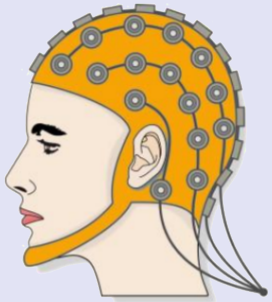
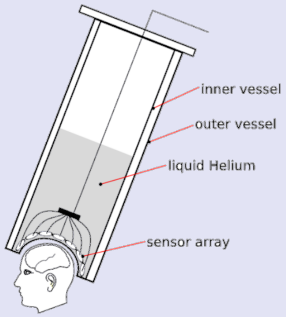
Using the Dictionary Structure for efficient Convolutional Dictionary Learning

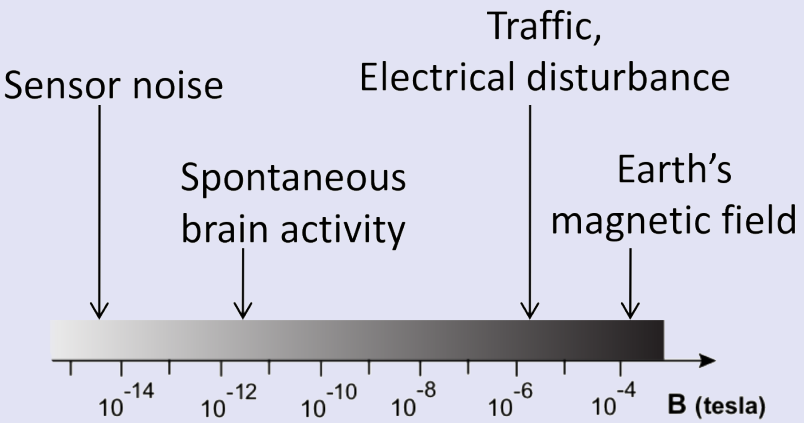
Thomas Moreau INRIA Saclay – parietal

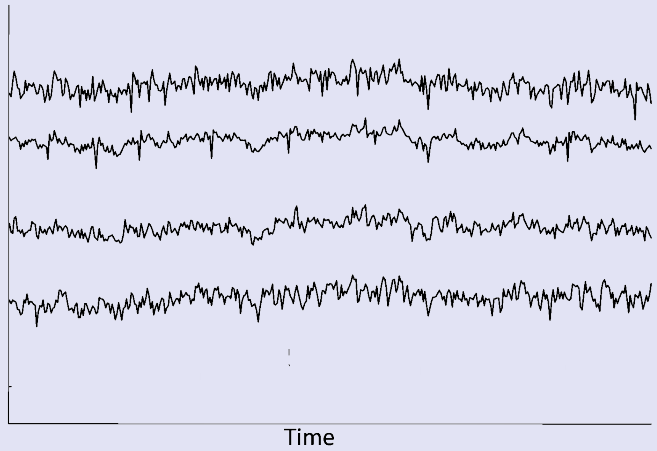
Studying brain activity through electromagnetic signals

- Brain (electrical) activity produces an electromagnetic field.
- This can be measured with EEG or MEG.









Goal: Study Oscillation in Neural Data

Oscillations are believed to play an important role in cognitive functions.

Many studies rely on Fourier or wavelet analyses:

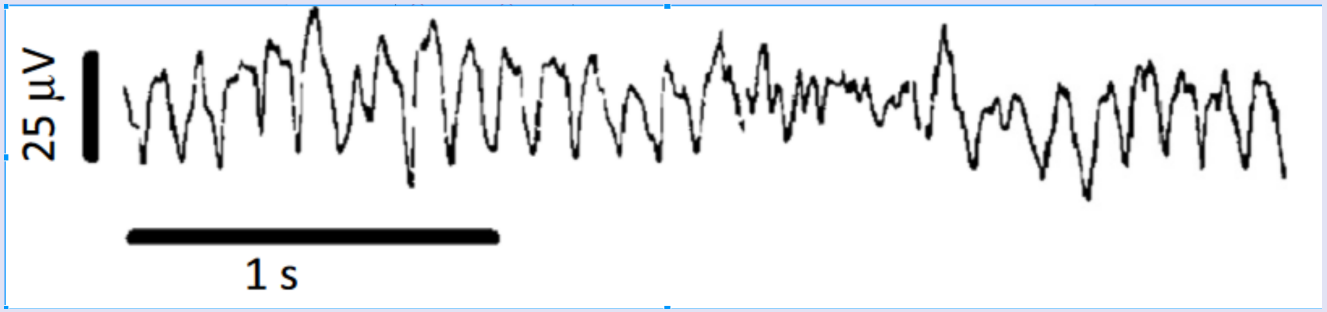
- Easy interpretation,
- Standard analysis *e.g.* canonical bands alpha, beta or theta.

[Buzsaki, 2006]

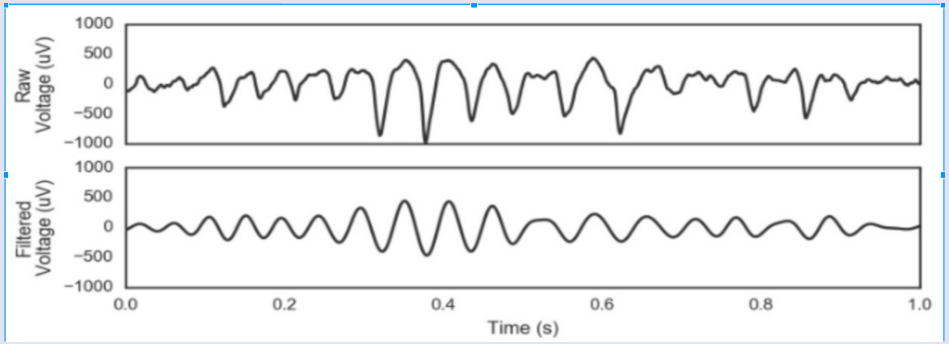
Goal: Study Oscillation in Neural Data

However, some brain rhythms are not sinusoidal, *e.g.* mu-waves.

[Hari, 2006]



and filtering degrades waveforms

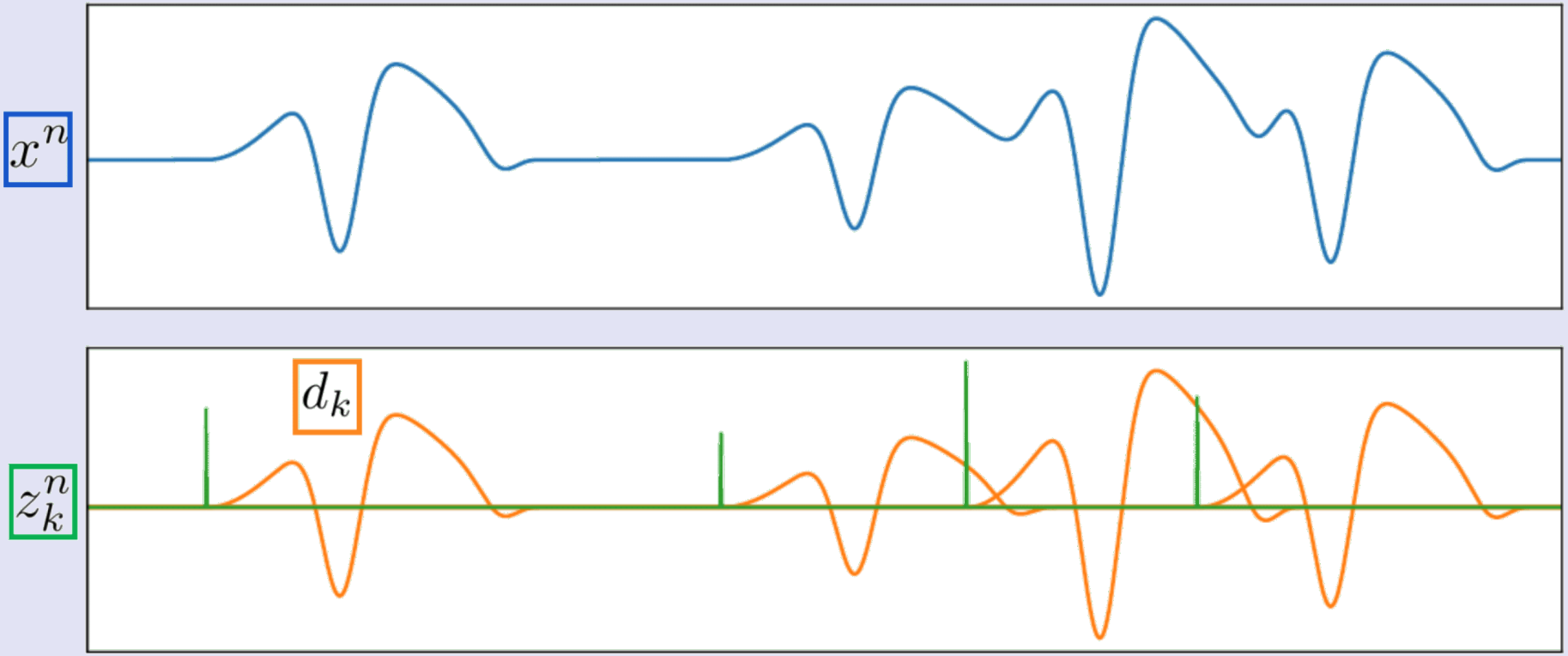


$\Rightarrow$ Can we do better with data-driven approach?

Convolutional representations

Extracting shift invariant patterns

Key idea: decouple the localization of the patterns and their shape



Extracting shift invariant patterns

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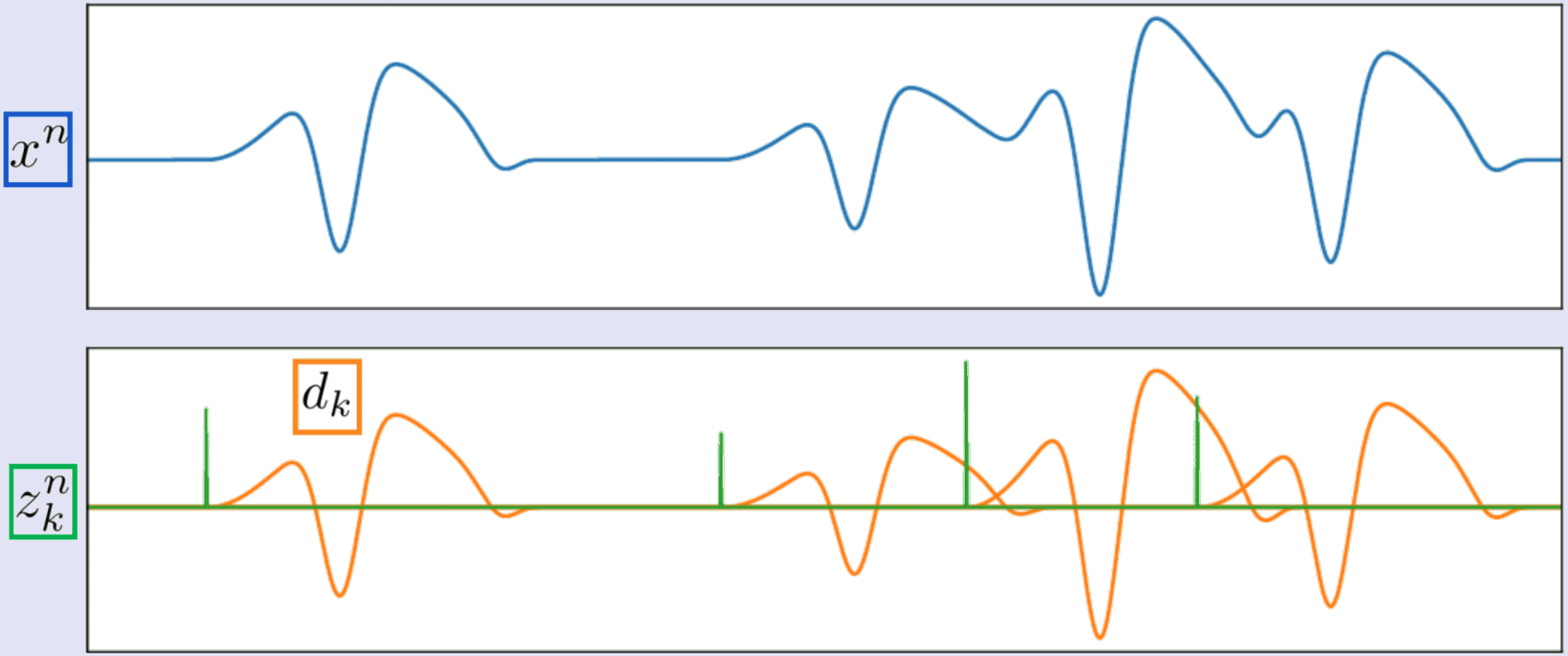
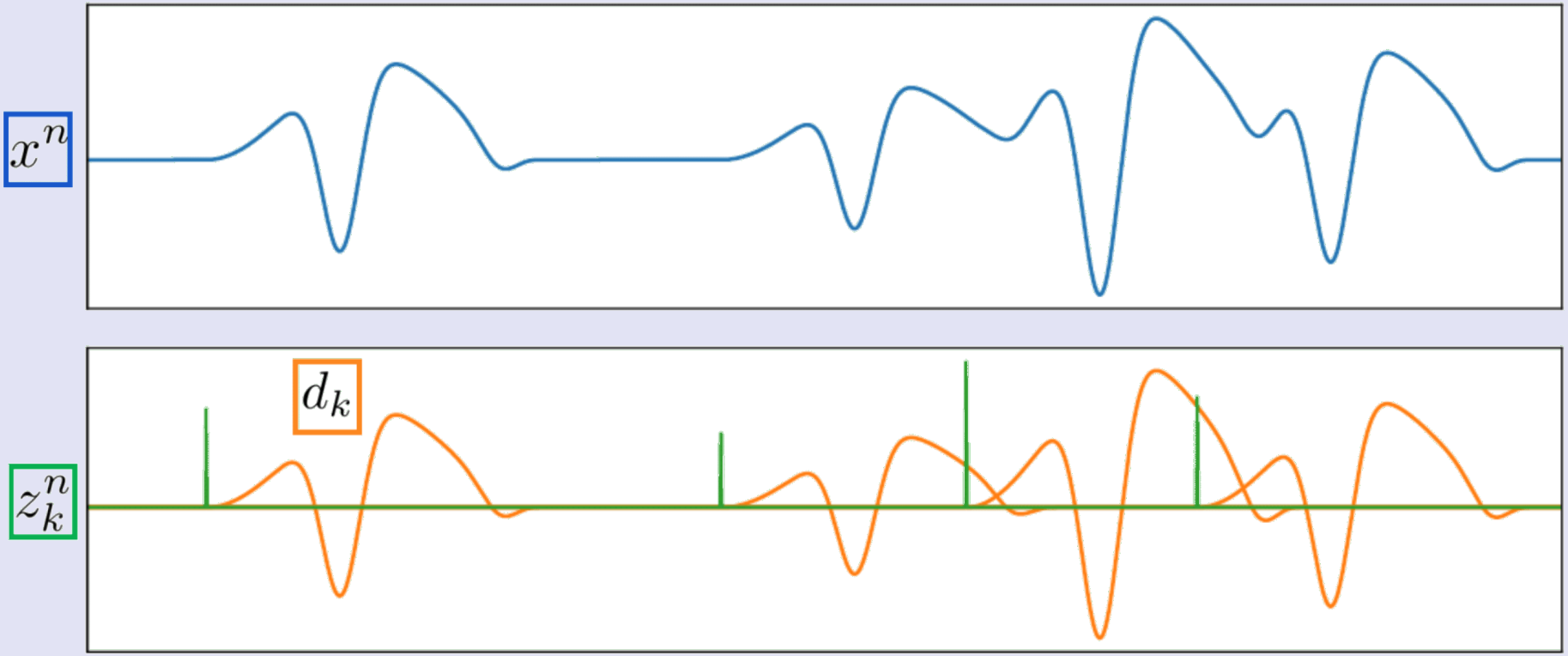


Convolutional Representation:

$$x^n[t] = \sum_{k=1}^{K} (z_k^n * d_k)[t] + \varepsilon[t]$$

Extracting shift invariant patterns

Key idea: decouple the localization of the patterns and their shape



Convolutional Dictionary Learning:

$$\min_{d,z} \sum_{n=1}^{N} \frac{1}{2} \left\| x^n - \sum_{k=1}^{K} z_k^n * d_k \right\|_2^2 + \lambda \sum_{k=1}^{K} \|z_k^n\|_1,$$

$$\text{s.t.} \quad \|d_k\|_2^2 \leq 1$$

Notation

Sparse Convolutional model:

$$X[t] = \sum_{k=1}^{K} (\boldsymbol{D}_k * Z_k)[t] + \mathcal{E}[t]$$

with Z sparse. Few of its coefficients are non-zero.

- X is a signal of length T
- $\mathcal{E}$ is a noise signal of length T
- $\boldsymbol{D}$ is a set of K patterns of length W
- Z is a signal of length $L = T - W + 1$ in $\mathbb{R}^K$

Convolutional Dictionary Learning

Dictionary learning optimization problem for $\{X^{[n]}\}_{n=1}^{N}$

$$(Z^*, \boldsymbol{D}^*) = \underset{Z, \boldsymbol{D}}{\operatorname{argmin}} \frac{1}{N} \sum_{n=1}^{N} \underbrace{\|X^{[n]} - \sum_{k=1}^{K} \boldsymbol{D}_k * Z_k^{[n]}\|_2^2}_{E(Z) \text{ data fit}} + \underbrace{\lambda \|Z^{[n]}\|_1 + \mathbf{1}_\Omega(D)}_{\text{penalizations}}$$

with a convex constraint set Ω and a regularization parameter $\lambda > 0$.
Classic constraint sets Ω are the ℓ_2-ball or sphere.

This problem is bi-convex and an approximate solution is obtained through **alternate minimization**. [Engan et al., 1999; Grosse et al., 2007]

D-step: Dictionary updates

$\rightarrow$ Z fixed, update $\boldsymbol{D}$

$$\boldsymbol{D}^* = \underset{\boldsymbol{D}}{\operatorname{argmin}} \frac{1}{N} \sum_{n=1}^{N} \|X^{[n]} - \sum_{k=1}^{K} \boldsymbol{D}_k * Z_k^{[n]}\|_2^2 + \mathbf{1}_\Omega(D)$$

Related Algorithms:

- Proximal Gradient Descent (PDG) [Rockafellar, 1976]
- Accelerated PGD [Nesterov, 1983]
- Block Coordinate Descent [Mairal et al., 2010]
- Alternated Direction Method of Multiplier (ADMM) [Gabay and Mercier, 1976]

Z-step: Sparse coding

$\rightarrow \boldsymbol{D}$ fixed, update Z

$$Z^{[n],*} = \underset{Z^{[n]}}{\operatorname{argmin}} \|X^{[n]} - \sum_{k=1}^{K} \boldsymbol{D}_k * Z_k^{[n]}\|_2^2 + \lambda \|Z^{[n]}\|_1$$

$$\Rightarrow \text{Independent for each } n \in [\![1, N]\!]$$

Related Algorithms:

- Iterative Soft-Thresholding Algorithm (ISTA)
  [Daubechies et al., 2004]
- Fast ISTA
  [Beck and Teboulle, 2009]
- Alternated Direction Method of Multiplier (ADMM)
  [Gabay and Mercier, 1976]
- Coordinate Descent (CD)
  [Friedman et al., 2007]

We have to solve N independent problems with a common structure $\boldsymbol{D}$,

$$Z^{[n],*} = \underset{Z^{[n]}}{\operatorname{argmin}} \|X^{[n]} - \sum_{k=1}^{K} \boldsymbol{D}_k * Z_k^{[n]}\|_2^2 + \lambda \|Z^{[n]}\|_1$$

Can we use this structure to accelerate the resolution?

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Can we use this structure to accelerate the resolution?

Yes, with the Learned ISTA. [Gregor and Le Cun, 2010]

Why does it work?

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Why does it work? Analysis in the context of sparse coding.

(no convolution)

Part II: Coordinate Descent for CSC

Coordinate descent only performs local updates at each iteration.

$\Rightarrow$ More efficient for convolutional model with long signals.

Can we improve it with the structure of our problem?

Improving CD efficiency for the convolutional structure:

- Locally Greedy Coordinate Descent.
- Asynchronous and distributed algorithms: DICOD and DiCoDiLe.

