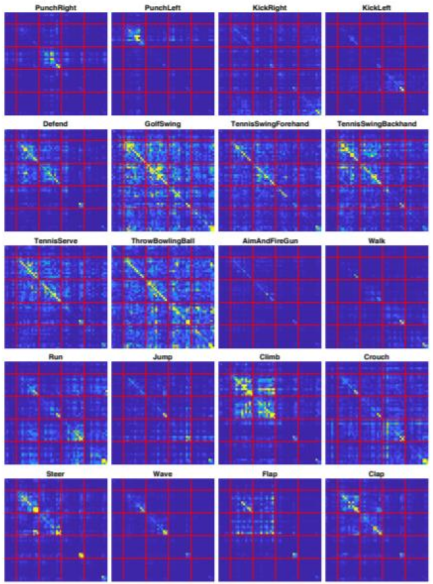


Figure 8: Averaged absolute S^2 matrices for all actions in G3D dataset.

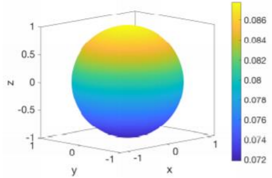


Figure 9: The von-Mises Fisher density on S^2 with mean direction $x = (0, 0, 1)$ and $\kappa = 0.1$.

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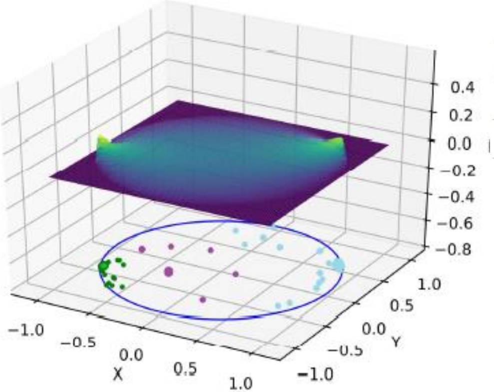
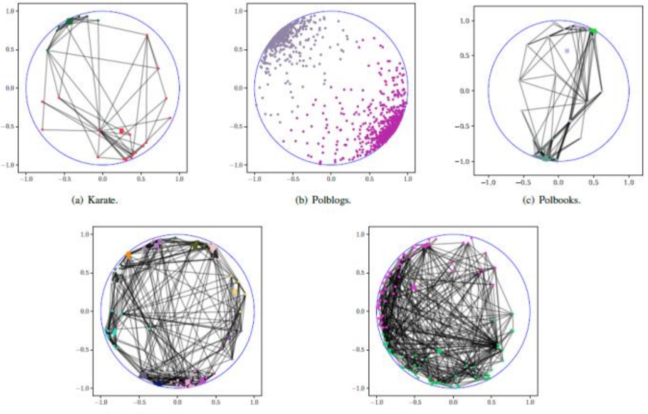
Motivation for Lie Group Machine Learning: Data in Homogenous Space where a Lie Group acts homogeneously

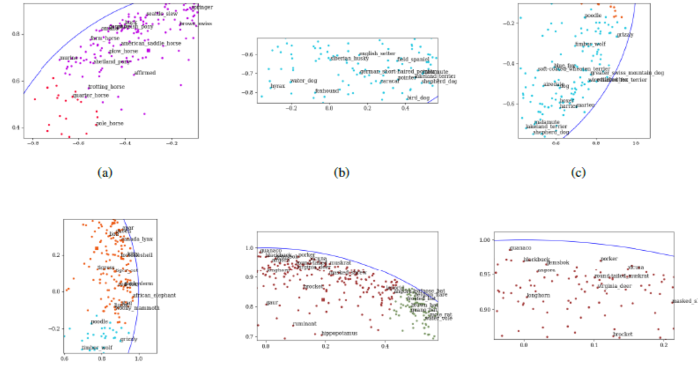
Poincaré/Hyperbolic Embedding in Poincaré Unit Disk for NLP (Natural Langage Processing)





(a) Karate. (b) Polblogs. (c) Polbooks.

(d) Football. (e) Adjnoun.



