Scketching: The Jonhson-Linderstrauss Theorem.

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Eigenvalues of random covariance matrices

Let X denote a matrix in $\mathbb{R}^{m \times n}$ whose entries $X_{i,j}$ are *i.i.d* with mean 0 and variance $\sigma^2 < \infty$. Consider its covariance matrix

$$Y = \frac{1}{n} X X^T$$

and its eigenvalues $\lambda_1, \ldots \lambda_m$.

What is the probability that $\lambda \in \mathbb{R}_+$ is an eigenvalue of Y?

If you know Y, this is given by the discrete probability law

$$\mu_m(\lambda) = \frac{1}{m} \sum_{i=1}^{m} \delta_{\lambda_i}(\lambda) \ .$$

Marchenko-Pastur Law gives the assymptotic value for this law.
When m and n go to $+\infty$.

Theorem - Marčenko & Pastur (1967)

Assume $m, n \to \infty$ such that $m/n \to \alpha$. Then $\mu_m \to \mu$ where

$$\mu(A) = \begin{cases} (1 - \frac{1}{\alpha})\delta_{0 \in A} + \nu(A) \ , & \text{if} \quad \alpha > 1 \\ \nu(A) \ , & \text{if} \quad 0 \leq \alpha \leq 1 \end{cases}$$

with

$$d\nu(x) = \frac{1}{2\pi\sigma^2} \frac{\sqrt{(\lambda_+ - x)(x - \lambda_-)}}{\lambda x} \delta\{x \in [\lambda_-, \lambda_+]\}$$

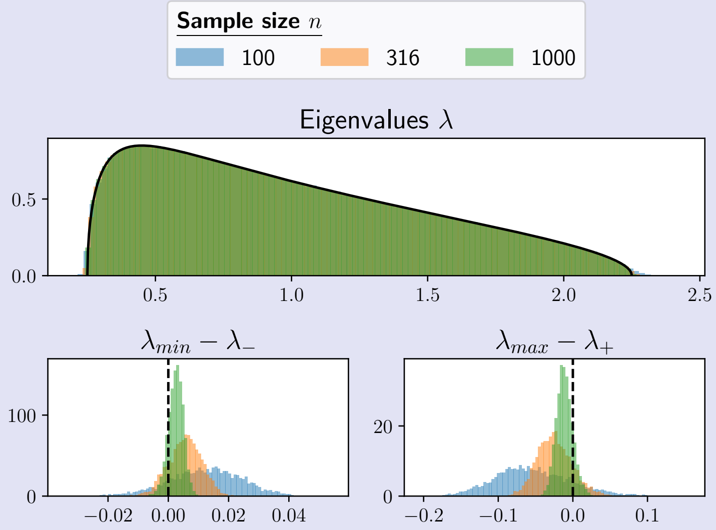
and $\lambda_\pm = \sigma^2(1 \pm \sqrt{\lambda})^2$.

The Marchenko Pastur Law - debuncked

- **Low rank case:** If $\alpha > 1$, Y is low rank and the probability of an eigenvalues to be 0 is $1 - \frac{1}{\alpha}$.
- **Semi circle law:** The support of $d\nu$ is on $[\lambda_-, \lambda_+]$. This means that assymptotically,
- **Bounds on the eigenvalues:** The support of $d\nu$ is on $[\lambda_-, \lambda_+]$. This means that assymptotically,

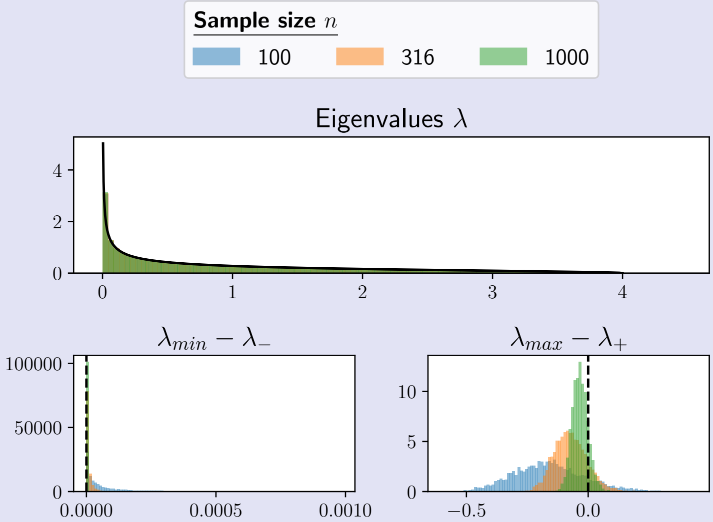
This law is quite accrurate even for low dimensions.

The Marchenko Pastur Law - empirically



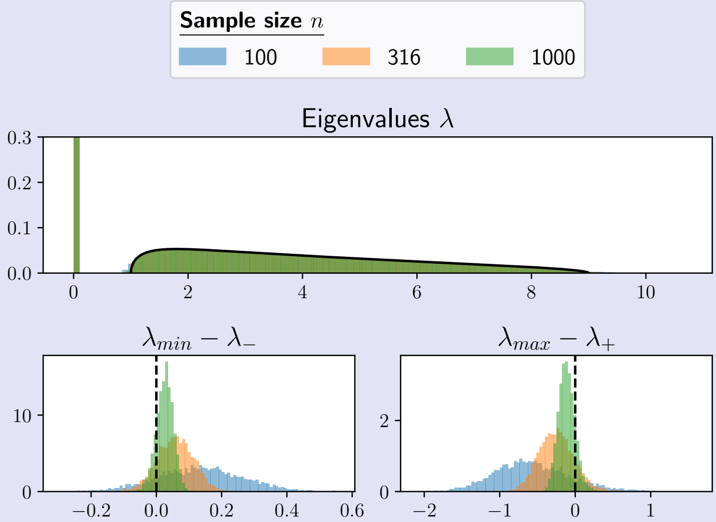
$$\sigma^2 = 1, \quad \alpha = 0.25, \quad \lambda_{min} = \lambda_- = 0.25, \quad \lambda_{max} = \lambda_+ = 2.25$$

The Marchenko Pastur Law - empirically



$$\sigma^2 = 1, \quad \alpha = 1, \quad \lambda_{min} = \lambda_- = 0, \quad \lambda_{max} = \lambda_+ = 4$$

The Marchenko Pastur Law - empirically



$$\sigma^2 = 1, \quad \alpha = 4, \quad \lambda_{min} = \lambda_- = 1, \quad \lambda_{max} = \lambda_+ = 9$$