For electrophisiological signals, the CDL reads,

$$\min_{Z,\boldsymbol{D}} \frac{1}{N} \sum_{n=1}^{N} \|X^{[n]} - \sum_{k=1}^{K} \boldsymbol{D}_k * Z_k^{[n]}\|_2^2 + \lambda \|Z^{[n]}\|_1 + \mathbf{1}_\Omega(D)$$

However, this model does not account for the physics of the problem.

Can we constrain the structure of the dictionary to learn more interpretable atoms?

$\Rightarrow$ Use rank-1 constraints to accomodate Maxwell's equations.

Adaptive Sparse Coding

References

- Moreau, T. and Bruna, J. (2017). Understanding Neural Sparse Coding with Matrix Factorization. In *International Conference on Learning Representation (ICLR)*

Vectorized model

- x is a vector in $\mathbb{R}^T$
- ϵ is a noise vector in $\mathbb{R}^T$
- D is a matrix in $\mathbb{R}^{T \times LK}$
- z is a coding vector in $\mathbb{R}^{LK}$

Sparse Linear model:

$$x = Dz + \epsilon$$

with z sparse.

Few of its coefficients are non-zero.

$$D = \begin{pmatrix} \bar{D} & \bar{0} & & 0 & \\ \bar{0} & \bar{D} & \bar{0} & & \\ & \bar{0} & \bar{D} & \bar{0} & \\ & & & \ddots & \\ & 0 & & \bar{0} & \bar{D} & \bar{0} \\ & & & & \bar{0} & \bar{D} & \bar{0} \\ & & & & & \bar{0} & \bar{D} \end{pmatrix}.$$

Adaptive Optimization

We have to solve N problems with a common structure $\boldsymbol{D}$.

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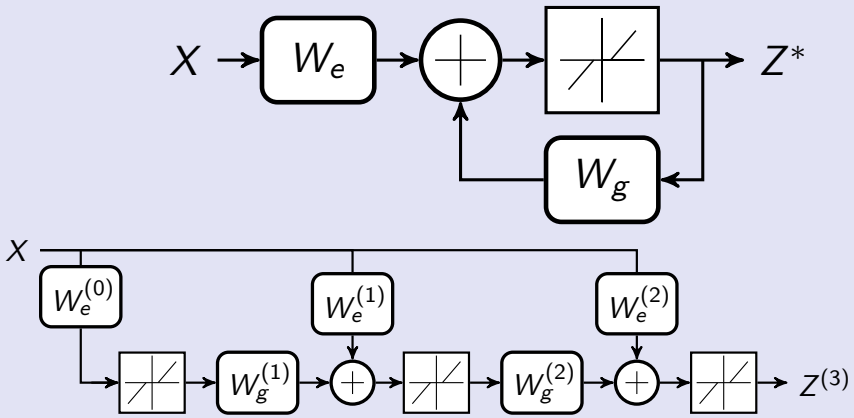

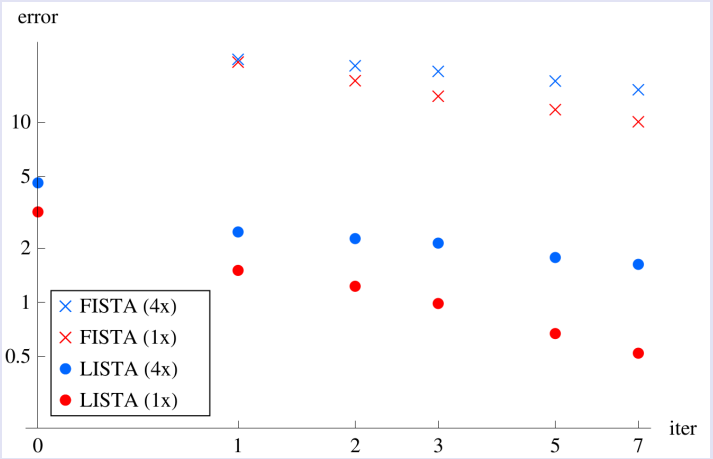

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Open problem: Why does it work?

- Can we leverage the structure of $\boldsymbol{D}$?
- Can we get a non-asymptotic acceleration of ISTA?
- How to explain LISTA performance?

[Giryes et al., 2018; Xin et al., 2016]

Notations

Consider the sparse coding problem with a dictionary D.

$$z^* = \operatorname*{argmin}_z F(z) = \underbrace{\|x - Dz\|_2^2}_{E(z)} + \lambda \|z\|_1$$

We denote $B = D^\mathsf{T} D$ is the Gram matrix of D.

We introduce a novel class of algorithms – FacNet – based on a sparse factorization of B.

Quadratic form: $Q_S(u, v) = \frac{1}{2}(u - v)^\mathsf{T} S(u - v) + \lambda \|u\|_1$.

Note that $F(z) = Q_B(z, D^\dagger x)$

For S is diagonal, $\operatorname{argmin}_u Q_S(u, v)$ can be efficiently minimized as the problem is separable on each coordinate.

Given an estimate $z^{(q)}$ of z^* at iteration q, we can write:

$$\begin{aligned} F(z) &= E(z) + \lambda \|z\|_1 \\ &= E(z^{(q)}) + \left\langle \nabla E(z^{(q)}), z - z^{(q)} \right\rangle + Q_B(\quad z, \quad z^{(q)}) \ , \end{aligned}$$

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ISTA: Replace B by diagonal matrix $S = \|B\|_2 \boldsymbol{I}_K$

$$\begin{aligned} F_q(z) &= E(z^{(q)}) + \left\langle \nabla E(z^{(q)}), z - z^{(q)} \right\rangle + Q_S\ (\ z,\ z^{(q)}) \ , \\ \min_z F_q(z) &\Leftrightarrow \min_z Q_S \left(\ z,\ z^{(q)} - S^{-1} \nabla E(z^{(q)}) \right) \end{aligned}$$

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ISTA: Replace B by diagonal matrix $S = \|B\|_2 \boldsymbol{I}_K$

FacNet: Replace B by $A^T S A$ (S diagonal, A unitary)

$$\widetilde{F}_q(z) = E(z^{(q)}) + \left\langle \nabla E(z^{(q)}), z - z^{(q)} \right\rangle + Q_{\boldsymbol{S}_q}(\boldsymbol{A}_q z, \boldsymbol{A}_q z^{(q)}) \ ,$$

$$\min_z \widetilde{F}_q(z) \Leftrightarrow \min_z Q_{\boldsymbol{S}_q}\left(\boldsymbol{A}_q z, \boldsymbol{A}_q z^{(q)} - \boldsymbol{S}_q^{-1}\boldsymbol{A}_q \nabla E(z^{(q)})\right)$$

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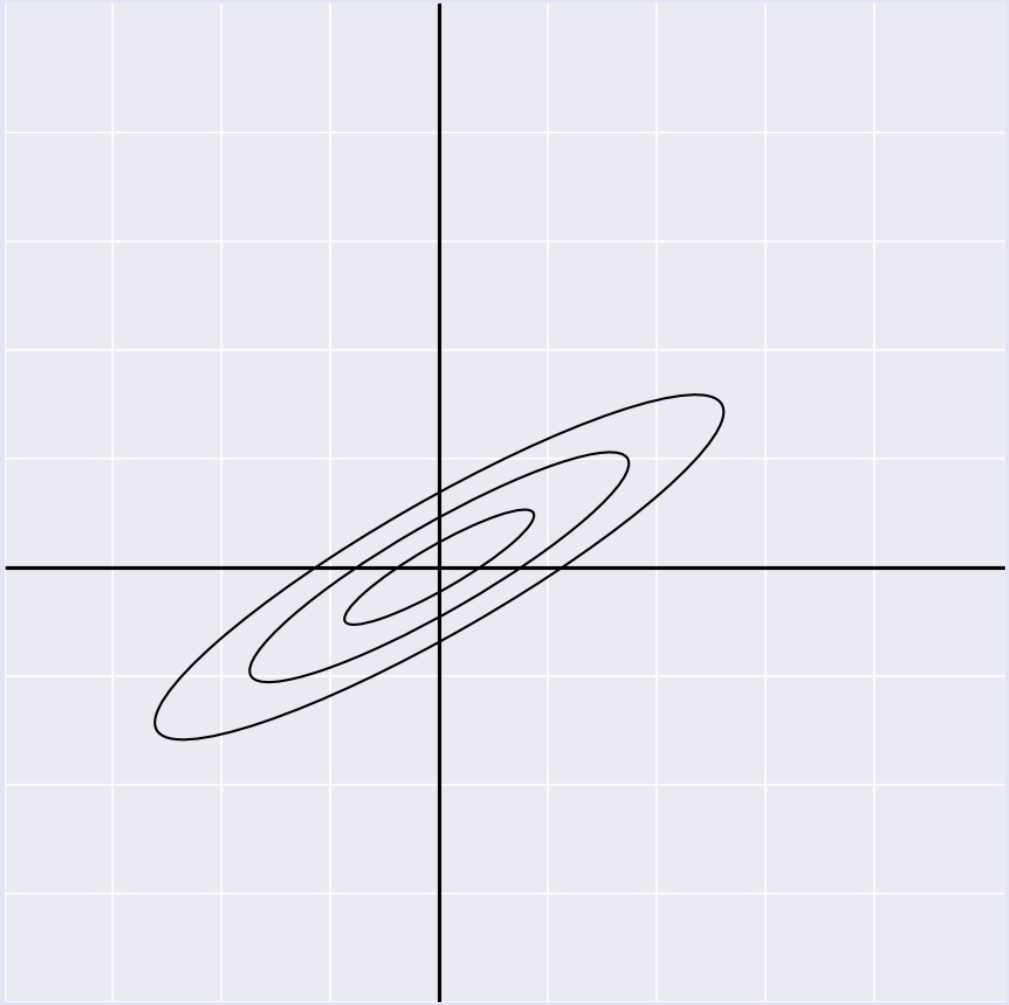
Can we choose A_q, S_q to accelerate the optimization compared to ISTA?

Toward and adaptive procedure



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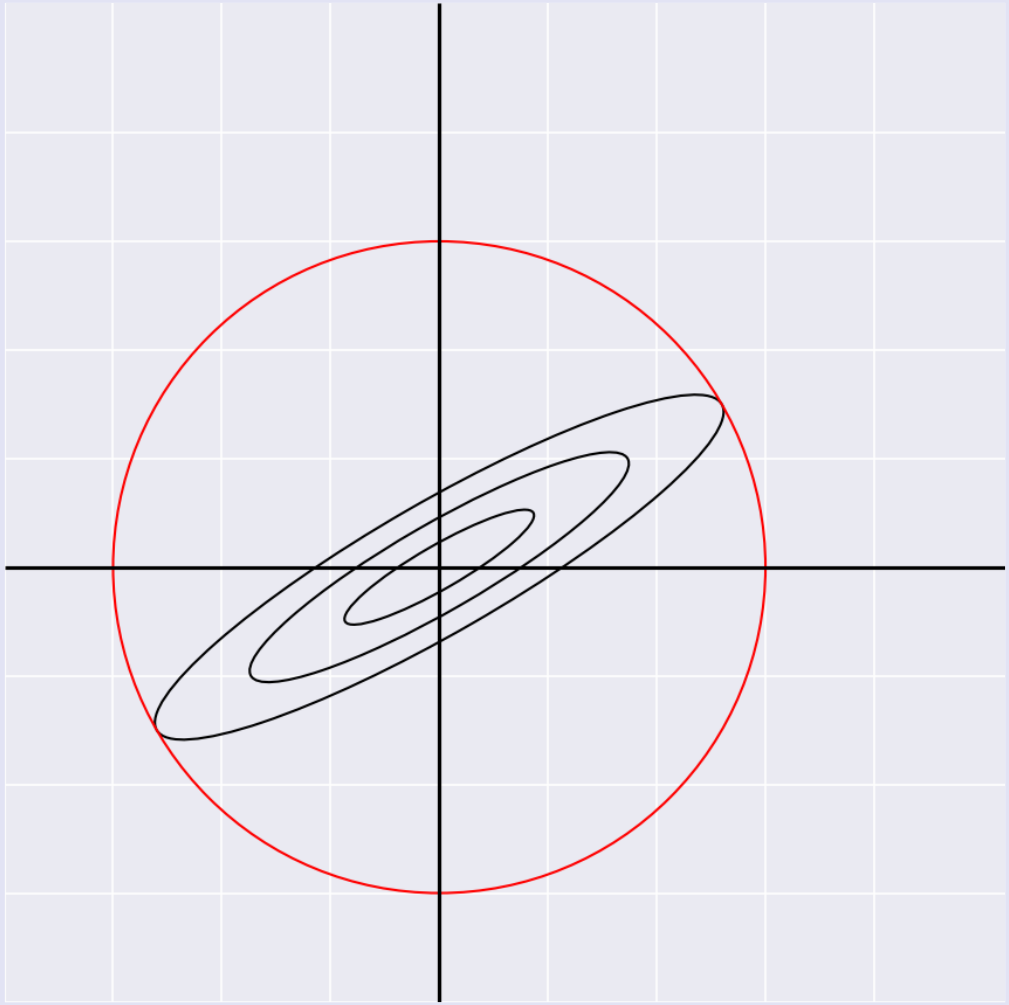

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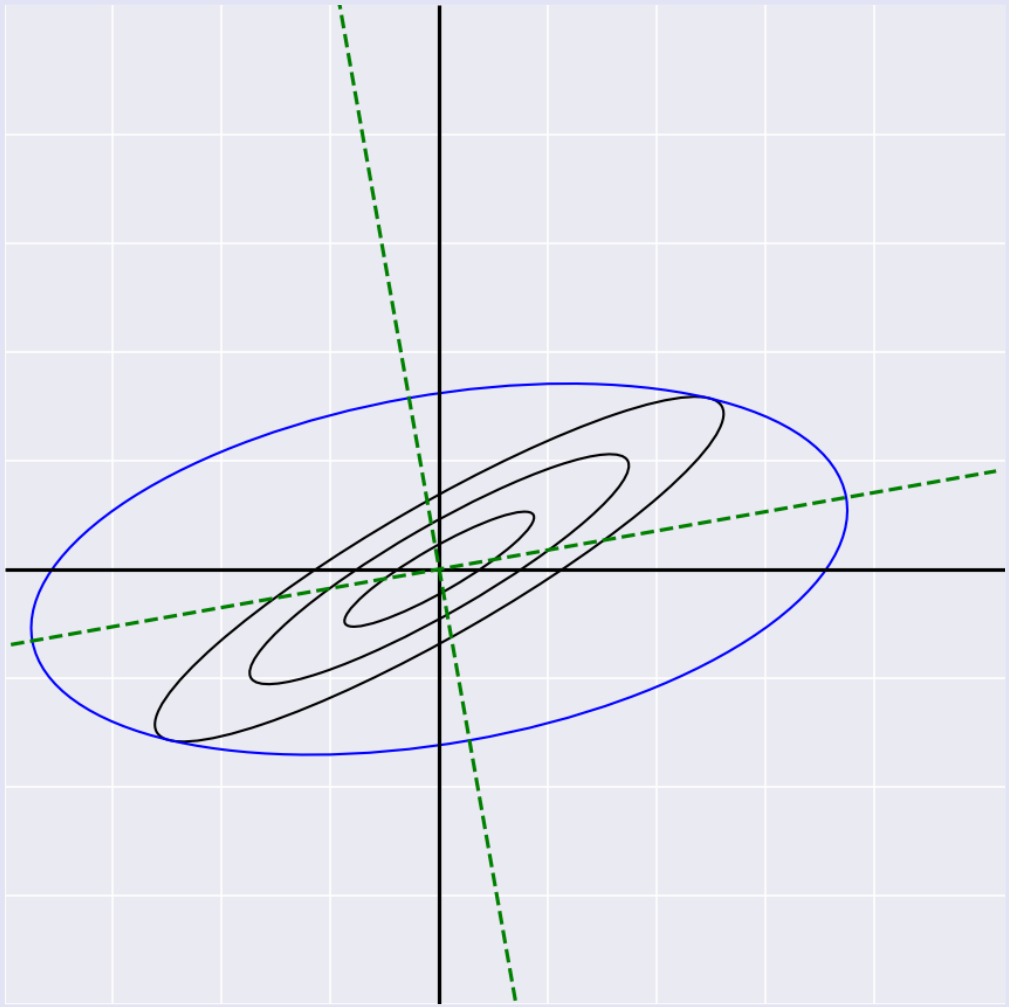


Toward an adaptive procedure

Similar iterative procedure with steps adapted to the problem topology.

$$\widetilde{F_q}(z) = F(z) + (z - z^{(q)})^{\mathsf{T}} R (z - z^{(q)}) + \delta_A(z)$$

Tradeoff between:

- Rotation to align the norm $\|\cdot\|_B$ and the norm $\|\cdot\|_1$, Computation

$$R = A^{\mathsf{T}} S A - B$$

- Deformation of the ℓ_1-norm with the rotation A . Accuracy

$$\delta_A(z) = \lambda \left(\|Az\|_1 - \|z\|_1 \right)$$

One step improvement

Suppose that $R = A^{\mathsf{T}} S A - B \succ 0$ is positive definite, and define

$$z^{(q+1)} = \arg\min_{z} \widetilde{F_q}(z) \ ,$$

Then

$$F(z^{(q+1)}) - F(z^*) \leq \frac{1}{2}(z^{(q)} - z^*)^{\mathsf{T}} R (z^{(q)} - z^*) + \delta_A(z^*) - \delta_A(z^{(q+1)}) \ .$$

We are interested in factorization (A, S) for which $\|R\|_2$ and δ_A are small.

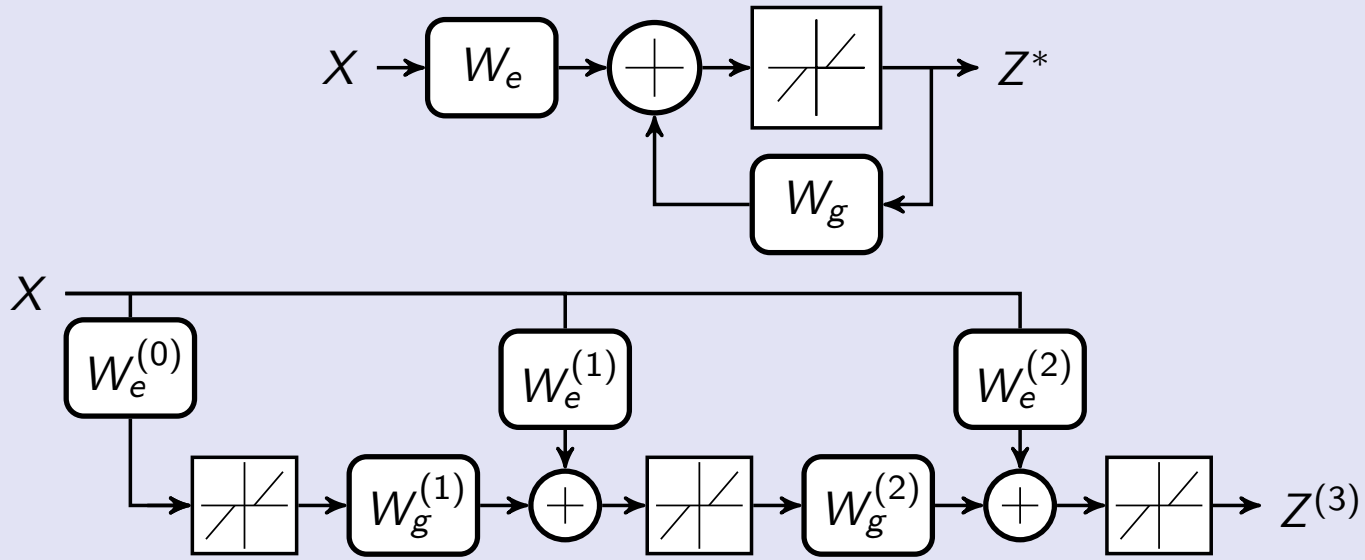
Theoretical results

- We showed that FacNet has the same asymptotic convergence rate as ISTA in $\mathcal{O}(\frac{1}{q})$.
- The constant factors are different and can be improved. If the factorization (A_q, S_q) at iteration q verifies

$$\|R_q\|_2 + 2\frac{L_{A_q}(z^{(q+1)})}{\|z^* - z^{(q)}\|_2} \leq \frac{\|B\|_2}{2}$$

and $A_p = \boldsymbol{I}_K, S_p = \|B\|_2 \boldsymbol{I}_K$ for $p > q$, then the procedure has improved convergence rate compared to ISTA.