(a) (b) (c)

(d) (e) (f)

$$G = SU(1,1) = \left\{ \begin{pmatrix} \alpha & \beta \\ \beta^* & \alpha^* \end{pmatrix} / |\alpha|^2 - |\beta|^2 = 1,\ \alpha, \beta \in C \right\}$$

$$D = \left\{ z = x + iy \in C / |z| < 1 \right\}$$

$$g(z) = (\alpha z + \beta) / (\beta^* z + \alpha^*)$$

H. Hajri, H. Zaatiti, and G. Hébrail. Learning graph-structured data using poincaré embeddings and riemannian k-means algorithms. CoRR, abs/1907.01662, 2019

M. Nickel and D. Kiela. Poincaré embeddings for learning hierarchical representations. In Advances in Neural Information Processing Systems 30, pages 6338–6347. Curran Associates, Inc., 2017

Federico López, Beatrice Pozzetti, Steve Trettel, Anna Wienhard, Hermitian Symmetric Spaces for Graph Embeddings, arXiv:2105.05275v1 [cs.LG]

SU(1,1) Equivariance and G-CNN

Use generalized convolution operator on SU(1,1)

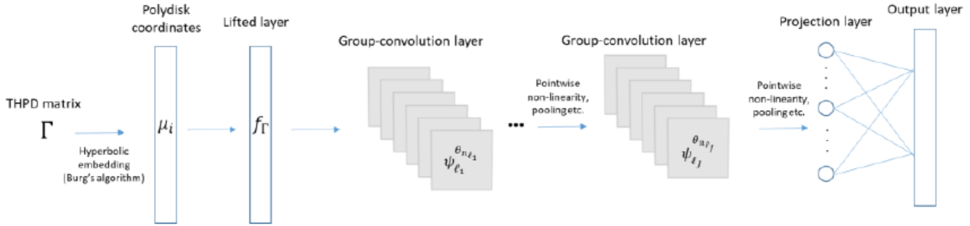
- $SU(1{,}1)$ is **not compact** → **localize the integral**
- $SU(1{,}1)$ is **not Euclidean** → **use an adequate kernel**

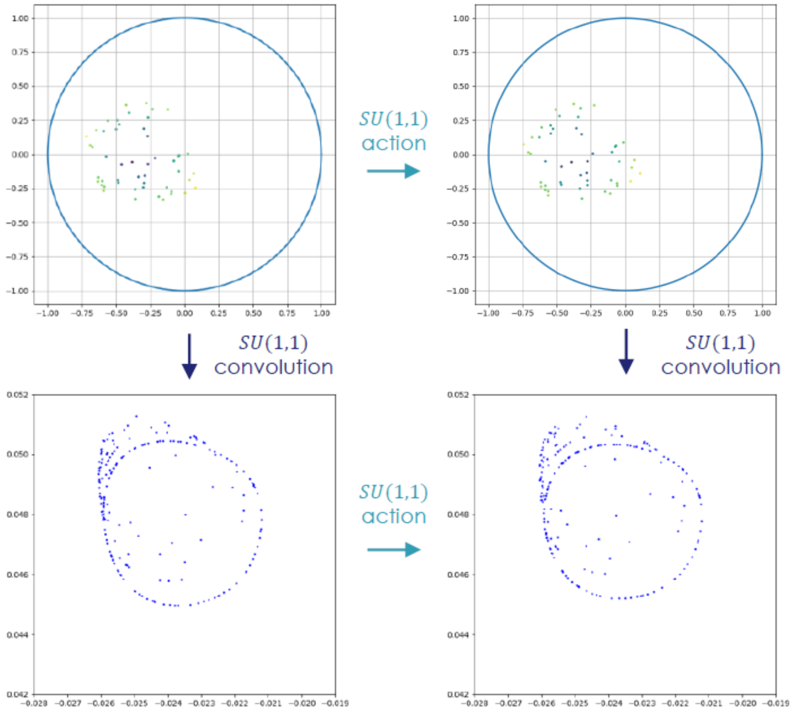
$$\boxed{\forall g \in SU(1{,}1), \psi^{\theta}(g) = (f \star_G K)(g) = \int_{h \in B_{\mathbb{D}}(g,M)} k_{\theta}(h^{-1}g) f(h) d\mu(h)}$$