$\Rightarrow$ There is a phase transition when $\|z^{(q)} - z^*\|_2 \to 0$



Network architecture for ISTA/LISTA. LISTA is the unfolded version of the RNN of ISTA, trainable with back-propagation.

With $W_e = \frac{D^\mathsf{T}}{\|B\|_2}$ and $W_g = I - \frac{B}{\|B\|_2}$, this network computes exactly 2 iterations of ISTA.

Specialization of LISTA

$$z^{(q+1)} = A^{\mathsf{T}} \underset{S}{\text{prox}}(Az^{(q)} - S^{-1}AB(z^{(q)} - y)) \ ,$$

with A unitary and S diagonal.
Same architecture with more constraints on the parameter space:

$$\begin{cases} W_e &= S^{-1}AD^{\mathsf{T}} \\ W_g &= A^{\mathsf{T}} - S^{-1}ABA^{\mathsf{T}} \end{cases}$$

$\Rightarrow$ LISTA can be at least as good as this model.

Generic Dictionary

- Generic dictionary uniformly sample in unit ball,

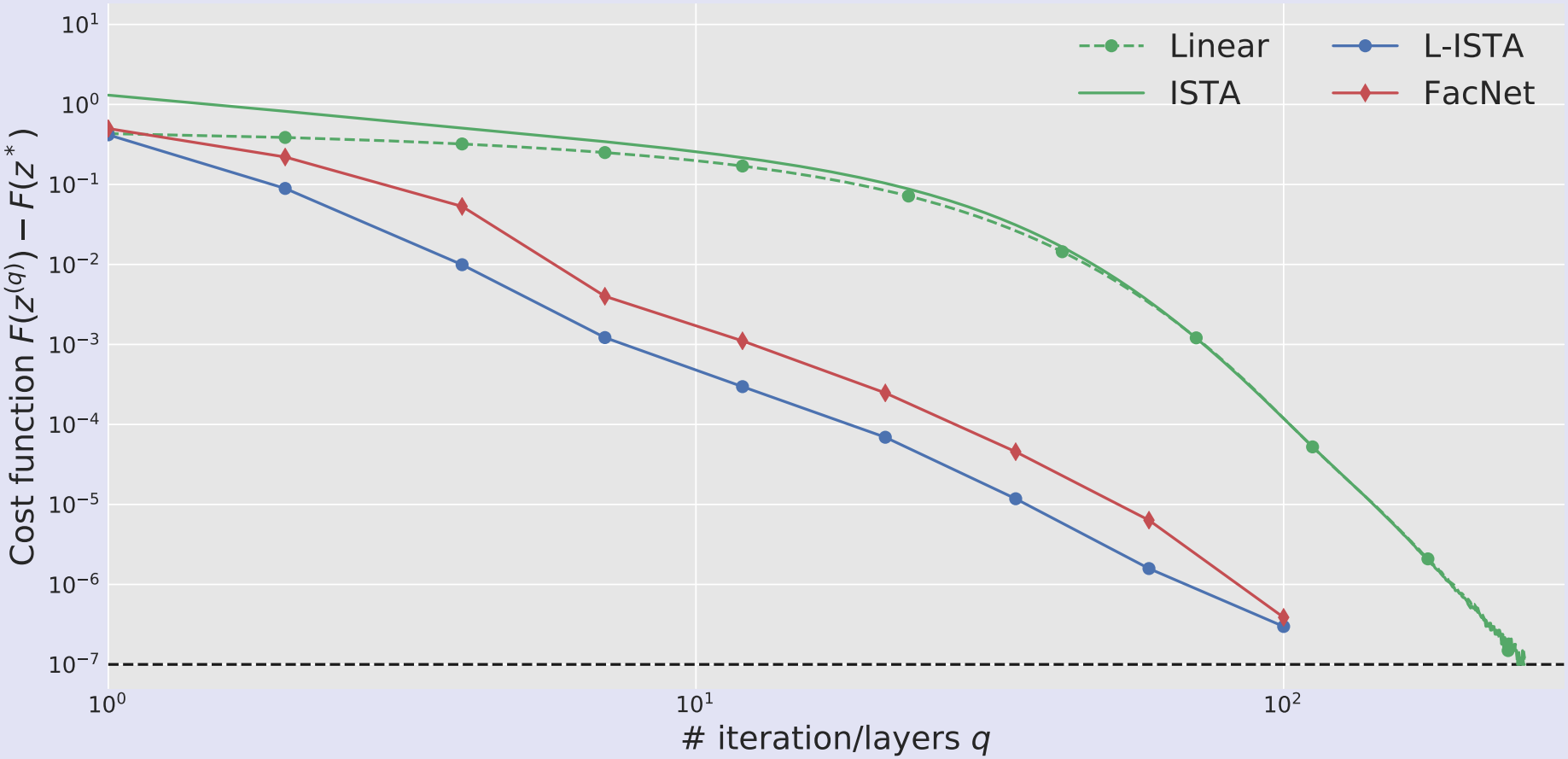
$$D \sim \mathcal{S}^{p-1} \ ,$$

- Sparse code generated with Bernouilli-Gaussian model, *s.t.*

$$z_k = b_k a_k, \qquad b_k \sim \mathcal{B}(\rho) \text{ and } a \sim \mathcal{N}(0, \sigma \boldsymbol{I}_K)$$

Fixed: $K = 100$, $P = 64$, $\sigma = 10$ and $\lambda = 0.01$

Generic Dictionary



$$\rho = {}^{1}/_{20}.$$

Evolution of the cost function $F(z^{(q)}) - F(z^*)$ with the number of layers/iterations q with a denser model

Adversarial dictionary

The dictionary is constructed such that it eigen-vectors are sampled from the Fourier basis, with

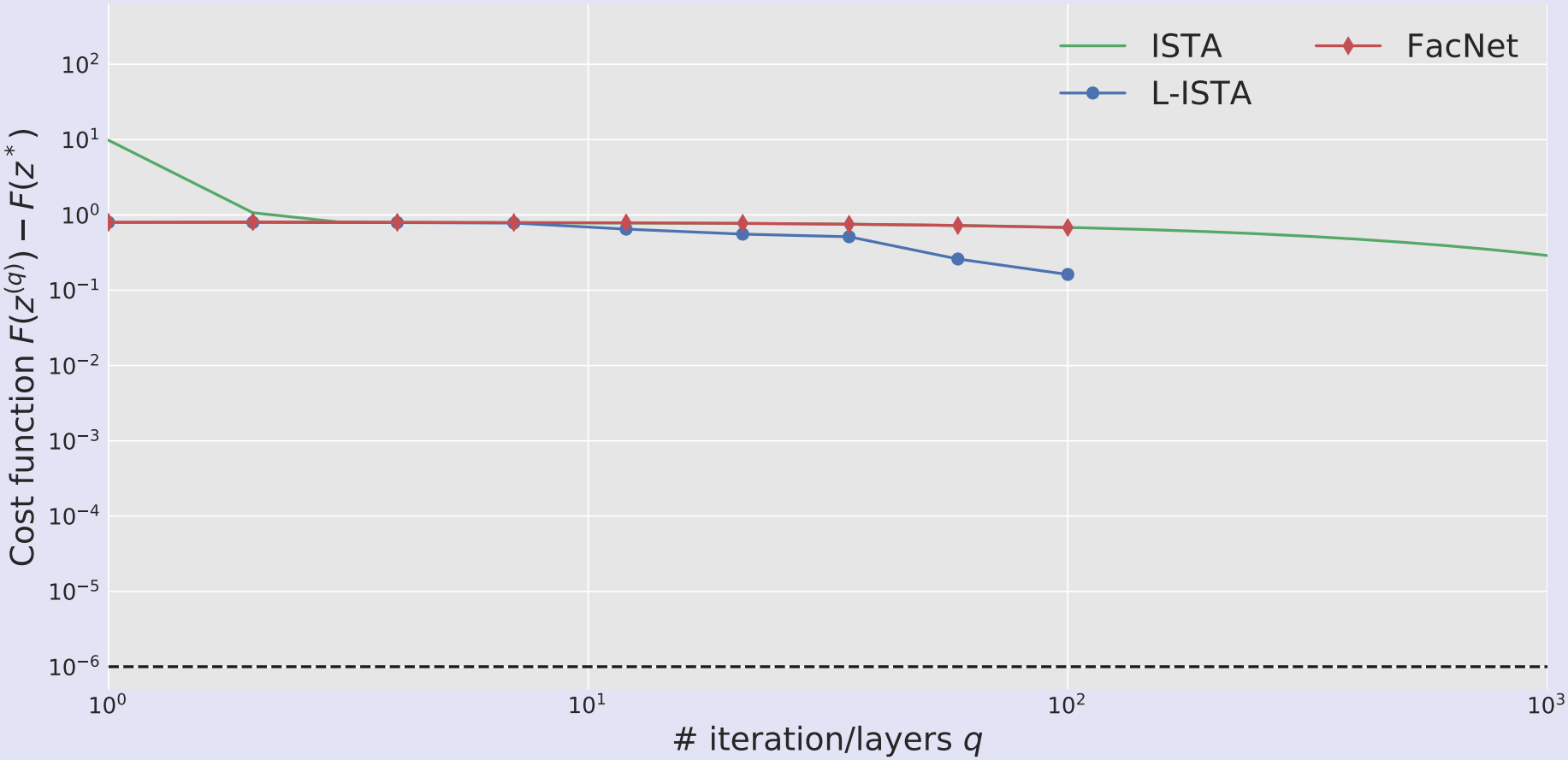
$$D_{k,j} = e^{-2i\pi k\zeta_j}$$

for a random subset of frequencies

$$\{\zeta_i\}_{0\leq i\leq p} \sim \mathcal{U}\left\{\frac{m}{K}; 0 \leq m \leq \frac{K}{2}\right\}$$

Diagonalizing B implies large deformation of the ℓ_1-norm.

Adversarial dictionary



Evolution of the cost function $F(z^{(q)}) - F(z^*)$ with the number of layers/iterations q with an adversarial dictionary.

Recap Part I

Take home message

- Non asymptotic acceleration of ISTA is possible based on the structure of $\boldsymbol{D}$,
- Sufficient analysis to explain LISTA acceleration,
- Empirically showed the structure of D is necessary for LISTA.

Ahead of us

- Improve the factorization formulation for direct optimization,
- Adaptation of the analysis to convolutional sparse coding,
- Explore the link with sparse eigenvectors of the gram matrix.

Adapting CD to Convolutional Sparse Coding

References

- Moreau, T., Oudre, L., and Vayatis, N. (2018). DICOD: Distributed Convolutional Sparse Coding. In *International Conference on Machine Learning (ICML)*, pages 3626–3634, Stockohlm, Sweden. PMLR (80)

Coordinate Descent (CD)

Minimize

$$Z^* = \underset{Z}{\operatorname{argmin}} \|X - \sum_{k=1}^{K} \boldsymbol{D}_k * Z_k\|_2^2 + \lambda \|Z\|_1$$

Update one coordinate at each iteration.

1. Select a coordinate (k_0, t_0) to update.

Three algorithms:

- Cyclic updates; $\mathcal{O}(1)$ [Friedman et al., 2007]
- Random updates; $\mathcal{O}(1)$ [Nesterov, 2010]
- Greedy updates; $\mathcal{O}(KL)$ [Osher and Li, 2009]

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Update one coordinate at each iteration.

1. Select a coordinate (k_0, t_0) to update.
2. Compute a new value $Z'_{k_0}[t_0]$ for this coordinate

For convolutional CD, we can use optimal updates:

$$Z'_{k_0}[t_0] = \frac{1}{\|\boldsymbol{D}_{k_0}\|_2^2} \mathbf{ST}(\beta_{k_0}[t_0], \lambda),$$

with $\mathbf{ST}(y, \lambda) = \operatorname{sign}(y)(|y| - \lambda)_+$. Kavukcuoglu et al. [2010] showed this can be done efficiently, with $\mathcal{O}(KW)$ operations.

Coordinate Descent (CD)

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Update one coordinate at each iteration.

1. Select a coordinate (k_0, t_0) to update.
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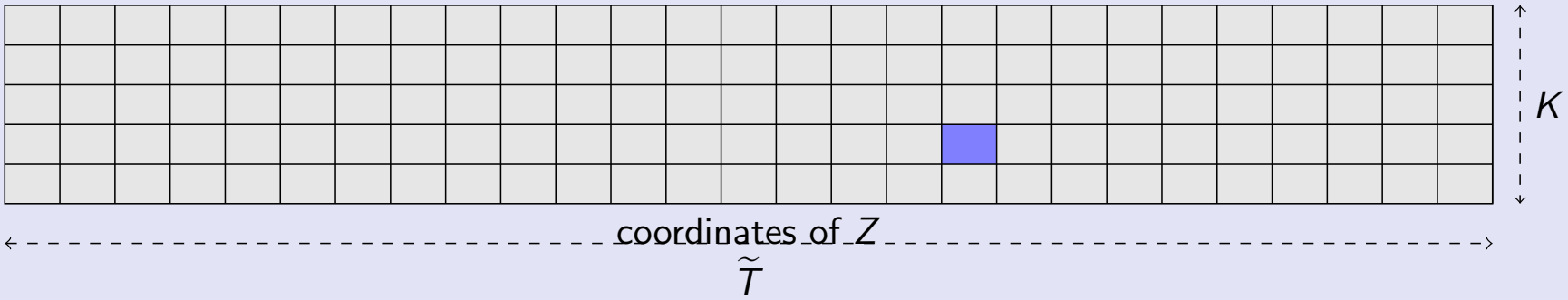
$\Rightarrow$ Converges to the optimal point for CSC problem in $\mathcal{O}\left(\frac{1}{q}\right)$ iterations.

Coordinate selection is a trade-off between cheap computational complexity (random/cyclic CD) and importance sampling with faster convergence (Greedy CD). [Nutini et al., 2015]

Locally greedy coordinate descent (LGCD) [Moreau et al., 2018]

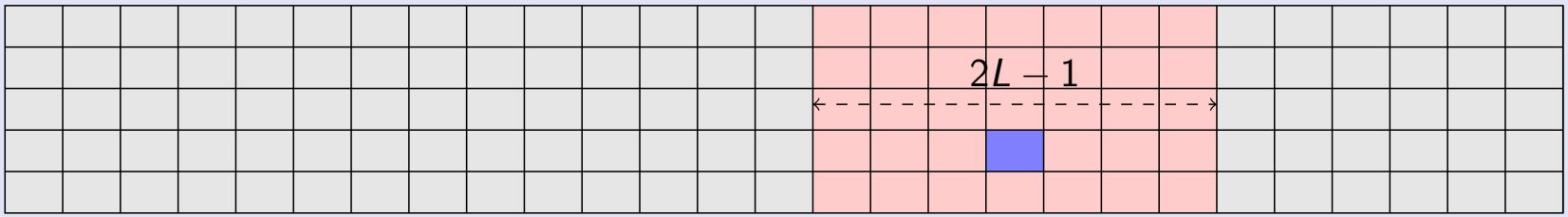
We introduced the LGCD method which is an extension of GCD.



GCD has $\mathcal{O}(K\widetilde{T})$ computational complexity.

Locally greedy coordinate descent (LGCD) [Moreau et al., 2018]

We introduced the LGCD method which is an extension of GCD.

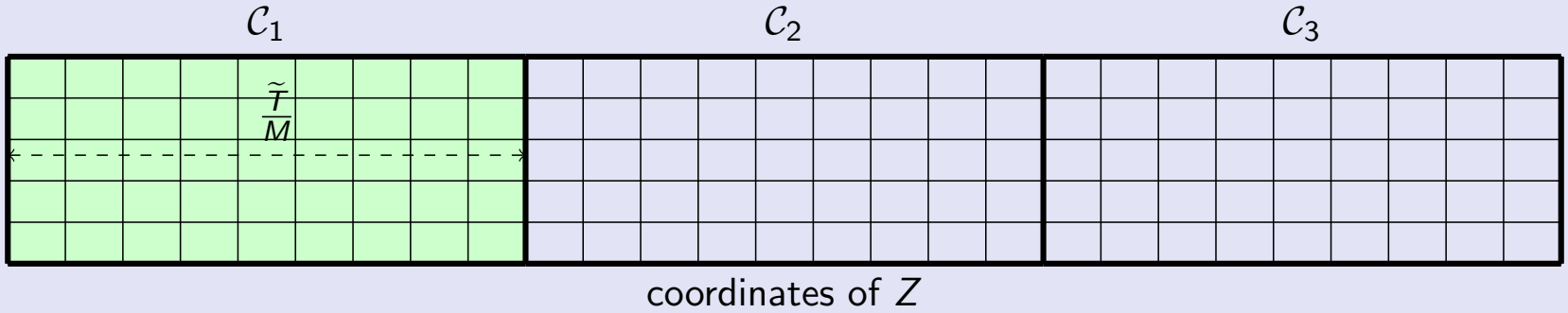


GCD has $\mathcal{O}(K\widetilde{T})$ computational complexity.

But the update itself has complexity $\mathcal{O}(KL)$

Locally greedy coordinate descent (LGCD) [Moreau et al., 2018]

We introduced the LGCD method which is an extension of GCD.



With a partition $\mathcal{C}_m$ of the signal domain $[1, K] \times [0, \widetilde{T}[$,

$$\mathcal{C}_m = [1, K] \times [\frac{(m-1)\widetilde{T}}{M}, \frac{m\widetilde{T}}{M}[$$

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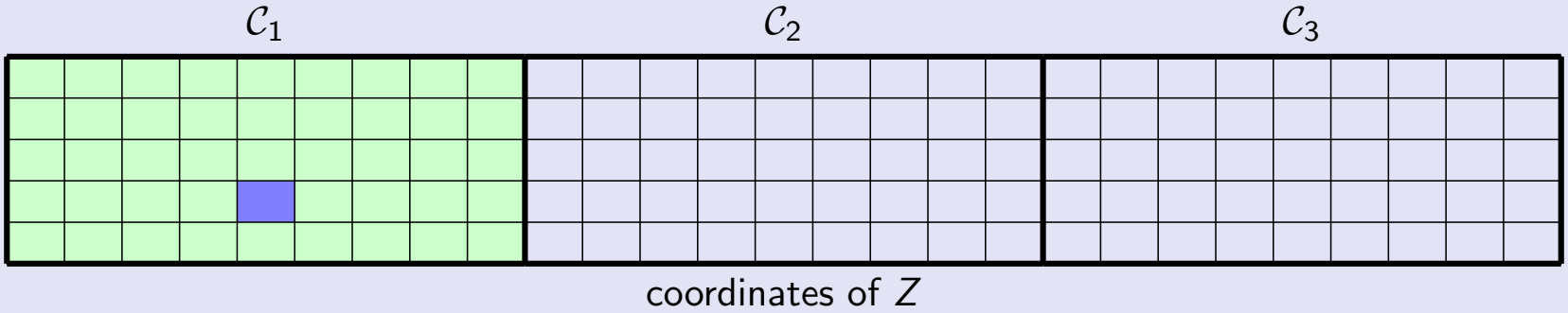

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$$\mathcal{C}_m = [1, K] \times [\frac{(m-1)\widetilde{T}}{M}, \frac{m\widetilde{T}}{M}[$$

The coordinate to update is chosen greedily on a sub-domain $\mathcal{C}_m$

$$\frac{\widetilde{T}}{M} = 2L - 1 \quad \Rightarrow \quad \mathcal{O}(\text{Coordinate selection}) = \mathcal{O}(\text{Coordinate Update})$$

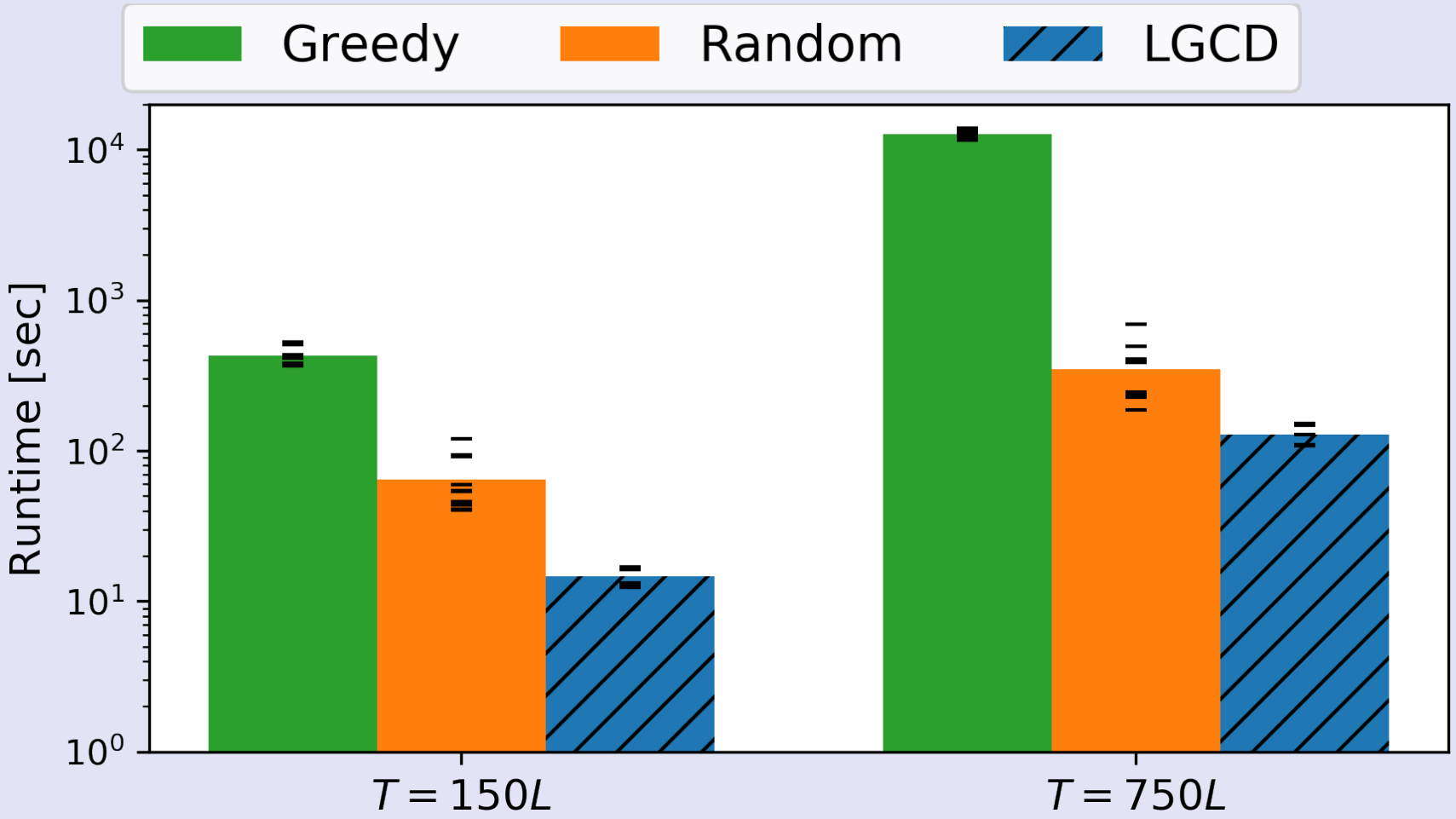
The overall iteration complexity is $\mathcal{O}(KL)$ instead of $\mathcal{O}(K\widetilde{T})$.

$$\Rightarrow \text{Efficient for sparse } Z$$

Fast optimization

Comparison of the coordinate selection strategy for CD on simulated signals

We set $K = 10$, $L = 150$, $\lambda = 0.1\lambda_{\max}$



Distributed optimization for CSC

References

- Moreau, T. and Gramfort, A. (2019). Distributed Convolutional Dictionary Learning (DiCoDiLe): Pattern Discovery in Large Images and Signals. *preprint ArXiv (to be submitted)*
- Moreau, T., Oudre, L., and Vayatis, N. (2018). DICOD: Distributed Convolutional Sparse Coding. In *International Conference on Machine Learning (ICML)*, pages 3626–3634, Stockohlm, Sweden. PMLR (80)

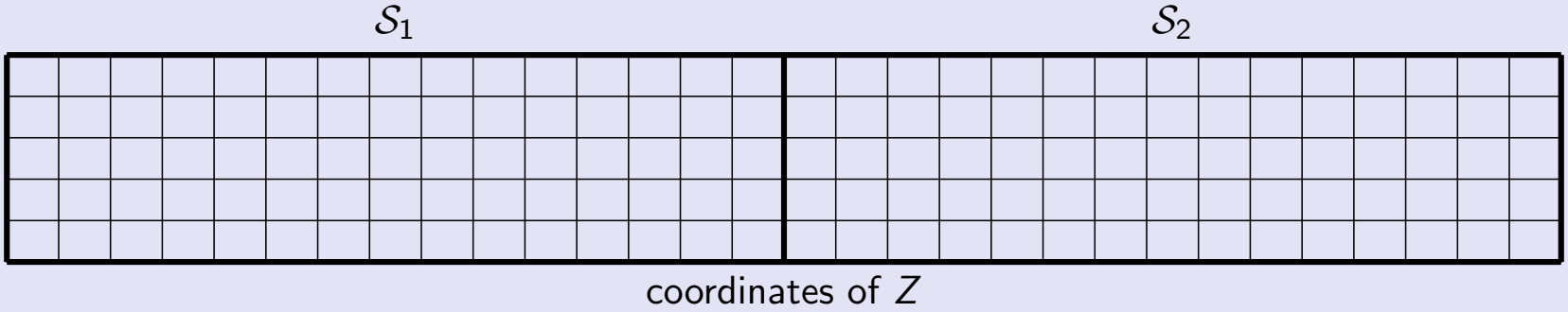
Weak dependence of the coordinate updates

The update of the W coordinates $(k_w, \omega_w)_{w=1}^{W}$ with additive update $\Delta Z_{k_w}[\omega_w]$ changes the cost by:

$$\Delta E = \overbrace{\sum_{i=1}^{W}}^{\text{iterative steps}} \Delta E_w \; \underbrace{- \sum_{w \neq w'} (d_{k_w} * d^{\curvearrowleft}_{k_{w'}})[\omega_{w'} - \omega_w] \Delta Z_{k_w}[\omega_w] \Delta Z_{k_{w'}}[\omega_{w'}]}_{\text{interference}},$$

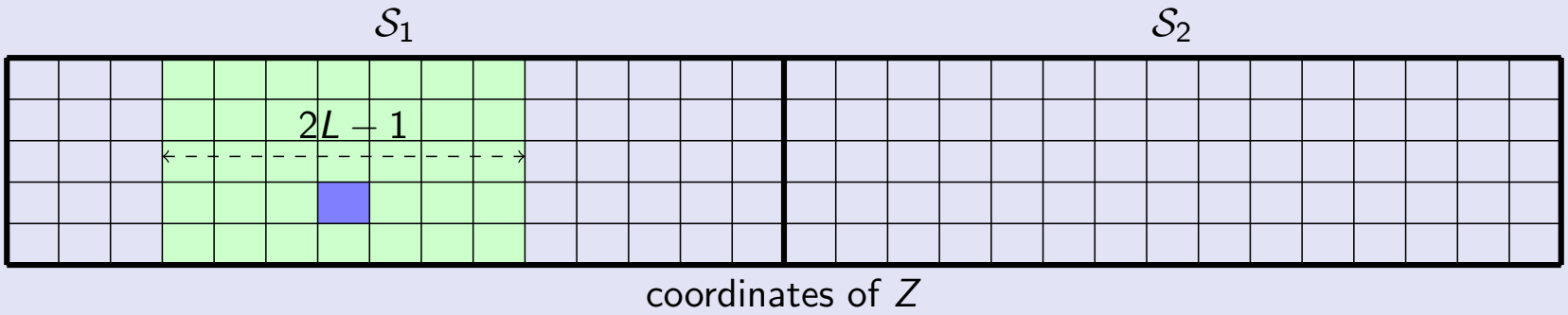
$\Rightarrow$ If the updates are far enough, they can be considered as independent.

Distributed Convolutional Coordinate Descent (DICOD)



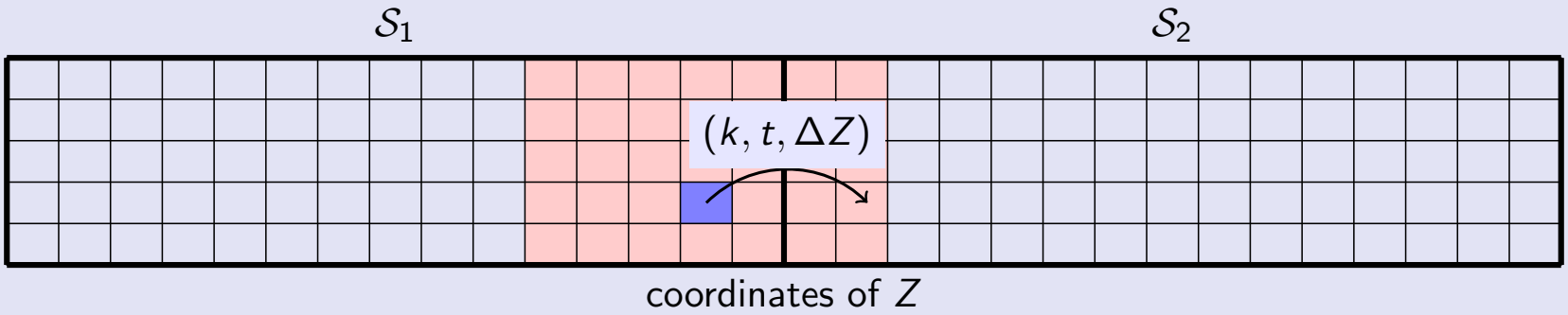
- Split the coordinates in continuous sub-segment $\mathcal{S}_w = \left[\frac{(w-1)T}{W}, \frac{wT}{W}\right[$.

Distributed Convolutional Coordinate Descent (DICOD)



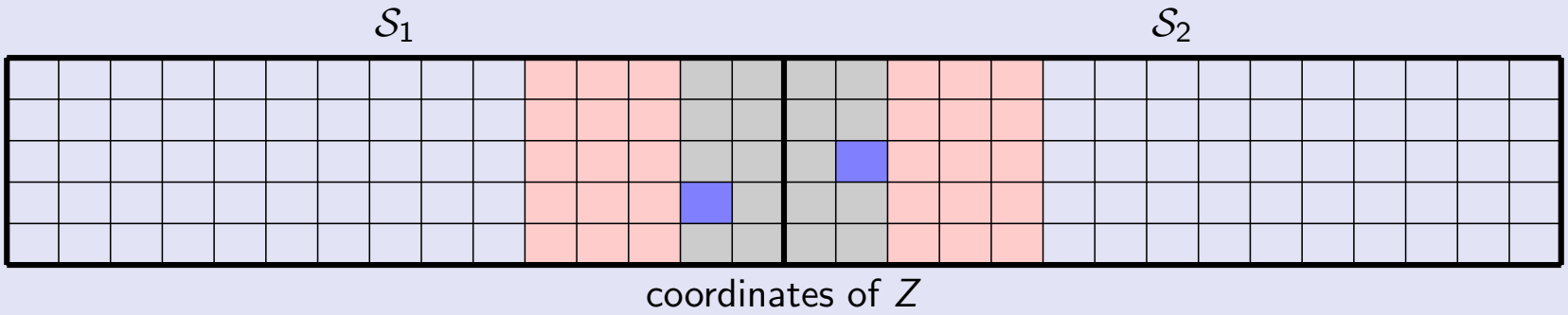
- Split the coordinates in continuous sub-segment $\mathcal{S}_w = \left[\frac{(w-1)T}{W}, \frac{wT}{W}\right[$.
- Use CD updates in parallel in each sub-segment.

Distributed Convolutional Coordinate Descent (DICOD)



- Split the coordinates in continuous sub-segment $\mathcal{S}_w = \left[\frac{(w-1)T}{W}, \frac{wT}{W}\right[$.
- Use CD updates in parallel in each sub-segment.
- Notify neighbor workers when the update is on the border of $\mathcal{S}_w$.

Distributed Convolutional Coordinate Descent (DICOD)



- Split the coordinates in continuous sub-segment $\mathcal{S}_w = \left[\frac{(w-1)T}{W}, \frac{wT}{W}\right[$.
- Use CD updates in parallel in each sub-segment.
- Notify neighbor workers when the update is on the border of $\mathcal{S}_w$.
- What do we do when two updates are interfering?

DICOD converges to the solution of the CSC for 1D signals without having a control mechanism on the interference.

Theorem (Convergence of DICOD)

We consider the following assumptions:

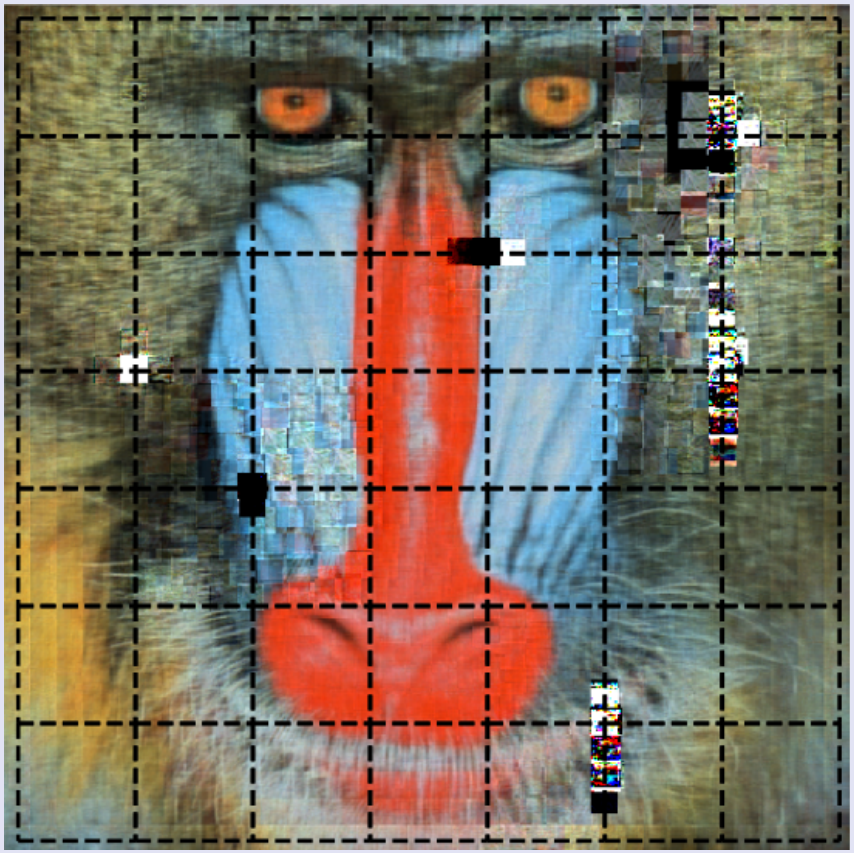
H1: If the cross correlation between atoms of $\boldsymbol{D}$ is strictly smaller than 1.

H2: No cores stop before all its coefficients are optimal.

H3: If the delay in communication between the processes is inferior to the update time.

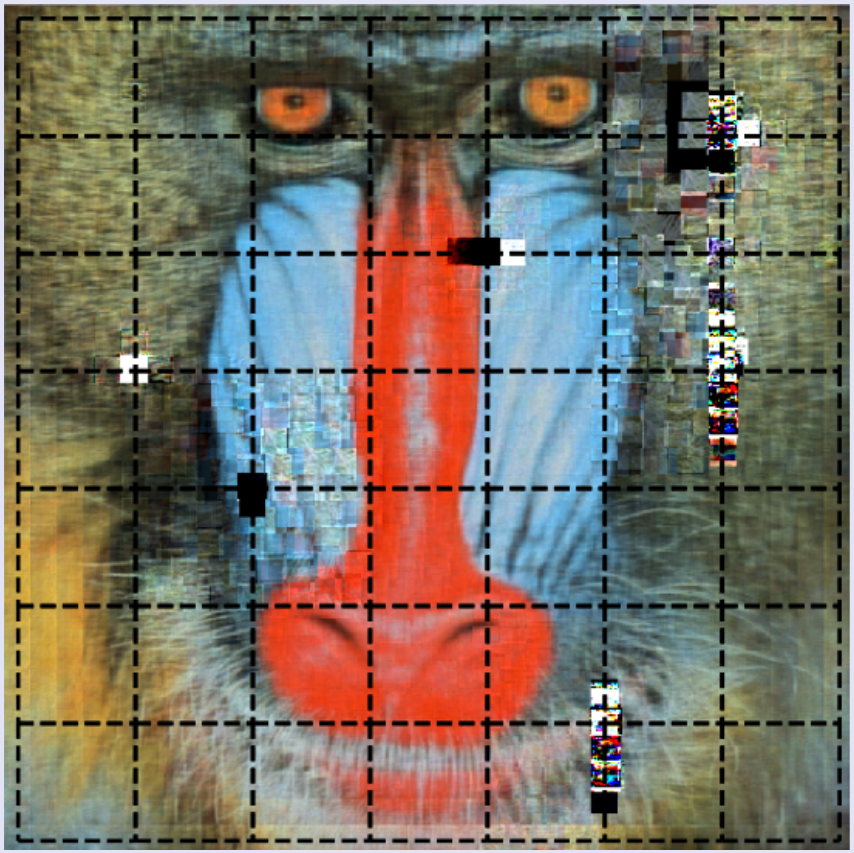
Under these assumptions, the DICOD algorithm converges asymptotically to the optimal solution Z^* of CSC.

Distributed Convolutional Dictionary Learning (DiCoDiLe-Z)



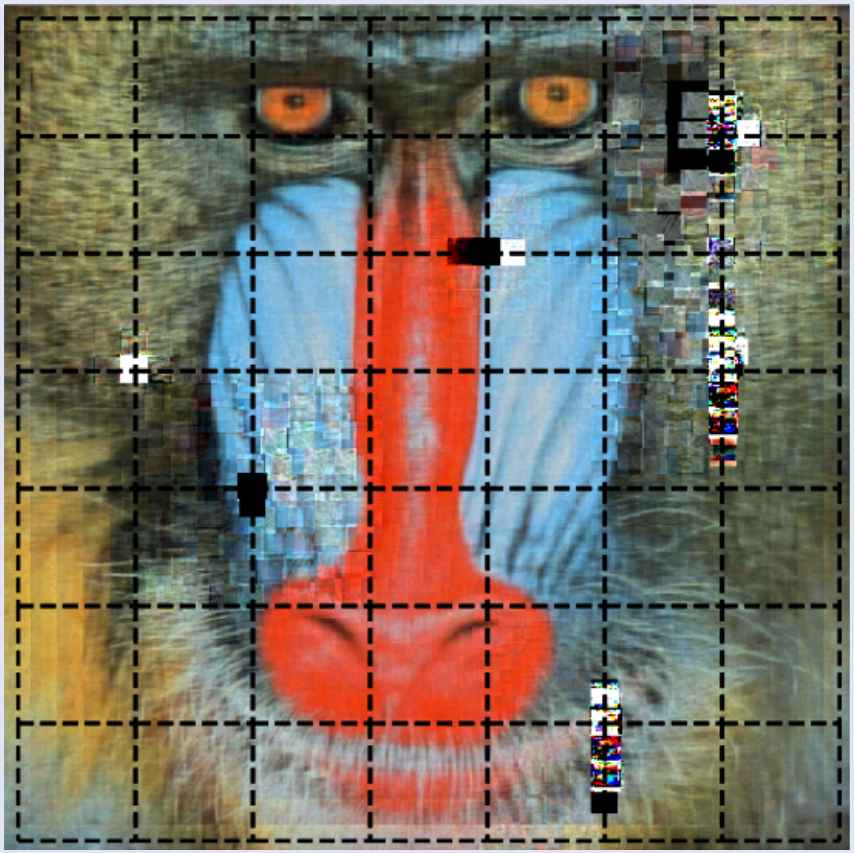
- DICOD does not work for higher dimensional signals.