$k_{\theta}(g) = \tilde{k}_{\theta}(\log_{\mathbb{D}}(g \circ 0_{\mathbb{D}}))$ $\quad \tilde{k}_{\theta}: \mathbb{R}^2 \to \mathbb{C}$ $\quad \log_{\mathbb{D}}: \mathbb{D} \to \mathbb{R}^2$ (Riemannian Logarithm)

$$B_{\mathbb{D}\,(g,M)} = \{h \in SU(1{,}1) \mid \rho_{\mathbb{D}}(g \circ 0_{\mathbb{D}}, h \circ 0_{\mathbb{D}}) \leq M\}$$

Use SU(1,1) convolution layers within a NN architecture





Equivariance of SU(1,1) convolution with a Gaussian Kernel

P.Y. Lagrave, F. Barbaresco and Y. Cabanes, SU(1, 1) Equivariant Neural Networks and Application to Robust Toeplitz Hermitian Positive Definite Matrix Classification, GSI'21 Paris, July 2021

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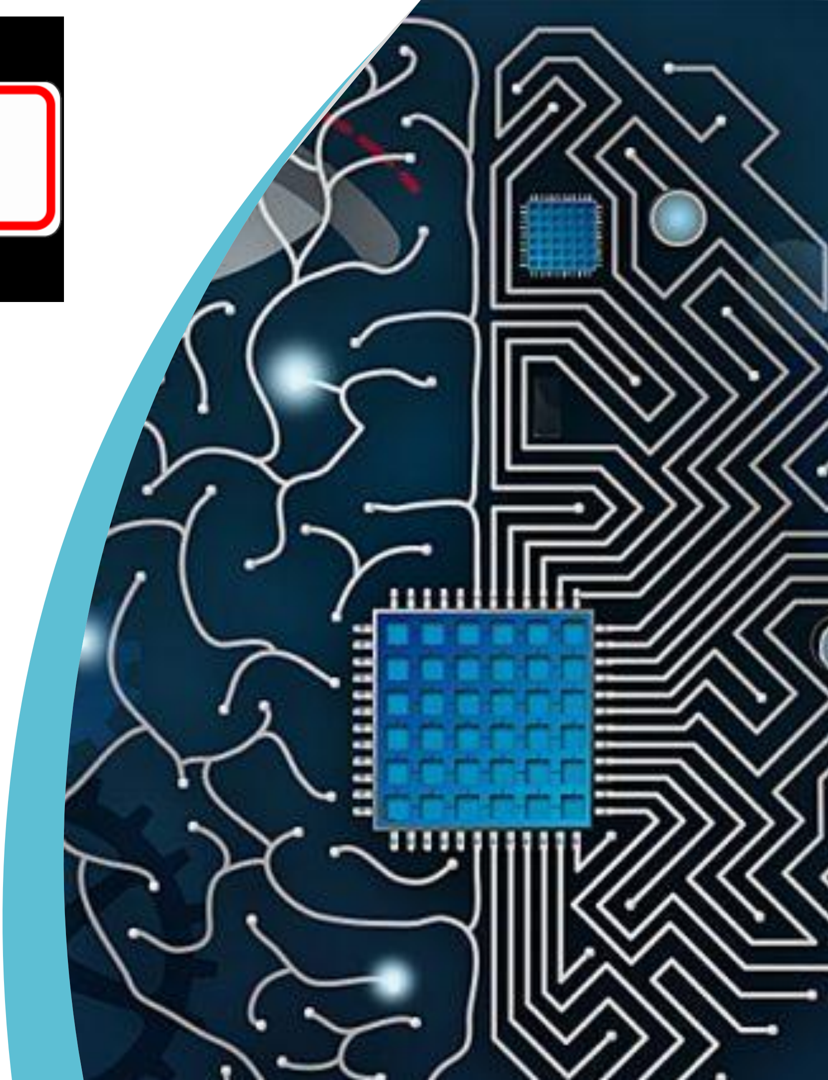
CONCLUSION



Que faut-il retenir après avoir tout oublié

Devant une bonne choucroute au jambon, ils oublièrent le pudding de graisse de phoque farci aux myrtilles ! — (Jean-Baptiste Charcot, Dans la mer du Groenland, 1928)

- **We present a model from Geometric Mechanics, developed by Jean-Marie Souriau as part of Mechanical Statistics, allowing to define an invariant Fisher-type metric and covariant statistical densities under Lie group action.**

- **The Souriau model makes it possible to define "Lie Group Statistics":**
  - **a Gibbs density of Maximum Entropy** on the Lie group coadjoint orbits (in the dual space of their Lie algebra)
  - with **coadjoint orbits** considered as a homogeneous symplectic manifold.