Distributed Convolutional Dictionary Learning (DiCoDiLe-Z)



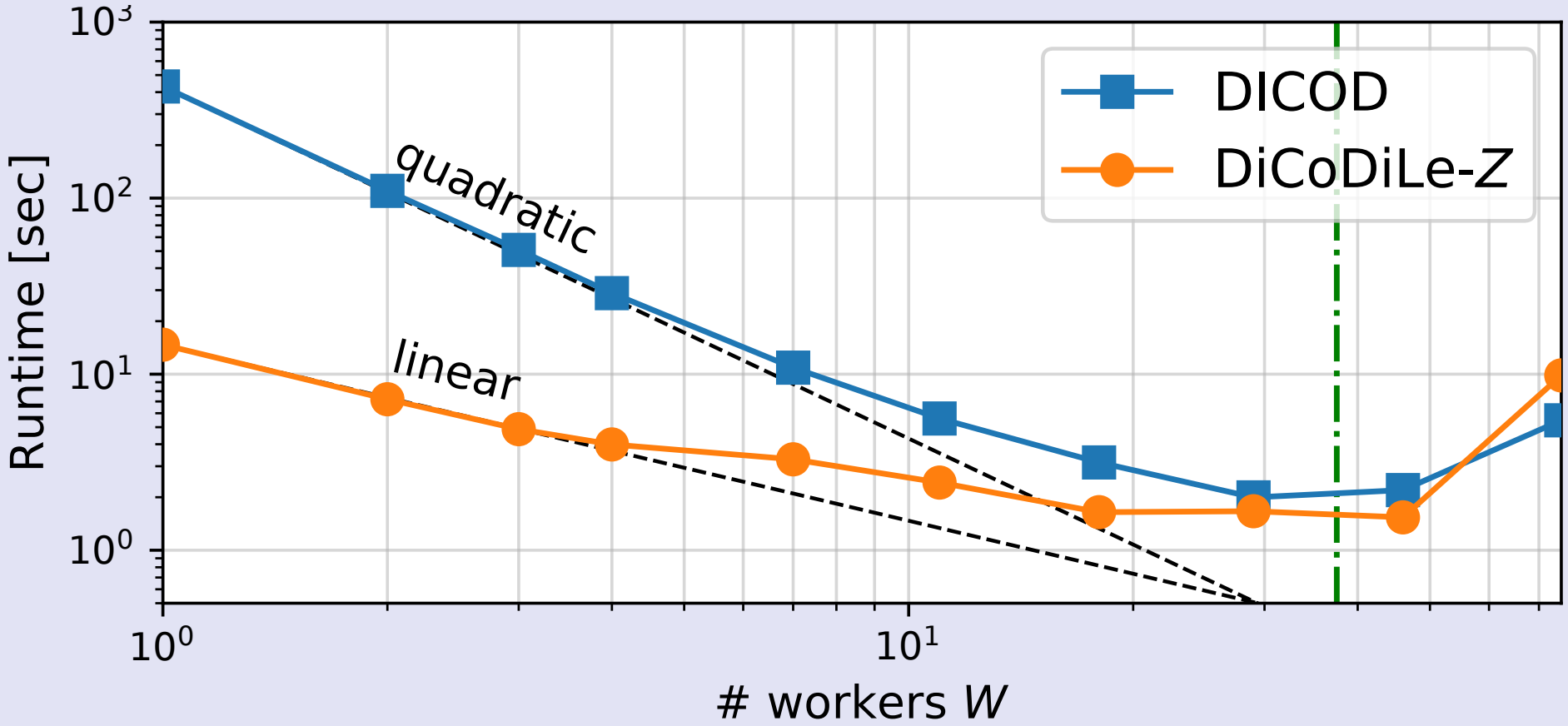
- DICOD does not work for higher dimensional signals.
- Extension require to control interferences.

Distributed Convolutional Dictionary Learning (DiCoDiLe-Z)



- DICOD does not work for higher dimensional signals.
- Extension require to control interferences.
- Use asynchronous mechanism: Soft-lock.

Numerical speed-up



Running time as a function fo the number of workers W.

Recap Part II

Take home message

- LGCD is a very efficient algorithm when working with CSC for long signals.
- Can be distributed efficiently for multi-dimensional signals,
- Good scaling properties with the number of workers W used to distribute the algorithm.

Ahead of us

- Extend this algorithm to local penalization such as Group LASSO.
- This algorithm could be used for algorithm such as MP for ℓ_0 or $\ell_{0,\infty}$ penalties.

Rank-1 Constrained Convolutional Dictionary Learning

References

- Dupré la Tour, T., Moreau, T., Jas, M., and Gramfort, A. (2018). Multivariate Convolutional Sparse Coding for Electromagnetic Brain Signals. In *Advances in Neural Information Processing Systems (NeurIPS)*, pages 3296–3306, Montreal, Canada

D-step: solving for the atoms

The dictionary update is performed by minimizing

$$\min_{\|\boldsymbol{D}_k\|_2 \leq 1} E(\{\boldsymbol{D}_k\}_k) \triangleq \sum_{n=1}^{N} \frac{1}{2} \|X^n - \sum_{k=1}^{K} z_k^n * \boldsymbol{D}_k\|_2^2 \quad . \tag{1}$$

Computing $\nabla_{\boldsymbol{D}_k} E(\{\boldsymbol{D}_k\}_k)$ can be done efficiently

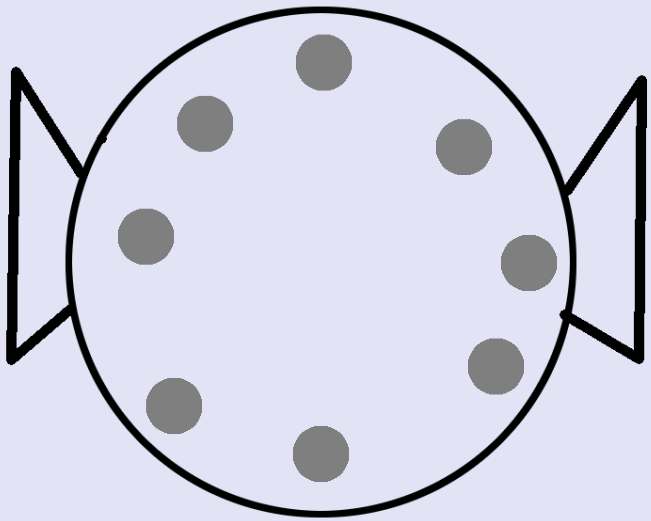
$$\nabla_{d_k} E(\{\boldsymbol{D}_k\}_k) = \sum_{n=1}^{N} (z_k^n)^{\leftharpoonup} * \left(x^n - \sum_{l=1}^{K} z_l^n * \boldsymbol{D}_l \right) = \Phi_k - \sum_{l=1}^{K} \Psi_{k,l} * \boldsymbol{D}_l \ ,$$

$\Rightarrow$ Save with Projected Gradient Descent (PGD) with an Armijo backtracking line-search for the D-step [Wright and Nocedal, 1999].

However, this model does not account for the physics of the problem.

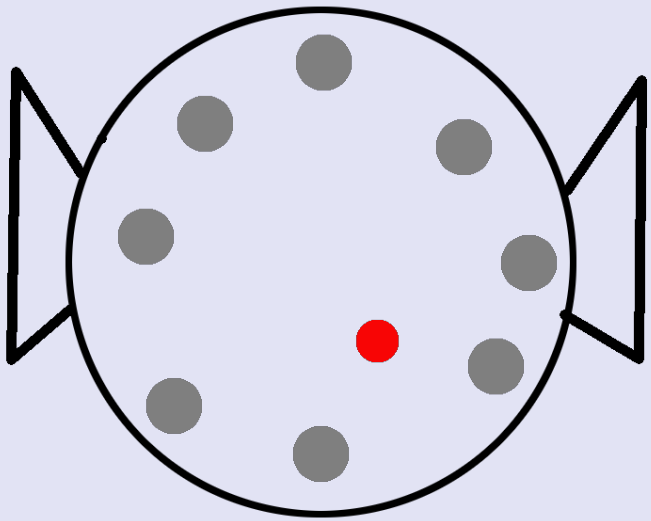
EM wave diffusion

- Recording here with 8 sensors



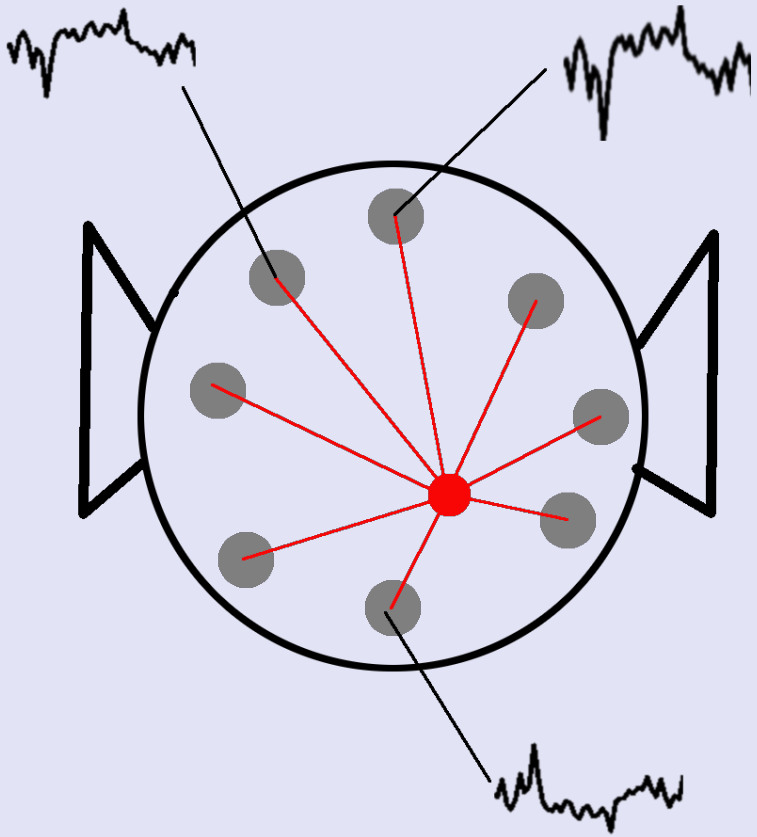
EM wave diffusion

- Recording here with 8 sensors
- EM activity in the brain



EM wave diffusion

- Recording here with 8 sensors
- EM activity in the brain
- The electric field is spread **linearly** and **instantaneously** over all sensors (Maxwell equations)



Multivariate CSC with rank-1 constraint

Idea: Impose a rank-1 constraint on the dictionary atoms D_k

To make the problem tractable, we decided to use auxiliary variables u_k and v_k *s.t.* $D_k = u_k v_k^\top$.

$$\min_{u_k, v_k, z_k^n} \sum_{n=1}^{N} \frac{1}{2} \left\| X^n - \sum_{k=1}^{K} z_k^n * (u_k v_k^\top) \right\|_2^2 + \lambda \sum_{k=1}^{K} \left\| z_k^n \right\|_1 , \tag{2}$$
$$\text{s.t.} \quad \|u_k\|_2^2 \leq 1 \ , \ \|v_k\|_2^2 \leq 1 \text{ and } z_k^n \geq 0 \ .$$

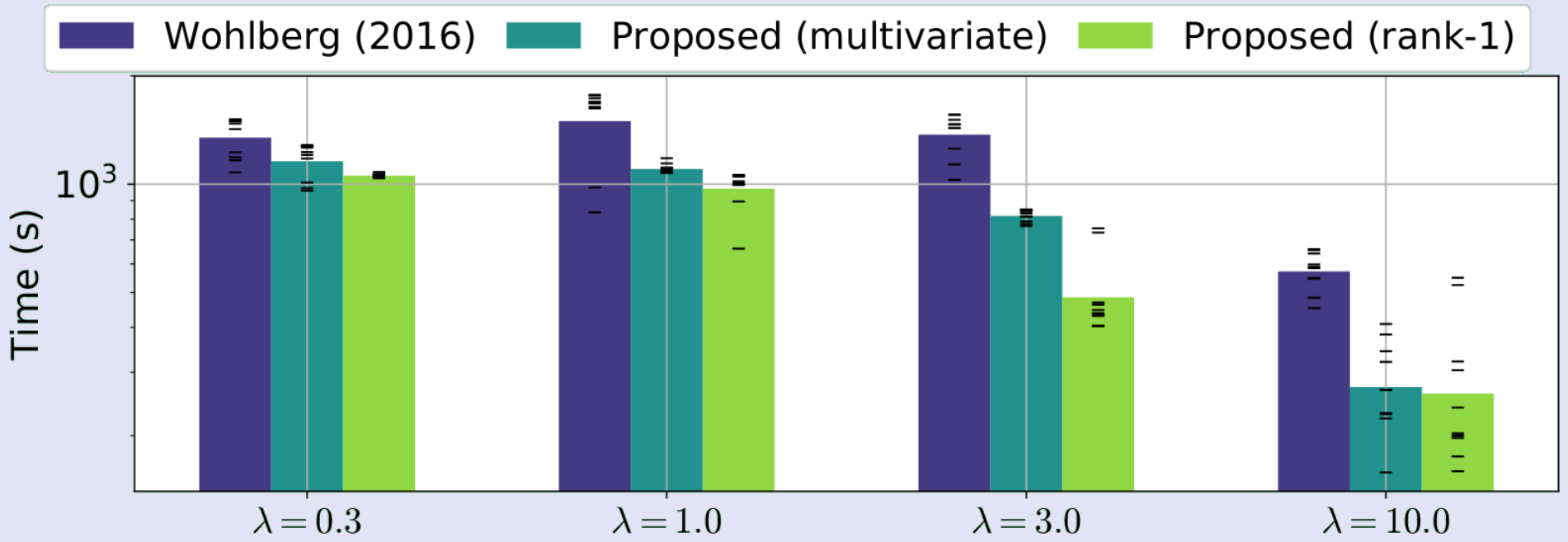
Here,

- $u_k \in \mathbb{R}^P$ is the spatial pattern of our atom
- $v_k \in \mathbb{R}^L$ is the temporal pattern of our atom

$\Rightarrow$ Tri-convex optimization problem , solved with alternate minimization.

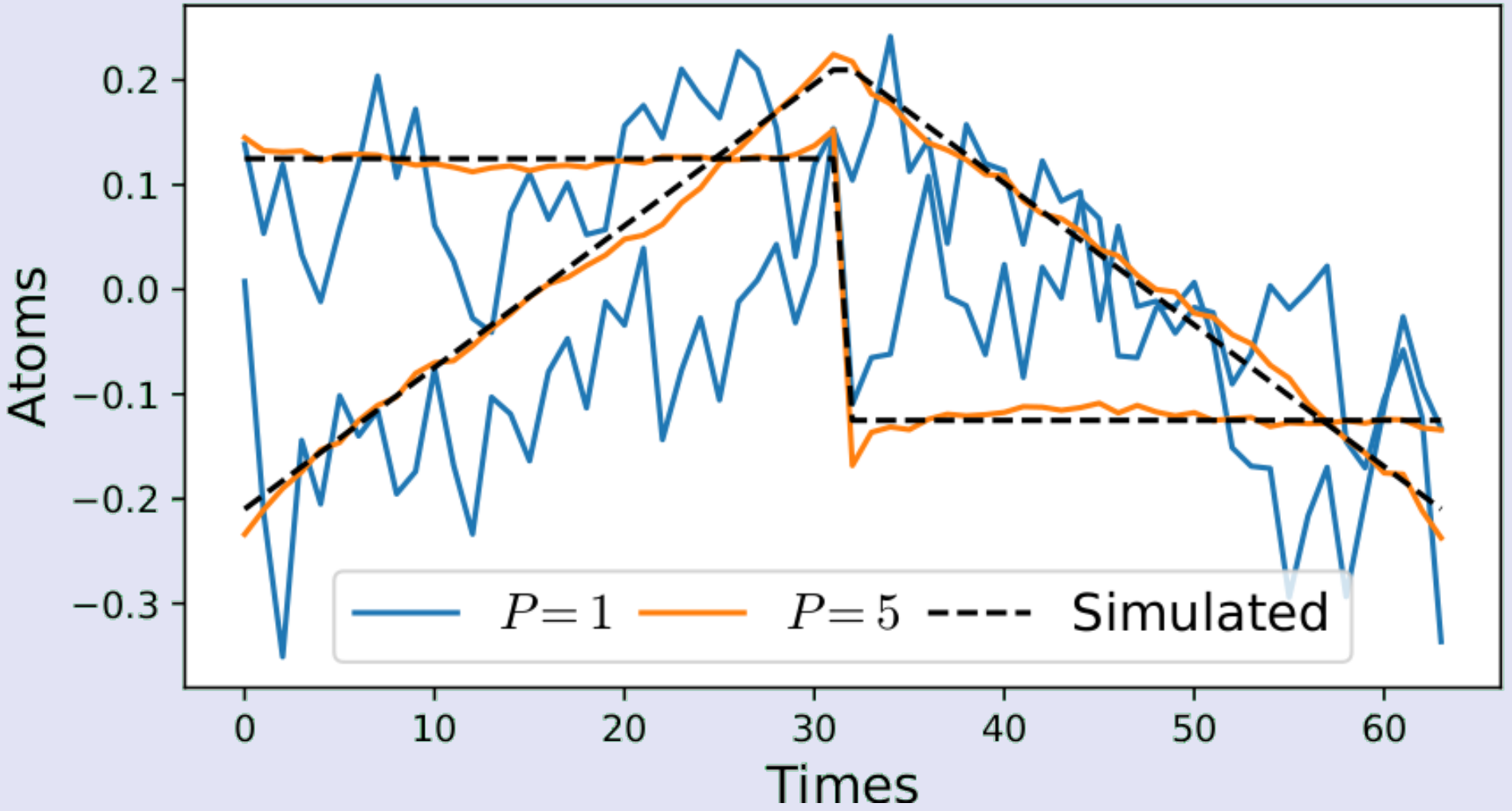
Fast optimization

Comparison with multivariate methods on somato dataset with $T = 134,700$, $K = 8$, $P = 5$ and $L = 128$



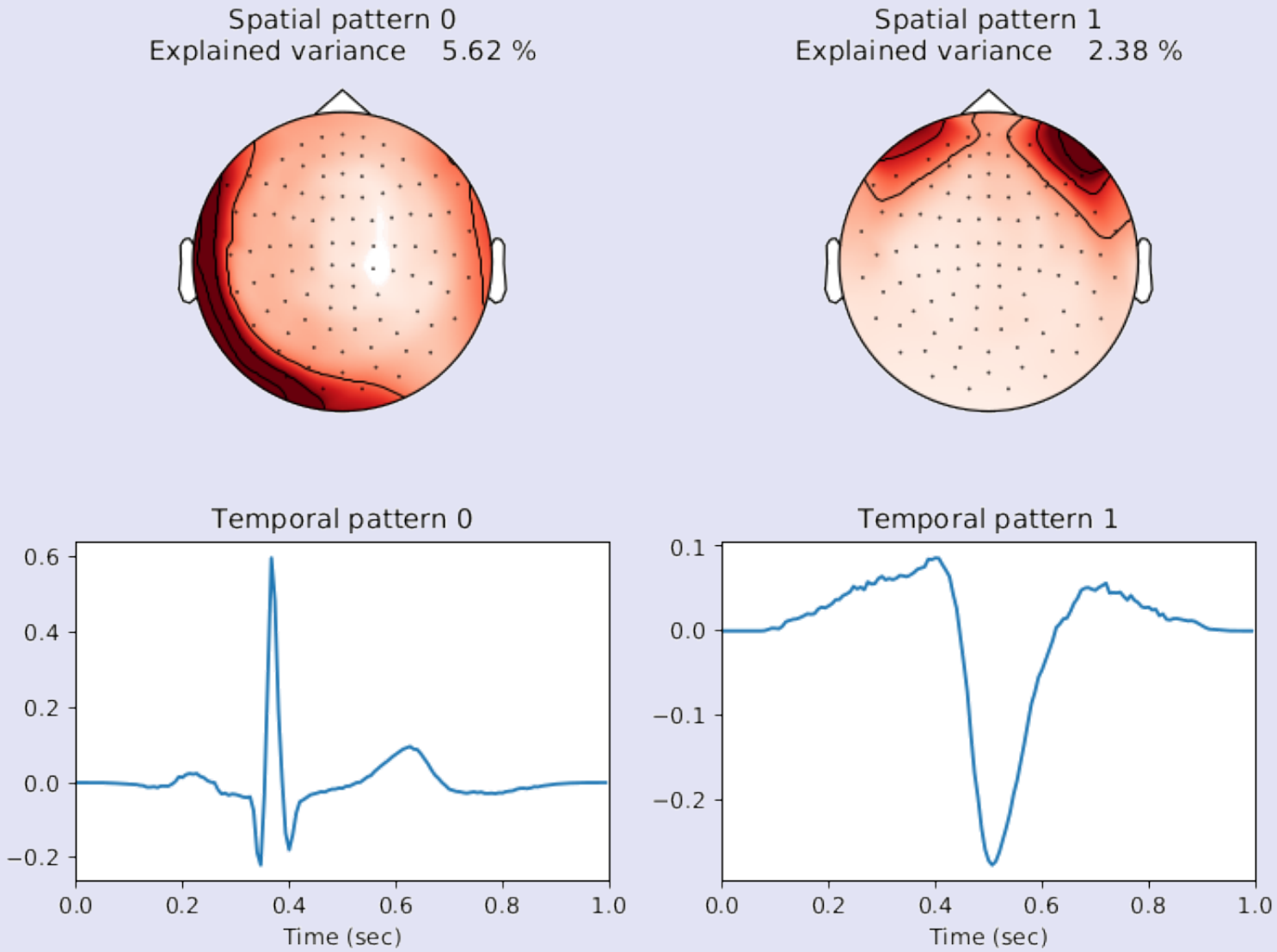
Pattern recovery

Patterns recovered with $P = 1$ and $P = 5$. The signals were generated with the two simulated temporal patterns and with $\sigma = 10^{-3}$.



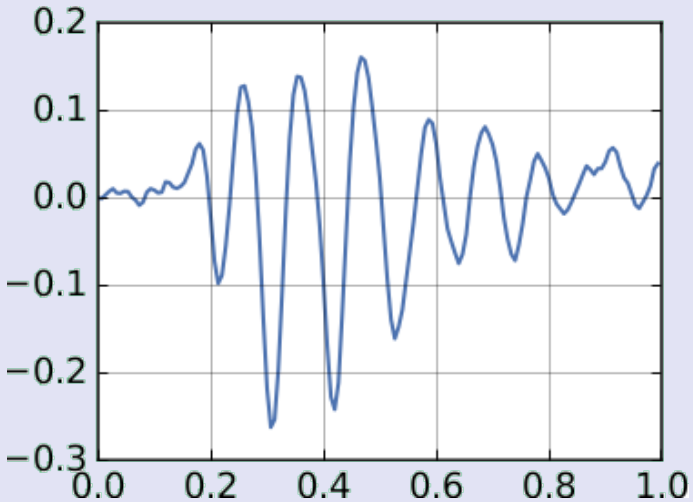
MNE somatosensory data

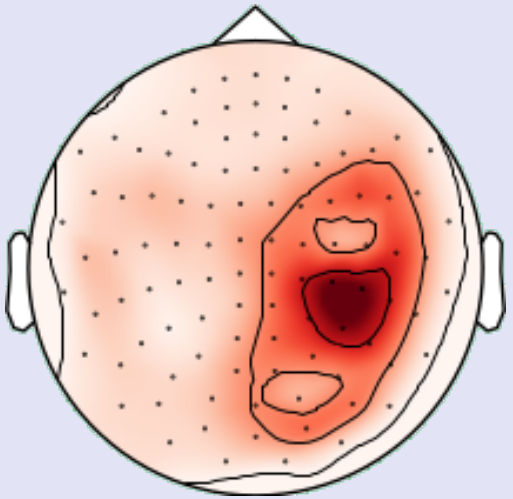
A selection of temporal waveforms of the atoms learned on the MNE sample dataset.

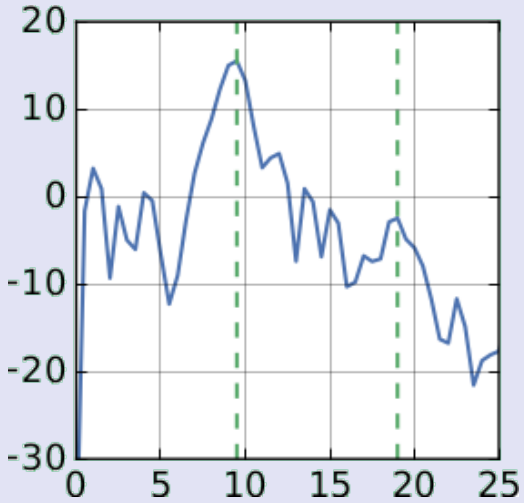


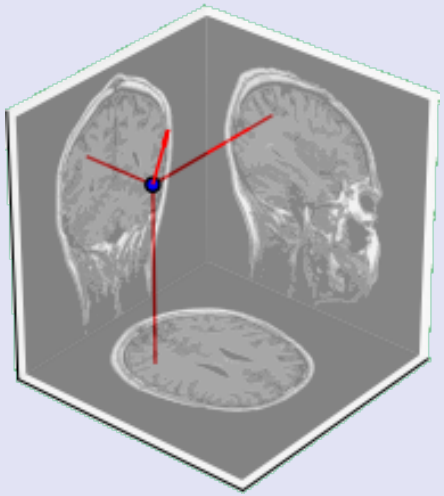
MNE somatosensory data

Atoms revealed using the MNE somatosensory data. Note the non-sinusoidal comb shape of the mu rhythm.









Recap Part III

Take home message

- The structure of the learned dictionary can be constrained to improve the interpretability of the recovered patterns.
- Can lead to more efficient algorithm and better recovery property.
- Open source package

Ahead of us

- Analysis of the patterns learned on large MEG database (HCP).
- Link between the learned waveforms and information propagation properties in the brain.
- Extension to scale invariant CDL to study frequency coupling in the brain.

Thanks!

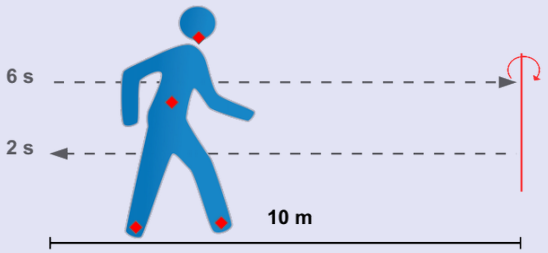
Code available online:

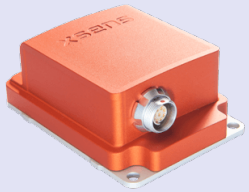
- **LISTA** : github.com/tommoral/AdaptiveOptim
- **DICOD** (& DiCoDiLe soon) : github.com/tommoral/dicod
- **alphacsc** : alphacsc.github.io

Slides are on my web page:

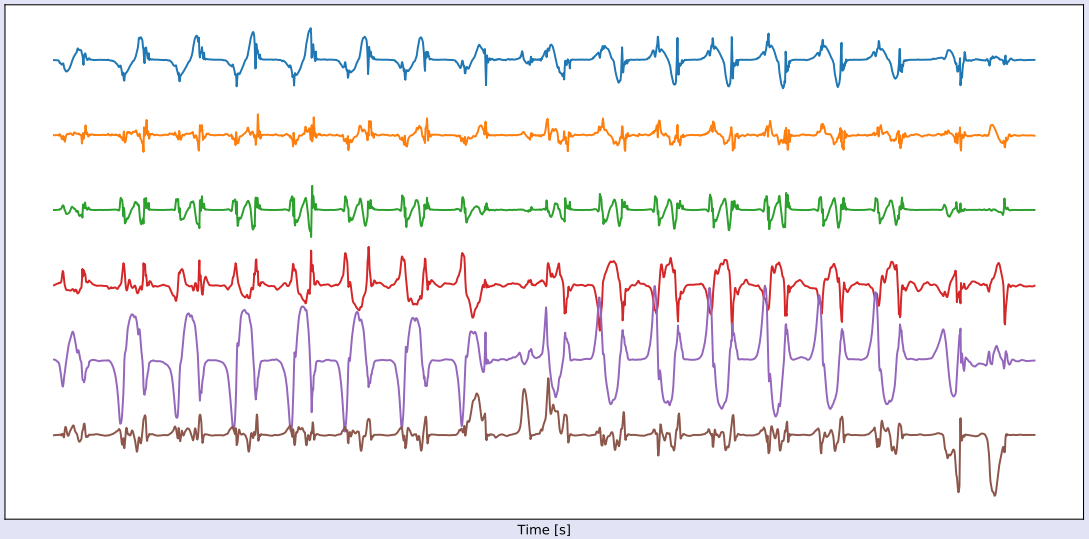
tommoral.github.io @tomamoral

Signals from human walking





Inertial captor



- Shift invariant patterns linked to steps,
- Manual segmentation of the signal is expensive.

$\Rightarrow$ Can we do better with data-driven approach?

Experiment

Create a dictionary with 25 Gaussian patterns ($W = 90$)

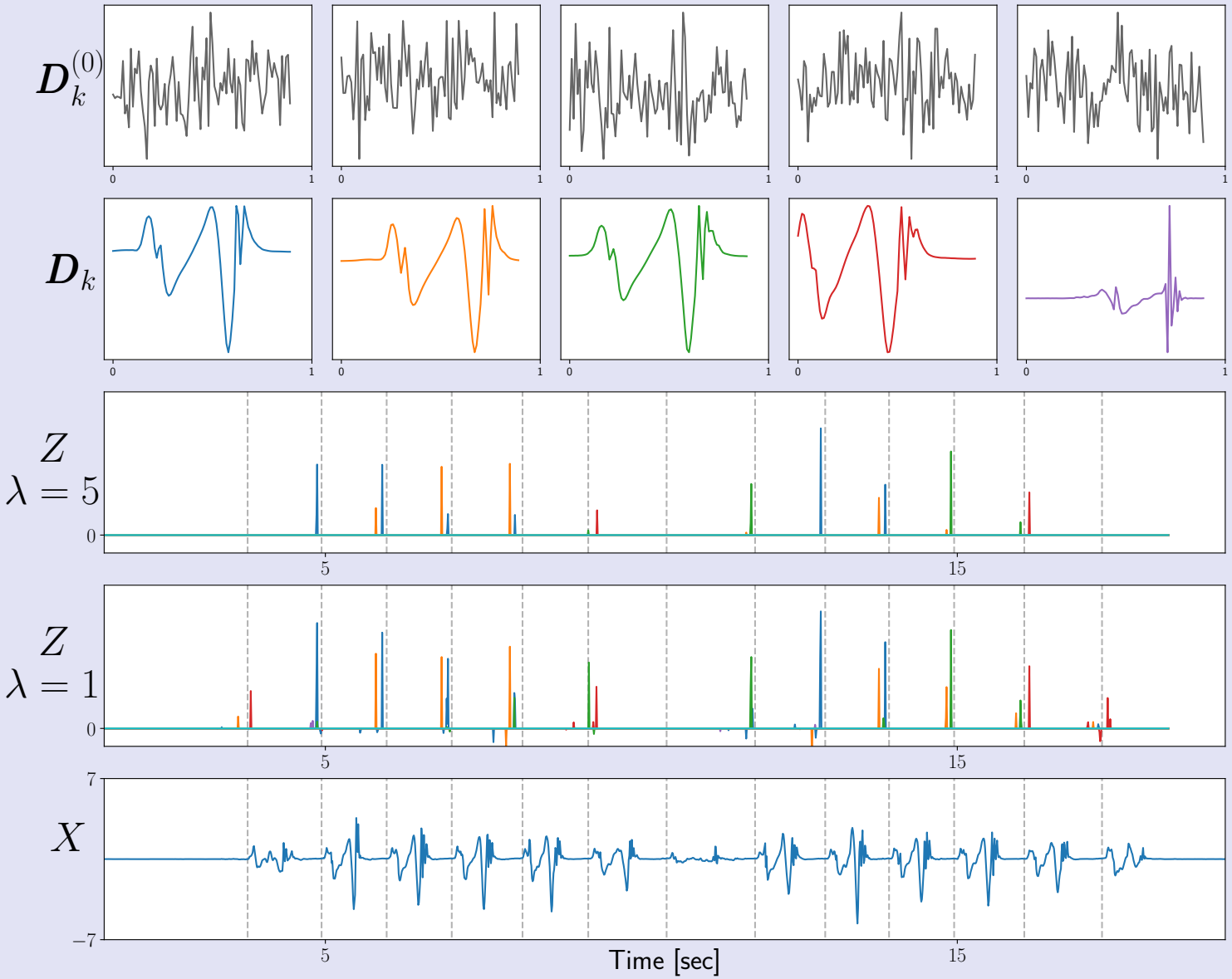
$$\boldsymbol{D}_k^{(0)} \sim \mathcal{N}(0, I_{90})$$

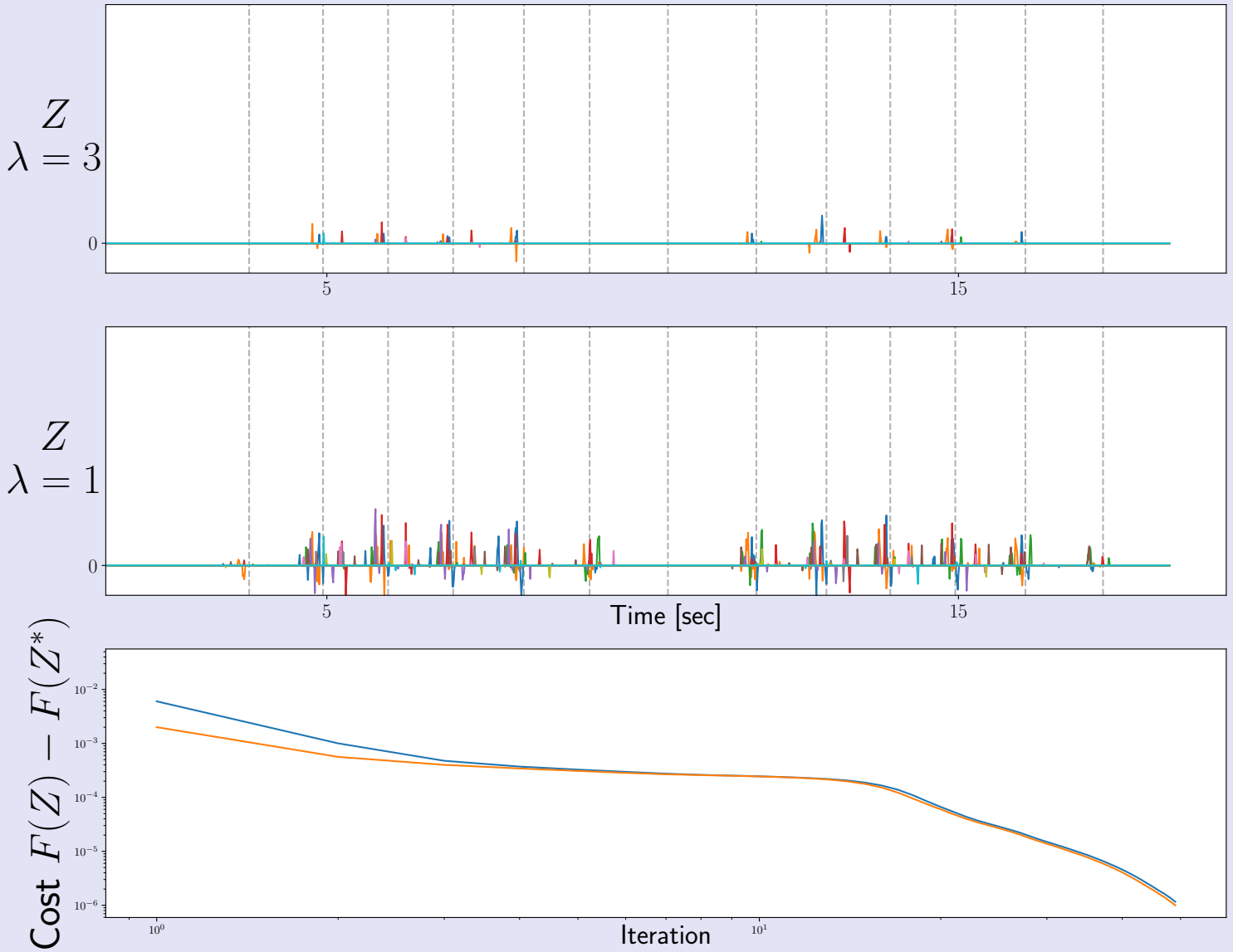
Use the Convolutional Dictionary Learning with DICOD to learn a dictionary $\boldsymbol{D}$ on a set of 50 recording of healthy subjects walking.

Challenges

- Alignment of the patterns,
- Detect steps of different amplitude,
- Handle multivariate signals.

Experiment




Z
$\lambda = 3$
0
5
15
Z
$\lambda = 1$
0
5
Time [sec]
15
Cost $F(Z) - F(Z^*)$
10^{-2}
10^{-3}
10^{-4}
10^{-5}
10^{-6}
10^{0}
10^{1}
Iteration

Related works

- Giryes et al. [2018]: Propose the inexact projected gradient descent and conjecture that LISTA accelerate the LASSO resolution by learning the sparsity pattern of the input distribution.

- Xin et al. [2016]: Study the Hard-thresholding Algorithm and its capacity to recover the support of a sparse vector.
  The paper relax the RIP conditions for the dictionary.

Generic Dictionaries

A dictionary $D \in \mathbb{R}^{p \times K}$ is a generic dictionary when its columns D_i are drawn uniformly over the ℓ_2 unit sphere $\mathcal{S}^{p-1}$.

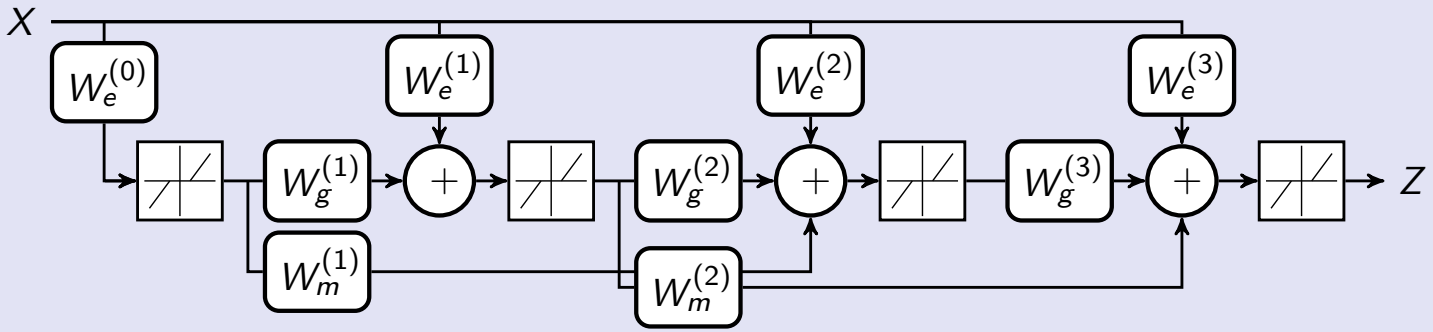
Theorem (Generic Acceleration)

In **expectation over the generic dictionary** D, the factorization algorithm using a diagonally dominant matrix $A \subset \mathcal{E}_\delta$, has better performance for iteration $q+1$ than the normal ISTA iteration – which uses the identity – when

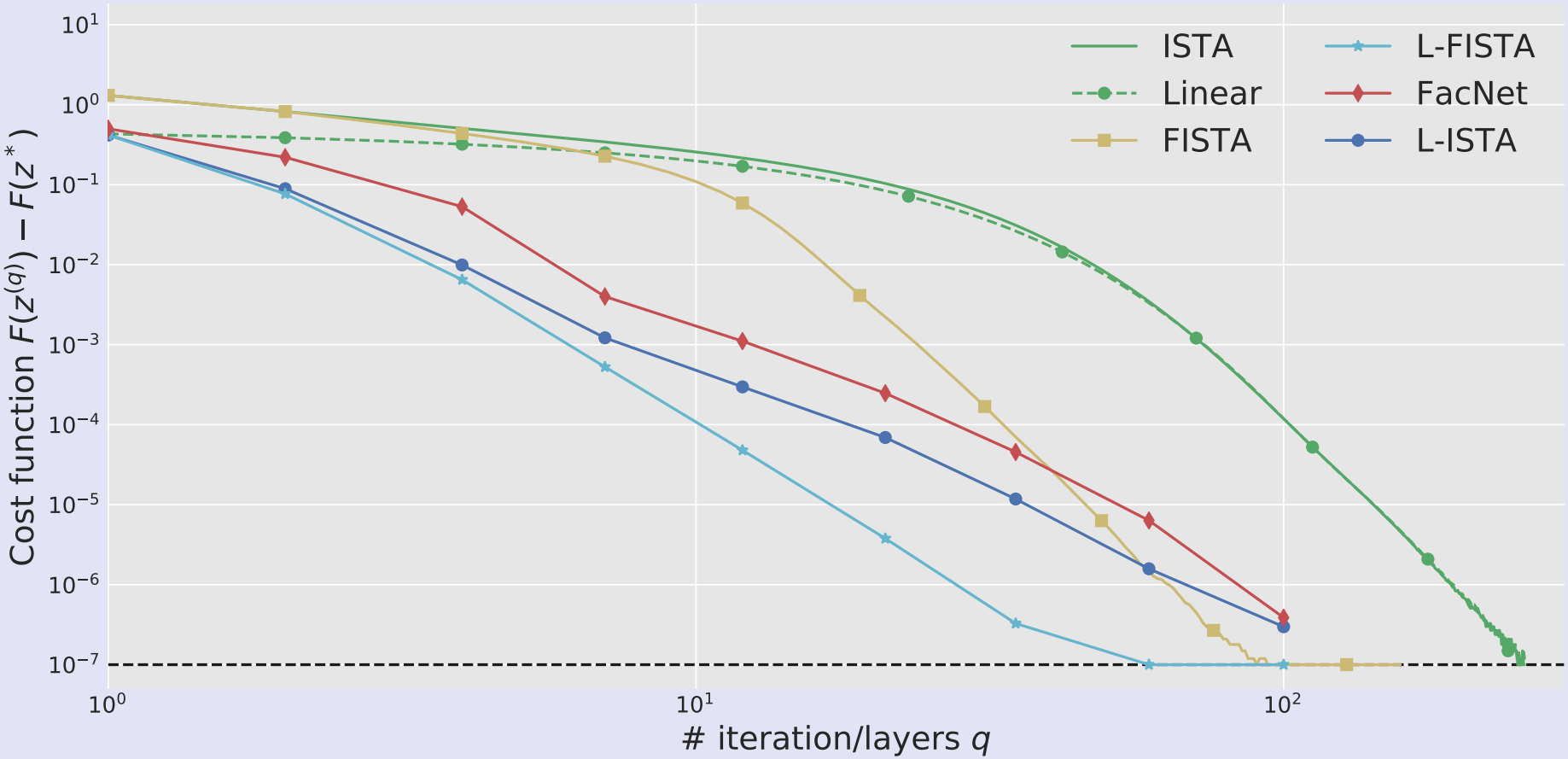
$$\lambda \mathbb{E}_z \left[\|z^{(q+1)}\|_1 + \|z^*\|_1 \right] \leq \sqrt{\frac{K(K-1)}{p}} \underbrace{\mathbb{E}_z \left[\|z^{(q)} - z^*\|_2^2 \right]}_{\substack{\text{expected resolution} \\ \text{at iteration } q}}$$

FacNet can improve the performances compared to ISTA when this is verified.