- **These densities are parameterized via the Souriau "Moment Map" :**
  - map from the symplectic manifold to the dual space of Lie algebra
  - tool geometrizing Noether's theorem
  - on which the group acts via the coadjoint operator

- **In this new model, Entropy is defined as an invariant Casimir function in coadjoint representation (this fact gives a natural geometrical definition of Entropy via the structural coefficients) and could be computed in abstract spaces.**

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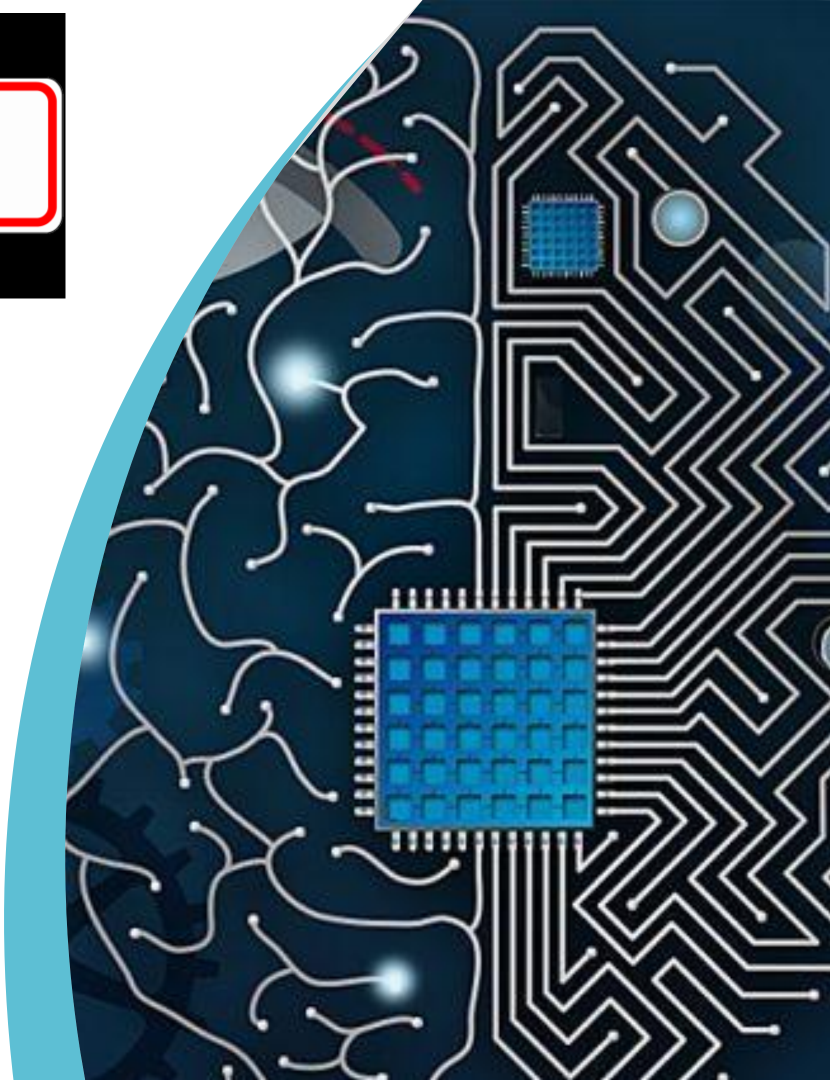
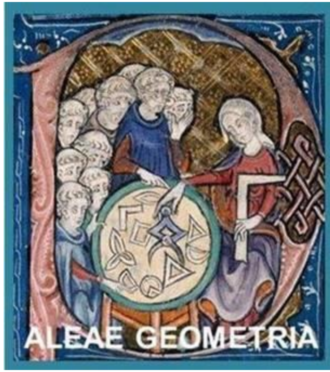
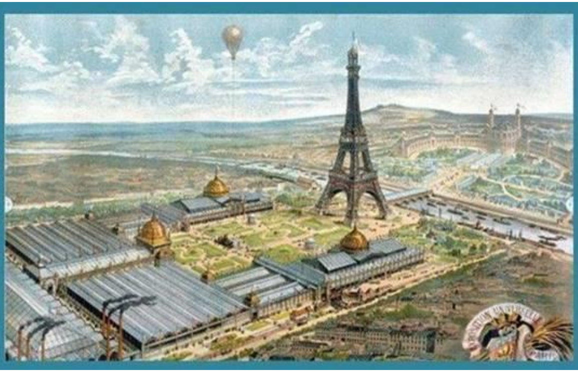
Conferences Cycle on Geometric Science of Information