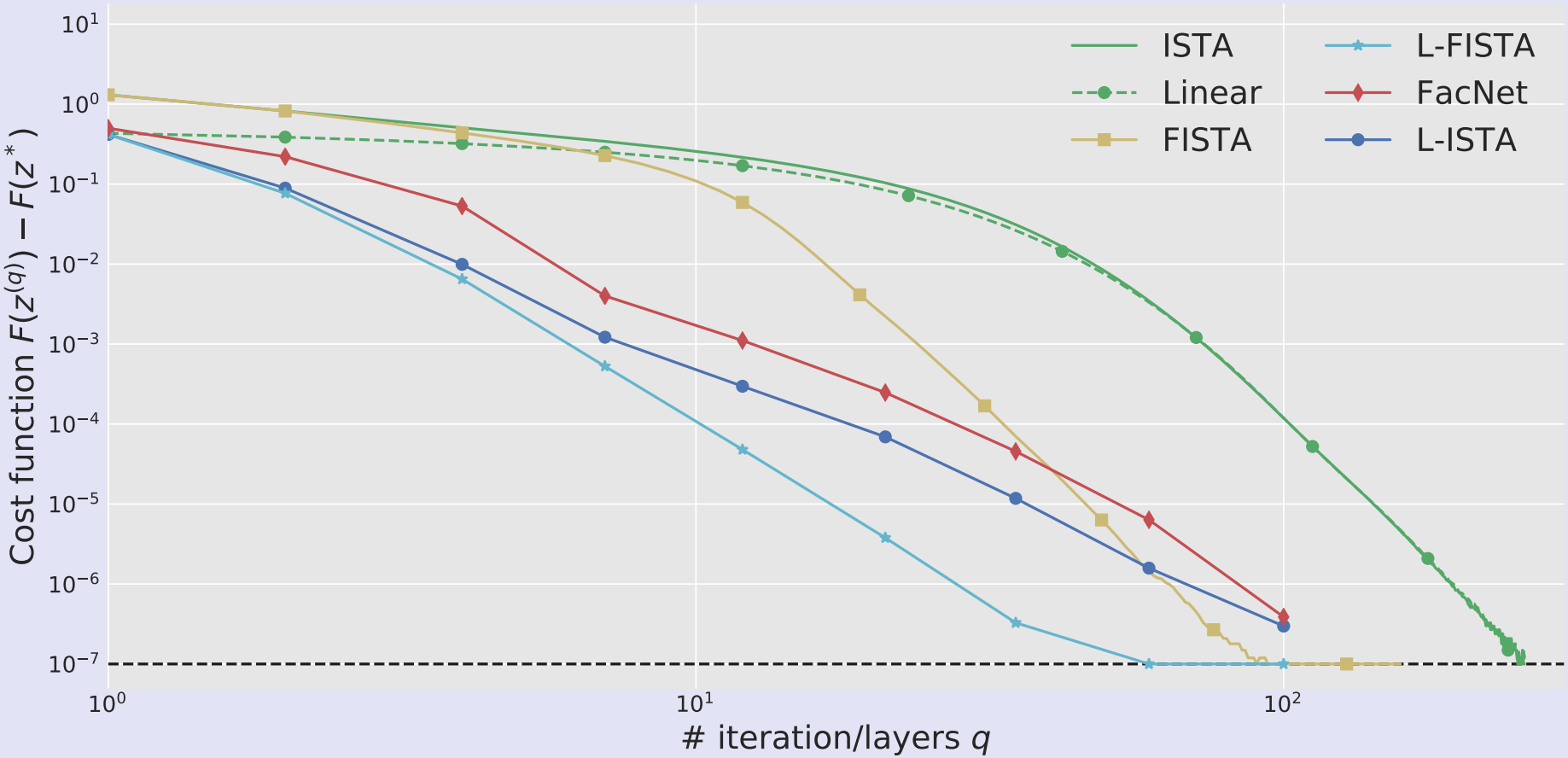
L-FISTA



Network architecture for L-FISTA.

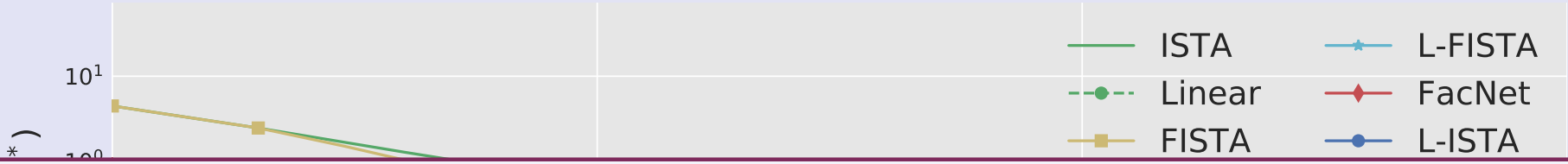


Evolution of the cost function $F(z^{(q)}) - F(z^*)$ with the number of layers/iterations q with a denser model

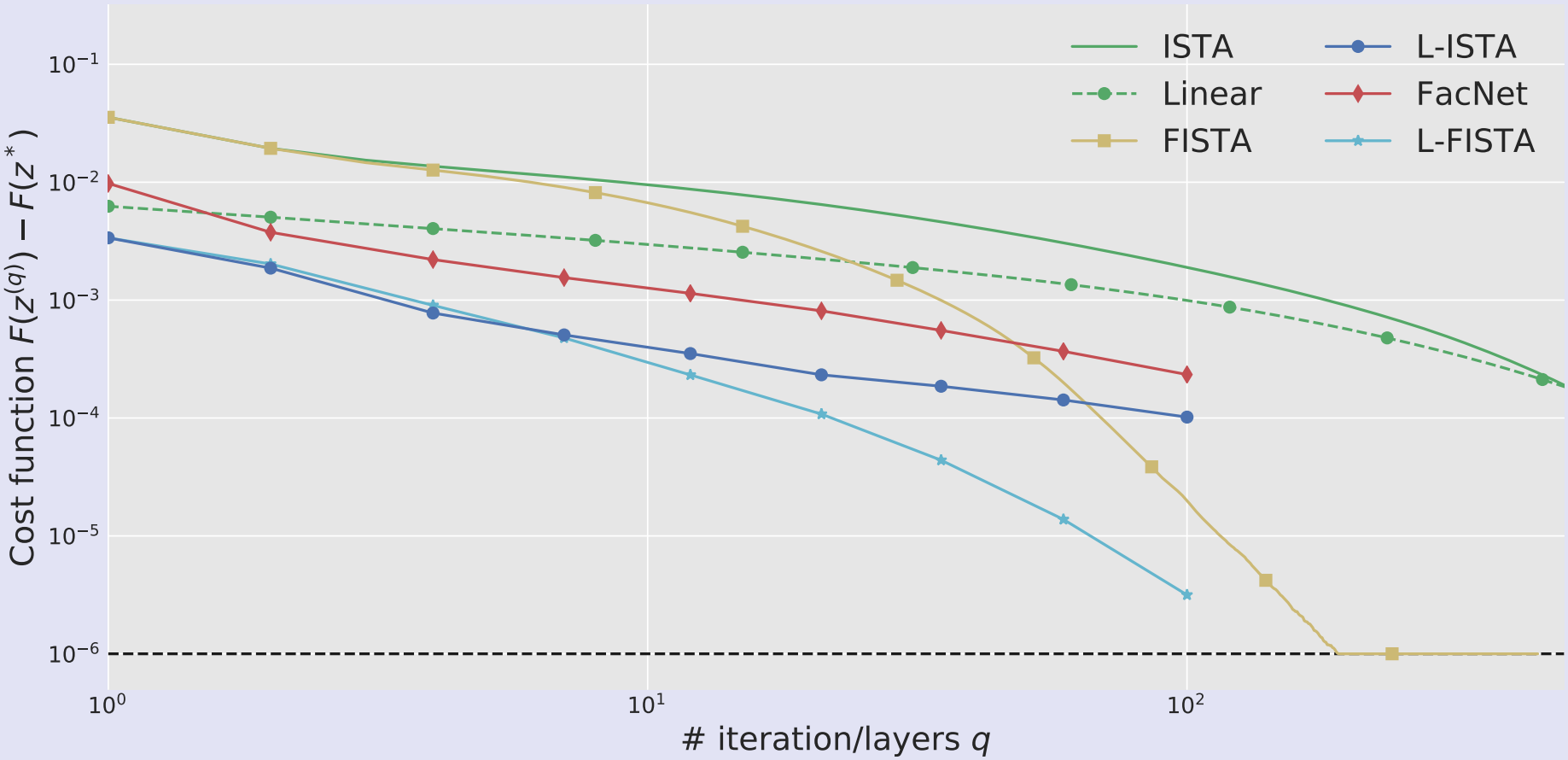


$$\rho = {}^{1}/_{20}.$$

Evolution of the cost function $F(z^{(q)}) - F(z^*)$ with the number of layers/iterations q with a denser model



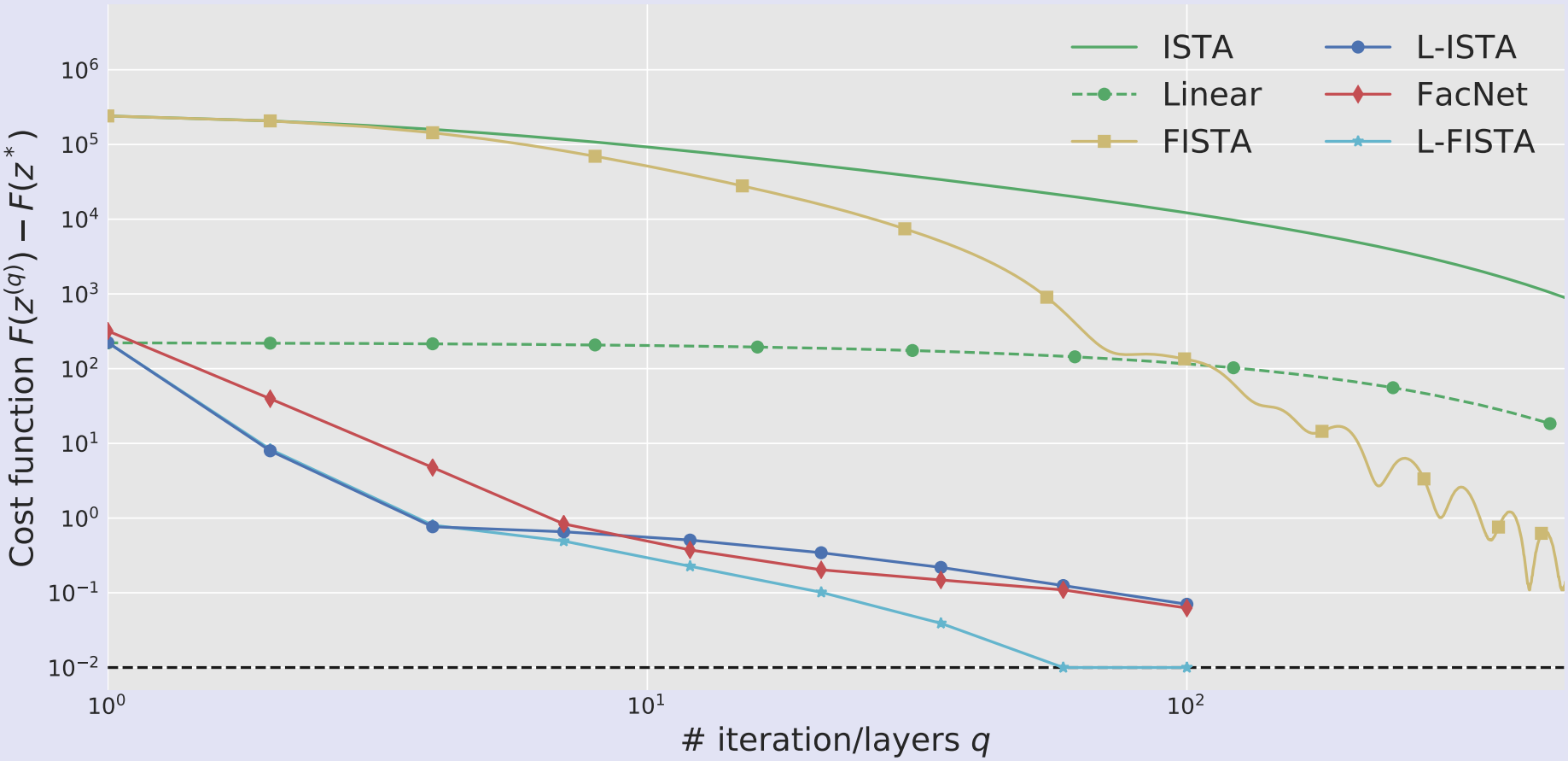
PASCAL 08



Evolution of the cost function $F(z^{(q)}) - F(z^*)$ with the number of layers

MNIST

Dictionary D with $K = 100$ atoms learned on 10 000 MNIST samples (17x17) with dictionary learning. LISTA trained with MNIST training set and tested on MNIST test set.



Finishing the process in a distributed environment

Non trivial point: **How to decide that the algorithm has converged?**

- Neighbors paused is not enough!
- Define a master 0 and send probes.
  Wait for M probes return.
- Uses the notion of message queue and network flow.
  Maybe we can have better way?

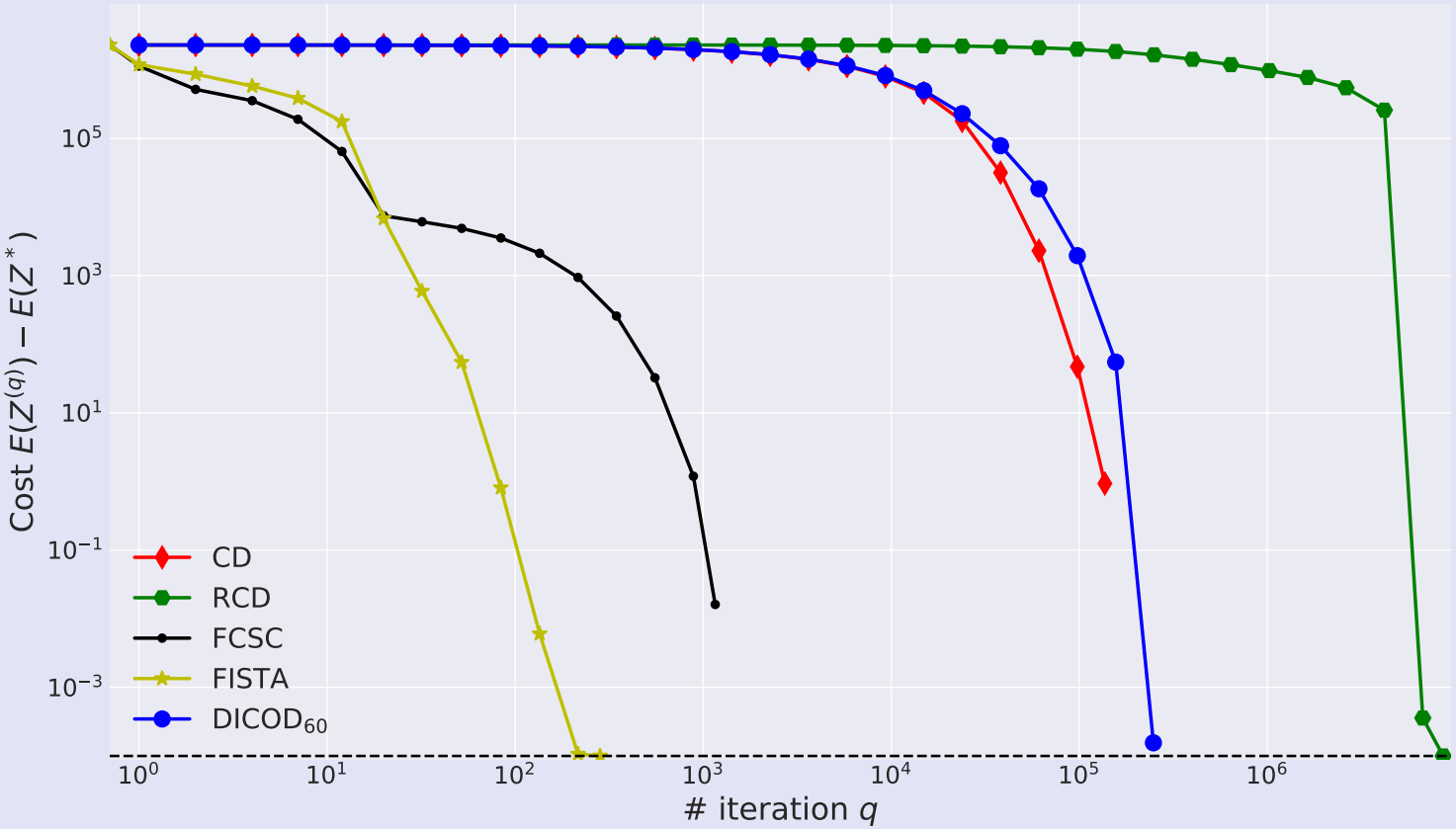
Numerical Experiments

Test on long signals generated with Bernoulli-Gaussian coding signal Z and a Gaussian dictionary $\boldsymbol{D}$. Fixed $K = 25$, $W = 200$ and $T = 600 * W$,