5th GSI'21 « Geometric Science of Information » Conference





GSI'21

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www.gsi2021.org

Special sessions on
« **Geometric Deep Learning** » chaired by **Erik J. Bekkers** & **Gabriel Peyré**

« **Lie Group Machine Learning** » chaired by **Frédéric Barbaresco & Gery de Saxcé**

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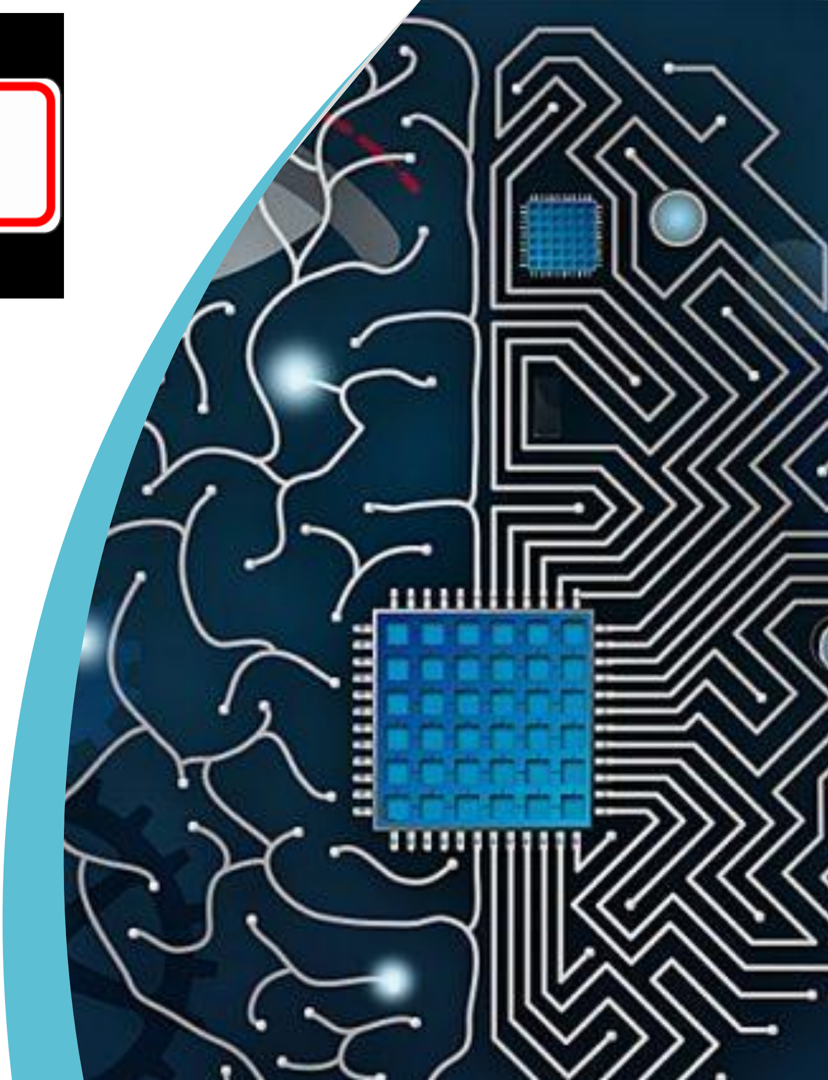
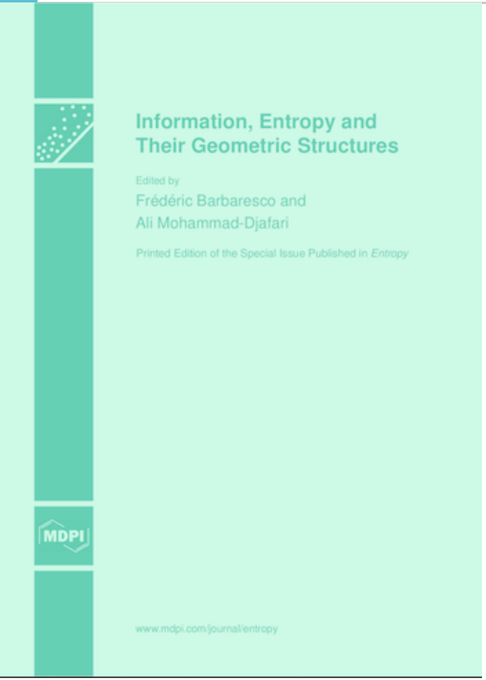
Main references