Algorithms implemented for benchmark

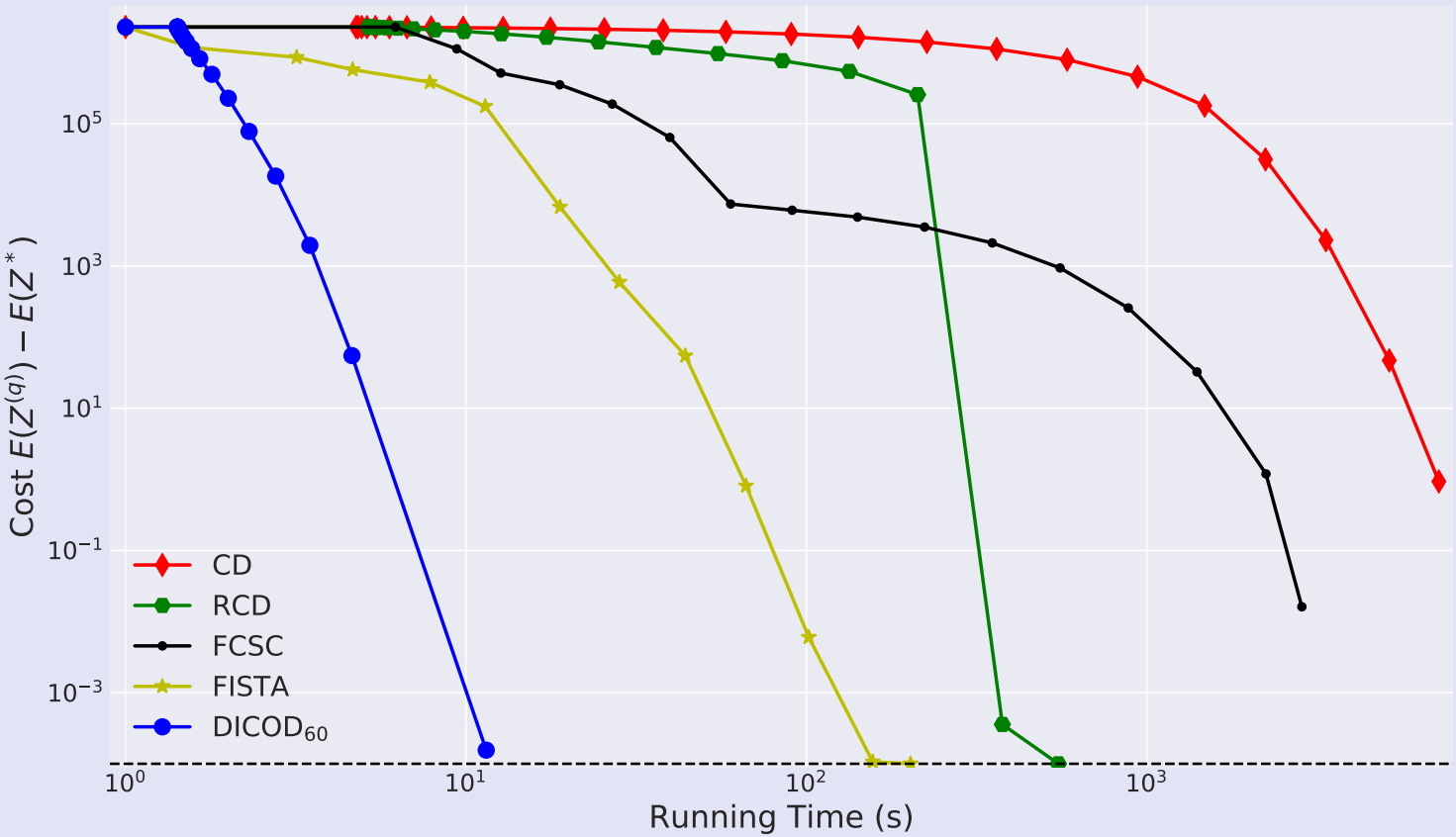
- Coordinate Descent (CD)
  [Kavukcuoglu et al., 2010]
- Randomized Coordinate Descent (RCD)
  [Nesterov, 2010]
- Fast Convolutional Sparse Coding (FCSC)
  [Bristow et al., 2013]
- Fast Ierative Soft-Thresholding Algorithm (FISTA)
  [Chalasani et al., 2013; Wohlberg, 2016]
- DICOD with 60 cores

Numerical convergence



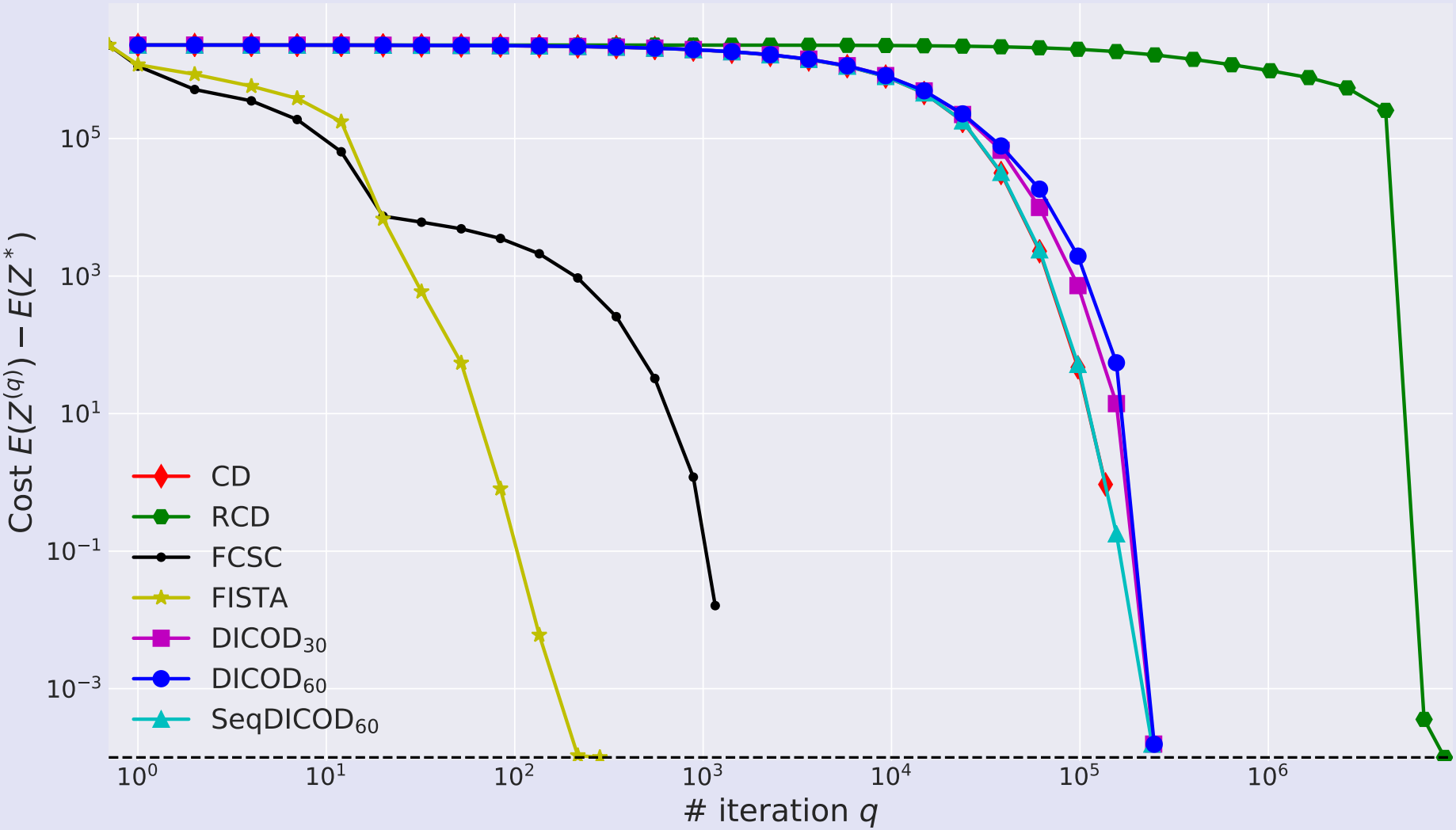
Cost as a function of the iterations

Numerical convergence



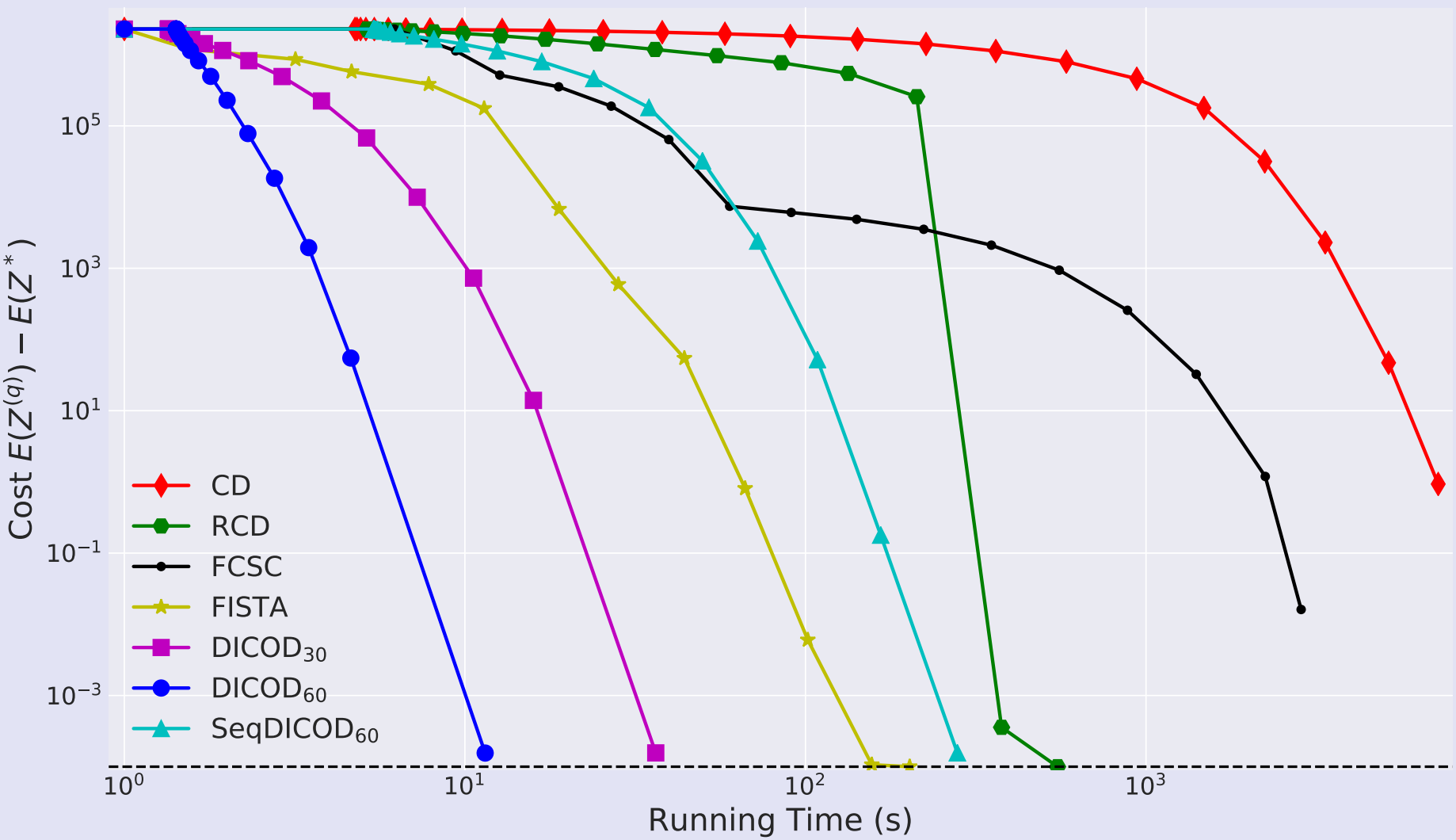
Cost as a function of the runtime

DICOD: numerical convergence



Cost as a function of the iterations

DICOD: numerical convergence



Cost as a function of the time

Complexity Analysis

Two sources of acceleration:

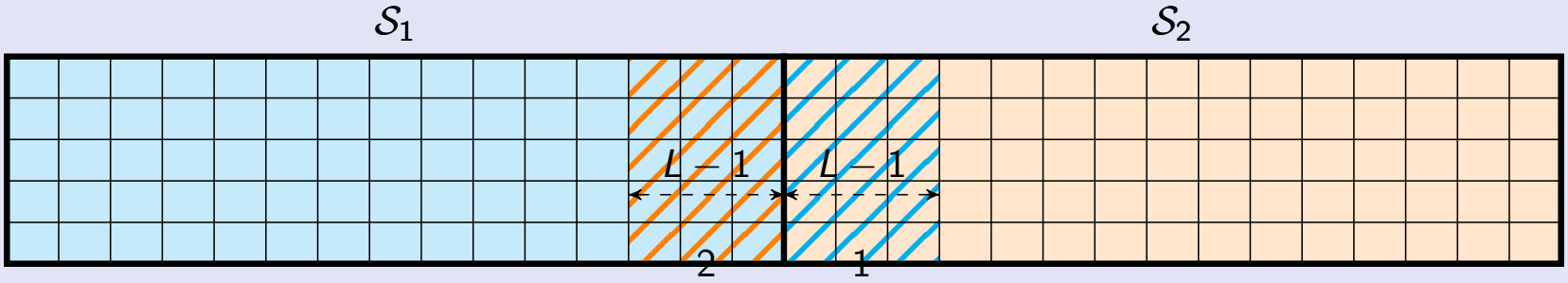
- Perform M updates in parallel,
- Each update is computed on a segment of size $\frac{L}{M}$
  Iteration complexity of $\mathcal{O}\left(K\frac{L}{M}\right)$ instead of $\mathcal{O}\left(KL\right)$

Limitations:

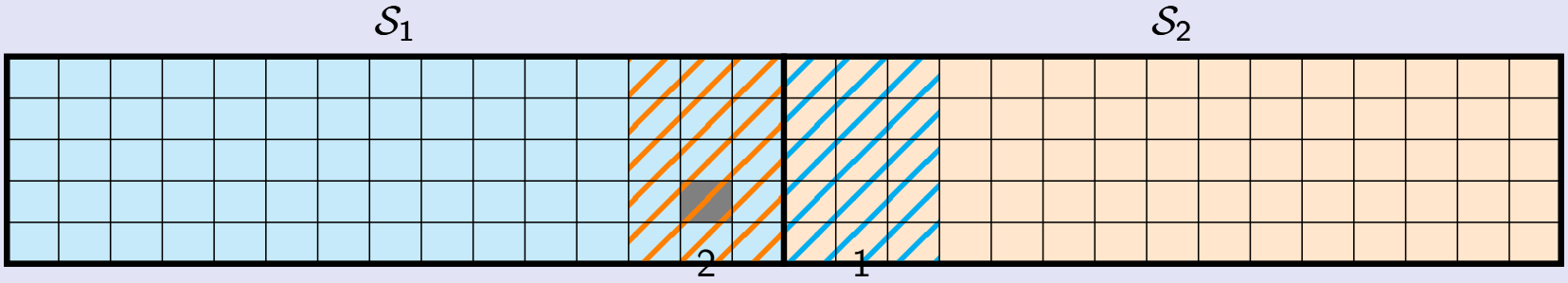
- Interfering updates, with probability $\alpha^2 = \left(\frac{WM}{T}\right)^2$

$$\mathbb{E}[Q_{dicod}] \underset{\alpha \to 0}{\gtrsim} M(1 - 2\alpha^2 M^2 + \mathcal{O}(\alpha^4 M^4)) \ .$$

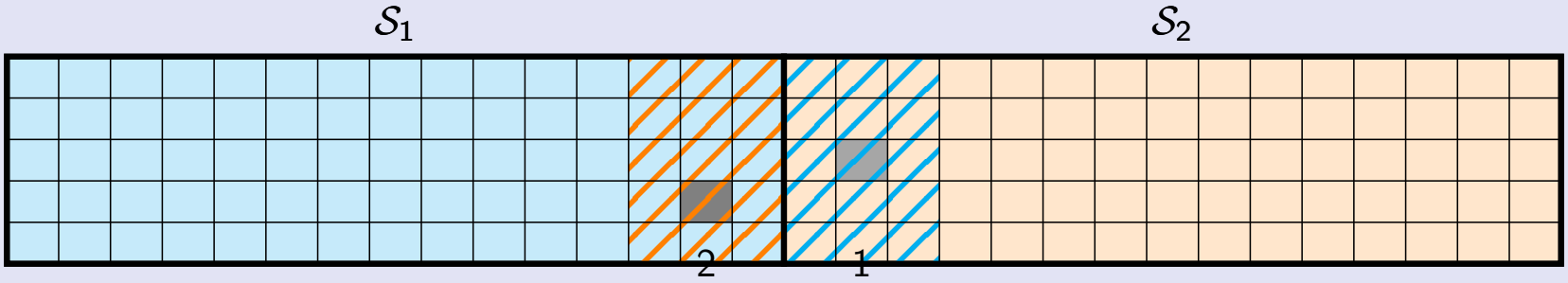
- Cost of the update of β in $\mathcal{O}\left(KW\right)$



- Keep track of the value of the optimal update in an extended zone of size $L-1$.

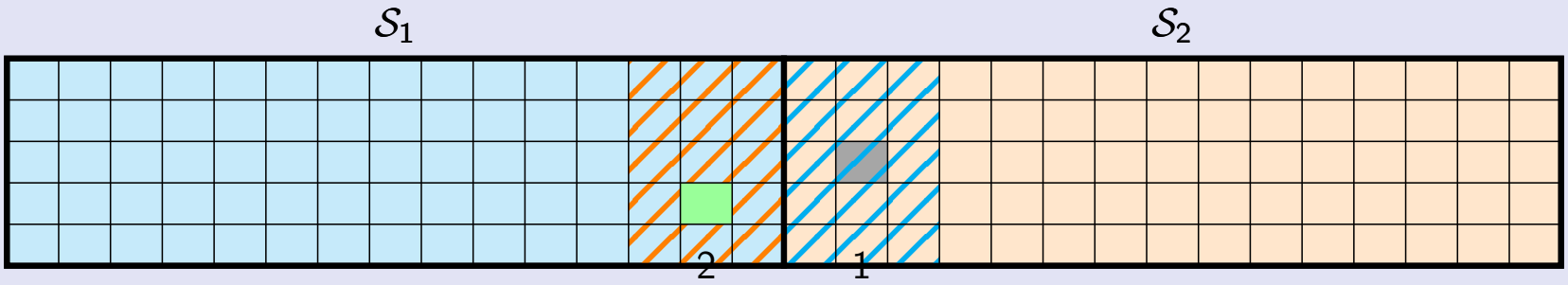


- Keep track of the value of the optimal update in an extended zone of size $L - 1$.
- Select an update candidate with LGCD.



- Keep track of the value of the optimal update in an extended zone of size $L - 1$.
- Select an update candidate with LGCD.
- If it is in the interfering zone, compare the value of the update with the value potential updates in the other worker.

Soft-Locks [Moreau and Gramfort, 2019]



- Keep track of the value of the optimal update in an extended zone of size $L - 1$.
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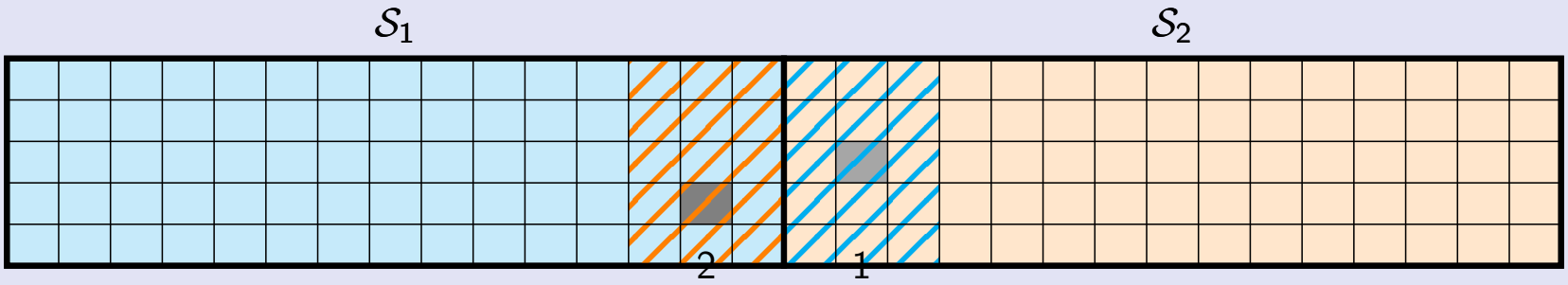
- Only perform the udpate if it is larger than the other update.



- Keep track of the value of the optimal update in an extended zone of size $L - 1$.
- Select an update candidate with LGCD.
- If it is in the interfering zone, compare the value of the update with the value potential updates in the other worker.
- Only perform the udpate if it is larger than the other update.

$\Rightarrow$ Give an update order asynchronously.

Pattern recovery

Evolution of the recovery loss with σ for different values of P. Using more channels improves the recovery of the original patterns.