Books



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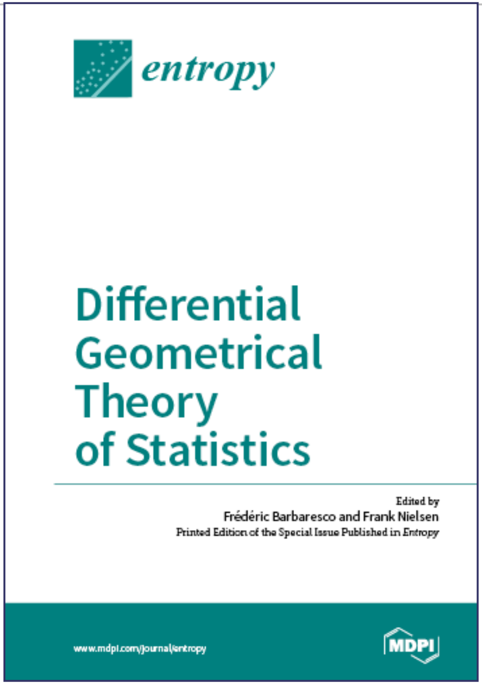

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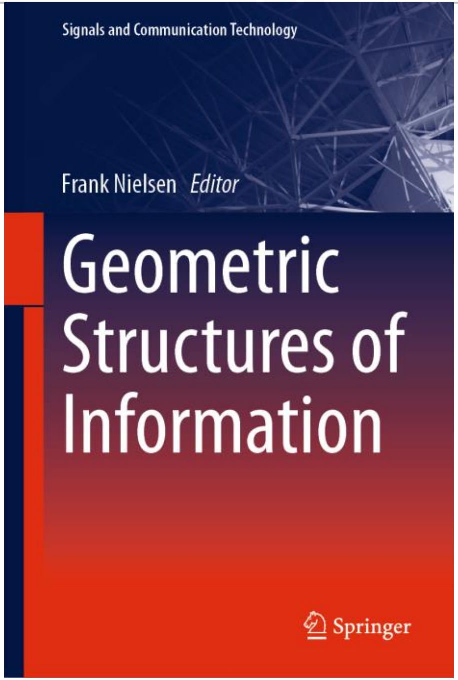

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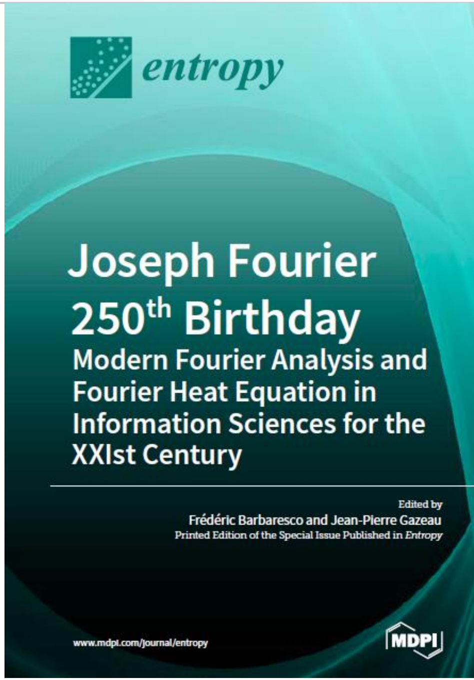

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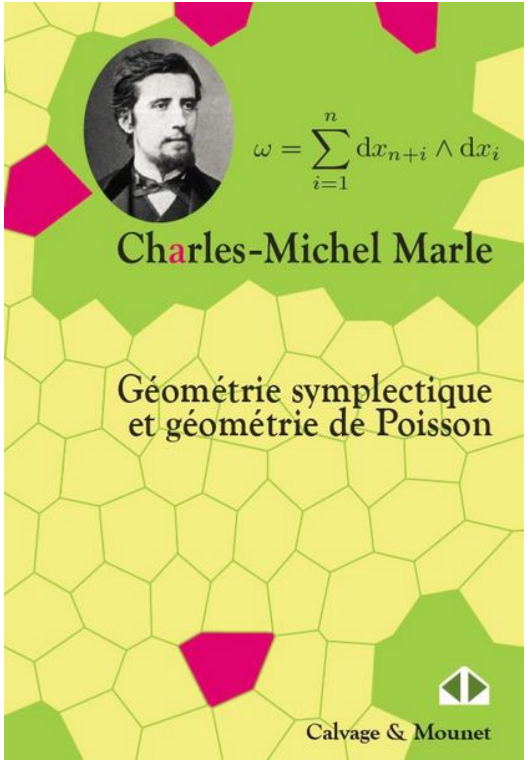

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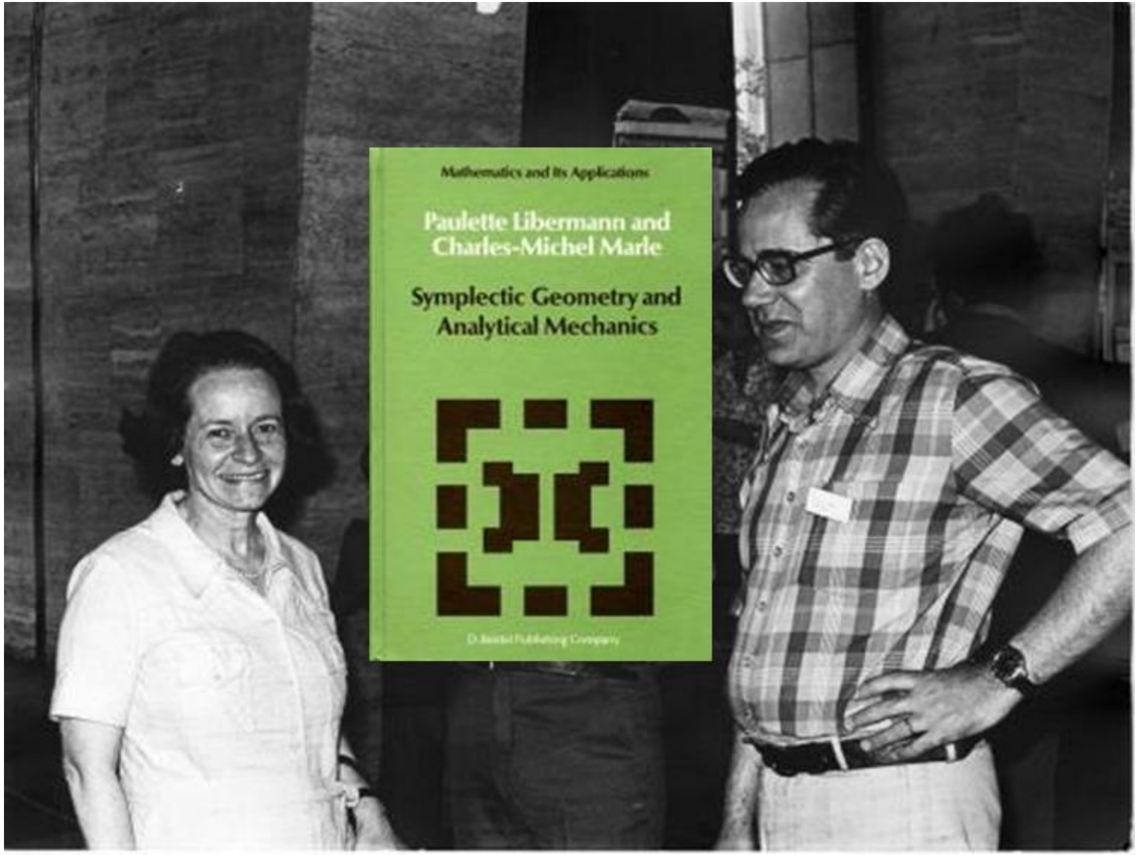

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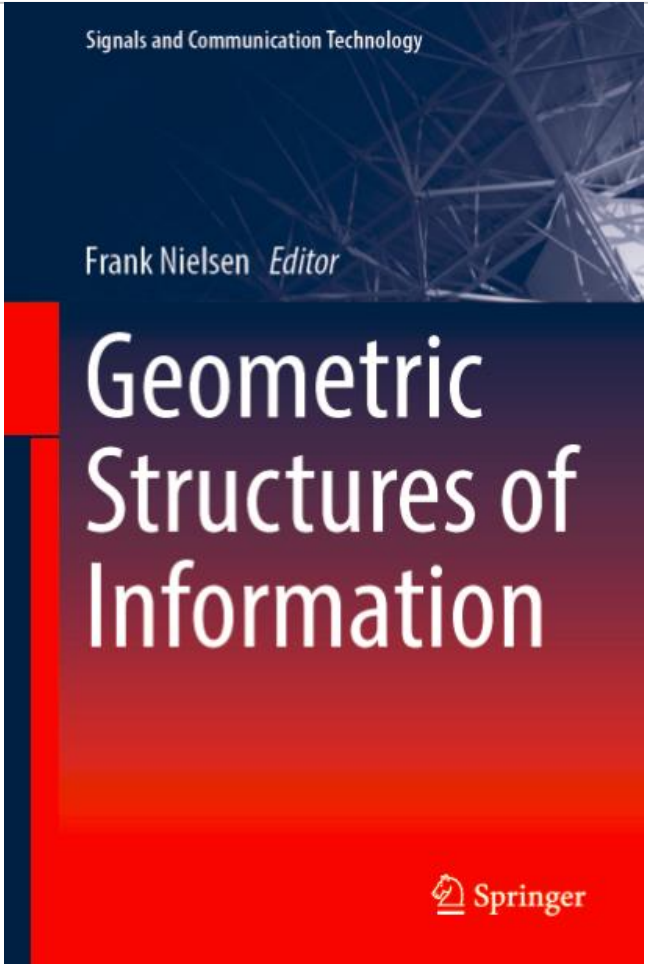
Geometric Structures of Information, SPRINGER



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Lie Group Machine Learning and Lie Group Structure Preserving Integrators

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Deadline for manuscript submissions:
6 January 2020

Message from the Guest Editors

Machine/deep learning explores use-case extensions for more abstract spaces as graphs and differential manifolds. Recent fruitful exchanges between geometric science of information and Lie group theory have opened new perspectives to extend machine learning on Lie groups to develop new schemes for processing structured data.

Structure-preserving integrators that preserve the Lie group structure have been studied from many points of view and with several extensions to a wide range of situations. Structure-preserving integrators are numerical algorithms that are specifically designed to preserve the geometric properties of the flow of the differential equation such as invariants, (multi)symplecticity, volume preservation, as well as the configuration manifold. They also naturally find applications in the extension of machine learning and deep learning algorithms to Lie group data.

This Special Issue will collect long versions of papers from contributions presented during the GSI'19 conference, but it will be not limited to these authors and is open to international communities involved in research on Lie group machine learning and Lie group structure-preserving integrators.

mdpi.com/si/30856

Special Issue

Lie Group Machine Learning and Lie Group Structure Preserving Integrators

Keywords

- Lie groups machine learning
- orbits method
- symplectic geometry
- geometric integrator
- symplectic integrator
- Hamilton's variational principle

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